Local $p$-Rank and Semi-Stable Reduction of Curves

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Abstract

In the present paper, we investigate the local $p$-ranks of coverings of stable curves. Let $G$ be a finite $p$-group, $f : Y \rightarrow X$ a morphism of stable curves over a complete discrete valuation ring with algebraically closed residue field of characteristic $p > 0$, $x$ a singular point of the special fiber $X_s$ of $X$. Suppose that the generic fiber $X_0$ of $X$ is smooth, and the morphism of generic fibers $f_0$ is a Galois étale covering with Galois group $G$. Write $Y'$ for the normalization of $X$ in the function field of $Y$, $\psi : Y' \rightarrow X$ for the resulting normalization morphism. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of $x$. Suppose that the inertia group $I_{y'} \subseteq G$ of $y'$ is an abelian $p$-group. Then we give an explicit formula for the $p$-rank of a connected component of $f^{-1}(x)$. Furthermore, we prove that the $p$-rank is bounded by $\#I_{y'} - 1$ under certain assumptions, where $\#I_{y'}$ denotes the order of $I_{y'}$. These results generalize the results of M. Saüdi concerning local $p$-ranks of coverings of curves to the case where $I_{y'}$ is an arbitrary abelian $p$-group.

Keywords: $p$-rank, semi-stable reduction, semi-stable covering, semi-graph with $p$-rank.

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1 Introduction and ideas

Let $R$ be a complete valuation ring with algebraically closed residue field $k$ of characteristic $p > 0$, $K$ the quotient field of $R$, and $K$ an algebraic closure of $K$. We use the notation $S$ to denote the spectrum of $R$. Write $\eta, \overline{\eta}$ and $s$ for the generic point, the geometric generic point, and the closed point corresponding to the natural morphisms $\text{Spec } K \to S$, $\text{Spec } K \to S$, and $\text{Spec } k \to S$, respectively. Let $X$ be a stable curve of genus $g_X$ over $S$. Write $X_\eta$, $X_{\overline{\eta}}$, and $X_s$ for the generic fiber, the geometric generic fiber, and the special fiber, respectively. Moreover, we suppose that $X_\eta$ is smooth over $\eta$.

Let $Y_\eta$ be a geometrically connected curve over $\eta$, $f_\eta : Y_\eta \to X_\eta$ a finite Galois étale covering over $\eta$ with Galois group $G$. By replacing $S$ by a finite extension of $S$, we may assume that $Y_\eta$ admits a stable model over $S$. Then $f_\eta$ extends uniquely to a $G$-stable covering (cf. Definition 3.3) $f : Y \to X$ over $S$ (cf. [L2, Theorem 0.2] or Remark 3.3.1 of the present paper). We are interested in understanding the structure of the special fiber $Y_s$ of $Y$. If the order $\sharp G$ of $G$ is prime to $p$, then by the specialization theorem for log étale fundamental groups, $f_s$ is an admissible covering (cf. [Y1]); thus, $Y_s$ may be obtained by gluing together tame coverings of the irreducible components of $X_s$. On the other hand, if $p\sharp G$, then $f_s$ is not a finite morphism in general. For example, if $\text{char}(K) = 0$ and $\text{char}(k) = p > 0$, then there exists a Zariski dense subset $Z$ of the set of closed points of $X$, which may in fact be taken to be $X$ when $k$ is an algebraic closure of $\mathbb{F}_p$, such that for any $x \in Z$, after possibly replacing $K$ by a finite extension of $K$, there exist a finite group $H$ and an $H$-stable covering $f_W : W \to X$ such that the fiber $(f_W)^{-1}(x)$ is not finite (cf. [T], [Y2]).

If $f^{-1}(x)$ is not finite, we shall call $x$ a vertical point associated to $f$ and call $f^{-1}(x)$ the vertical fiber associated to $x$ (cf. Definition 3.4). In order to investigate the properties of $Y_s$, we focus on the geometric invariant $\sigma(Y_s)$ which is called the $p$-rank of $Y_s$ (cf. Definition 3.1 and Remark 3.1.1). By the definition of the $p$-rank of a stable curve, to calculate $\sigma(Y_s)$, it suffices to calculate the rank of $H^1(\Gamma_Y, \mathbb{Z})$ (where $\Gamma_Y$ denotes the dual graph of $Y_s$), the $p$-ranks of the irreducible components of $Y_s$ which are finite over $X_s$, and the $p$-ranks of the vertical fibers of $f$. In the present paper, we study the $p$-rank of a vertical fiber and consider the following problem:

**Problem 1.1.** Let $G$ be a finite $p$-group, $x$ be a vertical point associated to the $G$-stable covering $f : Y \to X$, $f^{-1}(x)$ the vertical fiber associated to $x$.

(a) Does there exist a minimal bound on the $p$-rank $\sigma(f^{-1}(x))$ (note that $\sigma(f^{-1}(x))$ is always bounded by the genus of $Y_s$)?

(b) Does there exist an explicit formula for the $p$-rank $\sigma(f^{-1}(x))$?

We will answer Problem 1.1 under certain assumptions (cf. Theorem 1.5 and Theorem 1.10). First, let us review some well-known results concerning Problem 1.1.

If $x$ is a nonsingular point, M. Raynaud proved the following result (cf. [R, Théorème 1]):

**Theorem 1.2.** If $x$ is a non-singular point of $X_s$, and $G$ is an arbitrary $p$-group, then the $p$-rank $\sigma(f^{-1}(x))$ is equal to 0.
By Theorem 1.2, in order to resolve Problem 1.1, it is sufficient to consider the case where \( x \) is a singular point of \( X_s \). In order to explain the results obtained in the present paper, let us introduce some notations. Write \( X_1 \) and \( X_2 \) for the irreducible components of \( X_s \) which contain \( x \), \( \psi : Y' \to X \) for the normalization of \( X \) in the function field of \( Y \). Let \( y' \in \psi^{-1}(x) \) be a point in the inverse image of \( x \). Write \( I_{y'} \subseteq G \) for the inertia group of \( y' \). In order to calculate the \( p \)-rank of \( f^{-1}(x) \), since \( Y/I_{y'} \to X \) is finite étale over \( x \), by replacing \( X \) by the stable model of the quotient \( Y/I_{y'} \) (note that \( Y/I_{y'} \) is a semi-stable curve over \( S \) (cf. [R, Appendice, Corollaire])), we may assume that \( G \) is equal to \( I_{y'} \).

Thus, from the point of view of resolving Problem 1.1, we may assume without loss of generality that \( G = I_{y'} \). In the remainder of this section, we shall assume that \( G = I_{y'} \) is of order \( p^r \) for some positive integer \( r \). Then \( f^{-1}(x) \) is connected. With regard to Problem 1.1 (a), M. Saïd proved the following result (cf. [S, Theorem 1]), by applying Theorem 1.2:

**Theorem 1.3.** If \( G \) is a cyclic \( p \)-group, then we have \( \sigma(f^{-1}(x)) \leq \sharp G - 1 \), where \( \sharp G \) denotes the order of \( G \).

Furthermore, there is an open problem posed by Saïd as follows (cf. [S, Question]):

**Problem 1.4.** If \( G \) is an arbitrary \( p \)-group, does there exist a minimal bound on the \( p \)-rank \( \sigma(f^{-1}(x)) \) that depends only on the order \( \sharp G \)?

Let us introduce some notations. Suppose that \( G \) is an abelian \( p \)-group. Let

\[
\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_0 = G
\]

be a maximal filtration of \( G \) (i.e., \( G_i/G_{i+1} \cong \mathbb{Z}/p\mathbb{Z} \) for \( i = 0, \ldots, r - 1 \)). It follows from [R, Appendice, Corollaire], that for \( i = 0, \ldots, r \), \( Y_i := Y/G_i \) is a semi-stable curve over \( S \). Write \( X^{\text{sst}} \) for \( Y/G \) and \( g \) for the resulting morphism \( g : X^{\text{sst}} \to X \) induced by \( f \). Then we obtain a sequence of \( \mathbb{Z}/p\mathbb{Z} \)-semi-stable coverings (cf. Definition 3.3)

\[
\Phi_f : Y = Y_r \xrightarrow{d_r} Y_{r-1} \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_1} Y_0 = X^{\text{sst}} \xrightarrow{g} X.
\]

In the following, we use the subscript “\( \text{red} \)” to denote the reduced induced closed subscheme associated to a scheme. For each \( i = 1, \ldots, r \), write \( \phi_i : Y_i \to Y_0 \) for the composite morphism \( d_1 \circ \cdots \circ d_i \). For simplicity, we suppose that \( C := g^{-1}(x)_{\text{red}} = \bigcup_{j=1}^n P_j \), where, for each \( j = 1, \ldots, n, P_j \) is isomorphic to \( \mathbb{P}^1 \) and meets the other irreducible components of the special fiber \( X^{\text{sst}}_s \) of \( X^{\text{sst}} \) at precisely two points (i.e., a chain of \( \mathbb{P}^1 \)). Thus, the \( p \)-rank \( \sigma(f^{-1}(x)) \) is equal to \( \sigma(\phi_i^{-1}(C)) \). For each \( i = 1, \ldots, r \), we define a set of subcurves of \( C \) associated to \( \Phi_f \), which plays a key role in the present paper, as follows:

\[
\mathcal{E}_i^{\Phi_f} := \phi_i(\text{the étale locus of } d_1|_{\phi_i^{-1}(C)_{\text{red}}} : \phi_i^{-1}(C)_{\text{red}} \to \phi_{i-1}^{-1}(C)_{\text{red}}) \subset C.
\]

We shall call \( \mathcal{E}_i^{\Phi_f} \) the \( i \)-th étale-chain associated to \( \Phi_f \) and call the disjoint union

\[
\mathcal{E}_f^{\Phi_f} := \coprod_i \mathcal{E}_i^{\Phi_f}
\]
the étale-chain associated to \( \Phi_f \). For each connected component \( E \) of \( \mathcal{E}_i^{\Phi_f} \), we use the notation \( l(E) \) to denote the cardinality of the set of the irreducible components of \( E \) and call \( l(E) \) the length of \( E \).

We generalize Saïdi’s result as follows (see also Theorem 3.15):

**Theorem 1.5.** If \( G \) is an arbitrary abelian \( p \)-group, and \( \mathcal{E}_i \) is connected for each \( i = 1, \ldots, n \), then we have \( \sigma(f^{-1}(x)) \leq \sharp G - 1 \).

**Remark 1.5.1.** If \( \sharp G \) is equal to \( p \), then we may construct a \( \mathbb{Z}/p\mathbb{Z} \)-stable covering \( f : Y \rightarrow X \) such that there exists a singular vertical point \( x \) such that the \( p \)-rank of \( \sigma(f^{-1}(x)) \) is equal to \( p - 1 \) (cf. [Y4, Section 4]). Thus, at least in the case where \( \sharp G = p \), \( \sharp G - 1 \) is the minimal bound for \( \sigma(f^{-1}(x)) \).

Next, let us consider Problem 1.1 (b). Let \( \{V_i\}_{i=0}^{n+1} \) be a set of irreducible components of the special fiber \( Y_s \) of \( Y \) such that the following conditions are satisfied: (i) \( \phi_r(V_i) = P_i \) if \( i = 1, \ldots, n \); (ii) \( \phi_r(V_0) = X_1 \) and \( \phi_r(V_{n+1}) = X_2 \); (iii) the union \( \cup_{i=0}^{n+1} V_i \) is a connected semi-stable subcurve of the special fiber \( Y_s \) of \( Y \). Write \( I_P \subseteq G \) for the inertia subgroup of \( G \). Note that since \( G \) is an abelian \( p \)-group, \( I_P \) does not depend on the choices of \( V_i \).

If \( G \) is a cyclic \( p \)-group, Saïdi obtained an explicit formula of the \( p \)-rank \( \sigma(f^{-1}(x)) \) as follows (cf. [S, Proposition 1]):

**Theorem 1.6.** If \( G \) is a cyclic \( p \)-group, and \( I_{P_0} \) is equal to \( G \), then we have
\[
\sigma(f^{-1}(x)) = \sharp(G/I_{\text{min}}) - \sharp(G/I_{P_{n+1}}),
\]
where \( I_{\text{min}} \) denotes the group \( \cap_{i=0}^{n+1} I_{P_i} \).

For a \( G \)-covering of semi-graphs with \( p \)-rank, we develop a general method to compute the \( p \)-rank (cf. Theorem 2.8). As an application, we generalize Saïdi’s formula to the case where \( G \) is an arbitrary abelian \( p \)-group as follows (cf. Theorem 3.9 and Remark 3.9.1):

**Theorem 1.7.** If \( G \) is an arbitrary abelian \( p \)-group, then we have
\[
\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \sharp(G/I_{P_i}) - \sum_{i=1}^{n+1} \sharp(G/(I_{P_{i-1}} + I_{P_i})) + 1.
\]

Finally, I would mention that by using the theory of semi-graphs with \( p \)-rank, we can generalize Theorem 1.8 to the case where \( G \) is an arbitrary \( p \)-group. Furthermore, we can obtain a global \( p \)-rank formula for the special fiber \( Y_s \) (cf. [Y5]).

The present paper contains two parts. In Section 2, we develop the theory of semi-graphs with \( p \)-rank and calculate the \( p \)-ranks of \( G \)-coverings. In Section 3, we construct a semi-graph with \( p \)-rank from a vertical fiber of a \( G \)-stable covering in a natural way and apply the results of Section 2 to prove Theorem 1.5 and Theorem 1.8.

## 2 Semi-graphs with \( p \)-rank

In this section, we develop the theory of semi-graphs with \( p \)-rank. We always assume that \( G \) is an abelian \( p \)-group with order \( p^r \).
2.1 Definitions

We begin with some general remarks concerning semi-graphs (cf. [M]). A semi-graph $G$ consists of the following data: (i) A set $V_G$ whose elements we refer to as vertices; (ii) A set $E_G$ whose elements we refer to as edges. Any element $e \in E_G$ is a set of cardinality 2 satisfying the following property: For any $e \neq e' \in E_G$, we have $e \cap e' = \emptyset$; (iii) A set of maps $\{G^e\}_{e \in E_G}$ such that $\zeta_e : e \to V \cup \{V\}$ is a map from the set $e$ to the set $V \cup \{V\}$. For an edge $e \in E_G$, we shall refer to an element $b \in e$ as a branch of the edge $e$. An edge $e \in E_G$ is called closed (resp. open) if $\zeta_e^{-1}\{\{V\}\} = \emptyset$ (resp. $\zeta_e^{-1}(\{V\}) \neq 0$). A semi-graph will be called finite if both its set of vertices and its set of edges are finite. In the present paper, we only consider finite semi-graphs. Since a semi-graph can be regarded as a topological space, we shall call $G$ a connected semi-graph if $G$ is connected as a topological space.

Let $G$ be a semi-graph. Write $v(G)$ for the set of vertices of $G$, $e(G)$ for the set of closed edges of $G$, and $e'(G)$ for the set of open edges of $G$. For any element $v \in v(G)$, write $b(v)$ for the set of branches $\cup_{e \in e(G), e' \in e'(G)} \zeta_e^{-1}(v)$. For any element $e \in e(G) \cup e'(G)$, write $v(e)$ for the set which consists of the elements of $v(G)$ which are abutted by $e$. A morphism between semi-graphs $G \to \mathbb{H}$ is a collection of maps $v(G) \to v(\mathbb{H})$; $e(G) \cup e'(G) \to e(\mathbb{H}) \cup e'(\mathbb{H})$; and for each $e_G \in e(G) \cup e'(G)$ mapping to $e_H \in e(\mathbb{H}) \cup e'(\mathbb{H})$, a bijection $e_G \to e_H$; all of which are compatible with the $\{G^e\}_{e \in e(G) \cup e'(G)}$ and $\{G^e\}_{e \in e(G) \cup e'(G)}$.

A sub-semi-graph $G'$ of $G$ is a semi-graph satisfying the following properties: (i) $v(G')$ (resp. $e(G') \cup e'(G')$) is a subset of $v(G)$ (resp. $e(G) \cup e'(G)$); (ii) If $e \in e(G')$, then we have $\zeta_e^{G'}(e) = \zeta_e^G(e)$; (iii) If $e = \{b_1, b_2\}$ is an element of $e'(G')$ such that $\zeta_e^{G'}(b_1) \in v(G')$ and $\zeta_e^{G'}(b_2) \not\in v(G')$, then we have $\zeta_e^{G'}(b_1) = \zeta_e^G(b_1)$ and $\zeta_e^{G'}(b_2) = \{v(G')\}$.

Definition 2.1. Let $G'$ be a sub-semi-graph of a semi-graph $G$. We define a semi-graph $G \setminus G'$ as follows: (i) The set of vertices $v(G \setminus G')$ is $v(G) \setminus v(G')$; (ii) The set of closed edges $e(G \setminus G')$ is $e(G) \setminus e(G')$; (iii) The set of open edges $e'(G \setminus G')$ is $\{e \in e(G) \mid v(e) \cap v(G') \neq \emptyset \text{ in } G\}$; (iv) For any $e = \{b_1, \ldots, b_k\} \subset e(G \setminus G') \cup e'(G \setminus G')$, we have $\zeta_e^{G \setminus G'}(b_1) = \zeta_e^G(b_1)$ (resp. $\zeta_e^{G \setminus G'}(b_1) = \{v(G \setminus G')\}$) if $\zeta_e^G(b_1) \not\in v(G')$ (resp. $\zeta_e^G(b_1) \in v(G')$).

Definition 2.2. (a) Let $n$ be a positive natural number and $\mathbb{P}_n$ a semi-graph such that the following conditions hold: (i) $v(\mathbb{P}_n) = \{p_1, \ldots, p_n\}$, $e(\mathbb{P}_n) = \{e_{1,2}, \ldots, e_{n-1}\}$ and $e'(\mathbb{P}_n) = \{e_{0,1}, e_{n,n+1}\}$; (ii) $v(e_{i,i+1}) = \{p_i, p_{i+1}\}$; (iii) $v(e_{0,1}) = \{p_1\}$ and $v(e_{n,n+1}) = \{p_n\}$. We define $\mathcal{G}$ to be a triple $(G, \sigma_\mathcal{G}, \beta_\mathcal{G})$ which consists of a semi-graph $G$, a map $\sigma_\mathcal{G} : v(G) \to \mathbb{Z}$ and a morphism of semi-graphs $\beta_\mathcal{G} : G \to \mathbb{P}_n$. We shall call $\mathcal{G}$ a n-semigraph with p-rank. We shall refer to $G$ as the underlying semi-graph of $\mathcal{G}$, $\sigma_\mathcal{G}$ as the p-rank map of $\mathcal{G}$, $\beta_\mathcal{G}$ as the base morphism of $\mathcal{G}$, respectively. We define $\mathbb{P}_n = (\mathbb{P}_n, \sigma_\mathbb{P}_n, \beta_\mathbb{P}_n)$ as follows: $\sigma_\mathbb{P}_n(p_i)$ is equal to 0 for each $i = 1, \ldots, n$, and $\beta_\mathbb{P}_n = \text{id}_{\mathbb{P}_n}$ is an identity morphism of semi-graph $\mathbb{P}_n$. We shall call $\mathbb{P}_n$ a n-chain.

(b) We define the p-rank $\sigma(\mathcal{G})$ of $\mathcal{G}$ as follows:

$$\sigma(\mathcal{G}) := \sum_{v \in v(G)} \sigma(v) + \sum_{G_i \in \pi_0(\mathcal{G})} \text{rank}_\mathbb{Z} H^1(G_i, \mathbb{Z}),$$

where $\pi_0(-)$ denotes the set of connected components of $(-)$.

(c) $\mathcal{G}$ is called connected if the underlying semi-graph $G$ is a connected semi-graph.
From now on, we only consider connected $n$-semi-graphs with $p$-rank. Let $\mathcal{G}^1 := (G^1, \sigma_{G^1}, \beta_{G^1})$ and $\mathcal{G}^2 := (G^2, \sigma_{G^2}, \beta_{G^2})$ be two $n$-semi-graphs with $p$-rank. A morphism between $\mathcal{G}^1$ and $\mathcal{G}^2$ is defined by a morphism of the underlying semi-graphs $\beta : G^1 \rightarrow G^2$ such that $\beta_{G^2} \circ \beta = \beta_{G^1}$. We use the notation $b : \mathcal{G}^1 \rightarrow \mathcal{G}^2$ to denote the morphism of semi-graphs with $p$-rank determined by $\beta : G^1 \rightarrow G^2$ and call $\beta$ the underlying morphism of $b$. Note that for any $n$-semi-graph with $p$-rank $\mathcal{G} := (G, \sigma_\mathcal{G}, \beta_\mathcal{G})$, there is a natural morphism $b_\mathcal{G} : \mathcal{G} \rightarrow \mathcal{P}_n$ determined by the morphism of underlying semi-graphs $\beta_\mathcal{G} : G \rightarrow \mathbb{P}_n$.

Write $b_i^1$ (resp. $b_i^2$) for $\zeta_{i-1,i_1}^1(p_i)$ (resp. $\zeta_{i-1,i_2}^2(p_i)$). For any element $v_i \in \beta_{\mathcal{G}^1}^{-1}(p_i)$, write $b_{l}(v_i)$ (resp. $b_{r}(v_i)$) for the set

$$\{b \in b(v_i) \mid \beta_\mathcal{G}(b) = b_i^1\}$$

(resp. $\{b \in b(v_i) \mid \beta_\mathcal{G}(b) = b_i^2\}$).

**Definition 2.3.** Let $b : \mathcal{G}^1 := (G^1, \sigma_{G^1}, \beta_{G^1}) \rightarrow \mathcal{G}^2 := (G^2, \sigma_{G^2}, \beta_{G^2})$ be a morphism of $n$-semi-graphs with $p$-rank, $\beta$ the underlying morphism of $b$, $e \in e(G^1) \cup e'(G^1)$ an edge, $v_1$ a vertex of $G^1$ contained in $\beta_{\mathcal{G}^1}^{-1}(p_i)$, and $v_2 := \beta(v_1) \in \beta_{\mathcal{G}^2}^{-1}(p_i)$ the image of $v_1$.

1. We shall call $b$ $p$-étale (resp. $p$-purely inseparable) at $e$ if $\sharp \beta^{-1}(\beta(e)) = p$ (resp. $\sharp \beta^{-1}(\beta(e)) = 1$). We shall call $b$ $p$-generically étale at $v_1 \in \beta_{\mathcal{G}^1}^{-1}(p_i)$ if one of the following étale types holds:
   - **(Type-I)** $\sharp \beta^{-1}(v_2) = p$ and $\sigma_{\mathcal{G}^1}(v_1) = \sigma_{\mathcal{G}^2}(v_2)$;
   - **(Type-II)** $\sharp \beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = p \sharp b_l(v_2)$, $\sharp b_r(v_1) = p \sharp b_r(v_2)$, and $\sigma_{\mathcal{G}^1}(v_1) - 1 = p(\sigma_{\mathcal{G}^2}(v_2) - 1)$;
   - **(Type-III)** If $\sharp \beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = \sharp b_l(v_2)$, $\sharp b_r(v_1) = p \sharp b_r(v_2)$, and $\sigma_{\mathcal{G}^1}(v_1) - 1 = p(\sigma_{\mathcal{G}^2}(v_2) - 1) + (\sharp b_l(v_1))(p - 1)$;
   - **(Type-IV)** $\sharp \beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = p \sharp b_l(v_2)$, $\sharp b_r(v_1) = \sharp b_r(v_2)$, and $\sigma_{\mathcal{G}^1}(v_1) - 1 = p(\sigma_{\mathcal{G}^2}(v_2) - 1) + (\sharp b_r(v_1))(p - 1)$;
   - **(Type-V)** $\sharp \beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = \sharp b_l(v_2)$, $\sharp b_r(v_1) = \sharp b_r(v_2)$, and $\sigma_{\mathcal{G}^1}(v_1) - 1 = p(\sigma_{\mathcal{G}^2}(v_2) - 1) + (\sharp b_l(v_1) + \sharp b_r(v_1))(p - 1)$.

2. We shall call $b$ purely inseparable at $v_1 \in \beta_{\mathcal{G}^1}^{-1}(p_i)$ if $\sharp \beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = \sharp b_l(v_2)$, $\sharp b_r(v_1) = \sharp b_r(v_2)$, and $\sigma_{\mathcal{G}^1}(v_1) = \sigma_{\mathcal{G}^2}(v_2)$ hold.

3. We shall call $b$ a $p$-covering if the following conditions hold: (i) There exists a $\mathbb{Z}/p\mathbb{Z}$-action (which may be trivial) on $G^1$ (resp. a trivial $\mathbb{Z}/p\mathbb{Z}$-action on $G^2$), and the underlying morphism $\beta$ of $b$ is compatible with the $\mathbb{Z}/p\mathbb{Z}$-actions. Then the natural morphism $G^1/\mathbb{Z}/p\mathbb{Z} \rightarrow G^2$ induced by $b$ is an isomorphism; (ii) For any $v \in v(G^1)$, $b$ is either $p$-generically étale or purely inseparable at $v$; (iii) Let $e \in e(G^1)$ and $v(e) = \{v, v'\}$. If $b$ is $p$-generically étale at $v$ and $v'$, then $b$ is $p$-étale at $e$; (iv) For any $v \in v(G^1)$, then $\sigma_{\mathcal{G}^1}(v) = \sigma_{\mathcal{G}^1}(\tau(v))$ holds for each $\tau \in \mathbb{Z}/p\mathbb{Z}$. 

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Note that by the definition of $p$-covering, the identity morphism of a semi-graph with $p$-rank is a $p$-covering.

(d) We shall call $b$ a covering if $b$ is a composite of $p$-coverings.

(e) We shall call

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

an maximal filtration of $G$ if $G_j/G_{j+1} \cong \mathbb{Z}/p\mathbb{Z}$ for each $j = 1, \ldots, r - 1$. Suppose that $G^1$ (resp. $G^2$) admits a (resp. trivial) $G$-action (which may be trivial). Then for any maximal filtration $\Phi$ of $G$, there is a sequence of semi-graphs induced by $\Phi$:

$$G^1 = G_r \xrightarrow{\beta_r} G_{r-1} \xrightarrow{\beta_{r-1}} \cdots \xrightarrow{\beta_1} G_0,$$

where $G_j$ denotes the quotient of $G^1$ by $G_j$. We shall call $b$ a $G$-covering if for any maximal filtration $\Phi$ of $G$, there exists a set of $p$-coverings $\{b_j : G_j \twoheadrightarrow G_{j-1}, j = 1, \ldots, r\}$ such that the following conditions hold: (i) the underlying morphism $\beta$ of $b$ is compatible with the $G$-actions, and the natural morphism $G^1/G \twoheadrightarrow G^2$ induced by $\beta$ is an isomorphism; (ii) The underlying graph of $G_j$ is equal to $G_j$ for each $j = 0, \ldots, r$; (iii) The underlying morphism $G_j \twoheadrightarrow G_{j-1}$ of $b_j$ is equal to $\beta_j$ for each $j = 1, \ldots, r$; (iv) The composite morphism $b_1 \circ \cdots \circ b_r$ is equal to $b$. Then we obtain a sequence of $p$-coverings:

$$\Phi_{G} : G^1 = G_r \xrightarrow{b_r} G_{r-1} \xrightarrow{b_{r-1}} \cdots \xrightarrow{b_1} G_0 = G^2.$$

We shall call $\Phi_{G}$ a sequence of $p$-coverings induced by $\Phi$.

(f) Let $G$ be a $n$-semi-graph with $p$-rank. We shall call $G$ a covering (resp. $G$-covering) over $G$, if $b$ is a covering (resp. $G$-covering).

(g) Let $b : G^1 \twoheadrightarrow G^2$ be a $G$-covering, $v \in v(G)$ a vertex, and $e \in e(G) \cup e'(G)$ an edge. For any subgroup $H \subseteq G$, by Definition 2.3 (e), there exists a maximal filtration $\Phi^H$ and the sequence of $p$-coverings

$$\Phi^H_{G} : G^1 = G_r \xrightarrow{b^H_r} G_{r-1} \xrightarrow{b^H_{r-1}} \cdots \xrightarrow{b^H_1} G_0 = G^2$$

induced by $\Phi^H$ such that there exists $i$ such that the underlying graph of $G_i$ is isomorphic to $G^1/H$. We write $G^1/H$ for $G_i$. Thus, the natural morphism $b^H_1 \circ \cdots \circ b^H_r : G^1/H \twoheadrightarrow G^2$ is a covering. Then we define five subgroups of $G$ as follows:

$$D_v := \{\tau \in G \mid \tau(v) = v\},$$

$$I_v := \text{the maximal element of } \{H \subseteq G \mid G^1/H \text{ is purely inseparable at } v\},$$

$$I^l_v(b) := \{\tau \in D_v \mid \tau(b) = b \text{ for a branch } b \in b_l(v)\}/I_v,$$

$$I^l_v(b) := \{\tau \in D_v \mid \tau(b) = b \text{ for a branch } b \in b_r(v)\}/I_v,$$

$$I_e := \{\tau \in G \mid \tau(e) = e\}.$$
does not depend on the choice of \( b \in b_i(v) \) (resp. \( b \in b_r(v) \)), then we denote this group briefly by \( I_v^i \) (resp. \( I_v^r \)). Define

\[
D_v^e = D_v/(I_v^i / (I_v^i \cap I_v^r) \oplus I_v^r / (I_v^r \cap I_v^i) \oplus I_v^i \cap I_v^r \oplus I_v).
\]

Then we have the following exact sequence

\[
0 \longrightarrow I_v^i / (I_v^i \cap I_v^r) \oplus I_v^r / (I_v^r \cap I_v^i) \oplus I_v^i \cap I_v^r \oplus I_v \longrightarrow D_v \longrightarrow D_v^e \longrightarrow 0.
\]

**Remark 2.3.1.** Let \( \mathfrak{G} \) be a \( G \)-covering over \( \mathfrak{P}_n \) and \( v_i \in \beta_{\mathfrak{G}}^{-1}(p_i) \) a vertex of the underlying graph of \( \mathfrak{G} \). Then we have the following Deuring-Shafarevich type formula (cf. Proposition 3.2 for the Deuring-Shafarevich formula for curves)

\[
\sigma_{\mathfrak{G}}(v_i) - 1 = -\sharp D_{v_i}/I_{v_i} + \sharp((D_{v_i}/I_{v_i})/I_{v_i}^i)(\sharp I_{v_i}^r - 1) + \sharp((D_{v_i}/I_{v_i})/I_{v_i}^r)(\sharp I_{v_i}^i - 1).
\]

Let \( \mathfrak{G} \) be a \( G \)-covering over \( \mathfrak{P}_n \). By the definition of \( G \)-coverings, for any maximal filtration \( \Phi \) of \( G \), we have a sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank

\[
\Phi_{\mathfrak{G}} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{b_r} \mathfrak{G}_{r-1} \xrightarrow{b_{r-1}} \ldots \xrightarrow{b_1} \mathfrak{G}_0 = \mathfrak{P}_n
\]

induced by \( \Phi \). For each \( j = 1, \ldots, r \), we write \( \mathcal{V}_{j}^{\text{et}} \) for the set

\[
\{ v \in v(\mathfrak{G}_j) \mid b_j \text{ is étale at } v \},
\]

\( \mathcal{E}_{j}^{\text{et}} \) for the set

\[
\{ e \in e(\mathfrak{G}_j) \cup e'(\mathfrak{G}_j) \mid b_j \text{ is étale at } e \}.
\]

Since \( (\mathcal{V}_{j}^{\text{et}}, \mathcal{E}_{j}^{\text{et}}) \) admits a natural structure of semi-graph induced by \( \mathfrak{G}_j \), we may regard \( (\mathcal{V}_{j}^{\text{et}}, \mathcal{E}_{j}^{\text{et}}) \) as a sub-semi-graph of \( \mathfrak{G}_j \). Thus, the image \( \beta_{\Phi_j}((\mathcal{V}_{j}^{\text{et}}, \mathcal{E}_{j}^{\text{et}})) \) can be regarded as a sub-semi-graph of \( \mathfrak{P}_n \).

**Definition 2.4.** We shall call \( \mathbb{E}_{j}^{\Phi} := \beta_{\Phi_j}((\mathcal{V}_{j}^{\text{et}}, \mathcal{E}_{j}^{\text{et}})) \) (resp. the disjoint union \( \mathbb{E}^{\Phi} := \bigsqcup_{j} \mathbb{E}_{j}^{\Phi} \)) the \( j \)-th étale-chain (resp. the étale-chain) associated to \( \Phi_{\mathfrak{G}} \).

### 2.2 \( p \)-ranks and étale-chains of abelian coverings

Let \( \mathfrak{G} := (\mathfrak{G}, \sigma_{\mathfrak{G}}, \beta_{\mathfrak{G}}) \) be a \( G \)-covering over \( \mathfrak{P}_n \). We introduce two operators for \( \mathfrak{G} \).

**Operator I:** First, let us define a \( G \)-covering \( \mathfrak{G}^*[p_i] \) over \( \mathfrak{P}_n \). For any \( p_i \in v(\mathfrak{P}_n) \), let \( v_i \) be an element of \( \beta_{\mathfrak{G}}^{-1}(p_i) \).

If \( \sharp \beta_{\mathfrak{G}}^{-1}(p_i) = 1 \) (i.e., \( D_{v_i} = G \)), then we define \( \mathfrak{G}^*[p_i] \) to be \( \mathfrak{G} \); If \( \sharp \beta_{\mathfrak{G}}^{-1}(p_i) \neq 1 \), we define a new semi-graph \( \mathfrak{G}^*[p_i] \) as follows.

Define \( v(\mathfrak{G}^*[p_i]) \) (resp. \( e(\mathfrak{G}^*[p_i])\cup e'(\mathfrak{G}^*[p_i]) \)) to be the disjoint union \( (v(G)\setminus\beta_{\mathfrak{G}}^{-1}(p_i))\bigsqcup\{v^*\} \) (resp. \( e(G)\cup e'(G) \)).

The collection of maps \( \{\zeta_e^{\mathfrak{G}^*[p_i]}\}_e \) is as follows: (i) For any branch \( b \notin \cup_{v \in \beta_{\mathfrak{G}}^{-1}(p_i)} b(v) \), \( \zeta_e^{\mathfrak{G}^*[p_i]}(b) = \zeta_e^{\mathfrak{G}}(b) \) if \( b \in e \) and \( \zeta_e^{\mathfrak{G}^*[p_i]}(b) = \emptyset \) if \( b \notin e \); (ii) For any \( v \in \beta_{\mathfrak{G}}^{-1}(p_i) \) and any branch \( b \in b(v) \), \( \zeta_e^{\mathfrak{G}^*[p_i]}(b) = v^* \) if \( b \in e \) and \( \zeta_e^{\mathfrak{G}^*[p_i]}(b) = \emptyset \) if \( b \notin e \).
We define a map \( \sigma_{\ast}[p_i] : v(G^*[p_i]) \rightarrow \mathbb{Z} \) as follows: (i) If \( v^* \neq v \in v(G^*[p_i]) \), then we have \( \sigma_{\ast}[p_i](v) := \sigma(v) \); (ii) If \( v = v^* \), then we have

\[
\sigma_{\ast}[p_i](v^*) := -\sharp(G/I_{v}) + \sum_{v \in \beta_{\phi}^{-1}(p_i)} \sum_{b \in b_i(v)} (\sharp I^b_v - 1) + \sum_{v \in \beta_{\phi}^{-1}(p_i)} \sum_{b \in b_i(v)} (\sharp I^b_v - 1) + 1
\]

Thus, the triple \( \mathfrak{G}^*[p_i] := (G^*[p_i], \sigma_{\ast}[p_i], \beta_{\phi}[p_i]) \) is a \( n \)-semi-graph with \( p \)-rank.

Moreover, \( G^*[p_i] \) admits a natural \( G \)-action as follows: (i) The action of \( G \) on \( v(G^*[p_i]) \setminus \{v^*\} \) (resp. \( e(G^*[p_i]) \cup e'(G^*[p_i]) \)) is the action of \( G \) on \( v(G) \setminus \beta_{\phi}^{-1}(p_i) \) (resp. \( e(G) \cup e'(G) \)); (ii) For any \( \tau \in G \), we have \( \tau(v^*) = v^* \).

Let us explain that with the \( G \)-action defined above, \( \mathfrak{G}^*[p_i] \) is a \( G \)-covering over \( \mathfrak{P}_n \). Let

\[ \Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G \]

be an arbitrary maximal filtration of \( G \). Write

\[ \Phi_\theta : \mathfrak{G} = G_r \xrightarrow{b_r} G_{r-1} \xrightarrow{b_{r-1}} \cdots \xrightarrow{b_1} \mathfrak{G}_0 = \mathfrak{P}_n \]

for the sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank induced by \( \Phi \). Note that for each \( j = 0, \ldots, r \), \( \mathfrak{G}_j \) is a \( G/J_{G_j} \)-covering over \( \mathfrak{P}_n \). By the construction of \( \mathfrak{G}_j^*[p_i] \), we have

\[ \Phi_{\ast}^*[p_i] : \mathfrak{G}_j^*[p_i] = (\mathfrak{G}_j^*[p_i]_1 \xrightarrow{b_j[p_i]} \mathfrak{G}_{j-1}^*[p_i] \xrightarrow{b_{j-1}[p_i]} \cdots \xrightarrow{b_1[p_i]} \mathfrak{P}_n) \]

is a sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank. Thus, \( \mathfrak{G}^*[p_i] \) can be regarded as a \( G \)-covering over \( \mathfrak{P}_n \).

Note that by the construction of \( \mathfrak{G}^*[p_i] \), we see that \( E_j^\Phi = E_j^{\Phi_{\ast}^*[p_i]} \) for each \( j = 1, \ldots, r \).

**Operator II:** Let us define a \( G \)-covering \( \mathfrak{G}^*[p_i] \) over \( \mathfrak{P}_n \). For any \( p_i \in v(\mathfrak{P}_n) \), let \( v_i \) be an element of \( \beta_{\phi}^{-1}(p_i) \). Since \( G \) is a abelian group, we may write \( \{v_i^u\}_{u \in G/D_{v_i}} \) for \( \beta_{\phi}^{-1}(p_i) \), and \( \{v_i^u\}_{u \in G/D_{v_i}} \) admits an natural action of \( G \) on the index set \( G/D_{v_i} \). We define a new semi-graph \( G^*[p_i] \) as follows. If \( \beta_{\phi}^{-1}(p_i) \neq \emptyset \) (resp. \( e(G^*[p_i]) \cup e'(G^*[p_i]) \)), then \( G^*[p_i] \) to be \( G \). If \( \beta_{\phi}^{-1}(p_i) = \emptyset \), then we have \( \beta_{\phi}^{-1}(p_i)(e) = \beta_{\phi}(e) \).

Then \( \beta_{\phi}^{-1}(b_i) = \{b_i^u \}_{u \in G/D_{v_i}, s \in I_{v_i}^u \cap I_{v_i}^u \cap I_{v_i} \cap D_{v_i}} \) admits a natural action of \( G \) as follows:

\[
\tau(b_i^u, s) = b_i^u, \tau(s) \quad \text{if } \tau \in G/D_{v_i} \]

and

\[
\tau(b_i^u, s) = b_i^u, \tau(s) \quad \text{if } \tau \notin G/D_{v_i}. 
\]

Define \( v(G^*[p_i]) \) (resp. \( e(G^*[p_i]) \cup e'(G^*[p_i]) \)) to be the disjoint union \( v(G) \setminus \beta_{\phi}^{-1}(p_i) \)

\[ \prod \{v_i^u \}_{u \in G/D_{v_i}, s \in D_{v_i}} \]

(resp. \( e(G) \cup e'(G) \)). \{v_i^u \}_{u \in G/D_{v_i}, s \in D_{v_i}} \) admits a natural \( G \)-action.
as follows: For each \( \tau \in G \), \( \tau(v^*_{u,t}) = v^*_{\tau u,t} \) if \( \tau \not\in D_{vi} \), \( \tau(v^*_{u,t}) = v^*_{u,\tau t} \) if \( \tau \in D_{vi} \), and \( \tau(v^*_{u,t}) = v^*_{u,t} \) if \( \tau \in I_{vi}^l + I_{vi}^r + I_{vi} \).

The collection of maps \( \{ \zeta_{vi}^{G^*[p_1]} \}_e \) is as follows: (i) For any branch \( b \not\in \cup_{v \neq v_i} b(v) \), \( \zeta_{vi}^{G^*[p_1]}(b) = \zeta_i^{G}(b) \) if \( b \in e \) and \( \zeta_{vi}^{G^*[p_1]}(b) = \emptyset \) if \( b \not\in e \); (ii) \( \zeta_{vi}^{G^*[p_1]}(b) = v^*_{u,t} \) if \( b = b_i^{u,s,t} \in e \) (resp. \( \zeta_{vi}^{G^*[p_1]}(b) = 0 \) if \( b \not\in e \)).

We define a map \( \sigma_{G^*[p_1]} : v(G^*[p_1]) \to \mathbb{Z} \) as follows: If \( v^*_{u,t} \neq v \in v(G^*[p_1]) \), then we have \( \sigma_{G^*[p_1]}(v) = \sigma_{G}(v) \); If \( v = v^*_{u,t} \) then we have

\[
\sigma_{G^*[p_1]}(v) = -z(I_{vi}^l + I_{vi}^r) + \sharp((I_{vi}^l + I_{vi}^r)/I_{vi}^l)(\#I_{vi}^l - 1) + \sharp((I_{vi}^l + I_{vi}^r)/I_{vi}^l)(\#I_{vi}^l - 1) + 1.
\]

We define a morphism of semi-graphs \( \beta_{G^*[p_1]} : G^*[p_1] \to \mathbb{P}_n \) as follows: (i) For any \( v \in v(G^*[p_1]) \), then \( \beta_{G^*[p_1]}(v) = p_i \) if \( v \in \{ v_{u,t} \}_{u \in G/D_{vi}, t \in D_{vi}^e} \) and \( \beta_{G^*[p_1]}(v) = \beta_{G}(v) \) if \( v \not\in \{ v_{u,t} \}_{u \in G/D_{vi}, t \in D_{vi}^e} \); (ii) If \( e \in e(G^*[p_1]) \cup e'(G^*[p_1]) \), then we have \( \beta_{G^*[p_1]}(e) = \beta_{G}(e) \).

Thus, the triple \( G^*[p_1] := (G^*[p_1], \sigma_{G^*[p_1]}, \beta_{G^*[p_1]}) \) is a n-semi-graph with p-rank.

Moreover, \( G \) admits a natural G-action as follows: (i) the action of \( G \) on \( v(G^*[p_1]) \setminus \{ v_{u,t} \}_{u \in G/D_{vi}, t \in D_{vi}^e} \) (resp. \( e(G^*[p_1]) \cup e'(G^*[p_1]) \)) is the action of \( G \) on \( v(G) \setminus \beta^{-1}(p_1) \) (resp. \( e(G) \cup e'(G) \)); (ii) The action of \( G \) on \( \{ v_{u,t} \}_{u \in G/D_{vi}, t \in D_{vi}^e} \) is the action defined above.

Let us explain that with the G-action defined above, \( G^*[p_1] \) is a G-covering over \( \mathbb{P}_n \).

Let \( \Phi : \{ 1 \} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G \) be an arbitrary maximal filtration of \( G \). Write

\[
\Phi_{G} : G = G_r \xrightarrow{b_r} G_{r-1} \xrightarrow{b_{r-1}} \cdots \xrightarrow{b_1} G_0 = \mathbb{P}_n
\]

for the sequence of p-coverings of n-semi-graphs with p-rank induced by \( \Phi \). Note that for each \( j = 0, \ldots, r \), \( G_j \) is a G/G_j-covering over \( \mathbb{P}_n \). By the construction of \( G_j^*[p_1] \), we have

\[
\Phi_{G^*[p_1]} : G^*[p_1] = G^*[p_1] \xrightarrow{b_r^*[p_1]} G^*[p_1]_{r-1} \xrightarrow{b_{r-1}^*[p_1]} \cdots \xrightarrow{b_1^*[p_1]} \mathbb{P}_n
\]

is a sequence of p-coverings of n-semi-graphs with p-rank. Thus, \( G^*[p_1] \) can be regarded as a G-covering over \( \mathbb{P}_n \).

Note that by the construction of \( G^*[p_1] \), we see that \( E_j^{\Phi_{G^*[p_1]}} = E_j^{G^*[p_1]} \) for each \( j = 1, \ldots, r \).

**Definition 2.5.** Let \( G := (G, \sigma_G, \beta_G) \) be a G-covering over \( \mathbb{P}_n, p_i \) a vertex of \( v(\mathbb{P}_n) \). We define an operator \( \Rightarrow_{II} \) (resp. \( \Rightarrow_{I} \)) from a G-covering to a G-covering to be

\[
\Rightarrow_{II} (p_i)(G) = G^*[p_1]
\]

(resp. \( \Rightarrow_{I} (p_i)(G) := G^*[p_1] \)).

**Lemma 2.6.** Let \( G \) be a G-covering over \( \mathbb{P}_n \) and \( G \) the underlying semi-graph of \( G \). Let \( G^c \) be a semi-graph defined as follows: (i) \( v(G^c) = v(G) \cup \{ v_0, v_{n+1} \} \); (ii) \( e(G^c) = e(G) \cup e'(G) \) and \( e'(G') = 0 \); (iii) \( e^c_{G'} = e^c_{G} \) if \( \beta_G(e) \not\in (e_0, 1, e_{n+1}) \); (iv) \( e = \{ b', b' \} \) such that the image \( \beta_G(e) = e_0 + 1 \) and \( e^c_G(b') = v(G) \) (resp. the image \( \beta_G(e) = e_{n+1} + 1 \) and \( e^c_G(b') = v(G) \), we have \( e^c_G(b') = v_0 \) (resp. \( e^c_G(b') = v_{n+1} \)). Let \( I_{e_0} \) (resp. \( I_{e_{n+1}} \)) be the inertia group of an element of \( \beta_G^{-1}(e_0) \) (resp. \( \beta_G^{-1}(e_{n+1}) \)). Note that since \( G \) is...
an abelian group, $I_{e_0,1}$ (resp. $I_{e_{n+1}}$) does not depend on the choice of the elements of $\beta^{-1}_\varnothing(e_{0,1})$ (resp. $\beta^{-1}_\varnothing(e_{n,n+1})$). Then we have

$$\text{rank}_\mathbb{Z} \text{H}^1(G^c, \mathbb{Z}) - \text{rank}_\mathbb{Z} \text{H}^1(G, \mathbb{Z}) = \sharp G/I_{e_0,1} - 1 + \sharp G/I_{e_{n+1}} - 1.$$

**Proof.** The lemma follows from the construction of $G^c$ immediately. □

**Proposition 2.7.** Let $\mathfrak{G} := (G, \sigma, \beta)$ be a $G$-covering over $\mathfrak{P}_n$ and $p$, a vertex of $v(\mathfrak{P}_n)$. Then we have $\sigma(\mathfrak{G}) = \sigma(\mathfrak{G}^*[p])$ and $\sigma(\mathfrak{G}) = \sigma(\mathfrak{G}^*[p])$.

**Proof.** Let $v_1 \in \beta^{-1}_\varnothing(p)$. If $\sharp \beta^{-1}_\varnothing(p) = 1$ (resp. $\sharp \beta^{-1}_\varnothing(p) = \sharp G/I_{e_0,1}$), by the definition of Operator I (resp. Operator II), the proposition is trivial. Then we may assume that $\sharp \beta^{-1}_\varnothing(p) \neq 1$ (resp. $\sharp \beta^{-1}_\varnothing(p) \neq \sharp G/I_{e_0,1}$). Write $I_{e_0,1}$ (resp. $I_{e_{n+1}}$) for the inertia group of an element of $\beta^{-1}_\varnothing(e_{0,1})$ (resp. $\beta^{-1}_\varnothing(e_{n,n+1})$).

First, we will prove the proposition under the assumption that $I_{e_0,1} = I_{e_{n+1}} = G$ holds. Write $(-)$ for the rank of a semi-graph $(-)$ (i.e., the rank of $\text{H}^1((-), \mathbb{Z})$ as a free $\mathbb{Z}$-module). Thus, we have

$$\sigma(\mathfrak{G}) = \sum_{v \in \beta^{-1}_\varnothing(p)} \sigma(v) + \sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma(v) + \text{r}(G \setminus \beta^{-1}_\varnothing(p)),

\sigma(\mathfrak{G}^*[p]) = \sigma^*[p](v^*), + \sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma(v) + \text{r}(G \setminus \beta^{-1}_\varnothing(p))

$$

and

$$\sigma(\mathfrak{G}^*[p]) = \sum_{v \in \beta^{-1}_\varnothing(p)} \sigma^*[p](v) + \sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma(v) + \text{r}(G \setminus \beta^{-1}_\varnothing(p))

\sigma(\mathfrak{G}^*[p]) = \sum_{v \in \beta^{-1}_\varnothing(p)} \sigma^*[p](v) + \sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma(v) + \text{r}(G \setminus \beta^{-1}_\varnothing(p))

Note that we have $\text{r}(G \setminus \beta^{-1}_\varnothing(p)) = \text{r}(G^*[p] \setminus \beta^{-1}_\varnothing(p)) = \text{r}(G^*[p] \setminus \beta^{-1}_\varnothing(p))$ and $\sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma(v) = \sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma^*[p](v) = \sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma^*[p](v).

First, let us prove $\sigma(\mathfrak{G}) = \sigma(\mathfrak{G}^*[p])$. We follow the notations of Operator I. We have

$$\sigma(\mathfrak{G}) = \sum_{v \in \beta^{-1}_\varnothing(p)} \sigma(v) + \sum_{v \in (G \setminus \beta^{-1}_\varnothing(p))} \sigma(v) + \text{r}(G \setminus \beta^{-1}_\varnothing(p))

+ \sharp G/D_{v_1}(\sharp ((D_{v_1}/I_{v_1})/I_{v_1}^1) - 1 + \sharp ((D_{v_1}/I_{v_1})/I_{v_1}^1) - 1) + \sharp G/D_{v_1} - 1

= \sharp G/D_{v_1}(-\sharp D_{v_1}/I_{v_1} + \sharp ((D_{v_1}/I_{v_1})/I_{v_1}^1)(\sharp I_{v_1}^1 - 1) + \sharp ((D_{v_1}/I_{v_1})/I_{v_1}^1)(\sharp I_{v_1}^1 - 1) + 1)$$

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Thus, \( \sigma(\mathcal{G}) = \sigma(\mathcal{G}^{*}[p_i]) \) holds.

Suppose that either \( I_{e_{0,1}} \) or \( I_{e_{n,n+1}} \) is not equal to \( G \). By Lemma 2.6, we have

\[
\sigma(\mathcal{G}) = \sum_{v \in \mathcal{G}(p_i)} \sigma(\mathcal{G}) + \sum_{v \in \mathcal{G}(\beta^{-1}(p_i))} \sigma(\mathcal{G}) + r(\mathcal{G} \setminus \beta^{-1}(p_i))
\]

\[
+ \tau G/D_{v_i}\left(\tau((D_{v_i}/I_{v_i})/I^1_{v_i}) - 1 + \tau((D_{v_i}/I_{v_i})/I^r_{v_i}) - 1\right) + \tau G/D_{v_i} - 1
\]

\[
= \tau G/I_{v_i} - 1 + \sum_{v \in \mathcal{G}(\beta^{-1}(p_i))} \sigma(\mathcal{G}) + r(\mathcal{G} \setminus \beta^{-1}(p_i)).
\]

On the other hand, we have

\[
\sigma(\mathcal{G}^{*}[p_i]) = \sum_{v \in \mathcal{G}(\beta^{-1}(p_i))} \sigma(\mathcal{G}^{*}[p_i])(v^*) + \tau((G/I_{v_i})/I^1_{v_i}) - 1 + \tau((G/I_{v_i})/I^r_{v_i}) - 1
\]

\[
+ \tau((G/I_{v_i})/I^1_{v_i}) - 1 + \tau((G/I_{v_i})/I^r_{v_i}) - 1 + \tau G/D_{v_i} - 1
\]

\[
= \tau G/I_{v_i} - 1 + \sum_{v \in \mathcal{G}(\beta^{-1}(p_i))} \sigma(\mathcal{G}^{*}[p_i])(v^*) + r(\mathcal{G} \setminus \beta^{-1}(p_i)).
\]
We have

Thus, we have

Thus, we have

Thus, \( \sigma(\mathfrak{S}) = \sigma(\mathfrak{S}^*[p_i]) \) holds.

Next, let us compute \( \sigma(\mathfrak{S}^*[p_i]) \). First, suppose that \( I_{e_{0,1}} = I_{e_{n,1} + 1} = G \) holds. Write \( W \) for the group

We have

Thus, we have \( \sigma(\mathfrak{S}) = \sigma(\mathfrak{S}^*[p_i]) \).

Suppose that either \( I_{e_{0,1}} \) or \( I_{e_{n,1} + 1} \) is not equal to \( G \). By Lemma 2.6, we have

\[
\sigma(\mathfrak{S}^*[p_i]) = \sum_{v \in \beta^{-1}_\mathfrak{S}[p_i]} \sigma_{\mathfrak{S}^*[p_i]}(v) + \mathbb{Z}W - 1 + \mathbb{Z}(\mathbb{Z}(I_{V_{e_{0,1}}}^{I_{V_{e_{0,1}}}^I})/I_{V_{e_{0,1}}}^I - 1 + \mathbb{Z}(I_{V_{e_{0,1}}}^{I_{V_{e_{0,1}}}^I})/I_{V_{e_{0,1}}}^I) - 1)
\]
we obtain 

\[ \sharp W(-\sharp(I^l_{v_i} + I^r_{v_i}) + \sharp((I^l_{v_i} + I^l_{v_i})/I^l_{v_i})(\sharp I^l_{v_i} - 1) + \sharp((I^l_{v_i} + I^l_{v_i})/I^l_{v_i})(\sharp I^r_{v_i} - 1) + 1) + \sharp W - 1 + \sharp W(\sharp((I^l_{v_i} + I^l_{v_i})/I^l_{v_i}) - 1 + \sharp((I^l_{v_i} + I^l_{v_i})/I^l_{v_i}) - 1) \]

\[ + \sum_{v \in \gamma(G \setminus \beta^{-1}_\Phi(p_i))} \sigma_{\Phi^*}(v) + r(G \setminus \beta^{-1}_\Phi(p_i)) - \sharp G/I_{e_0,1} + 1 - \sharp G/I_{e_{n+1}} + 1 \]

\[ = \sharp G/I_{v_i} + \sum_{v \in \gamma(G \setminus \beta^{-1}_\Phi(p_i))} \sigma_{\Phi^*}(v) + r(G \setminus \beta^{-1}_\Phi(p_i)) - \sharp G/I_{e_0,1} - \sharp G/I_{e_{n+1}} + 1. \]

Thus, we have \( \sigma(\mathcal{G}) = \sigma(\mathcal{G}^*_{[p_i]}). \)

We complete the proof of the proposition. \( \square \)

**Remark 2.7.1.** Let \( \mathcal{G} \) be a \( G \)-covering over \( \mathcal{P}_n \). By the definition of coverings, for any maximal filtration of \( G \), there exists a sequence of \( p \)-coverings induced by the maximal filtration of \( G \):

\[ \mathcal{G} = \mathcal{G}_r \xrightarrow{b_r} \mathcal{G}_{r-1} \xrightarrow{b_{r-1}} \ldots \xrightarrow{b_1} \mathcal{G}_0 = \mathcal{P}_n. \]

By Proposition 2.7, for calculating the \( p \)-rank \( \sigma(\mathcal{G}) \), we may assume that \( b_i \) do not have either étale Type-I or Type-II for all \( i \).

**Theorem 2.8.** Let \( G \) be an abelian \( p \)-group with order \( p^r \), \( \Phi \) a maximal filtration of \( G \), and \( \mathcal{G} := (G, \sigma_\Phi, \beta_\Phi) \) a \( G \)-covering over \( \mathcal{P}_n \). Write

\[ \Phi_\Phi : \mathcal{G} = \mathcal{G}_r \xrightarrow{b_r} \mathcal{G}_{r-1} \xrightarrow{b_{r-1}} \ldots \xrightarrow{b_1} \mathcal{G}_0 = \mathcal{P}_n \]

for the sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank induced by \( \Phi \) and \( E^\Phi_{s,\Phi} \) for the étale-chain associated to \( \Phi_\Phi \). For each \( j = 1, \ldots, n \), write \( E^\Phi_{s,\Phi}(p_j) \) (resp. \( E^\Phi_{s,\Phi}(b^l_j), E^\Phi_{s,\Phi}(b^r_j) \)) for the disjoint union

\[ \prod_{s \text{ s.t. } p_j \in (E^\Phi_{s,\Phi})} E^\Phi_{s,\Phi}, \quad \prod_{s \text{ s.t. } b^l_j \in (E^\Phi_{s,\Phi}) \cup \epsilon'(E^\Phi_{s,\Phi})} E^\Phi_{s,\Phi}, \quad \prod_{s \text{ s.t. } b^r_j \in (E^\Phi_{s,\Phi}) \cup \epsilon'(E^\Phi_{s,\Phi})} E^\Phi_{s,\Phi}. \]

Then we have

\[ \sigma(\mathcal{G}) = \sum_{j=1}^n (p^{E^\Phi_{s,\Phi}(p_j)} - p^{E^\Phi_{s,\Phi}(b^l_j)} - p^{E^\Phi_{s,\Phi}(b^r_j)} + 1) + \sum_{j=2}^{n-1} (p^{E^\Phi_{s,\Phi}(b^r_j)} - 1). \]

\[ = \sum_{j=1}^n (p^{E^\Phi_{s,\Phi}(p_j)} - p^{E^\Phi_{s,\Phi}(b^l_j)} - p^{E^\Phi_{s,\Phi}(b^r_j)} + 1) + \sum_{j=2}^{n} (p^{E^\Phi_{s,\Phi}(b^l_j)} - 1). \]

**Proof.** By Remark 2.7.1, we may assume that \( b_j \) do not have étale Type-I for all \( j \). Thus, we obtain \( \gamma(G) = \{v_1, \ldots, v_n\} \), where for each \( j \), \( v_j \) denotes the unique vertex \( \beta^{-1}_\Phi(p_j) \).

Then for each \( j = 1, \ldots, n \), we have

\[ \sigma_{\Phi}(v_j) = -p^{E^\Phi_{s,\Phi}(p_j)} + p^{E^\Phi_{s,\Phi}(b^l_j)} (p^{E^\Phi_{s,\Phi}(p_j)} - p^{E^\Phi_{s,\Phi}(b^l_j)} - 1) + p^{E^\Phi_{s,\Phi}(b^r_j)} (p^{E^\Phi_{s,\Phi}(p_j)} - p^{E^\Phi_{s,\Phi}(b^r_j)} - 1) + 1 \]
\[ p^{E\Phi_e(p_j)} - p^{E\Phi_e(b'_j)} - p^{E\Phi_e(b''_j)} + 1. \]

On the other hand, the rank of \( H^1(G, \mathbb{Z}) \) as a free \( \mathbb{Z} \)-module is
\[
\sum_{j=1}^{n-1} (p^{E\Phi_e(b'_j)} - 1) = \sum_{j=2}^{n} (p^{E\Phi_e(b''_j)} - 1).
\]

Then we have
\[
\sigma(\mathcal{G}) = \sum_{j=1}^{n} (p^{E\Phi_e(p_j)} - p^{E\Phi_e(b'_j)} - p^{E\Phi_e(b''_j)} + 1) + \sum_{j=1}^{n-1} (p^{E\Phi_e(b'_j)} - 1)
\]
\[
= \sum_{j=1}^{n} (p^{E\Phi_e(p_j)} - p^{E\Phi_e(b'_j)} - p^{E\Phi_e(b''_j)} + 1) + \sum_{j=2}^{n} (p^{E\Phi_e(b''_j)} - 1).
\]

This completes the proof of the theorem.  

Corollary 2.9. Let \( G_i, i \in \{1, 2\} \) be an abelian \( p \)-group with order \( p^r \), \( \Phi^i \) a maximal filtration of \( G_i \), \( \mathcal{G}^i := (G^i, \sigma^{G_i}, \beta^{G_i}) \) a \( G_i \)-covering over \( \mathcal{P}_n \). Write
\[
\Phi^{G_i}_e : \mathcal{G}^i = \mathcal{G}^i \rightarrow \mathcal{G}^i_{r-1} \rightarrow \mathcal{G}^i_{r-2} \rightarrow \cdots \rightarrow \mathcal{G}^i_0 = \mathcal{P}_n
\]
for the sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank induced by \( \Phi^i \), and \( \mathcal{E}^{\Phi^{G_i}_e} \) for the \( \epsilon \)-chain associated to \( \Phi^{G_i}_e \). Suppose that \( \mathcal{E}^{\Phi^{G_i}_e} = \mathcal{E}^{\Phi^{G_i}_e} \) holds. Then we have \( \sigma(\mathcal{G}^1) = \sigma(\mathcal{G}^2) \).

Proof. Since \( \mathcal{E}^{\Phi^{G_i}_e} = \mathcal{E}^{\Phi^{G_i}_e} \) holds, we see that \( \mathcal{E}^{\Phi^{G_i}_e} = \mathcal{E}^{\Phi^{G_i}_e} = \mathcal{E}^{\Phi^{G_i}_e} \), and \( \mathcal{E}^{\Phi^{G_i}_e} = \mathcal{E}^{\Phi^{G_i}_e} \) for all \( j \). Thus, by Theorem 2.8, we obtain \( \sigma(\mathcal{G}^1) = \sigma(\mathcal{G}^2) \). This completes the proof of the corollary.

Theorem 2.10. Let \( G \) be an abelian \( p \)-group with order \( p^r \), \( \Phi_G \) a maximal filtration of \( G \), and \( \mathcal{G} \) a \( G \)-covering over \( \mathcal{P}_n \). Write
\[
\Phi^{\mathcal{G}} : \mathcal{G} = \mathcal{G}_r \rightarrow \mathcal{G}_{r-1} \rightarrow \mathcal{G}_{r-2} \rightarrow \cdots \rightarrow \mathcal{G}_0 = \mathcal{P}_n
\]
for the sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank induced by \( \Phi_G \), and \( \{E^{\Phi^{\mathcal{G}}}_j\}_{j \in J} \) for the set of \( j \)-th \( \epsilon \)-chains associated to \( \Phi^{\mathcal{G}} \). Let \( I := \{i_1, \ldots, i_r\} \) be a new index set. For each \( i = 1, \ldots, r \), write \( \mathcal{E}_i \) for \( \mathcal{E}^{\Phi^{\mathcal{G}}}_j \). Then there exist an elementary abelian group \( A \) with order \( p^r \), a maximal filtration \( \Phi_A \) of \( A \), and an \( A \)-covering \( \mathcal{F} \) over \( \mathcal{P}_n \) such that the \( i \)-th \( \epsilon \)-chain \( \mathcal{E}^{\Phi^{\mathcal{G}}}_i \) associated to the sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank \( \Phi^{\mathcal{G}}_A \) induced by \( \Phi_A \) is equal to \( \mathcal{E}_i \) for each \( i = 1, \ldots, r \).

Proof. Since the operator \( \Rightarrow^I H \) does not change the \( \epsilon \)-chain \( \mathcal{E}^{\Phi^{\mathcal{G}}}_e \), we may assume that \( b_i \) do not have \( \epsilon \)-Type-I for all \( i \). Let \( A_i, i \in \{1, \ldots, r\} \), be a cyclic abelian \( p \)-group with order \( p \). We construct a semi-graph with \( p \)-rank \( \mathcal{F} \) step by step.

\( \mathcal{F}_1 := (\mathcal{v}(\mathcal{F}_1), e(\mathcal{F}_1) \cup e'(\mathcal{F}_1), \{c^{\mathcal{F}_1}_e\}_e) \) is a semi-graph as follows:

(i) \( \mathcal{v}(\mathcal{F}_1) := \{v^1_1, \ldots, v^1_n\} \),
(ii) $e(F) \cup e'(F)$ consists of the following elements:

(a) $\{e_{i;i+1} := \{b_t(e_{i;i+1}^1), b_r(e_{i;i+1}^1)\}\}_{r \in A_1}$ is a set associated to $e_{i;i+1} \in e(E_1) \cup e'(E_1)$;

(b) $e_{i;i+1}^1 := \{b_t(e_{i;i+1}^1), b_r(e_{i;i+1}^1)\}$ is a set associated to $e_{i;i+1}$ if $e_{i;i+1} \not\in e(E_1) \cup e'(E_1)$;

(iii) $\zeta_e^F(b_t(e_{i;i+1}^1)) + v_{e+1}^i$ (resp. $\zeta_e^F(b_t(e_{i;i+1}^1)) = v_{e+1}^i$) if $i \neq 0$ and $\zeta_e^F(b_t(e_{i;i+1}^1)) = v(F)$ (resp. $\zeta_e^F(b_t(e_{i;i+1}^1)) = v(F_1)$) if $i = 0$;

(iv) $\zeta_e^F(b_r(e_{i;i+1}^1)) = v_{e+1}^i$ (resp. $\zeta_e^F(b_r(e_{i;i+1}^1)) = v_{e+1}^i$) if $i \neq n$ and $\zeta_e(b_r(e_{i;i+1}^1)) = v(F)$ (resp. $\zeta_e^F(b_r(e_{i;i+1}^1)) = v(F_1)$) if $i = n$.

We have a natural morphism $\beta_{\tilde{S}_1} : F_1 \rightarrow F_n$ defined as follows: (i) $\beta_{\tilde{S}_1}(v_1^i) = \rho_1$; (ii) $\beta_{\tilde{S}_1}(e_{i;i+1}^1) = e_{i;i+1}$ (resp. $\beta_{\tilde{S}_1}(b_t(e_{i;i+1}^1)) = e_{i;i+1}$).

Next, we define a $p$-rank map $\sigma_{\tilde{S}_1} : v(F_1) \rightarrow \mathbb{Z}$ as follows: (i) If $p_i \in v(E_1)$ and $\beta_{\tilde{S}_1}(b_i^t) = \beta_{\tilde{S}_1}^{-1}(b_i^r) = 1$, then we have

$$\sigma_{\tilde{S}_1}(v_1^i) = -p + p - 1 + p - 1 + 1 = p - 1;$$

(ii) If $p_i \not\in v(E_1)$, $\beta_{\tilde{S}_1}^{-1}(b_i^t) = 1$, and $\beta_{\tilde{S}_1}^{-1}(b_i^r) = p$, then we have

$$\sigma_{\tilde{S}_1}(v_1^i) = -p + p - 1 + 0 = 0;$$

(iii) If $p_i \not\in v(E_1)$, $\beta_{\tilde{S}_1}^{-1}(b_i^t) = p$, and $\beta_{\tilde{S}_1}^{-1}(b_i^r) = 1$, then we have

$$\sigma_{\tilde{S}_1}(v_1^i) = -p + p - 1 + 0 = 0;$$

(iv) If $p_i \not\in v(E_1)$ and $\beta_{\tilde{S}_1}^{-1}(b_i^t) = \beta_{\tilde{S}_1}^{-1}(b_i^r) = p$, then we have

$$\sigma_{\tilde{S}_1}(v_1^i) = -p + 1;$$

(v) If $p_i \not\in v(E_1)$, then we have

$$\sigma_{\tilde{S}_1}(v_1^i) = 0.$$

Moreover, $\tilde{S}_1$ admits a natural action of $A_1$ as follows: (i) The action of $A_1$ on $v(F_1)$ is trivial; (ii) For any $e \in e(F_1) \cup e'(F_1)$ and any element $\tau \in A_1$, $\tau e_{i;i+1} = e_{i;i+1}$ and $\tau(e_{i;i+1}^1) = e_{i;i+1}^1$ for all $\tau_1 \in A_1$.

Thus, with the action of $A_1$, $\tilde{S}_1 := (F_1, \sigma_{\tilde{S}_1}, \beta_{\tilde{S}_1})$ is an $A_1$-covering over $\mathcal{F}_n$. Next, let us construct $\mathcal{F}_2$.

$\mathcal{F}_2 := (v(F_2), e(F_2) \cup e'(F_2), \{e_{2}^F\}_e)$ is a semi-graph as follows:

(i) $v(F_2) := \{v_1^2, \ldots, v_n^2\}$;

(ii) $e(F_2) \cup e'(F_2)$ consists of the following elements:

(a) $\{e_{i;i+1}^2 := \{b_t(e_{i;i+1}^{1,2}), b_r(e_{i;i+1}^{1,2})\}\}_{r \in A_2}$ is a set associated to $e_{i;i+1}^1$ if $\beta_{\tilde{S}_1}(e_{i;i+1}^1) \in e(E_2) \cup e'(E_2)$;

(b) $\{e_{i;i+1}^2 := \{b_t(e_{i;i+1}^{1,2}), b_r(e_{i;i+1}^{1,2})\}\}_{r \in A_2}$ is a set associated to $e_{i;i+1}^1$ if $\beta_{\tilde{S}_1}(e_{i;i+1}^1) \in e(E_2) \cup e'(E_2)$;

(c) $e_{i;i+1}^{1,2} := \{b_t(e_{i;i+1}^{1,2}), b_r(e_{i;i+1}^{1,2})\}$ is a set associated to $e_{i;i+1}^1$ if $\beta_{\tilde{S}_1}(e_{i;i+1}^1) \not\in e(E_2) \cup e'(E_2)$;

(d) $\{e_{i;i+1}^{1,2} := \{b_t(e_{i;i+1}^{1,2}), b_r(e_{i;i+1}^{1,2})\}\}_{r \in A_2}$ is a set associated to $e_{i;i+1}^1$ if $\beta_{\tilde{S}_1}(e_{i;i+1}^1) \not\in e(E_2) \cup e'(E_2)$;
(iii) $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$ (resp. $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$) if $i \neq 0$ and $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$ (resp. $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$) if $i = 0$.

(iv) $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$ (resp. $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v_t^2$) if $i \neq n$ and $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$ (resp. $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$, $\zeta^2_e(b_t(e_{i,i+1}^{1,2})) = v(F_2)$) if $i = n$.

We have a natural morphism $\alpha_2 : F_2 \rightarrow F_1$ as follows: (i) $\alpha_2(v_t^2) = v_t^2$; (ii) $\alpha_2((e_{i,i+1}^{1,2})) = e_{i,i+1}^1$ (resp. $\alpha_2((e_{i,i+1}^{1,2})) = e_{i,i+1}^{1,2}$, $\alpha_2((e_{i,i+1}^{1,2})) = e_{i,i+1}^{1,2}$, $\alpha_2((e_{i,i+1}^{1,2})) = e_{i,i+1}^1$).

We define $\beta_{\tilde{\mathcal{S}}}$ to be the composite morphism $\beta_{\tilde{\mathcal{S}}} \circ \alpha_2$.

We define a $p$-rank map $\sigma_{\tilde{\mathcal{S}}} : v(F_2) \rightarrow \mathbb{Z}$ as follows: (i) If $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$ and $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$, then we have

$$\sigma_{\tilde{\mathcal{S}}}(v_t^2) - 1 = p(\sigma_{\tilde{\mathcal{S}}}(v_t^1) - 1);$$

(ii) If $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$ and $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$, we have

$$\sigma_{\tilde{\mathcal{S}}}(v_t^2) - 1 = p(\sigma_{\tilde{\mathcal{S}}}(v_t^1) - 1) + (\not\exists b_t(v_t^1))(p - 1);$$

(iii) If $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$ and $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$, we have

$$\sigma_{\tilde{\mathcal{S}}}(v_t^2) - 1 = p(\sigma_{\tilde{\mathcal{S}}}(v_t^1) - 1) + (\not\exists b_t(v_t^1))(p - 1);$$

(iv) If $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$ and $\not\exists b_t(v_t^2) = p\not\exists b_t(v_t^1)$, we have

$$\sigma_{\tilde{\mathcal{S}}}(v_t^2) - 1 = p(\sigma_{\tilde{\mathcal{S}}}(v_t^1) - 1) + (\not\exists b_t(v_t^1) + \not\exists b_t(v_t^1))(p - 1).$$

Moreover, there is a natural $A_1 \oplus A_2$-action on $\tilde{\mathcal{S}}$ defined as follows: (i) The action of $A_1 \oplus A_2$ on $v(F_2)$ is trivial; (ii) For any $e \in e(F_2) \cup e'(F_2)$ and any element $(\tau, \tau') \in A_1 \oplus A_2$, $(\tau, \tau').e_{i,i+1}^{1,2} = e_{i,i+1}^{1,2}$, $(\tau, \tau').e_{i,i+1}^{1,2} = e_{i,i+1}^{1,2}$, $(\tau, \tau').e_{i,i+1}^{1,2} = e_{i,i+1}^{1,2}$, and $(\tau, \tau').e_{i,i+1}^{1,2} = e_{i,i+1}^{1,2}$.

Thus, with the action of $A_1 \oplus A_2$, $\tilde{\mathcal{S}} := (F_2, \sigma_{\tilde{\mathcal{S}}}, \beta_{\tilde{\mathcal{S}}})$ is an $A_1 \oplus A_2$-covering over $\mathcal{P}_n$.

The maximal filtration

$$0 \subset A_2 \subset A_1 \oplus A_2$$

determines a sequence of $p$-coverings of $n$-semi-graphs with $p$-rank

$$\Phi_{\tilde{\mathcal{S}}} : \tilde{\mathcal{S}} \xrightarrow{a_2} \tilde{\mathcal{S}}_1 \xrightarrow{a_1} \tilde{\mathcal{S}}_0 = \mathcal{P}_n.$$

Furthermore, by the construction, we have $E_2^{\Phi_{\tilde{\mathcal{S}}} \circ \alpha_2} = E_2$ and $E_1^{\Phi_{\tilde{\mathcal{S}}} \circ \alpha_2} = E_1$.

By repeating the process above, we obtain an $A := \oplus_{i=1}^r A_i$-covering $\tilde{\mathcal{S}}_r$ over $\mathcal{P}_n$ and a maximal filtration

$$\Phi_A : 0 \subset A_n \subset A_n \oplus A_{n-1} \subset \cdots \subset \oplus_{i=1}^r A_i = A.$$

Then $\Phi_A$ induces a sequence of $p$-coverings of $n$-semi-graphs with $p$-rank

$$\Phi_{\tilde{\mathcal{S}}} : \tilde{\mathcal{S}} := \tilde{\mathcal{S}} \xrightarrow{a_r} \tilde{\mathcal{S}}_{r-1} \xrightarrow{a_{r-1}} \cdots \xrightarrow{a_1} \tilde{\mathcal{S}}_0 = \mathcal{P}_n.$$

By the construction, we have the $i$-th étale-chain $E_i^{\Phi_{\tilde{\mathcal{S}}} \circ \alpha_2}$ associated to $\Phi_{\tilde{\mathcal{S}}}$ is equal to $E_i$ for each $i = 1, \ldots, r$. We complete the proof of the theorem. □
Remark 2.10.1. For the sequence
\[
\Phi_\mathfrak{f} : \mathfrak{f} = \mathfrak{f}_r \xrightarrow{a_r} \mathfrak{f}_{r-1} \xrightarrow{a_{r-1}} \cdots \xrightarrow{a_1} \mathfrak{f}_0 = \mathfrak{P}_n
\]
constructed in Theorem 2.10, by Remark 2.7.1, we may assume that \( a_i \) do not have étale Type-II for all \( i \). Furthermore, by Corollary 2.9, we have \( \sigma(\mathfrak{S}) = \sigma(\mathfrak{f}) \).

2.3 Bounds of \( p \)-ranks of abelian coverings

Let \( G \) be a finite abelian \( p \)-group with order \( p^r \). In this subsection, we calculate a bound of \( p \)-rank of a \( G \)-covering over \( \mathfrak{P}_n \).

First, let us fix some notations. Let \( \mathfrak{G} \) be a \( G \)-covering over \( \mathfrak{P}_n \) and \( \Phi \) a maximal filtration of \( G \). Write

\[
\Phi_\mathfrak{G} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{b_r} \mathfrak{G}_{r-1} \xrightarrow{b_{r-1}} \cdots \xrightarrow{b_1} \mathfrak{G}_0 = \mathfrak{P}_n
\]

for the sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank induced by \( \Phi \) and \( \{ E_j^{\Phi_\mathfrak{G}} \}_j \) for the set of \( j \)-th étale-chains associated to \( \Phi_\mathfrak{G} \). If \( E_j^{\Phi_\mathfrak{G}} \) is empty, we have \( \sigma(\mathfrak{G}_j) = \sigma(\mathfrak{G}_{j-1}) \); thus, for calculating the bound of the \( p \)-rank \( \sigma(\mathfrak{G}) \), we may assume that \( E_j^{\Phi_\mathfrak{G}} \) are not empty for all \( j \). Moreover, by Remark 2.10.1, we may assume that for each \( j = 1, \ldots, r \) and each \( v \in v(\mathfrak{G}_j) \), \( b_j \) is not étale Type-II at \( v \).

Let \( e_0 \in \beta_0^{-1}(e_{0,1}) \) (resp. \( e_{n+1} \in \beta_0^{-1}(e_{n,n+1}) \)). Write \( I_{e_0} \) (resp. \( I_{e_{n+1}} \)) for the inertia group of \( e_0 \) (resp. \( e_{n+1} \)). Note that since \( G \) is an abelian group, the group \( I_{e_0} \) (resp. \( I_{e_{n+1}} \)) does not depend on the choice of the elements of \( \beta_0^{-1}(e_{0,1}) \) (resp. \( \beta_0^{-1}(e_{n,n+1}) \)). Moreover, according to Definition 2.3 (c-iii), \( G \) is generated by \( I_{e_0} \) and \( I_{e_{n+1}} \).

For each \( j = 1, \ldots, r \), since \( E_j^{\Phi_\mathfrak{G}} \) is a sub-semi-graph of \( \mathfrak{P}_n \), \( v(\mathfrak{E}_j^{\Phi_\mathfrak{G}}) \subseteq \{ p_1, \ldots, p_n \} = v(\mathfrak{P}_n) \) admits a natural order which is induced by the order of natural number \( \mathbb{N} \); then we may define the initial vertex and the terminal vertex for \( E_j^{\Phi_\mathfrak{G}} \). Write \( t(E_j^{\Phi_\mathfrak{G}}) \) (resp. \( t(E_j^{\Phi_\mathfrak{G}}) \)) for the initial (resp. the terminal) vertex of \( v(E_j^{\Phi_\mathfrak{G}}) \), \( l(E_j^{\Phi_\mathfrak{G}}) \) for \( v(E_j^{\Phi_\mathfrak{G}}) \). For an element \( p_i \in v(\mathfrak{P}_n) \), we shall say that \( p_i \) is \( \mathbb{A}_1 \)-type (resp. \( \mathbb{A}_1 \)-type; \( \mathbb{G}_m \)-type; \( \mathbb{P} \)-type; \( \mathbb{P} \)-type) at \( E_j^{\Phi_\mathfrak{G}} \) if \( p_i \) is equal to \( t(E_j^{\Phi_\mathfrak{G}}) \) (resp. \( l(E_j^{\Phi_\mathfrak{G}}) \)) (resp. \( t(E_j^{\Phi_\mathfrak{G}}) \)). Moreover, by Remark 2.10.1, we may assume that \( E_j^{\Phi_\mathfrak{G}} \) are not empty for all \( j \).

Lemma 2.11. Suppose that \( \pi_0(\mathfrak{E}_j^{\Phi_\mathfrak{G}}) = 1 \) for all \( j \), and either \( I_{e_0} \) or \( I_{e_{n+1}} \) is trivial. Then the \( p \)-rank \( \sigma(\mathfrak{G}) \) is equal to 0.
Proof. For each \( j = 1, \ldots, r \), write \( v^j_i \) for an element \( \beta_{G_j}^{-1}(p_i) \). Since \( \sharp \pi_0(\mathbb{E}_j^{\psi}) = 1 \) hold for all \( j \), and \( I_{c_0,1} \) (resp. \( I_{c_{n+1},1} \)) is trivial, \( b^i_j \) is not étale Type-III (resp. étale Type-IV) and étale Type-V at \( v^j_i \). Then we obtain \( \sigma_{G_j}(v^j_i) = 0 \) by applying Remark 2.3.1. Moreover, the underlying semi-graph of \( G_j \) is an tree. Thus, we obtain \( \sigma(\mathcal{G}_j) = 0 \). In particular, we have \( \sigma(\mathcal{G}) = 0 \). We complete the proof of the lemma.

Lemma 2.12. Let \( G_i \), \( i \in \{1, 2\} \) be an abelian \( p \)-group with order \( p^r \), \( \Phi^i \) a maximal filtration of \( G_i \), and \( \mathcal{G}^i := (\mathcal{G}^i, \sigma_{\Phi^i}, \beta_{\Phi^i}) \) a \( G_i \)-covering over \( \mathbb{P}_n \). Write

\[
\Phi_{\Phi^i} : \mathcal{G}^i = \mathcal{G}^i_0 \xrightarrow{b^i_0} \mathcal{G}^i_1 \xrightarrow{b^i_1} \cdots \xrightarrow{b^i_{r-1}} \mathcal{G}^i_r = \mathcal{G}
\]

for the sequence of \( p \)-coverings of \( n \)-semi-graphs with \( p \)-rank induced by \( \Phi^i \), and \( \mathbb{E}_j^{\Phi^i} \) for the étale-chain associated to \( \Phi_{\Phi^i} \). Suppose that for each \( j = 1, \ldots, r \), \( \sharp \pi_0(\mathbb{E}_j^{\Phi^i}) = 1 \), \( i(\mathbb{E}_j^{\Phi^i}) = i(\mathbb{E}_j^{\Phi^i-2}) \), and \( t(\mathbb{E}_j^{\Phi^i}) = t(\mathbb{E}_j^{\Phi^i-2}) \). Moreover, we suppose that \( \mathbb{E}_j^{\Phi^i} \) is equal to \( \mathbb{E}_j^{\Phi^i-2} \) if \( i(\mathbb{E}_j^{\Phi^i}) \neq 1 \) and \( t(\mathbb{E}_j^{\Phi^i}) \neq n \). Let \( e_0^1 \in \beta_{\Phi^1}^{-1}\mathcal{G}_0 \in \beta_{\Phi^2}^{-1}(e_0) \) (resp. \( e^1_{n+1} \in \beta_{\Phi^1}^{-1}(e_{n+1}) \) and \( e^2_{n+1} \in \beta_{\Phi^2}^{-1}(e_{n+1}) \)). Write \( I_{c_0} \) and \( I_{c_0} \) (resp. \( I_{c_{n+1},1} \) and \( I_{c_{n+1},1} \)) for the inertia groups of \( e^1_0 \) and \( e^2_0 \), respectively (resp. \( e^1_{n+1} \) and \( e^2_{n+1} \), respectively), \( D^1_{n+1} \) (resp. \( D^1_{n+1} \)) for \( G_1/I_{c_0} \) (resp. \( G_1/I_{c_{n+1},1} \)). Furthermore, we suppose that \( I_{c_0} \) and \( I_{c_{n+1},1} \) are equal to \( G_2 \). Then we have

\[
\sigma(\mathcal{G}^1) + \sharp D^0_1 - 1 + \sharp D^1_{n+1} - 1 = \sigma(\mathcal{G}^2).
\]

Proof. By Remark 2.7.1, we may assume that \( b^i_0 \) do not have étale Type-I. For any \( p_u \in v(\mathcal{P}_n) \), write \( v^j_i \) for the unique element of \( \beta_{G_j}^{-1}(p_u) \). Then \( D_{v^j_i} \) is equal to \( G_i \).

If \( n = 1 \), note that since \( \mathbb{E}_j^{\Phi^i} \) are not empty for all \( j \), both \( I_{c_0} \) and \( I_{c_{n+1},1} \) are trivial. Then we have

\[
\sigma(\mathcal{G}^1) = \sigma_{\Phi^1}(v^1_i) = -\sharp G_1 + \sharp D^1_1(\sharp I_{c_0} - 1) + \sharp D^1_{n+1}(\sharp I_{c_{n+1},1} - 1) + 1.
\]

On the other hand, since both \( I_{c_0} \) and \( I_{c_{n+1},1} \) are equal to \( G_2 \), we obtain

\[
\sigma(\mathcal{G}^2) = \sigma_{\Phi^2}(v^2_i) = -\sharp G_2 + \sharp G_2 - 1 + \sharp G_2 - 1 + 1 = \sharp G_2 - 1.
\]

Thus, we have

\[
\sigma(\mathcal{G}^1) + \sharp D^0_1 - 1 + \sharp D^1_{n+1} - 1 = \sigma(\mathcal{G}^2).
\]

If \( n > 1 \), by the assumptions of \( \mathbb{E}_j^{\Phi^i} \) and \( \mathbb{E}_j^{\Phi^i-2} \), we obtain

\[
\sum_{v \in v(\mathcal{G}^1) \setminus \{v^1_i, v^2_i\}} \sigma_{\Phi^1}(v) = \sum_{v \in v(\mathcal{G}^2) \setminus \{v^2_i, v^2_i\}} \sigma_{\Phi^2}(v)
\]

and

\[
\text{rank}_\mathbb{Z} H^1(\mathbb{G}^1, \mathbb{Z}) = \text{rank}_\mathbb{Z} H^1(\mathbb{G}^2, \mathbb{Z}).
\]

On the other hand, since both \( I_{c_0} \) and \( I_{c_{n+1},1} \) are equal to \( G_2 \), we have

\[
\sigma_{\Phi^2}(v^2_i) - \sigma_{\Phi^1}(v^1_i) = \sharp G_2 - 1 - \sharp D^0_1(\sharp I_{c_0} - 1) = \sharp G_1 - 1 - \sharp D^0_1(\sharp I_{c_0} - 1) = \sharp D^0_1 - 1
\]

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Thus, we obtain
\[ \sigma(\mathcal{G}^l) + \sharp D_0^1 - 1 + \sharp D_{n+1}^1 = \sum_{v \in \mathcal{G}(v^1) \setminus \{v_i, v_j\}} \sigma_{\mathcal{G}^l}(v) + \text{rank}_Z H^1(\mathcal{G}^l, \mathbb{Z}) + \sigma_{\mathcal{G}^l}(v_i^1) + \sigma_{\mathcal{G}^l}(v_j^1) + \sharp D_0^1 - 1 + \sharp D_{n+1}^1 - 1 = \sigma(\mathcal{G}^2). \]

We have the following theorem.

**Theorem 2.13.** Let $\mathcal{G}$ be a $G$-covering over $\mathfrak{P}_n$ and $\Phi$ a maximal filtration of $G$. Write

\[ \Phi_{\mathcal{G}} : \mathcal{G} = \mathcal{G}_0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_{r-1} \rightarrow \cdots \rightarrow \mathcal{G}_{r-1} \rightarrow \mathcal{G}_0 = \mathfrak{P}_n \]

for the sequence of $p$-coverings of $n$-semi-graphs with $p$-rank induced by $\Phi$, and \{\mathcal{E}_j^{\mathcal{G}}\}_j for the set of $j$-th etale-chains associated to $\Phi_{\mathcal{G}}$. Suppose that $\sharp \pi_0(\mathcal{E}_j^{\mathcal{G}}) = 1$ hold for all $j$. Then we have

\[ \sigma(\mathcal{G}) \leq p^r - 1. \]

**Proof.** We prove the theorem by induction. If $r = 1$, since $\pi_0(\mathcal{E}_1^{\mathcal{G}}) = 1$, let us check the theorem case by case. If either $I_{e_0}$ or $I_{e_{n+1}}$ is trivial, then by Lemma 2.11, we have $\sigma(\mathcal{G}) = 0$. If Both $I_{e_0}$ and $I_{e_{n+1}}$ are non-trivial, and $\ell(\mathcal{E}_1^{\mathcal{G}})$ is 1, we obtain rank$_Z H^1(\mathcal{G}, \mathbb{Z})$ is equal to 0; for each $v \in \mathcal{G}_1$, $\sigma(v)$ is equal to 0 if $\beta_{\mathcal{G}}(v)$ is not contained in $v(\mathcal{E}_1^{\mathcal{G}})$, and $\sigma(v)$ is equal to $p - 1$ if $\beta_{\mathcal{G}}(v)$ is contained in $v(\mathcal{E}_1^{\mathcal{G}})$; thus, we obtain $\sigma(\mathcal{G}) = p - 1$. If Both $I_{e_0}$ and $I_{e_{n+1}}$ are non-trivial, and $\ell(\mathcal{E}_1^{\mathcal{G}})$ is $\geq 2$, we have rank$_Z H^1(\mathcal{G}, \mathbb{Z})$ is equal to $p - 1$, and $\sigma(v)$ is equal to 0 for all $v \in \mathcal{G}$. Thus, we have $\sigma(\mathcal{G}) = p - 1$. This completes the proof of the theorem if $r = 1$. From now on, we assume that $r \geq 2$.

For each $i = 1, \ldots, n$, let $v_i$ be an element of $\beta_{\mathcal{G}}^{-1}(p_i) \subseteq v(\mathcal{G})$, $I_{v_i}$ the inertia group of $v_i$. Write $d$ for min\{i \mid I_{v_i} \neq G\}. If $d \neq 1$, then we have $\beta_{\mathcal{G}}|_{\beta_{\mathcal{G}}^{-1}(p_n \setminus \mathcal{P}_{d,n})} : \beta_{\mathcal{G}}^{-1}(P_n \setminus \mathcal{P}_{d,n}) \rightarrow \mathcal{P}_{d,n}$ is an isomorphism of semi-graphs. Then we have $\mathcal{G}' := (\mathcal{G} \setminus \beta_{\mathcal{G}}^{-1}(P_n \setminus \mathcal{P}_{d,n}), \mathcal{G}|_{\mathcal{G} \setminus \beta_{\mathcal{G}}^{-1}(P_n \setminus \mathcal{P}_{d,n})}, \beta_{\mathcal{G}}|_{\mathcal{G} \setminus \beta_{\mathcal{G}}^{-1}(P_n \setminus \mathcal{P}_{d,n})})$ is a $(n - d + 1)$-semi-graph with $p$-rank. Furthermore, $\mathcal{G} \setminus \beta_{\mathcal{G}}^{-1}(P_n \setminus \mathcal{P}_{d,n})$ admits a natural action of $G$ induced by the action of $G$ on $\mathcal{G}$. Thus, we may regard $\mathcal{G}'$ is a $G$-covering over $\mathfrak{P}_{d,n}$. Note that we have $\sigma(\mathcal{G}(v_i)) = 0$ for $i \leq d - 1$ and rank$_Z H^1(\beta_{\mathcal{G}}^{-1}(P_n \setminus \mathcal{P}_{d,n}), \mathbb{Z}) = 0$. Then $\sigma(\mathcal{G}')$ is equal to $\sigma(\mathcal{G})$. Thus, by replacing $\mathcal{G}$ (resp. $\mathfrak{P}_n$) by $\mathcal{G}'$ (resp. $\mathfrak{P}_{d,n}$), we may assume that $I_{v_i}$ is not equal to $G$. Similar arguments to the arguments given above imply that we may assume that $I_{v_i}$ is not equal to $G$.

Write $S_1$ (resp. $S_2, S_3, S_4, S_5$) for the set

\[ \{\mathcal{E}_j^{\mathcal{G}} \mid p_1 \text{ is } A_{v_i}^1 \text{-type at } \mathcal{E}_j^{\mathcal{G}}\} \text{ (resp. } \{\mathcal{E}_j^{\mathcal{G}} \mid p_1 \text{ is } A_{v_i}^1 \text{-type at } \mathcal{E}_j^{\mathcal{G}}\}) \]
\{E^\Phi_0 | p_1 \text{ is } G^1_m\text{-type at } E^\Phi_0, \} \quad \{E^\Phi_0 | p_1 \text{ is } P^1\text{-type at } E^\Phi_0 \}.

Write \( t \) for \( \max \{ t(E^\Phi_0) | i(E^\Phi_0) \in S_1 \cup S_2 \cup S_3 \cup S_5 \}, \) \( T_1 \) (resp. \( T_2, T_3, T_4, T_5 \)) for

\{E^\Phi_0 | p_t \text{ is } A^1_t\text{-type at } E^\Phi_0 \} \quad \{E^\Phi_0 | p_t \text{ is } A^1_t\text{-type at } E^\Phi_0 \},

\{E^\Phi_0 | p_t \text{ is } G^1_m\text{-type at } E^\Phi_0 \}, \{E^\Phi_0 | p_t \text{ is } P\text{-type at } E^\Phi_0 \},

\{E^\Phi_0 | p_t \text{ is } P^1\text{-type at } E^\Phi_0 \}).

For \( i \in \{1, 2, 3, 4, 5\} \), write \( n_i \) for \( \sharp S_i \). Write \( m_1 \) for \( \sharp T_1 \), \( m_2 \) for \( \sharp (S_1 \cap T_2) \), \( m_3 \) for \( \sharp T_3, m_4 \) for \( \sharp (S_4 \cap T_4) \), \( m_5 \) for \( \sharp T_5 \), \( a_1 \) for \( \sharp (S_1 \cap T_2) \), and \( a_2 \) for \( \sharp (S_1 \cap T_4) \). Write \( b_1 \) (resp. \( b_2 \)) for

\( \sharp \{E^\Phi_0 | S_4 \cap T_4 | i(E^\Phi_0) \geq 2 \text{ and } t(E^\Phi_0) \leq t - 1 \}

(\text{resp. } \sharp \{E^\Phi_0 | S_4 \cap T_4 | i(E^\Phi_0) \geq t + 1 \}).

Note that we have \( \sum_{i=1}^5 n_i = r \) and \( b_1 + b_2 = m_4 \). Since \( t \) is the maximal element of \( \{t(E^\Phi_0) | i(E^\Phi_0) \in S_1 \cup S_2 \} \) and \( \sharp \pi_0(E^\Phi_0) = 1 \), we obtain \( \sum_{i=1}^5 m_i = n_4 \) and \( a_1 + a_2 = n_1 \).

Let \( \{E_1, \ldots, E_r\} \) be a set of étale-chains associated to \( \Phi_0 \) with a new index set such that the following conditions: (i) \( T_5 = \{E_1, \ldots, E_{m_2}\} \); (ii) \( S_4 \cap T_4 = \{E_{m_5+1}, \ldots, E_{m_5+m_4}\} \); (iii) \( T_1 = \{E_{m_5+m_4+1}, \ldots, E_{m_5+m_4+m_1}\} \); (iv) \( S_1 \cap T_2 = \{E_{m_5+m_4+m_1+1}, \ldots, E_{m_5+m_4+m_1+m_2}\} \); (v) \( S_1 \cap T_2 = \{E_{m_5+m_4+m_1+m_2+1}, \ldots, E_{m_5+m_4+m_1+m_2+m_1}\} \); (vi) \( S_1 \cap T_4 = \{E_{m_5+m_4+m_1+m_2+1}, \ldots, E_{m_5+m_4+m_1+m_2+m_1}\} \); (vii) \( T_3 = \{E_{m_5+m_4+m_1+m_2+1}, \ldots, E_{m_5+m_4+m_1+m_2+m_1}\} \); (viii) \( S_2 = \{E_{m_5+m_1+1}, \ldots, E_{m_5+m_1}\} \); (ix) \( S_3 = \{E_{m_5+m_4+m_1+m_2+1}, \ldots, E_{m_5+m_4+m_1+m_2+m_1}\} \); (x) \( S_5 = \{E_{m_5+m_4+m_1+m_2+1}, \ldots, E_{m_5+m_4+m_1+m_2+m_1}\} \).

By Theorem 2.10, there exist an elementary abelian \( p \)-group \( A \), a maximal filtration \( \Phi_A \) of \( A \), an \( A \)-covering \( \mathcal{F} := (\mathbb{F}, \sigma_\mathcal{F}, \beta_\mathcal{F}) \) over \( \mathcal{P}_n \), and the sequence of \( p \)-coverings of \( n \)-semigraphs with \( p \)-rank induced by \( \Phi_A \)

\[ \Phi_\mathcal{F} : \mathcal{F} = \mathcal{F}_r \rightarrow \mathcal{F}_{r-1} \rightarrow \cdots \rightarrow \mathcal{F}_0 = \mathcal{P}_n \]

such that the \( j \)-th étale-chain \( E^\Phi_{j, \mathcal{F}} \) associated to \( \Phi_\mathcal{F} \) is equal to \( E_j \) for each \( j = 1, \ldots, r \).

Since \( \sigma(\mathcal{G}) \) is equal to \( \sigma(\mathcal{F}) \), in order to prove the theorem, it is sufficient to calculate the bound of \( \sigma(\mathcal{F}) \). Let \( u_1 \) be an element of \( \beta_\mathcal{F}^{-1}(p_1) \), \( e_0 \) (resp. \( e_{n+1} \)) be an element of \( \beta_\mathcal{F}^{-1}(e_{0,1}) \). Moreover, by Lemma 2.12, for calculating the bound of \( \sigma(\mathcal{F}) \), we may assume that \( G = I_{e_0} = I_{e_{n+1}} \) hold. Then we have \( n_2 = 0 \) and \( n_5 = 0 \). In particular, we have \( \sharp \beta_\mathcal{F}^{-1}(p_1) = \sharp \beta_\mathcal{F}^{-1}(p_n) = 1 \).

Case 1: If \( t = 1 \) and \( n = 1 \), since \( G = I_{e_0} = I_{e_{n+1}} \) hold, we obtain \( n_3 = r \) and

\[ \sigma(\mathcal{F}) = \sigma_\mathcal{F}(u_1) = (-1)p^{n_3} + 2(p^{n_3} - 1) + 1 = p^{n_3} - 1 = p^r - 1. \]

Thus, the theorem follows.

Case 2: If \( t = 1 \) and \( n \neq 1 \), since \( I_{e_0} \) is not trivial, \( \beta_\mathcal{F}|_{\beta_\mathcal{F}^{-1}(\mathbb{P}_{2,n})} : \beta_\mathcal{F}^{-1}(\mathbb{P}_{2,n}) \rightarrow \mathbb{P}_{2,n} \) is not an isomorphism. Write \( \mathcal{F}^{1,1} \) (resp. \( \mathcal{F}^{2,n} \)) for \( (\mathbb{F} \setminus \beta_\mathcal{F}^{-1}(\mathbb{F}_1), \sigma_\mathcal{F}|_{(\mathbb{F} \setminus \beta_\mathcal{F}^{-1}(\mathbb{F}_1))}, \beta_\mathcal{F}|_{\beta_\mathcal{F}^{-1}(\mathbb{F}_1)}) \).
(β_{\overline{1}}^{-1}(P_{2,n}), σ_{\overline{1}}|_{(β_{\overline{1}}^{-1}(P_{2,n}))}, β_{\overline{1}}|_{β^{-1}_0(P_{2,n}))}). \ 3^{1,1} (\text{resp. } 3^{2,n}) \text{ is a } G\text{-covering over } Ψ_{1,1} (\text{resp. } Ψ_{2,n}). \text{ Since } 3^{1,1}/D_{u_1} → Ψ_{1,1} (\text{resp. } 3^{2,n} → 3^{2,n}/D_{u_1}) \text{ is a composite of } p\text{-coverings which are purely inseparable, we see that } σ(3^{1,1}) = σ(3^{1,1}/D_{u_1}) (\text{resp. } σ(3^{2,n}) = σ(3^{2,n}/D_{u_1})). \text{ Moreover, } 3^{1,1} (\text{resp. } 3^{2,n}) \text{ can be regarded as a } D_{u_1}\text{-covering over } Ψ_{1,1} (\text{resp. } A/D_{u_1}\text{-covering over } Ψ_{2,n}). \text{ Since } σ(3) = σ(3^{1,1}) + σ(3^{2,n}), \#D_{u_1} < p^r, \text{ and } \#A/D_{u_1} < p^r, \text{ by induction, we have}
σ(3) ≤ \#D_{u_1} - 1 + \#A/D_{u_1} - 1 ≤ p^r - 1.

Thus, the theorem follows.

**Case 3:** If \( t = n \) and \( n \neq 1 \), write \( S' \) for the set \( \{E_j \mid i(E_j) = 1 \} \) and \( t(E_j) = n \), \( S'' \) for the complement \( \{E_1, \ldots, E_r \} \setminus S' \). Note that \( S' \) is not empty. Let \( \{E'_1, \ldots, E'_r \} \) be a set of \( \text{étale-chains} \) associated to \( Φ^r \) such that the following conditions: (i) \( S'' = \{E'_1, \ldots, E'_r, S'' \}; \) (ii) \( S' = \{E''_{S''+1}, \ldots, E'_r \}. \) By Theorem 2.10, there exist an elementary abelian \( p\)-group \( A' \), a maximal filtration \( Φ_{A'} \) of \( A' \), and an \( A'\)-covering \( 3' \) over \( Ψ_n \) such that the \( j\)-th \( \text{étale-chain} \) \( E''_{j'} \) associated to the sequence of \( p\)-coverings of \( n\)-semi-graphs with \( p\)-rank induced by \( Φ_{A'} \)

\[
Φ_{E'} : 3' = 3'_r → 3'_r → \cdots → 3'_0 = Ψ_n
\]
is equal to \( E'_j \) for each \( j = 1, \ldots, r \). Then since \( S'' \) is \( \leq r - 1 \), by induction, we have \( σ(3''_{E''}) ≤ p^{r/2} - 1 \). Note that since both \( I_{e_0} \) and \( I_{e_{n+1}} \) are equal to \( A' \), we write \( u'_{1}, (\text{resp. } u''_{1}, u'''_{1}) \) for the unique element of \( β_{\overline{1}}(p_1) \) (resp. \( β_{\overline{1}}^{-1}(p_1), β_{\overline{1}}^{-1}(p_1), β_{\overline{1}}^{-1}(p_1) \)). Then we have
\[
σ(3''_{E''_1}) = p^{r}σ_{3''_{E''_1}}(u''_{1}) + p^{r} - 1 + 1 = p^{r}σ_{3''_{E''_1}}(u''_{1})
\]
and
\[
σ(3''_{E''_1}) = p^{r}σ_{3''_{E''_1}}(u''_{1}) + p^{r} - 1 + 1 = p^{r}σ_{3''_{E''_1}}(u''_{1}).
\]
Thus, we have
\[
σ(3) = σ(3') = p^{r}σ(3''_{E''_1}) - σ_{3''_{E''_1}}(u''_{1}) + σ_{3''_{E''_1}}(u''_{1}) + σ_{3''_{E''_1}}(u''_{1}) + p^{r} - 1 ≤ p^r - 1.
\]
Thus, the theorem follows.

**Case 4:** If \( n \neq 1 \) and \( t \notin \{1, n\} \), we write \( 3[a_2] \) for \( 3_{m+m_4+m_1+m_2+n_1}, 3^{1,t-1}[a_2] \) (resp. \( 3^{t+1,n}[a_2] \)) for the \( (t-1)\)-semi-graph with \( p\)-rank \( (β_{\overline{1}}^{-1}[a_2](P_{1,t-1}), σ_{\overline{1}}[a_2]|_{β^{-1}_{\overline{1}}[a_2](P_{1,t-1})}] \), \( β_{\overline{1}}^{-1}(a_2)(P_{t+1,n}) \) (resp. the \( (n-t)\)-semi-graph with \( p\)-rank \( (β_{\overline{1}}^{-1}(a_2)(P_{t+1,n}), σ_{\overline{1}}[a_2]|_{β^{-1}_{\overline{1}}[a_2](P_{t+1,n})}] \), \( β_{\overline{1}}^{-1}(a_2)(P_{t+1,n}) \)). Similar arguments to the arguments given in the proof of Case 3 imply that
\[
σ(3^{1,t-1}[a_2]) ≤ p^{m_1+m_2+b_1+m_5} - 1
\]
(resp. \( σ(3^{t+1,n}[a_2]) ≤ p^{m_1+b_2+m_5} - 1 \)).

Moreover, by Lemma 2.12, we obtain
\[
σ(3^{1,t-1}[a_2]) ≤ p^{m_1+m_2+b_1+m_5} - p^{m_5+n_1} - m_2
\]
(resp. \( σ(3^{t+1,n}[a_2]) ≤ p^{m_1+b_2+m_5} - p^{m_5+n_1} \)).
Thus, we obtain
\[
\sigma(\mathcal{F}[a_2]) = \sigma(\mathcal{F}^{1,-1}[a_2]) + \sigma(\mathcal{F}^{1,n}[a_2]) + \sum_{v \in \beta^{-1}_{\mathcal{F}[a_2]}(p_1)} \sigma_{\mathcal{F}[a_2]}(v) + p^{m_5}(p^{m_2+n_1}-1+p^{m_1}-1)+p^{m_5}-1
\leq p^{n_1+m_2+b_1+m_5} + p^{m_1+b_2+m_5} - p^{m_5} - 1 + \sum_{v \in \beta^{-1}_{\mathcal{F}[a_2]}(p_1)} \sigma_{\mathcal{F}[a_2]}(v).
\]
Write \(v_1[a_2]\) for the unique element of \(\beta^{-1}_{\mathcal{F}[a_2]}(p_1)\). Note that \(\sigma_{\mathcal{F}[a_2]}(v_1[a_2])\) is equal to 0.

Write \(v_1\) (resp. \(v_t\)) for the unique (resp. an element) element of \(\beta^{-1}_{\mathcal{F}}(p_1)\) (\(\beta^{-1}_{\mathcal{F}}(p_t)\)). We have
\[
\sigma_{\mathcal{F}}(v_1) = -p^{n_1+n_3} + p^{n_1}(p^{n_3} - 1) + p^{n_1+n_3} - 1 + 1 = p^{n_1+n_3} - p^{n_1}
\]
and
\[
\sigma_{\mathcal{F}}(v_t) = -p^{m_1+m_2+m_3+a_1} + p^{a_1+m_2}(p^{m_1+m_3} - 1) + p^{m_1}(p^{a_1+m_2+m_3} - 1) + 1
= p^{m_1+m_2+m_3+a_1} - p^{m_2+a_1} - p^{m_1} + 1.
\]

Since we have
\[
\sigma(\mathcal{F}) - \sigma(\mathcal{F}) - \sum_{v \in \beta^{-1}_{\mathcal{F}[a_2]}(p_1)} \sigma_{\mathcal{F}}(v) = \sigma(\mathcal{F}[a_2]) - \sigma_{\mathcal{F}[a_2]}(v_1[a_2]) - \sum_{v \in \beta^{-1}_{\mathcal{F}[a_2]}(p_1)} \sigma_{\mathcal{F}[a_2]}(v)
\leq p^{n_1+m_2+b_1+m_5} + p^{m_1+b_2+m_5} - p^{m_5} - 1
\]
and \(\beta^{-1}_{\mathcal{F}}(p_t) = p^{m_5}\), we obtain
\[
\sigma(\mathcal{F}) \leq p^{n_1+m_2+b_1+m_5} + p^{m_1+b_2+m_5} - p^{m_5} - 1 + p^{n_1+n_3} - p^{n_1}
+ p^{m_5}(p^{m_1+m_2+m_3+a_1} - p^{m_2+a_1} - p^{m_1} + 1)
= p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3}
- p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1.
\]

By Lemma 4.1 in Appendix, we obtain
\[
\sigma(\mathcal{F}) \leq p^r - 1.
\]
Thus, we complete the proof of the theorem. \(\square\)

3 \( p \)-ranks of vertical fibers of abelian stable coverings

3.1 \( p \)-ranks and stable coverings

**Definition 3.1.** Let \(C\) be a disjoint union of projective curves over an algebraically closed field of characteristic \(p > 0\). We define the \( p \)-rank of \(C\) as follows:
\[
\sigma(C) := \dim_{\mathbb{F}_p} H^1_{\text{et}}(C, \mathbb{F}_p).
\]
**Remark 3.1.1.** Let $C$ be a semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write $\Gamma_C$ for the dual graph of $C$, $v(\Gamma_C)$ for the set of vertices of $\Gamma_C$. Then we have

$$\sigma(C) = \sum_{v \in v(\Gamma_C)} \sigma(\tilde{C}_v) + \text{rank}_\mathbb{Z} H^1(\Gamma_C, \mathbb{Z}),$$

where for $v \in v(\Gamma)$, $\tilde{C}_v$ denotes the normalization of the irreducible component of $C$ corresponding to $v$.

The $p$-rank of a $p$-Galois covering (i.e., the extension of function fields induced by the morphism of curves is a Galois extension, and the Galois group is a $p$-group) of a smooth projective curve can be calculated by the Deuring-Shafarevich formula as follows (cf. [C]):

**Proposition 3.2.** Let $h : C' \to C$ be a Galois covering (possibly ramified) of smooth projective curves over an algebraically closed field of characteristic $p > 0$, whose Galois group is a finite $p$-group $G$. Then we have

$$\sigma(C') - 1 = \sharp G(\sigma(C) - 1) + \sum_{c' \in (C')^{\text{cl}}} (e_{c'} - 1),$$

where $(C')^{\text{cl}}$ denotes the set of closed points of $C'$, $e_{c'}$ denotes the ramification index at $c'$, and $\sharp G$ denotes the order of $G$.

In the following of this subsection, let $R$ be a complete discrete valuation ring with algebraically closed residue field $k$ of characteristic $p > 0$, $K$ the quotient field, and $\overline{K}$ an algebraic closure of $K$. We use the notation $S$ to denote the spectrum of $R$, $\eta, \overline{\eta}$ and $s$ stand for the generic point, the geometric generic point, the closed point corresponding to the natural morphisms $\text{Spec} K \to S$, $\text{Spec} \overline{K} \to S$ and $\text{Spec} k \to S$, respectively. Let $X$ be a semi-stable curve over $S$. Write $X_\eta$, $X_{\overline{\eta}}$ and $X_\eta$ for the generic fiber, the geometric generic fiber and the special fiber, respectively. Moreover, we suppose that $X_\eta$ is smooth over $\eta$ and the genus $g_{X_\overline{\eta}}$ of $X_\overline{\eta}$ is $\geq 2$.

**Definition 3.3.** Let $f : Y \to X$ be a morphism of semi-stable curves over $S$, $G$ a finite group. Then $f$ is called a semi-stable covering (resp. $G$-semi-stable covering) over $S$ if the morphism of generic fibers $f_\eta$ is an étale covering (resp. an étale covering with Galois group $G$), and the following universal property is satisfied: if $g : Z \to X$ is a morphism of semi-stable curves over $S$ such that $Z_\eta = Y_\eta$ and $g_\eta = f_\eta$, then there exists a unique morphism $h : Z \to Y$ such that $f = g \circ h$ (cf. Remark 3.3.1 for the existence of $Y$). We call $f$ a stable covering (resp. $G$-stable covering) over $S$ if $f$ is a semi-stable covering, and $X$ is a stable curve. Note that by the construction of semi-stable coverings in Remark 3.3.1, if $f$ is a stable covering over $S$, then $Y$ is a stable curve over $S$.

**Remark 3.3.1.** Let $W$ be a semi-stable curve over $s$. We shall call a semi-stable subcurve $C \subseteq W$ a chain if all the irreducible components of $C$ are isomorphic to $\mathbb{P}^1$, the dual graph of $C$ is a tree, and for each irreducible component $C_i \subseteq C$, $C_i$ meets the other irreducible components of $W$ at at most two points.

Let $f_\eta : Y_\eta \to X_\eta$ be an étale covering. Suppose that $Y_\eta$ admits a semi-stable reduction over $S$. Write $Y'$ for the normalization of $X$ in the function field $K(Y)$, $Y'$. 

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for the unique minimal desingularization over \( S \) (cf. [L1, Proposition 9.3.32]) which is a semi-stable curve over \( S \). Then \( Y' \) (resp. \( Y' \)) admits an \( G \)-action induced by the action of \( G \) on \( Y_\eta \). We denote by \( f^1 : Y^1 \rightarrow X \) the composite of \( Y^1 \rightarrow Y' \) and the normalization morphism \( Y' \rightarrow X \). Write \( C^1_X \) for the set of the maximal elements (under the relationship “\( \subseteq \)”) of

\[
\{ C \text{ a chain of the special fiber } Y^1_s \text{ of } Y^1 \mid f^1(C) \text{ is a closed point of } X_s \}\text{.}
\]

Contracting \( C^1_X \), we obtain a semi-stable curve \( Y^2 \) over \( S \) (cf. [L1, Lemma 10. 3.31]). Moreover, we have a natural morphism \( f^2 : Y^2 \rightarrow X \) induced by \( f^1 \). Write \( C^2_X \) for the set of the maximal elements (under the relationship “\( \subseteq \)”) of

\[
\{ C \text{ a chain of the special fiber } Y^2_s \text{ of } Y^2 \mid f^2(C) \text{ is a closed point of } X_s \}\text{.}
\]

Contracting \( C^2_X \), we obtain a semi-stable curve \( Y^3 \) over \( S \) (cf. [L1, Lemma 10. 3.31]). Moreover, we have a natural morphism \( f^3 : Y^3 \rightarrow X \) induced by \( f^2 \). Repeating the process above, we obtain a contracting morphism \( c_Y : Y^1 \rightarrow Y \), and \( f_\eta \) extends to a morphism \( f : Y \rightarrow X \) over \( S \).

Next, let us prove that \( Y \) satisfies the universal property defined in Definition 3.3. Let \( Z \) be a semi-stable curve over \( S \) and \( g : Z \rightarrow X \) a morphism of semi-stable curves over \( S \) such that \( g_\eta = f_\eta \). If \( Z \) is regular, since \( Y^1 \) is the minimal desingularization over \( S \), we obtain a morphism \( Z \rightarrow Y^1 \). Thus, we have \( g \) factors through \( f \). If \( Z \) is not regular, write \( Z^{\text{reg}} \) for the minimal desingularization of \( Z \) over \( S \). Then we obtain a commutative diagram as follows:

\[
\begin{array}{ccc}
Z^{\text{reg}} & \xrightarrow{b} & Y^1 \\
\downarrow{r} & & \downarrow{}
\end{array}
\]

Write \( C_{Z^{\text{reg}}} \) for the set of \((-1)\)-curves of \( Z^{\text{reg}} \) whose images under the morphism \( b \) are closed points of \( Y^1_s \). Contracting \( r(C_{Z^{\text{reg}}}) \), we obtain a semi-stable curve \( Z' \) over \( S \), a morphism \( Y^1 \rightarrow Z' \), and the following commutative diagram:

\[
\begin{array}{ccc}
Z^{\text{reg}} & \xrightarrow{h} & Y^1 \\
\downarrow{r} & & \downarrow{r'} \\
Z & \xrightarrow{c_Z} & Z'.
\end{array}
\]

Write \( V_{c_Y} \) (resp. \( V_{r'} \)) for the set of irreducible components of \( Y^1_s \) such that for each element \( E \in V_{c_Y} \) (resp. \( E \in V_{r'} \)), \( c_Y(E) \) (resp. \( r'(E) \)) is a closed point of \( Y_s \) (resp. the special fiber \( Z'_s \) of \( Z' \)). By the constructions of \( Y \) and \( Z' \), we have \( V_{r'} \subseteq V_{c_Y} \). Then there is contracting morphism \( Z' \rightarrow Y \), and the following commutative diagram holds:

\[
\begin{array}{ccc}
Y^1 & \xrightarrow{c_Y} & Y^1 \\
\downarrow{r'} & & \downarrow{c_{Z'}} \\
Z' & \xrightarrow{c_{Z'}} & Y.
\end{array}
\]

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Then $g$ factors through $f$. Note that the uniqueness of contracting implies that the uniqueness of the morphism $h := c_{g'} \circ c_Z : Z \rightarrow Y$.

Note that if $f : Y \rightarrow X$ is a finite morphism of semi-stable curves over $S$, and the morphism of generic fibers $f_\eta$ is étale, then $f$ is a semi-stable covering.

**Definition 3.4.** Let $f : Y \rightarrow X$ be a semi-stable covering over $S$. Suppose that the morphism of special fibers $f_s : Y_s \rightarrow X_s$ is not finite. A closed point $x \in X$ is called a *vertical point associated to $f$*, or for simplicity, a *vertical point* when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the *vertical fiber associated to $x$*.

If a vertical point $x$ is nonsingular, the following result was proved by Raynaud (cf. [R, Théorème 1 and Proposition 1]).

**Proposition 3.5.** Let $G$ be a finite $p$-group, $f : Y \rightarrow X$ a $G$-semi-stable covering and $x$ a vertical point associated to $f$. If $x$ is a smooth point of $X_s$, then the $p$-rank of each connected component of the vertical fiber $f^{-1}(x)$ is equal to 0. On the other hand, by contracting the vertical fibers $f^{-1}(x)$, we obtain a curve $Y_c$ over $S$. Write $c : Y \rightarrow Y_c$ for the contracting morphism. Then the points $c(f^{-1}(x))$ are geometrically unibranch.

**Proposition 3.6.** Let $G$ be a finite group, $f : Y \rightarrow X$ a $G$-semi-stable covering, and $x$ a vertical point associated to $f$. If $x$ is a smooth point or a node which is contained in only one irreducible component (resp. a node which is contained in two different irreducible components), we use the notation $X_v$ (resp. $X_{v_1}$ and $X_{v_2}$) to denote the irreducible component which contains $x$ (resp. the irreducible components which contain $x$). Write $\psi : Y' \rightarrow X$ for the normalization of $X$ in the function field of $Y$. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of $x$, $Y'_v$ (resp. $Y'_{v_1}$ and $Y'_{v_2}$) for an irreducible component (resp. two irreducible components) of $Y'_v$ such that $\psi_s(Y'_v) = X_v$ and $y' \in Y'_v$ (resp. (i) $\psi_s(Y'_{v_1}) = X_{v_1}$ and $\psi_s(Y'_{v_2}) = X_{v_2}$; (ii) $y' \in Y'_{v_1}$ and $y' \in Y'_{v_2}$). Write $I_v \subseteq G$ (resp. $I_{v_1} \subseteq G$ and $I_{v_2} \subseteq G$) for the inertia subgroup of $Y'_v$ (resp. the inertia subgroups of $Y'_{v_1}$ and $Y'_{v_2}$, respectively).

Suppose that $G$ is a $p$-group (resp. an abelian group). Then we have $I_v \neq \{1\}$ (resp. $I_{v_1} \neq \{1\}$ or $I_{v_2} \neq \{1\}$). Moreover, write $I_{y'} \subseteq G$ for the inertia subgroup of $y'$, then $I_{y'}$ is equal to $I_v$ (resp. $I_{y'}$ is generated by $I_{v_1}$ and $I_{v_2}$).

**Proof.** Since $Y$ is normal, we obtain a natural morphism $\phi : Y \rightarrow Y'$. By using [BLR, 6.7 Proposition 4], we may contract the connected component of $f_s^{-1}(x)$ whose image under the morphism $\phi$ is $y'$. Thus, we obtain a contraction morphism $c : Y \rightarrow Y''$. Since $Y''$ is a blowing-up of $Y'$, $Y''$ is a fiber surface over $S$ (i.e., normal and flat over $S$) and there is natural commutative diagram as follows:

$$
\begin{array}{ccc}
Y_\eta & \longrightarrow & Y \\
\downarrow c_\eta & & \downarrow c \\
Y''_\eta & \longrightarrow & Y'' \\
\downarrow f''_\eta & & \downarrow f'' \\
X_\eta & \longrightarrow & X
\end{array}
$$
where $c_v$ is an identity morphism.

Write $Y''$ (resp. $Y''_1$ and $Y''_2$) for the unique irreducible component whose image under the natural morphism $Y'' \to Y'$ is $Y'_i$ (resp. $Y'_v$, $Y'_v'$), $y''$ for the image $c(\phi^{-1}(y'))$. Note that the inertia group of $Y''_v$ (resp. $Y''_v'$) is equal to $I_v$ (resp. $I_{v_1}, I_{v_2}$).

If $x$ is a smooth point, $G$ is a $p$-group, and $I_v$ is trivial, then $f''|_{Y''}$ is generically étale. By Proposition 3.5, we have $y''$ is geometrically unibranch. Thus, $y''$ is contained in only one irreducible component of $Y''$. By applying Zariski-Nagata purity, we have $f''|_{Y''}$ is étale at $y''$. Thus, $y''$ is a smooth point. Then $Y''$ is a semi-stable curve. This contradicts to the minimal properties of semi-stable coverings.

If $x$ is a node and $I_v$ (resp. $I_{v_1}$ and $I_{v_2}$) is (resp. are) trivial, since $G$ is abelian, $f''$ is étale over an open neighborhood of $x$. The completion of the local ring at $x$ is $\hat{\mathcal{O}}_{X,x} \cong \mathbb{R}[[u,v]]/(uv - \pi^{p^e})$, where $\pi$ denotes an uniformizer of $\mathcal{O}$ and $(n', p) = 1$. Since the étale fundamental group of $\text{Spec} \hat{\mathcal{O}}_{X,x} \setminus \{x\}$ is isomorphic to $\mathbb{Z}/n'\mathbb{Z}$ (cf. [T, Lemma 2.1 (iii)]), where $\hat{x}$ denotes the closed point of $\text{Spec} \hat{\mathcal{O}}_{X,x}$, we have $y''$ is a node. Then $Y''$ is a semi-stable model of $Y''_v$ over $S$ in either case, so that this contradicts to the minimal properties of semi-stable coverings. Thus, $I_v \neq \{1\}$ (resp. $I_{v_1} \neq \{1\}$ or $I_{v_2} \neq \{1\}$). This completes the proof of the proposition.

### 3.2 Semi-graphs with $p$-rank associated to vertical fibers

In this subsection, we construct a semi-graph with $p$-rank defined in Section 1 from a vertical fiber, and we apply the theory developed in Section 1 to calculate the bound of the $p$-rank of the vertical fiber.

First, we fix some notations. Let $G$ be a finite $p$-group, $f : Y \to X$ a $G$-stable covering over $S$, $x \in X_s$ a vertical point. Suppose that $x$ is a node contained in two irreducible components $X_1$ and $X_2$ (which may be equal) of $X_s$. Write $\psi : Y' \to X$ for the normalization of $X$ in the function field of $Y$. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of $x$. Write $I_{y'}$ for the inertia group of $y'$. Note that the natural morphism $Y/I_{y'} \to X$ induced by $f$ is finite étale over $x$. Thus, by replacing $X$ by the stable model of $Y/I_{y'}$, in order to calculate the $p$-rank of the vertical fiber $f^{-1}(x)$, we may assume that $I_{y'}$ is equal to $G$. From now on, we may assume that $G = I_{y'}$ is a $p$-group with order $p^r$. Then $f^{-1}(x)$ is connected.

Let $X^{\text{st}}_s$ be the quotient of $Y$ by $G$. By [R, Appendice, Corollaire], $X^{\text{st}}$ is a semi-stable curve with generic fiber $X_y$. Then we obtain a quotient morphism $h : Y \to X^{\text{st}}$ and a birational morphism $g : X^{\text{st}} \to X$ such that the composite morphism $g \circ h$ is equal to $f$. We still write $X_1$ and $X_2$ for the strict transforms of $X_1$ and $X_2$ under the birational morphism $g$, respectively. By the general theory of semi-stable curves, $g^{-1}(x)$ is a semi-stable subcurve of $X^{\text{st}}$ whose irreducible components are isomorphic to $\mathbb{P}^1_k$. Write $C$ for the semi-stable subcurve of $g^{-1}(x)$ which is a chain of projective lines $\cup_{i=1}^{p^r} P_i$ such that the following conditions: (i) $P_i$ is not equal to $P_j$ if $i \neq j$; (ii) $P_1 \cap X_1$ are $P_n \cap X_2$ are not empty; (iii) $P_i$ meets $P_{i+1}$ at only one point; (iv) $P_i \cap P_j$ is empty if $j$ is not equal to $i - 1, i$ or $i + 1$. Then we have

$$g^{-1}(x) = C \cup B,$$
where $B$ denotes the topological closure of $g^{-1}(x) \setminus C$ in $g^{-1}(x)$. Write $B_i$ for the union of the connected components of $B$ which intersect with $P_i$ are not empty.

**Lemma 3.7.** Let $V_i$ be an irreducible component of $h^{-1}(P_i)$, $I_{V_i} \subseteq G$ (resp. $D_{V_i} \subseteq G$) the inertia group (the decomposition group) of $V_i$, and $D_i$ for the image of $V_i$ under the quotient morphism $Y \to Y/I_{V_i}$. Write $h_i$ for the natural morphism $Y/I_{V_i} \to X_{\text{sst}}$. Then the branch locus of $h_i|_{D_i} : D_i \to P_i$ are contained in $P_i \cap (P_i+1 \cup P_i-1)$.

**Proof.** Write $E_i$ for the image of $D_i$ under the natural morphism $Y/I_{V_i} \to Y/D_{V_i}$. We have the restriction of $Y/D_{V_i} \to X_{\text{sst}}$ to $E_i$ is an identity morphism. Thus, by replacing $X_{\text{sst}}$ by $Y/D_{V_i}$, we may assume that $D_{V_i}$ is equal to $G$. Then $h_i$ is a $G/I_{V_i}$-semi-stable covering. Note that it is easy to see that the branch locus of $h_i|_{D_i}$ are contained in $P_i \cap (P_i+1 \cup P_i-1 \cup B_i)$.

By contracting $B_i$ (resp. $h_i^{-1}(B_i)$), we obtain a semi-stable curve $(X_{\text{sst}})'$ and a contraction morphism $c_{X_{\text{sst}}} : X_{\text{sst}} \to (X_{\text{sst}})'$ (resp. a fiber surface $(Y/I_{V_i})'$ and a contraction morphism $c_{Y/I_{V_i}} : Y/I_{V_i} \to (Y/I_{V_i})'$) over $S$. Moreover, $h_i$ induces a morphism $h_i' : (Y/I_{V_i})' \to (X_{\text{sst}})'$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
Y/I_{V_i} & \xrightarrow{c_{Y/I_{V_i}}} & (Y/I_{V_i})' \\
\downarrow{h_i} & & \downarrow{h_i'} \\
X_{\text{sst}} & \xrightarrow{c_{X_{\text{sst}}}} & (X_{\text{sst}})'.
\end{array}
$$

Since it follows from Proposition 3.5, $(h_i')^{-1}(c_{X_{\text{sst}}}(P_i \cap B)) \cap c_{Y/I_{V_i}}(D_i)$ are geometrically unibranch, $(h_i')^{-1}(c_{X_{\text{sst}}}(P_i \cap B))$ only are contained in one irreducible component of the special fiber of $(Y/I_{V_i})'$. Moreover, by applying Zariski-Nagata purity to $h_i$, $h_i'((h_i')^{-1}(c_{X_{\text{sst}}}(P_i))$ is contained in the étale locus of $h_i$. Thus, the set of branch points of $h_i'((h_i')^{-1}(c_{X_{\text{sst}}}(P_i))$ is contained in the set $c_{X_{\text{sst}}}(P_i \cap (P_i+1 \cup P_i-1))$. Moreover, $c_{Y/I_{V_i}}|_{D_i}$ is an isomorphism. Then we complete the proof of the lemma.\hfill\square

Next, we construct a semi-graph with $p$-rank from a vertical fiber. From now on, we assume that $G$ is an abelian $p$-group. Write $D_C$ for the set of points $C \cap (X_1 \cup X_2)$. Thus, we may regard $C := (C, D_C)$ as a pointed semi-stable curve over $s$. Write $\mathbb{P}_n$ for the dual graph associated to $C$, $\sigma_{\mathbb{P}_n}$ for the map satisfying the property $\sigma_{\mathbb{P}_n}(p_i) = \sigma(P_i)$. Then $\mathbb{P}_n := (\mathbb{P}_n, \sigma_{\mathbb{P}_n}, \text{id}_{\mathbb{P}_n})$ is a $n$-chain defined in Section 1.

Let

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset G_{r-2} \subset \cdots \subset G_1 \subset G_0 = G,$$

be a filtration of $G$ such that $G_j/G_{j+1} \cong \mathbb{Z}/p\mathbb{Z}$, $j = 0, \ldots, r - 1$. The filtration $\Phi$ induces a sequence of semi-stable coverings $\Phi_f$ as follows:

$$Y = Y_r \xrightarrow{d_r} Y_{r-1} \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_1} Y_0 = X_{\text{sst}},$$

where $Y_i, i = 0, \ldots, r$, denotes the semi-stable curve $Y/G_i$ over $S$.

For each $i = 0, \ldots, r$, write $\Gamma_i$ for the dual graph of the special fiber of $Y_i$. First, let us prove that the map $\beta_i : \Gamma_i \to \Gamma_{i-1}, 1 \leq i \leq r$, induced by $d_i$ is a morphism of semi-graphs. To verify $\beta_i$ is a morphism of semi-graphs, it is sufficient to prove that $\beta_i(e(\Gamma_i)) \subseteq e(\Gamma_{i-1})$,
where $e(-)$ denotes the set of edges of $(-)$. Let $y_i$ be a node of the special fiber $(Y_i)_s$ of $Y_i$. Write $Y^1_i$ and $Y^2_i$ for the irreducible components of $(Y_i)_s$, which contain $y_i$. $I_{Y^1_i} \subseteq G_{i-1}/G_i$ (resp. $I_{Y^2_i} \subseteq G_{i-1}/G_i$, $I_{y_i} \subseteq G_{i-1}/G_i$) for the inertia group of $Y^1_i$ (resp. $Y^2_i$, $y_i$). Write $I \subseteq G_{i-1}/G_i$ for the group generated by $I_{Y^1_i}$ and $I_{Y^2_i}$, $q_y$ for the quotient morphism $Y_i \to Y/I$. By the definitions, we obtain $I \subseteq I_y$. Applying Zariski-Nagata purity to Spec $\mathcal{O}_{Y/I,q_y}(y_i) \to \text{Spec} \mathcal{O}_{Y_i,d_i(y_i)}$, we have the morphism $Y/I \to Y_{i-1}$ induced by $d_i$ is étale at $q_y(y_i)$. This implies that $I = I_y$. Since for any element $\tau \in I$, we have $\tau(Y^1_i) = Y^1_i$ and $\tau(Y^2_i) = Y^2_i$, the proof of [R, Appendice, Proposition 5] (or [L1, Proposition 10.3.48]) implies that $q_y(y_i)$ is a node of $(Y/I)_s$. Thus, $d_i(y_i)$ is a node of the special fiber $(Y_{i-1})_s$ of $Y_{i-1}$. This means that $\beta_i$ is a morphism of semi-graphs.

Write $\phi_i, i = 1, \ldots, r$, for the composite morphism $d_1 \circ d_2 \circ \cdots \circ d_i$. Note that we have $h = \phi_r$. The semi-stable subcurve $\phi_i^{-1}(C) \cap \phi_j^{-1}(D_C)$ may be regarded as a pointed semi-stable curve over $s$. We use the notation $\mathcal{Y}_i$ to denote the resulting pointed semi-stable curve $(\phi_i^{-1}(C), \phi_i^{-1}(D_C))$. Write $\mathcal{G}_i$ for the dual graph of $\mathcal{Y}_i$, $\beta_{\mathcal{G}_i}$ for the natural morphism $\mathcal{G}_i \to \mathbb{P}_n$ induced by the morphism $\phi_i|_{\mathcal{Y}_i} : \mathcal{Y}_i \to \mathbb{C}$. For each $v \in v(\mathcal{G}_i)$, write $(\mathcal{Y}_i)_v$ for the irreducible component of $\mathcal{Y}_i$ corresponding to $v$. We define $\sigma_v$ to be the map satisfying the property $\sigma_v(v) = \sigma((\mathcal{Y}_i)_v)$ for all $v \in v(\mathcal{G}_i)$. Then $\mathcal{G}_i := (\mathcal{G}_i, \sigma_v, \beta_{\mathcal{G}_i})$ is a $n$-graph with $p$-rank. Moreover, $d_i|_{\mathcal{Y}_i}$ induces a natural morphism of semi-graphs with $p$-rank $b_i : \mathcal{G}_i \to \mathcal{G}_{i-1}$, and $\mathcal{G}$ admits a natural action of $G$ induced by the action of $G$ on $\mathcal{Y}_n$. Furthermore, $\Phi$ induces a sequence of morphisms of semi-graphs with $p$-rank

$$\Phi_\psi : \mathcal{G} := \mathcal{G}_r \to \mathcal{G}_{r-1} \to \cdots \to \mathcal{G}_1 \to \mathcal{G}_0 = \mathbb{P}_n.$$  

On the other hand, by Lemma 3.7 and Zariski-Nagata purity, it is easy to check that for each $i = 1, \ldots, r$, $b_i$ is a $p$-covering. Thus, $\mathcal{G}$ is a $G$-covering over $\mathbb{P}_n$. For each $i = 1, \ldots, r$, we write $E_i^{\Phi_\psi}$ for the $i$-th étale-chain associated to $\Phi_\psi$.

On the other hand, write $\{Y_i\}_j$ for the set of connected components contained in the étale locus of $d_i$ such that the image $\phi_j(Y_i)$ are contained in $g^{-1}(x)$ for all $j$, $y_i^{et}$ for the disjoint union $\bigsqcup_j Y_i^j$. Note that $\phi_j(Y_i^{et}) \setminus B$ is a disjoint union of semi-stable subcurve of $C$. For each connected component $E$ of $\phi_j(Y_i^{et}) \setminus B$, with the set of closed points $D_E := E \cap C \setminus E$, we may regard $E := (E, D_E)$ as a pointed semi-stable subcurve of $C$ over $s$. We define $E_i^{\Phi_\psi}$ as the disjoint union

$$\bigsqcup_{E \subseteq \phi_j(Y_i^{et}) \setminus B} E.$$  

We shall call $E_i^{\Phi_\psi}$ the $i$-th étale-chain associated to $\Phi_\psi$, and write $E_i$ for the disjoint union of the dual graph of the connected components of $E_i^{\Phi_\psi}$. We define $\check{E}_i^{\Phi_\psi}$ as the disjoint union

$$\bigsqcup_i \check{E}_i^{\Phi_\psi},$$  

and call $\check{E}_i^{\Phi_\psi}$ the étale-chain associated to $\Phi_\psi$. From the construction of $E_i$, it is easy to see that $E_i$ are equal to $E_i^{\Phi_\psi}$ for all $i$.

Note that $C \cap B$ are smooth points of $C$. By Proposition 3.5, we have the $p$-ranks of the connected components of $h^{-1}(B)$ are equal to 0. Thus, we have $\sigma(f^{-1}(x)) = \sigma(\phi_r^{-1}(C))$. Moreover, by applying Lemma 3.7, we obtain $\sigma(\phi_r^{-1}(C)) = \sigma(\mathcal{G})$.
Summarizing the discussion, we obtain the following proposition.

**Proposition 3.8.** Let $G$ be a finite abelian $p$-group with order $p^r$, $f : Y \rightarrow X$ a $G$-stable covering over $S$, $x \in X_S$ a vertical point. Write $\psi : Y' \rightarrow X$ for the normalization of $X$ in the function field of $Y$. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of $x$. Write $I_{y'}$ for the inertia group of $y'$. Suppose that $G = I_{y'}$. Let $\Phi$ be a maximal filtration of $G$. Write $\Phi_f$ for the sequence of semi-stable curves induced by $\Phi$ which was constructed in this subsection, $\rho_i^{\Phi_f}$ for the $i$-th étale-chain associated to $\Phi_f$ for each $i$. Then there exist a semi-graph with $p$-rank $\mathfrak{G}$ and a sequence of $p$-coverings of semi-graphs with $p$-rank $\Phi_{\mathfrak{G}}$ induced by $\Phi$ which was constructed in this subsection such that $\mathfrak{G}$ is a $G$-covering over $\mathfrak{P}_n$, and for each $i = 1, \ldots, r$, the $i$-th étale-chain $E_i^{\Phi_{\mathfrak{G}}}$ associated to $\Phi_{\mathfrak{G}}$ is equal to the dual graph of $\rho_i^{\Phi_f}$. Furthermore, we have $\sigma(f^{-1}(x)) = \sigma(\mathfrak{G})$.

### 3.3 $p$-ranks of vertical fibers

We follow the notations of Section 3.2. Let $\{Z_i\}_{i=0}^{n+1}$ a subset the set of irreducible components of the special fiber $Y_s$ of $Y$ such that the following conditions hold: (i) $\phi_i(Z_i) = P_i$ if $i \not\in \{0, n+1\}$; (ii) $\phi_r(Z_0) = X_1$ and $\phi_r(Z_{n+1}) = X_2$; (iii) the union $\bigcup_{i=0}^{n+1} Z_i$ is a connected semi-stable subcurve of the special fiber $Y_s$ of $Y$. Write $P_i \subset G$ for the inertia subgroup of $Z_i$. Note that since $G$ is an abelian $p$-group, $P_i$ does not depend on the choice of $Z_i$.

By using the theory of étale-chains, we obtain an explicit formula of $p$-rank of $f^{-1}(x)$ as follows:

**Theorem 3.9.** If $G$ is an abelian $p$-group, then we have

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \sharp(G/I_{P_i}) - \sum_{i=1}^{n+1} \sharp(G/(I_{P_{i-1}} + I_{P_i})) + 1.$$  

**Proof.** We follow the notations of Theorem 2.8. Note that by Zariski-Nagata purity, we have the inertia group of a point of $Z_{i-1} \cap Z_i$ (resp. $Z_i \cap Z_{i+1}$) is equal to $I_{P_{i-1}} + I_{P_i}$ (resp. $I_{P_i} + I_{P_{i+1}}$). Then we have $\sharp E^{\Phi_{\mathfrak{G}}}(P_i) = \log_p(\sharp G/I_{P_i})$ (resp. $\sharp E^{\Phi_{\mathfrak{G}}}(b_{P_i}) = \log_p(\sharp G/(I_{P_{i-1}} + I_{P_i}))$), $\sharp E^{\Phi_{\mathfrak{G}}}(b_{P_i}) = \log_p(\sharp G/(I_{P_i} + I_{P_{i+1}}))$. Thus, we have

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} (\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i}))) + 1 + \sum_{i=1}^{n+1} (\sharp(G/(I_{P_{i-1}} + I_{P_i})))$$

$$= \sum_{i=1}^{n} \sharp(G/I_{P_i}) - \sum_{i=1}^{n+1} \sharp(G/(I_{P_{i-1}} + I_{P_i}))) + 1.$$  

This completes the proof of the theorem. \qed

**Remark 3.9.1.** If $G$ is a cyclic $p$-group, since $G$ is generated by $P_0$ and $P_{n+1}$, we may assume that $I_{P_n} = G$. Follows Lemma 3.10 below, there exists $u$ such that

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_u} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$
Then we obtain
\[ \sharp(G/I_{P_1}) - \sharp(G/(I_{P_1} + I_{P_2})) - \sharp(G/(I_{P_1} + I_{P_2} + I_{P_3})) + 1 = -\sharp(G/(I_{P_1})) + 1 \]
(resp. \( \sharp(G/(I_{P_u} + I_{P_{u+1}})) - \sharp(G/(I_{P_{u+1}})) + 1 = -\sharp(G/(I_{P_{u+1}})) + 1 \))
if \( i < u \),
\[ \sharp(G/I_{P_1}) - \sharp(G/(I_{P_1} + I_{P_2})) - \sharp(G/(I_{P_1} + I_{P_2} + I_{P_3})) + 1 = -\sharp(G/(I_{P_{u+1}})) + 1 \]
(resp. \( \sharp(G/(I_{P_{u+1}} + I_{P_{u+2}})) - \sharp(G/I_{P_{u+1}})) + 1 = -\sharp(G/(I_{P_{u+1}})) + 1 \))
if \( i > u \) and
\[ \sharp(G/I_{P_1}) - \sharp(G/(I_{P_1} + I_{P_2})) - \sharp(G/(I_{P_1} + I_{P_2} + I_{P_3})) + 1 = -\sharp(G/(I_{P_{u+1}})) + 1 \]
(resp. \( \sharp(G/I_{P_{u+1}} + I_{P_{u+2}})) - \sharp(G/I_{P_{u+1}})) + 1 = -\sharp(G/(I_{P_{u+1}})) + 1 \))
if \( i = u \). Thus, by applying Theorem 3.9, we obtain
\[ \sigma(f^{-1}(x)) = \sharp(G/I_{P_1}) - \sharp(G/I_{P_{u+1}}). \]
This formula was first obtained by Saüdi (cf. [S, Proposition 1]).

**Lemma 3.10.** If \( G \cong \mathbb{Z}/p^n\mathbb{Z} \) is a cyclic group, then there exists \( 0 \leq u \leq n + 1 \) such that
\[ I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_u} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}. \]
In particular, \( \sharp \pi_0(\mathcal{E}_s^{\Phi^I}) \leq 1 \) hold for all \( i \), where \( \sharp \pi_0(\cdot) \) denotes the cardinality of the connected components of \( \cdot \).

*Proof.* If the lemma is not true, there exist \( s, t \) and \( v \) such that \( I_{P_s} \neq I_{P_t}, I_{P_t} \neq I_{P_i} \) and \( I_{P_r} \subseteq I_{P_{r+1}} = \cdots = I_{P_s} = \cdots = I_{P_u-1} \supsetneq I_{P_s} \). Since \( G \) is a cyclic group, we may assume \( I_{P_r} \supseteq I_{P_s} \).

Considering the quotient of \( Y \) by \( I_{P_s} \), we obtain a natural morphism of semi-stable curves \( h_s : Y/I_{P_s} \longrightarrow X^{\text{sst}} \) over \( S \). By contacting \( P_{s+1}, P_{s+2}, \ldots, P_{t-1}, B_{s+1}, \ldots, B_{t-1} \) (resp. \( h_s^{-1}(P_{s+1}), h_s^{-1}(P_{s+2}), \ldots, h_s^{-1}(P_{t-1}), h_s^{-1}(B_{s+1}), \ldots, h_s^{-1}(B_{t-1}) \)), we obtain a semi-stable curve \( (X^{\text{sst}})^{\prime} \) (resp. a fiber surface \( (Y/I_{P_s})^{\prime} \)) and a contacting morphism \( c_{X^{\text{sst}}} : X^{\text{sst}} \longrightarrow (X^{\text{sst}})^{\prime} \) (resp. \( c_{Y/I_{P_s}} : Y/I_{P_s} \longrightarrow (Y/I_{P_s})^{\prime} \)). The morphism \( h_s \) induces a morphism of fiber surfaces \( h_s^{\prime} : (Y/I_{P_s})^{\prime} \longrightarrow (X^{\text{sst}})^{\prime} \). Then we have the following commutative diagram as follows:

\[
\begin{array}{ccc}
Y/I_{P_s} & \xrightarrow{c_{Y/I_{P_s}}} & (Y/I_{P_s})^{\prime} \\
\downarrow{h_s} & & \downarrow{h_s^{\prime}} \\
X^{\text{sst}} & \xrightarrow{c_{X^{\text{sst}}}} & (X^{\text{sst}})^{\prime}.
\end{array}
\]

Write \( P_s' \) and \( P_s'' \) for the images \( c_{X^{\text{sst}}}(P_s) \) and \( c_{X^{\text{sst}}}(P_i) \), respectively, and \( x_{s,t}' \) for the closed point \( P_s' \cap P_t' \in (X^{\text{sst}})^{\prime} \). By Proposition 3.6, we have \( (Y/I_{P_s})^{\prime} \) is a semi-stable curve over \( S \), moreover, we have \( h_s^{\prime} \) is étale over \( x_{s,t}' \). Then the inertia groups of the closed points \((h_s^{\prime})^{-1}(x_{s,t}^{\prime}) \) of the special fiber \( (Y/I_{P_s})_s \) of \( (Y/I_{P_s})^{\prime} \) are trivial.
On the other hand, since \( I_{P_i} \) is a proper subgroup of \( I_{P_i} \), we obtain the natural action of \( G/I_{P_i} \) on the irreducible components of \( h_{s}^{-1}(\cup_{j=s+1}^{t-1} P_j) \) is trivial. Thus, the inertia groups of the closed points \( c_{Y/I_{P_i}}(h_{s}^{-1}(\cup_{j=s+1}^{t-1} P_j)) = (h_{s}^{-1}(x_{s}')) \) of the special fiber \( (Y/I_{P_i})' \) of \( (Y/I_{P_i})' \) are not trivial. This is a contradiction. Then we complete the proof of the lemma.

On the other hand, we obtain a bound of \( \sigma(f^{-1}(x)) \).

**Theorem 3.11.** If \( G \) is an abelian \( p \)-group with order \( p^r \), and \( \mathcal{E}_i \) is connected for each \( i = 1, \ldots, n \), then we have \( \sigma(f^{-1}(x)) \leq p^r - 1 \).

**Proof.** Together with Theorem 2.13 and Proposition 3.8, the theorem follows.

### 4 Appendix

In this appendix, we prove the following elementary lemma which is used in the proof of Theorem 2.13.

**Lemma 4.1.** Following the notations of the proof of Theorem 2.13, then we have

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1 \\
\leq p^r - 1.
\]

**Proof.** We will check this inequality case by case. We denote by \( M \) the maximal number

\[
\max\{n_1 + m_2 + b_1 + m_5, m_1 + m_2 + m_3 + a_1 + m_5, m_1 + m_5 + b_2, n_1 + n_3\}.
\]

If \( M = r \), we have the following cases.

If \( n_1 + m_2 + b_1 + m_5 = r \), then we have \( n_2 = n_3 = b_2 = m_1 = m_3 = 0, m_4 = b_1 \) and \( n_4 = m_2 + b_1 + m_5 \). Thus, we obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1 = p^r + p^{m_5+a_1} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_5} - p^{n_1} - 1 = p^r - 1.
\]

If \( m_1 + m_2 + m_3 + a_1 + m_5 = r \), then we have \( n_1 = a_1 \) and \( n_2 = n_3 = m_4 = b_1 = b_2 = 0 \). Thus, we obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1 = p^{n_1+m_2+m_5} + p^r + p^{a_1} - p^{a_1+m_2+m_5} - p^{m_1+m_5} - p^{a_1} - 1 = p^r - 1.
\]

If \( m_5 + b_2 + m_2 = r \), then we have \( n_1 = a_1 = a_2 = m_1 = m_3 = n_3 = b_1 = 0 \) and \( m_4 = b_2 \). Thus, we obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1 = p^{m_2+m_5} + p^{m_5} + p^r + 1 - p^{m_5+m_2} - p^{m_5} - 1 - 1 = p^r - 1.
\]
If \( n_1 + n_3 = r \), then we have \( m_1 = m_2 = m_3 = m_4 = m_5 = b_1 = b_2 = n_4 = n_2 = 0 \). Thus, we obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]

\[
= p^{n_1} + p^{a_1} + 1 + p^r - p^{a_1} - 1 - p^{n_1} - 1 = p^r - 1.
\]

Thus, it is sufficient to assume that \( M \leq r - 1 \).

If \( M \leq r - 2 \), then we have

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1 \leq 4p^{r-2} - 4.
\]

Since \( p \) is a prime number, we have \( p^r - 1 - 4p^{r-2} + 4 > 0 \). Thus, we obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1 \leq p^r - 1.
\]

Thus, we may assume that \( M = r - 1 \).

If \( n_1 + m_2 + b_1 + m_5 = r - 1 \), we obtain \( n_2 + n_3 + m_1 + m_3 + b_2 = 1 \). If \( n_2 = 1 \), then we have \( n_3 = m_1 = m_3 = b_2 = 0 \). We obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]

\[
= p^{r-1} + p^{m_2+a_1+m_5} + p^{m_1+m_5} + p^{n_1} - p^{m_2+a_1+m_5} - p^{m_5} - p^{n_1} - 1
\]

\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( n_3 = 1 \), then we have \( n_2 = m_1 = m_3 = b_2 = 0 \). We obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]

\[
= p^{r-1} + p^{m_2+a_1+m_5} + p^{m_5} + p^{n_1+1} - p^{m_2+m_5+a_1} - p^{m_5} - p^{n_1} - 1
\]

\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( m_1 = 1 \), then we have \( n_2 = n_3 = m_3 = b_2 = 0 \). We obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]

\[
= p^{r-1} + p^{m_1+m_2+a_1+m_5} + p^{m_5+m_1} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_5+m_1} - p^{n_1} - 1
\]

\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( m_3 = 1 \), then we have \( n_2 = n_3 = m_1 = b_2 = 0 \). We obtain

\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]

\[
= p^{r-1} + p^{m_3+m_2+a_1+m_5} + p^{m_5} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_5} - p^{n_1} - 1
\]

\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]
If $b_2 = 1$, then we have $n_2 = n_3 = m_1 = m_3 = 0$. We obtain
\[
p^{n_1 + m_2 + b_1 + m_5} + p^{m_3 + m_2 + m_3 + a_1 + m_5} + p^{m_5 + b_2 + m_1} + p^{n_1 + n_3} - p^{m_5 + m_2 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{r-1} + p^{m_2 + a_1 + m_5} + p^{m_5 + b_2 + p^{n_1} - p^{m_2 + m_5 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1}
\]
\[\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If $a_1 + m_1 + m_2 + m_3 + m_5 = r - 1$, we obtain $a_2 + n_2 + n_3 + b_1 + b_2 = 1$. If $a_2 = 1$, then we have $n_2 = n_3 = b_1 = b_2 = 0$. We obtain
\[
p^{n_1 + m_2 + b_1 + m_5} + p^{m_3 + m_2 + m_3 + a_1 + m_5} + p^{m_5 + b_2 + m_1} + p^{n_1 + n_3} - p^{m_5 + m_2 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{n_1 + m_2 + m_5} + p^{r-1} + p^{m_1 + m_5} + p^{n_1} - p^{m_2 + m_5 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If $n_2 = 1$, then we have $a_2 = n_3 = b_1 = b_2 = 0$. We obtain
\[
p^{n_1 + m_2 + b_1 + m_5} + p^{m_3 + m_2 + m_3 + a_1 + m_5} + p^{m_5 + b_2 + m_1} + p^{n_1 + n_3} - p^{m_5 + m_2 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{n_1 + m_2 + m_5} + p^{r-1} + p^{m_1 + m_5} + p^{n_1} - p^{m_2 + m_5 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{r-1} - 1 < p^r - 1.
\]

If $n_3 = 1$, then we have $a_2 = n_2 = b_1 = b_2 = 0$. We obtain
\[
p^{n_1 + m_2 + b_1 + m_5} + p^{m_3 + m_2 + m_3 + a_1 + m_5} + p^{m_5 + b_2 + m_1} + p^{n_1 + n_3} - p^{m_5 + m_2 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{n_1 + m_2 + m_5} + p^{r-1} + p^{m_1 + m_5} + p^{n_1} - p^{m_2 + m_5 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If $b_1 = 1$, then we have $a_2 = n_2 = n_3 = b_2 = 0$. We obtain
\[
p^{n_1 + m_2 + b_1 + m_5} + p^{m_3 + m_2 + m_3 + a_1 + m_5} + p^{m_5 + b_2 + m_1} + p^{n_1 + n_3} - p^{m_5 + m_2 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{n_1 + m_2 + b_1 + m_5} + p^{r-1} + p^{m_1 + m_5} + p^{n_1} - p^{m_2 + m_5 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If $b_3 = 1$, then we have $a_2 = n_2 = n_3 = b_1 = 0$. We obtain
\[
p^{n_1 + m_2 + b_1 + m_5} + p^{m_3 + m_2 + m_3 + a_1 + m_5} + p^{m_5 + b_2 + m_1} + p^{n_1 + n_3} - p^{m_5 + m_2 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{n_1 + m_2 + b_1 + m_5} + p^{r-1} + p^{m_1 + m_5} + p^{n_1} - p^{m_2 + m_5 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If $m_1 + b_2 + m_5 = r - 1$, we obtain $a_1 + a_2 + n_2 + n_3 + m_2 + m_3 + b_1 = 1$. If $a_1 = 1$, then we have $a_2 = n_2 = n_3 = m_2 = m_3 = b_1 = 0$. We obtain
\[
p^{n_1 + m_2 + b_1 + m_5} + p^{m_3 + m_2 + m_3 + a_1 + m_5} + p^{m_5 + b_2 + m_1} + p^{n_1 + n_3} - p^{m_5 + m_2 + a_1} - p^{m_1 + m_5} - p^{n_1} - 1
\]
\[= p^{n_1 + m_2 + b_1 + m_5} + p^{r-1} + p^{n_1} - p^{a_1 + m_5} - p^{m_1 + m_5} - p^{n_1} - 1
\]

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If $a_2 = 1$, then we have $a_1 = a_2 = n_3 = m_2 = m_3 = b_1 = 0$. We obtain
\[
\begin{align*}
&\leq 2p^{r-1} - 1 \leq p^r - 1.
\end{align*}
\]

If $n_2 = 1$, then we have $a_1 = a_2 = n_3 = m_2 = m_3 = b_1 = 0$. We obtain
\[
\begin{align*}
&\leq 2p^{r-1} - 1 \leq p^r - 1.
\end{align*}
\]

If $n_3 = 1$, then we have $a_1 = a_2 = n_2 = n_3 = m_3 = b_1 = 0$. We obtain
\[
\begin{align*}
&\leq 2p^{r-1} - 1 \leq p^r - 1.
\end{align*}
\]

If $m_2 = 1$, then we have $a_1 = a_2 = n_2 = n_3 = m_2 = b_1 = 0$. We obtain
\[
\begin{align*}
&\leq 2p^{r-1} - 1 \leq p^r - 1.
\end{align*}
\]

If $m_3 = 1$, then we have $a_1 = a_2 = n_2 = n_3 = m_2 = b_1 = 0$. We obtain
\[
\begin{align*}
&\leq 2p^{r-1} - 1 \leq p^r - 1.
\end{align*}
\]

If $b_1 = 1$, then we have $a_1 = a_2 = n_2 = n_3 = m_2 = m_3 = 0$. We obtain
\[
\begin{align*}
&\leq 2p^{r-1} - 1 \leq p^r - 1.
\end{align*}
\]

If $n_1 + n_3 = r - 1$, we obtain $n_2 + m_1 + m_2 + m_3 + m_4 + m_5 = 1$. If $n_2 = 1$, then we have $m_1 = m_2 = m_3 = b_1 = b_2 = m_5 = 0$. We obtain
\[
\begin{align*}
&\leq 2p^{r-1} - 1 \leq p^r - 1.
\end{align*}
\]
\[ = p^{r-1} - 1 < p^r - 1. \]

If \( m_1 = 1 \), then we have \( n_2 = m_2 = m_3 = b_1 = b_2 = m_5 = 0 \). We obtain
\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]
\[
= p^{n_1} + p^{a_1+m_1} + p^{m_1} + p^{r-1} - p^{a_1} - p^{m_1} - p^{n_1} - 1
\]
\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( m_2 = 1 \), then we have \( n_2 = m_1 = m_3 = b_1 = b_2 = m_5 = 0 \). We obtain
\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]
\[
= p^{n_1+m_2} + p^{a_1+m_2} + 1 + p^{r-1} - p^{a_1+m_2} - 1 - p^{n_1} - 1
\]
\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( m_3 = 1 \), then we have \( n_2 = m_1 = m_2 = b_1 = b_2 = m_5 = 0 \). We obtain
\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]
\[
= p^{n_1} + p^{a_1+m_3} + 1 + p^{r-1} - p^{a_1} - 1 - p^{n_1} - 1
\]
\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( b_1 = 1 \), then we have \( n_2 = m_1 = m_2 = m_3 = b_1 = m_5 = 0 \). We obtain
\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]
\[
= p^{n_1+b_1} + p^{a_1} + 1 + p^{r-1} - p^{a_1} - 1 - p^{n_1} - 1
\]
\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( b_2 = 1 \), then we have \( n_2 = m_1 = m_2 = m_3 = b_1 = b_2 = 0 \). We obtain
\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]
\[
= p^{n_1} + p^{a_1} + p^{b_2} + p^{r-1} - p^{a_1} - 1 - p^{n_1} - 1
\]
\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

If \( m_5 = 1 \), then we have \( n_2 = m_1 = m_2 = m_3 = b_1 = b_2 = 0 \). We obtain
\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]
\[
= p^{n_1+m_5} + p^{a_1+m_5} + p^{m_5} + p^{r-1} - p^{a_1+m_5} - p^{m_5} - p^{n_1} - 1
\]
\[
\leq 2p^{r-1} - 1 \leq p^r - 1.
\]

Thus, we obtain
\[
p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1
\]
\[
\leq p^r - 1.
\]

We complete the proof of the lemma. \( \square \)
References


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