The Combinatorial Mono-anabelian Geometry of Curves over Algebraically Closed Fields of Positive Characteristic I: Combinatorial Grothendieck Conjecture

Yu Yang

Abstract

In the present paper, we study anabelian geometry of curves over algebraically closed fields of positive characteristic. Let $X^\bullet = (X, D_X)$ be a pointed stable curve over an algebraically closed field of characteristic $p > 0$ and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. We prove that there exists a group-theoretical algorithm whose input datum is the admissible fundamental group $\Pi_{X^\bullet}$, and whose output data are the topological and the combinatorial structures associated to $X^\bullet$. This result can be regarded as a mono-anabelian version of the combinatorial Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$.

Keywords: pointed stable curve, fundamental group, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14G32; Secondary 14H30.

Contents

Introduction

In the present paper, we study the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. Let

$$X^\bullet = (X, D)$$

be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. Moreover, by choosing a suitable base point of $X^\bullet$, we have the admissible fundamental group (=geometric log étale fundamental group)

$$\Pi_{X^\bullet}$$

of $X^\bullet$ (cf. [Y3, Section 2] for the definitions of admissible coverings and admissible fundamental groups). The admissible fundamental group of a pointed stable curves is a natural generalization of tame fundamental group of a smooth pointed stable curve. In particular, if $X^\bullet$ is smooth over k, then $\Pi_{X^\bullet}$ is naturally isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(X^\bullet)$. The main question of interest in the anabelian geometry of curves is, roughly speaking, the following:

> *How much geometric information about the isomorphism class of a pointed stable curve is contained in the knowledge of its fundamental group?*

Suppose that the characteristic $\mathrm{char}(k)$ of k is 0. The structure of $\Pi_{X^\bullet}$ is well-known, which is isomorphic to the profinite completion of the topological fundamental group of a Riemann surface of type (g_X, n_X) (cf. [V, Théorème 2.2 (c)]). In particular, $\Pi_{X^\bullet}$ is a free profinite group with $2g_X + n_X - 1$ generators if $n_X > 0$. This means that the geometric information of $X^\bullet$ cannot be carried out by the isomorphism class of $\Pi_{X^\bullet}$ (i.e., no anabelian geometry exists in this situation).

On the other hand, when $\mathrm{char}(k) = p > 0$, the situation is quite different from that in characteristic 0, and the structure of $\Pi_{X^\bullet}$ is no longer known. In the remainder of the introduction, we assume that $\mathrm{char}(k) = p > 0$, and that $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$ in k. The admissible fundamental group $\Pi_{X^\bullet}$ is very mysterious. Since the late 1990s, some developments of F. Pop, M. Saïdi, M. Raynaud, A. Tamagawa, and the author (cf. [PS], [R], [T1], [T2], [T3], [Y1], [Y2], [Y4]) showed evidence for very strong anabelian phenomena for *smooth* pointed stable curves over *algebraically closed fields of characteristic* $p > 0$. In this situation, the Galois group of the base field is trivial, and the arithmetic fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field.

Let us explain some background about the theory of anabelian geometry of curves over algebracially closed fields of characteristic $p > 0$ from the point of view of moduli spaces. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over $\overline{\mathbb{F}}_p$ parameterizing pointed stable curves of type (g, n) and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of q, and $k(q) \subseteq k'$ an algebraically closed field. Then the natural morphism $\mathrm{Spec}\, k \to \overline{M}_{g,n}$ determines a pointed stable curve $X_q^\bullet$ of type (g, n) over k'. We denote by $\Pi_{X_q^\bullet}$ the admissible fundamental group of $X_q^\bullet$. Since the isomorphism class of $\Pi_{X_q^\bullet}$ does not depend on the choices of the base field k', we may write Π_q for $\Pi_{X_q^\bullet}$. We denote by $\overline{\Pi}_{g,n}$ the set of isomorphism classes of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$. Then we have a surjective map

$$\pi_{g,n}^{\mathrm{adm}} : \overline{M}_{g,n}/\sim_{fe} \twoheadrightarrow \overline{\Pi}_{g,n},\ [q] \mapsto [\Pi_q],$$

where $\sim_{fe}$ denotes the Frobenius equivalence (cf. [Y4, Definition 3.4]), $[q]$ denotes the image of q in $\overline{M}_{g,n}/\sim_{fe}$, and $[\Pi_q]$ denotes the isomorphism class of Π_q in $\overline{\Pi}_{g,n}$. *The weak Isom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic* $p > 0$ (=the Weak Isom-version Conjecture) says that $\pi_{g,n}^{\mathrm{adm}}$ is a *bijection* (i.e., the moduli spaces of curves can be reconstructed as *sets* from the isomorphism classes of the admissible fundamental groups of curves corresponding to points of the moduli spaces). This kinds of anabelian phenomena go beyond Grothendieck's anabelian geometry (cf. [G1], [G2]), and show that the admissible fundamental group of a pointed stable curve over an algebraically closed field of characteristic $p > 0$ must encode "*moduli*" of the curve. Moreover, this is the reason that we do not have an explicit description of the admissible (or tame) fundamental group of any pointed stable curve in positive characteristic.

Recently, in [Y5], the author introduced a topology on $\overline{\Pi}_{g,n}$ which is called *the moduli spaces of admissible fundamental groups of type* (g, n). Moreover, the author posed *the Homeomorphism Conjecture* which says that $\pi_{g,n}^{\mathrm{adm}}$ is a *homeomorphism*, where $\overline{M}_{g,n}/\sim_{fe}$ is a topological space whose topology is induced by the Zariski topology of $\overline{M}_{g,n}$. In [Y5], [Y6], the author proved that the Homeomorphism Conjecture holds when $\dim(\overline{M}_{g,n}) = 1$. The Homeomorphism Conjecture gives us a new insight into the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ based on the following philosophy: The anabelian properties of pointed stable curves over algebraically closed fields of characteristic p are equivalent to the topological properties of the topological space $\overline{\Pi}_{g,n}$.

In the case of moduli spaces of curves, we have the following important morphisms of moduli stacks (i.e., clutching morphisms and forgetting morphisms):

$$\overline{\mathcal{M}}_{g_1,n_1} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2-2},$$

$$\overline{\mathcal{M}}_{g_3,n_3} \to \overline{\mathcal{M}}_{g_3+1,n_3-2},$$

$$\overline{\mathcal{M}}_{g_4,n_4} \to \overline{\mathcal{M}}_{g_4,n_4-1}$$

which play important roles for studying the topological properties of moduli spaces of curves. Then we may ask whether or not there exist clutching morphisms and forgetting morphisms for moduli spaces of admissible fundamental groups. This question is essentially equivalent to the following anabelian problem:

Question 0.1. *Does there exists a group-theoretical algorithm whose input datum is an abstract topological group which is isomorphic to $\Pi_{X^\bullet}$, and the output data are the topological and the combinatorial structures associated to $X^\bullet$ (i.e., (g_X, n_X), the dual semi-graph $\Gamma_{X^\bullet}$ of $\Gamma_{X^\bullet}$, the admissible fundamental groups of smooth pointed stable curves associated to irreducible components of $X^\bullet$, the inertia subgroups of nodes of $X^\bullet$, and the inertia subgroups of marked points of $X^\bullet$)?*

In the present paper, the main theorem is as follows (cf. Theorem 3.11 for a more precise statement):

Theorem 0.2. *We maintain the notation introduced above. Then there exists a group-theoretical algorithm whose input datum is an abstract topological group which is isomorphic to $\Pi_{X^\bullet}$, and the output data are the topological and the combinatorial structures associated to $X^\bullet$*

This result implies that we can define clutching morphisms and forgetting morphisms for moduli spaces of admissible fundamental groups. Moreover, in [Y7], we will prove that clutching morphisms and forgetting morphisms for moduli spaces of admissible fundamental groups are continuous.

The present paper is organized as follows. In Section 1, we recall some notation and results which will be used in the present paper. In Section 2, we establish a correspondence between a subset of cohomology classes and the set of vertices (resp. the set of edges) of the dual semi-graph of a pointed stable curve. In Section 3, we prove Theorem 0.2.

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1 Preliminaries

In this section, we recall some notation and results which will be used in the present paper.

Definition 1.1. Let $\mathbb{G}$ be a semi-graph (cf. [M1, Section 1]).

(a) We shall denote by $v(\mathbb{G})$, $e^{\rm op}(\mathbb{G})$, and $e^{\rm cl}(\mathbb{G})$ the set of vertices of $\mathbb{G}$, the set of open edges of $\mathbb{G}$, and the set of closed edges of $\mathbb{G}$, respectively.

(b) The semi-graph $\mathbb{G}$ can be regarded as a topological space with natural topology induced by $\mathbb{R}^2$. We define an *one-point compactification* $\mathbb{G}^{\rm cpt}$ of $\mathbb{G}$ as follows: if $e^{\rm op}(\mathbb{G}) = \emptyset$, we put $\mathbb{G}^{\rm cpt} = \mathbb{G}$; otherwise, the set of vertices of $\mathbb{G}^{\rm cpt}$ is the disjoint union $v(\mathbb{G}^{\rm cpt}) \overset{\rm def}{=} v(\mathbb{G}) \sqcup \{v_\infty\}$, the set of closed edges of $\mathbb{G}^{\rm cpt}$ is $e^{\rm cl}(\mathbb{G}^{\rm cpt}) \overset{\rm def}{=} e^{\rm cl}(\mathbb{G}) \cup e^{\rm op}(\mathbb{G})$, the set of open edges of $\mathbb{G}^{\rm cpt}$ is empty, and every edge $e \in e^{\rm op}(\mathbb{G}) \subseteq e^{\rm cl}(\mathbb{G}^{\rm cpt})$ connects v_∞ with the vertex of $\mathbb{G}$ that is abutted by e.

(c) Let $v \in v(\mathbb{G})$. We shall say that $\mathbb{G}$ is 2-*connected* at v if $\mathbb{G} \setminus \{v\}$ is either empty or connected. Moreover, we shall say that $\mathbb{G}$ is 2-connected if $\mathbb{G}$ is 2-connected at each $v \in v(\mathbb{G})$. Note that, if $\mathbb{G}$ is connected, then $\mathbb{G}^{\rm cpt}$ is 2-connected at each $v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\rm cpt})$ if and only if $\mathbb{G}^{\rm cpt}$ is 2-connected.

(d) We put

$$b(v) \overset{\rm def}{=} \sum_{e \in e^{\rm op}(\mathbb{G}) \cup e^{\rm cl}(\mathbb{G})} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that e meets v. We put

$$v(\mathbb{G})^{b \leq 1} \overset{\rm def}{=} \{v \in v(\mathbb{G}) \mid b(v) \leq 1\},$$

and denote by $e^{\rm cl}(\mathbb{G})^{b \leq 1}$ the set of closed edges of $\mathbb{G}$ which meet a vertex of $v(\mathbb{G})^{b \leq 1}$. We put

$$b^{\rm cl}(v) \overset{\rm def}{=} \sum_{e \in e^{\rm cl}(\mathbb{G})} b_e(v).$$

We shall say that a vertex v is *terminal* if the following conditions are satisfied: (1) $\mathbb{G}$ is a connected semi-graph; (2) $\mathbb{G}$ is a tree (i.e., the Betti number of $\mathbb{G}$ is 0); (3) $b^{\mathrm{cl}}(v) \leq 1$.

Remark 1.1.1. Suppose that $\mathbb{G}$ is a connected semi-graph, and that $\mathbb{G}$ is a tree. Then $\mathbb{G}^{\mathrm{cpt}}$ is 2-connected if and only if one of the following holds: (i) $\#v(\mathbb{G}) = 1$; (ii) $\#v(\mathbb{G}) = 2$ and $\#e^{\mathrm{op}}(\mathbb{G}) = 0$; (iii) $\#v(\mathbb{G}) \geq 2$ and each terminal vertex of $\mathbb{G}$ meets some open edge of $\mathbb{G}$.

Let p be a prime number, and let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $\mathrm{char}(k) = p$, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ and $r_X \overset{\mathrm{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$. We shall write $\Pi_{X^\bullet}$, $\Pi^{\text{ét}}_{X^\bullet}$, and $\Pi^{\mathrm{top}}_{X^\bullet}$ for the admissible fundamental group of $X^\bullet$ (cf. [Y3, Section 2] for the definitions of admissible coverings and admissible fundamental groups), the étale fundamental group of X, and the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, respectively. Then we have the following natural surjections

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top}}_{X^\bullet}.$$

Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup. We write $X^\bullet_H$ for the pointed semi-stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H, $\Gamma_{X^\bullet_H}$ for the dual semi-graph of $X^\bullet_H$, and r_{X_H} for the Betti number of $\Gamma_{X^\bullet_H}$. Then we obtain an admissible covering

$$f^\bullet_H : X^\bullet_H \to X^\bullet$$

over k induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, and obtain a natural morphism of dual semi-graphs

$$f^{\mathrm{sg}}_H : \Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$$

induced by $f^\bullet_H$, where "sg" means "semi-graph". Moreover, if H is an open *normal* subgroup, then $\Gamma_{X^\bullet_H}$ admits an action of $\Pi_{X^\bullet}/H$ induced by the natural action of $\Pi_{X^\bullet}/H$ on $X^\bullet_H$. Note that the quotient of $\Gamma_{X^\bullet_H}$ by $\Pi_{X^\bullet}/H$ coincides with $\Gamma_{X^\bullet}$, and that H is isomorphic to the admissible fundamental group $\Pi_{X^\bullet_H}$ of $X^\bullet_H$. We also use the notation $H^{\text{ét}}$ and H^{top} to denote $\Pi^{\text{ét}}_{X^\bullet_H}$ and $\Pi^{\mathrm{top}}_{X^\bullet_H}$, respectively.

Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \overset{\mathrm{def}}{=} X_v \setminus X_v^{\mathrm{sing}}$, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$. We define a smooth pointed stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}^\bullet_v = (\widetilde{X}_v, D_{\widetilde{X}_v} \overset{\mathrm{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v)).$$

We shall say that $\widetilde{X}^\bullet_v$ *is the smooth pointed stable curve of type* (g_v, n_v) *associated to* v, or *the smooth pointed stable curve associated to* v for short. We denote by $\Pi_{\widetilde{X}^\bullet_v}$ the admissible fundamental group of $\widetilde{X}^\bullet_v$. Then we have the following natural outer injection

$$\Pi_{\widetilde{X}^\bullet_v} \hookrightarrow \Pi_{X^\bullet}.$$

We put

$$\widehat{X} \stackrel{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} X_H,\ D_{\widehat{X}} \stackrel{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} D_{X_H},\ \Gamma_{\widehat{X}^\bullet} \stackrel{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} \Gamma_{X^\bullet_H}.$$

We shall say that

$$\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}})$$

is the universal admissible covering corresponding to $\Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ is the dual semi-graph of $\widehat{X}^\bullet$. Note that we have that $\mathrm{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of $\Pi_{X^\bullet}$.

Let $v \in v(\Gamma_{X^\bullet})$, $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, $\widehat{v} \in v(\Gamma_{\widehat{X}^\bullet})$ a vertex over v, and $\widehat{e} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{\widehat{X}^\bullet})$ an edge over e. We denote by

$$\Pi_{\widehat{v}} \subseteq \Pi_{X^\bullet},\ I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$$

the stabilizer subgroups of $\widehat{v}$ and $\widehat{e}$, respectively. We see immediately that $\Pi_{\widehat{v}}$ is (outer) isomorphic to $\Pi_{\widetilde{X}^\bullet_v}$ of $\widetilde{X}^\bullet_v$, and that $I_{\widehat{e}}$ is (outer) isomorphic to an inertia subgroup associated to the closed point of X corresponding to e. Then we have that $I_{\widehat{e}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal pro-prime-to-p quotient of $(-)$. We put

$$\mathrm{Ver}(\Pi_{X^\bullet}) \stackrel{\mathrm{def}}{=} \{\Pi_{\widehat{v}}\}_{\widehat{v} \in v(\Gamma_{\widehat{X}^\bullet})},$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet}) \stackrel{\mathrm{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet})},$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet}) \stackrel{\mathrm{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\mathrm{cl}}(\Gamma_{\widehat{X}^\bullet})}.$$

Moreover, if $\widehat{e}$ abuts on $\widehat{v}$, then we have the following injections

$$I_{\widehat{e}} \hookrightarrow \Pi_{\widehat{v}} \hookrightarrow \Pi_{X^\bullet}.$$

Note that $\mathrm{Ver}(\Pi_{X^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})$, and $\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet})$ admit natural actions of $\Pi_{X^\bullet}$ (i.e., the conjugacy actions), and that we have the following natural bijections

$$\mathrm{Ver}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \xrightarrow{\sim} v(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

Let $t \in \mathbb{N}$ be an arbitrary positive natural number, K_{p^t-1} the kernel of the natural surjection

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z},$$

where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$. We put

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{p^t-1}\otimes \mathbb{F}_p)}{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/(p^t-1)\mathbb{Z})},$$

and shall say that $\mathrm{Avr}_p(\Pi_{X^\bullet})$ is *the limit of p-averages of* $\Pi_{X^\bullet}$. The following formula concerning $\mathrm{Avr}_p(\Pi_{X^\bullet})$ plays a fundamental role in the theory of (tame or admissible) anabelian geometry of curves over algebraically closed fields of characteristic $p>0$.

Theorem 1.2. *We maintain the notation introduced above. Suppose that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected. Then we have*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#v(\Gamma_{X^\bullet})^{b\leq 1} + \#e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1}.$$

Proof. We maintain the notation introduced in [Y3, Theorem 5.2]. Note that $\#v(\Gamma_{X^\bullet})^{b\leq 1} = \#V^{\mathrm{tre}}_{X^\bullet}$ and $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1} = \#E^{\mathrm{tre}}_{X^\bullet}$. Then the theorem follows immediately from [Y3, Theorem 5.2] (or [Y3, Remark 5.2.1]). □

Remark 1.2.1. Suppose that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected. Note that $\#v(\Gamma_{X^\bullet})^{b\leq 1}\neq 0$ if one of the following conditions holds: (i) $X^\bullet$ is smooth and $e^{\mathrm{op}}(\Gamma_{X^\bullet})\leq 1$; (ii) $D_X=\emptyset$ (i.e., $\#e^{\mathrm{op}}(\Gamma_{X^\bullet})=0$), $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet})=1$, and $\#v(\Gamma_{X^\bullet})=2$. Then if $\#v(\Gamma_{X^\bullet})^{b\leq 1}\neq 0$, we have $\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X-1$.

Remark 1.2.2. Let Δ be an arbitrary profinite group and $m, N\in\mathbb{N}$ positive natural numbers. We define the closed normal subgroup $D_N(\Delta)$ of Δ to be the topological closure of $[\Delta,\Delta]\Delta^N$, where $[\Delta,\Delta]$ denotes the commutator subgroup of Δ. Moreover, we define the closed normal subgroup $D^{(m)}_N(\Delta)$ of Δ inductively by $D^{(1)}_N(\Delta)\overset{\mathrm{def}}{=} D_N(\Delta)$ and $D^{(i+1)}_N(\Delta)\overset{\mathrm{def}}{=} D_N(D^{(i)}_N(\Delta))$, $i\in\{1,\dots,m-1\}$. Let $\ell\neq p$ be a prime number. We put

$$H \overset{\mathrm{def}}{=} D^{(3)}_\ell(\Pi_{X^\bullet}).$$

Then we see that the following conditions hold: (i) $(\#(\Pi_{X^\bullet}/H), p)=1$; (ii) the genus of $\widetilde{X}_{H,v}$ is positive for each $v\in v(\Gamma_{X^\bullet_H})$; (iii) $\Gamma_{X^\bullet_H}$ is 2-connected and $\#(v(\Gamma_{X^\bullet_H})^{b\leq 1})=0$.

Definition 1.3. Let $f^\bullet: Y^\bullet\to X^\bullet$ be an admissible covering over k, $f: Y\to X$ the underlying morphism induced by $f^\bullet$, and $\deg(f)$ the degree of f. For any $e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ (resp. $e\in e^{\mathrm{op}}(\Gamma_{X^\bullet})$), write x_e for the node (resp. marked point) of $X^\bullet$ corresponding to e. We put

$$e^{\mathrm{cl,ra}}_f \overset{\mathrm{def}}{=} \{e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid \#f^{-1}(x_e)=1\},$$

$$e^{\mathrm{cl,\acute{e}t}}_f \overset{\mathrm{def}}{=} \{e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid \#f^{-1}(x_e)=\deg(f)\},$$

$$e^{\mathrm{op,ra}}_f \overset{\mathrm{def}}{=} \{e\in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \mid \#f^{-1}(x_e)=1\},$$

$$v^{\mathrm{ra}}_f \overset{\mathrm{def}}{=} \{v\in v(\Gamma_{X^\bullet}) \mid \#\mathrm{Irr}(f^{-1}(X_v))=1\},$$

$$v^{\mathrm{sp}}_f \overset{\mathrm{def}}{=} \{v\in v(\Gamma_{X^\bullet}) \mid \#\mathrm{Irr}(f^{-1}(X_v))=\deg(f)\},$$

where $\mathrm{Irr}(-)$ denotes the set of irreducible components of $(-)$. If the Galois closure of $f^\bullet$ is a Galois admissible covering whose Galois group is a p-group, then the definition of admissible coverings implies that $\#e^{\mathrm{cl,ra}}_f = \#e^{\mathrm{op,ra}}_f = 0$.

Lemma 1.4. *Let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois admissible covering over k and $\Gamma_{Y^\bullet}$ the dual semi-graph of $Y^\bullet$. Suppose that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected, and that $\Gamma^{\mathrm{cpt}}_{Y^\bullet}$ is not 2-connected. Then there exists a unique vertex $v \in v(\Gamma_{X^\bullet})$ such that $f^{-1}(X_v)$ is irreducible.*

Proof. Let $f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ be the map of dual semi-graphs induced by $f^\bullet$ and $v \in v(\Gamma_{X^\bullet})$ an arbitrary vertex of $\Gamma_{X^\bullet}$. Since $\Gamma_{Y^\bullet}$ is not 2-connected, we have $\#v(\Gamma_{X^\bullet}) \geq 2$.

Suppose that $D_X \neq \emptyset$. Then we have $v(\Gamma^{\mathrm{cpt}}_{X^\bullet}) = v(\Gamma_{X^\bullet}) \cup \{v_{X,\infty}\}$ and $v(\Gamma^{\mathrm{cpt}}_{Y^\bullet}) = v(\Gamma_{Y^\bullet}) \cup \{v_{Y,\infty}\}$. Moreover, f^{sg} can be extended to a map

$$f^{\mathrm{sg,cpt}} : \Gamma^{\mathrm{cpt}}_{Y^\bullet} \to \Gamma^{\mathrm{cpt}}_{X^\bullet}$$

such that $f^{\mathrm{sg,cpt}}(v_{Y,\infty}) = v_{X,\infty}$. Since $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected, we obtain that $\Gamma^{\mathrm{cpt}}_{X^\bullet} \setminus \{v\}$ is connected, and that $(f^{\mathrm{sg,cpt}})^{-1}(\Gamma^{\mathrm{cpt}}_{X^\bullet} \setminus \{v\})$ is connected. Then, for each $w \in (f^{\mathrm{sg,cpt}})^{-1}(v)$, there exists a closed edge of $\Gamma^{\mathrm{cpt}}_{Y^\bullet}$ which meets w and $(f^{\mathrm{sg,cpt}})^{-1}(\Gamma^{\mathrm{cpt}}_{X^\bullet} \setminus \{v\})$. We obtain that $\Gamma^{\mathrm{cpt}}_{Y^\bullet}$ is 2-connected. This contradicts our assumptions. Then we may assume that $D_X = \emptyset$.

Since we assume that $D_X = \emptyset$, we have $\Gamma^{\mathrm{cpt}}_{X^\bullet} = \Gamma_{X^\bullet}$ and $\Gamma^{\mathrm{cpt}}_{Y^\bullet} = \Gamma_{Y^\bullet}$. Let v_1 and v_2 be vertices of $v(\Gamma_{X^\bullet})$ distinct from each other. Suppose that $(f^{\mathrm{sg}})^{-1}(v_1)$ and $(f^{\mathrm{sg}})^{-1}(v_2)$ are connected. Since $\Gamma_{X^\bullet}$ is 2-connected, $\Gamma_{X^\bullet} \setminus \{v\}$ is connected. If $v \notin \{v_1, v_2\}$, we see that $(f^{\mathrm{sg}})^{-1}(\Gamma_{X^\bullet} \setminus \{v\})$ is connected (since v_1 and v_2 are contained in $\Gamma_{X^\bullet} \setminus \{v\}$), and that, for each $w \in (f^{\mathrm{sg}})^{-1}(v)$, there exists a closed edge of $\Gamma_{Y^\bullet}$ which meets w and $(f^{\mathrm{sg}})^{-1}(\Gamma_{X^\bullet} \setminus \{v\})$. This means that $\Gamma_{Y^\bullet}$ is 2-connected at w for each $w \in (f^{\mathrm{sg}})^{-1}(v)$. Suppose that $v = v_1$. Then we have that $\Gamma_{X^\bullet} \setminus \{v_1\}$ is connected, and that $(f^{\mathrm{sg}})^{-1}(\Gamma_{X^\bullet} \setminus \{v_1\})$ is connected (since v_2 is contained in $\Gamma_{X^\bullet} \setminus \{v_1\}$). Thus $\Gamma_{Y^\bullet}$ is 2-connected at $(f^{\mathrm{sg}})^{-1}(v_1)$. Similar arguments to the arguments given in the proof above imply that $\Gamma_{Y^\bullet}$ is 2-connected at $(f^{\mathrm{sg}})^{-1}(v_2)$. Then $\Gamma_{Y^\bullet}$ is 2-connected. This contradicts our assumptions.

Suppose that $(f^{\mathrm{sg}})^{-1}(v)$ is not connected for each $v \in v(\Gamma_{X^\bullet})$. To verify the lemma, it is sufficient to prove that $\Gamma_{Y^\bullet}$ is 2-connected at w for each $w \in (f^{\mathrm{sg}})^{-1}(v)$. Let $C_1, C_2 \subseteq (f^{\mathrm{sg}})^{-1}(\Gamma_{X^\bullet} \setminus \{v\})$ be connected components distinct from each other. Since $f^\bullet$ is Galois, we obtain that there exist $w' \in (f^{\mathrm{sg}})^{-1}(v)$, $e_1 \in e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$, and $e_2 \in e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$ such that e_1 meets w' and C_1, and that e_2 meets w' and C_2. This implies that $\Gamma_{Y^\bullet} \setminus \{w\}$ is 2-connected for each $w \in (f^{\mathrm{sg}})^{-1}(v)$. We complete the proof of the lemma. $\square$

2 Cohomology classes, sets of vertices, and sets of edges

We maintain the notation introduced in Section 1. Let ℓ be a prime number. We put

$$v(\Gamma_{X^\bullet})^{>0,\ell} \stackrel{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \dim_{\mathbb{F}_\ell}(H^1_{\text{ét}}(\widetilde{X}_v, \mathbb{F}_\ell)) > 0\} \subseteq v(\Gamma_{X^\bullet}).$$

Write $M^{\text{ét}}_{X^\bullet}$ and $M^{\mathrm{top}}_{X^\bullet}$ for $\mathrm{Hom}(\Pi^{\text{ét}}_{X^\bullet}, \mathbb{F}_\ell)$ and $\mathrm{Hom}(\Pi^{\mathrm{top}}_{X^\bullet}, \mathbb{F}_\ell)$, respectively. Note that there is a natural injection $M^{\mathrm{top}}_{X^\bullet} \hookrightarrow M^{\text{ét}}_{X^\bullet}$ induced by the natural surjection $\Pi^{\text{ét}}_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top}}_{X^\bullet}$. Moreover, we put

$$M^{\mathrm{nt}}_{X^\bullet} \stackrel{\mathrm{def}}{=} \mathrm{coker}(M^{\mathrm{top}}_{X^\bullet} \hookrightarrow M^{\text{ét}}_{X^\bullet}),$$

where "nt" means that "non-top".

The elements of $M^{\text{ét}}_{X^\bullet}$ correspond to étale, Galois abelian coverings of $X^\bullet$ of degree ℓ. We denote by $V^*_{X,\ell} \subseteq M^{\text{ét}}_{X^\bullet}$ the subset of elements whose image in $M^{\text{nt}}_{X^\bullet}$ is not 0. Let $\alpha \in V^*_{X,\ell}$. We denote by

$$X^\bullet_\alpha = (X_\alpha, D_{X_\alpha}) \to X^\bullet$$

the étale covering (i.e., the morphism of underlying curves is étale) corresponding to the element α and denote by $\Gamma_{X^\bullet_\alpha}$ the dual semi-graph of $X^\bullet_\alpha$. Then we have a map

$$\iota : V^*_{X,\ell} \to \mathbb{Z},\ \alpha \mapsto \#v(\Gamma_{X^\bullet_\alpha}).$$

Furthermore, we put

$$V^\star_{X,\ell} \overset{\text{def}}{=} \{\alpha \in V^*_{X,\ell} \mid \iota \text{ attains its maximum}\}$$

$$= \{\alpha \in V^*_{X,\ell} \mid \iota(\alpha) = \ell\#v(\Gamma_{X^\bullet}) - \ell + 1\}$$

$$= \{\alpha \in V^*_{X,\ell} \mid \#v^{\text{ra}}_{f_\alpha} = 1\}.$$

For each $\alpha \in V^\star_{X,\ell}$, $\iota(\alpha) = \ell\#v(\Gamma_{X^\bullet}) - \ell + 1$ implies that there exists a unique irreducible component $Z \subseteq X_\alpha$ whose decomposition group under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is not trivial. Let $v_\alpha \in v(\Gamma_{X^\bullet})$ such that $X_{v_\alpha} = f_\alpha(Z)$. Then we have $v_\alpha \in v(\Gamma_{X^\bullet})^{>0,\ell}$. This means that $V^\star_{X,\ell} = \emptyset$ if and only if $v(\Gamma_{X^\bullet})^{>0,\ell} = \emptyset$.

On the other hand, let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Write $f^{\text{sg}}_H : \Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by the admissible covering $f^\bullet_H : X^\bullet_H \to X^\bullet$ over k corresponding to H. We define a map

$$f^{\text{ver},\ell}_H : v(\Gamma_{X^\bullet_H})^{>0,\ell} \to v(\Gamma_{X^\bullet})^{>0,\ell}$$

as follows: Let $v_H \in v(\Gamma_{X^\bullet_H})^{>0,\ell}$ and $v \overset{\text{def}}{=} f^{\text{sg}}_H(v_H) \in v(\Gamma_{X^\bullet_H})$. Then we have that $f^{\text{ver},\ell}_H(v_H) = v$ if $\dim_{\mathbb{F}_\ell}(\text{Hom}(\Pi^{\text{ét}}_{\widetilde{X}^\bullet_v}, \mathbb{F}_\ell)) \neq 0$; otherwise, $f^{\text{ver},\ell}_H(v_H) = \emptyset$. Moreover, if $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup, then $v(\Gamma_{X^\bullet_H})^{>0,\ell}$ admits a natural action of $\Pi_{X^\bullet}/H$. We have the following proposition.

Proposition 2.1. *We define a pre-equivalence relation $\sim$ on $V^\star_{X,\ell}$ as follows:*

> *Let $\alpha, \beta \in V^\star_{X,\ell}$. We have that $\alpha \sim \beta$ if $\lambda\alpha + \mu\beta \in V^\star_{X,\ell}$ for each $\lambda, \mu \in \mathbb{F}^\times_\ell$ for which $\lambda\alpha + \mu\beta \in V^*_{X,\ell}$.*

Then $\sim$ is an equivalence relation on $V^\star_{X,\ell}$. Moreover, we have a natural bijection

$$\kappa_{X,\ell} : V_{X,\ell} \overset{\text{def}}{=} V^\star_{X,\ell}/\sim \xrightarrow{\sim} v(\Gamma_{X^\bullet})^{>0,\ell},\ [\alpha] \mapsto v_\alpha,$$

where $[\alpha]$ denotes the image of α in $V_{X,\ell}$.

Proof. Since $V_{X,\ell} = \emptyset$ if and only if $v(\Gamma_{X^\bullet})^{>0,\ell} = \emptyset$, we may suppose that $v(\Gamma_{X^\bullet})^{>0,\ell} \neq \emptyset$. Let $\alpha, \beta \in V^\star_{X,\ell}$.

If $v_\alpha = v_\beta$, then, for each $\lambda, \mu \in \mathbb{F}^\times_\ell$ for which $\lambda\alpha + \mu\beta \neq 0$, we have $v_{\lambda\alpha+\mu\beta} = v_\alpha = v_\beta$. Thus, $\alpha \sim \beta$.

On the other hand, if $\alpha \sim \beta$, we have $v_\alpha = v_\beta$; otherwise, there exist two irreducible components of $X^\bullet_{\alpha+\beta}$ whose decomposition groups under the actions of $\mathbb{Z}/\ell\mathbb{Z}$ are not trivial (i.e., $\alpha + \beta \notin V^\star_{X,\ell}$). Thus, $\alpha \sim \beta$ if and only if $v_\alpha = v_\beta$. This means that $\sim$ is an equivalence relation on $V^\star_{X,\ell}$.

Next, we prove that the map

$$\kappa_{X,\ell} : V_{X,\ell} \to v(\Gamma_{X^\bullet})^{>0,\ell},\ [\alpha] \mapsto v_\alpha$$

is a bijection. It is easy to see that $\kappa_{X,\ell}$ is an injection. On the other hand, for any irreducible component $X_v \in v(\Gamma_{X^\bullet})^{>0,\ell}$, we see that there is a Galois étale covering $f^\bullet : Y^\bullet \to X^\bullet$ (i.e., f is étale) whose Galois group is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ such that X_v is the unique irreducible component of $X^\bullet$ whose inverse image $f^{-1}(X_v)$ is connected. Then the cardinality of the set of irreducible components of $Y^\bullet$ is equal to $\ell(\#v(\Gamma_{X^\bullet}) - 1) + 1$. Thus, $Y^\bullet$ induces an element of $V_{X,\ell}$. This implies that $\kappa_{X,\ell}$ is a surjection. We complete the proof of the proposition. $\square$

Remark 2.1.1. Let ℓ and ℓ' be prime numbers distinct from each other. Write

$$V_{X,\ell},\ V_{X,\ell'}$$

for the sets associated to ℓ and ℓ' defined above, respectively. Suppose that $v(\Gamma_{X^\bullet})^{>0,\ell} \subseteq v(\Gamma_{X^\bullet})^{>0,\ell'}$ (note that $v(\Gamma_{X^\bullet})^{>0,\ell} = v(\Gamma_{X^\bullet})^{>0,\ell'}$ if ℓ and ℓ' are not equal to p). Then we may define a natural injection

$$V_{X,\ell} \hookrightarrow V_{X,\ell'}$$

which fits into the following commutative diagram

$$\begin{array}{ccc} V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>,\ell} \\ \downarrow & & \downarrow \\ V_{X,\ell'} & \xrightarrow{\kappa_{X,\ell'}} & v(\Gamma_{X^\bullet})^{>,\ell'} \end{array}$$

as follows. For each $\alpha \in V_{X,\ell}$ and each $\alpha' \in V_{X,\ell'}$, we write $X^\bullet_\alpha \to X^\bullet$ and $X^\bullet_{\alpha'} \to X^\bullet$ for the Galois admissible coverings corresponding to α and α', respectively. We consider the following connected Galois admissible covering

$$X^\bullet_\alpha \times_{X^\bullet} X^\bullet_{\alpha'} \to X^\bullet$$

over k whose Galois group is isomorphic to $\mathbb{Z}/\ell\ell'\mathbb{Z}$, where $X^\bullet_\alpha \times_{X^\bullet} X^\bullet_{\alpha'}$ denotes the fiber product in the category of pointed stable curves. Then it is easy to see that $v_\alpha = v_{\alpha'}$ if and only if the cardinality of the set of irreducible components of $X^\bullet_\alpha \times_{X^\bullet} X^\bullet_{\alpha'}$ is equal to

$$\ell\ell'(\#v(\Gamma_{X^\bullet}) - 1) + 1.$$

Then we obtain a natural injection

$$V_{X,\ell} \hookrightarrow V_{X,\ell'},\ [\alpha] \mapsto [\alpha'],$$

where the cardinality of the set of irreducible components of $X^\bullet_\alpha \times_{X^\bullet} X^\bullet_{\alpha'}$ is equal to $\ell\ell'(\#v(\Gamma_{X^\bullet}) - 1) + 1$. In particular, if ℓ and ℓ' are not equal to p, then the injection $V_{X,\ell} \hookrightarrow V_{X,\ell'}$ constructed above is a bijection.

Remark 2.1.2. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup,

$$f_H^\bullet : X_H^\bullet = (X_H, D_{X_H}) \to X^\bullet$$

the admissible covering over k with degree $\deg(f_H)$ corresponding to H, $\Gamma_{X_H^\bullet}$ the dual semi-graph of $X_H^\bullet$, and ℓ a prime number such that $(\ell, \deg(f_H)) = 1$.

Write $V_{X_H,\ell}$ and $V_{X,\ell}$ for the sets defined above. We have a natural map

$$\gamma_H^{\mathrm{ver},\ell} : V_{X_H,\ell} \to V_{X,\ell}$$

defined as follows. We put

$$\gamma_H^{\mathrm{ver},\ell}([\alpha_{X_H}]) = [\alpha_X],\ \alpha_{X_H} \in V^\star_{X_H,\ell},$$

where $\alpha_X \in V^\star_{X,\ell}$ satisfies the following conditions:

(i) α_X induces an element $\alpha'_{X_H} = \sum_{\alpha \in L_{\alpha_X}} c_\alpha \alpha$ via the natural homomorphism

$$\mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{F}_\ell) \to \mathrm{Hom}(H, \mathbb{F}_\ell),$$

where L_{α_X} is a subset of $V_{X_H,\ell}$ such that $[\alpha_1] \neq [\alpha_2]$ if $\alpha_1, \alpha_2 \in L_{\alpha_X}$ are distinct from each other, and that $c_\alpha \neq 0$ for each $\alpha \in L_{\alpha_X}$;

(ii) there exists $\alpha \in L_{\alpha_X}$ such that $[\alpha] = [\alpha_{X_H}]$ (i.e., $\alpha \sim \alpha_{X_H}$).

It is easy to check that $\gamma_H^{\mathrm{ver},\ell}$ is well-defined, and that the following diagram

$$\begin{array}{ccc} V_{X_H,\ell} & \xrightarrow{\kappa_{X_H,\ell}} & v(\Gamma_{X_H^\bullet})^{>0,\ell} \\ {\scriptstyle \gamma_H^{\mathrm{ver},\ell}} \downarrow & & {\scriptstyle f_H^{\mathrm{ver},\ell}} \downarrow \\ V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>0,\ell} \end{array}$$

is commutative.

Moreover, suppose that H is *normal*. Then there is an action of $\Pi_{X^\bullet}/H$ on the set

$$V^\star_{X_H,\ell} \subseteq H^1_{\text{ét}}(X_H, \mathbb{F}_\ell) = \mathrm{Hom}(H^{\text{ét}}, \mathbb{F}_\ell)$$

via the natural outer representation

$$\Pi_{X^\bullet}/H \to \mathrm{Out}(H^{\text{ét}}) \to \mathrm{Aut}(H^{\text{ét,ab}})$$

induced by the following natural exact sequence

$$1 \to H^{\text{ét}} \to \Pi_{X^\bullet}/(\ker(H \twoheadrightarrow H^{\text{ét}})) \to \Pi_{X^\bullet}/H \to 1.$$

Let $\alpha, \alpha' \in V_{X_H,\ell}$. We see that, for each $\sigma \in \Pi_{X^\bullet}/H$, $\alpha \sim \alpha'$ if and only if $\sigma(\alpha) \sim \sigma(\alpha')$. Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on $V_{X_H,\ell}$. On the other hand, $v(\Gamma_{X_H^\bullet})^{>0,\ell}$ admits a natural action of $\Pi_{X^\bullet}/H$ induced by the action of $\Pi_{X^\bullet}/H$ on $X_H^\bullet$. We see immediately that the bijection $\kappa_{X_H,\ell} : V_{X_H,\ell} \xrightarrow{\sim} v(\Gamma_{X_H^\bullet})^{>0,\ell}$ is $\Pi_{X^\bullet}/H$-equivalent.

In the remainder of this section, suppose that the genus g_v of $\widetilde{X}^\bullet_v$ is *positive* for each $v \in v(\Gamma_{X^\bullet})$, and that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-*connected.* We shall say that

$$\mathfrak{T}_{X^\bullet} \stackrel{\mathrm{def}}{=} (\ell, d, f^\bullet_X : Y^\bullet \to X^\bullet)$$

is an *edge-triple* associated to $X^\bullet$ if the following conditions are satisfied:

(i) ℓ and d are prime numbers distinct from each other and from p;

(ii) $\ell \equiv 1 \pmod d$; this means that all dth roots of unity are contained in $\mathbb{F}_\ell$; moreover, we write $\mu_d \subseteq \mathbb{F}_\ell^\times$ for the subgroup of dth roots of unity;

(iii) $f^\bullet_X : Y^\bullet \stackrel{\mathrm{def}}{=} (Y, D_Y) \to X^\bullet$ is a Galois *étale* covering whose Galois group is isomorphic to μ_d such that $\# v^{\mathrm{sp}}_{f_X} = 0$ holds (note that since g_v, $v \in v(\Gamma_{X^\bullet})$, is positive, $f^\bullet_X$ exists).

We fix an edge-triple $\mathfrak{T}_{X^\bullet} \stackrel{\mathrm{def}}{=} (\ell, d, f^\bullet_X : Y^\bullet \to X^\bullet)$ associated to $X^\bullet$. Let $\Pi_{Y^\bullet} \subseteq \Pi_{X^\bullet}$ be the admissible fundamental group of $Y^\bullet$. Write $M^{\text{ét}}_{Y^\bullet}$ and $M_{Y^\bullet}$ for $\mathrm{Hom}(\Pi^{\text{ét}}_{Y^\bullet}, \mathbb{F}_\ell)$ and $\mathrm{Hom}(\Pi_{Y^\bullet}, \mathbb{F}_\ell)$, respectively. We obtain a natural injection $M^{\text{ét}}_{Y^\bullet} \hookrightarrow M_{Y^\bullet}$ induced by the natural surjection $\Pi_{Y^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{Y^\bullet}$. Then we have an exact sequence

$$0 \to M^{\text{ét}}_{Y^\bullet} \to M_{Y^\bullet} \to M^{\mathrm{ra}}_{Y^\bullet} \stackrel{\mathrm{def}}{=} \mathrm{coker}(M^{\text{ét}}_{Y^\bullet} \hookrightarrow M_{Y^\bullet}) \to 0$$

with a natural action of μ_d.

Let $M^{\mathrm{ra}}_{Y^\bullet, \mu_d} \subseteq M^{\mathrm{ra}}_{Y^\bullet}$ be the subset of elements on which μ_d acts via the character $\mu_d \subseteq \mathbb{F}_\ell^\times$ and $E^*_{\mathfrak{T}_{X^\bullet}} \subseteq M_{Y^\bullet}$ the subset of elements that map to nonzero elements of $M^{\mathrm{ra}}_{Y^\bullet, \mu_d}$. Let $\alpha \in E^*_{\mathfrak{T}_{X^\bullet}}$. Write

$$g^\bullet_\alpha : Y^\bullet_\alpha \to Y^\bullet$$

for the admissible covering corresponding to the element α and $\Gamma_{Y^\bullet_\alpha}$ for the dual semi-graph of $Y^\bullet_\alpha$. Then we obtain a map

$$\epsilon : E^*_{\mathfrak{T}_{X^\bullet}} \to \mathbb{Z},\ \alpha \mapsto \#(e^{\mathrm{op}}(\Gamma_{Y^\bullet_\alpha}) \cup e^{\mathrm{cl}}(\Gamma_{Y^\bullet_\alpha})).$$

We define two subsets of $E^*_{\mathfrak{T}_{X^\bullet}}$ as follows:

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}} \stackrel{\mathrm{def}}{=} \{\alpha \in E^*_{\mathfrak{T}_{X^\bullet}} \mid \# e^{\mathrm{cl,ra}}_{g_\alpha} = d, \# e^{\mathrm{op,ra}}_{g_\alpha} = 0\},$$

and

$$E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}} \stackrel{\mathrm{def}}{=} \{\alpha \in U^*_{\ell, Y^\bullet} \mid \# e^{\mathrm{cl,ra}}_{g_\alpha} = 0, \# e^{\mathrm{op,ra}}_{g_\alpha} = d\},$$

where "cl" means "closed edge", and "op" means "open edge". Note that $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$ and $E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}}$ are not empty. For each $\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ (resp. $\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$), since the image of α is contained in $M^{\mathrm{ra}}_{Y^\bullet, \mu_d}$, we obtain that the action of μ_d on the set

$$\{y_e\}_{e \in e^{\mathrm{cl,ra}}_{g_\alpha}} \subseteq \mathrm{Nod}(Y^\bullet)\ (\text{resp. } \{y_e\}_{e \in e^{\mathrm{op,ra}}_{g_\alpha}} \subseteq D_Y)$$

is transitive, where $\mathrm{Nod}(-)$ denotes the set of nodes of $(-)$, and y_e denotes the node (resp. the marked point) of $Y^\bullet$ corresponding to e. Then there exists a unique node (resp. marked point) x_α of $X^\bullet$ such that $f_X(y_e) = x_\alpha$ for every $y_e \in \{y_e\}_{e \in e^{\mathrm{cl,ra}}_{g_\alpha}}$ (resp. $y_e \in \{y_e\}_{e \in e^{\mathrm{op,ra}}_{g_\alpha}}$). We denote by $e_\alpha \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ the closed edge (resp. $e_\alpha \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ the open edge) corresponding to x_α. Moreover, we have the following proposition.

Proposition 2.2. *We maintain the notation introduced above. We define a pre-equivalence relation $\sim$ on $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$ (resp. $E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}}$) as follows:*

> *Let $\alpha, \beta \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$ (resp. $\alpha, \beta \in E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}}$). Then $\alpha \sim \beta$ if $\lambda\alpha + \mu\beta \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$ (resp. $E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}}$) for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in E^{*}_{\mathfrak{T}_{X^\bullet}}$.*

Then the pre-equivalence relation $\sim$ on $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$ (resp. $E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}}$) defined above is an equivalence relation. Moreover, we have a natural bijection

$$\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}} : E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}} \overset{\mathrm{def}}{=} E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}/\sim \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X^\bullet}),\ [\alpha] \mapsto e_\alpha$$

$$(\textit{resp. } \vartheta^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}} : E^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}} \overset{\mathrm{def}}{=} E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}}/\sim \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X^\bullet}),\ [\alpha] \mapsto e_\alpha),$$

where $[\alpha]$ denotes the image of α in $E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$ (resp. $E^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}}$).

Proof. Let $\alpha, \beta \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$. If $e^{\mathrm{cl,ra}}_{g_\alpha} = e^{\mathrm{cl,ra}}_{g_\beta}$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \neq 0$, we have $e^{\mathrm{cl,ra}}_{g_{\lambda\alpha+\mu\beta}} = e^{\mathrm{cl,ra}}_{g_\alpha} = e^{\mathrm{cl,ra}}_{g_\beta}$. Thus, $\alpha \sim \beta$.

On the other hand, if $\alpha \sim \beta$, then we have $e^{\mathrm{cl,ra}}_{g_\alpha} = e^{\mathrm{cl,ra}}_{g_\beta}$; otherwise, we obtain $\# e^{\mathrm{cl,ra}}_{g_{\alpha+\beta}} = 2d$. Thus, $\alpha \sim \beta$ if and only if $e^{\mathrm{cl,ra}}_{g_\alpha} = e^{\mathrm{cl,ra}}_{g_\beta}$. This means that $\sim$ is an equivalence relation on $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$.

Next, let us prove that $\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$ is a bijection. It is easy to see that $\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$ is an injection. On the other hand, for each $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, the structure of the maximal pro-ℓ admissible fundamental groups implies that there is a Galois admissible covering of $h^\bullet : Z^\bullet \to Y^\bullet$ such that the element corresponding to $h^\bullet$ is contained in $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$. Then $\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$ is a surjection.

Similar arguments to the arguments given in the proof above imply that the "resp" part holds. This completes the proof of the proposition. □

Remark 2.2.1. In this remark, we prove that the sets

$$E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}},\ E^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}}$$

do not depend on the choices of $\mathfrak{T}_{X^\bullet}$.

We only treat the case of closed edges. Let

$$\mathfrak{T}^*_{X^\bullet} \overset{\mathrm{def}}{=} (\ell^*, d^*, f^{\bullet,*}_X : Y^{\bullet,*} \to X^\bullet)$$

be an arbitrary edge-triple associated to $X^\bullet$. Hence we obtain $E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}}$ and a natural bijection

$$\vartheta^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}} : E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}} \to e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

We will show that there exists a bijection $E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}} \overset{\sim}{\to} E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$ which fits into the following commutative diagram

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}} & \xrightarrow{\vartheta^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \\ \downarrow & & \| \\ E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}} & \xrightarrow{\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}). \end{array}$$

First, suppose that $\ell \neq \ell^*$, and that $d \neq d^*$. Then we may define a bijection

$$E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}} \overset{\sim}{\to} E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$$

which is compatible with the bijections $\vartheta^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}}$ and $\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$ as follows. Let $\alpha \in E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$ and $\alpha^* \in E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}}$. Write $Y^\bullet_\alpha \to Y^\bullet$ and $Y^\bullet_{\alpha^*} \to Y^{\bullet,*}$ for the Galois admissible coverings corresponding to α and α^*, respectively. We consider the following connected Galois admissible covering

$$Y^\bullet_\alpha \times_{X^\bullet} Y^\bullet_{\alpha^*} \to X^\bullet$$

over k with Galois group $\mathbb{Z}/dd^*\ell\ell^*\mathbb{Z}$. Then we see that $e_\alpha = e_{\alpha^*}$ if and only if the cardinality of the set of nodes of $Y^\bullet_\alpha \times_{X^\bullet} Y^\bullet_{\alpha^*}$ is equal to

$$dd^*(\ell\ell^*(\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - 1) + 1).$$

Then we define a map

$$E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}} \overset{\sim}{\to} E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}},\ [\alpha^*] \mapsto [\alpha],$$

where the cardinality of the set of nodes of $Y^\bullet_\alpha \times_{X^\bullet} Y^\bullet_{\alpha^*}$ is equal to $dd^*(\ell\ell^*(\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - 1) + 1)$.

Next, let us prove the general case. Let

$$\mathfrak{T}^{**}_{X^\bullet} \overset{\mathrm{def}}{=} (\ell^{**}, d^{**}, f^{\bullet,**} : Y^{\bullet,**} \to X^\bullet)$$

be an edge-triple associated to $X^\bullet$ such that $\ell^{**} \neq \ell$, $\ell^{**} \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain $E^{\mathrm{cl}}_{\mathfrak{T}^{**}_{X^\bullet}}$ and a bijection $\vartheta^{\mathrm{cl}}_{\mathfrak{T}^{**}_{X^\bullet}} : E^{\mathrm{cl}}_{\mathfrak{T}^{**}_{X^\bullet}} \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. Then the proof above implies that there are two bijections

$$E^{\mathrm{cl}}_{\mathfrak{T}^{**}_{X^\bullet}} \overset{\sim}{\to} E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}} \text{ and } E^{\mathrm{cl}}_{\mathfrak{T}^{**}_{X^\bullet}} \overset{\sim}{\to} E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}}.$$

Thus, we obtain the desired map $E^{\mathrm{cl}}_{\mathfrak{T}^*_{X^\bullet}} \overset{\sim}{\to} E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$.

Remark 2.2.2. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup, $f^\bullet_H : X^\bullet_H \to X^\bullet$ the Galois admissible covering over k with degree $\deg(f_H)$ induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, and $\Gamma_{X^\bullet_H}$ the dual semi-graph of $X^\bullet_H$. Moreover, we have two natural maps

$$f^{\mathrm{cl}}_H : e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{cl}}(\Gamma_{X^\bullet}),$$

$$f^{\mathrm{op}}_H : e^{\mathrm{op}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{op}}(\Gamma_{X^\bullet})$$

induced by $f^\bullet_H$.

Let

$$\mathfrak{T}_{X^\bullet} \overset{\mathrm{def}}{=} (\ell, d, f^\bullet_X : Y^\bullet \to X^\bullet)$$

be an edge-triple associated to $X^\bullet$ such that $(\ell, \deg(f_H)) = (d, \deg(f_H)) = 1$. Then we obtain an edge-triple

$$\mathfrak{T}_{X^\bullet_H} \overset{\mathrm{def}}{=} (\ell, d, f^\bullet_{X_H} : Z^\bullet \overset{\mathrm{def}}{=} Y^\bullet \times_{X^\bullet} X^\bullet_H \to X^\bullet_H)$$

associated to $X_H^\bullet$ induced by the edge-triple $\mathfrak{T}_{X^\bullet}$. Moreover, we obtain two natural maps

$$f_H^{\mathrm{cl}} : e^{\mathrm{cl}}(\Gamma_{X_H^\bullet}) \to e^{\mathrm{cl}}(\Gamma_{X^\bullet}),$$

$$f_H^{\mathrm{op}} : e^{\mathrm{op}}(\Gamma_{X_H^\bullet}) \to e^{\mathrm{op}}(\Gamma_{X^\bullet})$$

induced by $f_H^\bullet$.

Write $\Pi_{Z^\bullet}$ and $\Pi_{Y^\bullet}$ for the open subgroups of $\Pi_{X^\bullet}$ corresponding to $Z^\bullet$ and $Y^\bullet$, respectively. Note that we have $\Pi_{Z^\bullet} \subseteq \Pi_{Y^\bullet}$. Then we have a natural map

$$\gamma^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet},H} : E^{\mathrm{cl}}_{\mathfrak{T}_{X_H^\bullet}} \to E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$$

defined as follows. We define

$$\gamma^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet},H}([\alpha_{X_H}]) = [\alpha_X], \ \alpha_{X_H} \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X_H^\bullet}}$$

where $\alpha_X \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$ satisfies the following conditions:

(i) α_X induces an element $\alpha_Z = \sum_{\beta \in J_{\alpha_X}} c_\beta \beta$ via the natural homomorphism

$$\mathrm{Hom}(\Pi_{Y^\bullet}, \mathbb{F}_\ell) \to \mathrm{Hom}(\Pi_{Z^\bullet}, \mathbb{F}_\ell),$$

where J_{α_X} is a subset of $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X_H^\bullet}}$ such that $[\beta_1] \neq [\beta_2]$ if $\beta_1, \beta_2 \in J_{\alpha_X}$ are distinct from each other, and that $c_\beta \neq 0$ for each $\beta \in J_{\alpha_X}$;

(ii) there exists $\beta \in J_{\alpha_X}$ such that $[\beta] = [\alpha_X]$ (i.e., $\beta \sim \alpha_X$).

By applying similar arguments to the arguments given above, we obtain a natural map

$$\gamma^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet},H} : E^{\mathrm{op}}_{\mathfrak{T}_{X_H^\bullet}} \to E^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}}.$$

It is easy to check that $\gamma^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet},H}$ and $\gamma^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet},H}$ are well-defined, and that the following diagrams

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}_{X_H^\bullet}} & \xrightarrow{\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X_H^\bullet}}} & e^{\mathrm{cl}}(\Gamma_{X_H^\bullet}) \\ \gamma^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet},H} \downarrow & & f_H^{\mathrm{cl}} \downarrow \\ E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}} & \xrightarrow{\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}), \end{array}$$

and

$$\begin{array}{ccc} E^{\mathrm{op}}_{\mathfrak{T}_{X_H^\bullet}} & \xrightarrow{\vartheta^{\mathrm{op}}_{\mathfrak{T}_{X_H^\bullet}}} & e^{\mathrm{op}}(\Gamma_{X_H^\bullet}) \\ \gamma^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet},H} \downarrow & & f_H^{\mathrm{op}} \downarrow \\ E^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}} & \xrightarrow{\vartheta^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}}} & e^{\mathrm{op}}(\Gamma_{X^\bullet}), \end{array}$$

are commutative.

Moreover, suppose that H is *normal*. We claim that $E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet_H}}$ and $E^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet_H}}$ admit natural actions of $\Pi_{X^\bullet}/H$, and that $\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet_H}}$ and $\vartheta^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet_H}}$ are $\Pi_{X^\bullet}/H$-equivalent.

We only treat the case of $E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet_H}}$. Note that $\Pi_{Z^\bullet}$ is an open normal subgroup of $\Pi_{X^\bullet}$ and

$$\Pi_{X^\bullet}/\Pi_{Z^\bullet} \cong \Pi_{X^\bullet}/H \times \mathbb{Z}/d\mathbb{Z}.$$

Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet_H}} \subseteq \mathrm{Hom}(\Pi_{Z^\bullet}, \mathbb{F}_\ell)$$

via the natural outer representation

$$\Pi_{X^\bullet}/H \hookrightarrow \Pi_{X^\bullet}/\Pi_{Z^\bullet} \to \mathrm{Out}(\Pi_{Z^\bullet})$$

induced by the following natural exact sequence

$$1 \to \Pi_{Z^\bullet} \to \Pi_{X^\bullet} \to \Pi_{X^\bullet}/\Pi_{Z^\bullet} \to 1.$$

Let $\beta, \beta' \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet_H}}$. We obverse that, for each $\tau \in \Pi_{X^\bullet}/H$, $\beta \sim \beta'$ if and only if $\tau(\beta) \sim \tau(\beta')$. Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on $E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet_H}}$. On the other hand, $e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})$ admits an action of $\Pi_{X^\bullet}/H$ induced by the action of $\Pi_{X^\bullet}/H$ on $X^\bullet_H$. We see immediately that $\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet_H}}$ is $\Pi_{X^\bullet}/H$-equivalent.

Let $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ (resp. $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$). We put

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}} \mid e = e_\alpha\}$$

$$(\text{resp. } E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}} \mid e = e_\alpha\}).$$

Then, for each $e, e' \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ (resp. $e, e' \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$) distinct from each other, we have

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet},e} \cap E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet},e'} = \emptyset \ (\text{resp. } E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet},e} \cap E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet},e'} = \emptyset).$$

Thus, we have

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}} = \bigsqcup_{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})} E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet},e} \ (\text{resp. } E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}} = \bigsqcup_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet},e}).$$

For each $m \in \mathbb{Z}_{\geq 0}$, we put

$$E^{\mathrm{cl},\star,m}_{\mathfrak{T}_{X^\bullet},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet},e} \mid \# v^{\mathrm{sp}}_{g_\alpha} = m\}$$

$$(\text{resp. } E^{\mathrm{op},\star,m}_{\mathfrak{T}_{X^\bullet},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet},e} \mid \# v^{\mathrm{sp}}_{g_\alpha} = m\}).$$

If e is a closed edge corresponding to a node which is contained in two irreducible components of $Y^\bullet$ distinct from each other, then $E^{\mathrm{cl},\star,m}_{\mathfrak{T}_{X^\bullet},e} = \emptyset$ for $m \geq \#v(\Gamma_{Y^\bullet}) - 1$. If e is a closed edge corresponding to a node which is contained in a unique irreducible component of $Y^\bullet$, then $E^{\mathrm{cl},\star,m}_{\mathfrak{T}_{X^\bullet},e} = \emptyset$ for $m \geq \#v(\Gamma_{Y^\bullet})$. If e is an open edge, then $E^{\mathrm{op},\star,m}_{\mathfrak{T}_{X^\bullet},e} = \emptyset$ for $m \geq \#v(\Gamma_{Y^\bullet})$.

We note that the edge-triple

$$\mathfrak{T}_{X^\bullet} \stackrel{\mathrm{def}}{=} (\ell, d, f^\bullet_X : Y^\bullet \to X^\bullet)$$

associated to $X^\bullet$ is equivalent to a triple

$$\mathfrak{T}_{\Pi_{X^\bullet}} \stackrel{\mathrm{def}}{=} (\ell, d, y),$$

where $y \in \mathrm{Hom}(\Pi^{\text{ét}}_{X^\bullet}, \mathbb{F}_d)$ corresponding to $f^\bullet_X$. We shall say that $\mathfrak{T}_{\Pi_{X^\bullet}}$ an edge-triple associated to $\Pi_{X^\bullet}$. Then we also use the notation

$$E^*_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ E^{\mathrm{op}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}, E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e},\ E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e},\ E^{\mathrm{cl},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}},e},\ E^{\mathrm{op},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}},e},\ \vartheta^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ \vartheta^{\mathrm{op}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

to denote $E^*_{\mathfrak{T}_{X^\bullet}}$, $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet}}$, $E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet}}$, $E^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$, $E^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}}$, $E^{\mathrm{cl},\star}_{\mathfrak{T}_{X^\bullet},e}$, $E^{\mathrm{op},\star}_{\mathfrak{T}_{X^\bullet},e}$, $E^{\mathrm{cl},\star,m}_{\mathfrak{T}_{X^\bullet},e}$, $E^{\mathrm{op},\star,m}_{\mathfrak{T}_{X^\bullet},e}$, $\vartheta^{\mathrm{cl}}_{\mathfrak{T}_{X^\bullet}}$, $\vartheta^{\mathrm{op}}_{\mathfrak{T}_{X^\bullet}}$, respectively.

3 Mono-anabelian combinatorial Grothendieck conjecture in positive characteristic

We maintain the notation introduced in previous sections. First, let us define the term "mono-anabelian reconstruction".

Definition 3.1. Let $i \in \{1, 2\}$, and let $\mathcal{F}_i$ be a geometric object and $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object $\mathcal{F}_i$.

Let $\mathrm{Inv}_{\mathcal{F}_i}$ be an invariant depending on the isomorphism class of $\mathcal{F}_i$ (in a certain category), we shall say that $\mathrm{Inv}_{\mathcal{F}_i}$ can be *mono-anabelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Inv}_{\mathcal{F}_i}$.

Let $\mathrm{Add}_{\mathcal{F}_i}$ be an additional structure (e.g. a family of subgroups, a family of quotient groups) on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on $\mathcal{F}_i$. We shall say that $\mathrm{Add}_{\mathcal{F}_i}$ can be *mono-anabelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_i}$.

We shall say that a map (or a morphism) $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$ can be *mono-anabelian reconstructed* from $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$.

Remark 3.1.1. The philosophy of "mono-anabelian" was introduced by S. Mochizuki (cf. [M2]). The classical point of view of anabelian geometry (i.e., the anabelian geometry considered in [G1], [G2]) focuses on a comparison between two geometric objects via their fundamental groups. Moreover, the term "group-theoretical", in the classical point of view, means that "preserved by an arbitrary isomorphism between the fundamental groups under consideration". The classical point of view is referred to as *bi-anabelian geometry.* On the other hand, mono-anabelian geometry focuses on the establishing a *group-theoretical algorithm* whose input datum is an abstract topological group which is isomorphic to the fundamental group of a given geometric object of interest (resp. a continuous homomorphism of abstract topological groups which are isomorphic to the fundamental groups of given geometric objects of interest), and whose output datum is a geometry object which is isomorphic to the given geometric object (resp. a morphism of geometric objects which is isomorphic to the given geometric objects of interest). In the point of view of mono-anabelian geometry, the term "group-theoretical algorithm" is used to mean that "the algorithm in a discussion is phrased in language that only depends on the topological group structure of the fundamental groups under consideration". Mono-anabelian results are the strongest form in the theory of anabelian geometry, then we have

$$\text{mono-anabelian results} \Rightarrow \text{bi-anabelian results.}$$

One of the main difficulties of establishing a group-theoretical algorithm for reconstructing the topological and the combinatorial structures associated to $X^\bullet$ is that, for each open subgroup $H \subseteq \Pi_{X^\bullet}$, we need to prove that the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet_H}$ and the étale fundamental group of the underlying curve of $X^\bullet_H$ (or the weight-monodromy filtration of the first ℓ-adic étale cohomology group of X_H, where $\ell \neq p$) can be mono-anabelian reconstructed from H. When the base field is an arithmetic field, the weight-monodromy filtration can be mono-anabelian reconstructed by applying the theory of "weight". In our situation (i.e., the base field is an algebraically closed field), we have the following key observation:

> The formula for $\mathrm{Avr}_p(H)$ of H plays a role of (outer) Galois representations in the theory of the combinatorial anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

We introduce the following condition for $X^\bullet$.

Condition A . We shall say that $X^\bullet$ *satisfies Condition A* if the following conditions hold: (i) g_v is positive for each $v \in v(\Gamma_{X^\bullet})$; (ii) $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected; (iii) $\#v(\Gamma_{X^\bullet})^{b\leq 1} = 0$ (then $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1} = 0$).

In the remainder of this section, we suppose that $X^\bullet$ *satisfies Condition A unless indicated otherwise.* Note that Theorem 1.2 and Condition A imply that

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X.$$

Lemma 3.2. *(i) The data $p \overset{\mathrm{def}}{=} \mathrm{char}(k)$, g_X, $n_X = \#e^{\mathrm{op}}(\Gamma_{X^\bullet})$, r_X, and $\Pi^{\mathrm{top},p}_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, where $\Pi^{\mathrm{top},p}_{X^\bullet}$ denotes the maximal pro-p quotient of $\Pi^{\mathrm{top}}_{X^\bullet}$.*

(ii) The set $v(\Gamma_{X^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

(iii) Let $H \subseteq \Pi_{X^\bullet}$ be any open normal subgroup. Suppose that $\Gamma^{\mathrm{cpt}}_{X^\bullet_H}$ is 2-connected. Then the natural map

$$v(\Gamma_{X^\bullet_H})^{>0,p} \to v(\Gamma_{X^\bullet})^{>0,p}$$

can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

(iv) The cardinality $\#v(\Gamma_{X^\bullet})$ of $v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

Proof. (i) If $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell) = \dim_{\mathbb{F}_{\ell'}}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_{\ell'})$ holds for any two prime numbers ℓ and ℓ', then $g_X = 2g_X + n_X - 1$ if $n_X > 0$, and $g_X = 2g_X$ if $n_X = 0$. Thus, either $(g_X, n_X) = (0,1)$ or $(g_X, n_X) = (0,0)$ holds. Since $\Pi_{X^\bullet}$ is the admissible fundamental group of a pointed stable curve, this is a contradiction. Thus, p is the unique prime number such that $\dim_{\mathbb{F}_p}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_p) \neq \dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell)$ holds for each prime number $\ell \neq p$.

Let H be any open normal subgroup of $\Pi_{X^\bullet}$. We note that, if $\Pi_{X^\bullet}/H$ is a p-group, then the decomposition subgroup of $\Pi_{X^\bullet}/H$ of every irreducible component of $X_H^\bullet$ is trivial if and only if

$$g_{X_H} - r_{X_H} = \#(\Pi_{X^\bullet}/H)(g_X - r_X).$$

Thus, Theorem 1.2 implies that we may detect whether the equality

$$g_{X_H} - r_{X_H} = \#(\Pi_{X^\bullet}/H)(g_X - r_X)$$

holds or not, group-theoretically from $\Pi_{X^\bullet}$ and H if $\Gamma_{X_H^\bullet}^{\mathrm{cpt}}$ is 2-connected. We put

$$\mathrm{Top}_p(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \{H \subseteq \Pi_{X^\bullet} \text{ open normal} \mid \Pi_{X^\bullet}/H \text{ is a } p\text{-group}$$
$$\text{and, for any characteristic subgroup } Q \subseteq \Pi_{X^\bullet},$$
$$g_{X_{H\cap Q}} - r_{X_{H\cap Q}} = \#(\Pi_{X_Q^\bullet}/(H \cap Q))(g_{X_Q} - r_{X_Q})\}.$$

Note that Lemma 1.4 implies that $\Gamma_{X_{H\cap Q}^\bullet}^{\mathrm{cpt}}$ is 2-connected. Then by applying Theorem 1.2, we have that $\mathrm{Top}_p(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Thus, we obtain that

$$\Pi_{X^\bullet}^{\mathrm{top},p} = \Pi_{X^\bullet}/(\bigcap_{H \in \mathrm{Top}_p(\Pi_{X^\bullet})} H)$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Moreover, we have that

$$r_X = \dim_{\mathbb{Q}}(\Pi_{X^\bullet}^{\mathrm{top},p,\mathrm{ab}} \otimes \mathbb{Q})$$

can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. By Theorem 1.2 again, the genus

$$g_X = \mathrm{Avr}_p(\Pi_{X^\bullet}) + r_X$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

Let $\ell \neq p$ be a prime number. If $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell) \neq 2g_X$, then we have

$$n_X = \dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell) - 2g_X + 1.$$

Suppose that $\dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell) = 2g_X$. Then $n_X = 0$ if, for any open normal subgroup $H \subseteq \Pi_{X^\bullet}$, $\dim_{\mathbb{F}_\ell}(H^{\mathrm{ab}} \otimes \mathbb{F}_\ell) = 2g_{X_H}$. Otherwise, we have $n_X = 1$. We complete the proof of (i).

(ii) Since each Galois admissible covering of degree p is étale, by applying (i), we obtain that $V_{X,p}^*$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then to verify (ii), Proposition 2.1 implies that it is sufficient to prove that $V_{X,p}^\star$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Let $\alpha \in V_{X,p}^*$, $X_\alpha^\bullet$ the Galois admissible covering corresponding to α, $\Gamma_{X^\bullet_\alpha}$ the dual semi-graph of $X^\bullet_\alpha$, and r_{X_α} the Betti number of $\Gamma_{X^\bullet_\alpha}$. Moreover, let $0 \neq \gamma \in \mathrm{Hom}(\Pi^{\mathrm{top},p}_{X^\bullet}, \mathbb{F}_p)$ if $\Pi^{\mathrm{top},p}_{X^\bullet}$ is not trivial, $X^\bullet_\gamma$ the Galois admissible covering corresponding to γ, $X^\bullet_{\alpha,\gamma}$ the pointed stable curve $X^\bullet_\alpha \times_{X^\bullet} X^\bullet_\gamma$, $\Gamma_{X^\bullet_{\alpha,\gamma}}$ the dual semi-graph of $X^\bullet_{\alpha,\gamma}$, $\Gamma_{X^\bullet_\gamma}$ the dual semi-graph of $X^\bullet_\gamma$, $r_{X_{\alpha,\gamma}}$ the Betti number of $\Gamma_{X^\bullet_{\alpha,\gamma}}$, and r_{X_γ} the Betti number of $\Gamma_{X^\bullet_\gamma}$. Then we have the following claim:

Claim:

$$\#v(\Gamma_{X^\bullet_\alpha}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if

$$r_{X_\alpha} = pr_X.$$

Moreover, suppose that $r_X \neq 0$. Then

$$\#v(\Gamma_{X^\bullet_\alpha}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if

$$r_{X_{\alpha,\gamma}} = pr_{X_\gamma} + p^2 - 2p + 1.$$

Let us prove the claim. Since $r_{X_\alpha} = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_\alpha}) - \#v(\Gamma_{X^\bullet_\alpha}) + 1$ and $r_X = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - \#v(\Gamma_{X^\bullet}) + 1$, we have that $r_{X_\alpha} = pr_X$ holds if and only if

$$\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_\alpha}) - \#v(\Gamma_{X^\bullet_\alpha}) = p\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - p(\#v(\Gamma_{X^\bullet}) - 1) - 1.$$

Since $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_\alpha}) = p\#e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, we have

$$\#v(\Gamma_{X^\bullet_\alpha}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if $r_{X_\alpha} = pr_X$.

Suppose that $r_X \neq 0$. Since $0 \neq \gamma \in \mathrm{Hom}(\Pi^{\mathrm{top},p}_{X^\bullet}, \mathbb{F}_p)$, we have

$$r_{X_{\alpha,\gamma}} = p\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_\alpha}) - p\#v(\Gamma_{X^\bullet_\alpha}) + 1.$$

Then

$$r_{X_{\alpha,\gamma}} = pr_{X_\gamma} + p^2 - 2p + 1 = p(p\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - p\#v(\Gamma_{X^\bullet}) + 1) + p^2 - 2p + 1$$

if and only if

$$\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_\alpha}) - \#v(\Gamma_{X^\bullet_\alpha}) = p\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - p(\#v(\Gamma_{X^\bullet}) - 1) - 1$$

if and only if

$$\#v(\Gamma_{X^\bullet_\alpha}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1.$$

This completes the proof of the claim.

If $r_X = 0$ (i.e., $\Gamma_{X^\bullet}$ is a tree), then by applying Remark 1.1.1 and Remark 1.2.1, Condition A implies that either each terminal vertex of $\Gamma_{X^\bullet}$ meets some open edge of $\Gamma_{X^\bullet}$ or $\#v(\Gamma_{X^\bullet}) = 1$. Then we observer that the one-point compactification of the dual semi-graph of each connected Galois admissible covering of $X^\bullet$ is 2-connected. Then by the first part of the claim above and (i), we obtain that $V^\star_{X,p}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. If $r_X \neq 0$, then $\Gamma^{\rm cpt}_{X^\bullet_\alpha}$ is *not* 2-connected in general. Moreover, Lemma 1.4 implies that $\Gamma^{\rm cpt}_{X^\bullet_{\alpha,\gamma}}$ and $\Gamma^{\rm cpt}_{X^\bullet_\gamma}$ are 2-connected. Then the "moreover" part of the claim above and (i) imply that $V^\star_{X,p}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Thus, by Proposition 2.1, the set $v(\Gamma_{X^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. This completes the proof of (ii).

(iii) Since $\Gamma^{\rm cpt}_{X^\bullet_H}$ is 2-connected, we obtain that $X^\bullet_H$ satisfies Condition A. Moreover, by replacing $X^\bullet$ by $X^\bullet_H$, (ii) implies that $v(\Gamma_{X^\bullet_H})^{>0,p}$ can be mono-anabelian reconstructed from H. Then (iii) follows from Remark 2.1.2 and (ii).

(iv) Since

$$V^\star_{X_Q,p} \subseteq \mathrm{Hom}(Q^{\rm ab}, \mathbb{F}_p)$$

for each open normal subgroup $Q \subseteq \Pi_{X^\bullet}$, $V^\star_{X_Q,p}$ admits a natural action of $\Pi_{X^\bullet}/Q$ via the natural outer representation

$$\Pi_{X^\bullet}/Q \to \mathrm{Out}(Q) \to \mathrm{Aut}(Q^{\rm ab})$$

induced by the natural exact sequence

$$1 \to Q \to \Pi_{X^\bullet} \to \Pi_{X^\bullet}/Q \to 1.$$

By Theorem 1.2, there is an open normal subgroup $Q \subseteq \Pi_{X^\bullet}$ such that the p-rank of $\widetilde{X}^\bullet_{Q,v}$ is positive for each $v \in v(\Gamma_{X^\bullet_Q})$. Moreover, we may assume that $X^\bullet_Q$ satisfies Condition A. Then $V_{X_Q,p} \overset{\sim}{\to} v(\Gamma_{X^\bullet_Q})^{>0,p} = v(\Gamma_{X^\bullet_Q})$. Thus, we obtain that

$$\#v(\Gamma_{X^\bullet}) = \max\{Q \subseteq \Pi_{X^\bullet} \text{ open normal} \mid \#(V_{X_Q,p}/(\Pi_{X^\bullet}/Q))\}.$$

This completes the proof of the lemma. □

Lemma 3.3. *The data* $\#e^{\rm cl}(\Gamma_{X^\bullet})$, $\Pi^{\rm top}_{X^\bullet}$, *and* $\Pi^{\rm \acute{e}t}_{X^\bullet}$ *can be mono-anabelian reconstructed from* $\Pi_{X^\bullet}$.

Proof. By Lemma 3.2 (i) (iv), we obtain that r_X and $\#v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then

$$\#e^{\rm cl}(\Gamma_{X^\bullet}) = r_X + \#v(\Gamma_{X^\bullet}) - 1$$

and

$$\#e^{\rm op}(\Gamma_{X^\bullet}) = n_X$$

can be also mono-anabelian reconstructed from $\Pi_{X^\bullet}$. We put

$$\mathrm{Et}(\Pi_{X^\bullet}) \overset{\rm def}{=} \{H \subseteq \Pi_{X^\bullet} \text{ open normal} \mid \text{for each proper characteristic open normal subgroup } Q \text{ such that } \#e^{\rm cl}(\Gamma_{X^\bullet_{H\cap Q}}) = \#(\Pi_{X^\bullet}/(H\cap Q))\#e^{\rm cl}(\Gamma_{X^\bullet}) \text{ and } \#e^{\rm op}(\Gamma_{X^\bullet_{H\cap Q}}) = \#(\Pi_{X^\bullet}/(H\cap Q))\#e^{\rm op}(\Gamma_{X^\bullet})\}.$$

Note that, for each proper characteristic open normal subgroup Q, since $\Gamma^{\rm cpt}_{X^\bullet}$ is 2-connected, Lemma 1.4 implies that $\Gamma^{\rm cpt}_{X^\bullet_{H\cap Q}}$ is 2-connected. Moreover, $X^\bullet_{H\cap Q}$ satisfies Condition A.

Then $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H\cap Q}})$ and $\#e^{\mathrm{op}}(\Gamma_{X^\bullet_{H\cap Q}})$ can be mono-anabelian reconstructed from $H \cap Q$. Thus $\mathrm{Et}(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. This implies that

$$\Pi^{\text{ét}}_{X^\bullet} = \Pi_{X^\bullet} / \bigcap_{H \in \mathrm{Et}(\Pi_{X^\bullet})} H$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. On the other hand, we put

$$\mathrm{Top}(\Pi_{X^\bullet}) \stackrel{\mathrm{def}}{=} \{H \subseteq \Pi^{\text{ét}}_{X^\bullet} \text{ open normal} \mid \text{for each proper characteristic open normal subgroup } Q,$$
$$g_{X_{H\cap Q}} - r_{X_{H\cap Q}} = \#(\Pi_{X^\bullet}/(H \cap Q))(g_X - r_X)\}.$$

Note that since $X^\bullet_{H\cap Q}$ satisfies Condition A, Lemma 3.2 (i) implies that $\mathrm{Top}(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Thus we have that

$$\Pi^{\mathrm{top}}_{X^\bullet} = \Pi^{\text{ét}}_{X^\bullet} / \bigcap_{H \in \mathrm{Top}(\Pi_{X^\bullet})} H$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. This completes the proof of the lemma. □

Lemma 3.4. *Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup. Then the data g_{X_H}, n_{X_H}, r_{X_H}, $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})$, and $\#v(\Gamma_{X^\bullet_H})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H. Furthermore, we have that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H.*

Proof. Suppose that $r_X = 0$. Then by applying Remark 1.1.1, Condition A implies that either each terminal vertex of $\Gamma_{X^\bullet}$ meets some open edge of $\Gamma_{X^\bullet}$ or $\#v(\Gamma_{X^\bullet}) = 1$ holds. Then we observer that the one-point compactification of the dual semi-graph of each connected Galois admissible covering of $X^\bullet$ is 2-connected. Then $X^\bullet_H$ satisfies Condition A. Thus, the lemma follows from Lemma 3.2 and Lemma 3.3.

Suppose that $r_X \neq 0$. Then $\Gamma^{\mathrm{cpt}}_{X^\bullet_H}$ is *not* 2-connected in general. Since p can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, we may choose a prime number $\ell \neq p$ such that $(\ell, \#(\Pi_{X^\bullet}/H)) = 1$. Let $0 \neq \gamma \in \mathrm{Hom}(\Pi^{\mathrm{top}}_{X^\bullet}, \mathbb{F}_\ell)$, H_γ the kernel of $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top}}_{X^\bullet} \stackrel{\gamma}{\twoheadrightarrow} \mathbb{F}_\ell$, $X^\bullet_{H_\gamma} \to X^\bullet$ the admissible covering corresponding to H_γ, $X^\bullet_{H\cap H_\gamma}$ the pointed stable curve $X^\bullet_H \times_{X^\bullet} X^\bullet_{H_\gamma}$, $\Gamma_{X^\bullet_{H\cap H_\gamma}}$ the dual semi-graph of $X^\bullet_{H\cap H_\gamma}$, and $r_{X_{H\cap H_\gamma}}$ the Betti number of $\Gamma_{X^\bullet_{H\cap H_\gamma}}$. By Lemma 1.4, $\Gamma^{\mathrm{cpt}}_{X^\bullet_{H\cap H_\gamma}}$ and $\Gamma^{\mathrm{cpt}}_{X^\bullet_{H_\gamma}}$ are 2-connected. Moreover, $X^\bullet_{H\cap H_\gamma}$ satisfies Condition A. Then $g_{X_{H\cap H_\gamma}}$, $n_{X_{H\cap H_\gamma}}$, $r_{X_{H\cap H_\gamma}}$, $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H\cap H_\gamma}})$, and $\#v(\Gamma_{X^\bullet_{H\cap H_\gamma}})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and $H \cap H_\gamma$.

We note that

$$n_{X_{H\cap H_\gamma}} = \ell n_{X_H},\ \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H\cap H_\gamma}}) = \ell \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}),\ \#v(\Gamma_{X^\bullet_{H\cap H_\gamma}}) = \ell \#v(\Gamma_{X^\bullet_H}),$$

$$r_{X_{H\cap H_\gamma}} = \ell r_{X_H} - \ell + 1, \text{ and } g_{X_{H\cap H_\gamma}} = \ell(g_{X_H} - 1) + 1.$$

Then g_{X_H}, n_{X_H}, r_{X_H}, $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})$, and $\#v(\Gamma_{X^\bullet_H})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H. Moreover, similar arguments to the arguments given in the proof of Lemma 3.3 imply that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H. □

Proposition 3.5. *(i) Let ℓ be an arbitrary prime number. Then the set*

$$V_{X,\ell}$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

(ii) Let ℓ', ℓ'' be prime numbers distinct from each other such that $\ell'' \neq p$. Then there is a natural injection

$$V_{X,\ell'} \hookrightarrow V_{X,\ell''}$$

which fits into the following commutative diagram

$$\begin{array}{ccc} V_{X,\ell'} & \xrightarrow{\kappa_{X,\ell'}} & v(\Gamma_{X^\bullet})^{>0,\ell'} \\ \downarrow & & \downarrow \\ V_{X,\ell''} & \xrightarrow{\kappa_{X,\ell''}} & v(\Gamma_{X^\bullet})^{>0,\ell''}. \end{array}$$

Moreover, the injection can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

(iii) The set of vertices $v(\Gamma_{X^\bullet})$ of $\Gamma_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

Proof. (i) By Lemma 3.3, we obtain that $\Pi^{\text{ét}}_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then similar arguments to the arguments given in the proof of Lemma 3.2 (ii) imply (i) holds.

(ii) Let $\alpha' \in V_{X,\ell'}$ and $\alpha'' \in V_{X,\ell''}$. Write $X^\bullet_{\alpha'}$ and $X^\bullet_{\alpha''}$ for the pointed stable curves corresponding to α' and α'', H_α and $H_{\alpha''}$ for the open subgroups of $\Pi_{X^\bullet}$ corresponding to $X^\bullet_{\alpha'}$ and $X^\bullet_{\alpha''}$ (i.e., the kernels of $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \overset{\alpha'}{\twoheadrightarrow} \mathbb{F}_{\ell'}$ and $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \overset{\alpha''}{\twoheadrightarrow} \mathbb{F}_{\ell''}$), respectively. Then we obtain that

$$X^\bullet_{\alpha'} \times_{X^\bullet} X^\bullet_{\alpha''}$$

is a connected pointed stable curve corresponding to the open normal subgroup $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Moreover, Lemma 3.4 implies that the cardinality of the set of irreducible components of $X^\bullet_{\alpha'} \times_{X^\bullet} X^\bullet_{\alpha''}$ can be mono-anabelian reconstructed from $H_{\alpha'} \cap H_{\alpha''}$ and $\Pi_{X^\bullet}$. Then (ii) follows from Remark 2.1.1.

(iii) Lemma 3.2 (i) implies that p can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then we may choose a prime number ℓ distinct from p. Moreover, since $X^\bullet$ satisfies Condition A, we have

$$v(\Gamma_{X^\bullet})^{>0,\ell} = v(\Gamma_{X^\bullet}).$$

Then (iii) follows from (i), (ii), and Proposition 2.1. □

Proposition 3.6. *Let $\mathfrak{T}_{\Pi_{X^\bullet}} \overset{\text{def}}{=} (\ell, d, y)$ be an arbitrary edge-triple associated to $\Pi_{X^\bullet}$, H_y the kernel of $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \overset{y}{\twoheadrightarrow} \mathbb{F}_d$, and $f^\bullet : Y^\bullet \to X^\bullet$ the Galois admissible covering corresponding to H_y. Then*

$$E^{\text{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ E^{\text{op}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H_y.

Proof. We only treat the case of $E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$. Note that $Y^\bullet$ satisfies Condition A. By the definition of $E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, Lemma 3.3 implies that the set $E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H_y. Hence, to verify the proposition, it is sufficient to prove that the set $E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \subseteq E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H_y. Let $\alpha \in E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, $H_\alpha \subseteq H_y$ the kernel of α, $Y^\bullet_\alpha \to Y^\bullet$ the admissible covering corresponding to H_α, and $\Gamma_{Y^\bullet_\alpha}$ the dual semi-graph of $Y^\bullet_\alpha$. We observe that

$$\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

if and only if

$$\#e^{\mathrm{cl}}(\Gamma_{Y^\bullet_\alpha}) = \ell(\#e^{\mathrm{cl}}(\Gamma_{Y^\bullet}) - d) + d,\ \#e^{\mathrm{op}}(\Gamma_{Y^\bullet_\alpha}) = \ell\#e^{\mathrm{op}}(\Gamma_{Y^\bullet}).$$

Since $H_\alpha \subseteq H_y$ (resp. $H_y \subseteq \Pi_{X^\bullet}$) is an open normal subgroup, by Lemma 3.4, we have that $\#e^{\mathrm{cl}}(\Gamma_{Y^\bullet_\alpha})$ and $\#e^{\mathrm{op}}(\Gamma_{Y^\bullet_\alpha})$ (resp. $\#e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$ and $\#e^{\mathrm{op}}(\Gamma_{Y^\bullet})$) can be mono-anabelian reconstructed from H_α and H_y (resp. H_y and $\Pi_{X^\bullet}$). Then we obtain that $E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H_y. This completes the proof of the proposition. □

Next, we generalize Lemma 3.4 to the case where H is an arbitrary open subgroup of $\Pi_{X^\bullet}$.

Proposition 3.7. *Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup. Then the data g_{X_H}, n_{X_H}, r_{X_H}, $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})$, and $\#v(\Gamma_{X^\bullet_H})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H. Furthermore, we have that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H.*

Proof. Let $N \subseteq H$ be a proper open characteristic subgroup of $\Pi_{X^\bullet}$. Then $X^\bullet_N$ satisfies Condition A. Since N is a normal open subgroup of $\Pi_{X^\bullet}$, Lemma 3.2 and Lemma 3.4 imply that the data g_{X_N}, n_{X_N}, r_{X_N}, $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_N})$, $\#v(\Gamma_{X^\bullet_N})$, N^{top}, and $N^{\text{ét}}$ can be mono-anabelian reconstructed from N. Moreover, by Proposition 3.5, we obtain that $v(\Gamma_{X^\bullet_N})$ can be mono-anabelian reconstructed from N, and that $v(\Gamma_{X^\bullet_N})$ admits a natural action of H/N. Then we obtain that

$$\#v(\Gamma_{X_H}) = \#(v(\Gamma_{X_N})/(H/N)).$$

Let $\mathfrak{T}_N \overset{\mathrm{def}}{=} (\ell, d, y)$ be an arbitrary edge-triple associated to N, N_y the kernel of $N \twoheadrightarrow N^{\text{ét}} \overset{y}{\twoheadrightarrow} \mathbb{F}_d$, and $f^\bullet : Y^\bullet_N \to X^\bullet_N$ the Galois admissible covering corresponding to N_y. Then Proposition 3.6 implies that

$$E^{\mathrm{cl}}_{\mathfrak{T}_N},\ E^{\mathrm{op}}_{\mathfrak{T}_N}$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and N_y. Moreover, $E^{\mathrm{cl}}_{\mathfrak{T}_N}$ and $E^{\mathrm{op}}_{\mathfrak{T}_N}$ admit natural actions of H/N. Then we obtain that

$$\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) = \#(E^{\mathrm{cl}}_{\mathfrak{T}_N}/(H/N)),\ n_{X_H} = \#e^{\mathrm{op}}(\Gamma_{X^\bullet_H}) = \#(E^{\mathrm{op}}_{\mathfrak{T}_N}/(H/N)).$$

Moreover, we have that

$$r_{X_H} = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) - \#v(\Gamma_{X^\bullet_H}) + 1.$$

On the other hand, since the ramification indexes of the Galois admissible covering $Y^\bullet_N \to X^\bullet_H$ at each marked points can be mono-anabelian reconstructed from N and H via the action of H/N on $E^{\mathrm{op}}_{\mathfrak{T}_N}$, the Riemann-Hurwitz formula implies that the genus g_{X_H} can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and N.

Similar arguments to the arguments given in the proof of Lemma 3.3 imply that H^{top} and $H^{\text{ét}}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H. This completes the proof of the proposition. $\square$

Proposition 3.8. *(i) Let* $\mathfrak{T}'_{\Pi_{X^\bullet}} \overset{\mathrm{def}}{=} (\ell', d', y')$ *and* $\mathfrak{T}''_{\Pi_{X^\bullet}} \overset{\mathrm{def}}{=} (\ell'', d'', y'')$ *be edge-triples associated to* $\Pi_{X^\bullet}$*,* $H_{y'}$ *and* $H_{y''}$ *the kernels of* $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \overset{y'}{\twoheadrightarrow} \mathbb{F}_{d'}$ *and* $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \overset{y''}{\twoheadrightarrow} \mathbb{F}_{d''}$*,* $f^{\bullet,\prime} : Y^{\bullet,\prime} \to X^\bullet$ *and* $f^{\bullet,\prime\prime} : Y^{\bullet,\prime\prime} \to X^\bullet$ *the Galois admissible coverings corresponding to* $H_{y'}$ *and* $H_{y''}$*, respectively. Then there are natural bijections*

$$E^{\mathrm{cl}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} \overset{\sim}{\to} E^{\mathrm{cl}}_{\mathfrak{T}''_{\Pi_{X^\bullet}}},\ E^{\mathrm{op}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} \overset{\sim}{\to} E^{\mathrm{op}}_{\mathfrak{T}''_{\Pi_{X^\bullet}}}$$

which fit into the following commutative diagrams

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta^{\mathrm{cl}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \\ \downarrow & & \| \\ E^{\mathrm{cl}}_{\mathfrak{T}''_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta^{\mathrm{cl}}_{\mathfrak{T}''_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}), \end{array}$$

$$\begin{array}{ccc} E^{\mathrm{op}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta^{\mathrm{op}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}}} & e^{\mathrm{op}}(\Gamma_{X^\bullet}) \\ \downarrow & & \| \\ E^{\mathrm{op}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta^{\mathrm{op}}_{\mathfrak{T}''_{\Pi_{X^\bullet}}}} & e^{\mathrm{op}}(\Gamma_{X^\bullet}), \end{array}$$

respectively. Moreover, the bijections can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$*,* $H_{y'}$*, and* $H_{y''}$*.*

(ii) The set of closed edges $e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ *of* $\Gamma_{X^\bullet}$ *and the set of open edges* $e^{\mathrm{op}}(\Gamma_{X^\bullet})$ *of* $\Gamma_{X^\bullet}$ *can be mono-anabelian reconstructed from* $\Pi_{X^\bullet}$*.*

Proof. We only treat the case of closed edges. (i) Let $\alpha' \in E^{\mathrm{cl},\star}_{\mathfrak{T}'_{\Pi_{X^\bullet}}}$ and $\alpha'' \in E^{\mathrm{cl},\star}_{\mathfrak{T}''_{\Pi_{X^\bullet}}}$. Write $Y^\bullet_{\alpha'} \to Y^{\bullet,\prime}$ and $Y^\bullet_{\alpha''} \to Y^{\bullet,\prime\prime}$ for the Galois admissible coverings corresponding to α' and α'', $H_{\alpha'}$ and $H_{\alpha''}$ for the open subgroups of $\Pi_{X^\bullet}$ corresponding to $Y^\bullet_{\alpha'}$ and $Y^\bullet_{\alpha''}$ (i.e., the kernels of α' and α''), respectively. Then we obtain that

$$Y^\bullet_{\alpha'} \times_{X^\bullet} Y^\bullet_{\alpha''}$$

is a connected pointed stable curve corresponding to the open subgroup $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Moreover, Proposition 3.7 implies that the cardinality of the set of nodes of $Y^\bullet_{\alpha'} \times_{X^\bullet} Y^\bullet_{\alpha''}$ can be mono-anabelian reconstructed from $H_{\alpha'} \cap H_{\alpha''}$ and $\Pi_{X^\bullet}$. Then (i) follows immediately from Proposition 3.6 and Remark 2.2.1.

(ii) By Lemma 3.2 (i) and Lemma 3.3, p and $\Pi^{\text{ét}}_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then there is an edge-triple

$$\mathfrak{T}'''_{\Pi_{X^\bullet}} \overset{\text{def}}{=} (\ell''', d''', y''')$$

associated to $\Pi_{X^\bullet}$ which can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, where $y''' \in \mathrm{Hom}(\Pi^{\text{ét}}_{X^\bullet}, \mathbb{F}_{d'''})$. Thus, (ii) follows from (i) and Proposition 2.2. This completes the proof of the proposition. □

Proposition 3.9. *Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup.*

(i) The natural maps

$$v(\Gamma_{X^\bullet_H}) \to v(\Gamma_{X^\bullet}),\ e^{\text{cl}}(\Gamma_{X^\bullet_H}) \to e^{\text{cl}}(\Gamma_{X^\bullet}),\ \textit{and}\ \ e^{\text{op}}(\Gamma_{X^\bullet_H}) \to e^{\text{op}}(\Gamma_{X^\bullet})$$

induced by the Galois admissible covering $X^\bullet_H \to X^\bullet$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

(ii) Suppose that H is normal. Then the natural action of $\Pi_{X^\bullet}/H$ on $v(\Gamma_{X^\bullet_H})$ (resp. $e^{\text{cl}}(\Gamma_{X^\bullet_H})$, $e^{\text{op}}(\Gamma_{X^\bullet_H})$) induced by the natural action of $\Pi_{X^\bullet}/H$ on $X^\bullet_H$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

Proof. (i) By Lemma 3.2 (i), we may choose a prime number ℓ such that $\ell \neq p$ and $(\ell, [\Pi_{X^\bullet} : H]) = 1$. By Proposition 3.5, we obtain that $V^\star_{X,\ell}$ and $V_{X,\ell}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Moreover, by Proposition 3.7 and by applying similar arguments to the arguments given in the proof of Proposition 3.5 (i), we obtain that $V^\star_{X_H,\ell}$ and $V_{X_H,\ell}$ can be also mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H.

For each $\alpha \in V^\star_{X,\ell}$ and each $\alpha_H \in V_{\ell,X^\bullet_H}$, we write $Q_\alpha \subseteq \Pi_{X^\bullet}$ and $Q_{\alpha_H} \subseteq H$ for the kernels of $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \overset{\alpha}{\twoheadrightarrow} \mathbb{F}_\ell$ and $H \twoheadrightarrow H^{\text{ét}} \overset{\alpha_H}{\twoheadrightarrow} \mathbb{F}_\ell$, respectively. Then, by Remark 2.1.2, we observe that

$$[\alpha_H] \mapsto [\alpha],$$

where $[\alpha]$ and $[\alpha_H]$ denote the images of α and α_H in $V_{X,\ell}$ and $V_{X_H,\ell}$, respectively, if and only if one of the following holds: (1) there exists $\alpha'_H \in V_{X_H,\ell}$ such that $\alpha_H \sim \alpha'_H$ and $Q_{\alpha'_H} = Q_\alpha \cap H$, where $Q_{\alpha'_H}$ denotes the kernel of $H \twoheadrightarrow H^{\text{ét}} \overset{\alpha'_H}{\twoheadrightarrow} \mathbb{F}_\ell$; (2) there exists $\alpha''_H \in V_{X_H,\ell}$ such that $\alpha_H \sim \alpha''_H$ and

$$\#v(\Gamma_{X^\bullet_{Q_\alpha \cap Q_{\alpha''_H}}}) = \ell \# v(\Gamma_{X^\bullet_{Q_\alpha \cap H}}),$$

where $Q_{\alpha''_H}$ denotes the kernel of $H \twoheadrightarrow H^{\text{ét}} \overset{\alpha''_H}{\twoheadrightarrow} \mathbb{F}_\ell$. Thus, Proposition 3.7 implies that the natural map $v(\Gamma_{X^\bullet_H}) \to v(\Gamma_X)$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

Next, let us prove that the natural maps of sets of edges can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. We only treat the case of closed edges. By Lemma 3.2 and Lemma 3.3, we may choose an edge-triple

$$\mathfrak{T}_{\Pi_{X^\bullet}} \overset{\text{def}}{=} (\ell, d, y)$$

associated to $\Pi_{X^\bullet}$ such that $(\ell, [\Pi_{X^\bullet} : H]) = (d, [\Pi_{X^\bullet} : H]) = 1$. Write $H_y \subseteq \Pi_{X^\bullet}$ for the kernel of $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \overset{y}{\twoheadrightarrow} \mathbb{F}_d$. Moreover, we denote by

$$y_H : H^{\text{ét}} \to H^{\text{ét}}/\text{Im}(H \cap H_y) \cong \mathbb{F}_d.$$

Then we obtain an edge-triple

$$\mathfrak{T}_H \overset{\text{def}}{=} (\ell, d, y_H)$$

associated to H.

By applying Proposition 3.8, we obtain that

$$E^{\text{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ E^{\text{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \overset{\sim}{\to} e^{\text{cl}}(\Gamma_{X^\bullet})$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H_y. Moreover, by Proposition 3.7 and by similar arguments to the arguments given in the proof of Proposition 3.8, we obtain that

$$E^{\text{cl},\star}_{\mathfrak{T}_H},\ E^{\text{cl}}_{\mathfrak{T}_H} \overset{\sim}{\to} e^{\text{cl}}(\Gamma_{X^\bullet_H})$$

can be mono-anabelian reconstructed from H and $H \cap H_y$.

For each $\beta \in E^{\text{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ and each $\beta_H \in E^{\text{cl},\star}_{\mathfrak{T}_H}$, we write $P_\beta \subseteq H_y \subseteq \Pi_{X^\bullet}$ and $P_{\beta_H} \subseteq H \cap H_y \subseteq H$ for the kernels of β and β_H, respectively. Then, by Remark 2.2.2, we observe that

$$[\beta_H] \mapsto [\beta],$$

where $[\beta_H]$ and $[\beta]$ denote the images of β_H and β in $E^{\text{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ and $E^{\text{cl}}_{\mathfrak{T}_H}$, respectively, if and only if one of the following holds: (1) there exists $\beta'_H \in E^{\text{cl},\star}_{\mathfrak{T}_H}$ such that $\beta_H \sim \beta'_H$; (2) there exists $\beta''_H \in E^{\text{cl},\star}_{\mathfrak{T}_H}$ such that $\beta''_H \sim \beta_H$ and

$$\#e^{\text{cl}}(\Gamma_{X^\bullet_{P_\beta \cap P_{\beta''_H}}}) = \ell \# e^{\text{cl}}(\Gamma_{X^\bullet_{P_\beta \cap H}}),$$

where $P_{\beta''_H}$ denotes the kernel of β''_H. Thus, Proposition 3.7 implies that the natural map $e^{\text{cl}}(\Gamma_{X^\bullet_H}) \to e^{\text{cl}}(\Gamma_X)$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

(ii) follows from (i), Remark 2.1.2, and Remark 2.2.2. This completes the proof of the proposition. □

Next, we prove that the dual semi-graphs can be mono-anabelian reconstructed from the admissible fundamental groups.

Proposition 3.10. *(i) The dual semi-graph $\Gamma_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.*

(ii) For each open subgroup $H \subseteq \Pi_{X^\bullet}$, the natural map of dual semi-graphs

$$\Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$$

can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. Moreover, if $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup, then the action of $\Pi_{X^\bullet}/H$ on $\Gamma_{X^\bullet_H}$ induced by the action of $\Pi_{X^\bullet}/H$ on $X^\bullet_H$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

Proof. By Lemma 3.2 and Lemma 3.3, we may choose an edge-triple

$$\mathfrak{T}_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell, d, y)$$

associated to $\Pi_{X^\bullet}$. Write H_y for the kernel of $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \stackrel{y}{\twoheadrightarrow} \mathbb{F}_d$ and $Y^\bullet$ for the pointed stable curve over k corresponding to H_y. Then the sets

$$E^{\text{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}},\ E^{\text{op}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

can be mono-anabelian reconstructed from H_y and $\Pi_{X^\bullet}$.

Let $e \stackrel{\text{def}}{=}\in e^{\text{cl}}(\Gamma_{X^\bullet}) \cup e^{\text{op}}(\Gamma_{X^\bullet})$ be an arbitrary edge and $v(e)$ the set of vertices on which e abuts. By Proposition 3.9, to verify the proposition, it is sufficient to prove that $v(e)$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. We only treat the case of closed edges.

Let $\beta \in E^{\text{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$. Write $Y^\bullet_\beta \to Y^\bullet$ for the Galois admissible covering corresponding to β, H_β for the kernel of β which is the open normal subgroup of H_y corresponding to $Y^\bullet_\beta$, and $\Gamma_{Y^\bullet_\beta}$ for the dual semi-graph of $Y^\bullet_\beta$. Let $m_1 = \#v(\Gamma_{X^\bullet}) - 2$, $m_2 = \#v(\Gamma_{X^\bullet}) - 1$, and $i \in \{1, 2\}$. We observe that $\beta \in E^{\text{cl},\star,m_i}_{\mathfrak{T}_{\Pi_{X^\bullet}},e_\beta}$ if and only if

$$\#v(\Gamma_{Y^\bullet_\beta}) = \#v(\Gamma_{Y^\bullet}) - m_i + \ell m_i = \#v(\Gamma_{X^\bullet}) - m_i + \ell m_i.$$

Since Proposition 3.9 implies that $v(\Gamma_{Y^\bullet_\beta})$ and $v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from H_α and $\Pi_{X^\bullet}$, we have that $E^{\text{cl},\star,m_i}_{\mathfrak{T}_{\Pi_{X^\bullet}},e_\beta}$ can be mono-anabelian reconstructed from H_y and $\Pi_{X^\bullet}$. On the other hand, for each $m \in \mathbb{Z}_{\geq 0}$, if $E^{\text{cl},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \neq \emptyset$, then the composition of maps

$$E^{\text{cl},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \hookrightarrow E^{\text{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \twoheadrightarrow E^{\text{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \stackrel{\sim}{\to} e^{\text{cl}}(\Gamma_{X^\bullet})$$

is a surjection. This means that $E^{\text{cl},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \neq \emptyset$ if $E^{\text{cl},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \neq \emptyset$.

Let $\alpha \in E^{\text{cl},\star,n}_{\mathfrak{T}_{\Pi_{X^\bullet}},e}$ be arbitrary element, where $n = m_2$ if $E^{\text{cl},\star,m_2}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \neq \emptyset$ (i.e., e is contained in a unique irreducible component of $X^\bullet$), and that $n = m_1$ if $E^{\text{cl},\star,m_2}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} = \emptyset$ (i.e., e is contained in two different irreducible components of $X^\bullet$). Proposition 3.9 (i) implies that the natural map

$$f^{\text{ver}}_{H_\alpha} : v(\Gamma_{Y^\bullet_\alpha}) \to v(\Gamma_{X^\bullet})$$

can be mono-anabelian reconstructed from $H_\alpha \hookrightarrow \Pi_{X^\bullet}$. Then we have

$$v(e) = \{v \in v(\Gamma_{X^\bullet}) \mid \#(f^{\text{ver}}_{H_\alpha})^{-1}(v) = 1\}.$$

This means that $\Gamma_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. This completes the proof of (i).

Similar arguments to the arguments given in the proof above imply that, for each open subgroup $H \subseteq \Pi_{X^\bullet}$, the dual semi-graph

$$\Gamma_{X^\bullet_H}$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$ and H. Then (ii) follows immediately from Proposition 3.9. □

Now, we prove the main theorem of the present paper.

Theorem 3.11. *Let $X^\bullet$ be an arbitrary pointed stable curve (i.e., we do not assume that $X^\bullet$ satisfies Condition A) of type (g_X, n_X) over an algebraically closed field of positive characteristic and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Then the data*

$$(g_X, n_X),\ \Gamma_{X^\bullet},\ \{(g_v, n_v)\}_{v\in v(\Gamma_{X^\bullet})},\ \mathrm{Ver}(\Pi_{X^\bullet}),\ \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet}),\ \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

Proof. Since $\Pi_{X^\bullet}$ is topologically finitely generated, there exists a set of open normal subgroups $\{H_i\}_{i\in\mathbb{N}}$ (e.g. characteristic subgroups) of $\Pi_{X^\bullet}$ such that the following conditions are satisfied: (1) $H_1 \overset{\mathrm{def}}{=} \Pi_{X^\bullet}$; (2) $H_i \supseteq H_{i+1}$ for each $i \in \mathbb{N}$; (3) $\varprojlim_{i\in\mathbb{N}} \Pi_{X^\bullet}/H_i = \Pi_{X^\bullet}$. By Remark 1.2.2, we may assume that $X^\bullet_{H_2}$ satisfies Condition A.

First, we claim the following:

(i) for each $i \in \mathbb{N}$, the dual semi-graph $\Gamma_{X^\bullet_{H_i}}$ of $X^\bullet_{H_i}$ corresponding to H_i can be mono-anabelian reconstructed from H_i;

(ii) for each $i \in \mathbb{N}$, the natural map of dual semi-graphs

$$\Gamma_{X^\bullet_{H_i}} \to \Gamma_{X^\bullet}$$

induced by the admissible covering $X^\bullet_H \to X^\bullet$ can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{X^\bullet}$, and the natural action of $\Pi_{X^\bullet}/H_i$ on $\Gamma_{X^\bullet_{H_i}}$ induced by the natural action of $\Pi_{X^\bullet}/H_i$ on $X^\bullet_{H_i}$ can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{X^\bullet}$.

Proposition 3.10 (i) implies that, for each $i \geq 2$, $\Gamma_{X^\bullet_{H_i}}$ can be mono-anabelian reconstructed from H_i. Moreover, Remark 2.1.2 and Remark 2.2.2 imply that, for each $i \geq 2$, the natural action of $\Pi_{X^\bullet}/H_i$ on $\Gamma_{X^\bullet_{H_i}}$ induced by the natural action of $\Pi_{X^\bullet}/H_i$ on $X^\bullet_{H_i}$ can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{X^\bullet}$. For each $i, j \geq 2$ such that $j > i$, by applying Proposition 3.10 (ii), we may identify naturally $\Gamma_{X^\bullet_{H_j}}/(H_i/H_j)$ with $\Gamma_{X^\bullet_{H_i}}$. Moreover, we may identify naturally $\Gamma_{X^\bullet_{H_j}}/H_j$ with $\Gamma_{X^\bullet_{H_i}}/H_i$. Thus, we may put

$$\Gamma_{X^\bullet} \overset{\mathrm{def}}{=} \Gamma_{X^\bullet_{H_2}}/H_2.$$

Then we obtain a natural map

$$\Gamma_{X^\bullet_i} \to \Gamma_{X^\bullet_i}/H_i = \Gamma_{X^\bullet_2}/H_2 = \Gamma_{X^\bullet},\ i \geq 2,$$

which can be mono-anabelian reconstructed from $H_i \hookrightarrow \Pi_{X^\bullet}$. This completes the proof of the claim.

The claim implies that the data

$$\Gamma_{X^\bullet},\ \mathrm{Ver}(\Pi_{X^\bullet}),\ \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet}),\ \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Since $\Gamma_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, r_X can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Moreover, since

$\mathrm{Ver}(\Pi_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, [T2, Theorem 0.2] implies that $\{(g_v, n_v)\}_{v\in v(\Gamma_{X^\bullet})}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then

$$g_X \overset{\mathrm{def}}{=} r_X + \sum_{v\in v(\Gamma_{X^\bullet})} g_v,\ n_X \overset{\mathrm{def}}{=} \#(\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})/\Pi_{X^\bullet})$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. We complete the proof of the theorem. □

Remark 3.11.1. The bi-anabelian version of Theorem 3.11 has been proven by the author (cf. [Y1, Theorem 1.2]). On the other hand, [Y1, Theorem 1.2] does not implies that the clutching morphisms and forgetting morphisms for moduli spaces of admissible fundamental groups exist.

Remark 3.11.2. In this remark, let k_i, $i \in \{1, 2\}$, be an algebraically closed field of characteristic 0, $X_i^\bullet$ an pointed stable curves over k_i, $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$, I_i an profinite group, and $\rho : I_i \to \mathrm{Out}(\Pi_{X_i^\bullet})$ an outer Galois representation, where $\mathrm{Out}(-)$ denotes $\mathrm{Aut}(-)/\mathrm{Inn}(-)$. A similar result was obtained by Y. Hoshi and Mochizuki as follows (see [HM, Theorem A] for a more precise statement):

> *Suppose that $\rho : I_i \to \mathrm{Out}(\Pi_{X_i^\bullet})$, $i \in \{1, 2\}$, is a certain outer Galois representations of NN-type (i.e., an outer Galois representation induced by the fundamental exact sequence of log étale fundamental groups arising from a stable log curve over a log point whose underlying scheme is an algebraically closed field, and whose log structure induced by the log structure of a node of a stable log curves (cf. [HM, Definition 2.4])), that $\alpha : \Pi_{X_1^\bullet} \overset{\sim}{\to} \Pi_{X_2^\bullet}$ and $\beta : I_1 \overset{\sim}{\to} I_2$ are continuous isomorphisms of profinite groups, and that the diagram*
>
> $$\begin{array}{ccc} I_1 & \xrightarrow{\rho_{I_1}} & \mathrm{Out}(\Pi_{X_1^\bullet}) \\ \beta\downarrow & & \downarrow\mathrm{out}(\alpha) \\ I_2 & \xrightarrow{\rho_{I_2}} & \mathrm{Out}(\Pi_{X_2^\bullet}), \end{array}$$
>
> *is commutative. Then the data appeared in Theorem 3.11 associated to $X_1^\bullet$ and $X_2^\bullet$ are same (more precisely, the semi-graphs of anabelioids of PSC-type associated to $X_1^\bullet$ and $X_2^\bullet$ are isomorphic).*

This result is called *the (bi-anabelian) combinatorial Grothendieck conjecture in characteristic* 0 which plays a central role for the theory of combinatorial anabelian geometry of curves in characteristic 0. The proof of the combinatorial Grothendieck conjecture in characteristic 0 is completely different from the proof of Theorem 3.11 which requires the use of *the highly non-trivial outer Galois representations* (e.g. by using weight-monodromy conjecture or Weil conjecture for curves). For more details on the theory of the combinatorial anabelian geometry of curves in characteristic 0 and its applications, see [HM].

We see that Theorem 3.11 can be regarded as a mono-anabelian version of combinatorial Grothendieck conjecture in positive characteristic.

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Yu Yang

Address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail: yuyang@kurims.kyoto-u.ac.jp