On a variant of the uniform boundedness conjecture for Drinfeld modules.

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This talk is based on the speaker's master thesis:
On the $\mathfrak{p}$-primary uniform boundedness conjecture for Drinfeld modules (2020).

Contents

1. Backgrounds and the main result.
2. Proof of the main result.
3. Future work.

Conjecture (The UBC for abelian varieties).

L : a finitely generated field over a prime field.
$d > 0$: an integer.
Then there exists a constant $C := C(L, d) \geq 0$ which depends on L and d s.t.

$$|X(L)_{\mathrm{tors}}| < C \quad \text{holds for every } d\text{-dim abelian variety } X \text{ over } L.$$

Known results.

- Mazur[a][b]: The UBC for $d = 1$ and $L = \mathbb{Q}$.
- Merel[c]: The UBC for $d = 1$.

[a] B. Mazur. "Modular curves and the Eisenstein ideal". In: Inst. Hautes Études Sci. Publ. Math. 47 (1977). With an appendix by Mazur and M. Rapoport, 33–186 (1978).

[b] B. Mazur. "Rational isogenies of prime degree (with an appendix by D. Goldfeld)". In: Invent. Math. 44.2 (1978), pp. 129–162.

[c] Loïc Merel. "Bornes pour la torsion des courbes elliptiques sur les corps de nombres". In: Invent. Math. 124.1-3 (1996), pp. 437–449.

The p-primary Uniform Boundedness Conjecture.

Conjecture (The pUBC for abelian varieties).

L : a finitely generated field over a prime field.
$d > 0$: an integer.
p : a prime.
Then there exists a constant $C := C(L, p, d) \geq 0$ which depends on L, p and d s.t.

$$|X[p^\infty](L)| < C \quad \text{holds for every } d\text{-dim abelian variety } X \text{ over } L.$$

Known results.

- Manin[a]:The pUBC for $d = 1$.
- Cadoret[b]: The pUBC for $d = 2$ with real multiplication assuming the Bombieri-Lang conj.
- Cadoret-Tamagawa[c]: The pUBC for every 1-dimensional family of abelian varieties.

[a] Ju. I. Manin. "The p-torsion of elliptic curves is uniformly bounded". In: Izv. Akad. Nauk SSSR Ser. Mat. 33 (1969), pp. 459–465.

[b] Anna Cadoret. "The ℓ-primary torsion conjecture for abelian surfaces with real multiplication". In: Algebraic number theory and related topics 2010. RIMS Kôkyûroku Bessatsu, B32. Res. Inst. Math. Sci. (RIMS), Kyoto, 2012, pp. 195–204.

[c] Anna Cadoret and Akio Tamagawa. "Uniform boundedness of p-primary torsion of abelian schemes". In: Invent. Math. 188.1 (2012), pp. 83–125.

Here is the result of Cadoret-Tamagwa*.

Theorem [Cadoret-Tamagawa].

L : a finitely generated field over a prime field.
S : a 1-dimensional scheme of finite type over L.
A : an abelian scheme over S.
p : a prime s.t. $p \neq \mathrm{ch}(L)$.
Then there exists a $N := N(L, S, A, p) \geq 0$ depends on L, S, A and p s.t.

$$A_s[p^\infty](L) \subset A_s[\mathfrak{p}^N](L)$$

holds for every $s \in S(L)$, i.e. every L-rational p-primary torsion point of A_s is annihilated by p^N.

Motivation.

Find a "Drinfeld-module analogue" of Cadoret-Tamagawa's result.

*Anna Cadoret and Akio Tamagawa. "Torsion of abelian schemes and rational points on moduli spaces". In: Algebraic number theory and related topics 2007. RIMS Kôkyûroku Bessatsu, B12. Res. Inst. Math. Sci. (RIMS), Kyoto, 2009, pp. 7–29.

What are Drinfeld modules ?

Drinfeld modules are function-field analogues of abelian varieties introduced by Drinfeld[a] under the name of "elliptic module".

[a] V. G. Drinfeld. "Elliptic modules". In: Mat. Sb. (N.S.) 94(136) (1974), pp. 594–627, 656.

Notation.

- p : a prime.
- q : a power of p.
- C : a smooth geometrically irreducible projective curve over $\mathbb{F}_q$.
- ∞ : a fixed closed point of C.
- K : the function field of C.
- $A := \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$.

Definition (Drinfeld A-modules).

L : an A-field (i.e. a field L with a homomorphism $\iota : A \to L$).
A Drinfeld A-module over L is a homomorphism $\phi : A \to \mathrm{End}(\mathbb{G}_{a,L})$ which satisfies the following two conditions:

1. $\phi(A) \not\subset L$
2. $A \xrightarrow{\phi} \mathrm{End}(\mathbb{G}_{a,L}) \xrightarrow{\delta} L$ equals ι where δ is the differentiation map of $\mathbb{G}_{a,L}$ at 0.

Remark.

If we denote the q-th Frobenius by τ, then
$\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,L}) = L\{\tau\} := \{\sum_i a_i\tau^i \text{ (finite sum)} \mid a_i \in L\}$ and $\delta(\sum_i a_i\tau^i) = a_0$.

- For every Drinfeld A-module ϕ over an A-field L, we can define the **rank** of ϕ which plays a similar role as the dimension of an abelian variety.

Example.

Assume $A = \mathbb{F}_q[T]$.
Then the **rank** of ϕ equals the **degree** of $\phi(T) \in L\{\tau\}$ as a polynomial in τ.

- Let I be an ideal of A.
  The I-torsion subgroup of ϕ is defined by:

$$\phi[I] \coloneqq \cap_{a \in I} \ker(\phi_a : \mathbb{G}_a \to \mathbb{G}_a).$$

  If the characteristic of L $(\coloneqq \ker(\iota))$ does not divide I, $\phi[I]$ is a finite étale group scheme which is étale-locally isomorphic to $(A/I)^d$ where d is the rank of ϕ.

- Let $\mathfrak{p}$ be a maximal ideal of A.
  The $\mathfrak{p}$-adic Tate module of ϕ is defined by:

$$T_{\mathfrak{p}}(\phi) \coloneqq \varprojlim \phi[\mathfrak{p}^n](\overline{L}).$$

  If the characteristic of L does not divide $\mathfrak{p}$, $T_{\mathfrak{p}}(\phi)$ is a free $A_{\mathfrak{p}}$-module of rank d.

Poonen proved the finiteness of torsion submodules of Drinfeld modules[†].

Theorem [Poonen].

Let L be a finitely generated A-field which contains K and ϕ a Drinfeld A-module over L. Then the set of L-rational torsion points of ϕ is **finite**.

Remark.

$L = \mathbb{G}_a(L)$ can be regarded as an A-module through ϕ. This A-module is never finitely generated. This shows that an analogue of the Mordell-Weil theorem for Drinfeld module does not hold.

[†] Bjorn Poonen. "Local height functions and the Mordell-Weil theorem for Drinfel'd modules". In: Compositio Math. 97.3 (1995), pp. 349–368.

Conjecture (The UBC for Drinfeld A-modules).

L : a finitely generated field over K.
$d > 0$: an integer.
Then there exists a constant $C := C(L, d) \geq 0$ which depends on L and d s.t.

$$|\phi(L)_{\mathrm{tors}}| < C \text{ holds for every Drinfeld } A\text{-module } \phi \text{ of rank } d \text{ over } L.$$

Known results.

- Poonen[a]: The UBC for $d = 1$.
- Pál[b]: If $A = \mathbb{F}_2[T]$, $Y_0(\mathfrak{p})(K)$ is empty for every $\mathfrak{p}$ with $\deg(\mathfrak{p}) \geq 3$.
- Armana[c]: If $A = \mathbb{F}_q[T]$, $Y_0(\mathfrak{p})(K)$ is empty for every $\mathfrak{p}$ with $\deg(\mathfrak{p}) = 3, 4$.

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[a] Bjorn Poonen. "Torsion in rank 1 Drinfeld modules and the uniform boundedness conjecture". In: Math. Ann. 308.4 (1997), pp. 571–586.

[b] Ambrus Pál. "On the torsion of Drinfeld modules of rank two". In: J. Reine Angew. Math. 640 (2010), pp. 1–45.

[c] Cécile Armana. "Torsion des modules de Drinfeld de rang 2 et formes modulaires de Drinfeld". In: Algebra & Number Theory 6.6 (2012), pp. 1239–1288.

Conjecture (The $\mathfrak{p}$UBC for Drinfeld A-modules).

L : a finitely generated field over K.
$d > 0$: an integer.
$\mathfrak{p}$: a maximal ideal of A.
Then there exists a $C := C(L, \mathfrak{p}, d) \geq 0$ which depends on L, $\mathfrak{p}$ and d s.t.

$$|\phi[\mathfrak{p}^\infty](L)| < C \text{ holds for every Drinfeld } A\text{-module } \phi \text{ of rank } d \text{ over } L.$$

Known results.

- Poonen : The $\mathfrak{p}$UBC for $d = 2$ and $A = \mathbb{F}_q[T]$.
- Cornelissen-Kato-Kool[a]: A strong version of the $\mathfrak{p}$UBC for $d = 2$.

[a] Gunther Cornelissen, Fumiharu Kato, and Janne Kool. "A combinatorial Li-Yau inequality and rational points on curves". In: Math. Ann. 361.1-2 (2015), pp. 211–258.

Main result.

The $\mathfrak{p}$UBC for every 1-dimensional family of Drinfeld modules of arbitrary rank.

Theorem [I].

L : a finitely generated field over K.
S : a 1-dimensional scheme of finite type over L.
ϕ : a Drinfeld A-module of rank d over S.
$\mathfrak{p}$: a maximal ideal of A.
Then there exists an integer $N := N(L, S, \phi, \mathfrak{p}) \geq 0$ which depends on L, S, ϕ and $\mathfrak{p}$ s.t.

$$\phi_s[\mathfrak{p}^\infty](L) \subset \phi_s[\mathfrak{p}^N](L)$$

holds for every $s \in S(L)$, i.e. every L-rational $\mathfrak{p}$-primary torsion point of ϕ_s is annihilated by $\mathfrak{p}^N$.

Corollary.

The theorem implies the $\mathfrak{p}$UBC for $d = 2$.
($\because$) Apply this theorem for Drinfeld modular curves.

A model case: $S = Y_1(\mathfrak{p})$ and ϕ is the universal Drinfeld A-module.

In this case, the claim of the theorem $\Leftrightarrow Y_1(\mathfrak{p}^n)(L) = \emptyset$ for $n \gg 0$.

1. First, consider the tower of modular curves
$$\cdots \to X_1(\mathfrak{p}^{n+1}) \to X_1(\mathfrak{p}^n) \to X_1(\mathfrak{p}^{n-1}) \to \cdots$$
and prove that the genus $g(X_1(\mathfrak{p}^n))$ goes to ∞ when $n \to \infty$.
2. Since $X_1(\mathfrak{p}^n)$ is not $\mathbb{F}_p$-isotrivial, $X_1(\mathfrak{p}^n)(L)$ is finite if $g(X_1(\mathfrak{p}^n)) \geq 2$ by a positive characteristic analogue of the Mordell conjecture proved by Samuel.
3. If $Y_1(\mathfrak{p}^n)(L) \neq \emptyset$, then $\varprojlim Y_1(\mathfrak{p})(L) \neq \emptyset$ for every n, which shows a Drinfeld module over L has infinitely many L-torsion points.

Hence $Y_1(\mathfrak{p}^n)(L)$ is empty for $n \gg 0$.

1. For a general S, we will define an analogue of the modular curve, and prove that the genus goes to infinity.
2. However, since we do not assume S is non-isotrivial, so we cannnot conclude the theorem as above.

We have divided the proof into three parts:

The strategy of the proof.

Assume that the assersion of the theorem does not hold.

- Step 1: Show that every component of a tower of finite connected étale covers of S (= an analogue of the modular tower) has an L-rational point.
- Step 2: Prove the genus of that tower goes to infinity.
- Step 3: Take nice models of $\mathrm{Spec}(L)$ and S over $\mathbb{F}_p$ and extends the Drinfeld module ϕ to such a model. Specializing the $\mathfrak{p}$-primary torsion subgroup of ϕ at some closed point of the model leads to a contradiction.

Notation.

$\eta : \mathrm{Spec}(L(S)) \to S$: the generic point of S.
$\overline{\eta} : \mathrm{Spec}(\overline{L(S)}) \to S$: a geometric generic point of S.
$\pi_1(S, \overline{\eta})$: the étale fundamental group of S.
$\Pi := \pi_1(S \times_L \overline{L}, \overline{\eta})$: the geometric fundamental group of S.

Since $\phi[\mathfrak{p}^n]$ is a finite étale group scheme, we have an action $\pi_1(S, \overline{\eta}) \curvearrowright \phi_\eta[\mathfrak{p}^n](\overline{L(S)})$.

Definition ($S_{v_n} \to S$).

$n > 0$: an integer.
For $v_n \in \phi_\eta[\mathfrak{p}^n]^*(\overline{L(S)}) := \phi_\eta[\mathfrak{p}^n](\overline{L(S)}) \setminus \mathfrak{p}\phi_\eta[\mathfrak{p}^n](\overline{L(S)})$, write S_{v_n} for the connected étale cover of S which correponds to the stabilizer of v_n w.r.t. $\pi_1(S) \curvearrowright \phi_\eta[\mathfrak{p}^n](\overline{L(S)})$.

S_{v_n} plays a role analogous to $Y_1(\mathfrak{p}^n)$.

Lemma.

Assume that the main result does not hold.
Then $\exists v := (v_n) \in T_{\mathfrak{p}}(\phi) \setminus \mathfrak{p}T_{\mathfrak{p}}(\phi)$ s.t.

$$S_{v_n}(L) \neq \emptyset$$

holds for every $n \geq 0$.
Hence we have a "modular tower" of geometrically connected finite étale covers of S:

$$\cdots \to S_{v_{n+1}} \to S_{v_n} \to S_{v_{n-1}} \to \cdots .$$

Observation.

If $\varprojlim S_{v_n}(L) \neq \emptyset$, we can conclude the proof.

$\because$Let (s_n) be an arbitary element of $\varprojlim S_{v_n}(L)$ and consider the specialization map $\phi_\eta[\mathfrak{p}^n](\overline{L(S)}) \to \phi_{s_0}[\mathfrak{p}^n](\overline{\kappa(s_0)})$ for every n. Then the image of v_n under this map is $\kappa(s_0)$-rational. This contradicts the finiteness of torsion points of ϕ_s. $\square$

In the following, we fix $v := (v_n) \in T_{\mathfrak{p}}(\phi)$ as in the lemma.
We may assume that $S_{v_n}(L)$ is infinite.

Next, we prove that the genus of $\{S_{v_n}\}$ goes to infinity.

Proposition.

$\overline{L}$: an algebraic closure of L.
g_{v_n} : the genus of the compactification of $S_{v_n} \times_L \overline{L}$.

$$\text{Then } \lim_{n\to\infty} g_{v_n} = \infty \text{ holds.}$$

Ingredients for the proof are:

1. The Riemann-Hurwitz formula.
2. Breuer-Pink's result on the image of the monodromy representation $\pi_1(S \times_L \overline{L}) \to \mathrm{GL}_d(K_{\mathfrak{p}})$ associated to $\mathfrak{p}$-adic Tate modules of Drinfeld A-modules.
3. Oesterlé's theorem on the reduction modulo $\mathfrak{p}^n$ of analytic closed subsets in $A_{\mathfrak{p}}^d$.

Breuer and Pink proved the following thorem[‡].

Theorem [Breuer-Pink].

k : finitely generated field over K.
X : variety over k with the generic point $\xi : \mathrm{Spec}(k(X)) \to X$.
ψ : Drinfeld A-module over X s.t. ψ_ξ is not isotrivial and $A = \mathrm{End}_{\overline{k(X)}}(\psi_\xi)$.
Then the image of the monodromy representation associated to the $\mathfrak{p}$-adic Tate module

$$\pi_1(X_{k^{\mathrm{sep}}}, \bar{\xi}) \to \mathrm{GL}(T_{\mathfrak{p}}(\psi_\xi))$$

is commensurable with $\mathrm{SL}(T_{\mathfrak{p}}(\psi_\xi))$.

In the following, for simplicity, we assume that

- $A = \mathrm{End}_{\overline{L(S)}}(\phi_\eta)$.
- ϕ_η is not L-isotrivial.

[‡] Florian Breuer and Richard Pink. "Monodromy groups associated to non-isotrivial Drinfeld modules in generic characteristic". In: Number fields and function fields—two parallel worlds. Vol. 239. Progr. Math. Birkhäuser Boston, Boston, MA, 2005, pp. 61–69.

Notation.

- $\lambda_{v_n} := \frac{2g_{v_n}-2}{\deg(S_{v_n}\to S)}$.
- $T_{\mathfrak{p}} := T_{\mathfrak{p}}(\phi_\eta)$: the $\mathfrak{p}$-adic Tate module of ϕ_η.
- $\rho : \Pi \to \mathrm{GL}_{A_{\mathfrak{p}}}(T_{\mathfrak{p}})$: the monodromy representation.
- $G := \mathrm{Im}(\rho)$.
- $\rho_n : \Pi \to \mathrm{GL}_{A/\mathfrak{p}^n}(T_{\mathfrak{p}}/\mathfrak{p}^n T_{\mathfrak{p}})$: the mod $\mathfrak{p}^n$ monodromy representation.
- $G_n := \mathrm{Im}(\rho_n)$.
- $P_1, \ldots, P_r$: the cusps of $S \times_L \overline{L}$.
- $I_{P_i}, \ldots, I_{P_r}$: the image of the inertia subgroup at P_i through ρ.

Property.

- $\{\lambda_{v_n}\}_n$ is increasing.
- $\lim_{n\to\infty} g_{v_n} = \infty \Leftrightarrow \lim_{n\to\infty} \lambda_{v_n} > 0$

- By the Riemann-Hurwitz formula, λ_{v_n} satisfies the following inequality:

$$\lambda_{v_n} \geq -2 + \sum_{1 \leq i \leq r} \left(1 - \frac{|I_{P_i} \backslash G_n v_n|}{|G_n v_n|} \right).$$

- By the affineness of the moduli spaces of Drinfeld modules and the non L-isotriviality of ϕ_η, we may assume that $\exists\, i$ s.t. $|I_{P_i}| = \infty$.

- Note that the number of cusps of S_{v_n} above P_i equals $|I_{P_i} \backslash G_n v_n|$.
  By Breuer-Pink's result and the quasi-unipotency of the action of inertia subgroups, one can show that $\lim_{n\to\infty} |I_{P_i} \backslash G_n v_n| = \infty$.
  Hence, by replacing S with S_{v_n}, we may assume that $|I_{P_i}| = \infty$ for at least three i's.

Hence it suffices to prove that

$$\frac{|I_{P_i}\backslash G_n v_n|}{|G_n v_n|} \to 0.$$

This follows from the following proposition:

Proposition.

M : a free $A_{\mathfrak{p}}$-module of finite rank.
$G \subset \mathrm{GL}(M)$: an analytic closed subgroup.
$v \in M \setminus \mathfrak{p}M$.
$I \subset G$: a closed subgroup.
Then, under some assumptions,

$$\lim_{n\to\infty} \frac{|I\backslash G_n v_n|}{|G_n v_n|} = \frac{1}{|I|} \qquad \text{holds.}$$

By using Oesterlé's theorem[§], the assertion of this proposition is reduced to the inequality $\dim(Gv)^I < \dim Gv$ (the dimension as an analytic space).

[§] Joseph Oesterlé. "Réduction modulo p^n des sous-ensembles analytiques fermés de $\mathbf{Z}_p^N$". In: Invent. Math. 66.2 (1982), pp. 325–341.

The strategy of the proof.

Assume that the assersion of the theorem does not hold.

- Step 1: Show that every component of a tower of finite connected étale covers of S (= an analogue of the modular tower) has an L-rational point.
- Step 2: Prove the genus of that tower goes to infinity.
- Step 3: Take nice models of $\mathrm{Spec}(L)$ and S over $\mathbb{F}_p$ and extends the Drinfeld module ϕ to such a model. Specializing the $\mathfrak{p}$-primary torsion subgroup of ϕ at some closed point of the model leads to a contradiction.

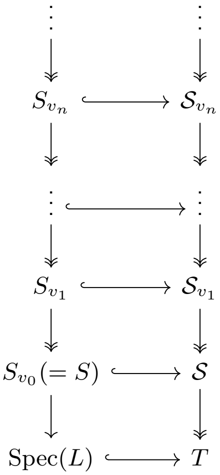
Lemma [Cadoret-Tamagawa].

L : a finitely generated extension of $\mathrm{ch}(L) = p > 0$.
C : a proper, normal and geometrically integral 1-dimensional scheme over L s.t. the genus of the normalization of $C \times_L \overline{L}$ is greater than 1.
S : a non-empty open subscheme of C.
If $S(L)$ is infinite, we moreover assume that S is $\mathbb{F}_p$-isotrivial.

Then there exists a $\mathbb{F}_p$-morphism $f : \mathcal{S} \to T$ between separated, integral and normal $\mathbb{F}_p$-schemes of finite type which satisfies the following properties:

1. The function field $\mathbb{F}_p(T)$ of T is $\mathbb{F}_p$-isomorphic to L.
2. Under the identification $\mathbb{F}(T) = L$, S is L-isomorphic to the generic fiber $\mathcal{S}_L$ of f.
3. Under the identification $S = \mathcal{S}_L$, we have $S(L) = \mathcal{S}(T)$ i.e. every L-points of S uniquely extends to a T-point of $\mathcal{S}$.

Step 3 (2/3)



- We show that a non-empty open subscheme of S is $\mathbb{F}_p$-isotrivial. Then we take a model $f : \mathcal{S} \to T$ as in the lemma. We may assume that ϕ extends to a Drinfeld A-module $\phi_{\mathcal{S}}$ over $\mathcal{S}$.

- As the same as S_{v_n}, the étale covering corresponding to the stabilizer of v_n w.r.t. $\pi_1(\mathcal{S}, \overline{\eta}) \curvearrowright \phi_\eta[\mathfrak{p}^n](\overline{L(S)})$ is denoted by $\mathcal{S}_{v_n} \to \mathcal{S}$.

- Then one can show that $S_{v_n}(L) = \mathcal{S}_{v_n}(T)$ by using $S(L) = \mathcal{S}(T)$ and the normality of $\mathcal{S}_{v_n}$. In particular, $\mathcal{S}_{v_n}(T) \neq \emptyset$ for every n.

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
S_{v_n}(L) & \underset{\sim}{\longleftarrow} & \mathcal{S}_{v_n}(T) & \longrightarrow & \mathcal{S}_{v_n}(\kappa(t)) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & \underset{\sim}{\longleftarrow} & \vdots & \longrightarrow & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
S_{v_1}(L) & \underset{\sim}{\longleftarrow} & \mathcal{S}_{v_1}(T) & \longrightarrow & \mathcal{S}_{v_1}(\kappa(t)) \\
\downarrow & & \downarrow & & \downarrow \\
S_{v_0}(L) & \underset{\sim}{\longleftarrow} & \mathcal{S}(T) & \longrightarrow & \mathcal{S}(\kappa(t))
\end{array}
$$

- **Hence, for a closed point $t \in T$, $\mathcal{S}_{v_n}(\kappa(t)) \neq \emptyset$ is finite and non-empty.**
- Take an $(x_n) \in \varprojlim \mathcal{S}_{v_n}(\kappa(t))$ and set $x := x_0 \in \mathcal{S}(\kappa(t))$.
- Then the image of v_n under the specialization map $\phi_\eta[\mathfrak{p}^n](\overline{L(S)}) \to (\phi_{\mathcal{S}})_x[\mathfrak{p}^n](\overline{\kappa(t)})$ is $\kappa(t)$-rational for every n.

  Hence $(\phi_{\mathcal{S}})_x$ has infinitely many $\kappa(t)$-rational torsion points. This contradicts Poonen's theorem. □

In general, $\mathrm{End}_{\overline{L(S)}}(\phi_\eta)$ may strictly contain A and ϕ_η may be L-isotrivial.

- If ϕ_η is L-isotrivial, we can prove the following stronger result:

Theorem.

L : a finitely generated field over K.
S : a normal integral scheme of finite type over L.
ϕ : a Drinfeld A-module over S s.t. ϕ_η is L-isotrivial.
$d > 0$: an integer.
Then there exists a constant $C = C(L, S, d) \geq 0$ which depends on L, S and ϕ s.t.

$$|\phi_s(L')_{\mathrm{tors}}| < C$$

holds for every extension L'/L with $[L' : L] \leq d$ and $s \in S(L')$.

- If ϕ_η is not L-isotrivial and $E = \mathrm{End}_{\overline{L(S)}}(\phi_\eta)$ strictly contains A, one can roughly regard ϕ as a Drinfeld E-module. Then almost the same proof as above works well.

Future work.

- <u>The $\mathfrak{p}$UBC for $d = 3$.</u>
  By the main result, we know that the set of rational points of the moduli space of Drinfeld modules of rank 3 with level $\mathfrak{p}^n$ (which is an affine surface) is empty for $n \gg 0$ or Zariski dense for every $n > 0$.

- <u>The UBC for $d = 2$.</u>
  This conjecture largely remains open since the formal immersion method (which Mazur and Merel used to prove the UBC for ellptic curves) is difficult to adapt to Drinfeld modular curves.

Thank you for your attention.