Glueability of Combinatorial Cuspidalizations

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"Combinatorial Anabelian Geometry"

In the present talks:

n: a nonnegative integer
(g, r): a pair of nonnegative integers s.t. $2-2g-r<0$
k: an algebraically closed field of characteristic zero
$S \stackrel{\text{def}}{=} \mathrm{Spec}(k)$
$S^{\log}$: the fs log scheme obtained by equipping S with
the fs log structure determined by $\mathbb{N} \to k$, $1 \mapsto 0$
$X^{\log}$: a stable log curve/$S^{\log}$ of type (g, r)
$\mathcal{G}$: the semi-graph of anabelioids det'd by the stable log curve $X^{\log}$ over $S^{\log}$

$E' \subseteq E \subseteq \{\text{positive integers} \leq n\}$

$X_E^{\log}$: the $\#E$-th log configuration space of $X^{\log}$,
where we think of the factors as being labeled by the elements of E
$p_{E/E'}^{\log} \colon X_E^{\log} \to X_{E'}^{\log}$: the natural projection morphism
$\Pi_E \stackrel{\text{def}}{=} \mathrm{Ker}(\pi_1(X_E^{\log}) \to \pi_1(S^{\log}))$
$p_{E/E'} \colon \Pi_E \twoheadrightarrow \Pi_{E'}$: the outer surjective continuous homomorphism induced by $p_{E/E'}^{\log}$

$0 \leq j \leq i \leq n$

$X_i^{\log} \stackrel{\text{def}}{=} X_{\{\text{positive integers } \leq i\}}^{\log}$
$p_{i/j}^{\log} \stackrel{\text{def}}{=} p_{\{\text{positive integers } \leq i\}/\{\text{positive integers } \leq j\}}^{\log} \colon X_i^{\log} \to X_j^{\log}$
$\Pi_i \stackrel{\text{def}}{=} \Pi_{\{\text{positive integers } \leq i\}}$
$p_{i/j} \stackrel{\text{def}}{=} p_{\{\text{positive integers } \leq i\}/\{\text{positive integers } \leq j\}} \colon \Pi_i \to \Pi_j$
$\Pi_{i/j} \stackrel{\text{def}}{=} \mathrm{Ker}(p_{i/j})$
$\Rightarrow \mathfrak{S}_n \curvearrowright X_n^{\log}$
$\Rightarrow \mathfrak{S}_n \stackrel{\text{out}}{\curvearrowright} \Pi_n$

$i \in E \subseteq \{\text{positive integers} \leq n\}$
$x\colon S \to X_n$: an S-valued geometric point

$$x_E\colon S_{E,x} \stackrel{\text{def}}{=} S \stackrel{x}{\to} X_n \stackrel{p^{\log}_{\{\text{positive integers} \leq n\}/E}}{\to} X_E$$

$x^{\log}_E\colon S^{\log}_{E,x} \to X^{\log}_E$: the strict morphism determined by
the morphism $x_E\colon S_{E,x} \to X_E$ and the log structure of $X^{\log}_E$

$X^{\log}_{i\in E,x}$: the stable log curve over $S^{\log}_{E\setminus\{i\}}$ obtained by forming
the fiber product of $x^{\log}_{E\setminus\{i\}}\colon S^{\log}_{E\setminus\{i\}} \to X^{\log}_{E\setminus\{i\}}$ and $p^{\log}_{E/(E\setminus\{i\})}\colon X^{\log}_E \to X^{\log}_{E\setminus\{i\}}$

$\mathcal{G}_{i\in E,x}$: the semi-graph of anabelioids det'd by the stable log curve $X^{\log}_{i\in E,x}$ over $S^{\log}_{E\setminus\{i\}}$

$\Rightarrow$ $\exists$a natural Π_E-conjugacy class of continuous isom. $\Pi_{\mathcal{G}_{i\in E,x}} \stackrel{\sim}{\to} \Pi_{E/(E\setminus\{i\})}$ $(\subseteq \Pi_E)$
Fix a cont. isom. $\Pi_{\mathcal{G}_{i\in E,x}} \stackrel{\sim}{\to} \Pi_{E/(E\setminus\{i\})}$ that is contained in this natural Π_E-conj. class

$v \in \mathrm{Vert}(\mathcal{G})$
$Z_v \subseteq X$: the irreducible component of X that corresponds to v
$X_v \stackrel{\mathrm{def}}{=}$ (the normalization of Z_v) $\setminus$ (the cusps and nodes)
Thus, by the definition of the notion of a point stable curve,
the open subscheme $X_v \subseteq X$ is a hyperbolic curve over k

Observe: $\exists$a natural closed immersion $(X_v)_n \hookrightarrow X_n$
which induces an outer <u>injective</u> continuous homomorphism

$$(\Pi_v)_n \stackrel{\mathrm{def}}{=} \text{“}\Pi_n\text{” for } X_v \quad \hookrightarrow \quad \Pi_n$$

$m \leq n \Rightarrow$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (\Pi_v)_{n/m} & \longrightarrow & (\Pi_v)_n & \xrightarrow{(p_v)_{n/m}} & (\Pi_v)_m & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{n/m} & \longrightarrow & \Pi_n & \xrightarrow[p_{n/m}]{} & \Pi_m & \longrightarrow & 1 \end{array}$$

(cf. the next page, i.e., the copy of p.17 of the slides for the 3rd and 4th talks)

Sketch of the proof of Lemma (TpdSych)

Lemma (TpdSych)

$T_b \subseteq \Pi_{n-1/n-2}$: a $\{1,\ldots,n-1\}$-tripod
ν: a cusp of the tripod from which T_b arises
$T_f \subseteq \Pi_{n/n-1}$: a $\{1,\ldots,n\}$-tripod that arises from ν
$\Rightarrow$

- The inclusions
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta]$
  hold
- $\exists$a "geometric" outer continuous isomorphism $\iota\colon T_f \xrightarrow{\sim} T_b$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] & \\ {}^{\mathfrak{T}_{T_f}}\swarrow & & \searrow^{\mathfrak{T}_{T_b}} \\ \mathrm{Out}(T_f) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_b) \end{array}$$

  commutes (which thus implies the inclusion
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta]$)

By replacing T_f by a suitable Π_n-conjugate of T_f, we may assume:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{n/n-1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow \text{(cmm. trm.)} \text{— cf. the next talk} & & \uparrow & & \\ 1 & \longrightarrow & \text{“}\Pi_{2/1}\text{ for a tpd”} & \longrightarrow & \text{“}\Pi_2\text{ for a tpd”} & \longrightarrow & T_b & \longrightarrow & 1 \\ & & \uparrow & & & & & & \\ & & T_f & & & & & & \end{array}$$

$\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|]$
By a sim. arg. to the arg. app'd in the pf of Lemma (ConfiGC), $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
Moreover, the outom. α also pres. the conj. class of "Π_2 for a tpd" (cf. the next talk)
Thus, we may assume: $(g,r,n) = (0,3,2)$

Observe: If one takes a suitable $\sigma \in \mathfrak{S}_5$,
then $T_f \hookrightarrow \Pi_2 \overset{\sigma}{\xrightarrow{\sim}} \Pi_2 \overset{p_{2/1}}{\twoheadrightarrow} \Pi_1$ determines a "geometric" $T_f \xrightarrow{\sim} T_b$
The resulting $\mathrm{Out}(T_f) \xrightarrow{\sim} \mathrm{Out}(T_b)$ maps $\alpha|_{T_f} \mapsto (\sigma\alpha\sigma^{-1})_1$
$\alpha|_{T_f} \in \mathrm{Out}(T_f)^{\Delta} \quad \Rightarrow \quad (\sigma\alpha\sigma^{-1})_1 \in \mathrm{Out}(T_b)^{\Delta}$
$\overset{\text{[CmbCsp]}}{\Rightarrow}$ $\sigma\alpha\sigma^{-1}$ centralizes $\mathfrak{S}_5$
$\Rightarrow \sigma\alpha\sigma^{-1} = \alpha$
$\Rightarrow$ The resulting $\mathrm{Out}(T_f) \xrightarrow{\sim} \mathrm{Out}(T_b)$ maps $\mathfrak{T}_{T_f}(\alpha) = \alpha|_{T_f} \mapsto \alpha_1 = \mathfrak{T}_{T_b}(\alpha)$, as desired

Combinatorial Cuspidalization

$\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$
injective? surjective? ....?

(cf. the talks by Minamide and Iijima)

Glueability of Combinatorial Cuspidalizations

Relationship between:

- the combinatorial cuspidalization for Π_n
  $\mathrm{Out}^{\mathrm{F}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{F}}(\Pi_n)$
- the combinatorial cuspidalizations for the $(\Pi_v)_n$'s
  $\mathrm{Out}^{\mathrm{F}}((\Pi_v)_{n+1}) \to \mathrm{Out}^{\mathrm{F}}((\Pi_v)_n)$

Review: the case of $\underline{n = 1}$
the relationship of $\mathrm{Out}(\Pi_1)$ and $\mathrm{Out}(\Pi_v)$'s

$\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G})$: the subgroup consisting of automorphisms of $\mathcal{G}$
that induce the identity automorphism on the underlying semi-graph

$\mathrm{Dehn}(\mathcal{G}) \subseteq \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$: the subgp consisting of α s.t. the autom. α_v of $\mathcal{G}_v$ is trivial

$\mathrm{Glu}(\mathcal{G}) \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Out}(\Pi_v)$: the subgroup consisting of $(\alpha_v)_v$ s.t.

- the outomorphism α_v fixes every conjugacy class of cuspidal subgroups in Π_v
- $\forall v,\, w \in \mathrm{Vert}(\mathcal{G})$
  the equality $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ holds

Theorem in [CbTpI] — cf. [CbTpI], Theorem B, (iii)

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G}) \longrightarrow 1$$

Theorem in [CbTpI]

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G}) \longrightarrow 1$$

Theorem (GlCmbCsp) — cf. [CbTpII], Theorem F

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & 1
\end{array}$$

where:

$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{FC}}(\Pi_n) \times_{\mathrm{Out}(\Pi_1)} \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$

$\Rightarrow$

$$\begin{array}{ccccccc}
\mathrm{Dehn}(\mathcal{G}) & \hookrightarrow & \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) & \hookrightarrow & \mathrm{Out}(\Pi_{\mathcal{G}}) & \xrightarrow{\sim} & \mathrm{Out}(\Pi_1) \\
 & \searrow & \uparrow & & & & \uparrow \text{[NodNon]} \\
 & & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} & & \hookrightarrow & & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)
\end{array}$$

Theorem (GlCmbCsp)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & 1 \end{array}$$

$\mathrm{Glu}(\Pi_n)^{|\mathrm{grph}|} \subseteq \prod_{v \in \mathrm{Vert}(\mathcal{G})} \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}$: the subgroup consisting of $(\alpha_v)_v$ s.t.:

- if $n = 1$:
  $\forall v, w \in \mathrm{Vert}(\mathcal{G})$
  the equality $\chi_v(\alpha_v) = \chi_w(\alpha_w)$ holds

- if $n = 2$, then:
  $\forall \nu$: a node of $X^{\log}$
  b_1, b_2: the distinct two branches of ν
  v_1, v_2: the irr. comp. of $X^{\log}$ to which b_1, b_2 abut, respectively
  $\Rightarrow$ a $\{1,2\}$-tripod T_1 of $(\Pi_{v_1})_2$ that arises from the "cusp b_1"
  $\sim_{\text{conj. in } \Pi_2}$ a $\{1,2\}$-tripod T_2 of $(\Pi_{v_2})_2$ that arises from the "cusp b_2"
  Observe: $\alpha_{v_i} \in \mathrm{Out}^{\mathrm{FC}}((\Pi_{v_i})_n)^{|\mathrm{grph}|} \overset{\text{Lemma (ConfiGC)}}{\Rightarrow} \alpha_{v_i} \in \mathrm{Out}^{\mathrm{FC}}((\Pi_{v_i})_n)[T_i]$

  the equality $\mathfrak{T}_{T_1}(\alpha_{v_1}) = \mathfrak{T}_{T_2}(\alpha_{v_2})$ holds

- if $n \geq 3$, then:
  $\forall v_1, v_2 \in \mathrm{Vert}(\mathcal{G})$
  $T_i \subseteq (\Pi_{v_i})_3$: a 3-central $\{1,2,3\}$-tripod of $(\Pi_{v_i})_n$
  $\Rightarrow$ the closed subgroup T_i is a 3-central $\{1,2,3\}$-tripod of Π_n
  $\Rightarrow T_1 \sim_{\text{conj. in } \Pi_3} T_2$

  the equality $\mathfrak{T}_{T_1}(\alpha_{v_1}) = \mathfrak{T}_{T_2}(\alpha_{v_2})$ holds

Theorem (GlCmbCsp)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & 1
\end{array}$$

(A) how to define $\rho_n \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \to \mathrm{Glu}(\Pi_n)^{|\mathrm{grph}|}$
(B) the compatibility of ρ_{n+1} and ρ_n (i.e., the commutativity of the right-hand square)
(C) the equality $\mathrm{Dehn}(\mathcal{G}) = \mathrm{Ker}(\rho_n)$
(D) the surjectivity of ρ_n

(A), (B)
If one proves
- $\forall \alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|}$ preserves the conjugacy class of $(\Pi_v)_n$
- the closed subgroup $(\Pi_v)_n \subseteq \Pi_n$ is commensurably terminal

then, by

Remark

G: a group
$H \subseteq G$: a subgroup

$\alpha \in \mathrm{Aut}(G)$
$\Rightarrow$ can define the restriction $\alpha|_H \in \mathrm{Aut}(H)$ whenever α preserves $H \subseteq G$

On the other hand:
$\alpha \in \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$
$\Rightarrow$ cannot define the "restriction" $\alpha|_H \in \mathrm{Out}(H)$ in general
even if α preserves the conjugacy class of $H \subseteq G$
the "natural restriction" is
not $\in \mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ but $\in \mathrm{Aut}(H)/\mathrm{Inn}(N_G(H))$

In particular:
- the outomorphism α preserves the conjugacy class of $H \subseteq G$
- the equality $N_G(H) = Z_G(H) \cdot H$ holds

$\Rightarrow$ can define the restriction $\alpha|_H \in \mathrm{Out}(H)$

Theorem in [CbTpI] — cf. [CbTpI], Corollary 3.9, (iv)

$\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G})$
$v,\, w \in \mathrm{Vert}(\mathcal{G})$
$\Rightarrow$ the equality $\chi_v(\alpha_{\mathcal{G}|_v}) = \chi_w(\alpha_{\mathcal{G}|_w})$ holds

Theorem ((≥ 3)-TpdSych) — cf. [CbTpII], Theorem 3.18, (ii)

Suppose: $n \geq 3$
$E,\, E' \subseteq \{\text{positive integers} \leq n\}$
$T \subseteq \Pi_E$: an E-tripod
$T' \subseteq \Pi_{E'}$: an E'-tripod
$\Rightarrow$ $\exists$a "geometric" outer continuous isomorphism $\iota\colon T \xrightarrow{\sim} T'$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T:|\mathrm{C}|] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T':|\mathrm{C}|] & \\ \swarrow^{\mathcal{T}_T} & & \searrow^{\mathcal{T}_{T'}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T') \end{array}$$

commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (\Pi_v)_{n/m} & \longrightarrow & (\Pi_v)_n & \xrightarrow{(p_v)_{n/m}} & (\Pi_v)_m & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{n/m} & \longrightarrow & \Pi_n & \xrightarrow[p_{n/m}]{} & \Pi_m & \longrightarrow & 1 \end{array}$$

(A), (B) OK

(A), (B)

- $\forall \alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|}$ preserves the conjugacy class of $(\Pi_v)_n$
- the closed subgroup $(\Pi_v)_n \subseteq \Pi_n$ is commensurably terminal

the 2nd •

by the induction on n, together with
  the commensurable terminality of $(\Pi_v)_{i+1/i} \subseteq \Pi_{i+1/i}$, where $0 \leq i \leq n-1$

the 1st •

by a similar arg. to the arg. applied in the proof of the surjectivity portion of

Theorem in [NodNon] — cf. [NodNon], Theorem B

$n \geq 1 \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is injective

$$n \geq n_{\mathrm{bij}} \overset{\mathrm{def}}{=} \begin{cases} 3 & r \neq 0 \\ 4 & r = 0 \end{cases}$$

$n \geq n_{\mathrm{bij}} \Rightarrow$ the homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1}) \to \mathrm{Out}^{\mathrm{FC}}(\Pi_n)$ is bijective

(cf. the talk by Minamide)

Sketch of the proof

For simplicity, suppose:

- $n = 2$
- $\exists$a cusp e of $X^{\log}$ on the irreducible component that corresponds to v

$\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{|\mathrm{grph}|}$
$\widetilde{\alpha} \in \mathrm{Aut}^{\mathrm{FC}}(\Pi_2)^{|\mathrm{grph}|}$: a lifting of α

$\widetilde{\alpha}((\Pi_v)_2) \sim^{?}_{\mathrm{conj.}} (\Pi_v)_2$

$\Pi_e \subseteq \Pi_1$: a cuspidal subgroup associated to e

$\overset{\text{[NodNon]}}{\Rightarrow}$ $\exists! \Pi_v \subseteq \Pi_1$: a verticial subgroup associated to v that $\underline{\text{contains}}$ Π_e

$\Rightarrow$ $\exists (\Pi_v)_2 \subseteq \Pi_2$ whose image by $p_{2/1}$ is given by $\Pi_v \subseteq \Pi_1$

$\exists T \subseteq (\Pi_v)_{2/1}$: a $\{1,2\}$-tripod that arises from e

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{2/1} & \longrightarrow & \Pi_2 & \xrightarrow{p_{2/1}} & \Pi_1 & \longrightarrow & 1\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & (\Pi_v)_{2/1} & \longrightarrow & (\Pi_v)_2 & \longrightarrow & \Pi_v & \longrightarrow & 1\\
 & & \uparrow & & & & \uparrow & & \\
 & & T & & & & \Pi_e & &
\end{array}$$

$\Rightarrow$ the log geometric fiber of $p^{\log}_{2/1} \colon X^{\log}_2 \to X^{\log}$ at e is obtained by glueing

- the tripod from which T arises and
- the "old/major" part (which is not necessarily an irr. comp.), say Y, "$\cong X^{\log}$"

Moreover, by a "van Kampen-type consideration",

for suitable $\Pi_{v^\circ} \subseteq \Pi_Y \subseteq \Pi_{2/1}$

— where we write v° for the irr. comp. of Y that corresponds to v —

- $J \overset{\text{def}}{=} T \cap \Pi_Y$ is $\underline{\text{nodal}}$ and $\underline{\text{coincides}}$ with $T \cap \Pi_{v^\circ}$
- $\varinjlim (T \hookleftarrow J \hookrightarrow \Pi_Y) \xrightarrow{\sim} \Pi_{2/1}$
- $\varinjlim (T \hookleftarrow J \hookrightarrow \Pi_{v^\circ}) \xrightarrow{\sim} (\Pi_v)_{2/1}$

$\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{|\mathrm{grph}|} \Rightarrow \widetilde{\alpha}_1 \curvearrowright \Pi_1$ $\underline{\text{preserves}}$ the conj. classes of $\Pi_e \subseteq \Pi_v \subseteq \Pi_1$

Thus, by replacing $\widetilde{\alpha}$ by a suitable Π_2-conjugate of $\widetilde{\alpha}$,

we may assume: $\widetilde{\alpha}_1(\Pi_e) = \Pi_e$, hence also $\widetilde{\alpha}_1(\Pi_v) = \Pi_v$

$\overset{\text{Lemma (ConfiGC)}}{\Rightarrow}$ $\widetilde{\alpha}_{2/1} \curvearrowright \Pi_{2/1}$ $\underline{\text{preserves}}$ the conj. classes of T, $\Pi_Y \subseteq \Pi_{2/1}$

Thus, by replacing $\widetilde{\alpha}$ by a suitable $\Pi_{2/1}$-conjugate of $\widetilde{\alpha}$,

we may assume: $\widetilde{\alpha}_{2/1}(J) = J$, hence also $\widetilde{\alpha}_{2/1}(T) = T$ and $\widetilde{\alpha}_{2/1}(\Pi_Y) = \Pi_Y$

Observe: the composite $\Pi_Y \hookrightarrow \Pi_{2/1} \hookrightarrow \Pi_2 \overset{p_{\{1,2\}/\{2\}}}{\twoheadrightarrow} \Pi_{\{2\}}$ is
an $\underline{\text{isomorphism induced by the natural identification “}Y \cong X^{\log}\text{”}}$

Thus, since $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{|\mathrm{grph}|}$, which thus implies that $\alpha_{\{2\}}(\Pi_v) \sim_{\mathrm{conj.}} \Pi_v$,
$\widetilde{\alpha}_{2/1}|_{\Pi_Y} \curvearrowright \Pi_Y$ $\underline{\text{preserves}}$ the conjugacy class of $\Pi_{v^\circ} \subseteq \Pi_Y$
Thus, since $\widetilde{\alpha}_{2/1}(J) = J$,
$\widetilde{\alpha}_{2/1}(\Pi_{v^\circ}) = \Pi_{v^\circ}$
$\Rightarrow$ $(\widetilde{\alpha}_{2/1}|_T, \widetilde{\alpha}_{2/1}|_{\Pi_{v^\circ}})$ determines a continuous automorphism $\widetilde{\alpha}_{v,2/1} \curvearrowright (\Pi_v)_{2/1}$

By a consideration related to the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & (\Pi_v)_{2/1} & \longrightarrow & (\Pi_v)_2 & \longrightarrow & \Pi_v & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}((\Pi_v)_{2/1}) & \longrightarrow & \mathrm{Aut}((\Pi_v)_{2/1}) & \longrightarrow & \mathrm{Out}((\Pi_v)_{2/1}) & \longrightarrow & 1, \end{array}$$

$(\widetilde{\alpha}_1|_{\Pi_v}, \widetilde{\alpha}_{v,2/1})$ determines a continuous outomorphism $\alpha_v \overset{\mathrm{out}}{\curvearrowright} (\Pi_v)_2$

(cf. the next page, i.e., the copy of p.17 of the slides for the 3rd and 4th talks)

Sketch of the proof of Lemma (TpdSych)

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$T_b \subseteq \Pi_{n-1/n-2}$: a $\{1,\ldots,n-1\}$-tripod
ν: a cusp of the tripod from which T_b arises
$T_f \subseteq \Pi_{n/n-1}$: a $\{1,\ldots,n\}$-tripod that arises from ν
$\Rightarrow$

- The inclusions
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta]$
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$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] & \\ \swarrow^{\mathfrak{T}_{T_f}} & & \searrow^{\mathfrak{T}_{T_b}} \\ \mathrm{Out}(T_f) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_b) \end{array}$$

commutes (which thus implies the inclusion
$\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta]$)

By replacing T_f by a suitable Π_n-conjugate of T_f, we may assume:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{n/n-1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow \ (\text{cmm. trm.}) \text{ — cf. the next talk} & & \uparrow & & \\ 1 & \longrightarrow & \text{“}\Pi_{2/1} \text{ for a tpd”} & \longrightarrow & \text{“}\Pi_2 \text{ for a tpd”} & \longrightarrow & T_b & \longrightarrow & 1 \\ & & \uparrow & & & & & & \\ & & T_f & & & & & & \end{array}$$

$\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|]$
By a sim. arg. to the arg. app'd in the pf of Lemma (ConfiGC), $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
Moreover, the outom. α also pres. the conj. class of "Π_2 for a tpd" (cf. the next talk)
Thus, we may assume: $(g, r, n) = (0, 3, 2)$

Observe: If one takes a suitable $\sigma \in \mathfrak{S}_5$,
then $T_f \hookrightarrow \Pi_2 \overset{\sigma}{\xrightarrow{\sim}} \Pi_2 \overset{p_{2/1}}{\twoheadrightarrow} \Pi_1$ determines a "geometric" $T_f \xrightarrow{\sim} T_b$
The resulting $\mathrm{Out}(T_f) \xrightarrow{\sim} \mathrm{Out}(T_b)$ maps $\alpha|_{T_f} \mapsto (\sigma\alpha\sigma^{-1})_1$
$\alpha|_{T_f} \in \mathrm{Out}(T_f)^{\Delta} \quad \Rightarrow \quad (\sigma\alpha\sigma^{-1})_1 \in \mathrm{Out}(T_b)^{\Delta}$
$\overset{\text{[CmbCsp]}}{\Rightarrow}$ $\sigma\alpha\sigma^{-1}$ centralizes $\mathfrak{S}_5$
$\Rightarrow \sigma\alpha\sigma^{-1} = \alpha$
$\Rightarrow$ The resulting $\mathrm{Out}(T_f) \xrightarrow{\sim} \mathrm{Out}(T_b)$ maps $\mathfrak{T}_{T_f}(\alpha) = \alpha|_{T_f} \mapsto \alpha_1 = \mathfrak{T}_{T_b}(\alpha)$, as desired

Theorem (GlCmbCsp)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & 1
\end{array}$$

(A) how to define $\rho_n \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \to \mathrm{Glu}(\Pi_n)^{|\mathrm{grph}|}$
(B) the compatibility of ρ_{n+1} and ρ_n
(C) the equality $\mathrm{Dehn}(\mathcal{G}) = \mathrm{Ker}(\rho_n)$
(D) the surjectivity of ρ_n

(C)

Theorem in [CbTpI]

$$1 \longrightarrow \mathrm{Dehn}(\mathcal{G}) \longrightarrow \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}) \longrightarrow \mathrm{Glu}(\mathcal{G}) \longrightarrow 1$$

$$\begin{array}{ccccccccc}
 & & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n})^{|\mathrm{grph}|} & \xrightarrow{\rho_n} & \mathrm{Glu}(\Pi_{n})^{|\mathrm{grph}|} & & \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{1})^{|\mathrm{grph}|} & \xrightarrow[\rho_1]{} & \mathrm{Glu}(\Pi_{1})^{|\mathrm{grph}|} & \longrightarrow & 1
\end{array}$$

the lower sequence is exact by Theorem in [CbTpI]
$\Rightarrow$ (C) OK

Thus, to verify Theorem (GlCmbCsp),
it suffices to verify: (D) the <u>surjectivity of $\rho_n \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|} \to \mathrm{Glu}(\Pi_n)^{|\mathrm{grph}|}$</u>

By applying (nontrivial) induction on

- n (i.e., by the passage from "Π_n" to "$\Pi_{n/m}$" and "Π_m") and
- $\#\mathrm{Node}(\mathcal{G})$ (i.e., by the passage from "$\mathcal{G}$" to "$\mathcal{G}_{\rightsquigarrow S}$"),

it suffices to verify:
the <u>surjectivity of $\rho_2 \colon \mathrm{Out}^{\mathrm{FC}}(\Pi_2)^{|\mathrm{grph}|} \to \mathrm{Glu}(\Pi_2)^{|\mathrm{grph}|}$ in the case where $\#\mathrm{Node}(\mathcal{G}) = 1$</u>

<u>Sketch of the proof</u>
For simplicity, suppose: the dual semi-graph of $\mathcal{G}$ is a <u>tree</u>
e: the unique node of $X^{\log}$
v, $w \in \mathrm{Vert}(\mathcal{G})$: distinct

$(\alpha_v, \alpha_w) \in \mathrm{Glu}(\Pi_2)^{|\mathrm{grph}|} \subseteq \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_2)^{|\mathrm{grph}|} \times \mathrm{Out}^{\mathrm{FC}}((\Pi_w)_2)^{|\mathrm{grph}|}$
$\widetilde{\alpha}_\square \in \mathrm{Aut}^{\mathrm{FC}}((\Pi_\square)_2)^{|\mathrm{grph}|}$: a lifting of $\alpha_\square$

$\Pi_e \subseteq \Pi_1$: a nodal subgroup associated to e
$\overset{\text{[NodNon]}}{\Rightarrow}$ $\exists!\Pi_v$, $\Pi_w \subseteq \Pi_1$: verticial subgroups associated to v, w that <u>contain</u> Π_e, resp.

Then: by a "van Kampen-type consideration",
$\varinjlim (\Pi_v \hookleftarrow \Pi_e \hookrightarrow \Pi_w) \overset{\sim}{\to} \Pi_1$

$\exists (\Pi_v)_2$, $(\Pi_w)_2 \subseteq \Pi_2$ s.t.

(a) the intersection $T \overset{\mathrm{def}}{=} (\Pi_v)_{2/1} \cap (\Pi_w)_{2/1}$ is a $\{1,2\}$-tpd that arises from the node e
(b) the images of $(\Pi_v)_2$, $(\Pi_w)_2 \subseteq \Pi_2$ by $p_{2/1}$ are given by Π_v, $\Pi_w \subseteq \Pi_1$, respectively

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{2/1} & \longrightarrow & \Pi_2 & \xrightarrow{p_{2/1}} & \Pi_1 & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & (\Pi_v)_{2/1} & \longrightarrow & (\Pi_v)_2 & \longrightarrow & \Pi_v & \longrightarrow & 1 \\
 & & \uparrow & & & & \uparrow & & \\
 & & T & & & & \Pi_e & & \\
 & & \downarrow & & & & \downarrow & & \\
1 & \longrightarrow & (\Pi_w)_{2/1} & \longrightarrow & (\Pi_w)_2 & \longrightarrow & \Pi_w & \longrightarrow & 1
\end{array}$$

$\alpha_\square \in \mathrm{Out}^{\mathrm{FC}}((\Pi_\square)_2)^{|\mathrm{grph}|} \Rightarrow (\widetilde{\alpha}_\square)_1 \curvearrowright \Pi_\square$ $\underline{\text{preserves}}$ the conjugacy class of $\Pi_e \subseteq \Pi_\square$
Thus, by replacing $\widetilde{\alpha}_\square$ by a suitable $(\Pi_\square)_2$-conjugate of $\widetilde{\alpha}_\square$,
we may assume: $(\widetilde{\alpha}_\square)_1(\Pi_e) = \Pi_e$
$(\alpha_v, \alpha_w) \in \mathrm{Glu}(\Pi_2)^{|\mathrm{grph}|} \Rightarrow (\widetilde{\alpha}_v)_1|_{\Pi_e} = (\widetilde{\alpha}_w)_1|_{\Pi_e}$ Note: Π_e is a "$\underline{\text{cyclotome}}$"
$\Rightarrow ((\widetilde{\alpha}_v)_1, (\widetilde{\alpha}_w)_1)$ determines a continuous automorphism $\widetilde{\alpha}_1 \curvearrowright \Pi_1$

$\alpha_\square \in \mathrm{Out}^{\mathrm{FC}}((\Pi_\square)_2)^{|\mathrm{grph}|}$
$\overset{\text{Lemma (ConfiGC)}}{\Rightarrow} (\widetilde{\alpha}_\square)_{2/1} \curvearrowright (\Pi_\square)_{2/1}$ $\underline{\text{preserves}}$ the conjugacy class of $T \subseteq (\Pi_\square)_{2/1}$
Thus, by replacing $\widetilde{\alpha}_\square$ by a suitable $(\Pi_\square)_{2/1}$-conjugate of $\widetilde{\alpha}_\square$,
we may assume: $(\widetilde{\alpha}_\square)_{2/1}(T) = T$

$(\alpha_v, \alpha_w) \in \mathrm{Glu}(\Pi_2)^{|\mathrm{grph}|} \Rightarrow$ by replacing $\widetilde{\alpha}_v$ by a suitable T-conjugate of $\widetilde{\alpha}_v$,
we may assume: $(\widetilde{\alpha}_v)_{2/1}|_T = (\widetilde{\alpha}_w)_{2/1}|_T$

Again by a "van Kampen-type consideration",
$\varinjlim ((\Pi_v)_{2/1} \hookleftarrow T \hookrightarrow (\Pi_w)_{2/1}) \xrightarrow{\sim} \Pi_{2/1}$
$\Rightarrow ((\widetilde{\alpha}_v)_{2/1}, (\widetilde{\alpha}_w)_{2/1})$ determines a continuous automorphism $\widetilde{\alpha}_{2/1} \curvearrowright \Pi_{2/1}$

By a consideration related to the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{2/1} & \longrightarrow & \Pi_2 & \longrightarrow & \Pi_1 & \longrightarrow & 1 \\
 & & \wr\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Pi_{2/1}) & \longrightarrow & \mathrm{Aut}(\Pi_{2/1}) & \longrightarrow & \mathrm{Out}(\Pi_{2/1}) & \longrightarrow & 1,
\end{array}$$

$(\widetilde{\alpha}_1, \widetilde{\alpha}_{2/1})$ determines a continuous outomorphism $\alpha \overset{\text{out}}{\curvearrowright} \Pi_2$

Corollary (FC-nonsurj) — cf. [CbTpII], Theorem A, (iii)

$(g, r) \notin \{(0,3), (1,1)\}$
$\Rightarrow$ the (injective) homomorphism $\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is not surjective

Proof

By considering a suitable specialization of $X^{\log}$,
we may assume: $\forall$vertex of $\mathcal{G}$ is a tripod

$(g, r) \notin \{(0,3), (1,1)\}$
$\Rightarrow \exists v$, $w \in \mathrm{Vert}(\mathcal{G})$: distinct
$\exists$a node of $\mathcal{G}$ that abuts to both v and w
$\alpha_v \in \mathrm{Out}^{|\mathrm{C}|}(\Pi_v)^{\Delta}$, $\alpha_w \in \mathrm{Out}^{|\mathrm{C}|}(\Pi_w)^{\Delta}$ s.t.

- $\alpha_v \neq \phi^{-1}\alpha_w\phi$ for $\forall$"geometric" isomorphism $\phi\colon \Pi_v \xrightarrow{\sim} \Pi_w$
- $\chi_v(\alpha_v) = \chi_w(\alpha_w)$
  (cf. "$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\text{a tpd})^{\Delta}$"
  cf. the talks by Yamashita and Minamide)

By

Theorem (GlCmbCsp)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n+1})^{|\mathrm{grph}|} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Dehn}(\mathcal{G}) & \longrightarrow & \mathrm{Out}^{\mathrm{FC}}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & \mathrm{Glu}(\Pi_{n})^{|\mathrm{grph}|} & \longrightarrow & 1
\end{array}$$

$\exists \alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_1)^{|\mathrm{grph}|}$ s.t. $\alpha|_{\Pi_v} = \alpha_v$, $\alpha|_{\Pi_w} = \alpha_w$

Assume: $\mathrm{Out}^{\mathrm{FC}}(\Pi_2) \hookrightarrow \mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is surjective
$\Rightarrow \exists \alpha_2 \in \mathrm{Out}^{\mathrm{FC}}(\Pi_2)$ whose image in $\mathrm{Out}^{\mathrm{FC}}(\Pi_1)$ is $= \alpha$
But this contradicts Theorem (2-TpdSych)

Theorem (2-TpdSych) — cf. [CbTpII], Theorem 3.17, (i), (ii)

Suppose: $n = 2$
$E \subseteq \{1, 2\}$
$T \subseteq \Pi_E$: an E-tripod
$T_0 \subseteq \Pi_1$: a $\{1\}$-tripod
If one of the following two conditions is satisfied,
then $\exists$a "geometric" outer continuous isomorphism $\iota\colon T \xrightarrow{\sim} T_0$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_0 : |\mathrm{C}|, \Delta] & \\ \swarrow^{\mathcal{T}_T} & & \searrow^{\mathcal{T}_{T_0}} \\ \mathrm{Out}(T) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_0) \end{array}$$

commutes

- $E = \{1, 2\}$
  $\exists$a cusp ν of the $\{1\}$-tripod from which T_0 arises
  s.t. the E-tripod T arises from ν
- $E = \{1\}$
  $\exists$a node of $X^{\log}$ that abuts to both the $\{1\}$-tripod from which T_0 arises
  and the $\{1\}$-tripod from which T arises

Corollary (GT) — cf. [CbTpII], Theorem C, (iv)

Suppose:

- $n \geq 3$
- either $r \neq 0$ or $n \geq 4$

$T \subseteq \Pi_3$: a <u>central</u> $\{1,2,3\}$-tripod of Π_n
$\Rightarrow$

$$\begin{array}{ccccc} \mathrm{Out}^{\mathrm{F}}(\Pi_n) & \overset{\text{Theorem (F-ctr)}}{=\!=\!=\!=\!=} & \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] & \xrightarrow{\mathfrak{T}_T} & \mathrm{Out}(T) \\ & \searrow & & & \uparrow \\ & & & & \mathrm{GT} \end{array}$$

moreover, the resulting $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{GT}$ is <u>surjective</u>

<u>Proof of $\exists$ of the factorization</u>

Theorem (F-ctr) — cf. [CbTpII], Theorem 3.16, (v)

T: a <u>central</u> E-tripod of Π_n
$\Rightarrow$ the <u>equality</u> $\mathrm{Out}^{\mathrm{F}}(\Pi_n) = \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T : \Delta]$ holds

by a similar argument to the argument applied in the proof of Lemma (TpdSych)
(cf. the next page, i.e., the copy of p.17 of the slides for the 3rd and 4th talks)

cf. also $\mathrm{GT} = \mathrm{Out}(T)^{\Delta} \cap \mathrm{Im}(\mathrm{Out}^{\mathrm{F}}(T_2) \hookrightarrow \mathrm{Out}(T))$
(cf. the talk by Minamide)

Sketch of the proof of Lemma (TpdSych)

Lemma (TpdSych)

$T_b \subseteq \Pi_{n-1/n-2}$: a $\{1, \ldots, n-1\}$-tripod
ν: a cusp of the tripod from which T_b arises
$T_f \subseteq \Pi_{n/n-1}$: a $\{1, \ldots, n\}$-tripod that arises from ν
$\Rightarrow$

- The inclusions
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta]$
  hold
- $\exists$a "geometric" outer continuous isomorphism $\iota \colon T_f \overset{\sim}{\to} T_b$ s.t.

$$\begin{array}{ccc} & \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] & \\ \swarrow^{\mathfrak{T}_{T_f}} & & \searrow^{\mathfrak{T}_{T_b}} \\ \mathrm{Out}(T_f) & \xrightarrow[\text{conjugation by } \iota]{\sim} & \mathrm{Out}(T_b) \end{array}$$

  commutes (which thus implies the inclusion
  $\mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|, \Delta] \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|] \subseteq \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|, \Delta]$)

By replacing T_f by a suitable Π_n-conjugate of T_f, we may assume:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{n/n-1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow \text{(cmm. trm.) — cf. the next talk} & & \uparrow & & \\ 1 & \longrightarrow & \text{“}\Pi_{2/1} \text{ for a tpd”} & \longrightarrow & \text{“}\Pi_2 \text{ for a tpd”} & \longrightarrow & T_b & \longrightarrow & 1 \\ & & \uparrow & & & & & & \\ & & T_f & & & & & & \end{array}$$

$\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_b : |\mathrm{C}|]$
By a sim. arg. to the arg. app'd in the pf of Lemma (ConfiGC), $\alpha \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)[T_f : |\mathrm{C}|]$
Moreover, the outom. α also pres. the conj. class of "Π_2 for a tpd" (cf. the next talk)
Thus, we may assume: $(g, r, n) = (0, 3, 2)$

Observe: If one takes a suitable $\sigma \in \mathfrak{S}_5$,
then $T_f \hookrightarrow \Pi_2 \overset{\sigma}{\underset{\sim}{\to}} \Pi_2 \overset{p_{2/1}}{\twoheadrightarrow} \Pi_1$ determines a "geometric" $T_f \overset{\sim}{\to} T_b$
The resulting $\mathrm{Out}(T_f) \overset{\sim}{\to} \mathrm{Out}(T_b)$ maps $\alpha|_{T_f} \mapsto (\sigma\alpha\sigma^{-1})_1$
$\alpha|_{T_f} \in \mathrm{Out}(T_f)^{\Delta} \quad \Rightarrow \quad (\sigma\alpha\sigma^{-1})_1 \in \mathrm{Out}(T_b)^{\Delta}$
$\overset{\text{[CmbCsp]}}{\Rightarrow} \sigma\alpha\sigma^{-1}$ centralizes $\mathfrak{S}_5$
$\Rightarrow \sigma\alpha\sigma^{-1} = \alpha$
$\Rightarrow$ The resulting $\mathrm{Out}(T_f) \overset{\sim}{\to} \mathrm{Out}(T_b)$ maps $\mathfrak{T}_{T_f}(\alpha) = \alpha|_{T_f} \mapsto \alpha_1 = \mathfrak{T}_{T_b}(\alpha)$, as desired

Corollary (GT)

Suppose:

- $n \geq 3$
- either $r \neq 0$ or $n \geq 4$

$T \subseteq \Pi_3$: a central $\{1,2,3\}$-tripod of Π_n

$\Rightarrow$

$$\begin{array}{ccccc} \mathrm{Out}^{\mathrm{F}}(\Pi_n) & \overset{\text{Theorem (F-ctr)}}{=\!=\!=} & \mathrm{Out}^{\mathrm{F}}(\Pi_n)[T] & \xrightarrow{\mathfrak{T}_T} & \mathrm{Out}(T) \\ & \searrow\!\cdots & & & \uparrow \\ & & & & \mathrm{GT} \end{array}$$

moreover, the resulting $\mathrm{Out}^{\mathrm{F}}(\Pi_n) \to \mathrm{GT}$ is surjective

Proof of the surjectivity

By considering a suitable specialization of $X^{\log}$,
we may assume: $\forall$vertex of $\mathcal{G}$ is a tripod

$\alpha \in \mathrm{GT}$

$\Rightarrow \forall v \in \mathrm{Vert}(\mathcal{G})$, $\exists \alpha_{v,n} \in \mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}$ whose image in $\mathrm{Out}(\Pi_v)$ is $= \alpha$

$\overset{\text{Theorem } (\geq 3\text{-TpdSych})}{\Rightarrow}$ $\forall v \in \mathrm{Vert}(\mathcal{G})$, $\mathfrak{T}_{\text{a ctrl tpd in } (\Pi_v)_3}(\alpha_{v,n}) = \alpha$

$\overset{\text{Theorem (GlCmbCsp)}}{\Rightarrow}$

$\exists \alpha_n \in \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{|\mathrm{grph}|}$ whose image in $\mathrm{Out}^{\mathrm{FC}}((\Pi_v)_n)^{|\mathrm{grph}|}$ is $= \alpha_{v,n}$ for $\forall v \in \mathrm{Vert}(\mathcal{G})$

$\overset{\text{Theorem } (\geq 3\text{-TpdSych})}{\Rightarrow}$ $\mathfrak{T}_T(\alpha_n) = \alpha$, as desired

Remark

One may regard this surjective homomorphism

$$\mathrm{Out}^{\mathrm{F}}(\Pi_n) \twoheadrightarrow \mathrm{GT}$$

may be regarded as
a combinatorial analogue of the natural outer surjective continuous isomorphism

$$\pi_1(\mathcal{M}_{g,r/\mathbb{Q}}) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

Corollary (Csp/FntFld) — cf. [CbTpII], Corollary 4.16

$\square \in \{\circ, \bullet\}$
p: a prime number
$l_\square$: a prime number that is $\neq p$
$k_\square$: a finite field of characteristic p
$S_\square \stackrel{\text{def}}{=} \mathrm{Spec}(k_\square)$
$S_\square^{\log}$: the fs log scheme obtained by equipping $S_\square$ with
the fs log structure determined by $\mathbb{N} \to k_\square$, $1 \mapsto 0$
$\Rightarrow$

$$\pi_1(S_\square^{\log}) \twoheadrightarrow \pi_1(S_\square)$$

$s_\square$: a splitting of $\pi_1(S_\square^{\log}) \twoheadrightarrow \pi_1(S_\square)$
$X_\square^{\log}$: a stable log curve/$S_\square^{\log}$

If n is a positive integer, then:
${}_nX_\square^{\log}$: the n-th log configuration space of $X_\square^{\log}$,
$\pi_1^{l_\square\text{-gm}}({}_nX_\square^{\log})$: the geometrically pro-$l_\square$ log fundamental group of ${}_nX_\square^{\log}$
$\Rightarrow$

$$\pi_1^{l_\square\text{-gm}}({}_nX_\square^{\log}) \twoheadrightarrow \pi_1(S_\square^{\log}) \twoheadrightarrow \pi_1(S_\square)$$

${}_n\Pi_\square \twoheadrightarrow \pi_1(S_\square)$: the pull-back of $\pi_1^{l_\square\text{-gm}}({}_nX_\square^{\log}) \twoheadrightarrow \pi_1(S_\square^{\log})$ by $s_\square$

Then an arbitrary continuous isomorphism ${}_1\Pi_\circ \xrightarrow{\sim} {}_1\Pi_\bullet$
extends to a continuous isomorphism ${}_n\Pi_\circ \xrightarrow{\sim} {}_n\Pi_\bullet$ for $\forall n \geq 1$