Semi-continuity of conductor divisors of ℓ-adic sheaves.

H. Hu. (Nanjing University)

2nd Kyoto-Hefei 20 Aug, 2020

1. Classical ramification theory
2. Ramification for higher dimensions
3. Bounding ramifications by restricting to curves
4. Semi-continuity for conductors
5. Ramification bound for nearby cycles

Part 1. Classical ramification theory

K : a local field (e.g. $\mathbb{Q}_p$, $\mathbb{F}_p((t))$ $\cdots$)

O_K : integer ring of K (e.g. $\mathbb{Z}_p$, $\mathbb{F}_p[[t]]$ $\cdots$)

F : $F = O_K/m_K$ the residue field

G_K : $\mathrm{Gal}(\overline{K}/K)$ an absolute Galois group of K

We assume that $\mathrm{char}(F) = p > 0$ and F is perfect.

<u>Thm</u> (Herbrand) There is a decreasing filtration $\{G^i_{K,\mathrm{up}}\}_{i \in \mathbb{Q}_{\geq 0}}$ of G_K by closed normal subgroups.

It is called the upper numbering filtration of G_K.

- $G^0_{K,\mathrm{up}} = I_K$ the inertia subgroup of G_K
- $G^{0+}_{K,\mathrm{up}} = \overline{\bigcup_{b>0} G^b_{K,\mathrm{up}}} = P_K$ the wild inertia subgroup of G_K

P_K is a very large pro-p-group. It is the heart of the ramification theory for local fields.

- $G^i_{K,\mathrm{up}} / G^{i+}_{K,\mathrm{up}} \cong \pi_1^{\mathrm{alg}}(\mathbb{A}^1_{\overline{F}})$ $\quad (i \in \mathbb{Q}_{>0})$

$\Lambda = \mathbb{F}_{\ell^n} \qquad (\ell, p) = 1$

M a f.g. Λ-module with a continuous G_K-action.

Slope decomposition: $M = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{(r)}$ (finitely many $r \geq 0$)

where $M^{(0)} = M^{P_K}$,

for $r > 0$, $M^{(r)} = (M^{(r)})^{G^r_{K,up}} = \{0\}$. $(M^{(r)})^{G^{r+}_{K,up}} = M^{(r)}$.

Slopes of M: $\{ r \in \mathbb{Q}_{\geq 0} \mid M^{(r)} \neq 0 \}$ ← finite set

upper numbering conductor of M:

$\tilde{c}_K(M)$ = the largest slope of M. $\in \mathbb{Q}_{\geq 0}$

Swan conductor of M: $sw_K(M) = \sum_{r \in \mathbb{Q}_{\geq 0}} r \cdot rk_\Lambda M^{(r)}$

Total dimension of M: $dt_K(M) = sw_K(M) + rk_\Lambda(M)$

Thm (Hasse-Arf) $sw_K(M)$ and $dt_K(M)$ are integers.

Ex: $K = \mathbb{F}_p((t))$, $L = K[x]/(x^p - x - \frac{1}{t^m})$ $(m,p)=1$

ρ: $\mathrm{Gal}(L/K) \cong \mathbb{Z}/p\mathbb{Z} \longrightarrow \Lambda^\times$ a non-trivial character.

$\tilde{c}_K(\rho) = \mathrm{sw}_K(\rho) = m$, $\mathrm{dt}_K(\rho) = m+1$

Thm (Grothendieck-Ogg-Shafarevich) Let k be an algebraically closed field of characteristic $p>0$, let C be a connected, proper and smooth k-curve, let $j: U \to C$ be an open immersion and let $\mathcal{F}$ be a locally constant and constructible sheaf of Λ-modules on U. Then, we have

$$\chi_c(U,\mathcal{F}) = \mathrm{rk}_\Lambda \mathcal{F} \cdot \chi_c(U,\Lambda) - \sum_{x \in X-U} \mathrm{Sw}_{K_x}(\mathcal{F}|_{\mathrm{Spec}\, K_x})$$

$K_x = K(\hat{\mathcal{O}}_{C,x})$

Question: $k=\bar{k}$ $\mathrm{char}\, k = p > 0$, X a smooth k-scheme, D an effective Cartier divisor on X, $\mathcal{F}$ a locally constant and constructible sheaf of Λ-modules on $U = X - D$.

How to describe the ramification of $\mathcal{F}$ along D?

Part 2 Ramification for higher dimensions

(1980s–)

(i) Deligne – Laumon, Esnault – Kerz, Wiesend ⋯ approach

$\exists$ an effective Cartier divisor $R(\mathcal{F})$ supported on D

s.t. $\forall$ smooth k-curve C, $i: C \to X$

$x = (i^* D)_{red} \in C$ is a closed point of C, we have

$m_x(i^* R(\mathcal{F})) \geq sw_x(\mathcal{F}|_{C-\{x\}})$ and the bound is sharp ??

Picture 1

$\xi \in D$ the generic point of an irreducible component of D,

$\hat{\mathcal{O}}_{X,\xi}$ Complete DVR with the residue field $k(\xi)$

char $k(\xi) = p > 0$ but $k(\xi)$ is not perfect in general.

$K = K(\hat{\mathcal{O}}_{X,\xi})$ discrete valuation field

$G_K = \mathrm{Gal}(\bar{K}/K)$ $\quad G_K^{ab} = G_K / \overline{[G_K, G_K]}$

(ii) $\mathrm{tr.deg}(k(\xi)/k) = n$.

(1980s –)

$K_{n+1}(K) \longrightarrow G_K^{ab}$

$\uparrow$

Milnor K-group

(K. Kato's higher local class field theory)

It gives rise to two filtrations $\{G_K^{ab,i}\}_{i\in\mathbb{Q}_{\geq 1}}$ $\{G_{K.\log}^{ab,i}\}_{\mathbb{Q}_{\geq 0}}$

on G_K^{ab}

(2000s –)

(iii). Abbes and Saito's ramification filtration.

Using rigid geometry and other geometric method, they constructed two decreasing filtrations $\{G_K^i\}_{i\in\mathbb{Q}_{\geq 1}}$ $\{G_{K.\log}^i\}_{i\in\mathbb{Q}_{\geq 0}}$ on G_K that generalize the upper numbering filtration for local fields.

- $G_K^1 = G_{K.\log}^0 = I_K$ inertia subgroup of G_K
- $G_K^{1+} = G_{K.\log}^{0+} = P_K$ wild inertia subgroup of G_K
- $\forall\ i \geq 0$, $G_K^{i+1} \subseteq G_{K.\log}^i \subseteq G_K^i$
- If the residue field $F = O_K/m_K$ is perfect, we have $G_K^{i+1} = G_{K.\log}^i = G_{K.up}^i$ $(i \geq 0)$

(iv) Kedlaya and Xiao's ramification filtrations of G_K using p-adic differential equations (2000s –)

(v) Beilinson's singular support and Saito's characteristic cycles of ℓ-adic sheaves on smooth varieties (2015, 2017)

$k = \bar{k}$ char $k = p > 0$, X smooth k-scheme

$\mathcal{G} \in \mathrm{ob}\, D^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z})$ $(\ell, p) = 1$.

$SS(\mathcal{G})$: a closed conical subset of T^*X of equi-dimension $\dim X$.

$CC(\mathcal{G})$: a cycle of T^*X supported on $SS(\mathcal{G})$

$$-\left(CC(\mathcal{G}),\ \text{zero section of } T^*X\right) \underset{\uparrow}{=} \chi(X, \mathcal{G})$$

when X is projective

- A generalization of Grothendieck-Ogg-Shafarevich formula

(v) SS + CC

$\Downarrow$

(iii) Abbes-Saito's ramification filtrations

$\Updownarrow$

(iv) Kedlaya-Xiao's ramification filtrations

$\Downarrow$

(ii) Kato's filtration on G_K^{ab}

??

(i). Ramification of $\mathcal{F}$ along D by restricting to curves

Part 3. Bounding ramifications by restricting to curves

k a perfect field of characteristic $p>0$,

X smooth k-scheme.

D a SNCD of D

$D = \sum_i D_i$ irreducible components, $\xi_i \in D_i$ generic point

$U = X - D$, $j: U \to X$ open immersion

$\mathcal{F}$ l.c.c. étale sheaf of Λ-modules on U $\left(\Lambda = \mathbb{F}_{\ell^n}, (\ell, p) = 1\right)$

$K_i = k(\hat{\mathcal{O}}_{X,\xi_i})$, $\overline{K_i}$ a separable closure of K_i

$G_{K_i} = \mathrm{Gal}(\overline{K_i}/K_i)$, $\{G^r_{K_i}\}_{r\in\mathbb{Q}_{\geq 1}}$ Abbes and Saito's ramification filtration. $\{G^r_{K_i,\log}\}_{r\in\mathbb{Q}_{\geq 0}}$ Abbes and Saito's logarthmic ramification filtration.

$$\begin{tikzcd} \mathrm{Spec}\, K_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \ni \mathcal{F} \arrow[d] \\ \mathrm{Spec}\, \hat{\mathcal{O}}_{X,\xi_i} \arrow[r] & X \end{tikzcd}$$

$M_i = \mathcal{F}|_{\mathrm{Spec}\, K_i}$ $\leftrightsquigarrow$ a finitely generated Λ-module with a continuous G_{K_i}-action.

Slope decomposition

$$M_i = \bigoplus_{r \geq 1} M_i^{(r)}$$

$$M_i^{(1)} = M_i^{P_{K_i}}$$

$$(M_i^{(r)})^{G^r_{K_i}} = \{0\}$$

$$(M_i^{(r)})^{G^{r+}_{K_i}} = M_i^{(r)} \quad (r>1)$$

Total dimension

$$dt_{K_i}(M_i) = \sum_r r \cdot \dim_\Lambda M_i^{(r)}$$

If $k(\xi_i)$ is perfect, then
$dt_{K_i}(M_i)$ here $=$
$dt_{K_i}(M_i)$ by upper numbering fil

Total dimension divisor

$$DT_X(\mathcal{F}) = \sum_i dt_{K_i}(M_i) \cdot D_i \quad \leftarrow \text{Effective Cartier divisor}$$

Thm (A)

(1). For any smooth k-curve C and any immersion $h: C \to X$ such that $x = C \cap D$ is a closed point of X, we have

$$m_x\left(h^*(DT_X(\mathcal{F}))\right) \geq dt_x(\mathcal{F}|_{C-\{x\}}) \qquad \text{[H.-Yang 2017]}$$

(2). There exists an open dense subset $D_0 \subseteq D$ such that the ramification of $\mathcal{F}$ along D_0 is non-degenerate. For a curve C and any immersion $h: C \to X$ such that $x = C \cap D \subseteq D_0$ and $dh: T_x^*X \to T_x^*C$ satisfies $\ker(dh) \cap (SS(j_!\mathcal{F}) \times_X x) \subseteq \{0\} \subseteq T_x^*X$, we have

$$m_x(h^*(DT_X(\mathcal{F}))) = dt_x(\mathcal{F}|_{C-\{x\}}) \qquad \text{[Saito 2017]}$$

Picture 1

Logarithmic slope decomposition

$$\mathcal{F}|_{\mathrm{Spec}\,K_i} = M_i = \bigoplus_{r\geq 0} M_{i,\log}^{(r)}$$

$$M_{i,\log}^{(0)} = M_i^{P_{K_i}}$$

$$\left(M_{i,\log}^{(r)}\right)^{G_{K_i,\log}^{r}} = \{0\}$$

$$(r>0)$$

$$\left(M_{i,\log}^{r}\right)^{G_{K_i,\log}^{r+}} = M_{i,\log}^{r}$$

Swan conductor

$$\mathrm{Sw}_{K_i}(M_i) = \sum_{r\geq 0} r\cdot \dim\left(M_{i,\log}^{(r)}\right)$$

When $k(\xi_i)$ is perefect $\mathrm{Sw}_{K_i}(M_i)$ here $=$ $\mathrm{Sw}_{K_i}(M_i)$ by upper numbering

Swan divisor

$$\mathrm{Sw}_X(\mathcal{F}) = \sum_i \mathrm{Sw}_{K_i}(M_i)\cdot D_i$$

Thm (B) [H. 2019]

(1). For any smooth k-curve C and any immersion $h: C \longrightarrow X$ such that $x = C\cap D$ is a closed point of X, we have

$$m_x\left(h^*\left(\mathrm{Sw}_X(\mathcal{F})\right)\right) \geq \mathrm{Sw}_x\left(\mathcal{F}|_{C-\{x\}}\right)$$

(2). Assume that D is a smooth irreducible Cartier divisor of X. Let $I(X,D)$ be the set of triples $(C,\ h: C\to X,\ x)$

where C is a smooth k-curve, $h: C \to X$ an immersion $x = C \cap D$ a closed point. Then we have

$$\mathrm{Sw}_D(\mathcal{F}) = \sup_{(C,\, h: C \to X,\, x) \in \mathcal{I}(X, D)} \frac{\mathrm{Sw}_x(\mathcal{F}|_{C-\{x\}})}{m_x(h^*D)}$$

[Picture 1]

Swan conductor of $\mathcal{F}$ at the generic point of D

Remark: C is a curve,

$$dt_x(\mathcal{F}|_{C-\{x\}}) = \mathrm{Sw}_x(\mathcal{F}|_{C-\{x\}}) + \mathrm{rk}_\Lambda(\mathcal{F}|_{C-\{x\}})$$

A locally constant and constructible sheaf $\mathcal{F}$ ramified along a SNCD on a smooth k-scheme,

Total dimension divisor (Abbes – Saito's ramification filtration) sharply bounds the total dimension of $\mathcal{F}$ after restricting to curves. (Thm A)

Swan divisor (Abbes - Saito's logarithmic ramification filtration) sharply bounds the swan conductor of $\mathcal{F}$ after restricting to curves, in an asymptotic way. (Thm B)

Q: Which divisor sharply bounds the upper numbering conductor of $\mathcal{F}$ after restricting to curves ?

$$\mathcal{F}|_{\operatorname{Spec} K_i} = M_i = \bigoplus_{r \geq 1} M_i^{(r)} \quad \text{(Slope decomposition)}$$

<u>conductor</u> : $c_{K_i}(M_i) = $ largest slope of M_i

<u>conductor divisor</u>: $C_X(\mathcal{F}) = \sum_i c_{K_i}(M_i) \cdot D_i \in \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

Thm (C) [H. 2020]

(1). For any smooth k-curve C and any immersion $h: C \to X$ such that $x = C \cap D$ is a closed point of X, we have

$$m_x(h^*(C_X(\mathcal{F}))) \geq \tilde{c}_x(\mathcal{F}|_{C-\{x\}}) + 1$$

upper numbering conductor of $\mathcal{F}|_{C-\{x\}}$ at x.

(2) under the same condition as in Thm (A) (2) we have $m_x(h^*(C_X(\mathcal{F}))) = \tilde{c}_x(\mathcal{F}|_{C-\{x\}}) + 1$

$$\mathcal{F}|_{\mathrm{Spec}\, K_i} = M_i = \bigoplus_{r\geq 0} M_{i,\log}^{(r)} \quad (\text{logarithmic slope decomposition})$$

<u>logarithmic conductor</u> $\tilde{c}_{K_i}(M_i) =$ largest logarithmic slope of M_i

<u>logarithmic conductor divisor</u> $\widetilde{C}_X(\mathcal{F}) = \sum_i \tilde{c}_{K_i}(M_i)\cdot D_i$

Remark: If the residue field of K_i is perfect

$$\hat{c}_{K_i}(M) = \text{largest logarithmic slope of } M_i$$

$$\| \qquad\qquad \|$$

largest upper numbering slope of M_i

Thm (D). [H. 2020]

(1). For any smooth k-curve C and any immersion $h: C \to X$ such that $x = C\cap X$ is a closed point of X, we have

$$m_x(h^*(\widetilde{C}_X(\mathcal{F}))) \geq \tilde{c}_x(\mathcal{F}|_{C-\{x\}})$$

(2). Assume that D is a smooth irreducible Cartier divisor of X. Let $I(X, D)$ be the set of triples $(C, h: C\to X, x)$

where C is a smooth k-curve, $h: C \to X$ an immersion $x = C \cap D$ a closed point. Then we have

$$\widetilde{C}_D(\mathcal{F}) = \sup_{(C,\, h: C \to X,\, x) \in \mathcal{I}(X, D)} \frac{\widetilde{c}_x(\mathcal{F}|_{C - \{x\}})}{m_x(h^* D)}$$

$\downarrow$

logarithmic conductor of $\mathcal{F}$ at the generic point of D.

- Thm (A) ~ (D) implies that there are strong connections between

(i). Ramification of $\mathcal{F}$ along D by restricting to curves

$\Updownarrow$

(iii) Abbes-Saito's ramification filtrations

- Key point of proving Thm (C):

For any rational number $r_1, r_2, \cdots, r_m > 1$ and any closed conical subset B of T^*X satisfying

1° $\pi(B) = D$, $\pi: T^*X \to X$.

2° $\forall\ \bar{x} \to D$, $B \times_X \bar{x} \subseteq T^*_{\bar{x}} X$ is a locally constant 1-dim $k(\bar{x})$-vector space.

(T^*X is a locally constant vector bundle of X)

We construct a l.c.c sheaf $\mathcal{L}$ on U such that

(1) $\mathcal{L}|_{\operatorname{Spec} K_i}$ is isoclinic of conductor r_i

(2). For any immersion $h: C \to X$ from a smooth k-curve to X s.t. $C \cap D = x$ and s.t.

$$B \times_X \bar{x} \hookrightarrow T^*_{\bar{x}} X \xrightarrow{dh} T^*_{\bar{x}} C \text{ is an bijection,}$$

we have $\mathcal{L}|_{C-\{x\}}$ is isoclinic of upper numbering conductor

$$\tilde{c}_x(\mathcal{L}|_{C-\{x\}}) = \sum_{i=1}^{m} r_i \cdot m_x(h^* D_i) - 1$$

Applying Thm (A) to $\mathcal{F} \otimes \mathcal{L}$ for all $\mathcal{L}$.

we get Thm (C) for $\mathcal{F}$.

Thm (C) $\Rightarrow$ Thm (D). uses an asymptotic method.

Part 4. Semi-continuity of conductors

Thm (Deligne – Laumon's semi-continuity of Swan conductors)

Let S be an excellent scheme, $f: X \to S$ a smooth morphism of relative dimension 1, $D \subseteq X$ a closed subscheme of X such that $f|_D: D \to S$ is flat and quasi-finite, and $U = X - D$.

Let Λ be a finite field invertible in S. $\mathcal{F}$ a l.c.c. étale sheaf of Λ-modules on U. The function

$$\begin{aligned} \varphi: \quad S &\longrightarrow \mathbb{Z} \\ s &\longmapsto \sum_{t \in |D_{\bar{s}}|} \left(sw_t (\mathcal{F}|_{U_{\bar{s}}}) + rk_\Lambda \mathcal{F} \right) \end{aligned}$$

is constructible and lower semi-continuous.

$\boxed{\text{Picture 2}}$

D-module theory.

Let Y be a smooth complex curve, $y \in Y$ a closed point, $\mathcal{M}$ holonomic meromorphic connection on Y with poles at y. If $\mathcal{M}$ is not regular, we have two invariants of $\mathcal{M}$:

irregularity: $\mathrm{Irr}_y(\mathcal{M})$ | Poincaré-Katz rank: $PK_y(\mathcal{M})$

similarities between holonomic $\mathbf{D}$-modules and ℓ-adic sheaves

Swan conductors / total dimension | upper numbering conductor

Thm [Y. André 2007]

Let $f: X \to S$ be a smooth of relative dimension 1 morphisms between smooth complex varieties, $D \subseteq X$ an effective Cartier divisor s.t. $f|_D: D \to S$ is flat and quasi-finite. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Then the following two functions.

$$\varphi : |S| \longrightarrow \mathbb{Z}$$
$$s \longmapsto \sum_{t \in D_s} \mathrm{Irr}_t(\mathcal{M}|_{X_s})$$

$$\psi : |S| \longrightarrow \mathbb{Z}$$
$$s \longmapsto \sum_{t \in D_s} \mathrm{Pk}_t(\mathcal{M}|_{X_s})$$

are constructible and lower semi-continuous.

- The formulation of André's result is motivated by ℓ-adic case (Deligne – Laumon's result)
- André proved the semi-continuity for Poincaré-Katz rank. However, there is a missing of semi-continuity for conductors in the ℓ-adic case.

Thm (E) [H. 2020, in progress]

Let k be a perfect field of characteristic $p>0$, $f: X \to S$ is a smooth and of relative dimension 1 morphism of smooth k-schemes, $D = \sum_i D_i$ a SNCD of X s.t. $f|_{D_i} : D_i \to S$ is flat and quasi-finite. Let Λ be a finite field of characteristic ℓ, $(\ell, p)=1$, $\mathcal{F}$ a l.c.c sheaf of Λ-modules on $U = X - D$.

Then, the following function

$$\varphi: |S| \longrightarrow \mathbb{Q}$$

$$s \longmapsto \sum_{t \in |D_{\bar{s}}|} \left(\tilde{c}_t(\mathcal{F}|_{U_{\bar{s}}}) + 1 \right)$$

Picture 2

is constructible and lower semi-continuous.

- This is the semi-continuity of conductors in the geometric situation.
- The constructiblity is proved in the same way as in Deligne and Laumon's theorem. The semi-continuity is proved using the key step of Thm (C).

Part 5. Ramification bound of nearby cycles

$(S, s, \eta, \bar\eta)$ strict local trait, $\mathrm{char}(s) = p > 0$

$f : X \longrightarrow S$ semi-stable morphism

$$\begin{tikzcd}
X_{\bar\eta} \arrow[r] \arrow[d, "f_{\bar\eta}"'] \arrow[rr, bend left, "\bar j"] \arrow[dr, phantom, "\square"] & X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X_s \arrow[l, "i"'] \arrow[d, "f_s"] \\
\bar\eta \arrow[r] & \eta \arrow[r] & S & s \arrow[l]
\end{tikzcd}$$

Semi-stable means f_η is smooth and X_s is a SNCD of X.

$\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ $\quad (\ell, p) = 1$

$\mathcal{F}$ l.c.c. sheaf of Λ-modules on X_η

<u>nearby cycles</u> $\quad R^n\Psi(\mathcal{F}, f) = i^* R^n \bar f_* \bar j^* \mathcal{F}$ $\circlearrowleft$ $\mathrm{Gal}(\bar\eta/\eta)$

<u>étale cohomologies</u> $\quad H^n(X_{\bar\eta}, \mathcal{F}|_{X_{\bar\eta}})$ $\circlearrowleft$

Prop (Grothendieck (SGA 7), Rapoport - Zink, Illusie)

Assume that $\mathcal{F}$ is tamely ramified along X_s. Then the action of $\mathrm{Gal}(\bar{\eta}/\eta)$ on each $R^n\Psi(\mathcal{F}, f)$ is tame.

In particular, under the assumption that $f: X \to S$ is proper, the action of $\mathrm{Gal}(\bar{\eta}/\eta)$ on each $H^n(X_{\bar{\eta}}, \mathcal{F}|_{\bar{\eta}})$ is tame.

Q: The case involving wildly ramifications?

Conj (Leal 2016) $\quad X_s = \sum_i D_i$ (D_i irreducible)

Let $\tilde{c}_i$ be the logarithmic conductor of $\mathcal{F}$ at the generic point of D_i, and $\tilde{c} = \mathrm{Max}_i \{\tilde{c}_i\}$. Assume $f: X \to S$ is proper, Then, the ramification of each $H^n(X_{\bar{\eta}}, \mathcal{F}|_{X_{\bar{\eta}}})$ is (upper numbering) bounded by $\tilde{c}$, i.e. the action of $\mathrm{Gal}(\bar{\eta}/\eta)^{\tilde{c}+}_{up}$ is trivial.

$\{\mathrm{Gal}(\bar{\eta}/\eta)^r_{up}\}_{r \in \mathbb{Q}_{\geq 0}}$ upper numbering filtration of $\mathrm{Gal}(\bar{\eta}/\eta)$

Thm (Leal 2016) The conj is true when S is equal characteristic, $f: X \to S$ is a relative curve and $\mathrm{rk}\,\mathcal{F} = 1$.

Global method: Kato–Saito's conductor formula.

Thm (F) [H. 2020] Assume S is henselization of a smooth curve of positive characteristic (geometric). Then, the action of $\mathrm{Gal}(\bar{\eta}/\eta)^{\tilde{c}+}_{\mathrm{up}}$ on each $R^n\Psi(\mathcal{F}, f)$ is trivial.

In particular, Leal's conjecture is true when S is geometric.

- If we add a condition that $f: X \to S$ is smooth in Thm (F), it was proved by (H.-Teyssier 2018)

- When $f: X \to S$ is proper

$$E_2^{mn} = H^m(X_s, R^n\Psi(\mathcal{F}, f)) \Rightarrow H^{m+n}(X_{\bar{\eta}}, \mathcal{F})$$

- The approach to Thm (F) is pure local.

key point of Thm (C) + Beilinson's SS

+ Saito's CC

Sketch of the proof:

Step 1. By dévissage, it is sufficient to show, for every lisse sheaf $\mathcal{N}$ on η of upper numbering isoclinic of slope $> \tilde{c}$, we have

$$j_!(\mathcal{F} \otimes f_\eta^* \mathcal{N}) = Rj_*(\mathcal{F} \otimes f_\eta^* \mathcal{N})$$

Step 2. Descend to the case where S is a smooth curve with a closed point s and $\eta = V = S - \{s\}$ is the complement of $s \in S$.

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \arrow[dr, phantom, "\square"] & D \arrow[l, "i"'] \arrow[d] \\ V \arrow[r] & S & \{s\} \arrow[l] \end{tikzcd}$$

$$j_!(\mathcal{F} \otimes f_V^* \mathcal{N}) \overset{?}{=} Rj_*(\mathcal{F} \otimes f_\eta^* \mathcal{N}) \quad (\star)$$

Step 3. We may replace X by a finite surjective radiciel cover $\pi : X' \to X$

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "\pi_U"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & D' \arrow[l, "i'"'] \arrow[d] \\ U \arrow[r, "j"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \arrow[dr, phantom, "\square"] & D \arrow[l, "i"'] \arrow[d] \\ V \arrow[r] & S & \{s\} \arrow[l] \end{tikzcd}$$

and reduced to check $Rj'_* \pi_u^*(\mathcal{F}\otimes f_v^* N) = j'_! \pi_u^*(\mathcal{F}\otimes f_v^* N)$

we choose a very "good" such cover $\pi: X' \to X$,

that makes $j'_! \pi_u^* f_v^* N$ satisfying the following condition.

(i). $\forall\, x' \in D'$, $\dim_{k(x')}(SS(j'_! \pi_u^* f_v^* N) \times_{X'} x') = 1$

(ii). $SS(j'_! \pi_u^* f_v^* N) = SS(j'_! \pi_u^*(\mathcal{F}\otimes f_v^* N))$

(iii) $\forall\, x' \in D'$, we can find an immersion

$h: C \to X'$ from a smooth curve to X' such that

$C \cap D' = x'$, that $(\mathcal{F}\otimes f_v^* N)|_{C-\{x'\}}$ has isoclinic

of upper numbering conductor >0 and that

$h: C \to X'$ is $SS(j'_! \pi_u^*(\mathcal{F}\otimes f_v^* N))$ - transversal.

[Thm (C) plays a crucial role in Step 3]

Step 4. For any curve C in Step 3 (iii).

we have

$$h^* Rj'_* \pi_u^*(\mathcal{F}\otimes f_v^* N) \overset{bc}{\cong} Rg_*\left((\mathcal{F}\otimes f_v^* N)|_{C-\{x'\}}\right)$$

$$\overset{\text{Thm (C)}}{\cong} g_!\left((\mathcal{F}\otimes f_v^* N)|_{C-\{x'\}}\right)$$

where $g: C-\{x'\} \to C$.

It implies that $\left(Rj'_* \pi_u^*(\mathcal{F} \otimes f_v^* \mathcal{N})\right)_{\bar{x}'} = 0 \quad (\forall\, x' \in D')$

$\rightsquigarrow \quad Rj'_* \pi_u^*(\mathcal{F} \otimes f_v^* \mathcal{N}) = j'_! \pi_u^*(\mathcal{F} \otimes f_v^* \mathcal{N}) \qquad \square.$

☆ Remark: From Step 3, we see finite surjective radiciel map does not change étale site, however changes ramifications in higher dimensions (ramification filtrations, singular supports and characteristic cycles ....) ....

Thank you !