On the Boundedness and Graph-theoreticity of p-Ranks of Coverings of Curves

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Abstract

In the present paper, we investigate the p-ranks of coverings of stable curves. Let G be a p-group, $f: Y \longrightarrow X$ a morphism of semi-stable curves over a complete discrete valuation ring R with algebraically closed residue field of characteristic $p > 0$. Write η for the generic point of $S := \operatorname{Spec} R$ and s for the closed point of S. Let x be a singular point of the special fiber X_s of X. Suppose that the generic fiber X_η of X is smooth over η, and that the morphism $f_\eta : Y_\eta \longrightarrow X_\eta$ induced by f on the generic fibers is a Galois étale covering whose Galois group is isomorphic to G. Write Y' for the normalization of X in the function field of Y, $\psi : Y' \longrightarrow X$ for the resulting normalization morphism. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of x. Write $I_{y'}$ for the inertia group of y'. We prove that if $I_{y'}$ is an abelian p-group, then there exists a bound on the p-rank of a connected component of $f^{-1}(x)$ which only depends on $\sharp I_{y'}$, where $\sharp I_{y'}$ denotes the order of $I_{y'}$. This result gives an answer to an open problem posed by M. Saïdi in the case where $I_{y'}$ is abelian. On the other hand, we prove that the p-rank of $f^{-1}(x)$ (resp. Y_s) is determined by a certain collection of **purely combinatorial data** associated to f and x (resp. associated to f and the p-ranks of the normalizations of the irreducible components of X_s).

Keywords: p-rank, semi-stable covering, vertical point, vertical fiber.

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Contents

1 Introduction **2**

2 p-ranks of G-semi-stable coverings **4**

2.1 Definitions 4

2.2 Formulas for p-ranks of G-semi-stable coverings 5

3 Bounds of p-ranks of vertical fibers of abelian G-semi-stable coverings **7**

4 Graphs and p-ranks of G-semi-stable coverings **9**

1 Introduction

Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$. Write K for the quotient field of R; $S := \operatorname{Spec} R$; $\eta : \operatorname{Spec} K \longrightarrow S$ and $s : \operatorname{Spec} k \longrightarrow S$ for the natural morphisms. Let $\overline{K}$ be an algebraic closure of K. Write $\overline{\eta} : \operatorname{Spec} \overline{K} \longrightarrow S$ for the natural morphism. Let G be a finite p-group and X a semi-stable curve of genus g_X over S. Write X_η, $X_{\overline{\eta}}$, and X_s for the result of base-changing X by η, $\overline{\eta}$, and s, respectively. Moreover, we suppose that X_η is a smooth curve over η.

Let Y_η be a geometrically connected curve over η and $f_\eta : Y_\eta \longrightarrow X_\eta$ a finite Galois étale covering over η whose Galois group is isomorphic to G. By replacing S by a finite extension of S (i.e., the spectrum of the normalization of R in a finite extension of K), we may assume that Y_η admits a semi-stable model over S. Then f_η extends uniquely to a G**-semi-stable covering** (cf. Definition 2.1) $f : Y \longrightarrow X$ over S (cf. [Y, Proposition 3.4]). We are interested in understanding the structure of the special fiber Y_s of Y. Note that the morphism $f_s : Y_s \longrightarrow X_s$ induced by f on the special fibers is not a finite morphism in general. Let x be a closed point of X_s. If $f^{-1}(x)$ is not finite, we shall call x a **vertical point associated to** f and call $f^{-1}(x)$ the **vertical fiber associated to** x (cf. Definition 2.2). In order to investigate the properties of Y_s (resp. $f^{-1}(x)$), we focus on a geometric invariant $\sigma(Y_s) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(Y_s, \mathbb{F}_p)$ (resp. $\sigma(f^{-1}(x)) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(f^{-1}(x), \mathbb{F}_p)$) which is called the p-rank of Y_s (resp. the p-rank of $f^{-1}(x)$). In the present paper, we apply the formulas for $\sigma(Y_s)$ and $f^{-1}(x)$ obtained in [Y] to study the boundedness and graph-theoreticity of p-ranks of G-semi-stable coverings.

First, let us consider the boundedness of p-ranks of G-semi-stable coverings. Note that we always have $\sigma(Y_s) \leq g_{Y_{\overline{\eta}}} = \sigma(Y_{\overline{\eta}})/2 := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{F}_p)/2$ if $\mathrm{char}(K) = 0$ and $\sigma(Y_s) \leq \sigma(Y_{\overline{\eta}}) \leq g_{Y_{\overline{\eta}}}$ if $\mathrm{char}(K) = p > 0$, where $g_{Y_{\overline{\eta}}}$ denotes the genus of $Y_{\overline{\eta}} := Y_\eta \times_\eta \overline{\eta}$. Moreover, $\sigma(Y_{\overline{\eta}})$ can be calculated by applying the Riemann-Hurwitz formula if $\mathrm{char}(K) = 0$ and the Deuring-Shafarevich formula (cf. [C]) if $\mathrm{char}(K) = p > 0$, respectively. Thus, $\sigma(Y_s)$ is bounded by a quantity which is completely determined by $\sharp G$ and $\sigma(X_{\overline{\eta}}) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(X_{\overline{\eta}}, \mathbb{F}_p)$. In the present paper, we consider the boundedness of $\sigma(f^{-1}(x))$. Note that $\sigma(f^{-1}(x))$ is always bounded by $g_{Y_{\overline{\eta}}}$. If x is a **smooth point** of X_s, M. Raynaud proved the following result (cf. [R, Théorème 2]):

Theorem 1.1. *If x is a smooth point of X_s, and G is an p-group, then the p-rank $\sigma(f^{-1}(x))$ is equal to* 0.

By Theorem 1.1, we only need to treat the case where x is a **singular point** of X_s. In order to explain our results, let us introduce some notations. Write $\psi : Y' \longrightarrow X$ for the normalization of X in the function field of Y. Let $y' \in \psi^{-1}(x)$ be a point in the inverse image of x. Write $I_{y'} \subseteq G$ for the inertia group of y'. [Y, Proposition 3.4] implies that the morphism $Y_\eta/I_{y'} \longrightarrow X_\eta$ over η induced by f extends to a semi-stable covering $Y_{I_{y'}} \longrightarrow X$ over S. In order to calculate the p-rank of $f^{-1}(x)$, since (by the definition of $I_{y'}$!) the morphism $Y_{I_{y'}} \longrightarrow X$ is finite étale over x, by replacing X by $Y_{I_{y'}}$, we may assume without loss of generality that G is equal to $I_{y'}$. In the remainder of this subsection, we shall assume that $G = I_{y'}$. Then $f^{-1}(x)$ is connected. If $I_{y'}$ is cyclic, M. Saïdi proved the following result (cf. [S, Theorem 1]), by applying Theorem 1.1:

Theorem 1.2. *If G is a cyclic p-group, then we have $\sigma(f^{-1}(x)) \leq \sharp G - 1$, where $\sharp G$ denotes the order of G.*

Furthermore, there is an open problem posed by Saïdi as follows (cf. [S, Question]):

Problem 1.3. *If G is an arbitrary p-group, does there exist a bound on the p-rank $\sigma(f^{-1}(x))$ that depends only on the order $\sharp G$?*

In the present paper, by applying a formula for p-ranks of vertical fibers obtained in [Y], we generalize Saïdi's result (i.e., Theorem 1.2) and give an answer to Problem 1.3 in the case where G is an abelian group as follows (cf. Theorem 3.4 and Remark 3.4.1):

Theorem 1.4. *If G is an abelian p-group, then we have (cf. Definition 3.2 for the definitions of $M(G)$ and $B(\sharp G)$)*

$$\sigma(f^{-1}(x)) \leq M(G) \cdot \sharp G - 1 \leq B(\sharp G) \cdot \sharp G - 1,$$

where $B(\sharp G)$ only depends on $\sharp G$. In particular, if G is a cyclic p-group, we have

$$\sigma(f^{-1}(x)) \leq \sharp G - 1.$$

Next, let us consider the graph-theoreticity of p-ranks of G-semi-stable coverings. We pose a problem as follows:

Problem 1.5. *Is $\sigma(Y_s)$ (resp. $\sigma(f^{-1}(x))$) completely determined by $\sharp G$ and a suitable collection of purely combinatorial data associated to f (resp. f and x)?*

By using the resolution of nonsingularites over marked points of pointed semi-stable coverings, we construct a semi-graph $\Gamma_{Y_s}^{f\text{-etd}}$ associated to f, which is called the **extended dual semi-graph of Y_s associated to f** (resp. a semi-graph $\Gamma_x^{f\text{-etd}}$ associated to x and f which is called the **extended dual semi-graph of $f^{-1}(x)$**). Moreover, we define a certain collection of purely combinatorial data

$$\mathfrak{Com}^f := (\Gamma_{Y_s}^{f\text{-etd}}, \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \beta_f^{f\text{-etd}} : \Gamma_{Y_s}^{f\text{-etd}} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \sharp G)$$

$$(\text{resp. } \mathfrak{Com}_x^f := (\Gamma_x^{f\text{-etd}}, \sharp G))$$

associated to f (resp. associated to x and f) which depends only on f (resp. f and x) (cf. Definition 4.2 (resp. Definition 4.7)), where $\mathscr{X}^{\mathrm{sst}}$ is a pointed semi-stable curve over S associated to Y/G (see Section 4 for the construction of $\mathscr{X}^{\mathrm{sst}}$), and $\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ denotes the dual semi-graph of the special fiber of $\mathscr{X}^{\mathrm{sst}}$. We give an answer to Problem 1.5 as follows (cf. Theorem 4.5, Corollary 4.6, Theorem 4.8, and Corollary 4.9):

Theorem 1.6. *We maintain the notations introduced above. Then the p-rank $\sigma(Y_s)$ is completely determined by $\mathfrak{Com}^f$ and $\{\sigma(\widetilde{X}_v)\}_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})}$, where $\widetilde{X}_v$ denotes the normalization of the irreducible component X_v of $\mathscr{X}_s^{\mathrm{sst}}$ corresponding to v. Let $f^{-1}(x)$ be the vertical fiber associated to the vertical point x. Then the p-rank $\sigma(f^{-1}(x))$ is completely determined by $\mathfrak{Com}_x^f$. Moreover, $\sigma(f^{-1}(x))$ is completely determined by any stem of $\Gamma_x^{f\text{-etd}}$ (cf. Definition 4.7) and $\sharp G$.*

Next, let $h : Z \longrightarrow W$ be an J-semi-stable covering over S. Suppose that $(\alpha_1, \alpha_2) : \mathfrak{Com}^f \overset{\sim}{\to} \mathfrak{Com}^h$ is an isomorphism of quadruples (cf. Definition 4.2) such that $\sigma(\widetilde{X}_v) = \sigma(\widetilde{W}_{\alpha_2(v)})$ for each $v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$, where $\widetilde{W}_{\alpha_2(v)}$ denotes the normalization of the irreducible component $W_{\alpha_2(v)}$ of $\mathscr{W}_s^{\mathrm{sst}}$ corresponding to $\alpha_2(v)$. Then we have

$$\sigma(Y_s) = \sigma(Z_s).$$

Let w be a vertical point associated to h. Suppose that w is a singular point of the special fiber W_s of W, that $h^{-1}(w)$ is connected, and that $\alpha : \mathfrak{Com}_x^f \overset{\sim}{\to} \mathfrak{Com}_w^h$ is an isomorphism of pairs (cf. Definition 4.7). Then we have

$$\sigma(f^{-1}(x)) = \sigma(h^{-1}(w)).$$

The present paper is organized as follows. In Section 2, we give some definitions and recall the formulas for $\sigma(Y_s)$ and $\sigma(f^{-1}(x))$ obtained in [Y]. In Section 3, by applying the general theory of semi-stable curves and the formula for $\sigma(f^{-1}(x))$, we prove Theorem 1.4. In Section 4, by applying the resolution of nonsingularites over marked points of pointed semi-stable coverings, we define the extended dual graphs associated to Y_s and $f^{-1}(x)$. Then we prove Theorem 1.6 by using the formulas for $\sigma(Y_s)$ and $\sigma(f^{-1}(x))$.

2 p-ranks of G-semi-stable coverings

2.1 Definitions

Let $\mathscr{W} := (W, E_W)$ be a pointed semi-stable curve over a scheme A. We shall call W the underlying curve of $\mathscr{W}$ and E_W the set of marked points of $\mathscr{W}$ (each of which is a section $A \longrightarrow W$ of $W \longrightarrow A$). Write Im_{E_W} for the scheme theoretic images of the elements of E_W; we identify E_W with Im_{E_W}.

From now on, let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$. Write K for the quotient field, S for the spectrum of R, η for the generic point corresponding to the natural morphism $\mathrm{Spec}\, K \longrightarrow S$, and s for the closed point corresponding to the natural morphism $\mathrm{Spec}\, k \longrightarrow S$. Let $\mathscr{X} := (X, E_X)$ be a pointed semi-stable curve over S. Write $\mathscr{X}_\eta := (X_\eta, E_{X_\eta})$ and $\mathscr{X}_s := (X_s, E_{X_s})$ for the generic fiber over η and the special fiber over s, respectively. Moreover, we suppose that $\mathscr{X}_\eta$ is a smooth pointed curve over η.

Definition 2.1. Let $f : \mathscr{Y} := (Y, E_Y) \longrightarrow \mathscr{X}$ be a morphism of pointed semi-stable curves over S and G a finite group. The morphism f is called a **pointed semi-stable covering** (resp. **G-pointed semi-stable covering**) over S if the morphism $f_\eta : \mathscr{Y}_\eta = (Y_\eta, E_{Y_\eta}) \longrightarrow \mathscr{X}_\eta = (X_\eta, E_{X_\eta})$ over η induced by f on generic fibers is a finite generically étale morphism (resp. a Galois covering whose Galois group is isomorphic to G) such that the following conditions are satisfied: (i) the branch locus of f_η is contained in E_{X_η}; (ii) $f_\eta^{-1}(E_{X_\eta}) = E_{Y_\eta}$; (iii) the following universal property holds: if $g : \mathscr{Z} \longrightarrow \mathscr{X}$ is a morphism of pointed semi-stable curves over S such that the generic fiber $\mathscr{Z}_\eta$ of $\mathscr{Z}$ and the morphism $g_\eta : \mathscr{Z}_\eta \longrightarrow \mathscr{X}_\eta$ induced by g on generic fibers are equal to $\mathscr{Y}_\eta$ and f_η, respectively, then there exists a unique morphism $h : \mathscr{Z} \longrightarrow \mathscr{Y}$ such that $f = g \circ h$.

We shall call f a **pointed stable covering** (resp. **G-pointed stable covering**) over S if f is a pointed semi-stable covering (resp. G-pointed semi-stable covering) over S, and $\mathscr{X}$ is a pointed stable curve. We shall call f a **semi-stable covering** (resp. **stable covering, G-semi-stable covering, G-stable covering**) over S if f is a pointed semi-stable covering (resp. pointed stable covering, G-pointed semi-stable covering, G-pointed stable covering) over S, and E_X is empty.

Definition 2.2. Let $f : \mathscr{Y} \longrightarrow \mathscr{X}$ be a semi-stable covering over S. A closed point $x \in X_s$ is called a **vertical point associated to f**, or for simplicity, a **vertical point** when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the **vertical fiber associated to x**.

Definition 2.3. Let C be a projective curve over an algebraically closed field of characteristic $p > 0$. We define the **p-rank of C** as follows:

$$\sigma(C) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(C, \mathbb{F}_p).$$

2.2 Formulas for p-ranks of G-semi-stable coverings

From now on, we assume that G is a finite p-group. Let $f : \mathscr{Y} \longrightarrow \mathscr{X}$ be a G-semi-stable covering over S and x a vertical point associated to f. For simplicity, we write Y and X for $\mathscr{Y}$ and $\mathscr{X}$, respectively. Write X^{sst} for the semi-stable curve Y/G over S (cf. [R, Appendice Corollaire]). Then we obtain two morphisms of semi-stable curves $h : Y \longrightarrow X^{\mathrm{sst}}$ and $g : X^{\mathrm{sst}} \longrightarrow X$ such that $g \circ h = f$. Write Γ_{X_s}, $\Gamma_{X_s^{\mathrm{sst}}}$, and Γ_{Y_s} for the dual graphs of the special fiber X_s of X, the special fiber X_s^{sst} of X^{sst}, and the special fiber Y_s of Y, respectively.

Let $\mathbb{G}$ be a semi-graph (cf. [M] or the beginning of Section 2.1 of [Y]). Write $v(\mathbb{G})$ (resp. $e^{\mathrm{cl}}(\mathbb{G})$, $e^{\mathrm{lp}}(\mathbb{G}) \subseteq e^{\mathrm{cl}}(\mathbb{G})$, $e^{\mathrm{op}}(\mathbb{G})$) for the set of vertices (resp. the set of closed edges, the set of loops, the set of open edges) of $\mathbb{G}$. For each $v \in v(\mathbb{G})$, write $e(v)$ (resp. $v(e)$, $e^{\mathrm{lp}}(v)$) for the set of edges which abut to v (resp. the set of vertices which are abutted by e, the set of loops which abut to v).

Let v be an element of $v(\Gamma_{X_s^{\mathrm{sst}}})$, X_v the irreducible component of X_s corresponding to v, and Y_v an irreducible component such that $h(Y_v) = X_v$. Write $I_{Y_v} \subseteq G$ for the inertia group of Y_v. Since $\sharp I_{Y_v}$ does not depend on the choices of Y_v, we use the notation $\sharp I_v$ to denote $\sharp I_{Y_v}$. For the p-rank $\sigma(Y_s)$, we have the following theorem (cf. [Y, Theorem 4.5]).

Theorem 2.4. *We follow the notations above. Then we have*

$$\sigma(Y_s) = \sum_{v \in v(\Gamma_{X_s^{\mathrm{sst}}})} (\sharp G/\sharp I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} \sharp G/\sharp I_{v_e}(\sharp I_{v_e}/\sharp I_v - 1) + 1)$$

$$+ \sum_{e \in e^{\mathrm{cl}}(\Gamma_{X_s^{\mathrm{sst}}}) \setminus e^{\mathrm{lp}}(\Gamma_{X_s^{\mathrm{sst}}})} (\sharp G/\sharp I_{v_e} - 1) + \sum_{v \in v(\Gamma_{X_s^{\mathrm{sst}}})} \sharp e^{\mathrm{lp}}(v)(\sharp G/\sharp I_v - 1) + \dim_{\mathbb{C}} \mathrm{H}^1(\Gamma_{X_s^{\mathrm{sst}}}, \mathbb{C}),$$

where $\widetilde{X}_v$ denotes the normalization of the irreducible component X_v of X_s^{sst} corresponding to v, $\sharp I_{v_e}$ denotes $\max\{\sharp I_v\}_{v \in v(e)}$.

Next, let us consider the p-rank of $f^{-1}(x)$. Write Y' for the normalization of X in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f, and ψ for the resulting normalization morphism $Y' \longrightarrow X$. Then Y' admits a natural action of G induced by the action of G on Y. Let $y' \in \psi^{-1}(x)$. Write $I_{y'} \subseteq G$ for the inertia group of y'. In order to calculate the p-rank $\sigma(f^{-1}(x))$, since $Y/I_{y'} \longrightarrow X$ is finite étale above x, by replacing X and G by the semi-stable curve $Y/I_{y'}$ and $I_{y'}$, we may assume that $G = I_{y'}$. In the remainder of this section, we shall assume that $G = I_{y'}$. Then $f^{-1}(x)$ is connected. On the other hand, if the vertical point x is a smooth point of X_s, then [R, Théorème 2] implies that $\sigma(f^{-1}(x))$ is 0. Then we only need to treat the case where x is a node of X_s and assume that x is a singular point of X_s.

Let X_1' and X_2' (which may be equal) be the irreducible components of X_s which contain x. Write X_1 and X_2 for the strict transforms of X_1' and X_2' under the birational morphism $g : X^{\text{sst}} \longrightarrow X$, respectively. By the general theory of semi-stable curves, $g^{-1}(x)_{\text{red}} \subseteq X_s^{\text{sst}}$ is a semi-stable curve over s whose irreducible components are isomorphic to $\mathbb{P}^1_k$, where $(-)_{\text{red}}$ denotes the reduced induced closed subscheme of $(-)$. Write C for the semi-stable subcurve of $g^{-1}(x)_{\text{red}}$ which is a chain of projective lines $\cup_{i=1}^{n} P_i$ such that the following conditions hold: (i) for any $s, t = 1, \ldots, n$, $P_s \cap P_t = \emptyset$ if $|s-t| \geq 2$ and $P_s \cap P_t$ is reduced to a point if $|s-t| = 1$; (ii) $P_1 \cap X_1$ (resp. $P_n \cap X_2$) is reduced to a point; (iii) $C \cap \overline{\{X^{\text{sst}} \setminus C\}} = (P_1 \cap X_1) \cup (P_n \cap X_2)$, where $\overline{\{X^{\text{sst}} \setminus C\}}$ denotes the closure of $X^{\text{sst}} \setminus C$ in X^{sst}.

Let $\{V_i\}_{i=0}^{n+1}$ be a set of irreducible components of the special fiber Y_s of Y such that the following conditions hold: (i) $h(V_i) = P_i$ for $i = 1, \ldots, n$; (ii) $h(V_0) = X_1$ and $h(V_{n+1}) = X_2$; (iii) the union $\cup_{i=0}^{n+1} V_i \subseteq Y_s$ is a connected semi-stable curve over s. Write $I_{V_i} \subseteq G, i = 0, \ldots, n+1$ for the inertia group of V_i. [Y, Corollary 4.4] implies that for any $i = 0, \ldots, n$, either $I_{V_i} \subseteq I_{V_{i+1}}$ or $I_{V_i} \supseteq I_{V_{i+1}}$ holds.

Let $(u, w) \in \{0, \ldots, n+1\} \times \{0, \ldots, n+1\}$ be a pair such that $u \leq w$. We shall call a group $I_{u,w}^{\min}$ a minimal element of $\{I_{V_i}\}_{i=0}^{n+1}$ if one of the following conditions holds: (i) $(u, w) = (0, n+1)$ and for any $I_{V_i}, i = 0, \ldots, n+1$, $I_{0,n+1}^{\min} = I_{V_i}$; (ii) $(u, w) = (0, w) \neq (0, n+1)$, $I_{0,w}^{\min} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \subset I_{V_{w+1}}$; (iii) $(u, w) = (u, n+1) \neq (0, n+1)$, $I_{V_{u-1}} \supset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = I_{u,n+1}^{\min}$; (iv) $u \neq 0$, $w \neq n+1$, and $I_{V_{u-1}} \supset I_{u,w}^{\min} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \subset I_{V_{w+1}}$. We shall call a group $J_{u,w}^{\max}$ a maximal element of $\{I_{V_i}\}_{i=0}^{n+1}$ if one of the following conditions hold: (i) $(u, w) = (0, n+1)$ and for any $I_{V_i}, i = 0, \ldots, n+1$, $J_{0,n+1}^{\max} = I_{V_i}$; (ii) $(u, w) = (0, w) \neq (0, n+1)$, $J_{0,w}^{\max} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \supset I_{V_{w+1}}$; (iii) $(u, w) = (u, n+1) \neq (0, n+1)$, $I_{V_{u-1}} \subset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = J_{u,n+1}^{\max}$; (iv) $u \neq 0$, $w \neq n+1$, and $I_{V_{u-1}} \subset J_{u,w}^{\max} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \supset I_{V_{w+1}}$. We define Min to be

$$\{I_{u,w}^{\min}\}_{(u,w) \in \{1,\ldots,n\} \times \{1,\ldots,n+1\}} \text{ or } \{I_{0,n+1}^{\min}\}$$

and Max to be

$$\{I_{u,w}^{\max}\}_{(u,w) \in \{0,\ldots,n+1\} \times \{0,\ldots,n+1\}}.$$

Note that Min may be an empty set. We have the following formula (cf. [Y, Theorem 4.7]).

Theorem 2.5. *We follows the notations above, we have*

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \sharp G/\sharp I_{V_i} - \sum_{i=1}^{n+1} \sharp G/\sharp\langle I_{V_{i-1}}, I_{V_i}\rangle + 1$$

$$= \sum_{i=1}^{n} \sharp G/\sharp I_{V_i} - \sum_{i=1}^{n+1} \sharp G/\sharp I_{i-1,i} + 1$$

where for each $i = 1, \ldots, n+1$, $\langle I_{V_{i-1}}, I_{V_i}\rangle$ denotes the subgroup of G generated by $I_{V_{i-1}}$ and I_{V_i}, and $\sharp I_{i-1,i}$ denotes $\max\{\sharp I_{V_{i-1}}, \sharp I_{V_i}\}$. Note that $\sharp I_{V_i}, i = 0, \ldots, n+1$, does not depend on the choices of V_i. Moreover, we have

$$\sigma(f^{-1}(x)) = \sum_{I \in \mathrm{Min}} \sharp G/\sharp I - \sum_{J \in \mathrm{Max}} \sharp G/\sharp J + 1, \text{ if } \mathrm{Min} \neq \{I_{0,n+1}^{\mathrm{min}}\},$$

and

$$\sigma(f^{-1}(x)) = 0 \text{ if } \mathrm{Min} = \{I_{0,n+1}^{\mathrm{min}}\}.$$

3 Bounds of p-ranks of vertical fibers of abelian G-semi-stable coverings

In this section, we follow the notations of Section 2.2. Moreover, we assume that G is an abelian p-group, and that $f^{-1}(x)$ is connected.

Since G is abelian, $I_{V_i}, i = 0, \ldots, n+1$, does not depend on the choices of V_i. Then we use the notation I_{P_i} to denote I_{V_i} for each $i = 0, \ldots, n+1$. First, we have the following key proposition.

Proposition 3.1. *Suppose that $\sharp\mathrm{Min} \geq 2$. Let I' and I'' be two different elements of* Min*. Then neither $I' \subseteq I''$ nor $I' \supseteq I''$ holds.*

Proof. Without loss of generality, we may assume that $I' = I_{P_a}$ and $I'' = I_{P_b}$ such that $0 \leq a < b \leq n+1$, $I_{P_a} \neq I_{P_{a+1}}$, and $I_{P_{b-1}} \neq I_{P_b}$. Note that by the definition of Min, $I_{P_{a+1}}$ (resp. $I_{P_{b-1}}$) contains I_{P_a} (resp. I_{P_b}).

If $I' \subseteq I''$, we consider the quotient curve Y/I''. Then we obtain two morphisms of semi-stable curves $\xi_1 : Y \longrightarrow Y/I''$ and $\xi_2 : Y/I'' \longrightarrow X^{\mathrm{sst}}$ such that $\xi_2 \circ \xi_1 = h$. Write V_a and V_b for the irreducible components of Y_s such that $h(V_a) = P_a$ and $h(V_c) = P_c$, respectively. By contracting $\cup_{i=a+1}^{b-1} P_i$ and $\xi_2^{-1}(\cup_{i=a+1}^{b-1} P_i)_{\mathrm{red}}$ (cf. [BLR, 6.7 Proposition 4]), we obtain two contracting morphisms $c_{X^{\mathrm{sst}}} : X^{\mathrm{sst}} \longrightarrow (X^{\mathrm{sst}})^*$ and $c_{Y/I''} : Y/I'' \longrightarrow (Y/I'')^*$. Moreover, ξ_2 induces a morphism $\xi_2^* : (Y/I'')^* \longrightarrow (X^{\mathrm{sst}})^*$ such that the following commutative diagram:

$$\begin{array}{ccc} Y/I'' & \xrightarrow{c_{Y/I''}} & (Y/I'')^* \\ {\scriptstyle \xi_2}\big\downarrow & & {\scriptstyle \xi_2^*}\big\downarrow \\ X^{\mathrm{sst}} & \xrightarrow{c_{X^{\mathrm{sst}}}} & (X^{\mathrm{sst}})^*. \end{array}$$

Note that $(X^{\mathrm{sst}})^*$ is a semi-stable curve over S.

Since $I' = I_{P_a} \subseteq I'' = I_{P_b}$, ξ_2^* is étale at the generic points of $c_{Y/I''} \circ \xi_1(V_a)$ and $c_{Y/I''} \circ \xi_1(V_b)$. Thus, by applying Zariski-Nagata purity and [T, Lemma 2.1 (iii)], we obtain that ξ_2^* is étale at $c_{Y/I''}(V_a) \cap c_{Y/I''}(V_b)$ (i.e., the inertia group of each point of $c_{Y/I''}(V_a) \cap c_{Y/I''}(V_b)$ is trivial). On the other hand, since $I_{P_{b-1}}$ contains I_{P_b}, we have the inertia group of each point of $c_{Y/I''}(V_a) \cap c_{Y/I''}(V_b)$ is $I_{P_{b-1}}/I''$. Then we obtain $I_{P_{b-1}} = I''$. This is a contradiction. Then I' is not contained in I''.

Similar arguments to the arguments given in the proof above imply that I'' is not contained in I'. Then we complete the proof of the proposition. $\square$

Remark 3.1.1. We follow the notations of Proposition 3.1. If there is an element $I \in \mathrm{Min}$ such that $I = \cap_{i=0}^{n+1} I_{P_i}$ (e.g. G is cyclic), then we have

$$\sigma(f^{-1}(x)) = \sharp G/\sharp I - \sharp G/\sharp I_{P_0} - \sharp G/\sharp I_{P_{n+1}} + 1.$$

Definition 3.2. Let N be a finite p-group and H a subgroup of N. We define $I(H)$ to be a maximal set satisfied the following conditions: (i) $H \in I(H)$; (2) for any two different elements H' and H'' of $I(H)$, neither $H' \subseteq H''$ nor $H' \supseteq H''$ holds. Write $\mathrm{Sub}(N)$ for the set of the subgroups of N. We set

$$M(N) := \max\{\sharp I(N')\}_{I(N'),\ N' \subseteq \mathrm{Sub}(N)}.$$

For any $1 \leq d \leq \sharp N$, write $C_d(N)$ for the set of the subgroups of N with order d. Let A be an elementary abelian p-group such that $\sharp A = \sharp N$. We set

$$B(\sharp N) := \sharp \mathrm{Sub}(A),$$

where $\mathrm{Sub}(A)$ denotes the set of the subgroups of A. Note that $B(\sharp N)$ depends only on $\sharp N$.

We have the following lemma.

Lemma 3.3. *Let A be an elementary abelian p-group with order $\sharp G$ and $1 \leq d \leq \sharp G$ an integer number. Then we have*

$$\sharp C_d(G) \leq \sharp C_d(A).$$

In particular, we have

$$M(N) \leq B(\sharp N).$$

Proof. Since G is a p-group, G has non-trivial central subgroup. Fix a central subgroup Z of order p in G. Write $C_d^Z(G)$ (resp. $C_d^{\backslash Z}(G)$) for the set of subgroups of order d which contain Z (resp. do not contain Z). If H is a subgroup of G/Z, let $C_d^{(Z,H)}(G)$ be the set of $L \in C_d^{(Z)}(G)$ whose projection on G/Z is H. Let $C_d^Z[G/Z]$ be the set of $H \in C_d(G/Z)$ for which $C_d^{(Z,H)}(G) \neq \emptyset$. If $H \in C_d^Z[G/Z]$, then there is a natural bijection from $C_d^{(Z,H)}(G)$ to $\mathrm{Hom}(H, Z)$. Denote $G^* = G/(G^p[G,G])$.

If $d = 1$, the lemma is trivial. Then we may assume that p divides d. We have

$$\begin{aligned}\sharp C_d(G) = \sharp C_d^Z(G) + \sharp C_d^{\backslash Z}(G) &= \sharp C_{d/p}(G/Z) + \sharp C_d^{\backslash Z}(G)\\ &= \sharp C_{d/p}(G/Z) + \sum_{H \in C_d^Z[G/Z]} \sharp C_d^{(Z,H)}(G)\\ &= \sharp C_{d/p}(G/Z) + \sum_{H \in C_d^Z[G/Z]} \sharp(\mathrm{Hom}(H^*, Z)).\end{aligned}$$

Thus, we obtain

$$\sharp C_d(G) \leq \sharp C_{d/p}(G/Z) + \sum_{H \in C_d^Z[G/Z]} \sharp(\mathrm{Hom}((G/Z)^*, Z))$$

$$= \sharp C_{d/p}(G/Z) + \sharp C_d^Z[G/Z] \sharp(\mathrm{Hom}((G/Z)^*, Z))$$
$$\leq \sharp C_{d/p}(G/Z) + \sharp C_d(G/Z) \sharp(\mathrm{Hom}((G/Z)^*, Z)).$$

Write $Z' \cong \mathbb{Z}/p\mathbb{Z}$ for a subgroup of A. By induction, we have $\sharp C_{d/p}(G/Z) \leq \sharp C_{d/p}(A/Z')$. Then we obtain

$$\sharp C_d(G) \leq \sharp C_{d/p}(A/Z) + \sharp C_d(G/Z) \sharp(\mathrm{Hom}((G/Z)^*, Z)) \leq \sharp C_d(A).$$

This completes the proof of the lemma. □

Theorem 3.4. *Let $f : Y \longrightarrow X$ be a G-semi-stable covering over S, and x a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected, and that G is an abelian p-group. Then we have*

$$\sigma(f^{-1}(x)) \leq M(G) \cdot \sharp G - 1 \leq B(\sharp G) \cdot \sharp G - 1.$$

Proof. If x is a smooth point of the special fiber X_s of X, then $\sigma(f^{-1}(x)) = 0$ (cf. Theorem 1.1). Thus, we may assume that x is a singular point of X_s.

If $\mathrm{Min} = \emptyset$, then Theorem 2.5 implies that $\sigma(f^{-1}(x)) = 0$. The theorem follows. If $\mathrm{Min} \neq \emptyset$, then we have $\sharp\mathrm{Max} \geq 2$. Thus, by applying Theorem 2.5, we obtain

$$\sigma(f^{-1}(x)) = \sum_{I \in \mathrm{Min}} \sharp G/\sharp I - \sum_{J \in \mathrm{Max}} \sharp G/\sharp J + 1$$

$$\leq \sharp\mathrm{Min} \cdot \sharp G - 1 \leq M(G) \cdot \sharp G - 1 \leq B(\sharp G) \cdot \sharp G - 1.$$

□

Remark 3.4.1. If G is a cyclic p-group, then by the definition of $M(G)$, we have $M(G) = 1$. Thus, if G is a cyclic p-group, we have

$$\sigma(f^{-1}(x)) \leq \sharp G - 1.$$

This is the main theorem of [S].

4 Graphs and p-ranks of G-semi-stable coverings

We follow the notations of Section 2.2. Let $f : Y \longrightarrow X$ be a G-semi-stable covering over S, x a vertical point associated to f, $h : Y \longrightarrow X^{\mathrm{sst}} := Y/G$ for the finite G-semi-stable covering over S induced by f, and $g : X^{\mathrm{sst}} \longrightarrow X$ the morphism of semi-stable curves over S induced by f such that $g \circ h = f$. Suppose that $f^{-1}(x)$ is connected. In this section, by using the resolution of nonsingularities over marked points, we introduce a semi-graph $\Gamma_{Y_s}^{f\text{-etd}}$ associated f and a semi-graph $\Gamma_x^{f\text{-etd}}$ associated to the vertical fiber $f^{-1}(x)$. We will see that together with some data of $X_s^{\rm sst}$, the p-rank $\sigma(Y_s)$ is determined by $\Gamma_{Y_s}^{f\text{-etd}}$. Moreover, the p-rank $\sigma(f^{-1}(x))$ is determined by a **sub-semi-graph** of $\Gamma_x^{f\text{-etd}}$.

First, let us treat the global case. Let $x_s^v, v \in v(\Gamma_{X_s^{\rm sst}})$, be a smooth point of X_v, where X_v denotes the irreducible component of $X_s^{\rm sst}$ corresponding v. By replacing S by a finite extension of S, there is a S-rational point $x_S^v \in X^{\rm sst}(S)$ such that $x_S^v|_s = x_s^v$. Moreover, by replacing S by a finite extension of S, we may assume that $f^{-1}(x_S^v)_{\rm red}|_\eta$ are η-rational points of the generic fiber Y_η of Y. Write $E_{X^{\rm sst}}$ for the set of S-rational points $\{x_S^v\}_{v\in v(\Gamma_{X_s^{\rm sst}})} \subseteq X^{\rm sst}(S)$. We define a pointed semi-stable curve $\mathscr{X}^{\rm sst}$ to be $(X^{\rm sst}, E_{X^{\rm sst}})$. Write $\mathscr{X}_\eta^{\rm sst} = (X_\eta^{\rm sst}, E_{X_\eta^{\rm sst}})$ for the generic fiber of $\mathscr{X}^{\rm sst}$, $\mathscr{X}_s^{\rm sst} = (X_s^{\rm sst}, E_{X_s^{\rm sst}})$ for the special fiber of $\mathscr{X}^{\rm sst}$, and $\Gamma_{\mathscr{X}_s^{\rm sst}}$ for the dual semi-graph of $\mathscr{X}_s^{\rm sst}$. Together with the set of η-rational points $E_{Y_\eta} := f_\eta^{-1}(E_{X_\eta^{\rm sst}})$, we obtain a pointed semi-stable curve (Y_η, E_{Y_η}) and a natural morphism of pointed semi-stable curves $h_\eta^\bullet : (Y_\eta, E_{Y_\eta}) \longrightarrow \mathscr{X}_\eta^{\rm sst}$ induced by h_η. Then $h_\eta^\bullet$ extends uniquely to a G-pointed semi-stable covering $h^\bullet : \mathscr{Y} := (Y^*, E_{Y^*}) \longrightarrow \mathscr{X}^{\rm sst}$ such that $h^\bullet|_\eta = h_\eta^\bullet$ (cf. [Y, Proposition 3.4]). Write $\mathscr{Y}_\eta := (Y_\eta^*, E_{Y_\eta^*}) = (Y_\eta, E_{Y_\eta})$ for the generic fiber of $\mathscr{Y}$, $\mathscr{Y}_s := (Y_s^*, E_{Y_s^*})$ for the special fiber of $\mathscr{Y}$, and $\Gamma_{\mathscr{Y}_s}$ for the dual semi-graph of $\mathscr{Y}_s$. Note that the morphism of the underlying curves of the generic fibers $\underline{h}_\eta^\bullet : Y_\eta^* \longrightarrow X_\eta^{\rm sst}$ coincides with $h_\eta : Y_\eta \longrightarrow X_\eta^{\rm sst}$ over η, and the morphism of the underlying curves of the special fibers $\underline{h}_s^\bullet : Y_s^* \longrightarrow X_s^{\rm sst}$ does not coincide with $h_s : Y_s \longrightarrow X_s$ over s in general.

Proposition 4.1. *Let $v \in v(\Gamma_{X_s^{\rm sst}})$, X_v the irreducible component of the special fiber $X_s^{\rm sst}$ corresponding to v, Y_v^* an irreducible component of the special fiber Y_s^* of Y^* such that $\underline{h}_s^\bullet(Y_v^*) = X_v$. Write $D_{Y_v^*} \subseteq G$ (resp. $I_{Y_v^*} \subseteq G$) for the decomposition group (resp. the inertia group) of Y_v^*. Let x_s be a closed point of X_v.*

(i) If $I_{Y_v^*} = \{1\}$ or $x_s \in X_v \setminus E_{X_s^{\rm sst}}$, then x_s is not a vertical point associated to $h^\bullet$.

(ii) If $I_{Y_v^*}$ is not trivial and $x_s \in X_v \cap E_{X_s^{\rm sst}}$, then x_s is a vertical point associated to $h^\bullet$. Moreover, if $x_s \in X_v \cap E_{X_s^{\rm sst}}$ is a vertical point associated to $h^\bullet$, we write V_v for the set of the connected components of $(\underline{h}^\bullet)^{-1}(x_s)_{\rm red}$ which intersect with Y_v is not empty. Then for each element $E \in V_v$ (i.e., a connected component of $(\underline{h}^\bullet)^{-1}(x_s)_{\rm red}$), we have $\sharp E \cap E_{Y_s^*} = \sharp I_{Y_v^*}$.

Proof. By the construction of $h^\bullet$, we observe that $\underline{h}_s^\bullet|_{Y_s^*\setminus(\underline{h}_s^\bullet)^{-1}(E_{X_s^{\rm sst}})} : Y_s^*\setminus(\underline{h}_s^\bullet)^{-1}(E_{X_s^{\rm sst}}) \longrightarrow X_s^{\rm sst}\setminus E_{X_s^{\rm sst}}$ coincides with $h_s|_{Y_s\setminus h_s^{-1}(E_{X_s^{\rm sst}})} : Y_s\setminus h_s^{-1}(E_{X_s^{\rm sst}}) \longrightarrow X_s\setminus E_{X_s^{\rm sst}}$. Then (i) follows.

Write $x_\eta \in E_{X_\eta^{\rm sst}}$ for the marked point of $\mathscr{X}_\eta^{\rm sst}$ such that the reduction of x_η is x_s. Write Y_v for an irreducible component of Y_s such that $h_s(Y_v) = X_v$, $D_{Y_v} \subseteq G$ (resp. $I_{Y_v} \subseteq G$) for the decomposition group (resp. the inertia group) of Y_v. Note that we have $\sharp D_{Y_v} = \sharp D_{Y_v^*}$ and $\sharp I_{Y_v} = \sharp I_{Y_v^*}$.

If $I_{Y_v^*}$ is not trivial and $x_s \in X_v \cap E_{X_s^{\rm sst}}$, then we have $\sharp h_s^{-1}(x)_{\rm red} = \sharp G/\sharp I_{Y_v^*}$; moreover, $Y_v \cap h_s^{-1}(x)_{\rm red} = \sharp D_{Y_v}/\sharp I_{Y_v} = \sharp D_{Y_v^*}/\sharp I_{Y_v^*}$. Since $\sharp h_\eta^{-1}(x_\eta) = \sharp G$, we obtain that $\underline{h}^\bullet$ does not coincide with h over x_s. This means that x_s is a vertical point associated to $h^\bullet$.

Since V_v admits a natural action of G induced by the action of G on $\mathscr{Y}$, we have $\sharp V_v = \sharp D_{Y_v^*}/\sharp I_{Y_v^*}$. On the other hand, we have $\sharp((\underline{h}_s^\bullet)^{-1}(x_s)_{\rm red} \cap (\underline{h}_s^\bullet)^{-1}(X_v)_{\rm red}) = \sharp D_{Y_v^*}$. Thus, for each $E \in V_v$, we obtain $\sharp(E \cap E_{Y_s^*}) = \sharp I_{Y_v^*}$. This completes the proof of the proposition. $\square$

Remark 4.1.1. Since all the vertical points associated to $h^\bullet$ are smooth, the dual semi-graph Γ_{Y_s} of Y_s can be regarded as a sub-semi-graph of $\Gamma_{\mathscr{Y}_s}$ in a natural way.

Write $V_{h^\bullet}$ for the set of the connected components of the vertical fibers associated to the vertical points associated to $h^\bullet$ (note that Proposition 4.1 implies that all the vertical points associated to $h^\bullet$ are contained in $E_{X_s^{\mathrm{sst}}}$). For each $v \in v(\Gamma_{Y_s}) \subseteq v(\Gamma_{\mathscr{Y}_s})$, write Y_v^* for the irreducible component of Y_s^* corresponding to v. Write M_E for the set $E_{Y_s^*} \cap E$ for each $E \in V_{h^\bullet}$. Proposition 4.1 implies that if $E \cap Y_v^* \neq \emptyset$, then $\sharp M_E = \sharp I_{Y_v^*}$.

We define a semi-graph $\Gamma_{Y_s}^{f\text{-etd}}$ as follows: (i) $v(\Gamma_{Y_s}^{f\text{-etd}}) := v(\Gamma_{Y_s}) \coprod \{v_E\}_{E \in V_{h^\bullet}}$; (ii) $e^{\mathrm{cl}}(\Gamma_Y^{\mathrm{etd}}) := e^{\mathrm{cl}}(\Gamma_{Y_s}) \coprod \{e_E\}_{E \in V_{h^\bullet}}$ and $e^{\mathrm{op}}(\Gamma_{Y_s}^{f\text{-etd}}) := e^{\mathrm{op}}(\Gamma_{\mathscr{Y}_s})$; (iii) for each $e \in e^{\mathrm{cl}}(\Gamma_Y^{\mathrm{etd}}) \setminus \{e_E\}_{E \in V_{h^\bullet}}$, $\zeta_e^{\Gamma_{Y_s}^{f\text{-etd}}} = \zeta_e^{\Gamma_{\mathscr{Y}_s}}$; (iv) for each $e = \{b_1^e, b_2^e\} \in \{e_E\}_{E \in V_{h^\bullet}}$, $\zeta_e^{\Gamma_{Y_s}^{f\text{-etd}}}(b_1^e) = \zeta_e^{\Gamma_{\mathscr{Y}_s}}(b_1^e)$ and $\zeta_e^{\Gamma_{Y_s}^{f\text{-etd}}}(b_2^e) = v_E$; (v) for each $e = \{b_1^e, b_2^e\} \in e^{\mathrm{op}}(\Gamma_{Y_s}^{\mathrm{etd}})$, write y_e for the closed point of Y_s^* corresponding to e; we set $\zeta_e^{\Gamma_{Y_s}^{f\text{-etd}}}(b_1^e) = v_E$ and $\zeta_e^{\Gamma_{Y_s}^{f\text{-etd}}}(b_2^e) = \{v(\Gamma_{Y_s}^{f\text{-etd}})\}$ if $y_e \in M_E$, and $\zeta_e^{\Gamma_{Y_s}^{f\text{-etd}}} = \zeta_e^{\Gamma_{\mathscr{Y}_s}}$ if $y_e \notin \cup_{E \in V_{h^\bullet}} M_E$.

Write $\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ for the dual semi-graph of $\mathscr{X}_s^{\mathrm{sst}}$. There is a natural map $\beta_f^\bullet : \Gamma_{\mathscr{Y}_s} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ of semi-graphs induced by $h^\bullet$. Note that since $h^\bullet$ is not finite, $\beta_f^\bullet$ is not a morphism of semi-graphs in general. Furthermore, $\beta_f^\bullet$ induces a map $\beta_f^{\mathrm{etd}} : \Gamma_{Y_s}^{f\text{-etd}} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ as follows: (i) for each $v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$, $\beta_f^{\mathrm{etd}}(v) := \beta_f^\bullet(v)$ if $v \notin \{v_E\}_{E \in V_{h^\bullet}}$, and if $v = v_E \in \{v_E\}_{E \in V_{h^\bullet}}$, $\beta_f^{\mathrm{etd}}(v)$ is equal to the open edge corresponding to the marked point of $\mathscr{X}_s$ which is the image of E; (ii) for each $e \in e^{\mathrm{cl}}(\Gamma_{Y_s}^{f\text{-etd}}) \cup e^{\mathrm{op}}(\Gamma_{Y_s}^{f\text{-etd}})$, $\beta_f^{\mathrm{etd}}(e) = \beta_f^\bullet(e)$ if $e \notin \cup_{E \in V_{h^\bullet}} e(v_E)$, and $\beta_f^{\mathrm{etd}}(e)$ is equal to the open edge corresponding to the marked point of $\mathscr{X}_s$ which is the image of E.

Note that it is easy to see that $\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ and $\Gamma_{Y_s}^{f\text{-etd}}$ do not depend on the choices of the set of marked points $E_{X_s^{\mathrm{sst}}}$.

Definition 4.2. Let $f : Y \longrightarrow X$ be a G-semi-stable covering over S and $\beta_f : \Gamma_{Y_s} \longrightarrow \Gamma_{X_s^{\mathrm{sst}}}$ the morphism of dual graphs induced by the morphism of semi-stable curves $h|_s : Y_s \longrightarrow X_s^{\mathrm{sst}}$ over s. We shall call the semi-graph $\Gamma_{Y_s}^{f\text{-etd}}$ (resp. the morphism of semi-graphs $\beta_f^{\mathrm{etd}} : \Gamma_{Y_s}^{f\text{-etd}} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$) constructed above the **extended dual semi-graph** of Y_s (resp. the **extended map** of β_f) associated to f . We define $\mathfrak{Com}^f$ associated to the G-semi-stable covering f to be the quadruple $(\Gamma_{Y_s}^{f\text{-etd}}, \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \beta_f^{f\text{-etd}} : \Gamma_{Y_s}^{f\text{-etd}} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \sharp G)$.

Let $\mathbb{G}_1^i$ and $\mathbb{G}_2^i$, $i \in \{1, 2\}$, be two semi-graphs, $\beta_i : \mathbb{G}_1^i \longrightarrow \mathbb{G}_2^i$ a map of semi-graphs, and m_i is a positive number. We shall call two quadruples $(\mathbb{G}_1^1, \mathbb{G}_2^1, \beta_1 : \mathbb{G}_1^1 \longrightarrow \mathbb{G}_2^1, m_1)$ and $(\mathbb{G}_1^2, \mathbb{G}_2^2, \beta_2 : \mathbb{G}_1^2 \longrightarrow \mathbb{G}_2^2, m_2)$ are isomorphic if $m_1 = m_2$ and there exist two isomorphism of semi-graphs $\alpha_1 : \mathbb{G}_1^1 \xrightarrow{\sim} \mathbb{G}_1^2$ and $\alpha_2 : \mathbb{G}_2^1 \xrightarrow{\sim} \mathbb{G}_2^2$ such that the following commutative diagram holds:

$$\begin{array}{ccc} \mathbb{G}_1^1 & \xrightarrow{\alpha_1} & \mathbb{G}_1^2 \\ {\scriptstyle\beta_1}\downarrow & & {\scriptstyle\beta_2}\downarrow \\ \mathbb{G}_2^1 & \xrightarrow{\alpha_2} & \mathbb{G}_2^2. \end{array}$$

We use the notation (α_1, α_2) to denote the isomorphism of quadruples defined above.

Note that by the definition of $\Gamma_{Y_s}^{f\text{-etd}}$, Γ_{Y_s} can be regarded as a sub-semi-graph of $\Gamma_{Y_s}^{f\text{-etd}}$. Moreover, we have the following lemma.

Lemma 4.3. *The dual semi-graph Γ_{Y_s} of the special fiber Y_s of Y can be reconstructed by $\sharp G$ and the extended dual semi-graph $\Gamma_{Y_s}^{f\text{-etd}}$ of Y_s associated to f in a purely graphic way. Moreover, the morphism of dual graphs $\beta_f : \Gamma_{Y_s} \longrightarrow \Gamma_{X_s^{\mathrm{sst}}}$ can be reconstructed by $\sharp G$ and the extended map $\beta_f^{\mathrm{etd}} : \Gamma_{Y_s}^{f\text{-etd}} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ associated to f.*

Proof. Write $\mathbb{G}$ and $\mathbb{H}$ for $\Gamma_{Y_s}^{f\text{-etd}}$ and Γ_{Y_s}, respectively. Let V be a subset of $v(\mathbb{G})$ defined as follows:

$$\{v \in v(\mathbb{G}) \mid \sharp e(v) \cap e^{\mathrm{op}}(\mathbb{G}) \neq \sharp G \text{ and there is only one vertex } v \neq v' \in v(\mathbb{G})$$

$$\text{such that there is an edge } e \text{ which links } v \text{ and } v' \ \}.$$

We define a sub-semi-graph $\mathbb{G}'$ as follows: (i) $v(\mathbb{G}') := v(\mathbb{G}) \setminus V$ (note that by Lemma 4.4 below, we obtain $v(\mathbb{G}')$ is not empty); (ii) $e^{\mathrm{cl}}(\mathbb{G}') := e^{\mathrm{cl}}(\mathbb{G}) \setminus \{e(v)\}_{v\in V}$; (iii) $e^{\mathrm{op}}(\mathbb{G}') = \emptyset$; (iv) For each $e \in e^{\mathrm{cl}}(\mathbb{G}')$, we set $\zeta_e^{\mathbb{G}'} := \zeta_e^{\mathbb{G}}$. It is easy to see that $\mathbb{G}' = \mathbb{H}$. Thus, Γ_{Y_s} can be reconstructed by $\Gamma_{Y_s}^{f\text{-etd}}$ and $\sharp G$.

Moreover, note that $\Gamma_{X_s^{\mathrm{sst}}}$ is equal to the image $\beta_f^{\mathrm{etd}}(\Gamma_{Y_s})$. Thus, $\beta_f : \Gamma_{Y_s} \longrightarrow \Gamma_{X_s^{\mathrm{sst}}}$ can be reconstructed by $\beta_f^{\mathrm{etd}} : \Gamma_{Y_s}^{f\text{-etd}} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ and $\sharp G$. This completes the proof of the lemma. $\square$

Lemma 4.4. *Let $f : Y \longrightarrow X$ be a G-semi-stable covering over S. Suppose that the special fiber X_s of X is irreducible, and the morphism of special fibers $f_s : Y_s \longrightarrow X_s$ over s is not generically étale over X_s. Then Y_s is not irreducible.*

Proof. If the lemma does not hold, we may assume that Y_s is irreducible. Since f_s is not generically étale, by replacing G by the inertia group $I_{Y_s} \subseteq G$ and replacing X by Y/I_{Y_s}, we may assume that $G = I_{Y_s}$. Then we obtain the genus $g(Y_s)$ of Y_s is equal to the genus $g(X_s)$ of X_s. On the other hand, since the morphism of generic fibers $f_\eta : Y_\eta \longrightarrow X_\eta$ is a connected étale covering with a non-trivial Galois group G, we obtain the genus $g(Y_\eta)$ of Y_η is strictly greater than the genus $g(X_\eta)$ of X_η. This is a contradiction. We complete the proof of the lemma. $\square$

Theorem 4.5. *We follow the notations above. The p-rank $\sigma(Y_s)$ is determined by $\mathfrak{Com}^f$ and $\{\sigma(\widetilde{X}_v)\}_{v\in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})}$, where $\widetilde{X}_v$ denotes the normalization of the irreducible component X_v of $\mathscr{X}_s^{\mathrm{sst}}$ corresponding to v.*

Proof. The theorem follows from Theorem 2.4, Proposition 4.1, and Lemma 4.3. $\square$

Moreover, we have the following corollary.

Corollary 4.6. *Let $f : Y \longrightarrow X$ (resp. $h : Z \longrightarrow W$) be a G-semi-stable covering (resp. J-semi-stable covering) over S, $h_f : Y \longrightarrow X^{\mathrm{sst}} := Y/G$ (resp. $h_h : Z \longrightarrow W^{\mathrm{sst}} := Z/G$) the quotient morphism, Γ_{Y_s} and $\Gamma_{X_s^{\mathrm{sst}}}$ (resp. Γ_{Z_s} and $\Gamma_{W_s^{\mathrm{sst}}}$) the dual graphs of the special fiber Y_s of Y (resp. Z_s of Z) and the special fiber X_s^{sst} of X^{sst} (resp. W_s^{sst} of W^{sst}), respectively, $\beta_f : \Gamma_{Y_s} \longrightarrow \Gamma_{X_s^{\mathrm{sst}}}$ the $\Gamma_{Y_s}^{f\text{-etd}}$ the extended dual semi-graph of Y_s associated to f (resp. $\Gamma_{Z_s}^{h\text{-etd}}$ the extended dual semi-graph of Z_s associated to h), and $\beta_f^{\mathrm{etd}} : \Gamma_{Y_s^{\mathrm{etd}}} \longrightarrow \Gamma_{\mathscr{X}_s^{\mathrm{sst}}}$ the extended map of $\beta_f : \Gamma_{Y_s} \longrightarrow \Gamma_{X_s^{\mathrm{sst}}}$ associated to f (resp. $\beta_h^{\mathrm{etd}} : \Gamma_{Z_s^{\mathrm{etd}}} \longrightarrow \Gamma_{\mathscr{W}_s^{\mathrm{sst}}}$ the extended map of $\beta_h : \Gamma_{Y_s} \longrightarrow \Gamma_{X_s^{\mathrm{sst}}}$ associated to h). Suppose that $(\alpha_1, \alpha_2) : \mathfrak{Com}^f \overset{\sim}{\to} \mathfrak{Com}^h$*

is an isomorphism of quadruples such that $\sigma(\widetilde{X}_v)=\sigma(\widetilde{W}_{\alpha_2(v)})$ for each $v\in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})$, where $\widetilde{X}_v$ and $\widetilde{W}_{\alpha_2(v)}$ denote the normalization of the irreducible components X_v and $W_{\alpha_2(v)}$ of $\mathscr{X}_s^{\mathrm{sst}}$ and $\mathscr{W}_s^{\mathrm{sst}}$ corresponding to v and $\alpha_2(v)$, respectively. Then we have

$$\sigma(Y_s)=\sigma(Z_s).$$

Next, let us treat the local case. We only treat the case where x is a singular point of X_s. Let X_1' and X_2' (which may be equal) be two irreducible components X_s which contain x. Write X_1 and X_2 for the strict transforms of X_1' and X_2' under the birational morphism $g:X^{\mathrm{sst}}\longrightarrow X$, $C:=\cup_{i=1}^n P_i\subseteq g^{-1}(x)_{\mathrm{red}}$ for the chain of $\mathbb{P}^1$, V_x for $h^{-1}(X_1\cup X_2\cup C)_{\mathrm{red}}$, and V_x^* for $(\underline{h}^\bullet)^{-1}(X_1\cup X_2\cup C)_{\mathrm{red}}$. Note that since $f^{-1}(x)$ is connected, V_x and V_x^* are connected too. We define a pointed semi-stable curve $\mathscr{V}_x$ to be $(V_x^*, E_{V_x^*}:=V_x^*\cap E_{Y_s^*})$. Write Γ_{V_x} and $\Gamma_{\mathscr{V}_x}$ for the dual graphs of V_x and $\mathscr{V}_x$, respectively. Then Γ_{V_x} can be regarded as a sub-semi-graph of $\Gamma_{\mathscr{V}_x}$ in a natural way. Write $V_{h^\bullet}^x$ for the set

$$\{E\in V_{h^\bullet}\mid E\subseteq V_x^*\}.$$

We define a semi-graph $\Gamma_x^{f\text{-etd}}$ as follows: (i) $v(\Gamma_x^{f\text{-etd}}):=v(\Gamma_{V_x})\coprod\{v_E\}_{E\in V_{h^\bullet}^x}$; (ii) $e^{\mathrm{cl}}(\Gamma_x^{f\text{-etd}}):=e^{\mathrm{cl}}(\Gamma_{V_x})\coprod\{e_E\}_{E\in V_{h^\bullet}^x}$ and $e^{\mathrm{op}}(\Gamma_x^{f\text{-etd}}):=e^{\mathrm{op}}(\Gamma_{\mathscr{V}_x})$; (iii) For each $e\in e^{\mathrm{cl}}(\Gamma_x^{f\text{-etd}})\setminus\{e_E\}_{E\in V_{h^\bullet}^x}$, $\zeta_e^{\Gamma_x^{f\text{-etd}}}=\zeta_e^{\Gamma_{\mathscr{V}_x}}$; (iv) For each $e=\{b_1^e,b_2^e\}\in\{e_E\}_{E\in V_{h^\bullet}^x}$, $\zeta_e^{\Gamma_x^{f\text{-etd}}}(b_1^e)=\zeta_e^{\Gamma_{\mathscr{V}_x}}(b_1^e)$ and $\zeta_e^{\Gamma_x^{f\text{-etd}}}(b_2^e)=v_E$; (v) For each $e=\{b_1^e,b_2^e\}\in e^{\mathrm{op}}(\Gamma_x^{f\text{-etd}})$, write y_e for the closed point of V_x^* corresponding to e. We set $\zeta_e^{\Gamma_x^{f\text{-etd}}}(b_1^e)=v_E$ and $\zeta_e^{\Gamma_x^{f\text{-etd}}}(b_2^e)=\{v(\Gamma_x^{f\text{-etd}})\}$ if $y_e\in M_E$, and $\zeta_e^{\Gamma_x^{f\text{-etd}}}=\zeta_e^{\Gamma_{\mathscr{Y}_s}}$ if $y_e\notin\cup_{E\in V_{h^\bullet}^x}M_E$.

Definition 4.7. Let $f:Y\longrightarrow X$ be a G-semi-stable covering over S and x a vertical point associated to f. Suppose that x is a singular point of the special fiber X_s, and that the vertical fiber $f^{-1}(x)$ associated to x is connected. We shall call the semi-graph $\Gamma_x^{f\text{-etd}}$ constructed above the **extended dual semi-graph** associated to the vertical fiber $f^{-1}(x)$. We shall call a connected sub-semi-graph $\mathbb{V}\subseteq\Gamma_x^{f\text{-etd}}$ a **stem** of $\Gamma_x^{f\text{-etd}}$ if the following conditions are satisfied:

(i) $v(\mathbb{V})=\{v_0,\dots,v_{n+1}\}\cup\{v\in\{v_E\}_{E\in V_{h^\bullet}^x}\mid$ there exist $e\in e^{\mathrm{cl}}(\Gamma_x^{f\text{-etd}})$ and $v'\in\{v_0,\dots,v_{n+1}\}$ such that e links v and $v'\}$;

(ii) for each $v_i\in v(\mathbb{V})$, the irreducible component $Y_{v_i}^*\subseteq V_x^*$ corresponding to v_i such that $\underline{h}_s^\bullet(Y_{v_i}^*)=P_i\subseteq C$ if $i\neq 0,n+1$, and $\underline{h}_s^\bullet(Y_{v_i}^*)=X_i\subseteq X_s^{\mathrm{sst}}$ if $i=0,n+1$;

(iii) $e^{\mathrm{cl}}(\mathbb{V})\cup e^{\mathrm{op}}(\mathbb{V}):=\{e=\{b_1^e,b_2^e\}\in e^{\mathrm{cl}}(\Gamma_x^{f\text{-etd}})\cup e^{\mathrm{op}}(\Gamma_x^{f\text{-etd}})\mid \zeta_e^{\Gamma_x^{f\text{-etd}}}(b_1^e)\in v(\mathbb{V})$ and $\zeta_e^{\Gamma_x^{f\text{-etd}}}(b_2^e)\in v(\mathbb{V})\}$.

We define $\mathfrak{Com}_x^f$ associated the G-semi-stable covering $f:Y\longrightarrow X$ over S and a vertical point x associated to f to be the pair $(\Gamma_x^{f\text{-etd}},\sharp G)$.

Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be two semi-graphs, and m_1 and m_2 two positive integer numbers. We shall call two pairs $(\mathbb{G}_1,m_1)$ and $(\mathbb{G}_2,m_2)$ are isomorphic if $m_1=m_2$ and there exists an isomorphism of semi-graphs $\alpha:\mathbb{G}_1\xrightarrow{\sim}\mathbb{G}_2$. We also use the notation α to denote this isomorphism of pairs.

Note that by the definition of $\Gamma_x^{f\text{-etd}}$, Γ_{V_x} can be regarded as a sub-semi-graph of $\Gamma_x^{f\text{-etd}}$ in a natural way. Similar arguments to the arguments given in the proof of Lemma 4.3, we have the following lemma.

Lemma 4.8. *The dual semi-graph Γ_{V_x} of V_x can be reconstructed by $\Gamma_x^{f\text{-etd}}$ and $\sharp G$ in a purely graphic way. Moreover, there exists a stem $\mathbb{V}$ of Γ_{V_x} which can be reconstructed by $\Gamma_x^{f\text{-etd}}$ and $\sharp G$.*

Theorem 4.9. *We follow the notations above. The p-rank $\sigma(f^{-1}(x))$ is determined by a stem of Γ_x^{etd}.*

Proof. The theorem follows from Theorem 2.4, Proposition 4.1, and Lemma 4.8. $\square$

Moreover, we have the following corollary.

Corollary 4.10. *Let $f : Y \longrightarrow X$ (resp. $h : Z \longrightarrow W$) be a G-semi-stable covering (resp. J-semi-stable covering) over S and x (resp. w) a vertical point associated to f (resp. h). Suppose that x (resp. w) is a singular point of the special fiber X_s of X (resp. W_s of W), and that $f^{-1}(x)$ (resp. $h^{-1}(w)$) is connected. Let $\Gamma_x^{f\text{-etd}}$ and $\Gamma_w^{h\text{-etd}}$ be the extended dual graphs associated to the vertical fiber $f^{-1}(x)$ and $h^{-1}(w)$, respectively, and $\alpha : \mathfrak{Com}_x^f \overset{\sim}{\to} \mathfrak{Com}_w^h$ an isomorphism of pairs. Then we have*

$$\sigma(f^{-1}(x)) = \sigma(h^{-1}(w)).$$

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