Resolution of non-singularities and absolute anabelian conjecture

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Outline

X, Y: hyperbolic curves over finite extension of $\mathbb{Q}_p$

- Absolute anabelian conjecture $AAC(X, Y)$:
  Isomorphisms of étale fundamental group $\Pi_X \simeq \Pi_Y$ come from isomorphisms $X \simeq Y$
  (*absolute*: not given with an augmentation map to G_K)
- Resolution of non-singularities (RNS_X):
  Every semistable model of X is dominated by the stable model of some finite étale cover of X
- Main result of this talk:

$$RNS_X \quad \& \quad RNS_Y \implies AAC(X, Y)$$

relative Grothendieck Anab. Conj. (Isom. version)

Theorem (S. Mochizuki)

X/K, Y/L : two hyperbolic curves over sub-p-adic fields.
Π_X, Π_Y: étale fundamental groups of X and Y.
G_K, G_L: absolute Galois groups of L and K.
Assume following commutative diagram:

$$\begin{array}{ccc} \Pi_X & \xrightarrow{\sim} & \Pi_Y \\ \downarrow & & \downarrow \\ G_K & \xrightarrow{\sim} & G_L \end{array}$$

such that $G_K \to G_L$ is induced by an isomorphism $K \xrightarrow{\sim} L$.
Then $\Pi_X \to \Pi_Y$ is induced by an isomorphism $X \xrightarrow{\sim} Y$

absolute Anab. Conj. (Isom version)

Conjecture (AAC(X,Y), S. Mochizuki)

X/K, Y/L : two hyperbolic curves over p-adic fields.
Π_X, Π_Y: étale fundamental groups of X and Y.
Assume we have an isomorphism:

$$\phi : \Pi_X \xrightarrow{\sim} \Pi_Y$$

Then $\Pi_X \to \Pi_Y$ is induced by an isomorphism $X \xrightarrow{\sim} Y$

Proposition

Under the same assumptions, ϕ induces an isomorphism $G_K \xrightarrow{\sim} G_L$ (but not known to be geometric in general)

Remark

Neukirch-Uchida + Rel. AC $\implies$ AAC over Number Fields

Curves of Quasi-Belyi type

Definition

A hyp. curve X is of Quasi-Belyi type if there are maps:

$$\begin{array}{ccc} X & \xleftarrow[f.\acute{e}t.]{} & Y \\ & & \Big\downarrow {\scriptstyle dominant} \\ & & \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{array}$$

Theorem (Mochizuki)

If X, Y are curves of quasi-Belyi type, then $AAC(X, Y)$ is true.

Intermediate steps

$\widetilde{X} = \varprojlim_{(S,s_0)\to(X,x_0)} S$: universal pro-finite étale cover of X.

Natural action $\Pi_X \curvearrowright \widetilde{X}$

Definition

- Let x closed point of X and $\tilde{x} \in \widetilde{X}$ a preimage of x.
  $D_{\tilde{x}} = \mathrm{Stab}_{\Pi_X}(\tilde{x}) \subset \Pi_X$: decomposition group of x
  $D_x =$ conjugacy class of $D_{\tilde{x}}$.
- An isomorphism $\phi : \Pi_X \xrightarrow{\sim} \Pi_Y$ is point-theoretic if $D \subset \Pi_X$ is a decomposition group if and only if $\phi(\Pi_X)$ is a decomposition group.

Proposition

Let $\phi : \Pi_X \xrightarrow{\sim} \Pi_Y$ be point-theoretic, then ϕ is induced by an isomorphism $X \xrightarrow{\sim} Y$

Characterization of decomposition groups via *curspidalization* techniques.

Resolution of Non-Singularities

Definition

- Let X be a hyperbolic curve over an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. X satisfies resolution of non-singularities (RNS_X) if for every semi-stable model $\mathfrak{X}$ of X, there exists a finite étale cover $f: Y \to X$ such that f extends to a morphism $\mathfrak{Y} \to \mathfrak{X}$ where $\mathfrak{Y}$ is the stable model of Y.

$$\begin{array}{ccc} Y & \xhookrightarrow{\text{st.model}} & \mathfrak{Y} \\ \downarrow{\scriptstyle \text{f. ét}} & & \vdots \\ X & \xhookrightarrow{\text{semi-st.model}} & \mathfrak{X} \end{array}$$

- Let X be a hyperbolic curve over a finite extension $K$of $\mathbb{Q}_p$. X satisfies resolution of non-singularities (RNS_X) if its pullback to an algebraic closure of K does.

Valuative version of RNS

Definition

A valuation v on $K(X)$ is of type 2 if it extends the valuation of $\mathbb{Q}_p$ and its residue field $\tilde{k}_v$ is of transcendance degree 1 over $\mathbb{F}_p$.

Example

If $\mathfrak{X}$ is a normal model of X and Z is a irreducible component of the special fiber $\mathfrak{X}_s$, then $v_z = mult_Z$ is a valuation of type 2 on $K(X)$.
A valuation of this form where $\mathfrak{X}$ is the stable model (if it exists) is called *skeletal*

Proposition

X satisfies resolution of non-singularities if and only if for every valuation v of type 2 on $K(X)$, there exists a finite étale cover $Y \to X$ and a skeletal valuation v' on $K(Y)$ such that $v = v'_{|K(X)}$.

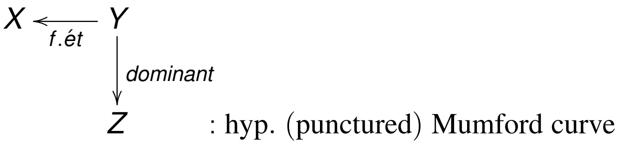
Example of Curves with RNS

A smooth curve over $\overline{\mathbb{Q}}_p$ is a Mumford curve if every normalized irreducible component of its stable model is isomorphic to $\mathbb{P}^1$.

Theorem

Let X, Y be two hyperbolic curves over $\overline{\mathbb{Q}}_p$ and assume that Y satisfies RNS.

1. *If there is a dominant map $f : X \to Y$, then X satisfies RNS.*
2. *If there is a finite étale cover $f : Y \to X$, then X satisfies RNS.*
3. *If X is a (punctured) Mumford curve, then X satisfies RNS.*
4. *If X is of Belyi type, then X satisfies RNS.*



Main Result

Theorem

Let X and Y be two hyperbolic curves over finite extensions of $\mathbb{Q}_p$ satisfying RNS. Then every isomorphism of fundamental groups $\phi : \Pi_X \xrightarrow{\sim} \Pi_Y$ is induced by an isomorphism $X \simeq Y$.

Remark

Includes some proper curves, contrary to the quasi-Belyi type result.

Sketch of the proof:

- Step 1: One just needs to show that ϕ is *point-theoretic*.
- Step 2: Recovery of the topological Berkovich space.
- Step 3: Characterization of rigid points.

Berkovich spaces

Let X be an alg. variety $/K$ non-archimedean field.
If $X = \operatorname{Spec} A$,

$$X^{\text{an}} = \{\text{mult. semi} - \text{norms } A \to \mathbb{R}_{\geqslant 0}, \text{extending norm of } K\}$$

topology: coarsest s.th. $\forall f \in A, x := |-(x)| \in X \mapsto |f(x)| \in \mathbb{R}$ cont.
$X \mapsto X^{\text{an}}$ functorial, maps open coverings to open coverings.
$\implies$ glues together for general X.
set theor., $X^{\text{an}} = \{(x, |-|); x \in X, |-| : \text{mult. norm on } k(x)\}$
Example of points:

- $\implies X_{cl} \hookrightarrow X^{an}$ (rigid points)
- $X(\hat{\overline{K}}) \to X^{an}$ (type 1 points ($\supset$ rigid points))
- If X is a smooth curve, valuations of type 2 on $K(X)$ induce points in X^{an} (type 2 points)

Example: the affine line

Let $X = \mathrm{Spec}(\mathbf{C}_p[T]) = \mathbb{A}^1_{\mathbf{C}_p}$.

If $a \in \mathbf{C}_p, r \in \mathbb{R}_{\geqslant 0}$, $|\sum_i a_i (T-a)^i|_{b_{a,r}} := \max_i(|a_i| r^i) \quad \rightsquigarrow b_{a,r} \in \mathbb{A}^{1,an}_{\mathbf{C}_p}$

- If r=0, $b_{a,r}$ of type 1.
- If $r \in |p|^{\mathbb{Q}}$, $b_{a,r}$ of type 2.
- If $r \notin |p|^{\mathbb{Q}}$, $b_{a,r}$ of type 3 ($\mathrm{rk}(|K(X)^\times|_{b_{a,r}}) = 2$).
- + points of type 4 corr. to decreasing sequences of balls with empty intersection, completion of $K(X)$ is an immediate extension of $\mathbf{C}_p$.

Berkovich curves

X/K: smooth curve over non-archimedean field
$\overline{X}$: smooth compactification of X
$\mathfrak{X}/O_K$: semi-stable model of X/K
$\mathbb{G}_{\mathfrak{X}}$: dual graph of the semi-stable curve $\mathfrak{X}_s$
There is a natural topological embedding ι and a strong deformation retraction π:

$$\mathbb{G}_{\mathfrak{X}} \underset{\pi}{\overset{\iota}{\rightleftarrows}} X^{an}$$

$X^{an} \backslash \iota(\mathbb{G}_{\mathfrak{X}})$: disjoint union of potential open disks (becomes a disk after finite extension of the base field).
By taking the inverse limit over all potential semi-stable models, they induce a homeomorphism

$$\overline{X}^{an} \xrightarrow{\sim} \varprojlim_{\mathfrak{X}/K'} \mathbb{G}_{\mathfrak{X}}$$

Sketch of the proof

- Step 2: Recovery of the topological Berkovich space.

Theorem (Mochizuki)

Let X/K and Y/L be two hyperbolic curves over finite extensions of $\mathbb{Q}_p$ that admit stable models $\mathfrak{X}/O_K$ and $\mathfrak{Y}/O_L$. They are naturally enriched as log-schemes $\mathfrak{X}^{log}$ and $\mathfrak{Y}^{log}$. Then every isomorphism $\Pi_X \xrightarrow{\sim} \Pi_Y$ induces an isomorphism of log-special fibers $\phi^{log} : \mathfrak{X}_s^{log} \xrightarrow{\sim} \mathfrak{Y}_s^{log}$.

In particular, it induces an isomorphism of dual graphs of the stable reduction: $\phi_{\mathbb{G}} : \mathbb{G}_X \xrightarrow{\sim} \mathbb{G}_Y$.

If X satisfies RNS, one gets a natural homeomorphism

$$\widetilde{X}^{\mathrm{an}} \subset \overline{\widetilde{X}}^{\mathrm{an}} := \varprojlim_{S} \overline{S} \xrightarrow{\sim} \varprojlim_{(S,s)} \mathbb{G}_S,$$

where S goes through pointed finite étale covers of X admitting stable reduction over their constant field ($\overline{S}$: smooth compactification of S).
Apply isom. $(-)_{\mathbb{G}}$ to open subgps of Π_X and Π_Y, $\rightsquigarrow$ homeomorphism

$$\tilde{\phi} : \widetilde{X}^{\mathrm{an}} \xrightarrow{\sim} \widetilde{Y}^{\mathrm{an}}$$

(compatible with the actions of Π_X and Π_Y and ϕ). Quotient by actions of the fundamental groups (resp. geom. fund. groups) $\rightsquigarrow$

$$\phi^{\mathrm{an}} : X^{\mathrm{an}} \xrightarrow{\sim} Y^{\mathrm{an}} \quad (\text{resp.} \quad \phi^{\mathrm{an}}_{\mathbb{C}_p} : X^{\mathrm{an}}_{\mathbb{C}_p} \xrightarrow{\sim} Y^{\mathrm{an}}_{\mathbb{C}_p}).$$

Action compatibility $\implies$ If $\tilde{x} \in \widetilde{X}^{\mathrm{an}}$, $\phi(D_{\tilde{x}}) = D_{\tilde{\phi}(\tilde{x})}$
$\rightsquigarrow$ To show point-theoreticity, it is enough to show that ϕ^{an} maps rigid points to rigid points.

Does every homeomorphism $X^{an} \to Y^{an}$ preserves rigid points?
No (cannot distinguish between type 1 and type 4 points).
Need of a stronger property about this homeomorphism.

- Step 3: Metric characterization of $\mathbb{C}_p$-points.
  Let $\mathfrak{X}$ be a semi-stable model of $X_{\mathbb{C}_p}$, x a node of $\mathfrak{X}_s$.
  Then $\mathfrak{X}$ étale loc. $\simeq \mathrm{Spec}\, O_{\mathbb{C}_p}[u,v]/(uv-a)$.
  Let e edge of dual graph $\mathbb{G}_{\mathfrak{X}}$ of $\mathfrak{X}$.
  Set $length(e) := v(a) \rightsquigarrow$ metric on $\mathbb{G}_{\mathfrak{X}}$.
  $X^{an}_{(2)} \simeq \mathrm{inj\,lim}_{\mathfrak{X}}\, V(\mathbb{G}_{\mathfrak{X}}) \rightsquigarrow$ natural metric d on $X^{an}_{(2)}$.

$$\phi^{log} : \mathfrak{X}_s^{log} \xrightarrow{\sim} \mathfrak{Y}_s^{log} \implies \phi^{an}_{(2)} : X^{an}_{\mathbb{C}_p,(2)} \to Y^{an}_{\mathbb{C}_p,(2)} \text{ is an isometry.}$$

Let $x_0 \in X^{an}_{(2)}$.

Proposition

Let $x \in X^{an}_{\mathbb{C}_p}$, then x is a $\mathbb{C}_p$-point (is of type 1) if and only if:

$$d(x_0, x) := \sup_U \inf_{z \in U_{(2)}} d(x_0, z) = +\infty$$

where U goes through open neighbourhood of x in $X^{an}_{\mathbb{C}_p}$.

Sketch:
Metric extends to $\iota(\mathbb{G}_{\mathfrak{X}}) \rightsquigarrow$ reduce to the case of a disk in $X^{an}_{\mathbb{C}_p} \backslash \iota(\mathbb{G}_{\mathfrak{X}})$.
In a disk, explicit description of the metric:

$$\begin{aligned} d(b_{a,r}, b_{a',r'}) &= |log_p(r) - log_p(r')| && \text{if } |a-a'| \leqslant max(r, r') \\ &= -2v(a-a') - log_p(r) - log_p(r')| && \text{if } |a-a'| \geqslant max(r, r') \end{aligned}$$

If $b_{a',r'} \to x$ of type 1, $r' \to 0$ so $d(b_{a,r}, b_{a',r'}) \to +\infty$.
If $(B(a_i, r_i))_{i \in \mathrm{N}}$ is a decreasing seq of balls s.t. $\bigcap_i B(a_i, r_i) = \varnothing$, $r_i \to r > 0$.

$\Longrightarrow \phi^{\text{an}}$ preserves points of type 1.
If x is a point of type 1, then x is rigid, if and only if the image $p(D_x)$ by the augmentation map $p : \Pi_X \to G_K$ is open.
$\Longrightarrow \phi^{\text{an}}$ preserves rigid points.

If X satisfies RNS and $\Pi_X \simeq \Pi_Y$, does Y satisfies RNS?
Not known in general...

Theorem (Mochizuki)

Let $\mathcal{B}$ (curves of Belyi type) be the smallest family of hyperbolic orbicurves over finite extensions of $\mathbb{Q}_p$ such that:

- $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ *belong to $\mathcal{B}$;*
- *If X belongs to $\mathcal{B}$ and $Y \to X$ is an open embedding, then Y belongs to $\mathcal{B}$;*
- *If X belongs to $\mathcal{B}$ and $Y \to X$ is finite étale, then Y belongs to $\mathcal{B}$;*
- *If X belongs to $\mathcal{B}$ and $X \to Y$ is finite étale, then Y belongs to $\mathcal{B}$.*
- *If X belongs to $\mathcal{B}$ and $Y \to X$ is a partial coarsification, then Y belongs to $\mathcal{B}$.*

If X and Y are hyperbolic orbicurves such that $X \in \mathcal{M}$, then every isomorphism $\Pi_X \tilde{\to} \Pi_Y$ comes from an isomorphism $X \tilde{\to} Y$

Corollary

Let $\mathcal{M}$ be the smallest family of hyperbolic orbicurves over finite extensions of $\mathbb{Q}_p$ such that:

- *Hyperbolic (punctured) Mumford curves belong to $\mathcal{M}$;*
- *If X belongs to $\mathcal{M}$ and $Y \to X$ is an open embedding, then Y belongs to $\mathcal{M}$;*
- *If X belongs to $\mathcal{M}$ and $Y \to X$ is finite étale, then Y belongs to $\mathcal{M}$;*
- *If X belongs to $\mathcal{M}$ and $X \to Y$ is finite étale, then Y belongs to $\mathcal{M}$.*
- *If X belongs to $\mathcal{M}$ and $Y \to X$ is a partial coarsification, then Y belongs to $\mathcal{M}$.*

If X and Y are hyperbolic orbicurves such that $X \in \mathcal{M}$, then every isomorphism $\Pi_X \xrightarrow{\sim} \Pi_Y$ comes from an isomorphism $X \xrightarrow{\sim} Y$

Sketch of RNS for Mumford curves

Let X/K be a proper Mumford curve.

- It is enough to show that the union of images of skeletal valuation by arbitrary finite étale morphisms are dense inside X^{an}.
- p-adic Hodge theory $\implies$ $H^1(X, \mathbb{Z}_p(1)) \xrightarrow{p_X} H^0(X, \Omega^1_X) \otimes_K \mathbb{C}_p$
  Local computation in a disk:

Lemma

If $c \in H^1(X, \mathbb{Z}_p(1))$, let $Y_{n,c} \xrightarrow{\phi_{n,c}} X$ be the finite étale μ_{p^n}-cover corr. to c. If $x \in X(\mathbb{C}_p)$ s.t. $\exists c \in H^1(X, \mathbb{Z}_p(1))$ s.t. $p_X(c) \neq 0$ and $\mathrm{mult}_x p_X(c)$ is not of the form $p^k - 1$ for any k, then $\exists z_n \in (Y_{n,c})_{K_n}$ s.t. $\phi_{n,c}(z_n) \xrightarrow{n} x$.

- Let X be a hyperbolic curve over K. Let $x \in X(K')$, where K' is a finite extension of K.

$$\begin{array}{ccccc}
H^1(X, \mathbb{Z}_p(1))/(\mathrm{Ker}\, p_X) & \hookrightarrow & H^0(X, \Omega^1_X) & \xrightarrow{ev_x} & K' \subset \mathbb{C}_p \\
\big\downarrow & & \big\downarrow & \nearrow_{ev'_x} & \\
H^1(X', \mathbb{Z}_p(1))/(\mathrm{Ker}\, p_{X'}) & \hookrightarrow & H^0(X', \Omega^1_{X'}) & &
\end{array}$$

where X' is any topological finite cover and $x' \in X'(K')$ is a preimage of x.
As $g(X') \to \infty$, $\dim_{\mathbb{Q}_p} H^1(X', \mathbb{Q}_p(1))/(\mathrm{Ker}\, p_{X'}) \to \infty$, but $\dim_{\mathbb{Q}_p} K'$ stays finite
$\implies \exists (X', x')$ and $c \in H^1(X, \mathbb{Z}_p(1)) \backslash (\mathrm{Ker}\, p'_X)$ s.t. $p_X(c)(x') \neq 0$.