ABC...L: The uniform *abc*-conjecture and zeros of Dirichlet *L*-functions

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2nd Kyoto–Hefei Workshop, August 2020

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Characters

Let $q \geq 1$ be an integer.

A **Dirichlet character** $\chi \pmod q$ is a function $\chi : \mathbb{Z} \to \mathbb{C}^*$ s.t.:

- $\chi(nm) = \chi(n)\chi(m)$ for every n, m;
- $\chi(n+q) = \chi(n)$ for every n;
- $\chi(n) = 0$ if $\gcd(n,q) > 1$.

Alternatively, χ is the lifting of a homeomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^*$.

- Primitive: $\nexists\, d \mid q\ (d \neq q)$ s.t.
$$\begin{array}{ccc} (\mathbb{Z}/q\mathbb{Z})^\times & \xrightarrow{\ \chi \pmod q\ } & \mathbb{C}^* \\ & \searrow \quad\quad \nearrow \chi' \pmod d & \\ & (\mathbb{Z}/d\mathbb{Z})^\times & \end{array}$$
- Principal: $(\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^*$ is trivial (i.e., $\chi_0(n) = \begin{cases} 1, \text{ if } (n,q) = 1 \\ 0, \text{ if } (n,q) > 1 \end{cases}$)
- Real: $\chi = \overline{\chi}$ $\quad (\iff$ Quadratic: $\chi^2 = \chi_0)$
- Even: $\chi(-1) = 1$, $\quad$ Odd: $\chi(-1) = -1$.

Real characters

$$\left\{\begin{matrix}\text{Real primitive}\\ \text{Dirichlet characters}\end{matrix}\right\} \longleftrightarrow \left\{\left(\frac{D}{\cdot}\right),\ \begin{matrix}D\text{ fundamental}\\ \text{discriminant}\end{matrix}\right\}$$

- A **fundamental discriminant** is an integer $D \in \mathbb{Z}$ s.t.:
  - $\exists\, K/\mathbb{Q}$ quadratic $\mid \Delta_K = D$; or, *equivalently*,
  - $\begin{cases} D \equiv 1 \pmod 4,\ D \text{ square-free; } \textbf{or} \\ D \equiv 0 \pmod 4,\ \text{s.t. } D/4 \equiv 2 \text{ or } 3 \pmod 4 \text{ and } D/4 \text{ square-free.} \end{cases}$
- The **Kronecker symbol** $\left(\frac{D}{\cdot}\right) : \mathbb{Z} \to \{-1, 0, 1\}$ is
  - Completely multiplicative (i.e., $\left(\frac{D}{m}\right)\left(\frac{D}{n}\right) = \left(\frac{D}{mn}\right),\ \forall m, n \in \mathbb{Z}$);
  - $\left(\frac{D}{p}\right) = \begin{cases} 1, & (p) \text{ splits in } \mathbb{Q}(\sqrt{D}) \\ -1, & (p) \text{ is inert } \cdots \\ 0, & (p) \text{ ramifies } \cdots \quad \text{(i.e., } p \mid D) \end{cases}$ $\qquad \left(\frac{D}{-1}\right) = \operatorname{sgn}(D).$

Writing $\chi_D := \left(\frac{D}{\cdot}\right)$, we have $\boxed{\chi_D \pmod{|D|} \text{ real, primitive}}$

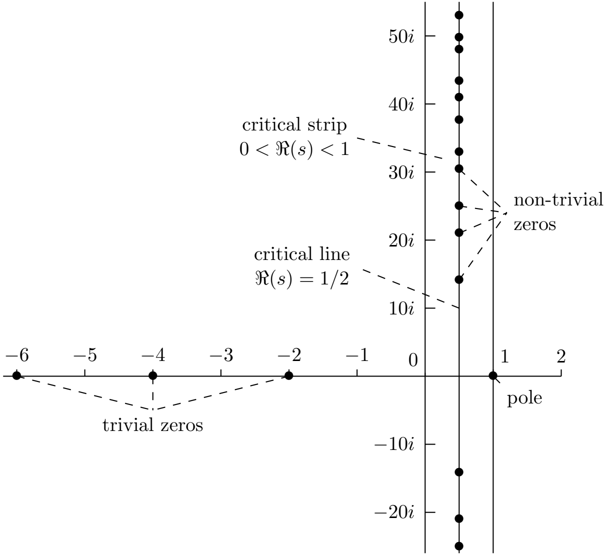
L-functions

The **Dirichlet L-function** associated to *non-principal* $\chi \pmod q$:

$$L(s,\chi) := \sum_{n\geq 1} \frac{\chi(n)}{n^s} \quad \left(= \prod_p \frac{1}{1-\chi(p)p^{-s}} \right), \qquad (\Re(s) > 1)$$

- Analytic continuation: (E.g.: $1 + z + z^2 + \cdots = \frac{1}{1-z}$)
  - $L(s,\chi)$ is entire;
- Functional equation: For $\mathfrak{a}_\chi := 0$ (if χ even) or 1 (if χ odd),
  - $L^*(s,\chi) := (\pi/q)^{-\frac{1}{2}(s+\mathfrak{a}_\chi)}\,\Gamma(\frac{1}{2}(s+\mathfrak{a}_\chi))\,L(s,\chi)$ is entire;
  - $L^*(s,\chi) = W(\chi)\,L^*(1-s,\overline{\chi})$, where $|W(\chi)| = 1$. Reflection
- Critical strip:
  - (Trivial zeros) Poles of $\Gamma(\frac{1}{2}(s+\mathfrak{a}_\chi))$, i.e.: $\begin{cases} 0,-2,-4\dots, & (\chi \text{ even}) \\ -1,-3,-5\dots, & (\chi \text{ odd}) \end{cases}$
  - (Non-trivial zeros) All other zeros are in $\{s \in \mathbb{C} \mid 0 < \Re(s) < 1\}$

Anatomy of $\zeta(s) = \sum_{n\geq 1} n^{-s}$



Classical (quasi) zero-free regions

[Gronwall 1913, Landau 1918, Titchmarsh 1933]

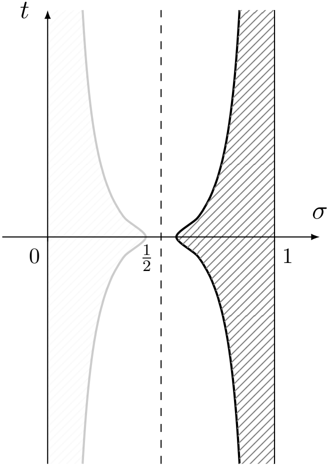
Write $s = \sigma + it$ ($\sigma = \Re(s)$, $t = \Im(s)$), and let $\chi \pmod q$ be a Dirichlet character.

There exists $c_0 > 0$ such that, in the region

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{c_0}{\log q(|t|+2)} \right\},$$

the function $L(s, \chi)$ has:

- (χ complex) no zeros;
- (χ real) at most <u>one</u> zero, which is necessarily *real* and *simple* – the so-called **Siegel zero**.



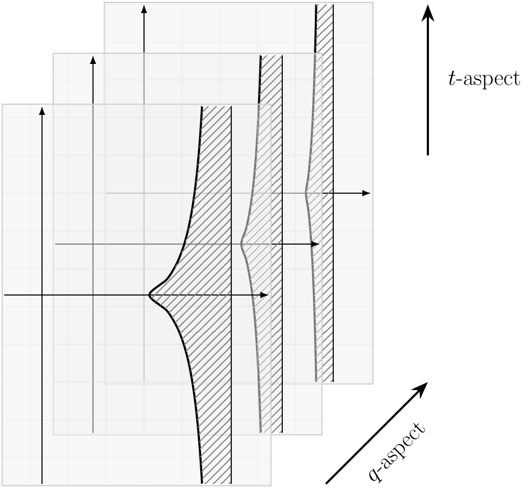
q-aspect vs. t-aspect

$$\sigma < 1 - \frac{c_0}{\log q(|t|+2)}$$

$$\Updownarrow$$

$$\boxed{\frac{1}{1-\sigma} \ll \log q(|t|+2)}$$

- <u>q-aspect:</u> $q \to +\infty$, $|t|$ bounded;
- <u>t-aspect:</u> q bounded, $|t| \to +\infty$.



Siegel zeros

Let:

- $D \in \mathbb{Z}$ be a fundamental discriminant
- β_D the largest real zero of $L(s, \chi_D)$

Conjecture ("no Siegel zeros")

$$\frac{1}{1-\beta_D} \ll \log|D|$$

Remark. Imprimitive case follows from primitive case.

- (Siegel, 1935) For every $\varepsilon > 0$, it holds that **Ineffective!**

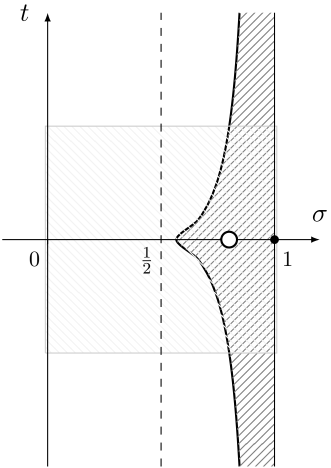
$$\frac{1}{1-\beta_D} \ll |D|^{\varepsilon} \qquad \left(\iff \begin{cases} h_{\mathbb{Q}(\sqrt{D})} \gg |D|^{\frac{1}{2}-\varepsilon}, & \text{for } D < 0 \\ h_{\mathbb{Q}(\sqrt{D})} \log \eta_D \gg D^{\frac{1}{2}-\varepsilon}, & \text{for } D > 0 \end{cases} \right)$$

- (GRH + Chowla) $\frac{1}{1-\beta_D} = \frac{1}{2}$ (if $D < 0$), $\frac{1}{1-\beta_D} = 1$ (if $D > 0$).

In summary (1/5)

Siegel zeros are...

- about <u>real primitive characters</u>
  - χ exceptional $\implies \chi = \chi_D$
  - *Exceptional* := violates (q-)ZFR by at most **one** (real, simple) zero
- a <u>q-aspect problem</u>
  - Box of height 1 ($|t| \leq 1$)
  - Zeros "very close" to $s = 1$
- related to <u>quadratic fields</u>
  - $\chi_D \longleftrightarrow \mathbb{Q}(\sqrt{D})$
  - $\beta_D \longleftrightarrow h_{\mathbb{Q}(\sqrt{D})}$



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Uniform $abc \implies \frac{1}{2}$ "no Siegel zeros"

Theorem (Granville–Stark, 2000)

Uniform abc-conj. $\implies$ *"No Siegel zeros" for* $\chi_D \pmod{|D|},\ D < 0$

Uniform abc-conj. $\implies \limsup_{D \to -\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log |D|} \leq 3$

$$\Downarrow$$

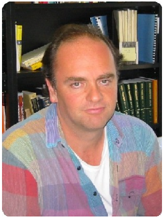
$$\limsup_{D \to -\infty} \frac{\frac{L'}{L}(1, \chi_D)}{\log |D|} < +\infty$$

$$\Updownarrow$$

($\frac{1}{2}$ "No Siegel zeros") $\exists \delta > 0 \mid \beta_D < 1 - \frac{\delta}{\log |D|}$

$\boxed{\frac{1}{1 - \beta_D}} \longleftrightarrow \boxed{\frac{L'}{L}(1, \chi_D)} \longleftrightarrow \boxed{\mathrm{ht}(j(\tau_D))}$

Key to the bridge!



A. Granville

H. Stark

Uniform *abc*-conjecture (1/2)

Let $K/\mathbb{Q}$ be a NF, $\mathcal{M}_K$ its places, $\mathcal{M}_K^{\text{non}} \subseteq \mathcal{M}_K$ non-arch. places

For a point $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n_K$, define:

- (naïve, abs, log) height $\operatorname{ht}(P)$

$$\boxed{\frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} \log\left(\max_i\{\|x_i\|_v\}\right)}$$

- (log) conductor $\mathcal{N}_K(P)$

$$\boxed{\frac{1}{[K:\mathbb{Q}]} \sum_{\substack{v \in \mathcal{M}_K^{\text{non}} \\ \exists i,j \leq n \text{ s.t.} \\ v(x_i) \neq v(x_j)}} f_v \log(p_v)}$$

$$\updownarrow$$

$$\begin{cases} v \sim \mathfrak{p} = \mathfrak{p}_v \\ p_v \sim \mathfrak{p}_v \cap \mathbb{Q} \\ f_v := [K_v : \mathbb{Q}_{p_v}] \end{cases}$$

For $a, b, c \in \mathbb{Z}$ coprime,

- $\operatorname{ht}([a:b:c]) = \log\max\{|a|, |b|, |c|\}$
- $\mathcal{N}_{\mathbb{Q}}([a:b:c]) = \log\left(\prod_{p \mid abc} p\right)$

For $\alpha \in \overline{\mathbb{Q}}$, $\operatorname{ht}(\alpha) := \operatorname{ht}([\alpha : 1])$.

$$\boxed{\alpha \text{ integral} \Rightarrow \operatorname{ht}(\alpha) = \frac{1}{|\mathcal{A}|} \sum_{\alpha^* \in \mathcal{A}} \log^+ |\alpha^*|}$$

$\mathcal{A} = \{\text{conjugates of } \alpha\}$

Uniform abc-conjecture (2/2)

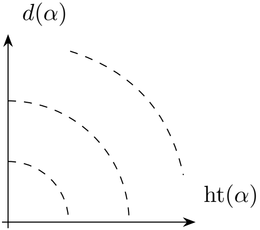
abc for number fields

Fix $K/\mathbb{Q}$ a number field. Then, for every $\varepsilon > 0$, there is $\mathcal{C}(K,\varepsilon) \in \mathbb{R}_+$ such that, $\forall a,b,c \in K \mid a+b+c=0$, we have

$$\mathrm{ht}([a:b:c]) < (1+\varepsilon)\Big(\mathcal{N}_K([a:b:c]) + \log(\mathrm{rd}_K)\Big) + \mathcal{C}(K,\varepsilon),$$

where $\mathrm{rd}_K := |\Delta_K|^{1/[K:\mathbb{Q}]}$ *is the* root-discriminant *of* K.

- Hermite: Δ_K bdd. $\Rightarrow \#\{K\} < \infty$
- Northcott: $d(\alpha)$, $\mathrm{ht}(\alpha)$ bdd. $\Rightarrow \#\{\alpha\} < \infty$



d-aspect
vs.
height-aspect

Uniform abc-conjecture (U-abc)

$$\mathcal{C}(K,\varepsilon) = \mathcal{C}(\varepsilon)$$

Vojta's general conjecture $\Longrightarrow$ U-abc

Singular moduli (1/2)

Let $\tau \in \mathfrak{h}$ $\;(\Im(\tau) > 0)$

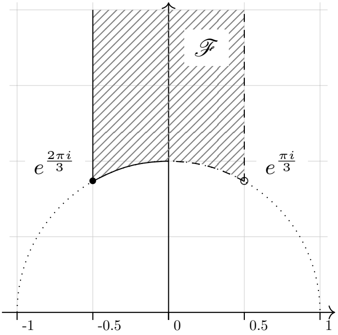
- CM-point: $\tau \mid A\tau^2 + B\tau + C = 0$ $\quad\begin{pmatrix} A, B, C \in \mathbb{Z},\ A > 0, \\ \gcd(A, B, C) = 1,\ \text{unique} \end{pmatrix}$
- Singular modulus: $j(\tau)$ $\quad$ ($j = j$-invariant, τ a CM-point)

$j : \mathfrak{h} \to \mathbb{C}$ is the **unique** function s.t.:

- is holomorphic;
- $j(i) = 1728,\ \ j(e^{2\pi i/3}) = 0,\ \ j(i\infty) = \infty;$
- $j\left(\dfrac{a\tau + b}{c\tau + d}\right) = j(\tau),\ \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$

$$\boxed{j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots}$$

q-expansion of the j-invariant ($q = e^{2\pi i\tau}$)



Singular moduli (2/2)

Heegner points Λ_D — Reduced bin. quad. forms of disc. D

$$\left\{\begin{matrix}\text{CM-points } \tau \in \mathscr{F}, \\ \operatorname{disc}(\tau) = D\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}(a,b,c) := ax^2 + bxy + cy^2 \\ \text{s.t. } b^2 - 4ac = D, \text{ and} \\ -a < b \leq a < c \text{ or } 0 \leq b \leq a = c\end{matrix}\right\}$$

$$\cup \qquad\qquad \cup$$

$$\boxed{\tau_D := \underbrace{\frac{\sqrt{D}}{2}}_{D\equiv 0(4)} \text{ or } \underbrace{\frac{-1+\sqrt{D}}{2}}_{D\equiv 1(4)}} \leftrightarrow \underbrace{\text{Principal form}}_{(1,0,-\frac{D}{4}) \text{ or } (1,1,\frac{1-D}{4})}$$

$$\mathbb{Z}[\tau_D] = \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$$

Write $\mathrm{H}_D :=$ Hilbert class field of $\mathbb{Q}(\sqrt{D})$.

- $\mathrm{H}_D = \mathbb{Q}(\sqrt{D}, j(\tau_D)) \quad \left([\mathrm{H}_D : \mathbb{Q}(\sqrt{D})] = [\mathbb{Q}(j(\tau_D)) : \mathbb{Q}] = h_{\mathbb{Q}(\sqrt{D})}\right)$
- $\{j(\tau) \mid \tau \in \Lambda_D\} = \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$-conjugates of $j(\tau_D)$
- $j(\tau_D)$ is an algebraic **integer!** $\boxed{\Rightarrow \operatorname{ht}(j(\tau_D)) = \frac{1}{h_{\mathbb{Q}(\sqrt{D})}} \sum_{\tau \in \Lambda_D} \log^+ |j(\tau)|}$

Main Theorems

Theorem 1 (Analytic)

$$\frac{1}{1-\beta_D} < \left(1 - \frac{1}{\sqrt{5}}\right)\frac{1}{2}\log|D| + \frac{L'}{L}(1,\chi_D) + \left(1 + \frac{2}{\sqrt{5}}\right)$$

Theorem 2 ("Bridge") [$\mathrm{ht}_{\mathrm{Fal}}$: Colmez, 1993]†

$$\frac{L'}{L}(1,\chi_D) = \frac{1}{6}\,\mathrm{ht}(j(\tau_D)) - \frac{1}{2}\log|D| + O(1)$$

Theorem 3 (Algebraic)

$$\text{U-}abc \implies \limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log|D|} = 3$$

†**Remark.** For CM ell. curves $E_{/\mathbb{C}}$, $\quad \mathrm{ht}_{\mathrm{Fal}}(E) = \frac{1}{12}\,\mathrm{ht}(j_E) + O(\log(\mathrm{ht}))$

Some consequences

Main corollary ($\frac{1}{2}$ "no Siegel zeros" w/ an explicit constant)

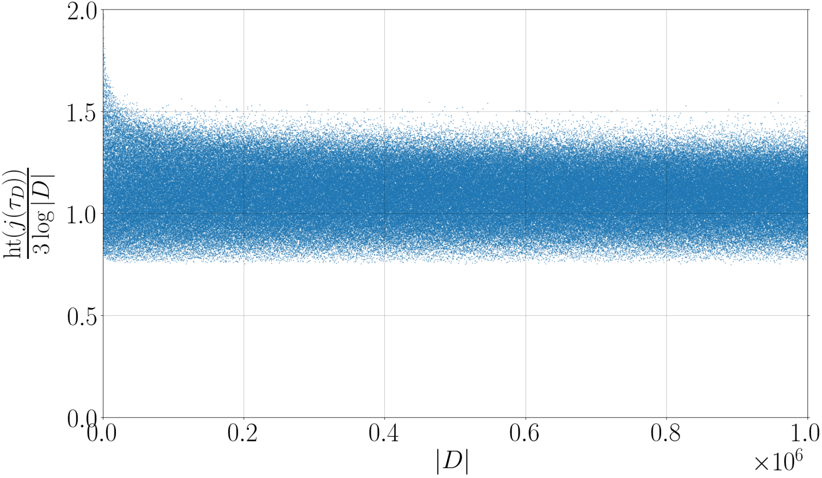
As $D \to -\infty$ through fundamental discriminants,

$$\text{U-}abc \implies \beta_D < 1 - \frac{(2+\varphi) - o(1)}{\log|D|} \quad \left(\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033\ldots\right)$$

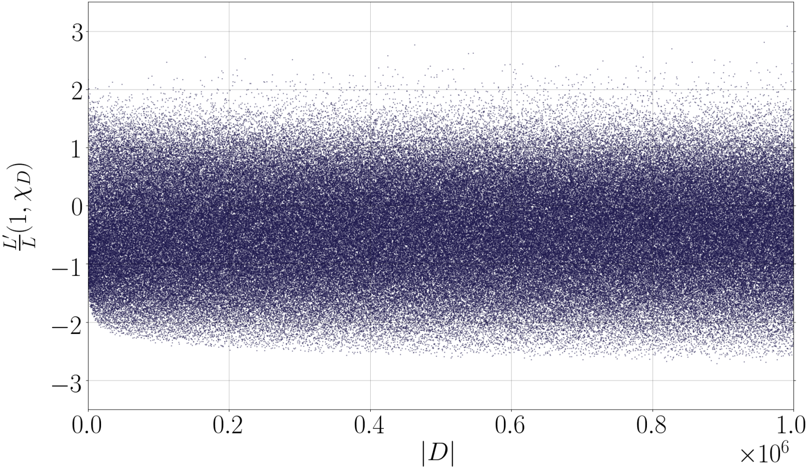
- $\left(\frac{1}{\sqrt{5}} + \underline{o(1)}\right) 3\log|D| \;\le\; \boxed{\mathrm{ht}(j(\tau_D))} \overset{\text{U-}abc}{\le} (1+o(1))\, 3\log|D|$
- $\left(\frac{1}{\sqrt{5}} + \underline{o(1)}\right) \frac{1}{2}\log|D| \;\le\; \boxed{\frac{L'}{L}(1,\chi_D) + \frac{1}{2}\log|D|} \overset{\text{U-}abc}{\le} (1+o(1))\, \frac{1}{2}\log|D|$
- $(1+o(1))\, \frac{\pi}{3}\frac{\sqrt{|D|}}{\log|D|} \sum_{(a,b,c)} \frac{1}{a} \overset{\text{U-}abc}{\le} \boxed{h_{\mathbb{Q}(\sqrt{D})}} \le \left(\sqrt{5} + \underline{o(1)}\right) \frac{\pi}{3}\frac{\sqrt{|D|}}{\log|D|} \sum_{(a,b,c)} \frac{1}{a}$

The three are **equivalent**, and the U-*abc* bounds **are attained!**

Graph of $\mathrm{ht}(j(\tau_D))$



Graph of $\frac{L'}{L}(1, \chi_D)$



In summary (2/5)

The three main theorems:

- Theorem 1
  - Analytic (zeros of L-functions)
  - Unconditional *lower bounds*
- Theorem 2
  - "Bridge" (connects Thms 1, 3)
  - $$\text{“}\tfrac{1}{\sqrt{5}}\text{” LB} \;\leq\; f(D) \;\overset{\text{U-}abc}{\leq}\; \text{UB}$$
- Theorem 3
  - Algebraic (height of $j(\tau_D)$)
  - U-*abc* conditional *upper bounds*
  - Best possible

$f(D)$ $(D < 0)$

- $\mathrm{ht}(j(\tau_D))$
- $\dfrac{L'}{L}(1, \chi_D)$
- avg $\Im(\tau)$ $(\tau \in \Lambda_D)$

- $h_{\mathbb{Q}(\sqrt{D})}$ $(\Leftrightarrow L(1, \chi_D))$
- $L'(1, \chi_D)$
- $\displaystyle\sum_{(a,b,c)} \frac{1}{a}$

Contents

The summation $\sum_{\varrho(\chi)} \frac{1}{\varrho}$

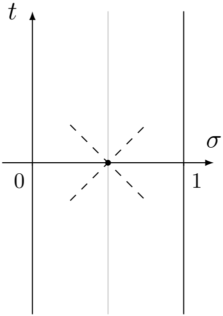
Let $D < 0$ be a fundamental discriminant.

- Classical formula (Functional Eq. + Hadamard product)

$$\frac{L'}{L}(s,\chi_D) = \left(\sum_{\varrho(\chi_D)} \frac{1}{s-\varrho} \right) - \frac{1}{2}\log\left(\frac{|D|}{\pi}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right)$$

- By the reflection formula:

$$L(\varrho,\chi) = 0 \implies \begin{cases} \varrho,\ 1-\overline{\varrho} \ \text{ zeros of } \ L(s,\chi) \\ \overline{\varrho},\ 1-\varrho \ \text{ zeros of } \ L(s,\overline{\chi}) \end{cases}$$



- Hence:

$$\boxed{\sum_{\varrho(\chi_D)} \frac{1}{\varrho} = \frac{1}{2}\log|D| + \frac{L'}{L}(1,\chi_D) - \frac{1}{2}\left(\gamma + \log\pi\right)}$$

Pairing up zeros (1/2)

In general, writing ($\varrho \in$ critical strip, $\varepsilon > 0$):

$$\Pi_\varepsilon(\varrho) := \frac{1}{\varrho + \varepsilon} + \frac{1}{\overline{\varrho} + \varepsilon} + \frac{1}{1 - \varrho + \varepsilon} + \frac{1}{1 - \overline{\varrho} + \varepsilon} \quad \underline{\text{(pairing function)}}$$

we get:

$$\boxed{\sum_{\varrho(\chi_D)} \frac{\Pi_{s-1}(\varrho)}{4} = \frac{1}{2}\log|D| + \frac{L'}{L}(s, \chi_D) - \frac{1}{2}\left(-\frac{\Gamma'}{\Gamma}\left(\frac{s+1}{2}\right) + \log \pi \right)}$$

Lemma 1

For $0 < \varepsilon < .85$, *we have:*

- $$0 < \sum_{\varrho(\chi_D)} \frac{\Pi_\varepsilon(\varrho)}{4} < \frac{1}{2}\log|D| + \frac{1}{\varepsilon}$$
- $$\left| \sum_{\varrho(\chi_D)} \frac{\Pi_\varepsilon(\varrho)}{4} - \frac{1}{2}\log|D| + \frac{1}{2}\Big(\gamma + \log \pi\Big) \right| < 1 + \frac{1}{\varepsilon}$$

Pairing up zeros (2/2)

- Goal: Estimate Π_0 in the critical strip $(=: \mathfrak{S})$
- Idea: Perturb ε in Π_ε

Lemma 2 (The pairing inequalities)

i *For every $s \in \mathfrak{S}$, we have:*

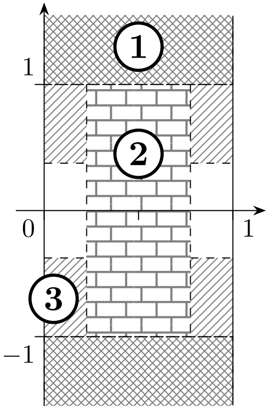
$$\Pi_0(s) > \frac{\Pi_{\varphi-1}(s)}{2\varphi - 1} \qquad \left(w/\ \varphi = \frac{1+\sqrt{5}}{2} \right)$$

ii *Take $0 < \varepsilon < 1$, $M \geq 2$, and consider*

$$\mathcal{B}_M := \left\{ s \in \mathfrak{S} \;\middle|\; \sigma > 1 - \frac{1}{M},\ |t| < \frac{1}{\sqrt{M}} \right\}$$

Then, in $\mathfrak{S} \setminus \big(\mathcal{B}_M \cup (1 - \mathcal{B}_M)\big)$, we have:

$$|\Pi_0(s) - \Pi_\varepsilon(s)| < 5M\varepsilon\,\Pi_\varepsilon(s)$$



$$\widetilde{\mathcal{B}} = \mathfrak{S} \setminus \big(① \cup ② \cup ③ \big)$$

Proof of Theorem 1

Take $z_0 \in \mathfrak{S}$ any non-trivial zero of $L(s, \chi_D)$.

$$\frac{L'}{L}(1,\chi_D) = \frac{\Pi_0(z_0)}{4} + \left(\sum_{\varrho(\chi_D)} \frac{\Pi_0(\varrho)}{4} - \frac{1}{2}\log q + \frac{1}{2}\Big(\gamma + \log \pi\Big) - \frac{\Pi_0(z_0)}{4} \right)$$

$$> \Re\left(\frac{1}{1-z_0}\right) + \frac{1}{2\varphi-1}\left(\sum_{\varrho(\chi_D)} \frac{\Pi_{\varphi-1}(\varrho)}{4} - \frac{1}{2}\log|D| + \frac{1}{2}\Big(\gamma+\log\pi\Big) - \frac{\Pi_{\varphi-1}(\varrho)}{4} \right) +$$

$$+ \left(1 - \frac{1}{2\varphi-1}\right)\left(-\frac{1}{2}\log|D| + \frac{1}{2}\Big(\gamma+\log\pi\Big)\right)$$

$$> \Re\left(\frac{1}{1-z_0}\right) - \frac{1}{2\varphi-1}\left(1 + \frac{2}{\varphi-1}\right) - \left(1-\frac{1}{2\varphi-1}\right)\frac{1}{2}\log|D|$$

$$= \Re\left(\frac{1}{1-z_0}\right) - \left(1-\frac{1}{\sqrt{5}}\right)\frac{1}{2}\log|D| - \left(1+\frac{2}{\sqrt{5}}\right)$$

$$\frac{1}{1-\beta_D} < \left(1-\frac{1}{\sqrt{5}}\right)\frac{1}{2}\log|D| + \frac{L'}{L}(1,\chi_D) + \left(1+\frac{2}{\sqrt{5}}\right) \qquad \square$$

In summary (3/5)

Since Theorem 1 $\implies \frac{L'}{L}(1,\chi_D) > -\left(1-\frac{1}{\sqrt{5}}\right)\frac{1}{2}\log|D| + O(1)$, we can derive, in particular, the *well-known* equivalence:

$$\text{“no Siegel zeros” for } D<0 \iff \boxed{\frac{L'}{L}(1,\chi_D) \ll \log|D|}$$

In this sense:

- $\frac{L'}{L}(1,\chi_D)$ encodes the Siegel zero
- The *pairing inequalities* yield **explicit estimates** for this encoding

Remark. (GRH bounds)

$$\text{GRH for } \chi_D\ (D<0) \implies \frac{L'}{L}(1,\chi_D) \ll \log\log|D|$$

Contents

Euler–Kronecker constants

The Dedekind ζ-function of $\mathbb{Q}(\sqrt{D})$: $(h(D) := h_{\mathbb{Q}(\sqrt{D})}$, $\mathrm{C}\ell(D) := \mathrm{C}\ell_{\mathbb{Q}(\sqrt{D})}$, etc.)

$$\zeta_{\mathbb{Q}(\sqrt{D})}(s) := \sum_{\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{D})}} \frac{1}{\mathrm{N}(\mathfrak{a})^s} \qquad \left(\begin{array}{l} = \zeta(s) L(s, \chi_D) \\ = \sum_{\mathscr{A} \in \mathrm{C}\ell(D)} \zeta(s, \mathscr{A}) \end{array} \right)$$

where $\zeta(s, \mathscr{A}) = \sum_{\substack{\mathfrak{a} \subseteq \mathscr{A} \\ \mathfrak{a} \text{ integral}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$ for $\mathscr{A} \in \mathrm{C}\ell(D)$ — (partial zeta function)

In general, as $s \to 1$:

- $\zeta_K(s) = \frac{c_{-1}}{s-1} + c_0 + O(s-1)$
- $\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + O(s-1)$
- $\zeta_K(s, \mathscr{A}) = \frac{\varkappa_K}{s-1} + \varkappa_K \mathfrak{K}(\mathscr{A}) + O(s-1)$

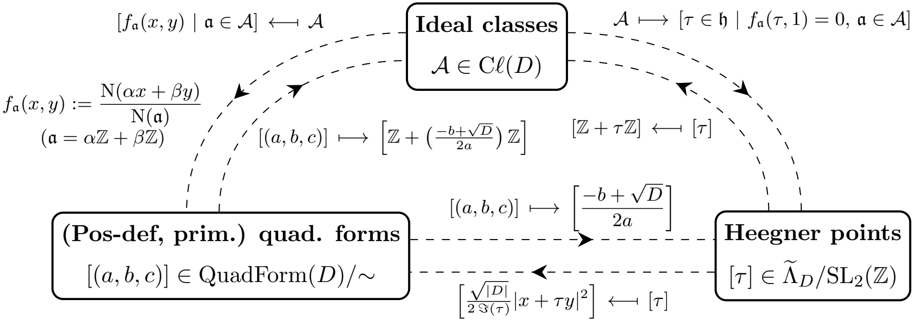
(Ihara, 2006)
Euler–Kronecker:
$\gamma_K := c_0 / c_{-1}$

Kronecker limits:
$\mathfrak{K}(\mathscr{A}),\ \mathscr{A} \in \mathrm{C}\ell_K$

$$\gamma_K = \frac{1}{h_K} \sum_{\mathscr{A} \in \mathrm{C}\ell_K} \mathfrak{K}(\mathscr{A})$$

$$\gamma + \frac{L'}{L}(1, \chi_D) = \frac{1}{h(D)} \sum_{\mathscr{A} \in \mathrm{C}\ell(D)} \mathfrak{K}(\mathscr{A})$$

Correspondence for $D < 0$ (Ideals–Forms–Points)



Partial zeta function		Epstein zeta function		real-analytic Eisenstein series
$\zeta(s, \mathscr{A})$	$\leftrightarrow$	$Z_{[(a,b,c)]}(s)$	$\leftrightarrow$	$E([\tau], s)$
$\displaystyle\sum_{\substack{\mathfrak{a} \subseteq \mathscr{A} \\ \mathfrak{a}\ \text{integral}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$		$\displaystyle\sum_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\neq(0,0)}} \frac{1}{(ax^2+bxy+cy^2)^s}$		$\displaystyle\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq 0}} \frac{\Im(\tau)^s}{\lvert m\tau+n\rvert^{2s}}$

Kronecker limits for $\mathbb{Q}(\sqrt{D})$ $(D < 0)$

$$\zeta(s,\mathscr{A}) = \frac{1}{w_D}\left(\frac{2}{\sqrt{|D|}}\right)^s E(\tau_{\mathscr{A}}, s)$$

$$\boxed{\mathscr{A} \leftrightarrow \begin{array}{c}\text{reduced}\\ (a,b,c)\end{array} \leftrightarrow \tau_{\mathscr{A}} = \frac{-b+\sqrt{D}}{2a}}$$

- For fixed $\tau \in \mathfrak{h}$, the Laurent expansion of E at $s = 1$ is:

$$\boxed{E(\tau,s) = \frac{\pi}{s-1} + \frac{\pi^2}{3}\Im(\tau) - \pi\log\Im(\tau) + \pi\,\mathcal{U}(\tau) + 2\pi\big(\gamma - \log(2)\big) + O(s-1)}$$

where:

$$\mathcal{U}(\tau) := 2\sum_{n\geq 1}\left(\sum_{d\,|\,n}\frac{1}{d}\right)\frac{\cos(2\pi n\,\Re(\tau))}{e^{2\pi n\,\Im(\tau)}} \qquad \left(= -\log(|\eta(\tau)|^2) - \frac{\pi}{6}\Im(\tau)\right)$$

Kronecker's (first) limit formula

$$\mathfrak{K}(\mathscr{A}) = \underbrace{\frac{\pi}{3}\Im(\tau_{\mathscr{A}}) - \log\Im(\tau_{\mathscr{A}}) + \mathcal{U}(\tau_{\mathscr{A}})}_{\mathscr{A}\textit{-dependent term}} \underbrace{-\frac{1}{2}\log|D|}_{\mathscr{A}\textit{-independent}} \underbrace{+2\gamma - \log(2)}_{\textit{constant term}}$$

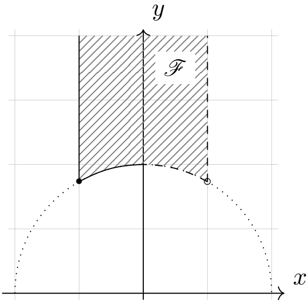
Duke's equidistribution theorem

Theorem (Duke, 1988)

$\Lambda_D = \{\tau_{\mathscr{A}} \mid \mathscr{A} \in \mathrm{C}\ell(D)\}$ *is equidistributed in* $\mathscr{F}$.

If $f : \mathscr{F} \to \mathbb{C}$ is Riemann-integrable, then:

$$\lim_{D\to-\infty} \frac{1}{h(D)} \sum_{\mathscr{A}\in\mathrm{C}\ell(D)} f(\tau_{\mathscr{A}}) = \int_{\mathscr{F}} f(z)\,\mathrm{d}\mu$$



$$z = x + iy$$

$$\mathrm{d}\mu = \frac{3}{\pi}\frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

(Normalized hyperbolic area element)

W. Duke

Proof of Theorem 2

KLF: $\mathfrak{K}(\mathscr{A}) = \frac{\pi}{3}\Im(\tau_{\mathscr{A}}) - \log \Im(\tau_{\mathscr{A}}) + \mathcal{U}(\tau_{\mathscr{A}}) - \frac{1}{2}\log|D| + 2\gamma - \log(2)$

- $\int_{\mathscr{F}} \mathcal{U}(z)\,\mathrm{d}\mu = 0.000151\ldots$
- $\int_{\mathscr{F}} \log(y)\,\mathrm{d}\mu = 0.952984\ldots$
- $\int_{\mathscr{F}} \left(\log^+|j(z)| - 2\pi y\right)\mathrm{d}\mu = -0.068692\ldots$

$\frac{1}{h(D)} \sum_{\mathscr{A}\in \mathrm{C}\ell(D)} \log \Im(\tau_{\mathscr{A}})$ is **hard** without Duke's theorem!

$\Downarrow$ (by Duke's theorem)

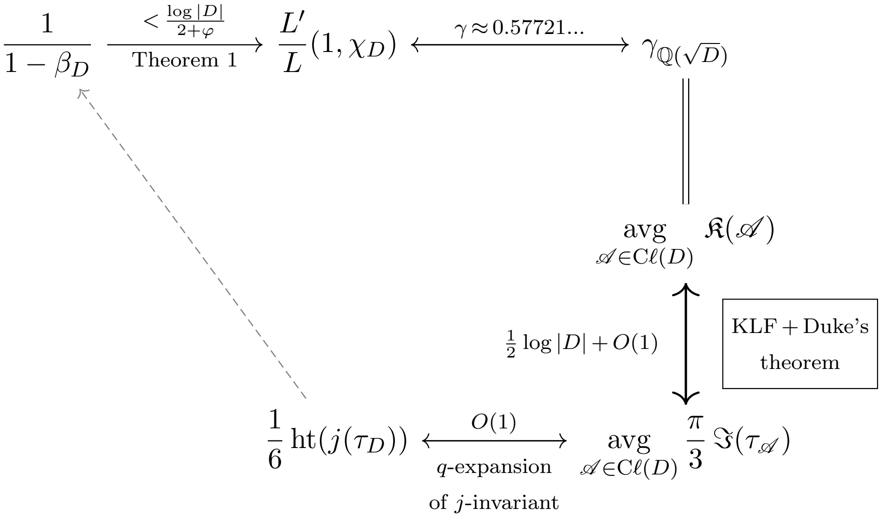
$$\gamma_{\mathbb{Q}(\sqrt{D})} = \frac{1}{6}\left(\frac{1}{h(D)} \sum_{\mathscr{A}\in \mathrm{C}\ell(D)} \log^+|j(\tau_{\mathscr{A}})|\right) - \frac{1}{2}\log|D| + O(1)$$

$\Downarrow$

$$\frac{L'}{L}(1,\chi_D) = \frac{1}{6}\,\mathrm{ht}(j(\tau_D)) - \frac{1}{2}\log|D| + O(1)$$

□

In summary (4/5)



Contents

Statement

Theorem 3

Let $D \in \mathbb{Z}$ denote negative fundamental discriminants. Then:

$$\text{U-}abc \implies \limsup_{D \to -\infty} \frac{\operatorname{ht}(j(\tau_D))}{\log |D|} = 3$$

- $\tau_D = \dfrac{\sqrt{D}}{2}$ (if $D \overset{4}{\equiv} 0$) or $\dfrac{-1+\sqrt{D}}{2}$ (if $D \overset{4}{\equiv} 1$)
- $j = j$-invariant function:

$$j(\tau) := \frac{\left(1 + 240 \sum_{n \geq 1} \left(\sum_{d|n} d^3\right) q^n\right)^3}{q \prod_{n \geq 1} (1-q^n)^{24}} = \frac{1}{q} + \sum_{n \geq 0} c(n) q^n \qquad \left(q = e^{2\pi i \tau}\right)$$

- ht = absolute logarithmic naïve (or Weil) height:

$$\operatorname{ht}(\alpha) := \frac{1}{\deg(\alpha)} \sum_{v \in \mathcal{M}_{\mathbb{Q}(\alpha)}} \log^+ \|\alpha\|_v \qquad \left(\deg(\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\right)$$

Two aspects of Theorem 3

The proof is divided into two parts:

- **Algebraic:** $\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log|D|} \overset{\text{U-}abc}{\leq} 3$ (Granville–Stark)
- **Analytic:** $\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log|D|} \geq 3$ (T.) [unconditional!]

Expected (e.g., from GRH)

$$\lim_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log|D|} = 3$$

Consequence of Theorem 1

$$\liminf_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log|D|} \geq \frac{3}{\sqrt{5}}$$

Granville–Stark's argument (1/2)

Consider the modular functions γ_2, γ_3 $\quad(\underline{\gamma_2^3 = j}, \ \underline{\gamma_3^2 = j - 1728})$, and the *abc*-type eq. $\boxed{\gamma_3(\tau_D)^2 - \gamma_2(\tau_D)^3 + 1728 = 0}$ in $\widetilde{\mathrm{H}}_D$ $\quad(\forall D < 0$ fund. disc.)

- *abc* for number fields implies: [Write $M := (1+\varepsilon)\log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D}) + \mathcal{C}(\widetilde{\mathrm{H}}_D, \varepsilon)$]

$$\mathrm{ht}\left(\left[\gamma_2(\tau_D)^3 : \gamma_3(\tau_D)^2 : 1728\right]\right) < (1+\varepsilon)\,\mathcal{N}_K\left(\left[\gamma_2(\tau_D)^3 : \gamma_3(\tau_D)^2 : 1728\right]\right) + M$$

- Then:

$$\begin{aligned} &\mathcal{N}_K\left(\left[\gamma_2(\tau_D)^3 : \gamma_3(\tau_D)^2 : 1728\right]\right) \\ &\le \frac{1}{3}\,\mathrm{ht}(\gamma_2(\tau_D)^3) + \frac{1}{2}\,\mathrm{ht}(\gamma_3(\tau_D)^2) + 1728 \\ &\le \frac{5}{6}\,\mathrm{ht}\left(\left[\gamma_2(\tau_D)^3 : \gamma_3(\tau_D)^2 : 1728\right]\right) + 1728 \end{aligned}$$

$$\Downarrow$$

$$\mathrm{ht}\left(\left[\gamma_2(\tau_D)^3 : \gamma_3(\tau_D)^2 : 1728\right]\right) < \frac{6}{1-5\varepsilon}M + O(1)$$

- $$\mathrm{ht}(j(\tau_D)) < \frac{6}{1-5\varepsilon}M + O(1)$$

$$\begin{array}{c} \widetilde{\mathrm{H}}_D \\ \gamma_2, \gamma_3(\tau_D) \ \big| \\ \mathrm{H}_D \\ j(\tau_D) \ \big| \\ \mathbb{Q}(\sqrt{D}) \\ \tau_D \ \big| \\ \mathbb{Q} \end{array}$$

Granville–Stark's argument (2/2)

Thus: [As $M := (1+\varepsilon)\log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D}) + \mathcal{C}(\widetilde{\mathrm{H}}_D, \varepsilon)$]

$$\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log|D|} \overset{abc}{\leq} \limsup_{D\to-\infty} \frac{6}{1-5\varepsilon}\frac{M}{\log|D|} \qquad (\forall\varepsilon>0)$$

$$\overset{\text{U-}abc}{\leq} \limsup_{D\to-\infty} \frac{6}{1-5\varepsilon}\frac{(1+\varepsilon)\log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D})}{\log|D|} \qquad (\forall\varepsilon>0)$$

$$\leq 6\cdot\limsup_{D\to-\infty} \frac{\log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D})}{\log|D|}$$

Main lemma [G–S, 2000]

$$\mathrm{rd}_{\widetilde{\mathrm{H}}_D} \ll \sqrt{|D|}$$

$$\Longrightarrow \quad \boxed{\limsup_{D\to-\infty} \frac{\log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D})}{\log|D|} \leq \frac{1}{2}}$$

Lower bounds for the lim sup

To complete the proof of Theorem 3, it remains to show that:

$$\limsup_{D\to-\infty} \frac{\mathrm{ht}(j(\tau_D))}{\log|D|} \geq 3$$

By Theorem 2 (the "bridge"), this is equivalent to:

$$\limsup_{D\to-\infty} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} \geq 0$$

Hence, it suffices to find a *subsequence* $\mathscr{D} \subseteq \{\text{fund. discriminants}\}$ s.t.:

$$\boxed{\limsup_{\substack{D\to-\infty \\ D\in\mathscr{D}}} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} \geq 0}$$

$q^{o(1)}$-smooth moduli

For $n \in \mathbb{Z}_{\geq 0}$, write $\mathcal{P}(n) := \max\{p \text{ prime} \mid p \text{ divides } n\}$.

- n is called *k-smooth* ($k \geq 2$) if $\mathcal{P}(n) \leq k$
- A set $\mathcal{S} \subseteq \mathbb{Z}_{\geq 0}$ is called *$n^{o(1)}$-smooth* if $\displaystyle\lim_{\substack{n \to +\infty \\ n \in \mathcal{S}}} \frac{\log \mathcal{P}(n)}{\log n} = 0$

$\left(\iff \mathcal{P}(n) = n^{o(1)} \text{ as } n \to +\infty \text{ through } \mathcal{S} \right)$

Chang's zero-free regions (2014)

For $\chi \pmod q$ primitive, $L(s, \chi)$ has no zeros (apart from possible Siegel zeros) in the region

$$\left\{ s \in \mathbb{C} \;\middle|\; \sigma \geq 1 - \frac{1}{f(q)},\ |t| \leq 1 \right\},$$

where $f : \mathbb{Z}_{\geq 2} \to \mathbb{R}$ satisfies:

$$f(q) = o(\log q) \text{ for } q^{o(1)}\text{-smooth moduli}$$

M.-C. Chang

Conclusion of Theorem 3

- Chang's ZFR + **the second pairing inequality**:

$$\boxed{\frac{L'}{L}(1, \chi_D) = \frac{1}{1-\beta_D} + O\Big(\sqrt{f(|D|)\,\log|D|}\Big)}$$

- Since $\dfrac{1}{1-\beta_D} > 0$, and $\sqrt{f(|D|)\,\log|D|} = o(\log|D|)$ for $|D|^{o(1)}$-smooth fundamental discriminants, it follows that

$$\limsup_{\substack{D \to -\infty \\ |D|^{o(1)}\text{-smooth}}} \frac{\frac{L'}{L}(1, \chi_D)}{\log|D|} \geq 0$$

$\square$

“no Siegel zeros” for arithmetic geometers

$$\boxed{\limsup_{D\to-\infty} \frac{\mathrm{ht}(j_E)}{\log|D|} < +\infty}$$

where $E_{/\mathbb{C}} = E_{/\mathbb{C}}(D)$ is CM by the maximal order in $\mathbb{Q}(\sqrt{D})$ (*full CM elliptic curves*)

U-*abc* type problems for analytic number theorists

$$\boxed{\limsup_{D\to-\infty} \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} = 0}$$

where $\chi_D \pmod{|D|}$ is the real primitive odd Dirichlet character modulo $|D|$

<u>naïve ht</u>: $\limsup = 3$

<u>Faltings ht</u>: $\limsup = 1/4$

$$\limsup \frac{\frac{L'}{L}(1,\chi_D)}{\log|D|} < +\infty$$

谢谢！

References

A. Granville and H. M. Stark,
ABC implies no "Siegel zeros" for *L*-functions of characters with negative discriminant
Invent. Math. **139** (2000), 509–523.

C. Táfula,
On Landau–Siegel zeros and heights of singular moduli
Submitted for publication. *Preprint* [arXiv:1911.07215]

C. Táfula,
On $\Re(\frac{L'}{L}(1,\chi))$ and zero-free regions near $s = 1$
Preprint [arXiv:2001.02405]