Synchronization of Tripods

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In the present talks:

n: a nonnegative integer
(g, r): a pair of nonnegative integers s.t. $2 - 2g - r < 0$
k: an algebraically closed field of characteristic zero
$S \stackrel{\text{def}}{=} \text{Spec}(k)$
$S^{\log}$: the fs log scheme obtained by equipping S with
the fs log structure determined by $\mathbb{N} \to k$, $1 \mapsto 0$
$X^{\log}$: a stable log curve/$S^{\log}$ of type (g, r)
$\mathcal{G}$: the semi-graph of anabelioids det'd by the stable log curve $X^{\log}$ over $S^{\log}$

$E' \subseteq E \subseteq \{\text{positive integers} \leq n\}$

$X_E^{\log}$: the $\#E$-th log configuration space of $X^{\log}$,
where we think of the factors as being labeled by the elements of E
$p_{E/E'}^{\log} \colon X_E^{\log} \to X_{E'}^{\log}$: the natural projection morphism
$\Pi_E \stackrel{\text{def}}{=} \text{Ker}(\pi_1(X_E^{\log}) \to \pi_1(S^{\log}))$
$p_{E/E'} \colon \Pi_E \twoheadrightarrow \Pi_{E'}$: the outer surjective continuous homomorphism induced by $p_{E/E'}^{\log}$

$0 \leq j \leq i \leq n$

$X_i^{\log} \stackrel{\text{def}}{=} X_{\{\text{positive integers } \leq i\}}^{\log}$
$p_{i/j}^{\log} \stackrel{\text{def}}{=} p_{\{\text{positive integers } \leq i\}/\{\text{positive integers } \leq j\}}^{\log} \colon X_i^{\log} \to X_j^{\log}$
$\Pi_i \stackrel{\text{def}}{=} \Pi_{\{\text{positive integers } \leq i\}}$
$p_{i/j} \stackrel{\text{def}}{=} p_{\{\text{positive integers } \leq i\}/\{\text{positive integers } \leq j\}} \colon \Pi_i \to \Pi_j$
$\Pi_{i/j} \stackrel{\text{def}}{=} \text{Ker}(p_{i/j})$
$\Rightarrow \mathfrak{S}_n \curvearrowright X_n^{\log}$
$\Rightarrow \mathfrak{S}_n \stackrel{\text{out}}{\curvearrowright} \Pi_n$

$i \in E \subseteq \{\text{positive integers} \leq n\}$
$x\colon S \to X_n$: an S-valued geometric point

$$x_E\colon S_{E,x} \stackrel{\text{def}}{=} S \stackrel{x}{\to} X_n \stackrel{p^{\log}_{\{\text{positive integers } \leq n\}/E}}{\to} X_E$$

$x_E^{\log}\colon S_{E,x}^{\log} \to X_E^{\log}$: the strict morphism determined by
the morphism $x_E\colon S_{E,x} \to X_E$ and the log structure of $X_E^{\log}$

$X_{i \in E,x}^{\log}$: the stable log curve over $S_{E\setminus\{i\}}^{\log}$ obtained by forming
the fiber product of $x_{E\setminus\{i\}}^{\log}\colon S_{E\setminus\{i\}}^{\log} \to X_{E\setminus\{i\}}^{\log}$ and $p_{E/(E\setminus\{i\})}^{\log}\colon X_E^{\log} \to X_{E\setminus\{i\}}^{\log}$

$\mathcal{G}_{i \in E,x}$: the semi-graph of anabelioids det'd by the stable log curve $X_{i\in E,x}^{\log}$ over $S_{E\setminus\{i\}}^{\log}$

$\Rightarrow$ $\exists$a natural Π_E-conjugacy class of continuous isom. $\Pi_{\mathcal{G}_{i\in E,x}} \stackrel{\sim}{\to} \Pi_{E/(E\setminus\{i\})}$ $(\subseteq \Pi_E)$
Fix a cont. isom. $\Pi_{\mathcal{G}_{i\in E,x}} \stackrel{\sim}{\to} \Pi_{E/(E\setminus\{i\})}$ that is contained in this natural Π_E-conj. class

Definition

$E \subseteq \{\text{positive integers} \leq n\}$

an <u>E-tripod</u> of $X_n^{\log} \overset{\text{def}}{\Leftrightarrow}$
an irreducible components of $X_{i\in E,x}^{\log}$ of type $(0,3)$ for some $i \in E$ and x

an <u>E-tripod</u> of $\Pi_n \overset{\text{def}}{\Leftrightarrow}$
a closed subgroup of $\Pi_{\mathcal{G}_{i\in E,x}} \xrightarrow{\sim} \Pi_{E/(E\setminus\{i\})}$ $(\subseteq \Pi_E)$ obtained by forming
a verticial subgroup associated to an E-tripod of $X_n^{\log}$

Tripod Synchronization

= synchronization among the various tripods of Π_n
$\Rightarrow$ an outer continuous automorphism of Π_n typically induces
the <u>same outer continuous automorphism</u> on the various tripods of Π_n

Definition

$E \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod

$\mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$: the subgp consisting of F-adm. conti. outom. of Π_n
s.t. the induced conti. outom. of Π_E preserve the Π_E-conjugacy class of $T \subseteq \Pi_E$

$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T] \overset{\text{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T]$

$\mathrm{Out}^{|\mathrm{C}|}(T) \subseteq \mathrm{Out}(T)$: the subgroup consisting of conti. outomorphisms of T that
induce the identity autom. on the set of conj. classes of cusp. inertia subgps of T

$\mathrm{Out}(T)^{\Delta} \overset{\text{def}}{=} Z_{\mathrm{Out}(T)}(\mathfrak{S}_3)$ Note: $\mathfrak{S}_3 \cong \mathrm{Aut}_S(\text{a tpd}_{/S}) \hookrightarrow \mathrm{Out}(\pi_1(\text{a tpd}_{/S}))$

$\mathrm{Out}^{|\mathrm{C}|}(T)^{\Delta} \overset{\text{def}}{=} \mathrm{Out}^{|\mathrm{C}|}(T) \cap \mathrm{Out}(T)^{\Delta}$

Example 1/3

an irreducible component of $X^{\log}$ of type (0,3) is a $\underline{\{*\}\text{-tripod}}$

Example-Definition 2/3

$E \subseteq \{\text{positive integers} \leq n\}$
i, $j \in E$: distinct

If the image of $x_{E\setminus\{i\}}$ is a cusp (resp. node) ν of $X^{\log}_{j\in E\setminus\{i\},x}$,
then $X^{\log}_{i\in E,x}$ has the "new/minor irreducible component", which is an $\underline{E\text{-tripod}}$

the $\underline{E\text{-tripod that arises from the cusp}}$ (resp. $\underline{\text{node}}$) $\underline{\nu}$

Example-Definition 3/3

i, j, $l \in \{\text{positive integers} \leq n\}$: distinct

$\Rightarrow$ the log stable curve $X^{\log}_{j\in\{j,l\},x}$ over $S^{\log}_{\{l\}}$ has the cusp
that arises from the diagonal divisor in the second log confi. space $X^{\log}_{\{j,l\}}$
$\Rightarrow$ the log stable curve $X^{\log}_{i\in\{i,j,l\},x}$ over $S^{\log}_{\{j,l\}}$ has the $\underline{\{i,j,l\}\text{-tripod}}$
that arises from this "diagonal cusp" of $X^{\log}_{j\in\{j,l\},x}$

the $\underline{i\text{-central } \{i,j,l\}\text{-tripod}}$

Theorem in [CmbCsp] — cf. [CmbCsp], Corollary 1.10, (i), (ii)

T: a central E-tripod of Π_n
Then:

- the equality $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T]$ holds
- the equalities $C_{\Pi_E}(T) = N_{\Pi_E}(T) = T \times Z_{\Pi_E}(T)$ hold

Review

In the proof of a theorem of [NodNon]:

$$\begin{array}{ccccc} & & \mathrm{Out}^{\mathrm{FC}}(\Pi_3) & \longrightarrow & \mathrm{Out}(\text{a central tripod}) \\ & \nearrow & \Big\downarrow \text{injective!} & & \\ \text{the abs. Gal. gp} & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_1) & & \end{array}$$

(cf. the talk by Minamide)

how to define the right-hand upper hor'l arrow "$\mathrm{Out}^{\mathrm{FC}}(\Pi_3) \to \mathrm{Out}(\text{a central tripod})$"

Remark

G: a group
$H \subseteq G$: a subgroup

$\alpha \in \mathrm{Aut}(G)$
$\Rightarrow$ can define the restriction $\alpha|_H \in \mathrm{Aut}(H)$ whenever α preserves $H \subseteq G$

On the other hand:
$\alpha \in \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$
$\Rightarrow$ cannot define the "restriction" $\alpha|_H \in \mathrm{Out}(H)$ in general
even if α preserves the conjugacy class of $H \subseteq G$
the "natural restriction" is
not $\in \mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ but $\in \mathrm{Aut}(H)/\mathrm{Inn}(N_G(H))$

In particular:

- the outomorphism α preserves the conjugacy class of $H \subseteq G$
- the equality $N_G(H) = Z_G(H) \cdot H$ holds

$\Rightarrow$ can define the restriction $\alpha|_H \in \mathrm{Out}(H)$

(cf. the talk by Iijima)

Theorem in [CmbCsp]

T: a central E-tripod of Π_n
Then:

- the equality $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T]$ holds
- the equalities $C_{\Pi_E}(T) = N_{\Pi_E}(T) = T \times Z_{\Pi_E}(T)$ hold

Theorem (weak-F-ctr) — cf. [CbTpII], Theorem 3.16, (v)

T: a central E-tripod of Π_n
$\Rightarrow$ the equality $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T]$ holds

Theorem (C(tpd)) — cf. [CbTpII], Theorem C, (i)

$E \subseteq \{$positive integers $\leq n\}$
$T \subseteq \Pi_E$: an E-tripod
$\Rightarrow$ the equalities $C_{\Pi_E}(T) = N_{\Pi_E}(T) = T \times Z_{\Pi_E}(T)$ hold

Corollary-Definition

$E \subseteq \{$positive integers $\leq n\}$
$T \subseteq \Pi_E$: an E-tripod

$\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] \to \mathrm{Out}(T)$: the restriction homomorphism
(well-defined by Theorem (C(tpd)))
the tripod homomorphism associated to T

$\Rightarrow$ If T is central, then we have $\mathfrak{T}_T \colon \mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{Out}(T)$ (cf. Theorem (weak-F-ctr))

$\mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T]$: the pull-back of $\mathrm{Out}^{|\mathrm{C}|}(T) \subseteq \mathrm{Out}(T)$ by $\mathfrak{T}_T$

$\mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : \Delta] \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T]$: the pull-back of $\mathrm{Out}(T)^{\Delta} \subseteq \mathrm{Out}(T)$ by $\mathfrak{T}_T$

$\mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : |\mathrm{C}|, \Delta] \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : |\mathrm{C}|] \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : \Delta]$

$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : \Box] \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : \Box]$

Proof of Theorem (F-ctr) 1/2

Theorem (weak-F-ctr) — cf. [CbTpII], Theorem 3.16, (v)

T: a central E-tripod of Π_n
$\Rightarrow$ the equality $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T]$ holds

More strongly:

Theorem (F-ctr) — cf. [CbTpII], Theorem 3.16, (v)

T: a central E-tripod of Π_n
$\Rightarrow$ the equality $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : \Delta]$ holds

similar to the case of "FC"
(cf. the talk by Minamide)

Proof of Theorem (F-ctr) 2/2

Theorem in [CbTpI] — cf. [CbTpI], Theorem A, (ii)

the image of $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$ is contained in $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) \subseteq \mathrm{Out}^{\mathrm{F}}(\Pi_n)$

Lemma (ConfiGC)

$1 \leq i \leq n-1$ $\quad c$, c': cusps of $X_{i/i-1}$
$I_\square \subseteq \Pi_{i/i-1}$: a cuspidal inertia subgroup associated to $\square$
$\mathcal{H}_\square$: the semi-graph of anabelioids of PSC-type det'd by the log geom. fiber at $\square$
$Y_\square \in \mathrm{Vert}(\mathcal{H}_\square)$: the vertex that corresponds to the "old/major irr. component"
$P_\square \in \mathrm{Vert}(\mathcal{H}_\square)$: the vertex that corresponds to the "new/minor irr. component"
Suppose: the autom. α_i and $\alpha_{i+1/i}$ fit into a commutative diagram

$$\begin{array}{ccccccc} I_c & \hookrightarrow & \Pi_i & \longrightarrow & \mathrm{Out}(\Pi_{i+1/i}) & \xrightarrow{\sim} & \mathrm{Out}(\Pi_{\mathcal{H}_c}) \\ \wr\,\vdots & & \wr\downarrow \alpha_i & & \wr\downarrow \text{conjugation by } \alpha_{i+1/i} & & \\ I_{c'} & \hookrightarrow & \Pi_i & \longrightarrow & \mathrm{Out}(\Pi_{i+1/i}) & \xrightarrow{\sim} & \mathrm{Out}(\Pi_{\mathcal{H}_{c'}}) \end{array}$$

$\Rightarrow$ the images $\alpha_{i+1/i}(\Pi_{Y_c})$, $\alpha_{i+1/i}(\Pi_{P_c})$ are $\Pi_{i+1/i}$-conjugates of $\Pi_{Y_{c'}}$, $\Pi_{P_{c'}}$, respectively

$\alpha \in \mathrm{Aut}^{\mathrm{F}}(\Pi_n)$
$I \subseteq \Pi_{2/1}$: a cuspidal inertia subgroup associated to the "diagonal cusp"
$T \subseteq \Pi_{3/2}$: a 3-central $\{1,2,3\}$-tripod
$\overset{\text{Theorem in [CbTpI]}}{\Rightarrow}$ $\alpha_{2/1} \curvearrowright \Pi_{2/1}$ is "compatible" with the cuspidal inertia subgps

Observe: if $J \subseteq \Pi_{2/1}$ is cuspidal, then:
$J \sim_{\text{conj.}} I \Leftrightarrow$ the image of J by $\Pi_{2/1} \hookrightarrow \Pi_2 \overset{p_{\{1,2\}/\{2\}}}{\twoheadrightarrow} \Pi_{\{2\}}$ is nontrivial
$\Rightarrow \alpha_{2/1}(I) \sim_{\text{conj.}} I$

By replacing α by a suitable Π_n-conjugate of α,
we may assume: $\alpha_{2/1}(I) = I$
$\overset{\text{Lemma (ConfiGC)}}{\Rightarrow}$ $\alpha_{3/2} \curvearrowright \Pi_{3/2}$ preserves the $\Pi_{3/2}$-conjugacy class of $T \subseteq \Pi_{3/2} \subseteq \Pi_3$

Moreover, by Theorem $(Z(\mathfrak{S}))$, α_3 centalizes $\mathfrak{S}_3 \Rightarrow \alpha_{3/2}|_T \in \mathrm{Out}(T)^\Delta$

Proof of Theorem (C(tpd))

Theorem (C(tpd))

$E \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
$\Rightarrow$ the equalities $C_{\Pi_E}(T) = N_{\Pi_E}(T) = T \times Z_{\Pi_E}(T)$ hold

Lemma (C(tpd)) — cf. [CbTpII], Lemma 3.8, (i), (ii)

$E \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
Then one of the following three conditions is satisfied:

(1) $\exists i \in E$ s.t. the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i\}}}{\twoheadrightarrow} \Pi_{\{i\}}$ is an $\{i\}$-tripod
(2) $\exists i,\ j \in E$: distinct $\exists \nu$: a cusp or node of $X^{\log}$ s.t.
the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i,j\}}}{\twoheadrightarrow} \Pi_{\{i,j\}}$ is an $\{i,j\}$-tripod that arises from ν
(3) $\exists i,\ j,\ l \in E$: distinct s.t.
the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i,j,l\}}}{\twoheadrightarrow} \Pi_{\{i,j,l\}}$ is a central $\{i,j,l\}$-tripod

proof, omit

Hint:
$i_T \in E$ be s.t. $T \subseteq \Pi_{E/(E\setminus\{i_T\})}$
$\Rightarrow$ $\exists$a natural bijective map
$\{\text{cusps of } X^{\log}_{i_T \in E,x}\} \xrightarrow{\sim} \{\text{cusps of } X^{\log}\} \cup (E \setminus \{i_T\})$

Theorem (C(tpd))

$E \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
$\Rightarrow$ the equalities $C_{\Pi_E}(T) = N_{\Pi_E}(T) = T \times Z_{\Pi_E}(T)$ hold

$C_{\Pi_E}(T) \subseteq N_{\Pi_E}(T)$

$i \in E$ be s.t. $T \subsetneq \Pi_{E/(E\setminus\{i\})}$
$\alpha \in C_{\Pi_E}(T)$
$\gamma \in C_{\Pi_{E/E\setminus\{i\}}}(T)$
$C_{\Pi_{E/E\setminus\{i\}}}(T) \subseteq C_{\Pi_E}(T) = C_{\Pi_E}(T \cap T^\alpha) = C_{\Pi_E}(T^\alpha)$
$\Rightarrow T^{\alpha\gamma} \sim_{\mathrm{cmm}} T^\alpha$
$\Rightarrow T^{\alpha\gamma\alpha^{-1}} \sim_{\mathrm{cmm}} T$
$\Rightarrow \alpha\gamma\alpha^{-1} \in C_{\Pi_{E/E\setminus\{i\}}}(T)$
$\Rightarrow C_{\Pi_E}(T) \subseteq N_{\Pi_E}(C_{\Pi_{E/E\setminus\{i\}}}(T)) \overset{\text{cmm. trm.}}{=} N_{\Pi_E}(T)$

it suffices to verify: $N_{\Pi_E}(T) \subseteq T \cdot Z_{\Pi_E}(T)$, or, equivalently

$$\begin{array}{ccc} N_{\Pi_E}(T) & \xrightarrow{\text{conjugation action}} & \mathrm{Aut}(T) \\ & \searrow & \uparrow \\ & & \mathrm{Inn}(T) \end{array}$$

Observe:
If the image T' of $T \hookrightarrow \Pi_E \overset{p_{E/E'}}{\twoheadrightarrow} \Pi_{E'}$ is an E'-tripod for $E' \subseteq E$, then:

$$\begin{array}{ccccc} N_{\Pi_E}(T) & \xrightarrow{\text{conjugation action}} & \mathrm{Aut}(T) & \hookleftarrow & \mathrm{Inn}(T) \\ \downarrow & & \downarrow \wr & & \downarrow \wr \\ N_{\Pi_{E'}}(T') & \xrightarrow[\text{conjugation action}]{} & \mathrm{Aut}(T') & \hookleftarrow & \mathrm{Inn}(T') \end{array}$$

Proof of: Lemma (C(tpd)) $\Rightarrow$ Theorem (C(tpd)) 2/4

Lemma (C(tpd))

$E \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
Then one of the following three conditions is satisfied:

(1) $\exists i \in E$ s.t. the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i\}}}{\twoheadrightarrow} \Pi_{\{i\}}$ is an $\{i\}$-tripod
(2) $\exists i, j \in E$: distinct $\quad \exists \nu$: a cusp or node of $X^{\log}$ s.t.
the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i,j\}}}{\twoheadrightarrow} \Pi_{\{i,j\}}$ is an $\{i,j\}$-tripod that arises from ν
(3) $\exists i, j, l \in E$: distinct s.t.
the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i,j,l\}}}{\twoheadrightarrow} \Pi_{\{i,j,l\}}$ is a central $\{i,j,l\}$-tripod

Thus, we may assume:
we are in the situation of one of the three cases of Lemma (C(tpd))

the case of (1):
the closed subgroup $T \subseteq \Pi_1$ is commensurably terminal
(cf. the talk by Yamashita)

the case of (3):
by

Theorem in [CmbCsp]

T: a central E-tripod of Π_n
Then:

- the equality $\mathrm{Out}^{\mathrm{FC}}(\Pi_n) = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T]$ holds
- the equalities $C_{\Pi_E}(T) = N_{\Pi_E}(T) = T \times Z_{\Pi_E}(T)$ hold

Thus, we may assume:
$E = \{1, 2\}$
$T \subseteq \Pi_{2/1}$
$\exists \nu$: a cusp or node of $X^{\log}$ from which T arises
By considering a suitable generization of $X^{\log}$,
we may assume: if ν is a node, then the set of nodes of $X^{\log}$ consists of only ν

x be s.t. the image of $x_{\{1\}}$ is ν
$\mathcal{H} \stackrel{\text{def}}{=} \mathcal{G}_{2\in\{1,2\},x}$
$\Pi_\nu \subseteq \Pi_1$: an edge-like subgroup associated to ν
$\Pi_2|_\nu \stackrel{\text{def}}{=} \Pi_2 \times_{\Pi_1} \Pi_\nu$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{2/1} & \longrightarrow & \Pi_2 & \xrightarrow{p_{2/1}} & \Pi_1 & \longrightarrow & 1 \\ & & \uparrow \wr & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Pi_{\mathcal{H}} & \longrightarrow & \Pi_2|_\nu & \longrightarrow & \Pi_\nu & \longrightarrow & 1 \end{array}$$

If ν is a cusp (resp. node),
then $\Pi_\nu \to \mathrm{Out}(\Pi_{\mathcal{H}})$ is of IPSC-type (resp. of SNN-type)

c_{dg}: the "diagonal cusp" of $X^{\log}_{2\in\{1,2\},x}$
$\Pi_{c_{\mathrm{dg}}} \subseteq \Pi_{2/1} \stackrel{\sim}{\leftarrow} \Pi_{\mathcal{H}}$: an edge-like subgp associated to c_{dg} contained in T

$p_{1\backslash 2}\colon \Pi_2 \twoheadrightarrow \Pi_{\{2\}}$, $\quad \Pi_{1\backslash 2} \stackrel{\text{def}}{=} \mathrm{Ker}(p_{1\backslash 2})$

$$1 \longrightarrow \Pi_{1\backslash 2} \longrightarrow \Pi_2 \xrightarrow{p_{1\backslash 2}} \Pi_{\{2\}} \longrightarrow 1$$

$D_{c_{\mathrm{dg}}} \stackrel{\text{def}}{=} Z_{\Pi_2}(\Pi_{c_{\mathrm{dg}}})$
$I_T|_\nu \stackrel{\text{def}}{=} Z_{\Pi_2|_\nu}(T) \subseteq D_T|_\nu \stackrel{\text{def}}{=} N_{\Pi_2|_\nu}(T)$
$\Rightarrow$
(a) the equality $D_T|_\nu = T \times I_T|_\nu$ holds
(b) $D_T|_\nu \subseteq \Pi_2|_\nu$: commensurably terminal
(c) the composite $I_T|_\nu \hookrightarrow \Pi_2|_\nu \twoheadrightarrow \Pi_\nu$ is an isomorphism
(cf. the talk by Minamide)

$D_{c_{\mathrm{dg}}} \overset{\mathrm{def}}{=} Z_{\Pi_2}(\Pi_{c_{\mathrm{dg}}})$
$I_T|_\nu \overset{\mathrm{def}}{=} Z_{\Pi_2|_\nu}(T) \subseteq D_T|_\nu \overset{\mathrm{def}}{=} N_{\Pi_2|_\nu}(T)$

(a) the equality $D_T|_\nu = T \times I_T|_\nu$ holds
(b) $D_T|_\nu \subseteq \Pi_2|_\nu$: commensurably terminal
(c) the composite $I_T|_\nu \hookrightarrow \Pi_2|_\nu \twoheadrightarrow \Pi_\nu$ is an isomorphism

Step 1:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{c_{\mathrm{dg}}} & \longrightarrow & D_{c_{\mathrm{dg}}} & \longrightarrow & \Pi_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow {\scriptstyle \text{conj. of diag.}} & & \\ 1 & \longrightarrow & \Pi_{2/1} \cap \Pi_{1\backslash 2} & \longrightarrow & \Pi_2 & \xrightarrow[(p_{2/1}, p_{1\backslash 2})]{} & \Pi_1 \times \Pi_{\{2\}} & \longrightarrow & 1 \end{array}$$

by a well-known fact concerning the decomposition groups associated to cusps

Step 2: $D_T|_\nu \subseteq \Pi_2$: normally terminal

$p_{2/1}(D_T|_\nu) \overset{(a)}{=} p_{2/1}(T \times I_T|_\nu) = p_{2/1}(I_T|_\nu) \overset{(c)}{=} \Pi_\nu$
$\Rightarrow p_{2/1}(N_{\Pi_2}(D_T|_\nu)) \subseteq N_{\Pi_1}(\Pi_\nu) \overset{\text{nrm. trm.}}{=} \Pi_\nu$
$\Rightarrow N_{\Pi_2}(D_T|_\nu) \subseteq \Pi_2|_\nu$
$\Rightarrow N_{\Pi_2}(D_T|_\nu) = N_{\Pi_2|_\nu}(D_T|_\nu) \overset{(b)}{=} D_T|_\nu$

Step 3: $Z_{\Pi_2}(T) = I_T|_\nu$

$p_{1\backslash 2}(T) \sim_{\text{conj.}} \Pi_\nu \subseteq \Pi_{\{2\}}$
$\Rightarrow p_{1\backslash 2}(Z_{\Pi_2}(T)) \subseteq Z_{\Pi_{\{2\}}}(\text{a conj. of } \Pi_\nu) \overset{\text{nrm. trm.}}{\subseteq}$ the conj. of Π_ν
Thus. since $Z_{\Pi_2}(T) \subseteq D_{c_{\mathrm{dg}}}$, by Step 1,
$p_{2/1}(Z_{\Pi_2}(T)) \subseteq$ a conj. of Π_ν
Thus, $I_T|_\nu \subseteq Z_{\Pi_2}(T)$ and $p_{2/1}(I_T|_\nu) \overset{(c)}{=} \Pi_\nu$,
$p_{2/1}(Z_{\Pi_2}(T)) = \Pi_\nu$, i.e., $Z_{\Pi_2}(T) \subseteq \Pi_2|_\nu$

$N_{\Pi_2}(T) \subseteq N_{\Pi_2}(Z_{\Pi_2}(T)) \overset{\text{Step 3}}{=} N_{\Pi_2}(I_T|_\nu)$
$\Rightarrow N_{\Pi_2}(T) \subseteq N_{\Pi_2}(T \cdot I_T|_\nu) \overset{(a)}{=} N_{\Pi_2}(D_T|_\nu) \overset{\text{Step 2}}{=} D_T|_\nu \overset{(a)}{=} T \cdot I_T|_\nu$

Tripod Synchronization

= synchronization among the various tripods of Π_n
  $\Rightarrow$ an outer continuous automorphism of Π_n typically induces
    the <u>same outer continuous automorphism</u> on the various tripods of Π_n

Theorem (2-TpdSych) — cf. [CbTpII], Theorem 3.17, (i), (ii)

Suppose: $n = 2$
$E \subseteq \{1, 2\}$
$T \subseteq \Pi_E$: an E-tripod
$T_0 \subseteq \Pi_1$: a $\{1\}$-tripod
If one of the following two conditions is satisfied,
  then $\exists$a "geometric" outer continuous isomorphism $\iota \colon T \xrightarrow{\sim} T_0$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_0 : |\mathrm{C}|, \Delta] & \\ {\swarrow}^{\mathcal{T}_T} & & {\searrow}^{\mathcal{T}_{T_0}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_0) \end{array}$$

commutes

- $E = \{1, 2\}$
  $\exists$a cusp ν of the $\{1\}$-tripod from which T_0 arises
    s.t. the E-tripod T arises from ν
- $E = \{1\}$
  $\exists$a node of $X^{\log}$ that abuts to both the $\{1\}$-tripod from which T_0 arises
    and the $\{1\}$-tripod from which T arises

Theorem ((≥ 3)-TpdSych) — cf. [CbTpII], Theorem 3.18, (ii)

Suppose: $n \geq 3$
E, $E' \subseteq \{$positive integers $\leq n\}$
$T \subseteq \Pi_E$: an E-tripod
$T' \subseteq \Pi_{E'}$: an E'-tripod
$\Rightarrow$ $\exists$a "geometric" outer continuous isomorphism $\iota \colon T \xrightarrow{\sim} T'$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T' : |\mathrm{C}|] & \\ {\swarrow}^{\mathcal{T}_T} & & {\searrow}^{\mathcal{T}_{T'}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T') \end{array}$$

commutes

Lemma (TpdSych)

$T_b \subseteq \Pi_{n-1/n-2}$: a $\{1, \ldots, n-1\}$-tripod
ν: a cusp of the tripod from which T_b arises
$T_f \subseteq \Pi_{n/n-1}$: a $\{1, \ldots, n\}$-tripod that arises from ν
$\Rightarrow$

- The inclusions
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta]$
  hold
- $\exists$a "geometric" outer continuous isomorphism $\iota \colon T_f \xrightarrow{\sim} T_b$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] & \\ \swarrow^{\mathcal{T}_{T_f}} & & \searrow^{\mathcal{T}_{T_b}} \\ \mathrm{Out}(T_f) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_b) \end{array}$$

  commutes (which thus implies the inclusion
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta]$)

by a similar arg. to the arg. applied in the proof of the surjectivity portion of

Theorem in [NodNon] — cf. [NodNon], Theorem B

$n \geq 1 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is injective

$$n \geq n_{\mathrm{bij}} \stackrel{\mathrm{def}}{=} \begin{cases} 3 & r \neq 0 \\ 4 & r = 0 \end{cases}$$

$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is bijective

(cf. the talk by Minamide)

Sketch of the proof of Lemma (TpdSych)

Lemma (TpdSych)

$T_b \subseteq \Pi_{n-1/n-2}$: a $\{1,\ldots,n-1\}$-tripod
ν: a cusp of the tripod from which T_b arises
$T_f \subseteq \Pi_{n/n-1}$: a $\{1,\ldots,n\}$-tripod that arises from ν
$\Rightarrow$

- The inclusions
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta]$
  hold
- $\exists$a "geometric" outer continuous isomorphism $\iota \colon T_f \xrightarrow{\sim} T_b$ s.t.

$$\begin{array}{ccccc} & & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] & & \\ & \swarrow^{\mathfrak{T}_{T_f}} & & \searrow^{\mathfrak{T}_{T_b}} & \\ \mathrm{Out}(T_f) & & \xrightarrow[\text{conjugation by } \iota]{\sim} & & \mathrm{Out}(T_b) \end{array}$$

  commutes (which thus implies the inclusion
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta]$)

By replacing T_f by a suitable Π_n-conjugate of T_f, we may assume:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{n/n-1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow \text{(cmm. trm.)} — \text{cf. the next talk} & & \uparrow & & \\ 1 & \longrightarrow & \text{"}\Pi_{2/1} \text{ for a tpd"} & \longrightarrow & \text{"}\Pi_2 \text{ for a tpd"} & \longrightarrow & T_b & \longrightarrow & 1 \\ & & \uparrow & & & & & & \\ & & T_f & & & & & & \end{array}$$

$\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|]$
By a sim. arg. to the arg. app'd in the pf of Lemma (ConfiGC), $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
Moreover, the outom. α also pres. the conj. class of "Π_2 for a tpd" (cf. the next talk)
Thus, we may assume: $(g, r, n) = (0, 3, 2)$

Observe: If one takes a suitable $\sigma \in \mathfrak{S}_5$,
then $T_f \hookrightarrow \Pi_2 \overset{\sigma}{\xrightarrow{\sim}} \Pi_2 \overset{p_{2/1}}{\twoheadrightarrow} \Pi_1$ determines a "geometric" $T_f \xrightarrow{\sim} T_b$
The resulting $\mathrm{Out}(T_f) \xrightarrow{\sim} \mathrm{Out}(T_b)$ maps $\alpha|_{T_f} \mapsto (\sigma\alpha\sigma^{-1})_1$
$\alpha|_{T_f} \in \mathrm{Out}(T_f)^{\Delta} \quad \Rightarrow \quad (\sigma\alpha\sigma^{-1})_1 \in \mathrm{Out}(T_b)^{\Delta}$
$\overset{\text{[CmbCsp]}}{\Rightarrow}$ $\sigma\alpha\sigma^{-1}$ centralizes $\mathfrak{S}_5$
$\Rightarrow \sigma\alpha\sigma^{-1} = \alpha$
$\Rightarrow$ The resulting $\mathrm{Out}(T_f) \xrightarrow{\sim} \mathrm{Out}(T_b)$ maps $\mathfrak{T}_{T_f}(\alpha) = \alpha|_{T_f} \mapsto \alpha_1 = \mathfrak{T}_{T_b}(\alpha)$, as desired

Proof of: Lemma (TpdSych) $\Rightarrow$ Theorem (2-TpdSych)

Theorem (2-TpdSych)

Suppose: $n = 2$
$E \subseteq \{1, 2\}$
$T \subseteq \Pi_E$: an E-tripod
$T_0 \subseteq \Pi_1$: a $\{1\}$-tripod
If one of the following two conditions is satisfied,
then $\exists$a "geometric" outer continuous isomorphism $\iota\colon T \xrightarrow{\sim} T_0$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_0 : |\mathrm{C}|, \Delta] & \\ \swarrow^{\mathcal{T}_T} & & \searrow^{\mathcal{T}_{T_0}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_0) \end{array}$$

commutes

- $E = \{1, 2\}$
  $\exists$a cusp ν of the $\{1\}$-tripod from which T_0 arises
  s.t. the E-tripod T arises from ν
- $E = \{1\}$
  $\exists$a node of $X^{\log}$ that abuts to both the $\{1\}$-tripod from which T_0 arises
  and the $\{1\}$-tripod from which T arises

Lemma (TpdSych)

$T_b \subseteq \Pi_{n-1/n-2}$: a $\{1, \ldots, n-1\}$-tripod
ν: a cusp of the tripod from which T_b arises
$T_f \subseteq \Pi_{n/n-1}$: a $\{1, \ldots, n\}$-tripod that arises from ν
$\Rightarrow$

- The <u>inclusions</u>
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta]$
  hold
- $\exists$a "geometric" outer continuous isomorphism $\iota\colon T_f \xrightarrow{\sim} T_b$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] & \\ \swarrow^{\mathcal{T}_{T_f}} & & \searrow^{\mathcal{T}_{T_b}} \\ \mathrm{Out}(T_f) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_b) \end{array}$$

  commutes (which thus implies the <u>inclusion</u>
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta]$)

Proof of: Lemma (TpdSych) $\Rightarrow$ Theorem ((≥ 3)-TpdSych) 1/3

Theorem ((≥ 3)-TpdSych)

Suppose: $n \geq 3$
E, $E' \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
$T' \subseteq \Pi_{E'}$: an E'-tripod
$\Rightarrow \exists$a "geometric" outer continuous isomorphism $\iota \colon T \xrightarrow{\sim} T'$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T' : |\mathrm{C}|] & \\ \swarrow^{\mathcal{T}_T} & & \searrow^{\mathcal{T}_{T'}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T') \end{array}$$

commutes

For tripods T, T' of Π_n,
$T \sim_{\mathrm{Sych}} T' \overset{\mathrm{def}}{\Leftrightarrow} \exists$a "geometric" outer continuous isomorphism $\iota \colon T \xrightarrow{\sim} T'$ s.t. ...

Observe:
an outer continuous isomorphism $\Pi_E \xrightarrow{\sim} \Pi_{E'}$ induced by a suitable element of $\mathfrak{S}_n$
det. an outer isom. (an i-central E-tripod) $\xrightarrow{\sim}$ (an i'-central E'-tripod) as in Theorem
Moreover: $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ centralizes with $\mathfrak{S}_n \subseteq \mathrm{Out}(\Pi_n)$ (cf. Theorem ($Z(\mathfrak{S})$))
$\Rightarrow$ an i-central E-tripod $\sim_{\mathrm{Sych}}$ an i'-central E'-tripod
Thus, it suffices to show: every tripod $\sim_{\mathrm{Sych}}$ a central tripod

Proof of: Lemma (TpdSych) $\Rightarrow$ Theorem ((≥ 3)-TpdSych) 2/3

Thus, it suffices to show: every tripod $\sim_{\text{Sych}}$ a central tripod

Thus, by

Lemma (C(tpd))

$E \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
Then one of the following three conditions is satisfied:

(1) $\exists i \in E$ s.t. the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i\}}}{\twoheadrightarrow} \Pi_{\{i\}}$ is an $\{i\}$-tripod
(2) $\exists i, j \in E$: distinct $\exists \nu$: a cusp or node of $X^{\log}$ s.t.
the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i,j\}}}{\twoheadrightarrow} \Pi_{\{i,j\}}$ is an $\{i, j\}$-tripod that arises from ν
(3) $\exists i, j, l \in E$: distinct s.t.
the image of $T \hookrightarrow \Pi_E \overset{p_{E/\{i,j,l\}}}{\twoheadrightarrow} \Pi_{\{i,j,l\}}$ is a central $\{i, j, l\}$-tripod

it suffices to show:
each of the tripods of (1), (2), (3) as in Lemma $\sim_{\text{Sych}}$ a central tripod

(a) a $\{1\}$-tripod $\overset{?}{\sim}_{\text{Sych}}$ a 3-central $\{1, 2, 3\}$-tripod
(b) a $\{1, 2\}$-tripod that arises from a cusp or node ν of $X^{\log}$
$\overset{?}{\sim}_{\text{Sych}}$ a 3-central $\{1, 2, 3\}$-tripod

Proof of: Lemma (TpdSych) ⇒ Theorem ((≥ 3)-TpdSych) 3/3

(a) a $\{1\}$-tripod $\sim^{?}_{\text{Sych}}$ a 3-central $\{1,2,3\}$-tripod

(b) a $\{1,2\}$-tripod that arises from a cusp or node ν of $X^{\log}$ $\sim^{?}_{\text{Sych}}$ a 3-central $\{1,2,3\}$-tripod

Theorem (F-ctr) — cf. [CbTpII], Theorem 3.16, (v)

T: a central E-tripod of Π_n
⇒ the equality $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : \Delta]$ holds

Lemma (TpdSych)

$T_b \subseteq \Pi_{n-1/n-2}$: a $\{1,\ldots,n-1\}$-tripod
ν: a cusp of the tripod from which T_b arises
$T_f \subseteq \Pi_{n/n-1}$: a $\{1,\ldots,n\}$-tripod that arises from ν
⇒

- The inclusions
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta]$
  hold
- ∃a "geometric" outer continuous isomorphism $\iota \colon T_f \xrightarrow{\sim} T_b$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] & \\ \swarrow^{\mathcal{T}_{T_f}} & & \searrow^{\mathcal{T}_{T_b}} \\ \mathrm{Out}(T_f) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_b) \end{array}$$

  commutes (which thus implies the inclusion
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta]$)

The case of (b)

Observe:
a 3-central $\{1,2,3\}$-tripod is a tripod that arises from a cusp of
a $\{1,2\}$-tripod that arises from a cusp or node ν of $X^{\log}$
$\overset{\text{Theorem (F-ctr), Lemma (TpdSych)}}{\Rightarrow}$ (b) OK
Moreover, the equality
$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[\text{such a tpd} : |\mathrm{C}|] = \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[\text{such a tpd} : |\mathrm{C}|, \Delta]$ holds

The case of (a)

a $\{1\}$-tripod $\sim^{\text{(b), Lemma (TpdSych)}}_{\text{Sych}}$
a $\{1,2\}$-tripod that arises from a suitable cusp or node of $X^{\log}$ $\sim^{\text{(b)}}_{\text{Sych}}$
a 3-central $\{1,2,3\}$-tripod
⇒ (a) OK

Tripod Synchronization

= synchronization among the various tripods of Π_n
⇒ an outer continuous automorphism of Π_n typically induces
the <u>same outer continuous automorphism</u> on the various tripods of Π_n

Theorem (2-TpdSych)

Suppose: $n = 2$
$E \subseteq \{1, 2\}$
$T \subseteq \Pi_E$: an E-tripod
$T_0 \subseteq \Pi_1$: a $\{1\}$-tripod
If one of the following two conditions is satisfied,
then ∃a "geometric" outer continuous isomorphism $\iota\colon T \xrightarrow{\sim} T_0$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_0 : |\mathrm{C}|, \Delta] & \\ \swarrow^{\mathcal{T}_T} & & \searrow^{\mathcal{T}_{T_0}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_0) \end{array}$$

commutes

- $E = \{1, 2\}$
  ∃a cusp ν of the $\{1\}$-tripod from which T_0 arises
  s.t. the E-tripod T arises form the "new/minor irreducible component"
  of $p^{\log}_{2/1}\colon X^{\log}_2 \to X^{\log}_1$ at the cusp ν
- $E = \{1\}$
  ∃a node of $X^{\log}$ that abuts to both the $\{1\}$-tripod from which T_0 arises
  and the $\{1\}$-tripod from which T arises

Theorem ((≥ 3)-TpdSych)

Suppose: $n \geq 3$
E, $E' \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
$T' \subseteq \Pi_{E'}$: an E'-tripod
⇒ ∃a "geometric" outer continuous isomorphism $\iota\colon T \xrightarrow{\sim} T'$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T' : |\mathrm{C}|] & \\ \swarrow^{\mathcal{T}_T} & & \searrow^{\mathcal{T}_{T'}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T') \end{array}$$

commutes