Counting rational points, the determinant method and the pseudo-effective threshold

arxiv: 1910.00306.

Work over $\mathbb{Q}$, $\mathbb{Z}$, Everything works over number fields.

$X \hookrightarrow \mathbb{P}^n_{\mathbb{Q}}$, $\dim(X) = d$, $\deg(X) = \delta$.

$\xi = [x_0 : \cdots : x_n] \in X(\mathbb{Q})$. primitive $\mathbb{Z}$-coordinate

Height. $H(\xi) = \max\{|x_0|, \cdots, |x_n|\}$. $h(\xi) = \log H(\xi)$.

$S(X;B) = \{\xi \in X(\mathbb{Q}) \mid H(\xi) \leq B\}$. $\# S(X;B) = N(X;B) < +\infty$.

Northcott's property.

- Counting rational points: understand $N(X;B)$.

Prop. $N(\mathbb{P}^n_{\mathbb{Q}}; B) = \frac{2^n}{\zeta(n+1)} B^{n+1} + o(B^{n+1})$ $B \to +\infty$.

$$\rightsquigarrow N(X;B) \ll_n \delta B^{d+1}$$

"$\sim$" when $\delta = 1$.

- $d = 1$. $N(X;B) \ll_{n,\delta,\varepsilon} B^{2/\delta + \varepsilon}$ $\forall \varepsilon > 0$ (Bombieri-Pila, 1989).

  Walsh (2015) $N(X;B) \ll_{n,\delta} B^{2/\delta}$.

  Castryck, Cluckers, Dittermann, Nguyen (2019).

  (Plane curve): $N(X;B) \ll \delta^4 B^{2/\delta}$.

  Conj (Heath-Brown) $N(X;B) \ll \delta^2 B^{2/\delta}$.

  genus ≥ 1. (Ellenberg, Venkatesh, 2013).

  $C \hookrightarrow \mathbb{P}^2_{\mathbb{Q}}$. $\exists\, c = c(\delta) > 0$ explicit, s.t. $N(C;B) \ll_\delta B^{2/\delta - c}$.

- $d \geq 2$. Conj (Heath-Brown, Serre). $\delta \geq 2$, $d \geq 2$. $\forall \varepsilon > 0$.

  $N(X;B) \ll_{n,\delta,\varepsilon} B^{d+\varepsilon}$

  C.C.D.N (2019). $\delta \geq 5$, $N(X;B) \ll_n \delta^{e(n)} B^d$.

  $\delta = 4$ OK. $\delta = 3$ open.

  $\delta = 2$ H-B, 2002, quadratic form.

Reformulated by Arakelov geometry.

$X \hookrightarrow \mathbb{P}^n_{\mathbb{Q}}$ $\dim(X)=d$, $\deg(X)=\delta$.

$\mathscr{X} \hookrightarrow \mathbb{P}^n_{\mathbb{Z}}$. schematic closure, $\overline{\mathscr{L}} - \mathscr{X}$ hermitian line bundle

$\xi \in X(\mathbb{Q})$ $P_\xi \in \mathscr{X}(\mathbb{Z})$ closure.

$$\begin{tikzcd} P_\xi^*\overline{\mathscr{L}} \arrow[r, dash] \arrow[d, dash] & \overline{\mathscr{L}} \arrow[d, dash] & \\ \operatorname{Spec}\mathbb{Z} \arrow[r, "P_\xi"] & \mathscr{X} \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

$$h_{\overline{\mathscr{L}}}(\xi) := \widehat{\deg}(P_\xi^*\overline{\mathscr{L}})$$
$$\parallel$$
$$-\sum_{v\in M_{\mathbb{Q}}} \log\|s\|_v .$$

$\mathbb{Z}$-primitive coordi

<u>Rmk</u>. $\boxed{\text{If}}$ $\mathcal{O}_{\mathbb{P}^n}(1) \leftarrow \ell^2$-norm. ~~$\forall$~~ for $v\in M_{\mathbb{Q},\infty}$ $\xi=[x_0:\cdots:x_n]$.

$\boxed{\text{Then}}$ $h(\xi) = \boxed{\cancel{\sum_p \log\max\{|x_i|_p\}}} + \log\sqrt{|x_0|^2+\cdots+|x_n|^2}$.

$H(\xi)=\exp(h(\xi))$.

$S(X;B)$ $\qquad N(X;B)=\#S(X;B)<+\infty$ $\mathscr{L}$: ample.

Determinant method.

Find hypersurfaces $\{H_i\}_{i\in I}$ in $\mathbb{P}^n$. s.t. $S(X;B)\subseteq \bigcup_i S(H_i;B)$

$\eta_X \notin \bigcup_i H_i$

come from matrix of monomials, Siegel's lemma, determinant.

⚠ (arithmetic part) Control $\deg(H_i)$ and $\# I$.

⚠ (geometric part). sub-varieties of X.

Viewpoint of the slope method

$D \in \mathbb{N}_+$. $F_D \subset H^0(\mathcal{X}, \mathcal{O}(1)|_{\mathcal{X}}^{\otimes D})$ $(D >> 0,\ F_D = \cdots)$.

- evaluation map (Bost). $\varphi_D : F_D \longrightarrow \bigoplus_{P \in S} P^* \mathcal{O}(1)|_{\mathcal{X}}^{\otimes D}$

$r_1(D) = rk(F_D)$. $\qquad s \longmapsto (s(P))_{P \in S}$.

Rmk $\boxed{\text{If}}$ φ_D not injective (bijective).

$\boxed{\text{Then}}$ $\exists\, s \in F_D \setminus \{0\}$. s.t. $\varphi_D(s) = 0 \Rightarrow \mathrm{div}(s) \supset S$.

Key point: choose S?

slope inequalities. $f : V \longrightarrow W$ linear map $/\mathbb{Q}$.

$\overline{V} \longrightarrow \overline{W}$ hermitian vector bundles on $\mathrm{Spec}\,\mathbb{Z}$.

$h(f) := \sum_{v \in M_{\mathbb{Q}}} \log \|f\|_v \longleftarrow$ operator norm.

Lemma. f: isom $\Longrightarrow \widehat{\deg}(\overline{V}) = \widehat{\deg}(\overline{W}) + h(\det(f))$.

$\Longrightarrow \hat{\mu}(\overline{V}) \leq \hat{\mu}_{max}(\overline{W}) + \frac{1}{r} h(\Lambda^r f)$ --- $(\bigstar)$.

$\hat{\mu}(\overline{V}) := \widehat{\deg}(\overline{V}) / rk(\overline{V})$.

Semi-global version (Salberger, 2015).

- $\{P_i\}_{i \in I} \subseteq X(\mathbb{Q})$ $(\mathcal{X}(\mathbb{Z}))$. ~~$P\mathcal{X}$~~ p: prime.

s.t. $\{P_i\}_{i \in I} \bmod p = \xi \in \mathcal{X}(\mathbb{F}_p)$.

m_ξ: maximal ideal of $\mathcal{O}_{\mathcal{X},\xi}$.

$R_\xi(F_D) := \sum_{k=1}^{\infty} \dim(F_D \cap m_\xi^k)$ Considered as $\mathbb{Z}_p$-module.

finite sum.

$(*)$ Thm. $X \longrightarrow \mathbb{P}^n_{\mathbb{Q}}$ integral.

$(R_i)_{i \in I} \subseteq X(\mathbb{Q})$. $(p_j)_{j \in J}$: primes.

$(R_i)_{i \in I} \bmod p_j = \xi_j \in \mathcal{X}(\mathbb{F}_{p_j})$ regular ⚠.

$\boxed{\text{If}}$ $\sup_i h(R_i) < \frac{\hat{\mu}(\bar{F}_D)}{D} - \frac{\log r_1(D)}{2D} + \sum_{j\in J} \frac{R_{\xi_j}(F_D)}{D\cdot r_1(D)} \log p_j$

$\boxed{\text{Then}}$ $\exists$ hypersurface H, $\deg(H) = D$. s.t. $\{R_i\}_{i\in I} \subseteq H$. $\eta_X \notin H$.

<u>sketch of proof</u>. If NOT. φ_0 is injective (bijective).

$(\star) \Longrightarrow \frac{\hat{\mu}(\bar{F}_D)}{D} \leq \sup_i h(R_i) + \frac{1}{D r_1(D)} h(\Lambda^{r_1(D)} \varphi_D)$.

- $v \in M_{\mathbb{Q},\infty}$. Hadamard's inequalities $\Longrightarrow \frac{1}{r_1(D)} \log \|\Lambda^{r_1(D)} \varphi_0\|_v \leq \log \sqrt{r_1(D)}$
- $v \in M_{\mathbb{Q},f}$. $v \to p$. then $\|\Lambda^{r_1(D)} \varphi_0\|_v \leq p^{-R_{\xi_j}(F_D)} \Longrightarrow \Sigma$ $\square$.

I) Understand $R_{\xi_j}(F_D)$

<u>Prop</u>. $\dim(F_D \cap m_\xi^k) = \dim \ker(F_{D,\mathbb{Q}} \longrightarrow H^0(X, \mathcal{O}(1)|_X^{\otimes D} \otimes \mathcal{O}_X / n_{\eta_j}^k))$.

$\eta_j \in X(\mathbb{Q})$ preimage of ξ_j. n_{η_j}: maximal ideal sheaf of η_j in $\mathcal{O}_X$.

$\eta \in X(\mathbb{Q})$. regular $\pi : \tilde{X} \longrightarrow X$ blow-up at η.

$E \subseteq \tilde{X}$. exceptional divisor. $I_E \subseteq \mathcal{O}_{\tilde{X}}$ ideal sheaf.

$$\begin{array}{ccc} H^0(X, \mathcal{O}(1)|_X^{\otimes D}) & \xrightarrow{f'} & H^0(X, \mathcal{O}(1)|_X^{\otimes D} \otimes \mathcal{O}_X / n_\eta^k) \\ \downarrow & & \downarrow \\ H^0(\tilde{X}, \pi^*\mathcal{O}(1)|_X^{\otimes D}) & \xrightarrow{\tilde{f}} & H^0(\tilde{X}, \pi^*\mathcal{O}(1)|_X^{\otimes D} \otimes \mathcal{O}_{\tilde{X}} / I_E^k) \end{array}$$

<u>Prop</u>. $\dim \ker f = \dim \ker \tilde{f}$.

H: Cartier divisor of X given by a hyperplane section of $\mathbb{P}^n$.

$\boxed{\text{Then}}$

$$R_\xi(F_D) = \sum_{k=1}^{\infty} h^0(\tilde{X}, D\pi^*H - kE)$$

$$= \frac{D^d}{d!} \sum_{k=1}^{\infty} \mathrm{vol}\left(\pi^*H - \frac{k}{D}E\right) + O(D^d)$$

$$= \frac{D^{d+1}}{d!} \underbrace{\int_0^\infty \mathrm{vol}(\pi^*H - \lambda E)\, d\lambda}_{\overset{!!}{I_X(H,\xi)}} + o(D^d)$$

$I_X(H, \xi) \leftarrow$ only depends on the reduction

(**) Thm $B = \sup_i H(P_i)$, $\varepsilon > 0$.

$\boxed{\text{If}}$ $\sum_{j \in J} \log p_j \gg_{n,\varepsilon,} \frac{\delta}{I_X(H,\xi)} \log B$.

$\boxed{\text{Then}}$ $\{P_i\}$ can be covered by a hypersurface of degree $O_{d,\delta,\varepsilon}(1)$.

control $S(X;B)$ $<$ i) $S(X^{sing};B)$, ii) $S(X^{reg};B)$.

i) Prop (H. Chen, 2012). X^{sing} can be covered by a hypersurface of degree $(\delta-1)(n-d)$ Chow form.

ii) $S(X^{reg};B) = \bigcup_{i=1}^{r} S(X^{reg};B,P_i)$.

$$\| $$

$$\bigcup_{\xi \in \mathcal{X}(\mathbb{F}_{p_i})} S(X^{reg};B,\xi) \leftarrow \text{reduction mod } p = \xi.$$

a control of r.

(***) Thm $S(X;B)$ can be covered by at most $\square$ hypersurfaces. of degree $O_{d,\delta,\varepsilon}(1)$.

$\square \ll_{\varepsilon,\delta,n} B^{(1+\varepsilon)d\delta/I_X(H,\xi)}$ $\longrightarrow$ taking lower bound.

II) Understand $I_X(H,\xi)$.

Prop (~~McKinnon~~ McKinnon-Roth, 2015, Salberger 2008).

$$I_X(H,\xi) \geq \frac{d\,vol(H)}{dt\ 1} \sqrt[d]{\frac{vol(H)}{\mu_\eta(X)}} \geq \frac{d}{d+1} \varepsilon_\eta(H)\, vol(H).$$

$$= \frac{d}{d+1} \delta^{1+\frac{1}{d}} \quad (\eta : \text{regular}) \Big\} \Longrightarrow \text{classic result}.$$

(**) Thm

Application. Ex (Salberger 2015). $X \hookrightarrow \mathbb{P}^n$ cubic surface.

U: complement of X of all lines. $N(U;B) \ll_\varepsilon B^{\frac{12}{7}+\varepsilon}$, $\forall \varepsilon > 0$.

better than $d\sqrt{\ }$. (classic method)

Conj (Manin) $N(U;B) \ll_\varepsilon B^{1+\varepsilon}$ $\forall \varepsilon > 0$