Absolute purity in motivic homotopy theory

Fangzhou Jin
joint work with F. Déglise, J. Fasel and A. Khan

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The absolute purity conjecture

Grothendieck's **absolute (cohomological) purity conjecture** (SGA5, Exposé I 3.1.4) is the following statement: if $i : Z \to X$ is a closed immersion between noetherian regular schemes of pure codimension c, $n \in \mathcal{O}(X)^*$ and $\Lambda = \mathbb{Z}/n\mathbb{Z}$, then the étale cohomology sheaf supported in Z with values in Λ can be computed as

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This conjecture has been solved by Gabber.

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  Based on Thomason's method + rigidity for algebraic K-theory

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- Construct Gysin morphisms and establish intersection theory.
- Study the coniveau spectral sequence.

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- Our work: study absolute purity in the framework of motivic homotopy theory.
- Main result: the absolute purity in motivic homotopy theory is satisfied with rational coefficients in mixed characteristic.

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- Can be used to study cohomology theories such as algebraic K-theory, Chow groups (motivic cohomology) and many others
- Advantage: has many a lot of structures coming from both topological and algebraic geometrical sides

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- Non-commutative geometry and singularity categories (Tabuada, Blanc-Robalo-Toën-Vezzosi)

Some topological background

- A **spectrum** $\mathbb{E}$ is a sequence $(E_n)_{n\in\mathbb{N}}$ of pointed spaces (e.g. CW-complexes or simplicial sets) together with continuous maps $\sigma_n : S^1 \wedge E_n \to E_{n+1}$ called **suspension maps**

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- Examples: Suspension spectra $\Sigma^\infty X$ for $X \in Top_\bullet$, in particular sphere spectrum S ; HA Eilenberg-Mac Lane spectrum for a ring A; MU complex cobordism spectrum
- From an ∞-categorical point of view, the category of spectra is the stabilization of the category of spaces, and is the universal stable (triangulated) category

The unstable motivic homotopy category

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- Bigraded $\mathbb{A}^1$-**homotopy sheaves**: for $X \in \mathrm{H}_\bullet(S)$, $\pi^{\mathbb{A}^1}_{a,b}(X)$ is the Nisnevich sheaf on Sm_S associated to the presheaf

$$U \mapsto [U \wedge S^{a-b} \wedge \mathbb{G}_m^b, X]_{\mathrm{H}_\bullet(S)}$$

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- $\mathrm{SH}(S)$ is the universal stable ∞-category which satisfies Nisnevich descent and $\mathbb{A}^1$-invariance (Robalo, Drew-Gallauer)

Motivic spectra

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- Milnor-Witt spectrum $\mathrm{H}_{MW}\mathbb{Z}$ represents Milnor-Witt motivic cohomology/higher Chow-Witt groups (Déglise-Fasel)

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- The 1-line is also computed (Röndigs-Spitzweck-Østvaer):

$$0 \to K^M_{2-n}/24 \to \pi_{n+1,n}(\mathbb{1}_k) \to \pi_{n+1,n} f_0(\mathrm{KQ})$$

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- For any morphism of schemes $f : X \to Y$, there is a pair of adjoint functors

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- They satisfy formal properties axiomatizing important theorems such as duality, base change and localization.

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- For oriented spectra, Déglise defined fundamental classes using Chern classes

Bivariant groups

- For $f : X \to S$ be a separated morphism of finite type, $v \in K_0(X)$ and $\mathbb{E} \in \mathrm{SH}(S)$, define the $\mathbb{E}$-**bivariant groups** (or **Borel-Moore** $\mathbb{E}$-**homology**) as

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- Its intersection theory is motivated by the intersection theory on Chow groups

Functoriality of bivariant groups

- Base change:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ T & \xrightarrow{p} & S \end{array}$$

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- Product: if $\mathbb{E}$ has a ring structure, $X \xrightarrow{f} Y \xrightarrow{g} S$

$$\mathbb{E}_m(X/Y, w) \otimes \mathbb{E}_n(Y/S, v) \to \mathbb{E}_{m+n}(X/S, w + f^*v)$$

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- The construction uses the *deformation to the normal cone*

Euler class and excess intersection formula

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- **Motivic Gauss-Bonnet formula** (Levine, Déglise-J.-Khan) For $p : X \to S$ a smooth and proper morphism

$$\chi(X/S) = p_* e(T_p)$$

  where $\chi(X/S)$ is the *categorical Euler characteristic*

The absolute purity property

- We say that an absolute spectrum $\mathbb{E}$ satisfies **absolute purity** if for any closed immersion $i : Z \to X$ between regular schemes, the purity transformation $\mathbb{E}_Z \otimes \mathrm{Th}(\tau_f) \to f^!\mathbb{E}_X$ is an isomorphism

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- From this property Cisinski-Déglise deduce that the rational motivic Eilenberg-Mac Lane spectrum $\mathrm{H}\mathbb{Q}$ also satisfies absolute purity, mainly because $\mathrm{H}\mathbb{Q}$ is a direct summand of $\mathrm{KGL}_{\mathbb{Q}}$ by the Grothendieck-Riemann-Roch theorem

The Main result

Theorem (Déglise-Fasel-J.-Khan):
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- The $+$-part $\mathbb{1}_{+,\mathbb{Q}}$ agrees with H$\mathbb{Q}$ (Cisinski-Déglise)
- Therefore it suffices to show that the minus part satisfies aboslute purity

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- Similar to the Chern character, the **Borel character** induces a decomposition of $\mathrm{KQ}_{\mathbb{Q}}$, where $\mathbb{1}_{-,\mathbb{Q}}$ can be identified as a direct summand
- This proves the absolute purity of $\mathbb{1}_{\mathbb{Q}}$ when 2 is invertible on the base scheme, since KQ is only well-defined in this case

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- For X a field of characteristic zero, ν_X is automatically an isomorphism; for X a field of positive characteristic, $\mathrm{SH}(X)_{-,\mathbb{Q}}$ vanishes by a theorem of Morel
- The key lemma then reduces the absolute purity of $\mathbb{1}_{-,\mathbb{Q}}$ in mixed characteristic to the case of $\mathbb{Q}$-schemes, which can be proved using Popescu's theorem: a closed immersion of affine regular schemes over a perfect field is a limit of closed immersions of smooth schemes

Some applications

- Our method can be used to deduce the following new results in mixed characteristic:
  - The six functors preserve constructible objects in the rational stable motivic homotopy category $\mathrm{SH}(\cdot, \mathbb{Q})$
  - The Grothendieck-Verdier duality holds for $\mathrm{SH}(\cdot, \mathbb{Q})$
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- The rational stable motivic homotopy category has a (unique) SL-orientation, that is, the Thom space of a vector bundle only depends on its determinant
- The rational bivariant groups $H_0^{\mathbb{A}^1}(X/S, v)_{\mathbb{Q}}$ agree with the rational Chow-Witt groups, and can be computed by the Gersten complex associated to Milnor-Witt K-theory
- Related work: absolute purity of the sphere spectrum over a Dedekind domain (Frankland-Nguyen-Spitzweck, work in progress)

Thank you!