Local p-Rank and Semi-Stable Reduction of Curves

Yu Yang

Abstract

In the present paper, we investigate the local p-ranks of coverings of stable curves. Let G be a finite p-group, $f : Y \longrightarrow X$ a morphism of stable curves over a complete discrete valuation ring with algebraically closed residue field of characteristic $p > 0$, x a singular point of the special fiber X_s of X. Suppose that the generic fiber X_η of X is smooth, and the morphism of generic fibers f_η is a Galois étale covering with Galois group G. Write Y' for the normalization of X in the function field of Y, $\psi : Y' \longrightarrow X$ for the resulting normalization morphism. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of x. Suppose that the inertia group $I_{y'} \subseteq G$ of y' is an abelian p-group. Then we give an explicit formula for the p-rank of a connected component of $f^{-1}(x)$. Furthermore, we prove that the p-rank is bounded by $\sharp I_{y'} - 1$ under certain assumptions, where $\sharp I_{y'}$ denotes the order of $I_{y'}$. These results generalize the results of M. Saïdi concerning local p-ranks of coverings of curves to the case where $I_{y'}$ is an arbitrary abelian p-group.

Keywords: p-rank, semi-stable reduction, semi-stable covering, semi-graph with p-rank.

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Contents

1 Introduction and ideas

Let R be a complete valuation ring with algebraically closed residue field k of characteristic $p > 0$, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R. Write $\eta, \overline{\eta}$ and s for the generic point, the geometric generic point, and the closed point corresponding to the natural morphisms $\mathrm{Spec}\, K \longrightarrow S$, $\mathrm{Spec}\, \overline{K} \longrightarrow S$, and $\mathrm{Spec}\, k \longrightarrow S$, respectively. Let X be a stable curve of genus g_X over S. Write X_η, $X_{\overline{\eta}}$, and X_s for the generic fiber, the geometric generic fiber, and the special fiber, respectively. Moreover, we suppose that X_η is smooth over η.

Let Y_η be a geometrically connected curve over η, $f_\eta : Y_\eta \longrightarrow X_\eta$ a finite Galois étale covering over η with Galois group G. By replacing S by a finite extension of S, we may assume that Y_η admits a stable model over S. Then f_η extends uniquely to a *G-stable covering* (cf. Definition 3.3) $f : Y \longrightarrow X$ over S (cf. [L2, Theorem 0.2] or Remark 3.3.1 of the present paper). We are interested in understanding the structure of the special fiber Y_s of Y. If the order $\sharp G$ of G is prime to p, then by the specialization theorem for log étale fundamental groups, f_s is an admissible covering (cf. [Y1]); thus, Y_s may be obtained by gluing together tame coverings of the irreducible components of X_s. On the other hand, if $p | \sharp G$, then f_s is not a finite morphism in general. For example, if $\mathrm{char}(K) = 0$ and $\mathrm{char}(k) = p > 0$, then there exists a Zariski dense subset Z of the set of closed points of X, which may in fact be taken to be X when k is an algebraic closure of $\mathbb{F}_p$, such that for any $x \in Z$, after possibly replacing K by a finite extension of K, there exist a finite group H and an H-stable covering $f_W : W \longrightarrow X$ such that the fiber $(f_W)^{-1}(x)$ is not finite (cf. [T], [Y2]).

If $f^{-1}(x)$ is not finite, we shall call x a *vertical point associated to* f and call $f^{-1}(x)$ the *vertical fiber associated to* x (cf. Definition 3.4). In order to investigate the properties of Y_s, we focus on a geometric invariant $\sigma(Y_s)$ which is called the p-rank of Y_s (cf. Definition 3.1 and Remark 3.1.1). By the definition of the p-rank of a stable curve, to calculate $\sigma(Y_s)$, it suffices to calculate the rank of $\mathrm{H}^1(\Gamma_{Y_s}, \mathbb{Z})$ (where Γ_{Y_s} denotes the dual graph of Y_s), the p-ranks of the irreducible components of Y_s which are finite over X_s, and the p-ranks of the vertical fibers of f. In the present paper, we study the p-rank of a vertical fiber and consider the following problem:

Problem 1.1. *Let G be a finite p-group, x be a vertical point associated to the G-stable covering $f : Y \longrightarrow X$, $f^{-1}(x)$ the vertical fiber associated to x.*

(a) Does there exist a minimal bound on the p-rank $\sigma(f^{-1}(x))$ (note that $\sigma(f^{-1}(x))$ is always bounded by the genus of Y_s)?

(b) Does there exist an explicit formula for the p-rank $\sigma(f^{-1}(x))$?

We will answer Problem 1.1 under certain assumptions (cf. Theorem 1.5 and Theorem 1.10). First, let us review some well-known results concerning Problem 1.1.

If x is a nonsingular point, M. Raynaud proved the following result (cf. [R, Théorème 1]):

Theorem 1.2. *If x is a non-singular point of X_s, and G is an arbitrary p-group, then the p-rank $\sigma(f^{-1}(x))$ is equal to 0.*

By Theorem 1.2, in order to resolve Problem 1.1, it is sufficient to consider the case where x is a *singular point* of X_s. In order to explain the results obtained in the present paper, let us introduce some notations. Write X_1 and X_2 for the irreducible components of X_s which contain x, $\psi : Y' \longrightarrow X$ for the normalization of X in the function field of Y. Let $y' \in \psi^{-1}(x)$ be a point in the inverse image of x. Write $I_{y'} \subseteq G$ for the inertia group of y'. In order to calculate the p-rank of $f^{-1}(x)$, since $Y/I_{y'} \longrightarrow X$ is finite étale over x, by replacing X by the stable model of the quotient $Y/I_{y'}$ (note that $Y/I_{y'}$ is a semi-stable curve over S (cf. [R, Appendice, Corollaire])), we may assume that G is equal to $I_{y'}$.

Thus, from the point of view of resolving Problem 1.1, we may assume without loss of generality that $G = I_{y'}$. In the remainder of this section, we shall assume that $G = I_{y'}$ is of order p^r for some positive integer r. Then $f^{-1}(x)$ is connected. With regard to Problem 1.1 (a), M. Saïdi proved the following result (cf. [S, Theorem 1]), by applying Theorem 1.2:

Theorem 1.3. *If G is a cyclic p-group, then we have $\sigma(f^{-1}(x)) \leq \sharp G - 1$, where $\sharp G$ denotes the order of G.*

Furthermore, there is an open problem posed by Saïdi as follows (cf. [S, Question]):

Problem 1.4. *If G is an arbitrary p-group, does there exist a minimal bound on the p-rank $\sigma(f^{-1}(x))$ that depends only on the order $\sharp G$?*

Let us introduce some notations. Suppose that G is an abelian p-group. Let

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_0 = G$$

be a maximal filtration of G (i.e., $G_i/G_{i+1} \cong \mathbb{Z}/p\mathbb{Z}$ for $i = 0, \ldots, r-1$). It follows from [R, Appendice, Corollaire], that for $i = 0, \ldots, r$, $Y_i := Y/G_i$ is a semi-stable curve over S. Write X^{sst} for Y/G and g for the resulting morphism $g : X^{\mathrm{sst}} \longrightarrow X$ induced by f. Then we obtain a sequence of $\mathbb{Z}/p\mathbb{Z}$*-semi-stable coverings* (cf. Definition 3.3)

$$\Phi_f : Y = Y_r \xrightarrow{d_r} Y_{r-1} \xrightarrow{d_{r-1}} \ldots \xrightarrow{d_1} Y_0 = X^{\mathrm{sst}} \xrightarrow{g} X.$$

In the following, we use the subscript "red" to denote the reduced induced closed subscheme associated to a scheme. For each $i = 1, \ldots, r$, write $\phi_i : Y_i \longrightarrow Y_0$ for the composite morphism $d_1 \circ \cdots \circ d_i$. For simplicity, we suppose that $C := g^{-1}(x)_{\mathrm{red}} = \cup_{j=1}^{n} P_j$, where, for each $j = 1, \ldots, n$, P_j is isomorphic to $\mathbb{P}^1$ and meets the other irreducible components of the special fiber X_s^{sst} of X^{sst} at precisely two points (i.e., a chain of $\mathbb{P}^1$). Thus, the p-rank $\sigma(f^{-1}(x))$ is equal to $\sigma(\phi_r^{-1}(C))$. For each $i = 1, \ldots, r$, we define a set of subcurves of C associated to Φ_f, which plays a key role in the present paper, as follows: ♦

$$\mathscr{E}_i^{\Phi_f} := \phi_i(\text{the étale locus of } d_i|_{\phi_i^{-1}(C)_{\mathrm{red}}} : \phi_i^{-1}(C)_{\mathrm{red}} \longrightarrow \phi_{i-1}^{-1}(C)_{\mathrm{red}}) \subset C.$$

We shall call $\mathscr{E}_i^{\Phi_f}$ the *i-th étale-chain associated to* Φ_f and call the disjoint union

$$\mathscr{E}^{\Phi_f} := \coprod_i \mathscr{E}_i^{\Phi_f}$$

the *étale-chain associated to* Φ_f. For each connected component E of $\mathscr{E}_i^{\Phi_f}$, we use the notation $l(E)$ to denote the cardinality of the set of the irreducible components of E and call $l(E)$ the length of E.

We generalize Saïdi's result as follows (see also Theorem 3.15):

Theorem 1.5. *If G is an arbitrary abelian p-group, and $\mathscr{E}_i$ is connected for each $i = 1, \dots, n$, then we have $\sigma(f^{-1}(x)) \leq \sharp G - 1$.*

Remark 1.5.1. If $\sharp G$ is equal to p, then we may construct a $\mathbb{Z}/p\mathbb{Z}$-stable covering $f : Y \longrightarrow X$ such that there exists a singular vertical point x such that the p-rank of $\sigma(f^{-1}(x))$ is equal to $p-1$ (cf. [Y4, Section 4]). Thus, at least in the case where $\sharp G = p$, $\sharp G - 1$ is the minimal bound for $\sigma(f^{-1}(x))$.

Next, let us consider Problem 1.1 (b). Let $\{V_i\}_{i=0}^{n+1}$ be a set of irreducible components of the special fiber Y_s of Y such that the following conditions are satisfied: (i) $\phi_r(V_i) = P_i$ if $i = 1, \dots, n$; (ii) $\phi_r(V_0) = X_1$ and $\phi_r(V_{n+1}) = X_2$; (iii) the union $\cup_{i=0}^{n+1} V_i$ is a connected semi-stable subcurve of the special fiber Y_s of Y. Write $I_{P_i} \subseteq G$ for the inertia subgroup of V_i. Note that since G is an abelian p-group, I_{P_i} does not depend on the choices of V_i.

If G is a cyclic p-group, Saïdi obtained an explicit formula of the p-rank $\sigma(f^{-1}(x))$ as follows (cf. [S, Proposition 1]):

Theorem 1.6. *If G is a cyclic p-group, and I_{P_0} is equal to G, then we have*

$$\sigma(f^{-1}(x)) = \sharp(G/I_{\min}) - \sharp(G/I_{P_{n+1}}),$$

where $I_{\min}$ denotes the group $\cap_{i=0}^{n+1} I_{P_i}$.

For a G-covering of semi-graphs with p-rank, we develop a general method to compute the p-rank (cf. Theorem 2.8). As an application, we generalize Saïdi's formula to the case where G is an arbitrary abelian p-group as follows (cf. Theorem 3.9 and Remark 3.9.1):

Theorem 1.7. *If G is an arbitrary abelian p-group, then we have*

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \sharp(G/I_{P_i}) - \sum_{i=1}^{n+1} \sharp(G/(I_{P_{i-1}} + I_{P_i})) + 1.$$

Finally, I would mention that by using the theory of semi-graphs with p-rank, we can generalize Theorem 1.8 to the case where G is an arbitrary p-group. Furthermore, we can obtain a global p-rank formula for the special fiber Y_s (cf. [Y5]).

The present paper contains two parts. In Section 2, we develop the theory of semi-graphs with p-rank and calculate the p-ranks of G-coverings. In Section 3, we construct a semi-graph with p-rank from a vertical fiber of a G-stable covering in a natural way and apply the results of Section 2 to prove Theorem 1.5 and Theorem 1.8.

2 Semi-graphs with p-rank

In this section, we develop the theory of semi-graphs with p-rank. We always assume that G is an abelian p-group with order p^r.

2.1 Definitions

We begin with some general remarks concerning semi-graphs (cf. [M]). A *semi-graph* $\mathbb{G}$ consists of the following data: (i) A set $\mathcal{V}_{\mathbb{G}}$ whose elements we refer to as vertices; (ii) A set $\mathcal{E}^{\mathbb{G}}$ whose elements we refer to as edges. Any element $e \in \mathcal{E}^{\mathbb{G}}$ is a set of cardinality 2 satisfying the following property: For any $e \neq e' \in \mathcal{E}^{\mathbb{G}}$, we have $e \cap e' = \emptyset$; (iii) A set of maps $\{\zeta_e^{\mathbb{G}}\}_{e \in \mathcal{E}^{\mathbb{G}}}$ such that $\zeta_e : e \longrightarrow \mathcal{V} \cup \{\mathcal{V}\}$ is a map from the set e to the set $\mathcal{V} \cup \{\mathcal{V}\}$. For an edge $e \in \mathcal{E}^{\mathbb{G}}$, we shall refer to an element $b \in e$ as a branch of the edge e. An edge $e \in \mathcal{E}^{\mathbb{G}}$ is called closed (resp. open) if $\zeta_e^{-1}(\{\mathcal{V}^{\mathbb{G}}\}) = \emptyset$ (resp. $\zeta_e^{-1}(\{\mathcal{V}^{\mathbb{G}}\}) \neq \emptyset$). A semi-graph will be called finite if both its set of vertices and its set of edges are finite. In the present paper, we only consider finite semi-graphs. Since a semi-graph can be regarded as a topological space, we shall call $\mathbb{G}$ a connected semi-graph if $\mathbb{G}$ is connected as a topological space.

Let $\mathbb{G}$ be a semi-graph. Write $v(\mathbb{G})$ for the set of vertices of $\mathbb{G}$, $e(\mathbb{G})$ for the set of closed edges of $\mathbb{G}$, and $e'(\mathbb{G})$ for the set of open edges of $\mathbb{G}$. For any element $v \in v(\mathbb{G})$, write $b(v)$ for the set of branches $\cup_{e \in e(\mathbb{G}) \cup e'(\mathbb{G})} \zeta_e^{-1}(v)$. For any element $e \in e(\mathbb{G}) \cup e'(\mathbb{G})$), write $v(e)$for the set which consists of the elements of $v(\mathbb{G})$ which are abutted by e. A morphism between semi-graphs $\mathbb{G} \longrightarrow \mathbb{H}$ is a collection of maps $v(\mathbb{G}) \longrightarrow v(\mathbb{H})$; $e(\mathbb{G}) \cup e'(\mathbb{G}) \longrightarrow e(\mathbb{H}) \cup e'(\mathbb{H})$; and for each $e_{\mathbb{G}} \in e(\mathbb{G}) \cup e'(\mathbb{G})$ mapping to $e_{\mathbb{H}} \in e(\mathbb{H}) \cup e'(\mathbb{H})$, a bijection $e_{\mathbb{G}} \xrightarrow{\sim} e_{\mathbb{H}}$; all of which are compatible with the $\{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G}) \cup e'(\mathbb{G})}$ and $\{\zeta_e^{\mathbb{H}}\}_{e \in e(\mathbb{H}) \cup e'(\mathbb{H})}$.

A sub-semi-graph $\mathbb{G}'$ of $\mathbb{G}$ is a semi-graph satisfying the following properties: (i) $v(\mathbb{G}')$ (resp. $e(\mathbb{G}') \cup e'(\mathbb{G}')$) is a subset of $v(\mathbb{G})$ (resp. $e(\mathbb{G}) \cup e'(\mathbb{G})$); (ii) If $e \in e(\mathbb{G}')$, then we have $\zeta_e^{\mathbb{G}'}(e) = \zeta_e^{\mathbb{G}}(e)$; (iii) If $e = \{b_1, b_2\}$ is an element of $e'(\mathbb{G}')$ such that $\zeta_e^{\mathbb{G}}(b_1) \in v(\mathbb{G}')$ and $\zeta_e^{\mathbb{G}}(b_2) \notin v(\mathbb{G}')$, then we have $\zeta_e^{\mathbb{G}'}(b_1) = \zeta_e^{\mathbb{G}}(b_1)$ and $\zeta_e^{\mathbb{G}'}(b_2) = \{v(\mathbb{G}')\}$.

Definition 2.1. Let $\mathbb{G}'$ be a sub-semi-graph of a semi-graph $\mathbb{G}$. We define a semi-graph $\mathbb{G} \backslash \mathbb{G}'$ as follows: (i) The set of vertices $v(\mathbb{G} \backslash \mathbb{G}')$ is $v(\mathbb{G}) \backslash v(\mathbb{G}')$; (ii) The set of closed edges $e(\mathbb{G} \backslash \mathbb{G}')$ is $e(\mathbb{G}) \backslash e(\mathbb{G}')$; (iii) The set of open edges $e'(\mathbb{G} \backslash \mathbb{G}')$ is $\{e \in e(\mathbb{G}) \mid v(e) \cap v(\mathbb{G} \backslash \mathbb{G}') \neq \emptyset \text{ in } \mathbb{G}\}$; (iv) For any $e = \{b_i\}_{i=\{1,2\}} \in e(\mathbb{G} \setminus \mathbb{G}') \cup e'(\mathbb{G} \setminus \mathbb{G}')$, we have $\zeta_e^{\mathbb{G} \backslash \mathbb{G}'}(b_i) = \zeta_e^{\mathbb{G}}(b_i)$ (resp. $\zeta_e^{\mathbb{G} \backslash \mathbb{G}'}(b_i) = \{v(\mathbb{G} \setminus \mathbb{G}')\}$) if $\zeta_e^{\mathbb{G}}(b_i) \notin v(\mathbb{G}')$ (resp. $\zeta_e^{\mathbb{G}}(b_i) \in v(\mathbb{G}')$).

Definition 2.2. (a) Let n be a positive natural number and $\mathbb{P}_n$ a semi-graph such that the following conditions hold: (i) $v(\mathbb{P}_n) = \{p_1, \ldots, p_n\}$, $e(\mathbb{P}_n) = \{e_{1,2}, \ldots, e_{n,n-1}\}$ and $e'(\mathbb{P}_n) = \{e_{0,1}, e_{n,n+1}\}$; (ii) $v(e_{i,i+1}) = \{p_i, p_{i+1}\}$; (iii) $v(e_{0,1}) = \{p_1\}$ and $v(e_{n,n+1}) = \{p_n\}$. We define $\mathfrak{G}$ to be a triple $(\mathbb{G}, \sigma_{\mathfrak{G}}, \beta_{\mathfrak{G}})$ which consists of a semi-graph $\mathbb{G}$, a map $\sigma_{\mathfrak{G}} : v(\mathbb{G}) \longrightarrow \mathbb{Z}$ and a morphism of semi-graphs $\beta_{\mathfrak{G}} : \mathbb{G} \longrightarrow \mathbb{P}_n$. We shall call $\mathfrak{G}$ a *n-semi-graph with p-rank*. We shall refer to $\mathbb{G}$ as the underlying semi-graph of $\mathfrak{G}$, $\sigma_{\mathfrak{G}}$ as the p-rank map of $\mathfrak{G}$, $\beta_{\mathfrak{G}}$ as the base morphism of $\mathfrak{G}$, respectively. We define $\mathfrak{P}_n := (\mathbb{P}_n, \sigma_{\mathfrak{P}_n}, \beta_{\mathfrak{P}_n})$ as follows: $\sigma_{\mathfrak{P}_n}(p_i)$ is equal to 0 for each $i = 1, \ldots, n$, and $\beta_{\mathfrak{P}_n} = \mathrm{id}_{\mathbb{P}_n}$ is an identity morphism of semi-graph $\mathbb{P}_n$. We shall call $\mathfrak{P}_n$ a *n-chain*.

(b) We define the *p-rank $\sigma(\mathfrak{G})$ of* $\mathfrak{G}$ as follows:

$$\sigma(\mathfrak{G}) := \sum_{v \in v(\mathbb{G})} \sigma(v) + \sum_{\mathbb{G}_i \in \pi_0(\mathbb{G})} \mathrm{rank}_{\mathbb{Z}} \mathrm{H}^1(\mathbb{G}_i, \mathbb{Z}),$$

where $\pi_0(-)$ denotes the set of connected components of $(-)$.

(c) $\mathfrak{G}$ is called connected if the underlying semi-graph $\mathbb{G}$ is a connected semi-graph.

From now on, we only consider connected n-semi-graphs with p-rank. Let $\mathfrak{G}^1 := (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}, \beta_{\mathfrak{G}^1})$ and $\mathfrak{G}^2 := (\mathbb{G}^2, \sigma_{\mathfrak{G}^2}, \beta_{\mathfrak{G}^2})$ be two n-semi-graphs with p-rank. A morphism between $\mathfrak{G}^1$ and $\mathfrak{G}^2$ is defined by a morphism of the underlying semi-graphs $\beta : \mathbb{G}^1 \longrightarrow \mathbb{G}^2$ such that $\beta_{\mathfrak{G}^2} \circ \beta = \beta_{\mathfrak{G}^1}$. We use the notation $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ to denotes the morphism of semi-graphs with p-rank determined by $\beta : \mathbb{G}^1 \longrightarrow \mathbb{G}^2$ and call β the underlying morphism of $\mathfrak{b}$. Note that for any n-semi-graph with p-rank $\mathfrak{G} := (\mathbb{G}, \sigma_{\mathfrak{G}}, \beta_{\mathfrak{G}})$, there is a natural morphism $\mathfrak{b}_{\mathfrak{G}} : \mathfrak{G} \longrightarrow \mathfrak{P}_n$ determined by the morphism of underlying semi-graphs $\beta_{\mathfrak{G}} : \mathbb{G} \longrightarrow \mathbb{P}_n$.

Write b_l^i (resp. b_r^i) for $\zeta_{e_{i-1,i}}^{-1}(p_i)$ (resp. $\zeta_{e_{i,i+1}}^{-1}(p_i)$). For any element $v_i \in \beta_{\mathfrak{G}}^{-1}(p_i)$, write $b_l(v_i)$ (resp. $b_r(v_i)$) for the set

$$\{b \in b(v_i) \mid \beta_{\mathfrak{G}}(b) = b_l^i\}$$

$$(\text{resp. } \{b \in b(v_i) \mid \beta_{\mathfrak{G}}(b) = b_r^i\}).$$

Definition 2.3. Let $\mathfrak{b} : \mathfrak{G}^1 := (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}, \beta_{\mathfrak{G}^1}) \longrightarrow \mathfrak{G}^2 := (\mathbb{G}^2, \sigma_{\mathfrak{G}^2}, \beta_{\mathfrak{G}^2})$ be a morphism of n-semi-graphs with p-rank, β the underlying morphism of $\mathfrak{b}$, $e \in e(\mathbb{G}^1) \cup e'(\mathbb{G}^1)$ an edge, v_1 a vertex of $\mathbb{G}^1$ contained in $\beta_{\mathfrak{G}^1}^{-1}(p_i)$, and $v_2 := \beta(v_1) \in \beta_{\mathfrak{G}^2}^{-1}(p_i)$ the image of v_1.

(a) We shall call $\mathfrak{b}$ *p-étale* (resp. *p-purely inseparable*) at e if $\sharp\beta^{-1}(\beta(e)) = p$ (resp. $\sharp\beta^{-1}(\beta(e)) = 1$). We shall call $\mathfrak{b}$ *p-generically étale* at $v_1 \in \beta_{\mathfrak{G}^1}^{-1}(p_i)$ if one of the following étale types holds:

(Type-I) $\sharp\beta^{-1}(v_2) = p$ and $\sigma_{\mathfrak{G}^1}(v_1) = \sigma_{\mathfrak{G}^2}(v_2)$;

(Type-II) $\sharp\beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = p\sharp b_l(v_2)$, $\sharp b_r(v_1) = p\sharp b_r(v_2)$, and

$$\sigma_{\mathfrak{G}^1}(v_1) - 1 = p(\sigma_{\mathfrak{G}^2}(v_2) - 1);$$

(Type-III) If $\sharp\beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = \sharp b_l(v_2)$, $\sharp b_r(v_1) = p\sharp b_r(v_2)$, and

$$\sigma_{\mathfrak{G}^1}(v_1) - 1 = p(\sigma_{\mathfrak{G}^2}(v_2) - 1) + (\sharp b_l(v_1))(p - 1);$$

(Type-IV) $\sharp\beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = p\sharp b_l(v_2)$, $\sharp b_r(v_1) = \sharp b_r(v_2)$, and

$$\sigma_{\mathfrak{G}^1}(v_1) - 1 = p(\sigma_{\mathfrak{G}^2}(v_2) - 1) + (\sharp b_r(v_1))(p - 1);$$

(Type-V) $\sharp\beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = \sharp b_l(v_2)$, $\sharp b_r(v_1) = \sharp b_r(v_2)$, and

$$\sigma_{\mathfrak{G}^1}(v_1) - 1 = p(\sigma_{\mathfrak{G}^2}(v_2) - 1) + (\sharp b_l(v_1) + \sharp b_r(v_1))(p - 1).$$

(b) We shall call $\mathfrak{b}$ *purely inseparable* at $v_1 \in \beta_{\mathfrak{G}^1}^{-1}(p_i)$ if $\sharp\beta^{-1}(v_2) = 1$, $\sharp b_l(v_1) = \sharp b_l(v_2)$, $\sharp b_r(v_1) = \sharp b_r(v_2)$, and $\sigma_{\mathfrak{G}^1}(v_1) = \sigma_{\mathfrak{G}^2}(v_2)$ hold.

(c) We shall call $\mathfrak{b}$ a *p-covering* if the following conditions hold: (i) There exists a $\mathbb{Z}/p\mathbb{Z}$-action (which may be trivial) on $\mathbb{G}^1$ (resp. a trivial $\mathbb{Z}/p\mathbb{Z}$-action on $\mathbb{G}^2$), and the underlying morphism β of $\mathfrak{b}$ is compatible with the $\mathbb{Z}/p\mathbb{Z}$-actions. Then the natural morphism $\mathbb{G}^1/\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}^2$ induced by $\mathfrak{b}$ is an isomorphism; (ii) For any $v \in v(\mathbb{G}^1)$, $\mathfrak{b}$ is either p-generically étale or purely inseparable at v; (iii) Let $e \in e(\mathbb{G}^1)$ and $v(e) = \{v, v'\}$. If $\mathfrak{b}$ is p-generically étale at v and v', then $\mathfrak{b}$ is p-étale at e; (iv) For any $v \in v(\mathbb{G}^1)$, then $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^1}(\tau(v))$ holds for each $\tau \in \mathbb{Z}/p\mathbb{Z}$.

Note that by the definition of p-covering, the identity morphism of a semi-graph with p-rank is a p-covering.

(d) We shall call $\mathfrak{b}$ a *covering* if $\mathfrak{b}$ is a composite of p-coverings.

(e) We shall call

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

an *maximal filtration* of G if $G_j/G_{j+1} \cong \mathbb{Z}/p\mathbb{Z}$ for each $j = 1, \ldots, r-1$. Suppose that $\mathbb{G}^1$ (resp. $\mathbb{G}^2$) admits a (resp. trivial) G-action (which may be trivial). Then for any maximal filtration Φ of G, there is a sequence of semi-graphs induced by Φ:

$$\mathbb{G}^1 = \mathbb{G}_r \xrightarrow{\beta_r} \mathbb{G}_{r-1} \xrightarrow{\beta_{r-1}} \ldots \xrightarrow{\beta_1} \mathbb{G}_0,$$

where $\mathbb{G}_j$ denotes the quotient of $\mathbb{G}^1$ by G_j. We shall call $\mathfrak{b}$ a *G-covering* if for any maximal filtration Φ of G, there exists a set of p-coverings $\{\mathfrak{b}_j : \mathfrak{G}_j \longrightarrow \mathfrak{G}_{j-1},\ j = 1, \ldots, r\}$ such that the following conditions hold: (i) the underlying morphism β of $\mathfrak{b}$ is compatible with the G-actions, and the natural morphism $\mathbb{G}^1/G \longrightarrow \mathbb{G}^2$ induced by β is an isomorphism; (ii) The underlying graph of $\mathfrak{G}_j$ is equal to $\mathbb{G}_j$ for each $j = 0, \ldots, r$; (iii) The underlying morphism $\mathbb{G}_j \longrightarrow \mathbb{G}_{j-1}$ of $\mathfrak{b}_j$ is equal to β_j for each $j = 1, \ldots, r$; (iv) The composite morphism $\mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is equal to $\mathfrak{b}$. Then we obtain a sequence of p-coverings:

$$\Phi_{\mathfrak{G}^1} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \ldots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

We shall call $\Phi_{\mathfrak{G}^1}$ *a sequence of p-coverings induced by* Φ.

(f) Let $\mathfrak{G}$ be a n-semi-graph with p-rank. We shall call $\mathfrak{G}$ a *covering* (resp. *G-covering*) *over* $\mathfrak{P}_n$ if $\mathfrak{b}_{\mathfrak{G}}$ is a covering (resp. G-covering).

(g) Let $\mathfrak{b} : \mathfrak{G}^1 \longrightarrow \mathfrak{G}^2$ be a G-covering, $v \in v(\mathbb{G})$ a vertex, and $e \in e(\mathbb{G}) \cup e'(\mathbb{G})$ an edge. For any subgroup $H \subseteq G$, by Definition 2.3 (e), there exists a maximal filtration Φ^H and the sequence of p-coverings

$$\Phi^H_{\mathfrak{G}^1} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}^H_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}^H_{r-1}} \ldots \xrightarrow{\mathfrak{b}^H_1} \mathfrak{G}_0 = \mathfrak{G}^2$$

induced by Φ^H such that there exists i such that the underlying graph of $\mathfrak{G}_i$ is isomorphic to $\mathbb{G}^1/H$. We write $\mathfrak{G}^1/H$ for $\mathfrak{G}_i$. Thus, the natural morphism $\mathfrak{b}^H_1 \circ \cdots \circ \mathfrak{b}^H_i : \mathfrak{G}^1/H \longrightarrow \mathfrak{G}^2$ is a covering. Then we define five subgroups of G as follows:

$$D_v := \{\tau \in G \mid \tau(v) = v\},$$

$$I_v := \text{the maximal element of } \{H \subseteq G \mid \mathfrak{G}^1 \longrightarrow \mathfrak{G}^1/H \text{ is purely inseparable at } v\},$$

$$I^l_v(b) := \{\tau \in D_v \mid \tau(b) = b \text{ for a branch } b \in b_l(v)\}/I_v,$$

$$I^r_v(b) := \{\tau \in D_v \mid \tau(b) = b \text{ for a branch } b \in b_r(v)\}/I_v,$$

$$I_e := \{\tau \in G \mid \tau(e) = e\}.$$

We shall call D_v (resp. I_v, $I^l_v(b)$, $I^r_v(b)$, I_e) *the decomposition group of* v (resp. *the inertia group of* v*, the inertia group of a left branch* b*, the inertia group of a right branch* b*, the inertia group of* e). Moreover, since G is an abelian p-group, the group $I^l_v(b)$ (resp. $I^r_v(b)$)

does not depend on the choice of $b \in b_l(v)$ (resp. $b \in b_r(v)$), then we denote this group briefly by I_v^l (resp. I_v^r). Define

$$D_v^e = D_v/(I_v^l/(I_v^l \cap I_v^r) \oplus I_v^r/(I_v^l \cap I_v^r) \oplus I_v^l \cap I_v^r \oplus I_v).$$

Then we have the following exact sequence

$$0 \longrightarrow I_v^l/(I_v^l \cap I_v^r) \oplus I_v^r/(I_v^l \cap I_v^r) \oplus I_v^l \cap I_v^r \oplus I_v \longrightarrow D_v \longrightarrow D_v^e \longrightarrow 0.$$

Remark 2.3.1. Let $\mathfrak{G}$ be a G-covering over $\mathfrak{P}_n$ and $v_i \in \beta_{\mathfrak{G}}^{-1}(p_i)$ a vertex of the underlying graph of $\mathfrak{G}$. Then we have the following Deuring-Shafarevich type formula (cf. Proposition 3.2 for the Deuring-Shafarevich formula for curves)

$$\sigma_{\mathfrak{G}}(v_i) - 1 = -\sharp D_{v_i}/I_{v_i} + \sharp((D_{v_i}/I_{v_i})/I_{v_i}^l)(\sharp I_{v_i}^l - 1) + \sharp((D_{v_i}/I_{v_i})/I_{v_i}^r)(\sharp I_{v_i}^r - 1).$$

Let $\mathfrak{G}$ be a G-covering over $\mathfrak{P}_n$. By the definition of G-coverings, for any maximal filtration Φ of G, we have a sequence of p-coverings of n-semi-graphs with p-rank

$$\Phi_{\mathfrak{G}} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n$$

induced by Φ. For each $j = 1, \dots, r$, we write $\mathcal{V}_j^{\text{ét}}$ for the set

$$\{v \in v(\mathbb{G}_j) \mid \mathfrak{b}_j \text{ is étale at } v\},$$

$\mathcal{E}_j^{\text{ét}}$ for the set

$$\{e \in e(\mathbb{G}_j) \cup e'(\mathbb{G}_j) \mid \mathfrak{b}_j \text{ is étale at } e\}.$$

Since $(\mathcal{V}_j^{\text{ét}}, \mathcal{E}_j^{\text{ét}})$ admits a natural structure of semi-graph induced by $\mathbb{G}_j$, we may regard $(\mathcal{V}_j^{\text{ét}}, \mathcal{E}_j^{\text{ét}})$ as a sub-semi-graph of $\mathbb{G}_j$. Thus, the image $\beta_{\mathfrak{G}_j}((\mathcal{V}_j^{\text{ét}}, \mathcal{E}_j^{\text{ét}}))$ can be regarded as a sub-semi-graph of $\mathbb{P}_n$.

Definition 2.4. We shall call $\mathbb{E}_j^{\Phi_{\mathfrak{G}}} := \beta_{\mathfrak{G}_j}((\mathcal{V}_j^{\text{ét}}, \mathcal{E}_j^{\text{ét}}))$ (resp. the disjoint union $\mathbb{E}^{\Phi_{\mathfrak{G}}} := \coprod_j \mathbb{E}_j^{\Phi_{\mathfrak{G}}}$) the *j-th étale-chain* (resp. the *étale-chain*) associated to $\Phi_{\mathfrak{G}}$.

2.2 p-ranks and étale-chains of abelian coverings

Let $\mathfrak{G} := (\mathbb{G}, \sigma_{\mathfrak{G}}, \beta_{\mathfrak{G}})$ be a G-covering over $\mathfrak{P}_n$. We introduce two operators for $\mathfrak{G}$.

Operator I: First, let us define a G-covering $\mathfrak{G}^*[p_i]$ over $\mathfrak{P}_n$. For any $p_i \in v(\mathbb{P}_n)$, let v_i be an element of $\beta_{\mathfrak{G}}^{-1}(p_i)$.

If $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) = 1$ (i.e., $D_{v_i} = G$), then we define $\mathbb{G}^*[p_i]$ to be $\mathbb{G}$; If $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) \neq 1$, we define a new semi-graph $\mathbb{G}^*[p_i]$ as follows.

Define $v(\mathbb{G}^*[p_i])$ (resp. $e(\mathbb{G}^*[p_i]) \cup e'(\mathbb{G}^*[p_i])$) to be the disjoint union $(v(\mathbb{G}) \backslash \beta_{\mathfrak{G}}^{-1}(p_i)) \coprod \{v^*\}$ (resp. $e(\mathbb{G}) \cup e'(\mathbb{G})$).

The collection of maps $\{\zeta_e^{\mathbb{G}^*[p_i]}\}_e$ is as follows: (i) For any branch $b \notin \cup_{v \in \beta_{\mathfrak{G}}^{-1}(p_i)} b(v)$, $\zeta_e^{\mathbb{G}^*[p_i]}(b) = \zeta_e^{\mathbb{G}}(b)$ if $b \in e$ and $\zeta_e^{\mathbb{G}^*[p_i]}(b) = \emptyset$ if $b \notin e$; (ii) For any $v \in \beta_{\mathfrak{G}}^{-1}(p_i)$ and any branch $b \in b(v)$, $\zeta_e^{\mathbb{G}^*[p_i]}(b) = v^*$ if $b \in e$ and $\zeta_e^{\mathbb{G}^*[p_i]}(b) = \emptyset$ if $b \notin e$.

We define a map $\sigma_{\mathfrak{G}^*[p_i]} : v(\mathbb{G}^*[p_i]) \longrightarrow \mathbb{Z}$ as follows: (i) If $v^* \neq v \in v(\mathbb{G}^*[p_i])$, then we have $\sigma_{\mathfrak{G}^*[p_i]}(v) := \sigma_{\mathfrak{G}}(v)$; (ii) If $v = v^*$, then we have

$$\sigma_{\mathfrak{G}^*[p_i]}(v^*) := -\sharp(G/I_{v_i}) + \sum_{v \in \beta_{\mathfrak{G}}^{-1}(p_i)} \sum_{b \in b_l(v)} (\sharp I_v^l(b) - 1) + \sum_{v \in \beta_{\mathfrak{G}}^{-1}(p_i)} \sum_{b \in b_r(v)} (\sharp I_v^r(b) - 1) + 1$$

$$= -\sharp(G/I_{v_i}) + \sharp((G/I_{v_i})/I_{v_i}^l)(\sharp I_{v_i}^l - 1) + \sharp((G/I_{v_i})/I_{v_i}^r)(\sharp I_{v_i}^r - 1) + 1.$$

We define a morphism of semi-graphs $\beta_{\mathfrak{G}^*[p_i]} : \mathbb{G}^*[p_i] \longrightarrow \mathbb{P}_n$ as follows: (i) For any $v \in v(\mathbb{G}^*[p_i])$, $\beta_{\mathfrak{G}^*[p_i]}(v) = p_i$ if $v = v^*$ and $\beta_{\mathfrak{G}^*[p_i]}(v) = \beta_{\mathfrak{G}}(v)$ if $v \notin \beta_{\mathfrak{G}}^{-1}(p_i)$; (ii) If $e \in e(\mathbb{G}^*[p_i]) \cup e'(\mathbb{G}^*[p_i])$, then we have $\beta_{\mathfrak{G}^*[p_i]}(e) = \beta_{\mathfrak{G}}(e)$.

Thus, the triple $\mathfrak{G}^*[p_i] := (\mathbb{G}^*[p_i], \sigma_{\mathfrak{G}^*[p_i]}, \beta_{\mathfrak{G}^*[p_i]})$ is a n-semi-graph with p-rank.

Moreover, $\mathbb{G}^*[p_i]$ admits a natural G-action as follows: (i) the action of G on $v(\mathbb{G}^*[p_i]) \setminus \{v^*\}$ (resp. $e(\mathbb{G}^*[p_i]) \cup e'(\mathbb{G}^*[p_i])$) is the action of G on $v(\mathbb{G}) \setminus \beta_{\mathfrak{G}}^{-1}(p_i)$ (resp. $e(\mathbb{G}) \cup e'(\mathbb{G})$); (ii) For any $\tau \in G$, we have $\tau(v^*) = v^*$.

Let us explain that with the G-action defined above, $\mathfrak{G}^*[p_i]$ is a G-covering over $\mathfrak{P}_n$. Let

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

be an arbitrary maximal filtration of G. Write

$$\Phi_{\mathfrak{G}} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ. Note that for each $j = 0, \dots, r$, $\mathfrak{G}_j$ is a G/G_j-covering over $\mathfrak{P}_n$. By the construction of $\mathfrak{G}_j^*[p_i]$, we have

$$\Phi_{\mathfrak{G}^*[p_i]} : \mathfrak{G}^*[p_i] = \mathfrak{G}_r^*[p_i] \xrightarrow{\mathfrak{b}_r^*[p_i]} \mathfrak{G}_{r-1}^*[p_i] \xrightarrow{\mathfrak{b}_{r-1}^*[p_i]} \dots \xrightarrow{\mathfrak{b}_1^*[p_i]} \mathfrak{P}_n.$$

is a sequence of p-coverings of n-semi-graphs with p-rank. Thus, $\mathfrak{G}^*[p_i]$ can be regarded as a G-covering over $\mathfrak{P}_n$.

Note that by the construction of $\mathfrak{G}^*[p_i]$, we see that $\mathbb{E}_j^{\Phi_{\mathfrak{G}}} = \mathbb{E}_j^{\Phi_{\mathfrak{G}^*[p_i]}}$ for each $j = 1, \dots, r$.

Operator II: Let us define a G-covering $\mathfrak{G}^\star[p_i]$ over $\mathfrak{P}_n$. For any $p_i \in v(\mathbb{P}_n)$, let v_i be an element of $\beta_{\mathfrak{G}}^{-1}(p_i)$, I_{v_i} the inertia group of v_i. Since G is a abelian group, we may write $\{v_i^u\}_{u \in G/D_{v_i}}$ for $\beta_{\mathfrak{G}}^{-1}(p_i)$, and $\{v_i^u\}_{u \in G/D_{v_i}}$ admits an natural action of G on the index set G/D_{v_i}. We define a new semi-graph $\mathbb{G}^\star[p_i]$ as follows. If $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) = \sharp(G/I_{v_i})$, we define $\mathbb{G}^\star[p_i]$ to be $\mathbb{G}$. If $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) \neq \sharp(G/I_{v_i})$, we have $\beta_{\mathfrak{G}}^{-1}(b_l^i) = \{b_l^{i,u,s,t}\}_{u \in G/D_{v_i}, s \in I_{v_i}^r/I_{v_i}^l \cap I_{v_i}^r, t \in D_{v_i}^e}$. Then $\beta_{\mathfrak{G}}^{-1}(b_l^i) = \{b_l^{i,u,s,t}\}_{u \in G/D_{v_i}, s \in I_{v_i}^r/I_{v_i}^l \cap I_{v_i}^r, t \in D_{v_i}^e}$ admits a natural action of G as follows: for $\tau \in G$, $\tau(b_l^{i,u,s,t}) = b_l^{i,\overline{\tau} \circ u,s,t}$ if $\tau \notin D_{v_i}$, where $\overline{\tau}$ denotes the image of τ under the quotient $G \longrightarrow G/D_{v_i}$, $\tau(b_l^{i,u,s,t}) = b_l^{i,u,\tau \circ s,t}$ if $\tau \in I_{v_i}^r/I_{v_i}^l \cap I_{v_i}^r$, $\tau(b_l^{i,u,s,t}) = b_l^{i,u,s,\overline{\tau} \circ t}$ if $\tau \notin I_{v_i}^l + I_{v_i}^r + I_{v_i}$, where $\overline{\tau}$ denotes the image of τ under the quotient $D_{v_i} \longrightarrow D_{v_i}^e$, and $\tau(b_l^{i,u,s,t}) = b_l^{i,u,s,t}$ if $\tau \in I_{v_i} + I_{v_i}^l$. Similarly, $\beta_{\mathfrak{G}}^{-1}(b_r^i) := \{b_r^{i,u,s,t}\}_{u \in G/D_{v_i}, s \in I_{v_i}^l/I_{v_i}^l \cap I_{v_i}^r, t \in D_{v_i}^e}$ also admits a natural action of G.

Define $v(\mathbb{G}^\star[p_i])$ (resp. $e(\mathbb{G}^\star[p_i]) \cup e'(\mathbb{G}^\star[p_i])$) to be the disjoint union $(v(\mathbb{G}) \setminus \beta_{\mathfrak{G}}^{-1}(p_i)) \coprod \{v_{u,t}^\star\}_{u \in G/D_{v_i}, t \in D_{v_i}^e}$ (resp. $e(\mathbb{G}) \cup e'(\mathbb{G})$). $\{v_{u,t}^\star\}_{u \in G/D_{v_i}, t \in D_{v_i}^e}$ admits a natural G-action as follows: For each $\tau \in G$, $\tau(v^{\star}_{u,t}) = v^{\star}_{\overline{\tau}\circ u,t}$ if $\tau \notin D_{v_i}$, $\tau(v^{\star}_{u,t}) = v^{\star}_{u,\overline{\tau}\circ t}$ if $\tau \in D^e_{v_i}$, and $\tau(v^{\star}_{u,t}) = v^{\star}_{u,t}$ if $\tau \in I^l_{v_i} + I^r_{v_i} + I_{v_i}$.

The collection of maps $\{\zeta^{\mathbb{G}^{\star}[p_i]}_e\}_e$ is as follows: (i) For any branch $b \notin \cup_{v\neq v_1} b(v)$, $\zeta^{\mathbb{G}^{\star}[p_i]}_e(b) = \zeta^{\mathbb{G}}_e(b)$ if $b \in e$ and $\zeta^{\mathbb{G}^{\star}[p_i]}_e(b) = \emptyset$ if $b \notin e$; (ii) $\zeta^{\mathbb{G}^{\star}[p_i]}_e(b) = v^{\star}_{u,t}$ if $b = b^{i,u,s,t}_l \in e$ (resp. $\zeta^{\mathbb{G}^{\star}[p_i]}_e(b) = v^{\star}_{u,t}$ if $b = b^{i,u,s,t}_r \in e$) and $\zeta^{\mathbb{G}^{\star}[p_i]}_e(b) = \emptyset$ if $b \notin e$.

We define a map $\sigma_{\mathfrak{G}^{\star}[p_i]} : v(\mathbb{G}^{\star}[p_i]) \longrightarrow \mathbb{Z}$ as follows: If $v^{\star}_{u,t} \neq v \in v(\mathbb{G}^{\star}[p_i])$, then we have $\sigma_{\mathfrak{G}^{\star}[p_i]}(v) := \sigma_{\mathfrak{G}}(v)$; If $v = v^{\star}_{u,t}$, then we have

$$\sigma_{\mathfrak{G}^{\star}[p_i]}(v^{\star}_{u,t}) := -\sharp(I^l_{v_i} + I^r_{v_i}) + \sharp((I^r_{v_i} + I^l_{v_i})/I^l_{v_i})(\sharp I^l_{v_i} - 1) + \sharp((I^r_{v_i} + I^l_{v_i})/I^r_{v_i})(\sharp I^r_{v_i} - 1) + 1.$$

We define a morphism of semi-graphs $\beta_{\mathfrak{G}^{\star}[p_i]} : \mathbb{G}^{\star}[p_i] \longrightarrow \mathbb{P}_n$ as follows: (i) For any $v \in v(\mathbb{G}^{\star}[p_i])$, then $\beta_{\mathfrak{G}^{\star}[p_i]}(v) = p_i$ if $v \in \{v^{\star}_{u,t}\}_{u\in G/D_{v_i}, t\in D^e_{v_i}}$ and $\beta_{\mathfrak{G}^{\star}[p_i]}(v) = \beta_{\mathfrak{G}}(v)$ if $v \notin \{v^{\star}_{u,t}\}_{u\in G/D_{v_i}, t\in D^e_{v_i}}$; (ii) If $e \in e(\mathbb{G}^{\star}[p_i]) \cup e'(\mathbb{G}^{\star}[p_i])$, then we have $\beta_{\mathfrak{G}^{\star}[p_i]}(e) = \beta_{\mathfrak{G}}(e)$.

Thus, the triple $\mathfrak{G}^{\star}[p_i] := (\mathbb{G}^{\star}[p_i], \sigma_{\mathfrak{G}^{\star}[p_i]}, \beta_{\mathfrak{G}^{\star}[p_i]})$ is a n-semi-graph with p-rank.

Moreover, $\mathbb{G}$ admits a natural G-action as follows: (i) the action of G on $v(\mathbb{G}^{\star}[p_i]) \setminus \{v^{\star}_{u,t}\}_{u\in G/D_{v_i}, t\in D^e_{v_i}}$ (resp. $e(\mathbb{G}^{\star}[p_i]) \cup e'(\mathbb{G}^{\star}[p_i])$) is the action of G on $v(\mathbb{G}) \setminus \beta^{-1}_{\mathfrak{G}}(p_i)$ (resp. $e(\mathbb{G}) \cup e'(\mathbb{G})$); (ii) The action of G on $\{v^{\star}_{u,t}\}_{u\in G/D_{v_i}, t\in D^e_{v_i}}$ is the action defined above.

Let us explain that with the G-action defined above, $\mathfrak{G}^{\star}[p_i]$ is a G-covering over $\mathfrak{P}_n$. Let

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

be an arbitrary maximal filtration of G. Write

$$\Phi_{\mathfrak{G}} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ. Note that for each $j = 0, \dots, r$, $\mathfrak{G}_j$ is a G/G_j-covering over $\mathfrak{P}_n$. By the construction of $\mathfrak{G}^{\star}_j[p_i]$, we have

$$\Phi_{\mathfrak{G}^{\star}[p_i]} : \mathfrak{G}^{\star}[p_i] = \mathfrak{G}^{\star}_r[p_i] \xrightarrow{\mathfrak{b}^{\star}_r[p_i]} \mathfrak{G}^{\star}_{r-1}[p_i] \xrightarrow{\mathfrak{b}^{\star}_{r-1}[p_i]} \dots \xrightarrow{\mathfrak{b}^{\star}_1[p_i]} \mathfrak{P}_n.$$

is a sequence of p-coverings of n-semi-graphs with p-rank. Thus, $\mathfrak{G}^{\star}[p_i]$ can be regarded as a G-covering over $\mathfrak{P}_n$.

Note that by the construction of $\mathfrak{G}^{\star}[p_i]$, we see that $\mathbb{E}^{\Phi_{\mathfrak{G}}}_j = \mathbb{E}^{\Phi_{\mathfrak{G}^{\star}[p_i]}}_j$ for each $j = 1, \dots, r$.

Definition 2.5. Let $\mathfrak{G} := (\mathbb{G}, \sigma_{\mathfrak{G}}, \beta_{\mathfrak{G}})$ be a G-covering over $\mathfrak{P}_n$, p_i a vertex of $v(\mathbb{P}_n)$. We define an operator $\rightleftharpoons^{I}_{II}$ (resp. $\rightleftharpoons^{II}_{I}$) from a G-covering to a G-covering to be

$$\rightleftharpoons^{I}_{II} (p_i)(\mathfrak{G}) = \mathfrak{G}^{*}[p_i]$$

$$(\text{resp. } \rightleftharpoons^{II}_{I} (p_i)(\mathfrak{G}) := \mathfrak{G}^{\star}[p_i]).$$

Lemma 2.6. *Let $\mathfrak{G}$ be a G-covering over $\mathfrak{P}_n$ and $\mathbb{G}$ the underlying semi-graph of $\mathfrak{G}$. Let $\mathbb{G}^{c}$ be a semi-graph defined as follows: (i) $v(\mathbb{G}^{c}) = v(\mathbb{G}) \cup \{v_0, v_{n+1}\}$; (ii) $e(\mathbb{G}^{c}) = e(\mathbb{G}) \cup e(\mathbb{G})$ and $e'(\mathbb{G}^{c}) = \emptyset$; (iii) $\zeta^{\mathbb{G}^{c}}_e = \zeta^{\mathbb{G}}_e$ if $\beta_{\mathfrak{G}}(e) \notin \{e_{0,1}, e_{n,n+1}\}$; (iv) If $e = \{b^l, b^r\}$ such that the image $\beta_{\mathfrak{G}}(e) = e_{0,1}$ and $\zeta^{\mathfrak{G}}_e(b^l) = \{v(\mathbb{G})\}$ (resp. the image $\beta_{\mathfrak{G}}(e) = e_{n,n+1}$ and $\zeta^{\mathfrak{G}}_e(b^r) = \{v(\mathbb{G})\}$), we have $\zeta^{\mathfrak{G}^{c}}_e(b^l) = v_0$ (resp. $\zeta^{\mathfrak{G}^{c}}_e(b^r) = v_{n+1}$). Let $I_{e_{0,1}}$ (resp. $I_{e_{n,n+1}}$) be the inertia group of an element of $\beta^{-1}_{\mathfrak{G}}(e_{0,1})$ (resp. $\beta^{-1}_{\mathfrak{G}}(e_{n,n+1})$). Note that since G is* *an abelian group, $I_{e_{0,1}}$ (resp. $I_{e_{n,n+1}}$) does not depend on the choice of the elements of $\beta_{\mathfrak{G}}^{-1}(e_{0,1})$ (resp. $\beta_{\mathfrak{G}}^{-1}(e_{n,n+1})$). Then we have*

$$\mathrm{rank}_{\mathbb{Z}}\mathrm{H}^1(\mathbb{G}^{\mathrm{c}}, \mathbb{Z}) - \mathrm{rank}_{\mathbb{Z}}\mathrm{H}^1(\mathbb{G}, \mathbb{Z}) = \sharp G/I_{e_{0,1}} - 1 + \sharp G/I_{e_{n,n+1}} - 1.$$

Proof. The lemma follows from the construction of $\mathbb{G}^{\mathrm{c}}$ immediately. □

Proposition 2.7. *Let $\mathfrak{G} := (\mathbb{G}, \sigma_{\mathfrak{G}}, \beta_{\mathfrak{G}})$ be a G-covering over $\mathfrak{P}_n$ and p_i a vertex of $v(\mathbb{P}_n)$. Then we have $\sigma(\mathfrak{G}) = \sigma(\mathfrak{G}^*[p_i])$ and $\sigma(\mathfrak{G}) = \sigma(\mathfrak{G}^\star[p_i])$.*

Proof. Let $v_i \in \beta_{\mathfrak{G}}^{-1}(p_i)$. If $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) = 1$ (resp. $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) = \sharp G/I_{v_i}$), by the definition of Operator I (resp. Operator II), the proposition is trivial. Then we may assume that $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) \neq 1$ (resp. $\sharp\beta_{\mathfrak{G}}^{-1}(p_i) \neq \sharp G/I_{v_i}$). Write $I_{e_{0,1}}$ (resp. $I_{e_{n,n+1}}$) for the inertia group of an element of $\beta_{\mathfrak{G}}^{-1}(e_{0,1})$ (resp. $\beta_{\mathfrak{G}}^{-1}(e_{n,n+1})$).

First, we will prove the proposition under the assumption that $I_{e_{0,1}} = I_{e_{n,n+1}} = G$ holds. Write $(-)$ for the rank of a semi-graph $(-)$ (i.e., the rank of $\mathrm{H}^1((-), \mathbb{Z})$ as a free $\mathbb{Z}$-module). Thus, we have

$$\sigma(\mathfrak{G}) = \sum_{v \in \beta_{\mathfrak{G}}^{-1}(p_i)} \sigma_{\mathfrak{G}}(v) + \sum_{v \in v(\mathbb{G} \setminus \beta_{\mathfrak{G}}^{-1}(p_i))} \sigma_{\mathfrak{G}}(v) + r(\mathbb{G} \setminus \beta_{\mathfrak{G}}^{-1}(p_i))$$
$$+ r(\mathbb{G}) - r(\mathbb{G} \setminus \beta_{\mathfrak{G}}^{-1}(p_i)),$$

$$\sigma(\mathfrak{G}^*[p_i]) = \sigma_{\mathfrak{G}^*[p_i]}(v^*) + \sum_{v \in v(\mathfrak{G}^*[p_i] \setminus \beta_{\mathfrak{G}^*[p_i]}^{-1}(p_i))} \sigma_{\mathfrak{G}^*[p_i]}(v) + r(\mathfrak{G}^*[p_i] \setminus \beta_{\mathfrak{G}^*[p_i]}^{-1}(p_i))$$
$$+ r(\mathfrak{G}^*[p_i]) - r(\mathfrak{G}^*[p_i] \setminus \beta_{\mathfrak{G}^*[p_i]}^{-1}(p_i)),$$

and

$$\sigma(\mathfrak{G}^\star[p_i]) = \sum_{v \in \beta_{\mathfrak{G}^\star[p_i]}^{-1}(p_i)} \sigma_{\mathfrak{G}^\star[p_i]}(v) + \sum_{v \in v(\mathfrak{G}^\star[p_i] \setminus \beta_{\mathfrak{G}^\star[p_i]}^{-1}(p_i))} \sigma_{\mathfrak{G}^\star[p_i]}(v) + r(\mathfrak{G}^\star[p_i] \setminus \beta_{\mathfrak{G}^\star[p_i]}^{-1}(p_i))$$
$$+ r(\mathfrak{G}^\star[p_i]) - r(\mathfrak{G}^\star[p_i] \setminus \beta_{\mathfrak{G}^\star[p_i]}^{-1}(p_i)).$$

Note that we have $r(\mathbb{G} \setminus \beta_{\mathfrak{G}}^{-1}(p_i)) = r(\mathfrak{G}^*[p_i] \setminus \beta_{\mathfrak{G}}^{-1}(p_i)) = r(\mathfrak{G}^\star[p_i] \setminus \beta_{\mathfrak{G}}^{-1}(p_i))$ and $\sum_{v \in v(\mathbb{G} \setminus \beta_{\mathfrak{G}}^{-1}(p_i))} \sigma_{\mathfrak{G}}(v) = \sum_{v \in v(\mathfrak{G}^*[p_i] \setminus \beta_{\mathfrak{G}^*[p_i]}^{-1}(p_i))} \sigma_{\mathfrak{G}^*[p_i]}(v) = \sum_{v \in v(\mathfrak{G}^\star[p_i] \setminus \beta_{\mathfrak{G}^\star[p_i]}^{-1}(p_i))} \sigma_{\mathfrak{G}^\star[p_i]}(v)$.

First, let us prove $\sigma(\mathfrak{G}) = \sigma(\mathfrak{G}^*[p_i])$. We follow the notations of Operator I. We have

$$\sigma(\mathfrak{G}) = \sum_{v \in \beta_{\mathfrak{G}}^{-1}(p_i)} \sigma_{\mathfrak{G}}(v) + \sum_{v \in v(\mathbb{G} \setminus \beta_{\mathfrak{G}}^{-1}(p_i))} \sigma_{\mathfrak{G}}(v) + r(\mathbb{G} \setminus \beta_{\mathfrak{G}}^{-1}(p_i))$$

$$+ \sharp G/D_{v_i}(\sharp((D_{v_i}/I_{v_i})/I_{v_i}^l) - 1 + \sharp((D_{v_i}/I_{v_i})/I_{v_i}^r) - 1) + \sharp G/D_{v_i} - 1$$

$$= \sharp G/D_{v_i}(-\sharp D_{v_i}/I_{v_i} + \sharp((D_{v_i}/I_{v_i})/I_{v_i}^l)(\sharp I_{v_i}^l - 1) + \sharp((D_{v_i}/I_{v_i})/I_{v_i}^r)(\sharp I_{v_i}^r - 1) + 1)$$

$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))$$

$$+\sharp G/D_{v_i}(\sharp((D_{v_i}/I_{v_i})/I_{v_i}^l)-1+\sharp((D_{v_i}/I_{v_i})/I_{v_i}^r)-1)+\sharp G/D_{v_i}-1$$

$$=\sharp G/I_{v_i}-1+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i)).$$

On the other hand, we have

$$\sigma(\mathfrak{G}^*[p_i])=\sigma_{\mathfrak{G}^*[p_i]}(v^*)+\sharp((G/I_{v_i})/I_{v_i}^l)-1+\sharp((G/I_{v_i})/I_{v_i}^r)-1$$

$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^*[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))$$

$$=-\sharp(G/I_{v_i})+\sharp((G/I_{v_i})/I_{v_i}^l)(\sharp I_{v_i}^l-1)+\sharp((G/I_{v_i})/I_{v_i}^r)(\sharp I_v^r-1)+1$$

$$+\sharp((G/I_{v_i})/I_{v_i}^l)-1+\sharp((G/I_{v_i})/I_{v_i}^r)-1$$

$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^*[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))$$

$$=\sharp G/I_{v_i}-1+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^*[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i)).$$

Thus, $\sigma(\mathfrak{G})=\sigma(\mathfrak{G}^*[p_i])$ holds.

Suppose that either $I_{e_{0,1}}$ or $I_{e_{n,n+1}}$ is not equal to G. By Lemma 2.6, we have

$$\sigma(\mathfrak{G})=\sum_{v\in\beta_{\mathfrak{G}}^{-1}(p_i)}\sigma_{\mathfrak{G}}(v)+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))$$

$$+\sharp G/D_{v_i}(\sharp((D_{v_i}/I_{v_i})/I_{v_i}^l)-1+\sharp((D_{v_i}/I_{v_i})/I_{v_i}^r)-1)+\sharp G/D_{v_i}-1-\sharp G/I_{e_{0,1}}-\sharp G/I_{e_{n,n+1}}$$

$$=\sharp G/D_{v_i}(-\sharp D_{v_i}/I_{v_i}+\sharp((D_{v_i}/I_{v_i})/I_{v_i}^l)(\sharp I_{v_i}^l-1)+\sharp((D_{v_i}/I_{v_i})/I_{v_i}^r)(\sharp I_{v_i}^r-1)+1)$$

$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))$$

$$+\sharp G/D_{v_i}(\sharp((D_{v_i}/I_{v_i})/I_{v_i}^l)-1+\sharp((D_{v_i}/I_{v_i})/I_{v_i}^r)-1)+\sharp G/D_{v_i}-1-\sharp G/I_{e_{0,1}}+1-\sharp G/I_{e_{n,n+1}}+1$$

$$=\sharp G/I_{v_i}+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))-\sharp G/I_{e_{0,1}}-\sharp G/I_{e_{n,n+1}}+1.$$

On the other hand, we have

$$\sigma(\mathfrak{G}^*[p_i])=\sigma_{\mathfrak{G}^*[p_i]}(v^*)+\sharp((G/I_{v_i})/I_{v_i}^l)-1+\sharp((G/I_{v_i})/I_{v_i}^r)-1$$

$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^*[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))-\sharp G/I_{e_{0,1}}+1-\sharp G/I_{e_{n,n+1}}+1$$

$$=-\sharp(G/I_{v_i})+\sharp((G/I_{v_i})/I^l_{v_i})(\sharp I^l_{v_i}-1)+\sharp((G/I_{v_i})/I^r_{v_i})(\sharp I^r_{v}-1)+1$$
$$+\sharp((G/I_{v_i})/I^l_{v_i})-1+\sharp((G/I_{v_i})/I^r_{v_i})-1$$
$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^*[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))-\sharp G/I_{e_{0,1}}+1-\sharp G/I_{e_{n,n+1}}+1$$

$$=\sharp G/I_{v_i}+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^*[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))-\sharp G/I_{e_{0,1}}-\sharp G/I_{e_{n,n+1}}+1.$$

Thus, $\sigma(\mathfrak{G})=\sigma(\mathfrak{G}^*[p_i])$ holds.

Next, let us compute $\sigma(\mathfrak{G}^\star[p_i])$. First, suppose that $I_{e_{0,1}}=I_{e_{n,n+1}}=G$ holds. Write W for the group

$$(G/I_{v_i})/(I^l_{v_i}+I^r_{v_i}).$$

We have

$$\sigma(\mathfrak{G}^\star[p_i])=\sum_{v\in\beta^{-1}_{\mathfrak{G}^\star[p_i]}(p_i)}\sigma_{\mathfrak{G}^\star[p_i]}(v)+\sharp W-1+\sharp W(\sharp((I^r_{v_i}+I^l_{v_i})/I^l_{v_i})-1+\sharp((I^r_{v_i}+I^l_{v_i})/I^r_{v_i})-1)$$
$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^\star[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))$$

$$=\sharp W(-\sharp(I^l_{v_i}+I^r_{v_i})+\sharp((I^r_{v_i}+I^l_{v_i})/I^l_{v_i})(\sharp I^l_{v_i}-1)+\sharp((I^r_{v_i}+I^l_{v_i})/I^r_{v_i})(\sharp I^r_{v_i}-1)+1)$$
$$+\sharp W-1+\sharp W(\sharp((I^r_{v_i}+I^l_{v_i})/I^l_{v_i})-1+\sharp((I^r_{v_i}+I^l_{v_i})/I^r_{v_i})-1)$$
$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^\star[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))$$
$$=\sharp G/I_{v_i}-1+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^\star[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i)).$$

Thus, we have $\sigma(\mathfrak{G})=\sigma(\mathfrak{G}^\star[p_i])$.

Suppose that either $I_{e_{0,1}}$ or $I_{e_{n,n+1}}$ is not equal to G. By Lemma 2.6, we have

$$\sigma(\mathfrak{G}^\star[p_i])=\sum_{v\in\beta^{-1}_{\mathfrak{G}^\star[p_i]}(p_i)}\sigma_{\mathfrak{G}^\star[p_i]}(v)+\sharp W-1+\sharp W(\sharp((I^r_{v_i}+I^l_{v_i})/I^l_{v_i})-1+\sharp((I^r_{v_i}+I^l_{v_i})/I^r_{v_i})-1)$$
$$+\sum_{v\in v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))}\sigma_{\mathfrak{G}^\star[p_i]}(v)+r(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(p_i))-\sharp G/I_{e_{0,1}}+1-\sharp G/I_{e_{n,n+1}}+1$$

$$= \sharp W(-\sharp(I^l_{v_i} + I^r_{v_i}) + \sharp((I^r_{v_i} + I^l_{v_i})/I^l_{v_i})(\sharp I^l_{v_i} - 1) + \sharp((I^r_{v_i} + I^l_{v_i})/I^r_{v_i})(\sharp I^r_{v_i} - 1) + 1)$$
$$+\sharp W - 1 + \sharp W(\sharp((I^r_{v_i} + I^l_{v_i})/I^l_{v_i}) - 1 + \sharp((I^r_{v_i} + I^l_{v_i})/I^r_{v_i}) - 1)$$
$$+ \sum_{v \in v(\mathbb{G}\setminus\beta^{-1}_{\mathfrak{G}}(p_i))} \sigma_{\mathfrak{G}^\star[p_i]}(v) + r(\mathbb{G} \setminus \beta^{-1}_{\mathfrak{G}}(p_i)) - \sharp G/I_{e_{0,1}} + 1 - \sharp G/I_{e_{n,n+1}} + 1$$

$$= \sharp G/I_{v_i} + \sum_{v \in v(\mathbb{G}\setminus\beta^{-1}_{\mathfrak{G}}(p_i))} \sigma_{\mathfrak{G}^\star[p_i]}(v) + r(\mathbb{G} \setminus \beta^{-1}_{\mathfrak{G}}(p_i)) - \sharp G/I_{e_{0,1}} - \sharp G/I_{e_{n,n+1}} + 1.$$

Thus, we have $\sigma(\mathfrak{G}) = \sigma(\mathfrak{G}^\star[p_i])$.

We complete the proof of the proposition. □

Remark 2.7.1. Let $\mathfrak{G}$ be a G-covering over $\mathfrak{P}_n$. By the definition of coverings, for any maximal filtration of G, there exists a sequence of p-coverings induced by the maximal filtration of G:

$$\mathfrak{G} = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n.$$

By Proposition 2.7, for calculating the p-rank $\sigma(\mathfrak{G})$, we may assume that $\mathfrak{b}_i$ do not have either étale Type-I for all i or Type-II for all i.

Theorem 2.8. *Let G be an abelian p-group with order p^r, Φ a maximal filtration of G, and $\mathfrak{G} := (\mathbb{G}, \sigma_{\mathfrak{G}}, \beta_{\mathfrak{G}})$ a G-covering over $\mathfrak{P}_n$. Write*

$$\Phi_{\mathfrak{G}} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ and $\mathbb{E}^{\Phi_{\mathfrak{G}}}$ for the étale-chain associated to $\Phi_{\mathfrak{G}}$. For each $j = 1, \dots, n$, write $\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j)$ (resp. $\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^l_j)$, $\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^r_j)$) for the disjoint union

$$\coprod_{s \text{ s.t. } p_j \in v(\mathbb{E}^{\Phi_{\mathfrak{G}}}_s)} \mathbb{E}^{\Phi_{\mathfrak{G}}}_s \quad (\textit{resp.} \coprod_{s \text{ s.t. } b^l_j \in e(\mathbb{E}^{\Phi_{\mathfrak{G}}}_s)\cup e'(\mathbb{E}^{\Phi_{\mathfrak{G}}}_s)} \mathbb{E}^{\Phi_{\mathfrak{G}}}_s, \coprod_{s \text{ s.t. } b^r_j \in e(\mathbb{E}^{\Phi_{\mathfrak{G}}}_s)\cup e'(\mathbb{E}^{\Phi_{\mathfrak{G}}}_s)} \mathbb{E}^{\Phi_{\mathfrak{G}}}_s).$$

Then we have

$$\sigma(\mathfrak{G}) = \sum_{j=1}^{n}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^l_j)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^r_j)} + 1) + \sum_{j=1}^{n-1}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^r_j)} - 1).$$

$$= \sum_{j=1}^{n}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^l_j)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^r_j)} + 1) + \sum_{j=2}^{n}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^l_j)} - 1).$$

Proof. By Remark 2.7.1, we may assume that $\mathfrak{b}_j$ do not have étale Type-I for all j. Thus, we obtain $v(\mathbb{G}) = \{v_1, \dots, v_n\}$, where for each j, v_j denotes the unique vertex $\beta^{-1}_{\mathfrak{G}}(p_j)$. Then for each $j = 1, \dots, n$, we have

$$\sigma_{\mathfrak{G}_i}(v_j) = -p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j)} + p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^l_j)}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j) - \sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^l_j)} - 1) + p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^r_j)}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j) - \sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b^r_j)} - 1) + 1$$

$$= p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^l)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^r)} + 1.$$

On the other hand, the rank of $\mathrm{H}^1(\mathbb{G}, \mathbb{Z})$ as a free $\mathbb{Z}$-module is

$$\sum_{j=1}^{n-1}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^r)} - 1) = \sum_{j=2}^{n}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^l)} - 1).$$

Then we have

$$\sigma(\mathfrak{G}) = \sum_{j=1}^{n}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^l)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^r)} + 1) + \sum_{j=1}^{n-1}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^r)} - 1)$$

$$= \sum_{j=1}^{n}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^l)} - p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^r)} + 1) + \sum_{j=2}^{n}(p^{\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_j^l)} - 1).$$

This completes the proof of the theorem. □

Corollary 2.9. *Let $G_i, i \in \{1, 2\}$ be an abelian p-group with order p^r, Φ^i a maximal filtration of G_i, $\mathfrak{G}^i := (\mathbb{G}^i, \sigma_{\mathfrak{G}^i}, \beta_{\mathfrak{G}^i})$ a G_i-covering over $\mathfrak{P}_n$. Write*

$$\Phi^i_{\mathfrak{G}^i} : \mathfrak{G}^i = \mathfrak{G}^i_r \xrightarrow{\mathfrak{b}^i_r} \mathfrak{G}^i_{r-1} \xrightarrow{\mathfrak{b}^i_{r-1}} \dots \xrightarrow{\mathfrak{b}^i_1} \mathfrak{G}^i_0 = \mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ^i, and $\mathbb{E}^{\Phi_{\mathfrak{G}^i}}$ for the étale-chain associated to $\Phi_{\mathfrak{G}^i}$. Suppose that $\mathbb{E}^{\Phi_{\mathfrak{G}^1}} = \mathbb{E}^{\Phi_{\mathfrak{G}^2}}$ holds. Then we have $\sigma(\mathfrak{G}^1) = \sigma(\mathfrak{G}^2)$.

Proof. Since $\mathbb{E}^{\Phi^1_{\mathfrak{G}^1}} = \mathbb{E}^{\Phi^2_{\mathfrak{G}^2}}$ holds, we see that $\sharp\mathbb{E}^{\Phi^1_{\mathfrak{G}^1}}(p_j) = \sharp\mathbb{E}^{\Phi^2_{\mathfrak{G}^2}}(p_j)$, $\sharp\mathbb{E}^{\Phi^1_{\mathfrak{G}^1}}(b_j^l) = \sharp\mathbb{E}^{\Phi^2_{\mathfrak{G}^2}}(b_j^l)$, and $\sharp\mathbb{E}^{\Phi^1_{\mathfrak{G}^1}}(b_j^r) = \sharp\mathbb{E}^{\Phi^2_{\mathfrak{G}^2}}(b_j^r)$ for all j. Thus, by Theorem 2.8, we obtain $\sigma(\mathfrak{G}^1) = \sigma(\mathfrak{G}^2)$. This completes the proof of the corollary. □

Theorem 2.10. *Let G be an abelian p-group with order p^r, Φ_G a maximal filtration of G, and $\mathfrak{G}$ a G-covering over $\mathfrak{P}_n$. Write*

$$\Phi_{\mathfrak{G}} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ_G, and $\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}_{j\in J}$ for the set of j-th étale-chains associated to $\Phi_{\mathfrak{G}}$. Let $I := \{j_1, \dots, j_r\}$ be a new index set. For each $i = 1, \dots, r$, write $\mathbb{E}_i$ for $\mathbb{E}_{j_i}^{\Phi_{\mathfrak{G}}}$. Then there exist an elementary abelian group A with order p^r, a maximal filtration Φ_A of A, and an A-covering $\mathfrak{F}$ over $\mathfrak{P}_n$ such that the i-th étale-chain $\mathbb{E}_i^{\Phi_{\mathfrak{F}}}$ associated to the sequence of p-coverings of n-semi-graphs with p-rank $\Phi_{\mathfrak{F}}$ induced by Φ_A is equal to $\mathbb{E}_i$ for each $i = 1, \dots, r$.

Proof. Since the operator $\rightleftharpoons_{II}^{I}$ does not change the étale-chain $\mathbb{E}^{\Phi_{\mathfrak{G}}}$, we may assume that $\mathfrak{b}_i$ do not have étale Type-I for all i. Let A_i, $i \in \{1, \dots, r\}$, be a cyclic abelian p-group with order p. We construct a semi-graph with p-rank $\mathfrak{F}$ step by step.

$\mathbb{F}_1 := (v(\mathbb{F}_1), e(\mathbb{F}_1) \cup e'(\mathbb{F}_1), \{\zeta_e^{\mathbb{F}_1}\}_e)$ is a semi-graph as follows:

(i) $v(\mathbb{F}_1) := \{v_1^1, \dots, v_n^1\}$;

(ii) $e(\mathbb{F}) \cup e'(\mathbb{F})$ consists of the following elements:

(a) $\{e^{\tau_1}_{i,i+1} := \{b_l(e^{\tau_1}_{i,i+1}), b_r(e^{\tau_1}_{i,i+1})\}\}_{\tau_1 \in A_1}$ is a set associated to $e_{i,i+1}$ if $e_{i,i+1} \in e(\mathbb{E}_1) \cup e'(\mathbb{E}_1)$;

(b) $e^1_{i,i+1} := \{b_l(e^1_{i,i+1}), b_r(e^1_{i,i+1})\}$ is a set associated to $e_{i,i+1}$ if $e_{i,i+1} \notin e(\mathbb{E}_1) \cup e'(\mathbb{E}_1)$;

(iii) $\zeta^{\mathbb{F}_1}_e(b_l(e^1_{i,i+1})) = v^1_i$ (resp. $\zeta^{\mathbb{F}_1}_e(b_l(e^{\tau_1}_{i,i+1})) = v^1_i$) if $i \neq 0$ and $\zeta^{\mathbb{F}_1}_e(b_l(e^1_{i,i+1})) = v(\mathbb{F})$ (resp. $\zeta^{\mathbb{F}_1}_e(b_l(e^{\tau_1}_{i,i+1})) = v(\mathbb{F}_1)$) if $i = 0$;

(iv) $\zeta^{\mathbb{F}_1}_e(b_r(e^1_{i,i+1})) = v^1_{i+1}$ (resp. $\zeta^{\mathbb{F}_1}_e(b_r(e^{\tau_1}_{i,i+1})) = v^1_{i+1}$) if $i \neq n$ and $\zeta_e(b_r(e^1_{i,i+1})) = v(\mathbb{F})$ (resp. $\zeta^{\mathbb{F}_1}_e(b_r(e^{\tau_1}_{i,i+1})) = v(\mathbb{F}_1)$) if $i = n$.

We have a natural morphism $\beta_{\mathfrak{F}_1} : \mathbb{F}_1 \longrightarrow \mathbb{P}_n$ defined as follows: (i) $\beta_{\mathfrak{F}_1}(v^1_i) = p_i$; (ii) $\beta_{\mathfrak{F}_1}((e^1_{i,i+1})) = e_{i,i+1}$ (resp. $\beta_{\mathfrak{F}_1}(b_l(e^{\tau_1}_{i,i+1})) = e_{i,i+1}$).

Next, we define a p-rank map $\sigma_{\mathfrak{F}_1} : v(\mathbb{F}_1) \longrightarrow \mathbb{Z}$ as follows: (i) If $p_i \in v(\mathbb{E}_1)$ and $\sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_l) = \sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_r) = 1$, then we have

$$\sigma_{\mathfrak{F}_1}(v^1_i) = -p + p - 1 + p - 1 + 1 = p - 1;$$

(ii) If $p_i \in v(\mathbb{E}_1), \sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_l) = 1$, and $\sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_r) = p$, then we have

$$\sigma_{\mathfrak{F}_1}(v^1_i) = -p + p - 1 + 1 = 0;$$

(iii) If $p_i \in v(\mathbb{E}_1), \sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_l) = p$, and $\sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_r) = 1$, then we have

$$\sigma_{\mathfrak{F}_1}(v^1_i) = -p + p - 1 + 1 = 0;$$

(iv) If $p_i \in v(\mathbb{E}_1)$ and $\sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_l) = \sharp\beta^{-1}_{\mathfrak{F}_1}(b^i_r) = p$, then we have

$$\sigma_{\mathfrak{F}_1}(v^1_i) = -p + 1;$$

(v) If $p_i \notin v(\mathbb{E}_1)$, then we have

$$\sigma_{\mathfrak{F}_1}(v^1_i) = 0.$$

Moreover, $\mathfrak{F}_1$ admits a natural action of A_1 as follows: (i) The action of A_1 on $v(\mathbb{F}_1)$ is trivial; (ii) For any $e \in e(\mathbb{F}_1) \cup e'(\mathbb{F}_1)$ and any element $\tau \in A_1$, $\tau.e^1_{i,i+1} = e^1_{i,i+1}$ and $\tau(e^{\tau_1}_{i,i+1}) = e^{\tau\circ\tau_1}_{i,i+1}$ for all $\tau_1 \in A_1$.

Thus, with the action of A_1, $\mathfrak{F}_1 := (\mathbb{F}_1, \sigma_{\mathfrak{F}_1}, \beta_{\mathfrak{F}_1})$ is an A_1-covering over $\mathfrak{P}_n$. Next, let us construct $\mathfrak{F}_2$.

$\mathbb{F}_2 := (v(\mathbb{F}_2), e(\mathbb{F}_2) \cup e'(\mathbb{F}_2), \{\zeta^{\mathbb{F}_2}_e\}_e)$ is a semi-graph as follows:

(i) $v(\mathbb{F}_2) := \{v^2_1, \dots, v^2_n\}$;

(ii) $e(\mathbb{F}_2) \cup e'(\mathbb{F}_2)$ consists of the following elements:

(a) $\{e^{1,\tau_2}_{i,i+1} := \{b_l(e^{1,\tau_2}_{i,i+1}), b_r(e^{1,\tau_2}_{i,i+1})\}\}_{\tau_2 \in A_2}$ is a set associated to $e^1_{i,i+1}$ if $\beta_{\mathfrak{F}_1}(e^1_{i,i+1}) \in e(\mathbb{E}_2) \cup e'(\mathbb{E}_2)$;

(b) $\{e^{\tau_1,\tau_2}_{i,i+1} := \{b_l(e^{\tau_1,\tau_2}_{i,i+1}), b_r(e^{\tau_1,\tau_2}_{i,i+1})\}_{\tau_2 \in A_2}\}_{\tau_1 \in A_1}$ is a set associated to $e^{\tau_1}_{i,i+1}$ if $\beta_{\mathfrak{F}_1}(e^{\tau_1}_{i,i+1}) \in e(\mathbb{E}_2) \cup e'(\mathbb{E}_2)$;

(c) $e^{1,2}_{i,i+1} := \{b_l(e^{1,2}_{i,i+1}), b_r(e^{1,2}_{i,i+1})\}$ is a set associated to $e^1_{i,i+1}$ if $\beta_{\mathfrak{F}_1}(e^1_{i,i+1}) \notin e(\mathbb{E}_2) \cup e'(\mathbb{E}_2)$;

(d) $\{e^{\tau_1,2}_{i,i+1} := \{b_l(e^{\tau_1,2}_{i,i+1}), b_r(e^{\tau_1,2}_{i,i+1})\}\}_{\tau_1 \in A_1}$ is a set associated to $e^{\tau_1}_{i,i+1}$ if $\beta_{\mathfrak{F}_1}(e^{\tau_1}_{i,i+1}) \notin e(\mathbb{E}_2) \cup e'(\mathbb{E}_2)$;

(iii) $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{1,2}))=v_i^2$ (resp. $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{\tau_1,2}))=v_i^2$, $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{1,\tau_2}))=v_i^2$, $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{\tau_1,\tau_2}))=v_i^2$) if $i\neq 0$ and $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{1,2}))=v(\mathbb{F}_2)$ (resp. $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{\tau_1,2}))=v(\mathbb{F}_2)$, $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{1,\tau_2}))=v(\mathbb{F}_2)$, $\zeta_e^{\mathbb{F}_2}(b_l(e_{i,i+1}^{\tau_1,\tau_2}))=v(\mathbb{F}_2)$) if $i=0$;

(iv) $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{1,2}))=v_i^2$ (resp. $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{\tau_1,2}))=v_i^2$, $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{1,\tau_2}))=v_i^2$, $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{\tau_1,\tau_2}))=v_i^2$) if $i\neq n$ and $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{1,2}))=v(\mathbb{F}_2)$ (resp. $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{\tau_1,2}))=v(\mathbb{F}_2)$, $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{1,\tau_2}))=v(\mathbb{F}_2)$, $\zeta_e^{\mathbb{F}_2}(b_r(e_{i,i+1}^{\tau_1,\tau_2}))=v(\mathbb{F}_2)$) if $i=n$.

We have a natural morphism $\alpha_2:\mathbb{F}_2\longrightarrow\mathbb{F}_1$ as follows: (i) $\alpha_2(v_i^2)=v_i^1$; (ii) $\alpha_2((e_{i,i+1}^{1,2}))=e_{i,i+1}^{1}$ (resp. $\alpha_2((e_{i,i+1}^{\tau_1,2}))=e_{i,i+1}^{\tau 1}$, $\alpha_2((e_{i,i+1}^{1,\tau_2}))=e_{i,i+1}^{1}$, $\alpha_2((e_{i,i+1}^{\tau_1,\tau_2}))=e_{i,i+1}^{\tau_1}$). We define $\beta_{\mathfrak{F}_2}$ to be the composite morphism $\beta_{\mathfrak{F}_1}\circ\alpha_2$.

We define a p-rank map $\sigma_{\mathfrak{F}_2}:v(\mathbb{F}_2)\longrightarrow\mathbb{Z}$ as follows: (i) If $\sharp b_l(v_i^2)=p\sharp b_l(v_i^1)$ and $\sharp b_r(v_i^2)=p\sharp b_r(v_i^1)$, then we have

$$\sigma_{\mathfrak{F}_2}(v_i^2)-1=p(\sigma_{\mathfrak{F}_1}(v_i^1)-1);$$

(ii) If $\sharp b_l(v_i^2)=\sharp b_l(v_i^1)$ and $\sharp b_r(v_i^2)=p\sharp b_r(v_i^1)$, we have

$$\sigma_{\mathfrak{F}_2}(v_i^2)-1=p(\sigma_{\mathfrak{F}_1}(v_i^1)-1)+(\sharp b_l(v_i^1))(p-1);$$

(iii) If $\sharp b_l(v_i^2)=p\sharp b_l(v_i^1)$ and $\sharp b_r(v_i^2)=\sharp b_r(v_i^1)$, we have

$$\sigma_{\mathfrak{F}_2}(v_i^2)-1=p(\sigma_{\mathfrak{F}_1}(v_i^1)-1)+(\sharp b_r(v_i^1))(p-1);$$

(iv) If $\sharp b_l(v_i^2)=\sharp b_l(v_i^2)$ and $\sharp b_r(v_i^2)=\sharp b_r(v_i^2)$, we have

$$\sigma_{\mathfrak{F}_2}(v_i^2)-1=p(\sigma_{\mathfrak{F}_1}(v_i^1)-1)+(\sharp b_l(v_i^1)+\sharp b_r(v_i^1))(p-1).$$

Moreover, there is a natural $A_1\oplus A_2$-action on $\mathfrak{F}_2$ defined as follows: (i) The action of $A_1\oplus A_2$ on $v(\mathbb{F}_2)$ is trivial; (ii) For any $e\in e(\mathbb{F}_2)\cup e'(\mathbb{F}_2)$ and any element $(\tau,\tau')\in A_1\oplus A_2$, $(\tau,\tau').e_{i,i+1}^{1,2}=e_{i,i+1}^{1,2}$, $(\tau,\tau').e_{i,i+1}^{\tau_1,2}=e_{i,i+1}^{\tau\circ\tau_1,2}$, $(\tau,\tau').e_{i,i+1}^{1,\tau_2}=e_{i,i+1}^{1,\tau'\circ\tau_2}$ and $(\tau,\tau').e_{i,i+1}^{\tau_1,\tau_2}=e_{i,i+1}^{\tau\circ\tau_1,\tau'\circ\tau_2}$.

Thus, with the action of $A_1\oplus A_2$, $\mathfrak{F}_2:=(\mathbb{F}_2,\sigma_{\mathfrak{F}_2},\beta_{\mathfrak{F}_2})$ is an $A_1\oplus A_2$-covering over $\mathfrak{P}_n$. The maximal filtration

$$0\subset A_2\subset A_1\oplus A_2$$

determines a sequence of p-coverings of n-semi-graphs with p-rank

$$\Phi_{\mathfrak{F}_2}:\mathfrak{F}_2\xrightarrow{\mathfrak{a}_2}\mathfrak{F}_1\xrightarrow{\mathfrak{a}_1}\mathfrak{F}_0=\mathfrak{P}_n.$$

Furthermore, by the construction, we have $\mathbb{E}_2^{\Phi_{\mathfrak{F}_2}}=\mathbb{E}_2$ and $\mathbb{E}_1^{\Phi_{\mathfrak{F}_1}}=\mathbb{E}_1$.

By repeating the process above, we obtain an $A:=\oplus_{i=1}^r A_i$-covering $\mathfrak{F}_r$ over $\mathfrak{P}_n$ and a maximal filtration

$$\Phi_A:0\subset A_n\subset A_n\oplus A_{n-1}\subset\cdots\subset\oplus_{i=1}^r A_i=A.$$

Then Φ_A induces a sequence of p-coverings of n-semi-graphs with p-rank

$$\Phi_{\mathfrak{F}_r}:\mathfrak{F}:=\mathfrak{F}_r\xrightarrow{\mathfrak{a}_r}\mathfrak{F}_{r-1}\xrightarrow{\mathfrak{a}_{r-1}}\dots\xrightarrow{\mathfrak{a}_1}\mathfrak{F}_0=\mathfrak{P}_n.$$

By the construction, we have the i-th étale-chain $\mathbb{E}_i^{\Phi_{\mathfrak{F}}}$ associated to $\Phi_{\mathfrak{F}_r}$ is equal to $\mathbb{E}_i$ for each $i=1,\dots,r$. We complete the proof of the theorem. $\square$

Remark 2.10.1. For the sequence

$$\Phi_{\mathfrak{F}} : \mathfrak{F} = \mathfrak{F}_r \xrightarrow{\mathfrak{a}_r} \mathfrak{F}_{r-1} \xrightarrow{\mathfrak{a}_{r-1}} \dots \xrightarrow{\mathfrak{a}_1} \mathfrak{F}_0 = \mathfrak{P}_n$$

constructed in Theorem 2.10, by Remark 2.7.1, we may assume that $\mathfrak{a}_i$ do not have étale Type-II for all i. Furthermore, by Corollary 2.9, we have $\sigma(\mathfrak{G}) = \sigma(\mathfrak{F})$.

2.3 Bounds of p-ranks of abelian coverings

Let G be a finite abelian p-group with order p^r. In this subsection, we calculate a bound of p-rank of a G-covering over $\mathfrak{P}_n$.

First, let us fix some notations. Let $\mathfrak{G}$ be a G-covering over $\mathfrak{P}_n$ and Φ a maximal filtration of G. Write

$$\Phi_{\mathfrak{G}} : \mathfrak{G} = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ and $\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}_j$ for the set of j-th étale-chains associated to $\Phi_{\mathfrak{G}}$. If $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$ is empty, we have $\sigma(\mathfrak{G}_j) = \sigma(\mathfrak{G}_{j-1})$; thus, for calculating the bound of the p-rank $\sigma(\mathfrak{G})$, we may assume that $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$ are not empty for all j. Moreover, by Remark 2.10.1, we may assume that for each $j = 1, \dots, r$ and each $v \in v(\mathbb{G}_j)$, $\mathfrak{b}_j$ is not étale Type-II at v.

Let $e_0 \in \beta_{\mathfrak{G}}^{-1}(e_{0,1})$ (resp. $e_{n+1} \in \beta_{\mathfrak{G}}^{-1}(e_{n,n+1})$). Write I_{e_0} (resp. $I_{e_{n+1}}$) for the inertia group of e_0 (resp. e_{n+1}). Note that since G is an abelian group, the group I_{e_0} (resp. $I_{e_{n+1}}$) does not depend on the choice of the elements of $\beta_{\mathfrak{G}}^{-1}(e_{0,1})$ (resp. $\beta_{\mathfrak{G}}^{-1}(e_{n,n+1})$). Moreover, according to Definition 2.3 (c-iii), G is generated by I_{e_0} and $I_{e_{n+1}}$.

For each $j = 1, \dots, r$, since $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$ is a sub-semi-graph of $\mathbb{P}_n$, $v(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \subseteq \{p_1, \dots, p_n\} = v(\mathbb{P}_n)$ admits a natural order which is induced by the order of natural number $\mathbb{N}$; then we may define the *initial vertex* and the *terminal vertex* for $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$. Write $i(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$ (resp. $t(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$) for the initial (resp. the terminal) vertex of $v(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$, $l(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$ for $\sharp v(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$. For an element $p_i \in v(\mathbb{P}_n)$, we shall say that p_i is $\mathbb{A}_r^1$-*type* (resp. $\mathbb{A}_l^1$-*type;* $\mathbb{G}_m^1$-*type;* P-*type;* $\mathbb{P}^1$-*type*) at $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$ if p_i is equal to $i(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$, and $\mathfrak{b}_j$ is étale at $\beta_{\mathfrak{G}_j}^{-1}(p_i)$ with Type-III (resp. p_i is equal to $t(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$, and $\mathfrak{b}_j$ is étale at $\beta_{\mathfrak{G}_j}^{-1}(p_i)$ with Type-IV; $v(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$ is equal to $\{p_i\}$, and $\mathfrak{b}_j$ is étale at $\beta_{\mathfrak{G}_j}^{-1}(p_i)$ with Type-V; $\mathfrak{b}_j$ is purely inseparable at $\beta_{\mathfrak{G}_j}^{-1}(p_i)$; p_i is contained in $v(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$, and $\mathfrak{b}_j$ is étale at $\beta_{\mathfrak{G}_j}^{-1}(p_i)$ with Type-I). Write $\sharp\pi_0(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})$ for the cardinality of the connected components of $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$. Note that since we assume that $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$ is not empty for each $j = 1, \dots, r$, we have $\sharp\pi_0(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \geq 1$.

We define a sub-semi-graph $\mathbb{P}_{x,y}$ of $\mathbb{P}_n$ as follows: (i) $v(\mathbb{P}_{x,y}) = \{p_x, \dots, p_y\}$; (ii) $e(\mathbb{P}_{x,y}) = \{e_{x,x+1}, \dots, e_{y-1,y}\}$ and $e'(\mathbb{P}_{x,y}) = \{e_{x-1,x}, e_{y,y+1}\}$; (iii) $\zeta_{e_{i,i+1}}^{\mathbb{P}_{x,y}}(b_l(e_{i,i+1})) = p_i$ and $\zeta_{e_{i,i+1}}^{\mathbb{P}_{x,y}}(b_r(e_{i,i+1})) = p_{i+1}$ if $i \notin \{x-1, y\}$ (iv) $\zeta_{e_{x-1,x}}^{\mathbb{P}_{x,y}}(b_l(e_{x-1,x})) = v(\mathbb{P}_{x,y})$ (resp. $\zeta_{e_{y,y+1}}^{\mathbb{P}_{x,y}}(b_r(e_{y,y+1})) = v(\mathbb{P}_{x,y})$) and $\zeta_{e_{x-1,x}}^{\mathbb{P}_{x,y}}(b_r(e_{x-1,x})) = p_x$ (resp. $\zeta_{e_{y,y+1}}^{\mathbb{P}_{x,y}}(b_r(e_{y,y+1})) = p_y$). Note that $\mathbb{P}_{x,y}$ is a sub-semi-graph of $\mathbb{P}_n$, and the semi-graph with p-rank $\mathfrak{P}_{x,y} := (\mathbb{P}_{x,y}, \sigma_{\mathfrak{P}_n}|_{v(\mathbb{P}_{x,y})}, \mathrm{id}_{\mathbb{P}_{x,y}})$ can be regarded as a $(y-x+1)$-chain.

Lemma 2.11. *Suppose that* $\sharp\pi_0(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) = 1$ *for all* j*, and either* $I_{e_{0,1}}$ *or* $I_{e_{n,n+1}}$ *is trivial. Then the* p*-rank* $\sigma(\mathfrak{G})$ *is equal to* 0.

Proof. For each $j = 1, \dots, r$, write v_i^j for an element $\beta_{\mathfrak{G}_j}^{-1}(p_i)$. Since $\sharp\pi_0(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) = 1$ hold for all j, and $I_{e_{0,1}}$ (resp. $I_{e_{n,n+1}}$) is trivial, $\mathfrak{b}_j$ is not étale Type-III (resp. étale Type-IV) and étale Type-V at v_i^j. Then we obtain $\sigma_{\mathfrak{G}_j}(v_i^j) = 0$ by applying Remark 2.3.1. Moreover, the underlying semi-graph of $\mathfrak{G}_j$ is an tree. Thus, we obtain $\sigma(\mathfrak{G}_j) = 0$. In particular, we have $\sigma(\mathfrak{G}) = 0$. We complete the proof of the lemma. □

Lemma 2.12. *Let $G_i, i \in \{1, 2\}$ be an abelian p-group with order p^r, Φ^i a maximal filtration of G_i, and $\mathfrak{G}^i := (\mathbb{G}^i, \sigma_{\mathfrak{G}^i}, \beta_{\mathfrak{G}^i})$ a G_i-covering over $\mathfrak{P}_n$. Write*

$$\Phi_{\mathfrak{G}^i} : \mathfrak{G}^i = \mathfrak{G}_r^i \xrightarrow{\mathfrak{b}_r^i} \mathfrak{G}_{r-1}^i \xrightarrow{\mathfrak{b}_{r-1}^i} \dots \xrightarrow{\mathfrak{b}_1^i} \mathfrak{G}_0^i = \mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ^i, and $\mathbb{E}^{\Phi_{\mathfrak{G}^i}}$ for the étale-chain associated to $\Phi_{\mathfrak{G}^i}$. Suppose that for each $j = 1, \dots, r$, $\sharp\pi_0(\mathbb{E}_j^{\Phi_{\mathfrak{G}^i}}) = 1$, $i(\mathbb{E}_j^{\Phi_{\mathfrak{G}^1}}) = i(\mathbb{E}_j^{\Phi_{\mathfrak{G}^2}})$, and $t(\mathbb{E}_j^{\Phi_{\mathfrak{G}^1}}) = t(\mathbb{E}_j^{\Phi_{\mathfrak{G}^2}})$. Moreover, we suppose that $\mathbb{E}_j^{\Phi_{\mathfrak{G}^1}}$ is equal to $\mathbb{E}_j^{\Phi_{\mathfrak{G}^2}}$ if $i(\mathbb{E}_j^{\Phi_{\mathfrak{G}^1}}) \neq 1$ and $t(\mathbb{E}_j^{\Phi_{\mathfrak{G}^1}}) \neq n$. Let $e_0^1 \in \beta_{\mathfrak{G}^1}^{-1}(e_{0,1})$ and $e_0^2 \in \beta_{\mathfrak{G}^2}^{-1}(e_{0,1})$ (resp. $e_{n+1}^1 \in \beta_{\mathfrak{G}^1}^{-1}(e_{n,n+1})$ and $e_{n+1}^2 \in \beta_{\mathfrak{G}^2}^{-1}(e_{n,n+1})$). Write $I_{e_0^1}$ and $I_{e_0^2}$ (resp. $I_{e_{n+1}^1}$ and $I_{e_{n+1}^2}$) for the inertia groups of e_0^1 and e_0^2, respectively (resp. e_{n+1}^1 and e_{n+1}^2, respectively), D_0^1 (resp. D_{n+1}^1) for $G_1/I_{e_0^1}$ (resp. $G_1/I_{e_{n+1}^1}$). Furthermore, we suppose that $I_{e_0^2}$ and $I_{e_{n+1}^2}$ are equal to G_2. Then we have

$$\sigma(\mathfrak{G}^1) + \sharp D_0^1 - 1 + \sharp D_{n+1}^1 - 1 = \sigma(\mathfrak{G}^2).$$

Proof. By Remark 2.7.1, we may assume that $\mathfrak{b}_j^i$ do not have étale Type-I. For any $p_u \in v(\mathbb{P}_n)$, write v_u^i for the unique element of $\beta_{\mathfrak{G}^i}^{-1}(p_u)$. Then $D_{v_u^i}$ is equal to G_i.

If $n = 1$, note that since $\mathbb{E}_j^{\Phi_{\mathfrak{G}}}$ are not empty for all j, both $I_{v_1^1}$ and $I_{v_1^2}$ are trivial. Then we have

$$\sigma(\mathfrak{G}^1) = \sigma_{\mathfrak{G}^1}(v_1^1) = -\sharp G_1 + \sharp D_0^1(\sharp I_{e_0^1} - 1) + \sharp D_{n+1}^1(\sharp I_{e_{n+1}^1} - 1) + 1.$$

On the other hand, since both $I_{e_0^2}$ and $I_{e_{n+1}^2}$ are equal to G_2, we obtain

$$\sigma(\mathfrak{G}^2) = \sigma_{\mathfrak{G}^1}(v_1^2) = -\sharp G_2 + \sharp G_2 - 1 + \sharp G_2 - 1 + 1 = \sharp G_2 - 1.$$

Thus, we have

$$\sigma(\mathfrak{G}^1) + \sharp D_0^1 - 1 + \sharp D_{n+1}^1 - 1 = \sigma(\mathfrak{G}^2).$$

If $n > 1$, by the assumptions of $\{\mathbb{E}_j^{\Phi_{\mathfrak{G}^1}}\}_j$ and $\{\mathbb{E}_j^{\Phi_{\mathfrak{G}^2}}\}_j$, we obtain

$$\sum_{v \in v(\mathbb{G}^1) \setminus \{v_1^1, v_n^1\}} \sigma_{\mathfrak{G}^1}(v) = \sum_{v \in v(\mathbb{G}^1) \setminus \{v_1^2, v_n^2\}} \sigma_{\mathfrak{G}^2}(v)$$

and

$$\mathrm{rank}_{\mathbb{Z}} \mathrm{H}^1(\mathbb{G}^1, \mathbb{Z}) = \mathrm{rank}_{\mathbb{Z}} \mathrm{H}^1(\mathbb{G}^2, \mathbb{Z}).$$

On the other hand, since both $I_{e_0^2}$ and $I_{e_{n+1}^2}$ are equal to G_2, we have

$$\sigma_{\mathfrak{G}^2}(v_1^2) - \sigma_{\mathfrak{G}^1}(v_1^1) = \sharp G_2 - 1 - \sharp D_0^1(\sharp I_{e_0^1} - 1) = \sharp G_1 - 1 - \sharp D_0^1(\sharp I_{e_0^1} - 1) = \sharp D_0^1 - 1$$

and

$$\sigma_{\mathfrak{G}^2}(v_n^2)-\sigma_{\mathfrak{G}^1}(v_n^1)=\sharp G_2-1-\sharp D_{n+1}^1(\sharp I_{e_{n+1}^1}-1)=\sharp G_1-1-\sharp D_{n+1}^1(\sharp I_{e_{n+1}^1}-1)=\sharp D_{n+1}^1-1.$$

Thus, we obtain

$$\sigma(\mathfrak{G}^1)+\sharp D_0^1-1+\sharp D_{n+1}^1-1$$

$$=\sum_{v\in v(\mathbb{G}^1)\setminus\{v_1^1,v_n^1\}}\sigma_{\mathfrak{G}^1}(v)+\mathrm{rank}_{\mathbb{Z}}\mathrm{H}^1(\mathbb{G}^1,\mathbb{Z})+\sigma_{\mathfrak{G}^1}(v_1^1)+\sigma_{\mathfrak{G}^1}(v_n^1)+\sharp D_0^1-1+\sharp D_{n+1}^1-1$$

$$=\sum_{v\in v(\mathbb{G}^2)\setminus\{v_1^2,v_n^2\}}\sigma_{\mathfrak{G}^2}(v)+\mathrm{rank}_{\mathbb{Z}}\mathrm{H}^1(\mathbb{G}^2,\mathbb{Z})+\sigma_{\mathfrak{G}^2}(v_1^2)+\sigma_{\mathfrak{G}^2}(v_n^2)=\sigma(\mathfrak{G}^2).$$

□

We have the following theorem.

Theorem 2.13. *Let $\mathfrak{G}$ be a G-covering over $\mathfrak{P}_n$ and Φ a maximal filtration of G. Write*

$$\Phi_{\mathfrak{G}}:\mathfrak{G}=\mathfrak{G}_r\xrightarrow{\mathfrak{b}_r}\mathfrak{G}_{r-1}\xrightarrow{\mathfrak{b}_{r-1}}\dots\xrightarrow{\mathfrak{b}_1}\mathfrak{G}_0=\mathfrak{P}_n$$

for the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ, and $\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}_j$ for the set of j-th étale-chains associated to $\Phi_{\mathfrak{G}}$. Suppose that $\sharp\pi_0(\mathbb{E}_j^{\Phi_{\mathfrak{G}}})=1$ hold for all j. Then we have

$$\sigma(\mathfrak{G})\leq p^r-1.$$

Proof. We prove the theorem by induction. If $r=1$, since $\pi_0(\mathbb{E}_1^{\Phi_1})=1$, let us check the theorem case by case. If either I_{e_0} or $I_{e_{n+1}}$ is trivial, then by Lemma 2.11, we have $\sigma(\mathfrak{G})=0$. If Both I_{e_0} and $I_{e_{n+1}}$ are non-trivial, and $l(\mathbb{E}_1^{\Phi_1})$ is 1, we obtain $\mathrm{rank}_{\mathbb{Z}}\mathrm{H}^1(\mathbb{G},\mathbb{Z})$ is equal to 0; for each $v\in v(\mathbb{G})$, $\sigma_{\mathfrak{G}}(v)$ is equal to 0 if $\beta_{\mathfrak{G}}(v)$ is not contained in $v(\mathbb{E}_1^{\Phi_{\mathfrak{G}}})$, and $\sigma_{\mathfrak{G}}(v)$ is equal to $p-1$ if $\beta_{\mathfrak{G}}(v)$ is contained in $v(\mathbb{E}_1^{\Phi_{\mathfrak{G}}})$; thus, we obtain $\sigma(\mathfrak{G})=p-1$. If Both I_{e_0} and $I_{e_{n+1}}$ are non-trivial, and $l(\mathbb{E}_1^{\Phi_1})$ is ≥ 2, we have $\mathrm{rank}_{\mathbb{Z}}\mathrm{H}^1(\mathbb{G},\mathbb{Z})$ is equal to $p-1$, and $\sigma_{\mathfrak{G}}(v)$ are equal to 0 for all $v\in v(\mathbb{G})$. Thus, we have $\sigma(\mathfrak{G})=p-1$. This completes the proof of the theorem if $r=1$. From now on, we assume that r is ≥ 2.

For each $i=1,\dots,n$, let v_i be an element of $\beta_{\mathfrak{G}}^{-1}(p_i)\subseteq v(\mathfrak{G})$, I_{v_i} the inertia group of v_i. Write d for $\min\{i\mid I_{v_i}\neq G\}$. If $d\neq 1$, then we have $\beta_{\mathfrak{G}}|_{\beta_{\mathfrak{G}}^{-1}(\mathbb{P}_n\setminus\mathbb{P}_{d,n})}:\beta_{\mathfrak{G}}^{-1}(\mathbb{P}_n\setminus\mathbb{P}_{d,n})\longrightarrow\mathbb{P}_{d,n}$ is an isomorphism of semi-graphs. Then we have $\mathfrak{G}':=(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(\mathbb{P}_n\setminus\mathbb{P}_{d,n}),\sigma_{\mathfrak{G}}|_{v(\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(\mathbb{P}_n\setminus\mathbb{P}_{d,n}))},\beta_{\mathfrak{G}}|_{\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(\mathbb{P}_n\setminus\mathbb{P}_{d,n})})$ is a $(n-d+1)$-semi-graph with p-rank. Furthermore, $\mathbb{G}\setminus\beta_{\mathfrak{G}}^{-1}(\mathbb{P}_n\setminus\mathbb{P}_{d,n})$ admits a natural action of G induced by the action of G on $\mathbb{G}$. Thus, we may regard $\mathfrak{G}'$ is a G-covering over $\mathfrak{P}_{d,n}$. Note that we have $\sigma_{\mathfrak{G}}(v_i)=0$ for $i\leq d-1$ and $\mathrm{rank}_{\mathbb{Z}}\mathrm{H}^1(\beta_{\mathfrak{G}}^{-1}(\mathbb{P}_n\setminus\mathbb{P}_{d,n}),\mathbb{Z})=0$. Then $\sigma(\mathfrak{G}')$ is equal to $\sigma(\mathfrak{G})$. Thus, by replacing $\mathfrak{G}$ (resp. $\mathfrak{P}_n$) by $\mathfrak{G}'$ (resp. $\mathfrak{P}_{d,n}$), we may assume that I_{v_1} is not equal to G. Similar arguments to the arguments given above imply that we may assume that I_{v_n} is not equal to G.

Write S_1 (resp. S_2, S_3, S_4, S_5) for the set

$$\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}}\mid p_1\text{ is }\mathbb{A}_r^1\text{-type at }\mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}\text{ (resp. }\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}}\mid p_1\text{ is }\mathbb{A}_l^1\text{-type at }\mathbb{E}_j^{\Phi_{\mathfrak{G}}}\},$$

$$\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_1 \text{ is } \mathbb{G}_m^1\text{-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}, \{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_1 \text{ is P-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\},$$
$$\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_1 \text{ is } \mathbb{P}^1\text{-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}).$$

Write t for $\max\{t(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \mid i(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \in S_1 \cup S_2 \cup S_3 \cup S_5\}$, T_1 (resp. T_2, T_3, T_4, T_5) for

$$\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_t \text{ is } \mathbb{A}_r^1\text{-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\} \text{ (resp. } \{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_t \text{ is } \mathbb{A}_l^1\text{-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\},$$

$$\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_t \text{ is } \mathbb{G}_m^1\text{-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}, \{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_t \text{ is P-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\},$$
$$\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \mid p_t \text{ is } \mathbb{P}^1\text{-type at } \mathbb{E}_j^{\Phi_{\mathfrak{G}}}\}).$$

For $i \in \{1,2,3,4,5\}$, write n_i for $\sharp S_i$. Write m_1 for $\sharp T_1$, m_2 for $\sharp(S_4 \cap T_2)$, m_3 for $\sharp T_3$, m_4 for $\sharp(S_4 \cap T_4)$, m_5 for $\sharp T_5$, a_1 for $\sharp(S_1 \cap T_2)$, and a_2 for $\sharp(S_1 \cap T_4)$. Write b_1 (resp. b_2) for

$$\sharp\{\mathbb{E}_j^{\Phi_{\mathfrak{G}}} \in S_4 \cap T_4 \mid i(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \geq 2 \text{ and } t(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \leq t-1\}$$

$$(\text{resp. } \sharp\{E_j^{\Phi_{\mathfrak{G}}} \in S_4 \cap T_4 \mid i(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \geq t+1\}).$$

Note that we have $\sum_{i=1}^5 = r$ and $b_1 + b_2 = m_4$. Since t is the maximal element of $\{l(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \mid i(\mathbb{E}_j^{\Phi_{\mathfrak{G}}}) \in S_1 \cup S_2\}$ and $\sharp\pi_0(\mathbb{E}^{\Phi_{\mathfrak{G}}}) = 1$, we obtain $\sum_{i=1}^5 m_i = n_4$ and $a_1 + a_2 = n_1$.

Let $\{\mathbb{E}_1, \ldots, \mathbb{E}_r\}$ be a set of étale-chains associated to $\Phi_{\mathfrak{G}}$ with a new index set such that the following conditions: (i) $T_5 = \{\mathbb{E}_1, \ldots, \mathbb{E}_{m_5}\}$; (ii) $S_4 \cap T_4 = \{\mathbb{E}_{m_5+1}, \ldots, \mathbb{E}_{m_5+m_4}\}$; (iii) $T_1 = \{\mathbb{E}_{m_5+m_4+1}, \ldots, \mathbb{E}_{m_5+m_4+m_1}\}$; (iv) $S_4 \cap T_2 = \{\mathbb{E}_{m_5+m_4+m_1+1}, \ldots, \mathbb{E}_{m_5+m_4+m_1+m_2}\}$; (v) $S_1 \cap T_2 = \{\mathbb{E}_{m_5+m_4+m_1+m_2+1}, \ldots, \mathbb{E}_{m_5+m_4+m_1+m_2+a_1}\}$; (vi) $S_1 \cap T_4 = \{\mathbb{E}_{m_5+m_4+m_1+m_2+a_1+1}, \ldots, \mathbb{E}_{m_5+m_4+m_1+m_2+n_1}\}$; (vii) $T_3 = \{\mathbb{E}_{m_5+m_4+m_1+m_2+n_1+1}, \ldots, \mathbb{E}_{n_1+n_4}\}$; (viii) $S_2 = \{\mathbb{E}_{n_1+n_4+1}, \ldots, \mathbb{E}_{n_1+n_2+n_4}\}$; (ix) $S_3 = \{\mathbb{E}_{n_1+n_2+n_4}, \ldots, \mathbb{E}_{n_1+n_2+n_3+n_4}\}$; (x) $S_5 = \{\mathbb{E}_{n_1+n_2+n_3+n_4+1}, \ldots, \mathbb{E}_r\}$. By Theorem 2.10, there exist an elementary abelian p-group A, a maximal filtration Φ_A of A, an A-covering $\mathfrak{F} := (\mathbb{F}, \sigma_{\mathfrak{F}}, \beta_{\mathfrak{F}})$ over $\mathfrak{P}_n$, and the sequence of p-coverings of n-semi-graphs with p-rank induced by Φ_A

$$\Phi_{\mathfrak{F}} : \mathfrak{F} = \mathfrak{F}_r \xrightarrow{\mathfrak{a}_r} \mathfrak{F}_{r-1} \xrightarrow{\mathfrak{a}_{r-1}} \ldots \xrightarrow{\mathfrak{a}_1} \mathfrak{F}_0 = \mathfrak{P}_n$$

such that the j-th étale-chain $\mathbb{E}_j^{\Phi_{\mathfrak{F}}}$ associated to $\Phi_{\mathfrak{F}}$ is equal to $\mathbb{E}_j$ for each $j = 1, \ldots, r$. Since $\sigma(\mathfrak{G})$ is equal to $\sigma(\mathfrak{F})$, in order to prove the theorem, it is sufficient to calculate the bound of $\sigma(\mathfrak{F})$. Let u_i be an element of $\beta_{\mathfrak{F}}^{-1}(p_i)$, e_0 (resp. e_{n+1}) an element of $\beta_{\mathfrak{F}}^{-1}(e_{0,1})$ (resp. $\beta_{\mathfrak{F}}^{-1}(e_{n,n+1})$). Moreover, by Lemma 2.12, for calculating the bound of $\sigma(\mathfrak{F})$, we may assume that $G = I_{e_0} = I_{e_{n+1}}$ hold. Then we have $n_2 = 0$ and $n_5 = 0$. In particular, we have $\sharp\beta_{\mathfrak{G}}^{-1}(p_1) = \sharp\beta_{\mathfrak{G}}^{-1}(p_n) = 1$.

Case 1: If $t = 1$ and $n = 1$, since $G = I_{e_0} = I_{e_{n+1}}$ hold, we obtain $n_3 = r$ and

$$\sigma(\mathfrak{F}) = \sigma_{\mathfrak{F}}(u_1) = (-1)p^{n_3} + 2(p^{n_3} - 1) + 1 = p^{n_3} - 1 = p^r - 1.$$

Thus, the theorem follows.

Case 2: If $t = 1$ and $n \neq 1$, since I_{v_n} is not trivial, $\beta_{\mathfrak{F}}|_{\beta_{\mathfrak{F}}^{-1}(\mathbb{P}_{2,n})} : \beta_{\mathfrak{G}}^{-1}(\mathbb{P}_{2,n}) \longrightarrow \mathbb{P}_{2,n}$ is not an isomorphism. Write $\mathfrak{F}^{1,1}$ (resp. $\mathfrak{F}^{2,n}$) for $(\mathbb{F} \setminus \beta_{\mathfrak{F}}^{-1}(\mathbb{P}_{1,1}), \sigma_{\mathfrak{F}}|_{v(\mathbb{F}\setminus\beta_{\mathfrak{F}}^{-1}(\mathbb{P}_{1,1}))}, \beta_{\mathfrak{F}}|_{\mathbb{F}\setminus\beta_{\mathfrak{F}}^{-1}(\mathbb{P}_{1,1})})$

$(\beta_{\mathfrak{F}}^{-1}(\mathbb{P}_{2,n}), \sigma_{\mathfrak{F}}|_{v(\beta_{\mathfrak{F}}^{-1}(\mathbb{P}_{2,n}))}, \beta_{\mathfrak{F}}|_{\beta_{\mathfrak{F}}^{-1}(\mathbb{P}_{2,n})}))$. $\mathfrak{F}^{1,1}$ (resp. $\mathfrak{F}^{2,n}$) is a G-covering over $\mathfrak{P}_{1,1}$ (resp. $\mathfrak{P}_{2,n}$). Since $\mathfrak{F}^{1,1}/D_{u_1} \longrightarrow \mathfrak{P}_{1,1}$ (resp. $\mathfrak{F}^{2,n} \longrightarrow \mathfrak{F}^{2,n}/D_{u_1}$) is a composite of p-coverings which are purely inseparable, we see that $\sigma(\mathfrak{F}^{1,1}) = \sigma(\mathfrak{F}^{1,1}/D_{u_1})$ (resp. $\sigma(\mathfrak{F}^{2,n}) = \sigma(\mathfrak{F}^{2,n}/D_{u_1})$). Moreover, $\mathfrak{F}^{1,1}$ (resp. $\mathfrak{F}^{2,n}$) can be regarded as a D_{u_1}-covering over $\mathfrak{P}_{1,1}$ (resp. a A/D_{u_1}-covering over $\mathfrak{P}_{2,n}$). Since $\sigma(\mathfrak{F}) = \sigma(\mathfrak{F}^{1,1}) + \sigma(\mathfrak{F}^{2,n}), \sharp D_{u_1} < p^r$, and $\sharp A/D_{u_1} < p^r$, by induction, we have

$$\sigma(\mathfrak{F}) \leq \sharp D_{u_1} - 1 + \sharp A/D_{u_1} - 1 \leq p^r - 1.$$

Thus, the theorem follows.

Case 3: If $t = n$ and $n \neq 1$, write S' for the set $\{\mathbb{E}_j \mid i(\mathbb{E}_j) = 1 \text{ and } t(\mathbb{E}_j) = n\}$, S'' for the complement $\{\mathbb{E}_1, \dots, \mathbb{E}_r\} \setminus S'$. Note that S' is not empty. Let $\{\mathbb{E}'_1, \dots, \mathbb{E}'_r\}$ be a set of étale-chains associated to $\Phi_{\mathfrak{F}}$ such that the following conditions: (i) $S'' = \{\mathbb{E}'_1, \dots, \mathbb{E}'_{\sharp S''}\}$; (ii) $S' = \{\mathbb{E}'_{\sharp S''+1}, \dots, \mathbb{E}'_r\}$. By Theorem 2.10, there exist an elementary abelian p-group A', a maximal filtration $\Phi_{A'}$ of A', and an A'-covering $\mathfrak{F}$ over $\mathfrak{P}_n$ such that the j-th étale-chain $\mathbb{E}_j^{\Phi_{\mathfrak{F}'}}$ associated to the sequence of p-coverings of n-semi-graphs with p-rank induced by $\Phi_{A'}$

$$\Phi_{\mathfrak{F}'} : \mathfrak{F}' = \mathfrak{F}'_r \xrightarrow{\mathfrak{a}'_r} \mathfrak{F}'_{r-1} \xrightarrow{\mathfrak{a}'_{r-1}} \dots \xrightarrow{\mathfrak{a}'_1} \mathfrak{F}'_0 = \mathfrak{P}_n$$

is equal to $\mathbb{E}'_j$ for each $j = 1, \dots, r$. Then since $\sharp S''$ is $\leq r-1$, by induction, we have $\sigma(\mathfrak{F}'_{\sharp S''}) \leq p^{\sharp S''} - 1$. Note that since both I_{e_0} and $I_{e_{n+1}}$ are equal to A', we write u'_1 (resp. u'_n, u''_1, u''_n) for the unique element of $\beta_{\mathfrak{F}'}^{-1}(p_1)$ (resp. $\beta_{\mathfrak{F}'}^{-1}(p_n)$, $\beta_{\mathfrak{F}'_{\sharp S''}}^{-1}(p_1)$, $\beta_{\mathfrak{F}'_{\sharp S''}}^{-1}(p_n)$). Then we have

$$\sigma_{\mathfrak{F}'}(u'_1) = p^{\sharp S'}(\sigma_{\mathfrak{F}'_{\sharp S''}}(u''_1) - 1) + p^{\sharp S'} - 1 + 1 = p^{\sharp S'}\sigma_{\mathfrak{F}'_{\sharp S''}}(u''_1)$$

and

$$\sigma_{\mathfrak{F}'}(u'_n) = p^{\sharp S'}(\sigma_{\mathfrak{F}'_{\sharp S''}}(u''_n) - 1) + p^{\sharp S'} - 1 + 1 = p^{\sharp S'}\sigma_{\mathfrak{F}'_{\sharp S''}}(u''_n).$$

Thus, we have

$$\sigma(\mathfrak{F}) = \sigma(\mathfrak{F}') = p^{\sharp S'}(\sigma(\mathfrak{F}'_{\sharp S''}) - \sigma_{\mathfrak{F}'_{\sharp S''}}(u''_1) - \sigma_{\mathfrak{F}'_{\sharp S''}}(u''_n)) + \sigma_{\mathfrak{F}'}(u'_1) + \sigma_{\mathfrak{F}'}(u'_n) + p^{\sharp S'} - 1 \leq p^r - 1.$$

Thus, the theorem follows.

Case 4: If $n \neq 1$ and $t \notin \{1, n\}$, we write $\mathfrak{F}[a_2]$ for $\mathfrak{F}_{m_5+m_4+m_1+m_2+n_1}$, $\mathfrak{F}^{1,t-1}[a_2]$ (resp. $\mathfrak{F}^{t+1,n}[a_2]$) for the $(t-1)$-semi-graph with p-rank $(\beta_{\mathfrak{F}[a_2]}^{-1}(\mathbb{P}_{1,t-1}), \sigma_{\mathfrak{F}[a_2]}|_{v(\beta_{\mathfrak{F}[a_2]}^{-1}(\mathbb{P}_{1,t-1}))},$ $\beta_{\mathfrak{F}[a_2]}|_{\beta_{\mathfrak{F}[a_2]}^{-1}(\mathbb{P}_{1,t+1})})$ (resp. the $(n-t)$-semi-graph with p-rank $(\beta_{\mathfrak{F}[a_2]}^{-1}(\mathbb{P}_{t+1,n}), \sigma_{\mathfrak{F}[a_2]}|_{v(\beta_{\mathfrak{F}[a_2]}^{-1}(\mathbb{P}_{t+1,n}))},$ $\beta_{\mathfrak{F}[a_2]}|_{\beta_{\mathfrak{F}[a_2]}^{-1}(\mathbb{P}_{t+1,n})}))$. Similar arguments to the arguments given in the proof of Case 3 imply that

$$\sigma(\mathfrak{F}^{1,t-1}[a_2]) \leq p^{n_1+m_2+b_1+m_5} - 1$$
$$(\text{resp. } \sigma(\mathfrak{F}^{t+1,n}[a_2]) \leq p^{m_1+b_2+m_5} - 1).$$

Moreover, by Lemma 2.12, we obtain

$$\sigma(\mathfrak{F}^{1,t-1}[a_2]) \leq p^{n_1+m_2+b_1+m_5} - p^{m_5+n_1+m_2}$$

$$(\text{resp. } \sigma(\mathfrak{F}^{t+1,n}[a_2]) \leq p^{m_1+b_2+m_5} - p^{m_5+m_1}).$$

Thus, we obtain

$$\sigma(\mathfrak{F}[a_2]) = \sigma(\mathfrak{F}^{1,t-1}[a_2])+\sigma(\mathfrak{F}^{t+1,n}[a_2])+\sum_{v\in\beta^{-1}_{\mathfrak{F}[a_2]}(p_t)}\sigma_{\mathfrak{F}[a_2]}(v)+p^{m_5}(p^{m_2+n_1}-1+p^{m_1}-1)+p^{m_5}-1$$

$$\leq p^{n_1+m_2+b_1+m_5}+p^{m_1+b_2+m_5}-p^{m_5}-1+\sum_{v\in\beta^{-1}_{\mathfrak{F}[a_2]}(p_t)}\sigma_{\mathfrak{F}[a_2]}(v).$$

Write $v_1[a_2]$ for the unique element of $\beta^{-1}_{\mathfrak{F}[a_2]}(p_1)$. Note that $\sigma_{\mathfrak{F}[a_2]}(v_1[a_2])$ is equal to 0.

Write v_1 (resp. v_t) for the unique (resp. an element) element of $\beta^{-1}_{\mathfrak{F}}(p_1)$ $(\beta^{-1}_{\mathfrak{F}}(p_t))$. We have

$$\sigma_{\mathfrak{F}}(v_1) = -p^{n_1+n_3}+p^{n_1}(p^{n_3}-1)+p^{n_1+n_3}-1+1 = p^{n_1+n_3}-p^{n_1}$$

and

$$\sigma_{\mathfrak{F}}(v_t) = -p^{m_1+m_2+m_3+a_1}+p^{a_1+m_2}(p^{m_1+m_3}-1)+p^{m_1}(p^{a_1+m_2+m_3}-1)+1$$

$$= p^{m_1+m_2+m_3+a_1}-p^{m_2+a_1}-p^{m_1}+1.$$

Since we have

$$\sigma(\mathfrak{F})-\sigma_{\mathfrak{F}}(v_1)-\sum_{v\in\beta^{-1}_{\mathfrak{F}}(p_t)}\sigma_{\mathfrak{F}}(v) = \sigma(\mathfrak{F}[a_2])-\sigma_{\mathfrak{F}[a_2]}(v_1[a_2])-\sum_{v\in\beta^{-1}_{\mathfrak{F}[a_2]}(p_t)}\sigma_{\mathfrak{F}[a_2]}(v)$$

$$\leq p^{n_1+m_2+b_1+m_5}+p^{m_1+b_2+m_5}-p^{m_5}-1$$

and $\sharp\beta^{-1}_{\mathfrak{F}}(p_t) = p^{m_5}$, we obtain

$$\sigma(\mathfrak{F}) \leq p^{n_1+m_2+b_1+m_5}+p^{m_1+b_2+m_5}-p^{m_5}-1+p^{n_1+n_3}-p^{n_1}$$

$$+p^{m_5}(p^{m_1+m_2+m_3+a_1}-p^{m_2+a_1}-p^{m_1}+1)$$

$$= p^{n_1+m_2+b_1+m_5}+p^{m_1+m_2+m_3+a_1+m_5}+p^{m_5+b_2+m_1}+p^{n_1+n_3}$$

$$-p^{m_5+m_2+a_1}-p^{m_1+m_5}-p^{n_1}-1.$$

By Lemma 4.1 in Appendix, we obtain

$$\sigma(\mathfrak{F}) \leq p^r-1.$$

Thus, we complete the proof of the theorem. □

3 p-ranks of vertical fibers of abelian stable coverings

3.1 p-ranks and stable coverings

Definition 3.1. Let C be a disjoint union of projective curves over an algebraically closed field of characteristic $p > 0$. We define the *p-rank of C* as follows:

$$\sigma(C) := \dim_{\mathbb{F}_p}\mathrm{H}^1_{\text{ét}}(C, \mathbb{F}_p).$$

Remark 3.1.1. Let C be a semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write Γ_C for the dual graph of C, $v(\Gamma_C)$ for the set of vertices of Γ_C. Then we have

$$\sigma(C) = \sum_{v \in v(\Gamma_C)} \sigma(\widetilde{C_v}) + \mathrm{rank}_{\mathbb{Z}} \mathrm{H}^1(\Gamma_C, \mathbb{Z}),$$

where for $v \in v(\Gamma)$, $\widetilde{C}_v$ denotes the normalization of the irreducible component of C corresponding to v.

The p-rank of a p-Galois covering (i.e., the extension of function fields induced by the morphism of curves is a Galois extension, and the Galois group is a p-group) of a smooth projective curve can be calculated by the Deuring-Shafarevich formula as follows (cf. [C]):

Proposition 3.2. *Let $h : C' \longrightarrow C$ be a Galois covering (possibly ramified) of smooth projective curves over an algebraically closed field of characteristic $p > 0$, whose Galois group is a finite p-group G. Then we have*

$$\sigma(C') - 1 = \sharp G(\sigma(C) - 1) + \sum_{c' \in (C')^{\mathrm{cl}}} (e_{c'} - 1),$$

where $(C')^{\mathrm{cl}}$ denotes the set of closed points of C', $e_{c'}$ denotes the ramification index at c', and $\sharp G$ denotes the order of G.

In the following of this subsection, let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$, K the quotient field, and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R, $\eta, \overline{\eta}$ and s stand for the generic point, the geometric generic point, the closed point corresponding to the natural morphisms $\mathrm{Spec}\, K \longrightarrow S$, $\mathrm{Spec}\, \overline{K} \longrightarrow S$ and $\mathrm{Spec}\, k \longrightarrow S$, respectively. Let X be a semi-stable curve over S. Write X_η, $X_{\overline{\eta}}$ and X_s for the generic fiber, the geometric generic fiber and the special fiber, respectively. Moreover, we suppose that X_η is smooth over η and the genus $g_{X_{\overline{\eta}}}$ of $X_{\overline{\eta}}$ is ≥ 2.

Definition 3.3. Let $f : Y \longrightarrow X$ be a morphism of semi-stable curves over S, G a finite group. Then f is called a *semi-stable covering* (resp. *G-semi-stable covering*) over S if the morphism of generic fibers f_η is an étale covering (resp. an étale covering with Galois group G), and the following universal property is satisfied: if $g : Z \longrightarrow X$ is a morphism of semi-stable curves over S such that $Z_\eta = Y_\eta$ and $g_\eta = f_\eta$, then there exists a unique morphism $h : Z \longrightarrow Y$ such that $f = g \circ h$ (cf. Remark 3.3.1 for the existence of Y). We call f a *stable covering* (resp. *G-stable covering*) over S if f is a semi-stable covering, and X is a stable curve. Note that by the construction of semi-stable coverings in Remark 3.3.1, if f is a stable covering over S, then Y is a stable curve over S.

Remark 3.3.1. Let W be a semi-stable curve over s. We shall called a semi-stable subcurve $C \subseteq W$ a *chain* if all the irreducible components of C are isomorphic to $\mathbb{P}^1$, the dual graph of C is a tree, and for each irreducible component $C_i \subseteq C$, C_i meets the other irreducible components of W at at most two points.

Let $f_\eta : Y_\eta \longrightarrow X_\eta$ be an étale covering. Suppose that Y_η admits a semi-stable reduction over S. Write Y' for the normalization of X in the function field $K(Y)$, Y^1

for the unique minimal desingularization over S (cf. [L1, Proposition 9.3.32]) which is a semi-stable curve over S. Then Y' (resp. Y^1) admits an G-action induced by the action of G on Y_η. We denote by $f^1 : Y^1 \longrightarrow X$ the composite of $Y^1 \longrightarrow Y'$ and the normalization morphism $Y' \longrightarrow X$. Write C_X^1 for the set of the maximal elements (under the relationship "$\subseteq$") of

$$\{C \text{ a chain of the special fiber } Y_s^1 \text{ of } Y^1 \mid f^1(C) \text{ is a closed point of } X_s\}.$$

Contracting C_X^1, we obtain a semi-stable curve Y^2 over S (cf. [L1, Lemma 10. 3.31]). Moreover, we have a natural morphism $f^2 : Y^2 \longrightarrow X$ induced by f^1. Write C_X^2 for the set of the maximal elements (under the relationship "$\subseteq$") of

$$\{C \text{ a chain of the special fiber } Y_s^2 \text{ of } Y^2 \mid f^2(C) \text{ is a closed point of } X_s\}.$$

Contracting C_X^2, we obtain a semi-stable curve Y^3 over S (cf. [L1, Lemma 10. 3.31]). Moreover, we have a natural morphism $f^3 : Y^3 \longrightarrow X$ induced by f^2. Repeating the process above, we obtain a semi-stable curve of Y over S, a contracting morphism $c_Y : Y^1 \longrightarrow Y$, and f_η extends to a morphism $f : Y \longrightarrow X$ over S.

Netx, let us prove that Y satisfies the universal property defined in Definition 3.3. Let Z be a semi-stable curve over S and $g : Z \longrightarrow X$ a morphism of semi-stable curves over S such that $g_\eta = f_\eta$. If Z is regular, since Y^1 is the minimal desingularization over S, we obtain a morphism $Z \longrightarrow Y^1$. Thus, we have g factors through f. If Z is not regular, write Z^{reg} for the minimal desingularization of Z over S. Then we obtain a commutative diagram as follows:

$$\begin{tikzcd} Z^{\mathrm{reg}} \arrow[r, "b"] \arrow[d, "r"'] & Y^1 \\ Z & \end{tikzcd}.$$

Write $C_{Z^{\mathrm{reg}}}$ for the set of (-1)-curves of Z^{reg} whose images under the morphism b are closed points of Y_s^1. Contracting $r(C_{Z^{\mathrm{reg}}})$, we obtain a semi-stable curve Z' over S, a morphism $Y^1 \longrightarrow Z'$, and the following commutative diagram:

$$\begin{tikzcd} Z^{\mathrm{reg}} \arrow[r, "h"] \arrow[d, "r"'] & Y^1 \arrow[d, "r'"'] \\ Z \arrow[r, "c_Z"] & Z'. \end{tikzcd}$$

Write V_{c_Y} (resp. $V_{r'}$) for the set of irreducible components of Y_s^1 such that for each element $E \in V_{c_Y}$ (resp. $E \in V_{r'}$), $c_Y(E)$ (resp. $r'(E)$) is a closed point of Y_s (resp. the special fiber Z_s' of Z'). By the constructions of Y and Z', we have $V_{r'} \subseteq V_{c_Y}$. Then there is contracting morphism $Z' \longrightarrow Y$, and the following commutative diagram holds:

$$\begin{tikzcd} Y^1 \arrow[r, equal] \arrow[d, "r'"'] & Y^1 \arrow[d, "c_Y"'] \\ Z' \arrow[r, "c_{Z'}"] & Y. \end{tikzcd}$$

Then g factors through f. Note that the uniqueness of contracting implies that the uniqueness of the morphism $h := c_{Z'} \circ c_Z : Z \longrightarrow Y$.

Note that if $f : Y \longrightarrow X$ is a finite morphism of semi-stable curves over S, and the morphism of generic fibers f_η is étale, then f is a semi-stable covering.

Definition 3.4. Let $f : Y \longrightarrow X$ be a semi-stable covering over S. Suppose that the morphism of special fibers $f_s : Y_s \longrightarrow X_s$ is not finite. A closed point $x \in X$ is called a *vertical point associated to* f, or for simplicity, a *vertical point* when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the *vertical fiber associated to* x.

If a vertical point x is nonsingular, the following result was proved by Raynaud (cf. [R, Théorème 1 and Proposition 1]).

Proposition 3.5. *Let G be a finite p-group, $f : Y \longrightarrow X$ a G-semi-stable covering and x a vertical point associated to f. If x is a smooth point of X_s, then the p-rank of each connected component of the vertical fiber $f^{-1}(x)$ associated to x is equal to 0. On the other hand, by contracting the vertical fibers $f^{-1}(x)$, we obtain a curve Y^{c} over S. Write $c : Y \longrightarrow Y^{\mathrm{c}}$ for the contracting morphism. Then the points $c(f^{-1}(x))$ are geometrically unibranch.*

Proposition 3.6. *Let G be a finite group, $f : Y \longrightarrow X$ a G-semi-stable covering, and x a vertical point associated to f. If x is a smooth point or a node which is contained in only one irreducible component (resp. a node which is contained in two different irreducible components), we use the notation X_v (resp. X_{v_1} and X_{v_2}) to denote the irreducible component which contains x (resp. the irreducible components which contain x). Write $\psi : Y' \longrightarrow X$ for the normalization of X in the function field of Y. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of x, Y'_v (resp. Y'_{v_1} and Y'_{v_2}) for an irreducible component (resp. two irreducible components) of Y'_s such that $\psi_s(Y'_v) = X_v$ and $y' \in Y'_v$ (resp. (i) $\psi_s(Y'_{v_1}) = X_{v_1}$ and $\psi_s(Y'_{v_2}) = X_{v_2}$; (ii) $y' \in Y'_{v_1}$ and $y' \in Y'_{v_2}$). Write $I_v \subseteq G$ (resp. $I_{v_1} \subseteq G$ and $I_{v_2} \subseteq G$) for the inertia subgroup of Y'_v (resp. the inertia subgroups of Y'_{v_1} and Y'_{v_2}, respectively).*

Suppose that G is a p-group (resp. an abelian group). Then we have $I_v \neq \{1\}$ (resp. $I_{v_1} \neq \{1\}$ or $I_{v_2} \neq \{1\}$). Moreover, write $I_{y'} \subseteq G$ for the inertia subgroup of y', then $I_{y'}$ is equal to I_v (resp. $I_{y'}$ is generated by I_{v_1} and I_{v_2}).

Proof. Since Y is normal, we obtain a natural morphism $\phi : Y \longrightarrow Y'$. By using [BLR, 6.7 Proposition 4], we may contract the connected component of $f_s^{-1}(x)$ whose image under the morphism ϕ is y'. Thus, we obtain a contraction morphism $c : Y \longrightarrow Y''$. Since Y'' is a blowing-up of Y', Y'' is a fiber surface over S (i.e., normal and flat over S) and there is natural commutative diagram as follows:

$$\begin{array}{ccc} Y_\eta & \longrightarrow & Y \\ {\scriptstyle c_\eta}\big\downarrow & & {\scriptstyle c}\big\downarrow \\ Y''_\eta & \longrightarrow & Y'' \\ {\scriptstyle f''_\eta}\big\downarrow & & {\scriptstyle f''}\big\downarrow \\ X_\eta & \longrightarrow & X, \end{array}$$

where c_η is an identity morphism.

Write Y''_v (resp. Y''_{v_1} and Y''_{v_2}) for the unique irreducible component whose image under the natural morphism $Y'' \longrightarrow Y'$ is Y'_v (resp. Y'_{v_1}, Y'_{v_2}), y'' for the image $c(\phi^{-1}(y'))$. Note that the inertia group of Y''_v (resp. Y''_{v_1}, Y''_{v_2}) is equal to I_v (resp. I_{v_1}, I_{v_2}).

If x is a smooth point, G is a p-group, and I_v is trivial, then $f''_s|_{Y''_v}$ is generically étale. By Proposition 3.5, we have y'' is geometrically unibranch. Thus, y'' is contained in only one irreducible component of Y''_s. By applying Zariski-Nagata purity, we have $f''_s|_{Y''_v}$ is étale at y''. Thus, y'' is a smooth point. Then Y'' is a semi-stable curve. This contradicts to the minimal properties of semi-stable coverings.

If x is a node and I_v (resp. I_{v_1} and I_{v_2}) is (resp. are) trivial, since G is abelian, f''_s is étale over an open neighborhood of x. The completion of the local ring at x is $\hat{\mathcal{O}}_{X,x} \cong R[[u,v]]/(uv - \pi^{p^e n'})$, where π denotes an uniformizer of R and $(n', p) = 1$. Since the étale fundamental group of $\mathrm{Spec}\, \hat{\mathcal{O}}_{X,x} \setminus \{\hat{x}\}$ is isomorphic to $\mathbb{Z}/n'\mathbb{Z}$ (cf. [T, Lemma 2.1 (iii)]), where $\hat{x}$ denotes the closed point of $\mathrm{Spec}\, \hat{\mathcal{O}}_{X,x}$, we have y'' is a node. Then Y'' is a semi-stable model of Y''_η over S in either case, so that this contradicts to the minimal properties of semi-stable coverings. Thus, $I_v \neq \{1\}$ (resp. $I_{v_1} \neq \{1\}$ or $I_{v_2} \neq \{1\}$). This completes the proof of the proposition. □

3.2 Semi-graphs with p-rank associated to vertical fibers

In this subsection, we construct a semi-graph with p-rank defined in Section 1 from a vertical fiber, and we apply the theory developed in Section 1 to calculate the bound of the p-rank of the vertical fiber.

First, we fix some notations. Let G be a finite p-group, $f: Y \longrightarrow X$ a G-stable covering over S, $x \in X_s$ a vertical point. Suppose that x is a node contained in two irreducible components X_1 and X_2 (which may be equal) of X_s. Write $\psi : Y' \longrightarrow X$ for the normalization of X in the function field of Y. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of x. Write $I_{y'}$ for the inertia group of y'. Note that the natural morphism $Y/I_{y'} \longrightarrow X$ induced by f is finite étale over x. Thus, by replacing X by the stable model of $Y/I_{y'}$, in order to calculate the p-rank of the vertical fiber $f^{-1}(x)$, we may assume that $I_{y'}$ is equal to G. From now on, we may assume that $G = I_{y'}$ is a p-group with order p^r. Then $f^{-1}(x)$ is connected.

Let X^{sst} be the quotient of Y by G. By [R, Appendice, Corollaire], X^{sst} is a semi-stable curve with generic fiber X_η. Then we obtain a quotient morphism $h : Y \longrightarrow X^{\mathrm{sst}}$ and a birational morphism $g : X^{\mathrm{sst}} \longrightarrow X$ such that the composite morphism $g \circ h$ is equal to f. We still write X_1 and X_2 for the strict transforms of X_1 and X_2 under the birational morphism g, respectively. By the general theory of semi-stable curves, $g^{-1}(x)$ is a semi-stable subcurve of X^{sst}_s whose irreducible components are isomorphic to $\mathbb{P}^1_k$. Write C for the semi-stable subcurve of $g^{-1}(x)$ which is a chain of projective lines $\cup_{i=1}^n P_i$ such that the following conditions: (i) P_i is not equal to P_j if $i \neq j$; (ii) $P_1 \cap X_1$ are $P_n \cap X_2$ are not empty; (iii) P_i meets P_{i+1} at only one point; (iv) $P_i \cap P_j$ is empty if j is not equal to $i-1, i$ or $i+1$. Then we have

$$g^{-1}(x) = C \cup B,$$

where B denotes the topological closure of $g^{-1}(x) \setminus C$ in $g^{-1}(x)$. Write B_i for the union of the connected components of B which intersect with P_i are not empty.

Lemma 3.7. *Let V_i be an irreducible component of $h^{-1}(P_i)$, $I_{V_i} \subseteq G$ (resp. $D_{V_i} \subseteq G$) the inertia group (the decomposition group) of V_i, and D_i for the image of V_i under the quotient morphism $Y \longrightarrow Y/I_{V_i}$. Write h_i for the natural morphism $Y/I_{V_i} \longrightarrow X^{sst}$. Then the branch locus of $h_i|_{D_i} : D_i \longrightarrow P_i$ are contained in $P_i \cap (P_{i+1} \cup P_{i-1})$.*

Proof. Write E_i for the image of D_i under the natural morphism $Y/I_{V_i} \longrightarrow Y/D_{V_i}$. We have the restriction of $Y/D_{V_i} \longrightarrow X^{\mathrm{sst}}$ to E_i is an identity morphism. Thus, by replacing X^{sst} by Y/D_{V_i}, we may assume that D_{V_i} is equal to G. Then h_i is a G/I_{V_i}-semi-stable covering. Note that it is easy to see that the branch locus of $h_i|_{D_i}$ are contained in $P_i \cap (P_{i+1} \cup P_{i-1} \cup B_i)$

By contracting B_i (resp. $h_i^{-1}(B_i)$), we obtain a semi-stable curve $(X^{\mathrm{sst}})'$ and a contraction morphism $c_{X^{\mathrm{sst}}} : X^{\mathrm{sst}} \longrightarrow (X^{\mathrm{sst}})'$ (resp. a fiber surface $(Y/I_{V_i})'$ and a contraction morphism $c_{Y/I_{V_i}} : Y/I_{V_i} \longrightarrow (Y/I_{V_i})'$) over S. Moreover, h_i induces a morphism $h'_i : (Y/I_{V_i})' \longrightarrow (X^{\mathrm{sst}})'$. Then we have the following commutative diagram:

$$\begin{array}{ccc} Y/I_{V_i} & \xrightarrow{c_{Y/I_{V_i}}} & (Y/I_{V_i})' \\ {\scriptstyle h_i}\downarrow & & {\scriptstyle h'_i}\downarrow \\ X^{\mathrm{sst}} & \xrightarrow{c_{X^{\mathrm{sst}}}} & (X^{\mathrm{sst}})'. \end{array}$$

Since it follows from Proposition 3.5, $(h'_i)^{-1}(c_{X^{\mathrm{sst}}}(P_i \cap B)) \cap c_{Y/I_{V_i}}(D_i)$ are geometrically unibranch, $(h'_i)^{-1}(c_{X^{\mathrm{sst}}}(P_i \cap B))$ only are contained in one irreducible component of the special fiber of $(Y/I_{V_s})'$. Moreover, by applying Zariski-Nagata purity to h'_i, $h'_i|_{(h'_i)^{-1}(c_{X^{\mathrm{sst}}}(P_i))}$ is contained in the étale locus of h'_i. Thus, the set of branch points of $h'_i|_{(h'_i)^{-1}(c_{X^{\mathrm{sst}}}(P_i))}$ is contained in the set $c_{X^{\mathrm{sst}}}(P_i \cap (P_{i+1} \cup P_{i-1}))$. Moreover, $c_{Y/I_{V_i}}|_{D_i}$ is an isomorphism. Then we complete the proof of the lemma. □

Next, we construct a semi-graph with p-rank from a vertical fiber. From now on, we assume that G is an abelian p-group. Write D_C for the set of points $C \cap (X_1 \cup X_2)$. Thus, we may regard $\mathcal{C} := (C, D_C)$ as a pointed semi-stable curve over s. Write $\mathbb{P}_n$ for the dual graph associated to $\mathcal{C}$, $\sigma_{\mathfrak{P}_n}$ for the map satisfying the property $\sigma_{\mathfrak{P}_n}(p_i) = \sigma(P_i)$. Then $\mathfrak{P}_n := (\mathbb{P}_n, \sigma_{\mathfrak{P}_n}, \mathrm{id}_{\mathbb{P}_n})$ is a n-chain defined in Section 1.

Let
$$\Phi : \{1\} = G_r \subset G_{n-1} \subset G_{n-2} \subset \cdots \subset G_1 \subset G_0 = G,$$
be a filtration of G such that $G_j/G_{j+1} \cong \mathbb{Z}/p\mathbb{Z}, j = 0, \ldots, r-1$. The filtration Φ induces a sequence of semi-stable coverings Φ_f as follows:

$$Y = Y_r \xrightarrow{d_r} Y_{r-1} \xrightarrow{d_{r-1}} \ldots \xrightarrow{d_1} Y_0 = X^{\mathrm{sst}},$$

where $Y_i, i = 0, \ldots, r$, denotes the semi-stable curve Y/G_i over S.

For each $i = 0, \ldots, r$, write Γ_i for the dual graph of the special fiber of Y_i. First, let us prove that the map $\beta_i : \Gamma_i \longrightarrow \Gamma_{i-1}, 1 \leq i \leq r$, induced by d_i is a morphism of semi-graphs. To verify β_i is a morphism of semi-graphs, it is sufficient to prove that $\beta_i(e(\Gamma_i)) \subseteq e(\Gamma_{i-1})$, where $e(-)$ denotes the set of edges of $(-)$. Let y_i be a node of the special fiber $(Y_i)_s$ of Y_i. Write Y_i^1 and Y_i^2 for the irreducible components of $(Y_i)_s$ which contain y_i, $I_{Y_i^1} \subseteq G_{i-1}/G_i$ (resp. $I_{Y_i^2} \subseteq G_{i-1}/G_i$, $I_{y_i} \subseteq G_{i-1}/G_i$) for the inertia group of Y_i^1 (resp. Y_i^2, y_i). Write $I \subseteq G_{i-1}/G_i$ for the group generated by $I_{Y_i^1}$ and $I_{Y_i^2}$, q_{y_i} for the quotient morphism $Y_i \longrightarrow Y/I$. By the definitions, we obtain $I \subseteq I_{y'}$. Applying Zariski-Nagata purity to $\mathrm{Spec}\,\mathcal{O}_{Y/I, q_{y_i}(y_i)} \longrightarrow \mathrm{Spec}\,\mathcal{O}_{Y_{i-1}, d_i(y_i)}$, we have the morphism $Y/I \longrightarrow Y_{i-1}$ induced by d_i is étale at $q_{y_i}(y_i)$. This implies that $I = I_{y'}$. Since for any element $\tau \in I$, we have $\tau(Y_i^1) = Y_i^1$ and $\tau(Y_i^2) = Y_i^2$, the proof of [R, Appendice, Proposition 5] (or [L1, Proposition 10.3.48]) implies that $q_{y_i}(y_i)$ is a node of $(Y_i/I)_s$. Thus, $d_i(y_i)$ is a node of the special fiber $(Y_{i-1})_s$ of Y_{i-1}. This means that β_i is a morphism of semi-graphs.

Write $\phi_i, i = 1, \dots, r$, for the composite morphism $d_1 \circ d_2 \circ \cdots \circ d_i$. Note that we have $h = \phi_r$. The semi-stable subcurve $\phi_i^{-1}(C)$ with $\phi_i^{-1}(D_C)$ may be regarded as a pointed semi-stable curve over s. We use the notation $\mathcal{Y}_i$ to denote the resulting pointed semi-stable curve $(\phi_i^{-1}(C), \phi_i^{-1}(D_C))$. Write $\mathbb{G}_i$ for the dual graph of $\mathcal{Y}_i$, $\beta_{\mathfrak{G}_i}$ for the natural morphism $\mathbb{G}_i \longrightarrow \mathbb{P}_n$ induced by the morphism $\phi_i|_{\mathcal{Y}_i} : \mathcal{Y}_i \longrightarrow \mathcal{C}$. For each $v \in v(\mathbb{G}_i)$, write $(Y_i)_v$ for the irreducible component of $\mathcal{Y}_i$ corresponding to v. We define $\sigma_{\mathfrak{G}_i}$ to be the map satisfying the property $\sigma_{\mathfrak{G}_i}(v) = \sigma((Y_i)_v)$ for all $v \in v(\mathbb{G}_i)$. Then $\mathfrak{G}_i := (\mathbb{G}_i, \sigma_{\mathfrak{G}_i}, \beta_{\mathfrak{G}_i})$ is a n-semi-graph with p-rank. Moreover, $d_i|_{\mathcal{Y}_i}$ induces a natural morphism of n-semi-graphs with p-rank $\mathfrak{b}_i : \mathfrak{G}_i \longrightarrow \mathfrak{G}_{i-1}$, and $\mathfrak{G}$ admits a natural action of G induced by the action of G on $\mathcal{Y}_n$. Furthermore, Φ induces a sequence of morphisms of semi-graphs with p-rank

$$\Phi_{\mathfrak{G}} : \mathfrak{G} := \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{P}_n.$$

On the other hand, by Lemma 3.7 and Zariski-Nagata purity, it is easy to check that for each $i = 1, \dots, r$, $\mathfrak{b}_i$ is a p-covering. Thus, $\mathfrak{G}$ is a G-covering over $\mathfrak{P}_n$. For each $i = 1, \dots, r$, we write $\mathbb{E}_i^{\Phi_{\mathfrak{G}}}$ for the i-th étale-chain associated to $\Phi_{\mathfrak{G}}$.

On the other hand, write $\{Y_i^j\}_j$ for the set of connected components contained in the étale locus of d_i such that the image $\overline{\phi_i(Y_i^j)}$ are contained in $g^{-1}(x)$ for all j, $Y_i^{\text{ét}}$ for the disjoint union $\coprod_j Y_i^j$. Note that $\overline{\phi_i(Y_i^{\text{ét}}) \setminus B}$ is a disjoint union of semi-stable subcurve of C. For each connected component E of $\overline{\phi_i(Y_i^{\text{ét}}) \setminus B}$, with the set of closed points $D_E := E \cap \overline{C \setminus E}$, we may regard $\mathcal{E} := (E, D_E)$ as a pointed semi-stable subcurve of $\mathcal{C}$ over s. We define $\mathscr{E}_i^{\Phi_f}$ as the disjoint union

$$\coprod_{E \subseteq \overline{\phi_i(Y_i^{\text{ét}}) \setminus B}} \mathcal{E}.$$

We shall call $\mathscr{E}_i^{\Phi_f}$ the *i-th étale-chain associated to* Φ_f, and write $\mathbb{E}_i$ for the disjoint union of the dual graph of the connected components of $\mathscr{E}_i^{\Phi_f}$. We define $\mathscr{E}^{\Phi_f}$ as the disjoint union

$$\coprod_i \mathscr{E}_i^{\Phi_f},$$

and call $\mathscr{E}^{\Phi_f}$ the *étale-chain associated to* Φ_f. From the construction of $\mathbb{E}_i$, it is easy to see that $\mathbb{E}_i$ are equal to $\mathbb{E}_i^{\Phi_{\mathfrak{G}}}$ for all i.

Note that $C \cap B$ are smooth points of C. By Proposition 3.5, we have the p-ranks of the connected components of $h^{-1}(B)$ are equal to 0. Thus, we have $\sigma(f^{-1}(x)) = \sigma(\phi_r^{-1}(C))$. Moreover, by applying Lemma 3.7, we obtain $\sigma(\phi_r^{-1}(C)) = \sigma(\mathfrak{G})$.

Summarizing the discussion, we obtain the following proposition.

Proposition 3.8. *Let G be a finite abelian p-group with order p^r, $f : Y \longrightarrow X$ a G-stable covering over S, $x \in X_s$ a vertical point. Write $\psi : Y' \longrightarrow X$ for the normalization of X in the function field of Y. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of x. Write $I_{y'}$ for the inertia group of y'. Suppose that $G = I_{y'}$. Let Φ be a maximal filtration of G. Write Φ_f for the sequence of semi-stable curves induced by Φ which was constructed in this subsection, $\mathscr{E}_i^{\Phi_f}$ for the i-th étale-chain associated to Φ_f for each i. Then there exist a semi-graph with p-rank $\mathfrak{G}$ and a sequence of p-coverings of semi-graphs with p-rank $\Phi_{\mathfrak{G}}$ induced by Φ which was constructed in this subsection such that $\mathfrak{G}$ is a G-covering over $\mathfrak{P}_n$, and for each $i = 1, \ldots, r$, the i-th étale-chain $\mathbb{E}_i^{\Phi_{\mathfrak{G}}}$ associated to $\Phi_{\mathfrak{G}}$ is equal to the dual graph of $\mathscr{E}_i^{\Phi_f}$. Furthermore, we have $\sigma(f^{-1}(x)) = \sigma(\mathfrak{G})$.*

3.3 p-ranks of vertical fibers

We follow the notations of Section 3.2. Let $\{Z_i\}_{i=0}^{n+1}$ a subset the set of irreducible components of the special fiber Y_s of Y such that the following conditions hold: (i) $\phi_r(Z_i) = P_i$ if $i \notin \{0, n+1\}$; (ii) $\phi_r(Z_0) = X_1$ and $\phi_r(Z_{n+1}) = X_2$; (iii) the union $\cup_{i=0}^{n+1} Z_i$ is a connected semi-stable subcurve of the special fiber Y_s of Y. Write $I_{P_i} \subseteq G$ for the inertia subgroup of Z_i. Note that since G is an abelian p-group, I_{P_i} does not depend on the choice of Z_i.

By using the theory of étale-chains, we obtain an explicit formula of p-rank of $f^{-1}(x)$ as follows:

Theorem 3.9. *If G is an abelian p-group, then we have*

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \sharp(G/I_{P_i}) - \sum_{i=1}^{n+1} \sharp(G/(I_{P_{i-1}} + I_{P_i})) + 1.$$

Proof. We follow the notations of Theorem 2.8. Note that by Zariski-Nagata purity, we have the inertia group of a point of $Z_{i-1} \cap Z_i$ (resp. $Z_i \cap Z_{i+1}$) is equal to $I_{P_{i-1}} + I_{P_i}$ (resp. $I_{P_i} + I_{P_{i+1}}$). Then we have $\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(p_j) = \log_p(\sharp G/I_{P_i})$ (resp. $\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_{v_j}^l) = \log_p(\sharp G/(I_{P_{i-1}} + I_{P_i}))$, $\sharp\mathbb{E}^{\Phi_{\mathfrak{G}}}(b_{v_j}^r) = \log_p(\sharp G/(I_{P_i} + I_{P_{i+1}}))$). Thus, we have

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n}(\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i})) + 1) + \sum_{i=1}^{n-1}(\sharp(G/(I_{P_{i+1}} + I_{P_i})) - 1)$$

$$= \sum_{i=1}^{n} \sharp(G/I_{P_i}) - \sum_{i=1}^{n+1} \sharp(G/(I_{P_{i-1}} + I_{P_i})) + 1.$$

This completes the proof of the theorem. □

Remark 3.9.1. If G is a cyclic p-group, since G is generated by I_{P_0} and $I_{P_{n+1}}$, we may assume that $I_{P_0} = G$. Follows Lemma 3.10 below, there exists u such that

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_u} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$

Then we obtain

$$\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i})) + 1 = -\sharp(G/(I_{P_{i-1}})) + 1$$

$$(\text{resp. } \sharp(G/(I_{P_{i+1}} + I_{P_i})) - 1 = \sharp(G/(I_{P_i})) - 1)$$

if $i < u$,

$$\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i})) + 1 = -\sharp(G/(I_{P_{i+1}})) + 1$$

$$(\text{resp. } \sharp(G/(I_{P_{i+1}} + I_{P_i})) - 1 = \sharp(G/(I_{P_{i+1}})) - 1)$$

if $i > u$ and

$$\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i})) + 1 = \sharp(G/I_{P_t}) - \sharp(G/I_{P_{t-1}}) - \sharp(G/(I_{P_{t+1}}) + 1$$

$$(\text{resp. } \sharp(G/(I_{P_{i+1}} + I_{P_i})) - 1 = \sharp(G/(I_{P_{t+1}})) - 1)$$

if $i = u$. Thus, by applying Theorem 3.9, we obtain

$$\sigma(f^{-1}(x)) = \sharp(G/I_{P_u}) - \sharp(G/I_{P_{n+1}}).$$

This formula was first obtained by Saïdi (cf. [S, Proposition 1]).

Lemma 3.10. *If $G \cong \mathbb{Z}/p^n\mathbb{Z}$ is a cyclic group, then there exists $0 \leq u \leq n+1$ such that*

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_i} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$

In particular, $\sharp\pi_0(\mathscr{E}_i^{\Phi_f}) \leq 1$ hold for all i, where $\sharp\pi_0(-)$ denotes the cardinality of the connected components of $(-)$.

Proof. If the lemma is not true, there exist s, t and v such that $I_{P_v} \neq I_{P_s}$, $I_{P_v} \neq I_{P_t}$ and $I_{P_s} \subset I_{P_{s+1}} = \cdots = I_{P_v} = \cdots = I_{P_{t-1}} \supset I_{P_t}$. Since G is a cyclic group, we may assume $I_{P_s} \supseteq I_{P_t}$.

Considering the quotient of Y by I_{P_s}, we obtain a natural morphism of semi-stable curves $h_s : Y/I_{P_s} \longrightarrow X^{\mathrm{sst}}$ over S. By contacting $P_{s+1}, P_{s+2}, \ldots, P_{t-1}, B_{s+1}, \ldots, B_{t-1}$ (resp. $h_s^{-1}(P_{s+1}), h_s^{-1}(P_{s+2}), \ldots, , h_s^{-1}(P_{t-1}), h_s^{-1}(B_{s+1}), \ldots, h_s^{-1}(B_{t-1})$), we obtain a semi-stable curve $(X^{\mathrm{sst}})'$ (resp. a fiber surface $(Y/I_{P_s})'$) and a contacting morphism $c_{X^{\mathrm{sst}}} : X^{\mathrm{sst}} \longrightarrow (X^{\mathrm{sst}})'$ (resp. $c_{Y/I_{P_s}} : Y/I_{P_s} \longrightarrow (Y/I_{P_s})'$). The morphism h_s induces a morphism of fiber surfaces $h'_s : (Y/I_{P_s})' \longrightarrow (X^{\mathrm{sst}})'$. Then we have the following commutative diagram as follows:

$$\begin{array}{ccc}
Y/I_{P_s} & \xrightarrow{c_{Y/I_{P_s}}} & (Y/I_{P_s})' \\
{\scriptstyle h_s}\big\downarrow & & {\scriptstyle h'_s}\big\downarrow \\
X^{\mathrm{sst}} & \xrightarrow{c_{X^{\mathrm{sst}}}} & (X^{\mathrm{sst}})'.
\end{array}$$

Write P'_s and P'_t for the images $c_{X^{\mathrm{sst}}}(P_s)$ and $c_{X^{\mathrm{sst}}}(P_t)$, respectively, and x'_{st} for the closed point $P'_s \cap P'_t \in (X^{\mathrm{sst}})'_s$. By Proposition 3.6, we have $(Y/I_{P_s})'$ is a semi-stable curve over S, moreover, we have h'_s is étale over x'_{st}. Then the inertia groups of the closed points $(h'_s)^{-1}(x'_{st})$ of the special fiber $(Y/I_{P_s})'_s$ of $(Y/I_{P_s})'$ are trivial.

On the other hand, since I_{P_s} is a proper subgroup of I_{P_v}, we obtain the natural action of G/I_{P_s} on the irreducible components of $h_s^{-1}(\cup_{j=s+1}^{t-1} P_j)$ is trivial. Thus, the inertia groups of the closed points $c_{Y/I_{P_s}}(h_s^{-1}(\cup_{j=s+1}^{t-1} P_j)) = (h'_s)^{-1}(x'_{st})$ of the special fiber $(Y/I_{P_s})'_s$ of $(Y/I_{P_s})'$ are not trivial. This is a contradiction. Then we complete the proof of the lemma. $\square$

On the other hand, we obtain a bound of $\sigma(f^{-1}(x))$.

Theorem 3.11. *If G is an abelian p-group with order p^r, and $\mathscr{E}_i$ is connected for each $i = 1, \ldots, n$, then we have $\sigma(f^{-1}(x)) \leq p^r - 1$.*

Proof. Together with Theorem 2.13 and Proposition 3.8, the theorem follows. $\square$

4 Appendix

In this appendix, we prove the following elementary lemma which is used in the proof of Theorem 2.13.

Lemma 4.1. *Following the notations of the proof of Theorem 2.13, then we have*

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1.$$

$$\leq p^r - 1.$$

Proof. We will check this inequality case by case. We denote by M the maximal number

$$\max\{n_1 + m_2 + b_1 + m_5, m_1 + m_2 + m_3 + a_1 + m_5, m_1 + m_5 + b_2, n_1 + n_3\}.$$

If $M = r$, we have the following cases.

If $n_1 + m_2 + b_1 + m_5 = r$, then we have $n_2 = n_3 = b_2 = m_1 = m_3 = 0$, $m_4 = b_1$ and $n_4 = m_2 + b_1 + m_5$. Thus, we obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^r + p^{m_2+m_5+a_1} + p^{m_5} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_5} - p^{n_1} - 1 = p^r - 1.$$

If $m_1 + m_2 + m_3 + a_1 + m_5 = r$, then we have $n_1 = a_1$ and $n_2 = n_3 = m_4 = b_1 = b_2 = 0$. Thus, we obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{a_1+m_2+m_5} + p^r + p^{m_1+m_5} + p^{a_1} - p^{a_1+m_2+m_5} - p^{m_1+m_5} - p^{a_1} - 1 = p^r - 1.$$

If $m_5 + b_2 + m_2 = r$, then we have $n_1 = a_1 = a_2 = m_1 = m_3 = n_3 = b_1 = 0$ and $m_4 = b_2$. Thus, we obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{m_2+m_5} + p^{m_5} + p^r + 1 - p^{m_5+m_2} - p^{m_5} - 1 - 1 = p^r - 1.$$

If $n_1 + n_3 = r$, then we have $m_1 = m_2 = m_3 = m_4 = m_5 = b_1 = b_2 = n_4 = n_2 = 0$. Thus, we obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{n_1} + p^{a_1} + 1 + p^r - p^{a_1} - 1 - p^{n_1} - 1 = p^r - 1.$$

Thus, it is sufficient to assume that $M \leq r - 1$.

If $M \leq r - 2$, then we have

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 4p^{r-2} - 4.$$

Since p is a prime number, we have $p^r - 1 - 4p^{r-2} + 4 > 0$. Thus, we obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq p^r - 1.$$

Thus, we may assume that $M = r - 1$.

If $n_1 + m_2 + b_1 + m_5 = r - 1$, we obtain $n_2 + n_3 + m_1 + m_3 + b_2 = 1$. If $n_2 = 1$, then we have $n_3 = m_1 = m_3 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{r-1} + p^{m_2+a_1+m_5} + p^{m_1+m_5} + p^{n_1} - p^{m_2+a_1+m_5} - p^{m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $n_3 = 1$, then we have $n_2 = m_1 = m_3 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{r-1} + p^{m_2+a_1+m_5} + p^{m_5} + p^{n_1+1} - p^{m_2+m_5+a_1} - p^{m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_1 = 1$, then we have $n_2 = n_3 = m_3 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{r-1} + p^{m_1+m_2+a_1+m_5} + p^{m_5+m_1} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_5+m_1} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_3 = 1$, then we have $n_2 = n_3 = m_1 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{r-1} + p^{m_3+m_2+a_1+m_5} + p^{m_5} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $b_2 = 1$, then we have $n_2 = n_3 = m_1 = m_3 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{r-1} + p^{m_2+a_1+m_5} + p^{m_5+b_2} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $a_1 + m_1 + m_2 + m_3 + m_5 = r - 1$, we obtain $a_2 + n_2 + n_3 + b_1 + b_2 = 1$. If $a_2 = 1$, then we have $n_2 = n_3 = b_1 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{n_1+m_2+m_5} + p^{r-1} + p^{m_1+m_5} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $n_2 = 1$, then we have $a_2 = n_3 = b_1 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{a_1+m_2+m_5} + p^{r-1} + p^{m_1+m_5} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{r-1} - 1 < p^r - 1.$$

If $n_3 = 1$, then we have $a_2 = n_2 = b_1 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{a_1+m_2+m_5} + p^{r-1} + p^{m_1+m_5} + p^{n_1+n_3} - p^{m_2+m_5+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $b_1 = 1$, then we have $a_2 = n_2 = n_3 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{a_1+m_2+b_1+m_5} + p^{r-1} + p^{m_1+m_5} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $b_2 = 1$, then we have $a_2 = n_2 = n_3 = b_1 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{a_1+m_2+m_5} + p^{r-1} + p^{m_1+m_5+b_2} + p^{n_1} - p^{m_2+m_5+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_1 + b_2 + m_5 = r - 1$, we obtain $a_1 + a_2 + n_2 + n_3 + m_2 + m_3 + b_1 = 1$. If $a_1 = 1$, then we have $a_2 = n_2 = n_3 = m_2 = m_3 = b_1 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{a_1+m_5} + p^{m_1+a_1+m_5} + p^{r-1} + p^{n_1} - p^{a_1+m_5} - p^{m_1+m_5} - p^{n_1} - 1$$

$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $a_2 = 1$, then we have $a_1 = n_2 = n_3 = m_2 = m_3 = b_1 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{a_2+m_5} + p^{m_1+m_5} + p^{r-1} + p^{n_1} - p^{m_5} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $n_2 = 1$, then we have $a_1 = a_2 = n_3 = m_2 = m_3 = b_1 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{m_5} + p^{m_1+m_5} + p^{r-1} + p^{n_1} - p^{m_5} - p^{m_1+m_5} - p^{n_1} - 1$$
$$= p^{r-1} - 1 < p^r - 1.$$

If $n_3 = 1$, then we have $a_1 = a_2 = n_2 = m_2 = m_3 = b_1 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{m_5} + p^{m_1+m_5} + p^{r-1} + p^{n_1+n_3} - p^{m_5} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_2 = 1$, then we have $a_1 = a_2 = n_2 = n_3 = m_3 = b_1 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{m_2+m_5} + p^{m_1+m_2+m_5} + p^{r-1} + p^{n_1} - p^{m_2+m_5} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_3 = 1$, then we have $a_1 = a_2 = n_2 = n_3 = m_2 = b_1 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{m_5} + p^{m_1+m_3+m_5} + p^{r-1} + p^{n_1} - p^{m_5} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $b_1 = 1$, then we have $a_1 = a_2 = n_2 = n_3 = m_2 = m_3 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{b_1+m_5} + p^{m_1+m_5} + p^{r-1} + p^{n_1} - p^{m_5} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $n_1 + n_3 = r - 1$, we obtain $n_2 + m_1 + m_2 + m_3 + m_4 + m_5 = 1$. If $n_2 = 1$, then we have $m_1 = m_2 = m_3 = b_1 = b_2 = m_5 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{n_1} + p^{a_1} + 1 + p^{r-1} - p^{a_1} - 1 - p^{n_1} - 1$$

$$= p^{r-1} - 1 < p^r - 1.$$

If $m_1 = 1$, then we have $n_2 = m_2 = m_3 = b_1 = b_2 = m_5 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{n_1} + p^{a_1+m_1} + p^{m_1} + p^{r-1} - p^{a_1} - p^{m_1} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_2 = 1$, then we have $n_2 = m_1 = m_3 = b_1 = b_2 = m_5 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{n_1+m_2} + p^{a_1+m_2} + 1 + p^{r-1} - p^{a_1+m_2} - 1 - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_3 = 1$, then we have $n_2 = m_1 = m_2 = b_1 = b_2 = m_5 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{n_1} + p^{a_1+m_3} + 1 + p^{r-1} - p^{a_1} - 1 - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $b_1 = 1$, then we have $n_2 = m_1 = m_2 = m_3 = b_2 = m_5 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{n_1+b_1} + p^{a_1} + 1 + p^{r-1} - p^{a_1} - 1 - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $b_2 = 1$, then we have $n_2 = m_1 = m_2 = m_3 = b_1 = m_5 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{n_1} + p^{a_1} + p^{b_2} + p^{r-1} - p^{a_1} - 1 - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

If $m_5 = 1$, then we have $n_2 = m_1 = m_2 = m_3 = b_1 = b_2 = 0$. We obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$

$$= p^{n_1+m_5} + p^{a_1+m_5} + p^{m_5} + p^{r-1} - p^{a_1+m_5} - p^{m_5} - p^{n_1} - 1$$
$$\leq 2p^{r-1} - 1 \leq p^r - 1.$$

Thus, we obtain

$$p^{n_1+m_2+b_1+m_5} + p^{m_1+m_2+m_3+a_1+m_5} + p^{m_5+b_2+m_1} + p^{n_1+n_3} - p^{m_5+m_2+a_1} - p^{m_1+m_5} - p^{n_1} - 1$$
$$\leq p^r - 1.$$

We complete the proof of the lemma.

□

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Yu Yang

Address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail: yuyang@kurims.kyoto-u.ac.jp