2nd Kyoto-Hefei Workshop on Arithmetic Geometry

- Higher residue pairing for projective smooth complete intersections.

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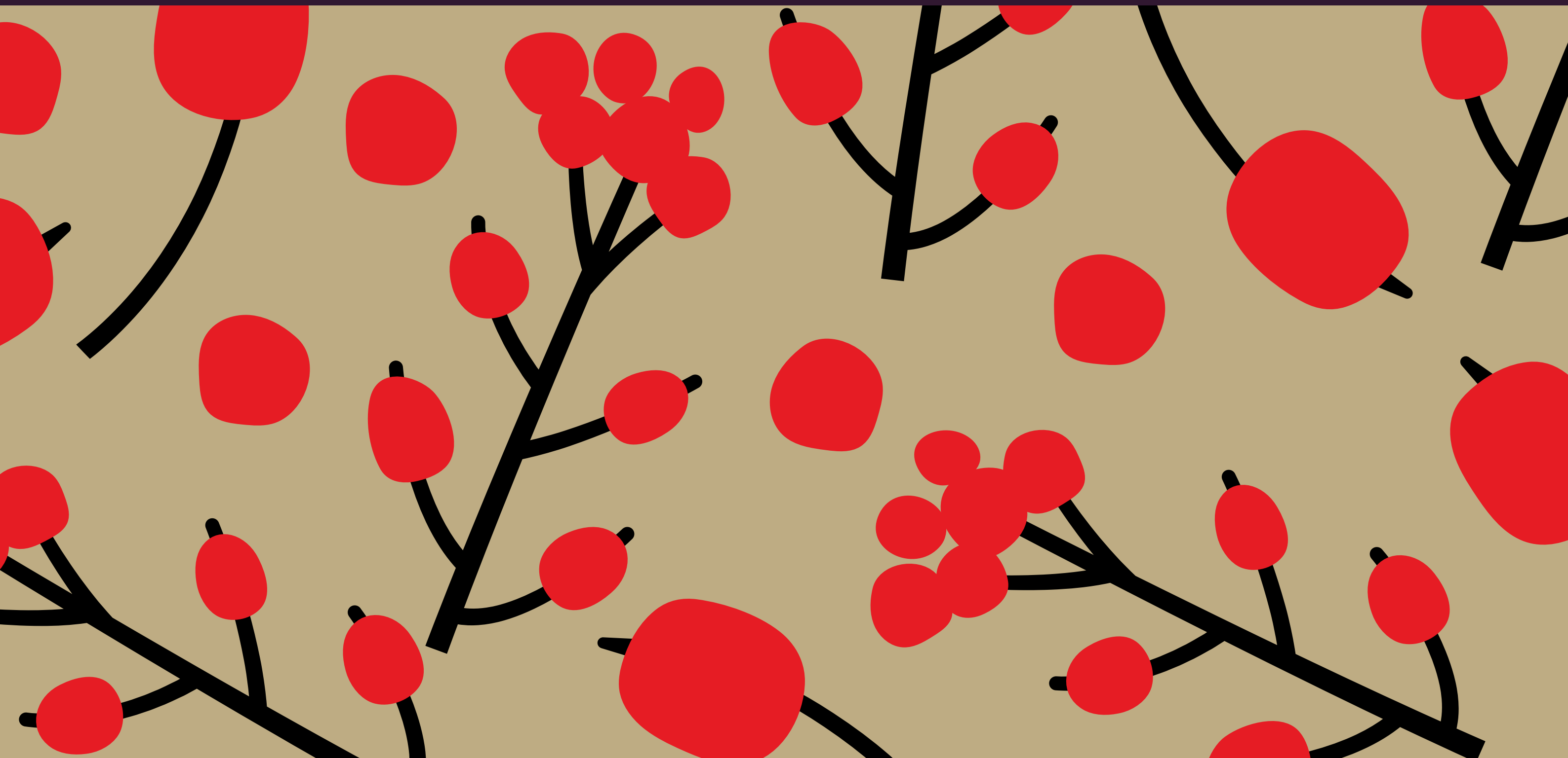

Higher residue pairing for projective smooth complete intersections.

Joint with Jaehyun Yim.

$\mathbb{P}^n \quad [x_1, \cdots, x_n] = \underline{x}$

$G_1(\underline{x}), \cdots, G_k(\underline{x})$ homog. poly

$\deg G_i(x) = d_i, \quad i = 1, \cdots, k$

$X_{\underline{G}}$ = a smooth projective complete intersection variety. defined by $G_1(\underline{x}) = \cdots = G_k(\underline{x}) = 0$

$X := X_{\underline{G}}(\mathbb{C}) \subsetneq \mathbb{P}^n(\mathbb{C})$

Consider $H^{n-k}_{prim}(X, \mathbb{C}) =: \mathbb{H}$

+ the cup product

$\mathbb{H} \times \mathbb{H} \to H^{2(n-k)}(X, \mathbb{C}) \simeq \mathbb{C}$

Higher residue pairing:

A quantized version of the cup product.

Caley's trick (C.I. ⇝ hypersurface)

$$\begin{tikzcd}
\mathbb{P}^{k-1} \arrow[r] & \mathbb{P}(\mathcal{E}) \supseteq X_S \arrow[d] \\
& \mathbb{P}^n \supseteq X_{\underline{G}}
\end{tikzcd}$$

$A := \mathbb{C}[y_1, \cdots, y_k, x_0, \cdots, x_n]$

$\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_k)$

$\mathbb{C}[x_0, \cdots, x_n]$

$\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z} \quad (ch)$

$\mathrm{Pic}(\mathbb{P}(\mathcal{E})) \simeq \mathbb{Z}^2 \quad (ch, wt)$

where $S := \sum_{i=1}^{k} y_i G_i(\underline{x})$

$$\begin{tikzcd}[column sep=small] X_0, & X_1, & \cdots, X_n, & y_1, & \cdots, y_k \\ \underline{q} = q_1 \arrow[u, phantom, "\shortparallel"] & q_2 \arrow[u, phantom, "\shortparallel"] & q_{n+1} \arrow[u, phantom, "\shortparallel"] & q_{n+2} \arrow[u, phantom, "\shortparallel"] & q_N \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

where $N = n+k+1$.

$\mathrm{ch}(x_i) = 1 \qquad \mathrm{ch}(y_i) = -d_i$

$\mathrm{wt}(x_i) = 0 \qquad \mathrm{wt}(y_i) = 1$

$$\mathbb{P}(\varepsilon)(\mathbb{C}) = \frac{\mathbb{C}^N - (\mathbb{C}^{n+1} \times 0) \cup (0 \times \mathbb{C}^k)}{\mathbb{C}^\times \times \mathbb{C}^\times}$$

$\mathbb{C}^\times \times \mathbb{C}^\times$ acts on $\mathbb{C}^N$

$$(c, \omega) \cdot (q_1, \cdots, q_N)$$
$$= c \cdot X_0, \cdots, c \cdot X_N, c^{-d_1} \cdot \omega \cdot y_1, \cdots, c^{-d_k} \cdot \omega \cdot y_k)$$

The background charge $c_X = \sum_{i=1}^{k} d_i - (n+1)$

$$= -\sum_{i=1}^{N} \mathrm{ch}(q_i)$$

$\mathrm{ch}(S) = 0$. $\mathrm{wt}(S) = 1$.

$$\rho : \overset{\text{abelian Lie alg}}{\mathfrak{g}} \longrightarrow \mathrm{End}(A)$$
$$d_i \longmapsto \frac{\partial}{\partial q_i} + \frac{\partial S}{\partial q_i}$$
$$:= e^{-S} \frac{\partial}{\partial q_i} \cdot e^S$$

The $\overset{\text{(dual)}}{\text{Chevalley}}$-Eilenberg complex for Lie alg. homology:

$$\begin{tikzcd} & & & A \\ 0 \arrow[r] & \mathcal{A}^{-N} \arrow[r] & \cdots \arrow[r] & \mathcal{A}^{-1} \arrow[r, "Q_S+\Delta \\ =: K_S"'] & \mathcal{A}^0 \arrow[u, phantom, "\shortparallel"] \arrow[r] & 0 \end{tikzcd}$$

$$Q_S = \sum_{i=1}^{N} \frac{\partial S}{\partial q_i} \cdot \frac{\partial}{\partial \eta_i}, \qquad \mathrm{ch}(\eta_i) + \mathrm{ch}(q_i) = 0$$

$$\Delta = \sum_{i=1}^{N} \frac{\partial}{\partial q_i} \cdot \frac{\partial}{\partial \eta_i}. \qquad \mathrm{wt}(\eta_i) + \mathrm{wt}(q_i) = 1$$

$$\mathcal{A}^{-1} := \mathcal{A} \cdot \eta_1 \oplus \cdots \oplus A \cdot \eta_N$$
$$\mathcal{A}^{-2} := \mathcal{A}\, \eta_1 \wedge \eta_2 \oplus \cdots \oplus A \cdot \eta_{N-1} \wedge \eta_N$$

$\mathcal{A}^r_{c_X}$ = the charge c_X-part of $\mathcal{A}^r_{c_X}$

$\mathcal{A}^r_{(\omega)}$ = the weight ω-part of $\mathcal{A}^r_{c_X}$

$(\Omega^N_{\mathbb{C}[\underline{q}]}, d+dS\wedge)$ $\qquad$ $(\Omega^N_{\mathbb{C}[\underline{q},S^{-1}]}, d)$

$$\begin{cases} ch(dq_i) - ch(q_i) = 0 \\ wt(dq_i) - wt(q_i) = 0 \end{cases}$$

$$\begin{tikzcd}
\mathcal{A}^\circ_{C_X} \arrow[r, "\mu"] & (\Omega^N_{\mathbb{C}[\underline{q}]})_0 \arrow[d, "\rho"] \\
& (\Omega^N_{\mathbb{C}[\underline{q},S^{-1}]})_{0,(0)} \arrow[d, "\theta_{wt}\circ\theta_{ch}"] \\
& \Omega^{N-2}_{\mathbb{C}[\underline{q},S^{-1}]_{0,(0)}} \arrow[d, "quotient"] \\
& \Omega^{N-2}_B / d(\Omega^{N-3}_B) = H^{N-2}_{dR}(\mathbb{P}(\mathcal{E})\setminus X_S) \arrow[d, "\simeq"'] \\
& H^{N-2}_{dR}(\mathbb{P}^n\setminus X_{\underline{G}}) \arrow[d, "\simeq"', "Res_{\underline{G}}"] \\
& H^{n-k}_{prim}(X_{\underline{G}}, \mathbb{C})
\end{tikzcd}$$

$N = n+k+1$

$A = \mathbb{C}[\underline{q}]$

$B := \mathbb{C}[\underline{q}, S^{-1}]_{0,(0)}$

$\mathbb{P}(\mathcal{E})\setminus X_S$
$= \mathrm{Spec}(B)$

Let $\varphi : \mathcal{A}^\circ_{C_X} \longrightarrow H^{n-k}_{prim}(X_{\underline{G}}, \mathbb{C})$ be the composition of the above $\mathbb{C}$-linear maps.

Lemma 1 φ is surjective and

$$\ker\varphi \simeq (Q_S + A)(\mathcal{A}^{-1}_{C_X}).$$

Thus

$$\mathcal{A}^\circ_{C_X} \Big/ (Q_S+A)(\mathcal{A}^{-1}_{C_X}) \xrightarrow[\sim]{\varphi} H^{n-k}_{prim}(X_{\underline{G}}, \mathbb{C})$$

Lemma 2 We have

$$\mathcal{A}^\circ_{C_X} \Big/ (Q_S+A)(\mathcal{A}^{-1}_{C_X}) \xleftarrow[\simeq]{\Phi} \mathcal{A}^\circ_{C_X} \Big/ Q_S(\mathcal{A}^{-1}_{C_X})$$

as $\mathbb{C}$-vector spaces.

Rmk.

(1) $$u : (\mathcal{A}^{\bullet}_{Cx}, Q_S, \Delta) \xrightarrow{u} ((\Omega^{\bullet}_{A})_0[-N], dS, d)$$

is a cochain isomorphism.

(2) $$\rho : ((\Omega^{\bullet}_{A})_0, dS + d) \xrightarrow[\rho]{} ((\Omega^{\bullet}_{A[\frac{1}{S}]})_{0,(0)}, d)$$

is a cochain quasi-isomorphism.

$$(\rho \circ u)(\underline{y}^{\underline{\nu}} \cdot \underline{x}^{\underline{u}})$$

$$= (-1)^{|\underline{\nu}|+k-1} (|\nu|+k-1)! \frac{\underline{y}^{\underline{\nu}} \cdot \underline{x}^{\underline{u}}}{S^{|\nu|+k}} d\underline{q}$$

for $\underline{y}^{\underline{\nu}} \cdot \underline{x}^{\underline{u}} \in \mathcal{A}^{\circ}_{Cx}$

(3) $$\mathrm{ch}(q_i) = \mathrm{ch}(dq_i), \quad i = 1, 2, \cdots, N$$
$$\mathrm{wt}(q_i) = \mathrm{wt}(dq_i), \quad i = 1, 2, \cdots, N$$

Θ_{ch} is the contraction operator with the vector field $\sum_{i=1}^{N} \mathrm{ch}(q_i) \cdot q_i \cdot \frac{\partial}{\partial q_i}$

Θ_{wt} is the contraction operator with the vector field $\sum_{i=1}^{N} \mathrm{wt}(q_i) \cdot q_i \cdot \frac{\partial}{\partial q_i}$

(4) $$\begin{tikzcd} \mathbb{P}(\mathcal{E}) \setminus X_S \arrow[d, "pr_1"] \\ \mathbb{P}^n \setminus X_{\underline{G}} \arrow[u, bend left, "s"] \end{tikzcd}$$

is an affine bundle of dimension $k-1$.

Thus pr_1 is a homotopy equivalence.

Define

$$S(\underline{x}) = (\underline{x}, G_1(\underline{x})^{e_1 - 1} \cdot \overline{G_1(\underline{x})}^{e_1}, \cdots, G_k(\underline{x})^{e_k - 1} \cdot \overline{G_k(\underline{x})}^{e_k})$$

$$e_i = \frac{\mathrm{lcm}(d_1, \cdots, d_k)}{d_i}$$

Then $S(\underline{x})$ is a well-defined section of pr_1.

(5) $\mathrm{Res}_{\underline{G}}$ is defined on the differential forms (not only on the cohomology classes).

(See [Dimca], Lemma 12)

Lemma 2 suggests the following quantization complex (The quantization complex $(\mathcal{A}^\bullet_{Cx}((\hbar)), Q_S+\hbar\Delta)$ can be viewed as a perturbative expansion of $(\mathcal{A}^\bullet_{Cx}, Q_S)$.

$(\mathcal{A}^\bullet_{Cx}((\hbar)), Q_S+\hbar\Delta)$ with the formal parameter $\hbar$ satisfying

$$ch(\hbar) = 0$$
$$wt(\hbar) = 1$$
$$gh(\hbar) = 0$$

Note that $\begin{cases} wt(Q_S) = 0 \\ wt(\Delta) = -1 \end{cases}$ implies that

$$wt(\hbar\Delta) = 0. \qquad gh(Q_S+\hbar\Delta)=1$$

Therefore $wt(Q_S+\hbar\Delta) = 0$

$ch(Q_S) = ch(\Delta) = 0$ also implies that

$$ch(Q_S+\hbar\Delta) = 0.$$

$(\mathcal{A}^\bullet_{Cx}((\hbar)), Q_S+\hbar\Delta)$ is a $\mathbb{Z}$-filtered complex:

$$(\mathcal{A}^\bullet_{Cx}[\![\hbar]\!]\hbar^{-m}, Q_S+\hbar\Delta) \overset{def}{=:} (\mathcal{A}^\bullet_{Cx}((\hbar))^{(m)}, Q_S+\hbar\Delta)$$

for $m \in \mathbb{Z}$.

We define a cochain map $r^{(0)}$

$$(\mathcal{A}^\bullet_{Cx}[\![\hbar]\!], Q_S+\hbar\Delta) \xrightarrow{r^{(0)}} (\mathcal{A}^\bullet_{Cx}. Q_S)$$

by taking the quotient by $\mathcal{A}^\bullet_{Cx}((\hbar))^{(-1)}$.

(i.e sending "$\hbar = 0$"

: the classical limit of the quantization complex.)

$$0 \to \mathcal{A}^\bullet_{Cx}((\hbar))^{(-1)} \hookrightarrow \mathcal{A}^\bullet_{Cx}((\hbar))^{(0)} \xrightarrow{r^{(0)}} \mathcal{A}^\bullet_{Cx} \to 0$$

<u>Def</u>:

$$\mathbb{H}_S := \frac{\mathcal{A}^0_{Cx}((\hbar))}{(Q_S+\hbar\Delta)(\mathcal{A}^{-1}_{Cx}((\hbar)))}$$

$$\mathbb{H}_S^{(m)} := \frac{\mathcal{A}^0_{Cx}((\hbar))^{(m)}}{(Q_S+\hbar\Delta)(\mathcal{A}^{-1}_{Cx}((\hbar))^{(m)})} = \frac{\mathcal{A}^0_{Cx}[\![\hbar]\!]\cdot\hbar^{-m}}{(Q_S+\hbar\Delta)(\mathcal{A}^{-1}_{Cx}((\hbar))^{(m)})}$$

for $m \in \mathbb{Z}$.

We also have an exact seq:

$$0 \to \mathbb{H}_S^{(-1)} \to \mathbb{H}_S^{(0)} \xrightarrow{r^{(0)}} \mathcal{A}^0_{Cx} \Big/ Q_S(\mathcal{A}^{-1}_{Cx}) \to 0$$

Prop. The triple

$$\left(\mathcal{A}^{\bullet}_{c_X}((\hbar)),\ Q_S + \hbar\Delta\ ,\ l_2^{\Delta}(\cdot\ ,\ \cdot) \right)$$

is a shifted dgla.

(differential graded Lie algebra)

Assume that $c_X := -(n+1) + \sum_{i=1}^{k} d_i = 0$

($X_{\underline{G}}$ is Calabi-Yau)

Prop. The quadraple

$\left(\mathcal{A}^{\bullet}_{0}((\hbar)),\ \bullet,\ Q_S,\ l_2^{\Delta} \right)$ is a DGBV algebra.

$\mathbb{H} := H^{n-k}_{prim}(X_{\underline{G}}, \mathbb{C})$

From the previous discussion, we have a $\mathbb{C}$-linear Bom

$$\begin{tikzcd} r^{(0)}(\mathbb{H}_S^{(0)}) \arrow[r, "\sim", "\mathfrak{q}"'] & \dfrac{\mathcal{A}_0^{0}}{(Q_S+\Delta)(\mathcal{A}_0^{-1})} \arrow[r, "\sim", "\varphi"'] & \mathbb{H} \end{tikzcd}$$

" non-canonical"
defined in the level of cohomologies

defined in the level of cochains.

Def. $|I| = \mu$. $\dim_{\mathbb{C}} \mathbb{H} =: \mu$

$T_{\mathbb{H}} \overset{def}{:=}$ the space of formal tangent vector fields on $\mathbb{H}$.

If we let $\{t^{\alpha} : \alpha \in I\}$ the coordinate of the affine manifold $\mathbb{H}$, then

$$T_{\mathbb{H}} = \mathbb{H}[[\underline{t}]].$$

$$\xleftarrow[\varphi \circ \mathfrak{q}]{\sim} \left(\mathcal{A}_0^{0} \big/ Q_S(\mathcal{A}_0^{-1}) \right)[[\underline{t}]]$$

where $\varphi \circ \mathfrak{q}$ is assumed to be $\underline{t}$-linear.

Note:

$Q_S(\mathcal{A}_0^{-1})$ is the Jacobian ideal of the commutative ring $\mathcal{A}_0^{0} = \mathbb{C}[\underline{q}]_0$ generated by $\frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_{6N}}$.

We have an exact sequence of $\mathbb{C}[\![\underline{t}]\!]((\hbar))$-modules

$$0 \to Q_S(\mathcal{A}_0^{-1}[\![\underline{t}]\!]((\hbar))) \hookrightarrow \mathcal{A}_0^{0}[\![\underline{t}]\!]((\hbar)) \xrightarrow[\varphi\circ q]{} \!\!\!\!\rightarrow T_{\mathbb{H}}((\hbar)) \longrightarrow 0$$

$$T_{\mathbb{H}}((\hbar)) = \mathbb{H}[\![\underline{t}]\!]((\hbar))$$

Notation:

$$f(\hbar) = \sum_{n\in\mathbb{Z}} {}^{n}u \cdot \hbar^{n} \Rightarrow f^{*}(\hbar) := \sum_{n\in\mathbb{Z}} {}^{n}u(-\hbar)^{n}$$

for ${}^{n}u \in T_{\mathbb{H}}$ or ${}^{n}u \in \mathcal{A}_0^{0}[\![\underline{t}]\!]$

Choose a subset $\{u_\alpha : \alpha \in I\} \subseteq \mathcal{A}_0^{0}$ s.t.

$u_\alpha + \mathrm{Im}(Q_S)$ is a $\mathbb{C}$-basis of $\mathcal{A}_0^{0}/Q_S(\mathcal{A}_0^{-1})$ $= \mathrm{Jac}_S$

Let $L := \sum_{\alpha\in I} u_\alpha \cdot t^{\alpha}$.

Using the isomorphism

$$\frac{\mathcal{A}_0^{0}[\![\underline{t}]\!]((\hbar))}{Q_S(\mathcal{A}_0^{-1}[\![\underline{t}]\!]((\hbar)))} \xrightarrow[\varphi\circ q]{\sim} T_{\mathbb{H}}((\hbar))$$

We define "the Gauss-Manin connection" on $T_{\mathbb{H}}((\hbar))$ over $\mathbb{H}$:

①

For each $\alpha \in I$, we define a connection

$$\nabla_\alpha^{S+L} : T_{\mathbb{H}}((\hbar)) \longrightarrow T_{\mathbb{H}}((\hbar))$$

by
$$\nabla_\alpha^{S+L}(\omega) = \left(e^{-S+L/\hbar} \cdot \frac{\partial}{\partial t^\alpha}\left(e^{S+L/\hbar} \cdot \omega\right)\right)$$

$$= \frac{\partial}{\partial t^\alpha}(\omega) + \frac{u_\alpha}{\hbar} \cdot \omega$$

for $\omega \in \mathcal{A}_0^{0}[\![\underline{t}]\!]((\hbar))$.

②

We also define a connection

$$\nabla_{\frac{1}{\hbar}}^{S+L} : T_{\mathbb{H}}((\hbar)) \longrightarrow T_{\mathbb{H}}((\hbar))$$

by
$$\nabla_{\frac{1}{\hbar}}^{S+L}(\omega) = \left(e^{-S+L/\hbar} \cdot \frac{\partial}{\partial(\frac{1}{\hbar})}\left(e^{S+L/\hbar} \cdot \omega\right)\right)$$

$$= \frac{\partial}{\partial \frac{1}{\hbar}}(\omega) + (S+L) \cdot \omega$$

for $\omega \in \mathcal{A}_0^{0}[\![\underline{t}]\!]((\hbar))$.

These are well-defined on cohomology classes, since Q_S commutes with ∇_α^{S+L} and $\nabla_{\frac{1}{\hbar}}^{S+L}$.

These are called the Gauss-Manin connections

Def. $[\mathfrak{u}] := \mathfrak{u} + \mathrm{Im}(Q_s)$

$$\eta : T_{\mathbb{H}} \times T_{\mathbb{H}} \longrightarrow \mathbb{C}[\![\underline{t}]\!]$$

$$\eta([\omega_1],[\omega_2]) := A^{\max}_{\mathfrak{u}_1\mathfrak{u}_2}, \quad \underset{[\varphi\circ q(\mathfrak{u}_i)]=[\omega_i]}{\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{A}^0_0[\![\underline{t}]\!]}$$

where $A^{\max}_{\mathfrak{u}_1\mathfrak{u}_2} \in \mathbb{C}[\![\underline{t}]\!]$ is uniquely determined by

$$\mathfrak{u}_1 \cdot \mathfrak{u}_2 = \sum_{\rho\in I} A^{\rho}_{\mathfrak{u}_1\mathfrak{u}_2} u_\rho + Q_s(\Lambda_{\mathfrak{u}_1\mathfrak{u}_2})$$

where $A^{\rho}_{\mathfrak{u}_1\mathfrak{u}_2} \in \mathbb{C}[\![\underline{t}]\!]$

$\Lambda_{\mathfrak{u}_1\mathfrak{u}_2} \in \mathcal{A}^{-1}_0[\![\underline{t}]\!]$.

Prop* The following diagram commutes up to sign:

$$\begin{tikzcd}
\mathbb{H} \times \mathbb{H} \arrow[rr, "\eta|_{\underline{t}=0}"] \arrow[dr, "\text{cup product}"'] & \arrow[d, phantom, "\circlearrowright"] & \mathbb{C} \arrow[d, equal] \\
& \mathbb{H}^{2(n-k)}(X_{\underline{G}},\mathbb{C}) \arrow[r, "\sim"] & \mathbb{C}
\end{tikzcd}$$

The Higher residue pairing is an extension of the cup product of $\mathbb{H}$ to the quantization cohomology $T_{\mathbb{H}}((\hbar))$ so that it is compatible with the Gauss–Manin connections.

Def. (Higher residue pairing)

A higher residue pairing for $T_{\mathbb{H}}$ is a $\mathbb{C}[\![\underline{t}]\!]$-bilinear symmetric pairing

$$K : T_{\mathbb{H}}((\hbar)) \times T_{\mathbb{H}}((\hbar)) \longrightarrow \mathbb{C}[\![\underline{t}]\!]((\hbar))$$

$$\left(K(\omega_1, \omega_2) \overset{\text{Notation}}{=} \sum_{r\in\mathbb{Z}} K^{(r)}(\omega_1,\omega_2)\cdot \hbar^r . \right)$$

s.t. ① $K(\omega_1, \omega_2) = K(\omega_2, \omega_1)^*$

② $g(\hbar)\cdot K(\omega_1,\omega_2) = K(g(\hbar)\cdot\omega_1, \omega_2)$

$= K(\omega_1, g^*(\hbar)\cdot\omega_2)$

for $g(\hbar) \in \mathbb{C}[\![\underline{t}]\!]((\hbar))$

③ $\partial_\alpha(K(\omega_1,\omega_2)) = K(\nabla^{SH}_\alpha\omega_1, \omega_2) + K(\omega_1, \nabla^{SH}_\alpha\omega_2)$

④ $\partial_{\frac{1}{\hbar}}(K(\omega_1,\omega_2)) = K(\nabla^{SH}_{\frac{1}{\hbar}}\omega_1, \omega_2) - K(\omega_1, \nabla^{SH}_{\frac{1}{\hbar}}\omega_2)$

⑤ The following diagram commutes:

$$\begin{tikzcd}
T_{\mathbb{H}}[\![\hbar]\!] \times T_{\mathbb{H}}[\![\hbar]\!] \arrow[r, "K^{(0)}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}[\![\underline{t}]\!][\![\hbar]\!] \arrow[d] \\
T_{\mathbb{H}} \times T_{\mathbb{H}} \arrow[r, "\eta"] & \mathbb{C}[\![\underline{t}]\!]
\end{tikzcd}$$

where η is a $\mathbb{C}[\![\underline{t}]\!]$-bilinear symmetric pairing coming from the cup-product pairing of $\underset{H^{n-k}_{prim}(X_{\underline{G}};\mathbb{C})}{\underset{\shortparallel}{\mathbb{H}}}$

Lemma If $\{[u_\alpha] : \alpha \in I\}$: Q_S-basis of Jac_S,

then for every $\Phi \in \mathcal{A}_0^0[\![\underline{t}]\!]((\hbar))$

there exists a unique $\mathbb{A}_\Phi^\gamma \in \mathbb{C}[\![\underline{t}]\!]((\hbar))$

$\gamma \in I$ s.t.

$$\Phi = \sum_{\gamma \in I} \mathbb{A}_\Phi^\gamma \cdot u_\gamma + Q_S(\Lambda_\Phi)$$

for some $\Lambda_\Phi \in \mathcal{A}_0^{-1}[\![\underline{t}]\!]((\hbar))$.

Thm

If we define

$$K([\omega_1], [\omega_2]) := \mathbb{A}_{u_1 \cdot u_2^*}^{max},$$

for $[u_i] = [(\varphi \circ \mathfrak{f})(\omega_i)]$,
$i = 1, 2$

then $K : T_{HH}((\hbar)) \times T_{HH}((\hbar)) \longrightarrow \mathbb{C}[\![\underline{t}]\!]((\hbar))$

is the higher residue pairing.

Pf) ①, ② : obvious by defn.

③, ④ follows from direct computations.

⑤ : follow from Prop*