Degeneration of Period Matrices of Stable Curves

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Abstract

In the present paper, we study the extent to which linear combinations of period matrices arising from stable curves are degenerate (i.e., as bilinear forms). We give a criterion to determine whether a stable curve admits such a degenerate linear combination of period matrices. In particular, this criterion can be interpreted as a certain analogue of the weight-monodromy conjecture for non-degenerate elements of pro-ℓ log étale fundamental groups of certain log points associated to the log stack $\overline{\mathcal{M}}_g^{\log}$.

Keywords: period matrix, stable curve, log étale fundamental group.
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Contents

Introduction

The anabelian geometry of hyperbolic curves concerns the problem of reconstructing hyperbolic curves from their fundamental groups. In order to understand these fundamental groups, many techniques of algebraic geometry are applied. On the other hand, in the case of stable curves over algebraically closed fields, an introduction of some ideas of a *combinatorial* nature allows one to prove some results in much greater generality under very weak hypotheses (cf. [6], [7], [15], [16]). By applying this point of view, we are able to discuss not only phenomena that arise scheme-theoretically but also phenomena that arise purely group-theoretically. Before we explain the main question that motivated the theory developed in the present paper, let us recall some basic facts concerning period matrices.

Let X be a stable curve of genus g over an algebraically closed field k, Γ_X the dual graph of X, and $\ell \neq \mathrm{char}(k)$ a prime number. Then one has a natural exact sequence of free $\mathbb{Z}_\ell$-modules (cf. [15] Definition 1.1 (ii) and [15] Remark 1.1.3)

$$0 \to M_X^{\mathrm{ver}} \to M_X \to M_X^{\mathrm{top}} \to 0,$$

where $M_X := \pi_1^{\ell\text{-adm}}(X)^{\mathrm{ab}}, M_X^{\mathrm{top}} := \pi_1^\ell(\Gamma_X)^{\mathrm{ab}}, M_X^{\mathrm{ver}} := \mathrm{Im}(\bigoplus_{v\in v(\Gamma_X)} \pi_1^\ell(X_v \setminus \mathrm{Node}(X))^{\mathrm{ab}} \to M_X)$ (cf. Notations and the beginning of Section 2.1), and $\mathrm{Node}(X)$ denotes the set of nodes of X. The stable curve X determines a morphism from $s := \mathrm{Spec}\, k$ to the moduli stack $\overline{\mathcal{M}}_g$, and the pull-back log structure of the natural log structure on $\overline{\mathcal{M}}_g$ determines a log structure on $\mathrm{Spec}\, k$; denote the resulting log scheme by $s^{\log}$ which admits a chart $(\mathrm{Spec}\, k, \bigoplus_{e\in e(\Gamma_X)} \mathbb{N})$. The pro-$\ell$ log étale fundamental group $\pi_1^\ell(s^{\log})$ is naturally isomorphic to $\bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)$. Therefore, we obtain a natural action of $\bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ on the extension $0 \to M_X^{\mathrm{ver}} \to M_X \to M_X^{\mathrm{top}} \to 0$. This extension determines an extension class $[M_X]$, which may be regarded as a homomorphism, which we refer to as the *pro-ℓ period matrix morphism* of X (cf. Proposition 2.3, Definition 2.4, and the surrounding discussion)

$$f_X : \pi_1^\ell(s^{\log}) \cong \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1) \to \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)).$$

For each element $a \in \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)$, we refer to $f_X(a)$ as the *pro-ℓ period matrix* associated to a.

If $a = (a_e)_e \in \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e$ is a *positive definite* element (cf. Definition 2.5), then the closed subgroup generated by a can be regard as the image of the the maximal pro-ℓ quotient of the inertia group of a p-adic local field (cf. the discussion after Remark 2.5.1). Thus, by applying Faltings-Chai's theory (or the weight-monodromy conjecture for curves), we know that the pro-ℓ period matrix $f_X(a)$ is positive definite, hence also *non-degenerate*. This non-degeneracy property of pro-ℓ period matrices is the most non-trivial part in S. Mochizuki's proof of the combinatorial version of the Grothendieck conjecture (=ComGC) for semi-graphs of anabelioids in the case of outer representations of IPSC-type (cf. [15] Corollary 2.8). More precisely, Mochizuki proved that the pro-ℓ period matrix associated to a positive definite element of any finite admissible covering $X' \to X$ of X is non-degenerate. Moreover, Mochizuki gave a criterion to determine whether or not an isomorphism between fundamental groups of semi-graphs of anabelioids that is compatible with the respective outer Galois actions by inertia groups is *graphic* (i.e., the isomorphism preserves verticial subgroups and edge-like subgroups). By considering the pro-ℓ log étale fundamental groups which arise from *cusps* and applying the ComGC in the IPSC-type case, Mochizuki gave an algebraic alternative proof of an injectivity theorem in the affine case due to M. Matsumoto (cf. [16]). But if one wants to extend Matsumoto's theorem to the projective case, it is natural to attempt to prove the ComGC in the case of outer representations of NN-type case (i.e., the outer Galois action arising from a non-degenerate (= all the coordinates of the element are nonzero) $a = (a_e)_e \in \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ (cf. [6] Definition 2.4 (iii))). On the other hand, if one attempts to imitate the proof of the ComGC in the IPSC-type case, one has to consider whether or not the pro-ℓ period matrix arising from a *node* is non-degenerate. Y. Hoshi and S. Mochizuki proved a version of the ComGC in the NN-type case under certain assumptions. By applying this version of the ComGC, they successfully extended the injectivity theorem to the projective case (cf. [6]).

More generally, in the theory of combinatorial anabelian geometry, in order to extend results (e.g., the ComGC) in the IPSC-type case to the NN-type case, one has to consider whether or not the pro-ℓ period matrix arising from a *non-degenerate* element of $\pi_1^\ell(s^{\log}) \cong \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ is degenerate. It is difficult to determine in general whether or not the pro-ℓ period matrix associated to a given *non-degenerate* element is *degenerate*. But at least we can ask which stable curves admit a *non-degenerate* element that gives rise to a *degenerate* pro-ℓ period matrix. This question may be formulated as follows:

Question 0.1. *Does there exist a criterion to determine whether or not the given stable curve X admits an element $a = (a_e)_e \in \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ such that $a_e \neq 0$ for each e and, moreover, the pro-ℓ period matrix $f_X(a)$ is degenerate?*

Our main theorem of the present paper is a criterion as follows (cf. Theorem 2.9):

Theorem 0.2. *Let X be a stable curve over an algebraically closed field k and Γ_X the dual graph of X. Then X is a pro-ℓ period matrix degenerate curve (cf. Definition 2.6) if and only if the maximal untangled subgraph $\Gamma_X^{\not o}$ (cf. Definition 2.8) of Γ_X is not a tree (i.e., $r(\Gamma_X^{\not o}) := \mathrm{rank}(\mathrm{H}^1(\Gamma_{\mathrm{X}}^{\not o}, \mathbb{Z})) \neq 0$).*

The weight-monodromy conjecture for curves may be formulated as the assertion that the pro-ℓ period matrix associated to an element of the inertia group associated with every stable curve is non-degenerate. Thus, our main theorem may also be interpreted as a certain analogue of the weight-monodromy conjecture for non-degenerate elements of $\pi_1^\ell(s^{\log})$ (cf. Corollary 2.11).

In Section 1, we recall some basic facts concerning log structures and log étale fundamental groups of stable curves.

In Section 2, we discuss the topic of degeneracy of pro-ℓ period matrices of stable curves and prove Theorem 0.2. Finally, we explain the relationship between Theorem 0.2 and the weight-monodromy conjecture.

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Notations

Numbers:

If k is a field, we shall write $(\mathrm{char}(k), n) = 1$ if $\mathrm{char}(k)$ and n are relatively prime or $\mathrm{char}(k) = 0$. Write $\mathbb{Z}$ for the ring of rational integers, and $\mathbb{Q}$ for the rational field. We always use the notation ℓ to denote a prime number such that $\ell \neq \mathrm{char}(k)$. The notations $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$ denote the ℓ-adic completions of $\mathbb{Z}$ and $\mathbb{Q}$, respectively.

Curves and their moduli stacks:

Let r and g be non-negative integers such that $2g - 2 + r > 0$. A *pointed stable curve* (X, D_X) of type (g, r) over a scheme S consists of a flat, proper morphism $\pi : X \to S$, together with a set of r distinct sections $D_X := \{s_i : S \to X\}_{i=1}^r$ such that for each geometric point $\overline{s}$ of S:

(i) The geometric fiber $X_{\overline{s}}$ is a reduced and connected curve of genus g with at most ordinary double points (i.e., nodes).

(ii) $X_{\overline{s}}$ is smooth at the points of $s_i(\overline{s})$ $(1 \leq i \leq r)$.

(iii) $s_i(\overline{s}) \neq s_j(\overline{s})$ for $i \neq j$.

(iv) For every nonsingular rational component E of $X_{\overline{s}}$, the sum of the number of points of E where E meets another component of $X_{\overline{s}}$ and the number of points in $\{s_i(\overline{s})\}_{i=1}^r$ included in E is at least 3.

Let (X, D_X) be a pointed stable curve of type (g, r) over S. We shall call D_X the set of marked points of (X, D_X) and X the underlying scheme of (X, D_X). We shall say that (X, D_X) is *smooth* if the morphism of schemes $\pi : X \to S$ is smooth. We shall say that (X, D_X) is a *stable curve* over S if $D_X = \emptyset$ (i.e., $r = 0$). If (X, D_X) is a stable curve over S, for simplicity we also use the notation X to denote the pointed stable curve (X, D_X).

Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of pointed stable curves of type (g, r) over $\mathrm{Spec}\,\mathbb{Z}$ (cf. [10]) and $\mathcal{M}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing pointed smooth curves with the natural open immersion $j : \mathcal{M}_{g,r} \to \overline{\mathcal{M}}_{g,r}$. Then $\overline{\mathcal{M}}_{g,r}^{\log}$ is the log stack obtained by equipping $\overline{\mathcal{M}}_g$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subset \overline{\mathcal{M}}_{g,r}$ relative to $\mathrm{Spec}\,\mathbb{Z}$ (i.e., the log structure determined by the sheaf of monoids $j_*\mathcal{O}^\times_{\mathcal{M}_{g,r}} \bigcap \mathcal{O}_{\overline{\mathcal{M}}_{g,r}}$). Let $\overline{\mathcal{X}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ be the underlying stack of the universal pointed stable curve over $\overline{\mathcal{M}}_{g,r}$. It is shown in [10] that $\overline{\mathcal{X}}_{g,r}$ may be naturally identified with $\overline{\mathcal{M}}_{g,r+1}$. Let us denote by $\overline{\mathcal{X}}_{g,r}^{\log}$ the log stack obtained by pulling back the log structure on $\overline{\mathcal{M}}_{g,r+1}^{\log}$ relative to this identification. Thus, we obtain a morphism of log stacks $\overline{\mathcal{X}}_{g,r}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$. In particular, if $r = 0$ (i.e., in the stable curve case), we use the notation $\overline{\mathcal{M}}_g$ (resp. $\overline{\mathcal{M}}_g^{\log}$, $\overline{\mathcal{X}}_g$, $\overline{\mathcal{X}}_g^{\log}$) to denote $\overline{\mathcal{M}}_{g,0}$ (resp. $\overline{\mathcal{M}}_{g,0}^{\log}$, $\overline{\mathcal{X}}_{g,0}$, $\overline{\mathcal{X}}_{g,0}^{\log}$).

For more details on stable curves, pointed stable curves and their moduli stacks, see [3], [10].

Galois categories and their fundamental groups:

We denote the Galois categories of finite étale, finite Kummer log étale, and finite admissible coverings of "$(-)$" by $\mathrm{Cov}(-)$, $\mathrm{Cov}((-)^{\log})$, and $\mathrm{Cov}_{\mathrm{adm}}(-)$, respectively. For any Galois category $(-)$, write $(-)^\ell$ for the subcategory of $(-)$ defined as follows: (i) the objects of $(-)^\ell$ are either empty object or the objects of $(-)$ such that the Galois groups of their Galois closures are ℓ-groups; (ii) for any $A, B \in (-)^\ell$, $\mathrm{Hom}_{(-)^\ell}(A, B) := \mathrm{Hom}_{(-)}(A, B)$.

The notations $\pi_1(-), \pi_1((-)^{\log})$, and $\pi_1^{\mathrm{adm}}(-)$ will be used to denote the étale, log étale, and admissible fundamental groups of "$(-)$", respectively; the notations $\pi_1^\ell(-), \pi_1^\ell((-)^{\log})$, and $\pi_1^{\ell\text{-adm}}(-)$ will be used to denote the pro-ℓ étale, pro-ℓ log étale, and pro-ℓ admissible fundamental groups, respectively (i.e., the maximal pro-ℓ quotients of $\pi_1(-), \pi_1((-)^{\log})$, and $\pi_1^{\mathrm{adm}}(-)$, respectively); the notation $(-)^{\mathrm{ab}}$ denotes the abelianization of a profinite group $(-)$ (i.e., the quotient of $(-)$ by the closure of the commutator subgroup of $(-)$).

For more details on Kummer log étale coverings, admissible coverings, log admissible coverings, and their fundamental groups for pointed stable curves, see [8], [13], [18].

1 Review of log étale fundamental groups of stable curves

In this section, we recall some basic facts concerning log structures and log étale fundamental groups of stable curves.

1.1 Log structures on stable curves

In this subsection, we will recall some basic facts concerning log structures of stable curves; for generalities on log schemes, see [8], [9].

Let X be a generically smooth stable curve over a complete discrete valuation ring $(R, \mathfrak{m}_R)$ with algebraically closed residue field $k := R/\mathfrak{m}_R$ and π a uniformizer of R. Write K for the quotient field of R and X_s (resp. X_η) for the special fiber (resp. generic fiber) of X over R. Then the stable curve $X \to S_2 := \operatorname{Spec} R$ induces a morphism $\phi_X : S_2 \to \overline{\mathcal{M}}_g \times_{\mathbb{Z}} R$. The completion of the local ring of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} R$ at the point $\phi_{X_s} : s := \operatorname{Spec} k \to \overline{\mathcal{M}}_g \times_{\mathbb{Z}} R$ is isomorphic to $R[\![t_1, ..., t_{3g-3}]\!]$, where the $t_1, \dots, t_{3g-3}$ are indeterminates (cf. [3]).

If we denote the number of nodes of X_s by m and assign labels $i = 1, \dots, m$ to each of the nodes, then the completion of the local ring of X at the node labeled i is isomorphic to $R[\![x_i, y_i]\!]/(x_i y_i - \pi^{n_i})$, and the indeterminate t_i may be chosen so as to correspond to the deformations of the node of X_s labeled i. Then the log structure on $S_1 :=\operatorname{Spec} R[\![t_1, ..., t_m, t_{m+1}, ..., t_{3g-3}]\!]$ induced by the log structure of $\overline{\mathcal{M}}_g^{\log} \times_{\mathbb{Z}} R$ may be described as the log structure associated to the following chart:

$$\mathbb{N}^m \to R[\![t_1, ..., t_m, t_{m+1}, ..., t_{3g-3}]\!],$$

where $(a_i)_i \mapsto \prod_{i \le m} t_i^{a_i}$. We denote the resulting log scheme by $S_1^{\log}$. Moreover, we also obtain a log structure on the closed point of S_1 by restricting the log structure of $S_1^{\log}$; we denote the resulting log scheme by $s_1^{\log}$. On the other hand, the closed point of S_2 determines a log structure on S_2, which admits a chart

$$\begin{array}{ccc} \mathbb{N} & \to & R \\ 1 & \mapsto & \pi. \end{array}$$

We denote the resulting log scheme by $S_2^{\log}$. Write $s_2^{\log}$ for the log scheme obtained by restricting the log structure of $S_2^{\log}$ to the closed point of S_2. Thus, we obtain a cartesian commutative diagram

$$\begin{array}{ccccc} X_2^{\log} & \longrightarrow & X_1^{\log} & \longrightarrow & \overline{\mathcal{X}}_g^{\log} \\ \downarrow & & \downarrow & & \downarrow \\ S_2^{\log} & \longrightarrow & S_1^{\log} & \longrightarrow & \overline{\mathcal{M}}_g^{\log} \end{array}$$

— where $X_1^{\log}$ (resp. $X_2^{\log}$) is defined so as to render the right-hand (resp. left-hand) square in the diagram cartesian; the underlying scheme of $X_1^{\log}$ (resp. $X_2^{\log}$) may be identified with $\overline{\mathcal{X}}_g \times_{\overline{\mathcal{M}}_g} \operatorname{Spec} R[[t_1, ..., t_{3g-3}]]$ (resp. X); for suitable choices of the indeterminates $t_1, \ldots, t_m$, the lower horizontal arrow in the left-hand square of the diagram may be described as follows: the morphism of underlying schemes is

$$\begin{array}{ccc} S_2 = \operatorname{Spec} R & \to & S_1 = \operatorname{Spec} R[[t_1, ..., t_{3g-3}]] \\ \pi^{n_i} & \leftarrow\!\shortmid & t_i \ (1 \leq i \leq m) \\ 0 & \leftarrow\!\shortmid & t_j \ (m+1 \leq j \leq 3g-3), \end{array}$$

and the morphism of charts is

$$\begin{array}{ccc} \mathbb{N} & \leftarrow & \mathbb{N}^m \\ \sum_i a_i n_i & \leftarrow\!\shortmid & (a_i). \end{array}$$

Note that $S_1^{\log}$ and $S_2^{\log}$ are log regular.

1.2 Log étale fundamental groups

For more details on the definition of the notion of a finite Kummer log étale covering, see [8] Section 3. Let $Y^{\log}$ be a connected fs log scheme and let $\overline{y}^{\log} \to Y^{\log}$ be a strict geometric point (cf. [5] Section 2, Definition 1). Then there is a natural log geometric point $\widetilde{y}^{\log} \to Y^{\log}$, which is constructed as in [8] p284, associated to $\overline{y}^{\log} \to Y^{\log}$. We shall call $\widetilde{y}^{\log} \to Y^{\log}$ the *strict log geometric point* associated to $\overline{y}^{\log} \to Y^{\log}$. Note that there is a natural morphism of log schemes $\widetilde{y}^{\log} \to \overline{y}^{\log}$ which induces the identity on the underlying schemes. Then the strict log geometric point $\widetilde{y}^{\log} \to \overline{y}^{\log}$ determines an associated log étale fundamental group $\pi_1(Y^{\log})$.

Let ℓ be a prime number that is $\neq \operatorname{char}(k)$. For a proof of the following specialization theorem for log étale fundamental groups, see [18] Theorem 2.2.

Proposition 1.1. *Suppose that* $X_2^{\log}$ *is as above. Let* $\overline{\eta} := \operatorname{Spec} \overline{K} \to \operatorname{Spec} K$ *be a geometric point of* $\operatorname{Spec} K$. *Write* K^t *for the maximal tamely ramified extension of* K *in* $\overline{K}$, R_{K^t} *for the integral closure of* R *in* K^t, $\eta^t := \operatorname{Spec} K^t$, $(\operatorname{Spec} R_{K^t})^{\log}$ *for the log scheme obtained by equipping* $\operatorname{Spec} R_{K^t}$ *with the log structure determined by the sheaf of nonzero regular functions, and* $\widetilde{t}_2^{\log}$ *for the log scheme*

$$\operatorname{Spec} k \times_{\operatorname{Spec} R_{K^t}} (\operatorname{Spec} R_{K^t})^{\log}$$

— *where we identify the residue field of R_{K^t} with k. Thus, we obtain a natural strict log geometric point $\widetilde{t}_2^{\log} \to S_2^{\log}$ induced by $\overline{\eta}$. Write $\widetilde{s}_2^{\log} \to S_2^{\log}$ for the strict log geometric point associated to $s_2^{\log} \to S_2^{\log}$. Note that $\widetilde{t}_2^{\log} \to S_2^{\log}$ is isomorphic to $\widetilde{s}_2^{\log} \to S_2^{\log}$. Then there is a natural isomorphism between the pro-ℓ log étale fundamental groups at the respective fibers of $X_2^{\log}$ over $\overline{\eta}$ and $\widetilde{s}_2^{\log}$, which is well-defined up to composition with an inner automorphism, as follows:*

$$\pi_1^{\ell}((X_2^{\log})_{\overline{\eta}}) \cong \pi_1^{\ell}((X_2^{\log})_{\eta^t}) \to \pi_1^{\ell}((X_2^{\log})_{\widetilde{s}_2^{\log}}) := \varprojlim_{\lambda} \pi_1^{\ell}(X_2^{\log} \times_{S_2^{\log}} (s_2^{\log})_{\lambda}),$$

where the projective limit is over all reduced covering points $(s_2^{\log})_{\lambda} \to s_2^{\log}$ (cf. [5] Definition 1 (ii)).

Next, let U_i, $i = 1, 2$, be the interior (i.e., the largest open subset (possibly empty) of the underlying scheme of a log scheme on which the log structure is trivial) of $S_i^{\log}$. Write X_i, $i = 1, 2$, for the underlying scheme of $X_i^{\log}$. For any $u_i \in U_i$, by the ℓ-adic stable reduction criterion, we obtain that the image of the natural morphism $\pi_1(U_i) \to \mathrm{Aut}(\mathrm{H}^1_{\text{ét}}(X_i \times_{S_i} \overline{u}_i, \mathbb{F}_\ell))$ arising from $X_i \times_{S_i} U_i \to U_i$ is an ℓ-group, where $\overline{u}_i$ is a geometric point over u_i. Thus, [17] Proposition 2.2 (iii) implies the following exact sequence:

$$1 \to \pi_1^{\ell}(X_i \times_{S_i} \overline{u}_i) \to \pi_1^{\ell}(X_i \times_{S_i} U_i) \to \pi_1^{\ell}(U_i) \to 1.$$

Since, for $i = 1, 2$, $S_i^{\log}$ is a log regular log scheme, by applying the theorem of log purity and the deformation theory of log schemes (cf. [5] Section 4, Corollary 1), we obtain a homotopy exact sequence as follows:

Corollary 1.2. *Suppose that $X_i^{\log} \to S_i^{\log}$, where $i \in \{1, 2\}$, is the morphism discussed above. Let $s_i^{\log} \to S_i^{\log}$ be the strict geometric point defined in Section 1.1, and $\widetilde{s}_i^{\log} \to S_i^{\log}$ the strict log geometric point associated to $s_i^{\log} \to S_i^{\log}$. Then the following sequence is exact:*

$$1 \to \pi_1^{\ell}((X_i^{\log})_{\widetilde{s}_i^{\log}}) := \varprojlim_{\lambda} \pi_1^{\ell}(X_i^{\log} \times_{S_i^{\log}} (s_i^{\log})_{\lambda}) \to \pi_1^{\ell}(X_i^{\log} \times_{S_i^{\log}} s_i^{\log}) \to \pi_1^{\ell}(s_i^{\log}) \to 1,$$

where the projective limit is over all reduced covering points $(s_i^{\log})_{\lambda} \to s_i^{\log}$ (cf. [5] Definition 1 (ii)).

On the other hand, there is a classical scheme-theoretic description of the group $\pi_1^{\ell}((X_i^{\log})_{\widetilde{s}_i^{\log}})$ that does not require one to apply the theory of log schemes, namely, by means of the *pro-ℓ admissible fundamental group*. We use the notation $\pi_1^{\ell\text{-adm}}(X_s)$ to denote the pro-ℓ admissible fundamental group of the special fiber X_s. We have a proposition as follows.

Proposition 1.3. *Let $i \in \{1, 2\}$. Suppose that X_s, $X_i^{\log}$, and $\widetilde{s}_i^{\log}$ are as in Corollary 1.2. Fix a strict log geometric point $\widetilde{x}_i^{\log} \to (X_i^{\log})_{s_i^{\log}} := X_i^{\log} \times_{S_i^{\log}} s_i^{\log}$ associated to a strict geometric point whose image is a smooth point of the underlying scheme of $(X_i^{\log})_{s_i}$. Then there is a natural isomorphism of fundamental groups, which is well-defined up to composition with an inner automorphism, as follows:*

$$\pi_1^{\ell\text{-}adm}(X_s) \cong \pi_1^{\ell}((X_i^{\log})_{\widetilde{s}_i^{\log}})$$

— where $\pi_1^{\ell}(-)$ is taken with respect to the base point determined by the strict log geometric point $\widetilde{x}_i^{\log} \to (X_i^{\log})_{s_i^{\log}}$; $\pi_1^{\ell\text{-}adm}(-)$ is taken with respect to the base point determined by the morphism of underlying schemes of $\widetilde{x}_i^{\log} \to (X_i^{\log})_{s_i^{\log}}$.

Proof. Write $(s_1)_n^{\log}$ (resp. $(s_2)_n^{\log}$) for the log scheme determined by the morphism of monoids

$$\begin{array}{ccc} \frac{1}{n} \cdot \mathbb{N}^m & \to & k \\ a \neq 0 & \mapsto & 0 \\ 0 & \mapsto & 1 \end{array}$$

(resp.

$$\begin{array}{ccc} \frac{1}{n} \cdot \mathbb{N} & \to & k \\ a \neq 0 & \mapsto & 0 \\ 0 & \mapsto & 1), \end{array}$$

where n is a positive integer such that $(\mathrm{char}(k), n) = 1$. If n' and n'' are positive integers such that n' divides n'', then we consider the morphism of log schemes $(s_1)_{n''}^{\log} \to (s_1)_{n'}^{\log}$ (resp. $(s_2)_{n''}^{\log} \to (s_2)_{n'}^{\log}$) determined by the morphism of monoids

$$\begin{array}{ccc} \frac{1}{n'} \cdot \mathbb{N}^m & \to & \frac{1}{n''} \cdot \mathbb{N}^m \\ a & \mapsto & a \end{array}$$

(resp.

$$\begin{array}{ccc} \frac{1}{n'} \cdot \mathbb{N} & \to & \frac{1}{n''} \cdot \mathbb{N} \\ a & \mapsto & a). \end{array}$$

If we allow n' and n'' to vary, then these morphisms determine an inductive system, and the reduced log scheme associated to the inductive limit is easily seen to be isomorphic to $\widetilde{s}_1^{\log}$ (resp. $\widetilde{s}_2^{\log}$). In the following, we shall fix one such isomorphism, which we shall use to *identify* this inductive limit with $\widetilde{s}_1^{\log}$ (resp. $\widetilde{s}_2^{\log}$).

To complete the proof of the proposition, it suffices to construct, in a natural way, an equivalence between the Galois categories $\mathrm{Cov}_{\mathrm{adm}}(X_s)^{\ell}$ and $\mathrm{Cov}((X_1^{\log})_{\widetilde{s}_1^{\log}})^{\ell}$ (resp. $\mathrm{Cov}((X_2^{\log})_{\widetilde{s}_2^{\log}})^{\ell}$). Here, we note that $\mathrm{Cov}((X_1^{\log})_{\widetilde{s}_1^{\log}})^{\ell}$ (resp. $\mathrm{Cov}((X_2^{\log})_{\widetilde{s}_2^{\log}})^{\ell}$) may be identified with $\varinjlim_n \mathrm{Cov}((X_1^{\log}) \times_{s_1^{\log}} (s_1)_n^{\log})^{\ell}$ (resp. $\varinjlim_n \mathrm{Cov}((X_2^{\log}) \times_{s_2^{\log}} (s_2)_n^{\log})^{\ell}$). Since any finite Kummer log étale covering of $(X_1^{\log}) \times_{s_1^{\log}} (s_1)_n^{\log}$ (resp. $(X_2^{\log}) \times_{s_2^{\log}} (s_2)_n^{\log}$) determines a multi-log admissible covering (i.e., a disjoint union of log admissible coverings) after base-change to $(s_1)_m^{\log}$ (resp. $(s_2)_m^{\log}$) for some sufficiently large positive integer m, the proposition follows immediately from [12] Proposition 3.11. □

Remark 1.3.1. The isomorphism $\pi_1^{\ell\text{-adm}}((X_2)_s) \cong \pi_1^{\ell}((X_2^{\log})_{\widetilde{s}_2^{\log}})$ can be also deduced by applying the log purity theorem, the specialization theorem for log étale fundamental groups, and the specialization theorem for admissible fundamental groups.

2 Degeneration of period matrices of stable curves

In this section, let k be an algebraically closed field.

2.1 Pro-ℓ period matrices of stable curves and their functorial properties

In this subsection, we give the definition of the pro-ℓ period matrix morphism associated to a stable curve over k.

Let X be a stable curve of genus g over k. Write Γ_X for the dual graph of X, $v(\Gamma_X)$ for the set of vertices of Γ_X, $e(\Gamma_X)$ for the set of edges of Γ_X, and $\Pi_X := \pi_1^{\ell\text{-adm}}(X)$ for the pro-ℓ admissible fundamental group of X. We use the notation X_v to denote the irreducible component of X corresponding to $v \in v(\Gamma_X)$. Thus, for each $v \in v(\Gamma_X)$, $U_v := X_v \setminus \mathrm{Node}(X)$ is an open subscheme of X_v, where $\mathrm{Node}(X)$ denotes the set of nodes of X; the pro-ℓ étale fundamental group of U_v, which we denote by $\Pi_v := \pi_1^{\ell}(U_v)$, may be regarded as the decomposition group $\subseteq \Pi_X$ (which is well-defined up to Π_X-conjugation) associated to X_v (cf. [14] Proposition 2.5 and [14] Example 2.10). For $e \in e(\Gamma_X)$, write Π_e ($\cong \mathbb{Z}_\ell(1)$) for the decomposition group $\subseteq \Pi_X$ (which is well-defined up to Π_X-conjugation) associated to the node corresponding to e. Write $\pi_1^{\ell}(\Gamma_X)$ for the pro-ℓ completion of the topological fundamental group of the dual graph Γ_X. Finally, we use the notation M_X (resp. M_X^{top}, $M_X^{\mathrm{ver}}, M_X^{\mathrm{edge}}$) to denote the abelianization of Π_X (resp. the abelianization of $\pi_1^{\ell}(\Gamma_X)$, $\mathrm{Im}(\bigoplus_{v\in v(\Gamma_X)} \Pi_v^{\mathrm{ab}} \to M_X)$, $\mathrm{Im}(\bigoplus_{e\in e(\Gamma_X)} \Pi_e^{\mathrm{ab}} \to M_X)$).

By the definitions given above, we obtain a filtration as follows:

$$0 \subseteq M_X^{\mathrm{edge}} \subseteq M_X^{\mathrm{ver}} \subseteq M_X.$$

Moreover, there are two natural exact sequences:

$$0 \to M_X^{\mathrm{ver}} \to M_X \to M_X^{\mathrm{top}} \to 0,$$

$$0 \to M_X^{\mathrm{edge}} \to M_X^{\mathrm{ver}} \to M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}} \to 0.$$

For more details on the first exact sequence, see [15] Definition 1.1 and [15] Remark 1.1.4. Furthermore, we have the following proposition which can be proved by using the structure of Picard schemes of stable curves (cf. [1] Section 9.2, Example 8) and the theory of Raynaud extensions (cf. [4] Chapter II, Section 1). On the other hand, for a purely group-theoretic proof, see [6] Lemma 1.4.

Proposition 2.1. *For $v \in v(\Gamma_X)$, write X_v' for the normalization of X_v, $J(X_v')$ for the Jacobian of X_v', and $(\Delta_v^{\mathrm{cpt}})^{\mathrm{ab}}$ for the pro-ℓ étale fundamental group of $J(X_v')$ (i.e., the ℓ-adic Tate module associated to $J(X_v')$). Then, we have*

$$M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}} \cong \bigoplus_v (\Delta_v^{\mathrm{cpt}})^{\mathrm{ab}}.$$

The stable curve $X \to \operatorname{Spec} k$ determines a classifying morphism $\operatorname{Spec} k \to \overline{\mathcal{M}}_g$ to the moduli stack $\overline{\mathcal{M}}_g$. Thus, we obtain a log structure on $\operatorname{Spec} k$, naturally associated to the stable curve X, by restricting the log structure of $\overline{\mathcal{M}}_g^{\log}$; denote the resulting log scheme by $s_X^{\log}$. Thus, we have an isomorphism $I_{s_X^{\log}} := \pi_1^{\ell}(s_X^{\log}) \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e$. We also obtain a stable log curve (for the definition of stable log curves, see [7] Section 0) $X^{\log} := \overline{\mathcal{X}}_g^{\log} \times_{\overline{\mathcal{M}}_g^{\log}} s_X^{\log}$ over $s_X^{\log}$ whose underlying scheme is X. Furthermore, there are natural actions of $I_{s_X^{\log}}$ on the exact sequences $0 \to M_X^{\mathrm{ver}} \to M_X \to M_X^{\mathrm{top}} \to 0$ and $0 \to M_X^{\mathrm{edge}} \to M_X^{\mathrm{ver}} \to M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}} \to 0$. Denote the extension class corresponding to M_X by

$$[M_X] \in \mathrm{Ext}^1_{I_{s_X^{\log}}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}).$$

By [11] Example 0.8, there is a spectral sequence converging to

$$\mathrm{Ext}^{p+q}_{I_{s_X^{\log}}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})$$

whose E_2-term is given by $\mathrm{H}^p(I_{s_X^{\log}}, \mathrm{Ext}^q_{\mathbb{Z}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}))$. In particular, we obtain a long exact sequence as follows:

$$0 \to \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}_{\mathbb{Z}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})) \to \mathrm{Ext}^1_{I_{s_X^{\log}}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})$$

$$\to \mathrm{H}^0(I_{s_X^{\log}}, \mathrm{Ext}^1_{\mathbb{Z}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})).$$

Since $M_X, M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}, M_X^{\mathrm{edge}}$ are free $\mathbb{Z}_\ell$-modules of finite rank, we thus conclude that the morphism $\mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}_{\mathbb{Z}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})) \to \mathrm{Ext}^1_{I_{s_X^{\log}}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})$ is an isomorphism. Thus, the extension class $[M_X]$ may be regarded as an element of $\mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}_{\mathbb{Z}}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}))$.

Here, we observe that, for any two finitely generated free $\mathbb{Z}_\ell$-modules M, N, we have natural isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}}(M, N) \cong \varprojlim_n \mathrm{Hom}_{\mathbb{Z}/\ell^n\mathbb{Z}}(M/\ell^n M, N/\ell^n N) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(M, N).$$

Thus, we shall use the notation $\mathrm{Hom}(-,-)$ to denote $\mathrm{Hom}_{\mathbb{Z}_\ell}(-,-)$.

Proposition 2.2. *In the notation of the above discussion, the actions of $I_{s_X^{\log}}$ on M_X^{top}, M_X^{ver}, M_X^{edge}, and M_X/M_X^{edge} are trivial.*

Proof. First, we have two exact sequences as follows:

$$0 \to M_X^{\mathrm{edge}} \to M_X \to M_X/M_X^{\mathrm{edge}} \longrightarrow 0$$

and

$$0 \to M_X^{\mathrm{ver}} \to M_X \to M_X^{\mathrm{top}} \longrightarrow 0.$$

By the Poincaré duality (cf. [15] Proposition 1.3), we have natural isomorphisms

$$M_X^{\mathrm{edge}} \cong \mathrm{Hom}(M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1))$$

and

$$M_X^{\mathrm{ver}} \cong \mathrm{Hom}(M_X/M_X^{\mathrm{edge}}, \mathbb{Z}_\ell(1)).$$

Thus, to complete the proof of our proposition, it suffices to show (since $M_X^{\mathrm{edge}} \subseteq M_X^{\mathrm{ver}}$, and $I_{s_X^{\log}}$ acts trivially on $\mathbb{Z}_\ell(1)$) that the action of $I_{s_X^{\log}}$ on M_X^{ver} (or M_X/M_X^{edge}) is trivial. Next, let us write $X_1 \to S_1$ for the restriction of the tautological curve $\overline{\mathcal{X}}_g$ over the moduli stack $\overline{\mathcal{M}}_g$ to the spectrum of the completion of the local ring at the point of $\overline{\mathcal{M}}_g$ corresponding to X. For each vertex v of $v(\Gamma_X)$, write $U_v := X_v \setminus \mathrm{Node}(X)$ and M_v for the image in M_X^{ver} of the decomposition group associated to X_v. Since every open subgroup of M_v corresponds to an abelian étale covering of the curve U_v, and every étale covering of U_v lifts uniquely (up to unique isomorphism), without base change, to an étale covering of the formal neighborhood of U_v in X_1, the action of $I_{s_X^{\log}}$ on M_X^{ver} is trivial. Then the proposition follows.

Alternatively, the proposition may be verified by observing that every open subgroup of M_X/M_X^{edge} corresponds to an abelian étale covering of the stable curve X, and every étale covering of X lifts uniquely (up to unique isomorphism) to an étale covering of X_1 without base change.

This completes the proof of our proposition. $\square$

By using Proposition 2.2, we can prove a proposition as follows:

Proposition 2.3. *In the notation of the above discussion, the natural map*

$$\mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}})) \to \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}))$$

is injective, and (if, by abuse of notation, we identify the domain of this injection with its image via the injection, then) the extension class $[M_X]$ is contained in

$$\mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}})).$$

Proof. The short exact sequence $0 \to M_X^{\mathrm{edge}} \to M_X^{\mathrm{ver}} \to M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}} \to 0$ of $I_{s_X^{\log}}$-modules determines a long exact sequence

$$0 \to \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}})^{I_{s_X^{\log}}} \to \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})^{I_{s_X^{\log}}}$$

$$\to \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}})^{I_{s_X^{\log}}} \to \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}}))$$

$$\to \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})) \to \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}})) \to \ldots$$

— where the superscript "$I_{s_X^{\log}}$" denotes the submodule of $I_{s_X^{\log}}$-invariants. Since the functor $\mathrm{Hom}(M_X^{\mathrm{top}}, -)$ is exact, and the actions of $I_{s_X^{\log}}$ on $M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}$, and $M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}}$ are trivial (cf. Proposition 2.2), the morphism

$$\mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})^{I_{s_X^{\log}}} \to \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}})^{I_{s_X^{\log}}}$$

is a surjection. Thus, the morphism

$$\mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}})) \to \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}))$$

is an injection.

Since the action of $I_{s_X^{\log}}$ on M_X/M_X^{edge} is trivial (cf. Proposition 2.2), it follows formally that the image of the extension class $[M_X]$ via the morphism $\mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}})) \to \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{ver}}/M_X^{\mathrm{edge}}))$ is 0. This implies that

$$[M_X] \in \mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}})).$$

This completes the proof of the proposition. □

Remark 2.3.1. Let $Y^\bullet := (Y, D)$ be a pointed stable curve over $\mathrm{Spec}\, k$. Then just as in the non-pointed case, we have a filtration as follows:

$$0 \subseteq M_{Y^\bullet}^{\mathrm{cusp}} \subseteq M_{Y^\bullet}^{\mathrm{edge}} \subseteq M_{Y^\bullet}^{\mathrm{ver}} \subseteq M_{Y^\bullet} \twoheadrightarrow M_{Y^\bullet}^{\mathrm{top}} := M_{Y^\bullet}/M_{Y^\bullet}^{\mathrm{ver}},$$

where $M_{Y^\bullet}$ denotes the abelianization of $\pi_1^{\ell\text{-adm}}(Y^\bullet)$; $M_{Y^\bullet}^{\mathrm{ver}}$ (resp. $M_{Y^\bullet}^{\mathrm{edge}}$, $M_{Y^\bullet}^{\mathrm{cusp}}$) denotes the closed subgroup of $M_{Y^\bullet}$ generated by the subgroups that arise from the irreducible components (resp. nodes and cusps, cusps). Similar arguments to the arguments given in the proofs of Proposition 2.2 and Proposition 2.3 imply that the actions of $I_{s_{Y^\bullet}^{\log}}$ on $M_{Y^\bullet}^{\mathrm{top}}, M_{Y^\bullet}^{\mathrm{ver}}, M_{Y^\bullet}^{\mathrm{edge}}, M_{Y^\bullet}/M_{Y^\bullet}^{\mathrm{edge}}$ are trivial, and, moreover, that we obtain a corresponding extension class

$$[M_{Y^\bullet}] \in \mathrm{H}^1(I_{s_{Y^\bullet}^{\log}}, \mathrm{Hom}(M_{Y^\bullet}^{\mathrm{top}}, M_{Y^\bullet}^{\mathrm{edge}})).$$

Since $\mathrm{H}^1(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}})) \cong \mathrm{Hom}(I_{s_X^{\log}}, \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}}))$ (cf. Proposition 2.2), by the Poincaré duality (cf. [15] Proposition 1.3), the extension class $[M_X]$ corresponds to a continuous group homomorphism

$$f_X : I_{s_X^{\log}} \to \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)).$$

Definition 2.4. We shall refer to the morphism f_X discussed above as the *pro-ℓ period matrix morphism associated to* X. For an element $a \in I_{s_X^{\log}}$, we shall refer to the quadratic form $f_X(a)$ on M_X^{top} as the *pro-ℓ period matrix associated to a*. Note that $f_X(a)$ is a symmetric quadratic form on M_X^{top} for each $a \in I_{s_X^{\log}}$ (cf. [4] Chapter III Section 8).

In the next two remarks, we will explain the functorial properties of period matrices.

Remark 2.4.1. We discuss a certain functorial property that relates the pro-ℓ period matrix morphism associated to a stable curve to the corresponding morphism associated to a stable "sub-curve".

Let X be a stable curve over $s := \mathrm{Spec}\, k$ which is *sturdy* (i.e., the genus of the normalization of each irreducible component of X is ≥ 2), Γ_X the dual graph of X, and V a subset of $v(\Gamma_X) \bigcup e(\Gamma_X)$. Suppose that $U_V := X \setminus ((\bigcup_{v \in V} X_v) \bigcup (\bigcup_{e \in V} e))$ is a *connected* curve. Write X_V for the compactification of U_V (i.e., the closure of U_V in the scheme obtained by normalizing the closure of U_V in X at the nodes of X contained in $X \setminus U_V$). Thus, the pair $(X_V, X_V \setminus U_V)$ determines a pointed stable curve $X_V^\bullet$ of type (g_V, r_V), which may be regarded as associated to V. Thus, if we write $s_X^{\log}$ (resp. $s_V^{\log}$; $(s_V^U)^{\log}$) for the log scheme whose underlying scheme is s, and whose log structure is obtained by pulling back the log structure of the log stack $\overline{\mathcal{M}}_g^{\log}$ (resp. $\overline{\mathcal{M}}_{g_V}^{\log}$; $\overline{\mathcal{M}}_{g_V,r_V}^{\log}$) via the classifying morphism σ (resp. σ_V; σ_V^U) associated to $X \to s$ (resp. $X_V \to s$; $X_V^\bullet \to s$, i.e., for a suitable choice of ordering of the cusps), then we obtain a stable log curve

$$X^{\log} \to s_X^{\log} \text{ (resp. } X_V^{\log} \to s_V^{\log};\ X_V^{\bullet \log} \to (s_V^U)^{\log})$$

by pulling back the morphism of log stacks $\overline{\mathcal{X}}_g^{\log} \to \overline{\mathcal{M}}_g^{\log}$ (resp. $\overline{\mathcal{X}}_{g_V}^{\log} \to \overline{\mathcal{M}}_{g_V}^{\log}$; $\overline{\mathcal{X}}_{g_V,r_V}^{\log} \to \overline{\mathcal{M}}_{g_V,r_V}^{\log}$). If $\mathcal{S}$ is a Deligne-Mumford stack over $\operatorname{Spec}\mathbb{Z}$, write $\mathcal{S}_s$ for the stack $\mathcal{S} \times_{\operatorname{Spec}\mathbb{Z}} s$ over s. Then the geometry of the stable curve X, together with the original choice of a subset V of $v(\Gamma_X)\bigcup e(\Gamma_X)$, determine a clutching morphism of moduli stacks (i.e., for a suitable choice of ordering of the cusps):

$$\psi : \mathcal{N} := (\overline{\mathcal{M}}_{g_V,r_V})_s \times_s \prod_{v \in V} (\overline{\mathcal{M}}_{g_v,r_v})_s \to (\overline{\mathcal{M}}_g)_s.$$

Let $\mathcal{N}^{\log}$ be the log stack whose underlying stack is $\mathcal{N}$, and whose log structure is the pull-back of the log structure of $(\overline{\mathcal{M}}_g)_s^{\log}$ by ψ. On the other hand, we also have a log structure determined by the divisor given by the union of pull-backs to $\mathcal{N}$ of the divisors at infinity of each of the factors $(\overline{\mathcal{M}}_{g_V,r_V})_s$ and $(\overline{\mathcal{M}}_{g_v,r_v})_s$, for $v \in V$; write $\mathcal{N}_V^{\log}$ for the resulting log stack, which, as is easily verified, is isomorphic to the log stack $(\overline{\mathcal{M}}_{g_V,r_V})_s^{\log} \times_s \prod_{v \in V} (\overline{\mathcal{M}}_{g_v,r_v})_s^{\log}$. We have a natural morphism between the two log stacks $\mathcal{N}^{\log}$ and $(\overline{\mathcal{M}}_{g_V})_s^{\log}$ obtained by composing the following three morphisms:

$$\mathcal{N}^{\log} \to \mathcal{N}_V^{\log} \to (\overline{\mathcal{M}}_{g_V,r_V})_s^{\log} \to (\overline{\mathcal{M}}_{g_V})_s^{\log}.$$

Here, the first morphism of log stacks is obtained by forgetting the portion of the log structure of $\mathcal{N}^{\log}$ that arises from the irreducible components of the divisor $(\overline{\mathcal{M}}_g)_s \setminus (\mathcal{M}_g)_s$ which contain the image of $(\overline{\mathcal{M}}_{g_V,r_V})_s \times_s \prod_{v \in V} (\overline{\mathcal{M}}_{g_v,r_v})_s$. The second morphism of log stacks is the natural projection. The third morphism of log stacks is obtained by forgetting the marked points.

Next, let us describe the local structure of the morphisms $\mathcal{N}^{\log} \to (\overline{\mathcal{M}}_{g_V,r_V})_s^{\log} \to (\overline{\mathcal{M}}_{g_V})_s^{\log}$. First, let us observe that the geometry of X determines a morphism $\tau : s \to \mathcal{N}$ such that $\sigma = \psi \circ \tau$. Then for suitable charts defined over étale neighborhoods of τ, σ_V^U and σ_V, the morphisms $\mathcal{N}^{\log} \to (\overline{\mathcal{M}}_{g_V,r_V})_s^{\log} \to (\overline{\mathcal{M}}_{g_V})_s^{\log}$ may be described in terms of morphisms of monoids as follows:

$$\bigoplus_{e \in \mathrm{Node}(X_V)} \mathbb{N}_e \to \bigoplus_{e \in \mathrm{Node}(U_V)} \mathbb{N}_e \to \bigoplus_{e \in \mathrm{Node}(X)} \mathbb{N}_e.$$

Here, the first arrow is induced by the natural bijection $\mathrm{Node}(U_V) \xrightarrow{\sim} \mathrm{Node}(X_V)$; the second arrow is the assignment $(a_e)_{e \in \mathrm{Node}(U_V))} \longmapsto ((a_e)_{e \in \mathrm{Node}(U_V))}, 0, ..., 0)$ induced by the natural inclusion $\mathrm{Node}(U_V) \hookrightarrow \mathrm{Node}(X)$. Thus, the associated morphisms of pro-ℓ log étale fundamental groups may be written as follows:

$$\begin{aligned}\pi_1^\ell(s_X^{\log}) \ \cong \bigoplus_{e\in \mathrm{Node}(X)} \mathbb{Z}_\ell(1)_e \ \to \ \pi_1^\ell((s_V^U)^{\log}) \ \cong \bigoplus_{e\in \mathrm{Node}(U_V)} \mathbb{Z}(1)_e \\ \xrightarrow{\sim} \ \pi_1^\ell(s_V^{\log}) \ \cong \bigoplus_{e\in \mathrm{Node}(X_V)} \mathbb{Z}_\ell(1)_e,\end{aligned}$$

where the morphisms are the natural projections.

Write $(X_V^{\log})_{s_X^{\log}}$ for the stable log curve $X_V^{\log} \times_{s_V^{\log}} s_X^{\log}$. Write $(U_V)^{\log}$ for the log scheme over $s_X^{\log}$ whose underlying scheme is U_V, and whose log structure is the pull-back of the log structure of $X^{\log}$. Thus, we have a commutative diagram of log schemes as follows:

$$\begin{array}{ccccccc} X_V^{\log} & \longleftarrow & (X_V^{\log})_{s_X^{\log}} & \longleftarrow & (U_V)^{\log} & \longrightarrow & X^{\log} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ s_V^{\log} & \longleftarrow & s_X^{\log} & = & s_X^{\log} & = & s_X^{\log} \end{array}$$

Choose a strict log geometric point $\widetilde{s}_X^{\log}$ (resp. $\widetilde{s}_V^{\log}$) over $s_X^{\log} \to \overline{\mathcal{M}}_g^{\log}$ (resp. $s_V^{\log} \to \overline{\mathcal{M}}_{g_V}^{\log}$) (cf. Section 1.2). Thus, by a similar argument to the argument given in the proof of Proposition 1.3, we have a natural (outer) isomorphism $\pi_1^\ell(((X_V^{\log})_{s_X^{\log}})_{\widetilde{s}_X^{\log}}) \cong \pi_1^\ell((X_V^{\log})_{\widetilde{s}_V^{\log}})$ induced by the morphism of log schemes $(X_V^{\log})_{s_X^{\log}} \to X_V^{\log}$. Moreover, the natural (outer) homomorphism $\pi_1^\ell((U_V)^{\log}_{\widetilde{s}_X^{\log}}) \to \pi_1^\ell(((X_V^{\log})_{s_X^{\log}})_{\widetilde{s}_X^{\log}})$ induced by the morphism of log schemes $(U_V)^{\log} \to (X_V^{\log})_{s_X^{\log}}$ is a surjection. Note that since $\pi_1^\ell((U_V)^{\log}_{\widetilde{s}_X^{\log}}) \cong \pi_1^\ell((X_V^{\bullet \log})_{\widetilde{s}_X^{\log}})$, we use the notation M_{U_V} (resp. $M_{U_V}^{\mathrm{ver}}$, $M_{U_V}^{\mathrm{edge}}$, $M_{U_V}^{\mathrm{top}}$, $M_{U_V}^{\mathrm{cusp}}$) to denote the group $M_{X_V^\bullet}$ (resp. $M_{X_V^\bullet}^{\mathrm{ver}}$, $M_{X_V^\bullet}^{\mathrm{edge}}$, $M_{X_V^\bullet}^{\mathrm{top}}$, $M_{X_V^\bullet}^{\mathrm{cusp}}$) defined in Remark 2.3.1.

By considering the right-hand square of the commutative diagram discussed above, together with the natural projection $M_X^{\mathrm{edge}} \to M_{U_V}^{\mathrm{edge}}$ (cf. also Remark 2.3.1) and the natural morphism $M_{U_V}^{\mathrm{top}} \to M_X^{\mathrm{top}}$ induced by the natural open immersion $U_V \hookrightarrow X$, we obtain a commutative diagram:

$$\begin{array}{ccc} \pi_1^\ell(s_X^{\log}) & \longrightarrow & \mathrm{Hom}(M_X^{\mathrm{top}}, M_X^{\mathrm{edge}}) \\ \| & & \downarrow \\ \pi_1^\ell(s_X^{\log}) & \longrightarrow & \mathrm{Hom}(M_{U_V}^{\mathrm{top}}, M_{U_V}^{\mathrm{edge}}). \end{array}$$

Note that the natural open immersion $U_V \hookrightarrow X_V$ induces natural isomorphisms $M_{U_V}^{\mathrm{top}} \xrightarrow{\sim} M_{X_V}^{\mathrm{top}}$ and $M_{U_V}^{\mathrm{edge}} \xrightarrow{\sim} M_{X_V}^{\mathrm{edge}} \bigoplus M_{U_V}^{\mathrm{cusp}}$. Thus, by applying a similar argument to the argument applied to obtain the commutative diagram of the preceding display, we obtain a commutative diagram

$$\begin{array}{ccc}
\pi_1^{\ell}(s_X^{\log}) & \longrightarrow & \mathrm{Hom}(M_{U_V}^{\mathrm{top}}, M_{U_V}^{\mathrm{edge}}) \\
\| & & \downarrow \\
\pi_1^{\ell}(s_X^{\log}) & \longrightarrow & \mathrm{Hom}(M_{X_V}^{\mathrm{top}}, M_{X_V}^{\mathrm{edge}}) \\
\| & & \downarrow \\
\pi_1^{\ell}(s_X^{\log}) & \longrightarrow & \mathrm{Hom}(M_{X_V}^{\mathrm{top}} \otimes M_{X_V}^{\mathrm{top}}, \mathbb{Z}_\ell(1)),
\end{array}$$

where the lower vertical arrow on the right-hand side $\mathrm{Hom}(M_{X_V}^{\mathrm{top}}, M_{X_V}^{\mathrm{edge}}) \to \mathrm{Hom}(M_{X_V}^{\mathrm{top}} \otimes M_{X_V}^{\mathrm{top}}, \mathbb{Z}_\ell(1))$ is the isomorphism induced by the Poincaré duality.

On the other hand, since the actions of $\pi_1^{\ell}(s_X^{\log})$ and $\pi_1^{\ell}(s_V^{\log})$ on $0 \to M_{X_V}^{\mathrm{ver}} \to M_{X_V} \to M_{X_V}^{\mathrm{top}} \to 0$ are compatible, we thus obtain a commutative diagram

$$\begin{array}{ccc}
\pi_1^{\ell}(s_X^{\log}) & \longrightarrow & \mathrm{Hom}(M_{X_V}^{\mathrm{top}} \otimes M_{X_V}^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \\
\downarrow & & \| \\
\pi_1^{\ell}(s_V^{\log}) & \longrightarrow & \mathrm{Hom}(M_{X_V}^{\mathrm{top}} \otimes M_{X_V}^{\mathrm{top}}, \mathbb{Z}_\ell(1)),
\end{array}$$

where the lower horizontal arrow is the pro-ℓ period matrix morphism (cf. Definition 2.4) associated to X_V. So we have a functorial property of pro-ℓ period matrix morphism as follows:

$$\begin{array}{ccc}
\pi_1^{\ell}(s_X^{\log}) & \xrightarrow{f_X} & \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \\
{\scriptstyle u_V}\downarrow & & {\scriptstyle v_V}\downarrow \\
\pi_1^{\ell}(s_V^{\log}) & \xrightarrow{f_{X_V}} & \mathrm{Hom}(M_{X_V}^{\mathrm{top}} \otimes M_{X_V}^{\mathrm{top}}, \mathbb{Z}_\ell(1)),
\end{array}$$

where the vertical morphism of the left-hand side is the natural projective, and the vertical morphism of the right-hand side is the morphism determined by the pro-ℓ completion of the natural morphism of topological fundamental groups $\pi_1(\Gamma_{X_V}) \to \pi_1(\Gamma_X)$ which is induced by the embedding $\Gamma_{X_V} \hookrightarrow \Gamma_X$.

Remark 2.4.2. In this remark, we will explain a functorial property that relates the various pro-ℓ period matrix morphisms associated to deformations of a stable curve.

First, let us explain how to deform a stable curve along a set of nodes. Let R be a complete discrete valuation ring with algebraically closed residue field k, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. Write $S := \mathrm{Spec}\, R$ for the spectrum of R and $\eta := \mathrm{Spec}\, K \hookrightarrow S$ (resp. $s := \mathrm{Spec}\, k \hookrightarrow S$) for the subscheme determined by the generic point (resp. closed point) of S. Let X be a stable curve over s of genus g, Γ_X the dual graph of X, and $m := \sharp e(\Gamma_X)$.

Let L be a subset of $e(\Gamma_X)$. We claim that we can deform the stable curve X along L to obtain a new stable curve over $\overline{\eta} := \mathrm{Spec}\, \overline{K}$ such that the set of edges of the dual graph of the new stable curve may be naturally identified with $e(\Gamma_X) \setminus L$. Write $\phi_s : s \to \overline{\mathcal{M}}_g$ for the classifying morphism determined by $X \to s$. Thus the completion of the local ring of the moduli stack $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} R$ over R at $s \to \overline{\mathcal{M}}_g \times_{\mathbb{Z}} R$ induced by ϕ_s is isomorphic to $R[\![t_1, ..., t_{3g-3}]\!]$. Furthermore, the indeterminates $t_1, ..., t_m$ may be chosen so as to correspond to the deformations of the nodes of X. Write $\{t_1, ..., t_d\}$ for the subset of $\{t_1, ..., t_m\}$ corresponding to the subset $L \subseteq e(\Gamma_X)$. Now fix a morphism $S \to \mathrm{Spec}\, R[\![t_1, ..., t_{3g-3}]\!]$ over S such that $t_{d+1}, ..., t_m \mapsto 0 \in R$, but $t_1, ..., t_d$ map to nonzero elements of R. Then the composite morphism ${}_{\backslash L}\phi : S \to \mathrm{Spec}\, R[\![t_1, ..., t_{3g-3}]\!] \to \overline{\mathcal{M}}_g$ determines a stable curve ${}_{\backslash L}\mathcal{X}$ over S. Moreover, the special fiber of ${}_{\backslash L}\mathcal{X}$ is naturally isomorphic to X over s. Write ${}_{\backslash L}X$ for the geometric generic fiber ${}_{\backslash L}\mathcal{X} \times_\eta \overline{\eta}$ and $\Gamma_{{}_{\backslash L}X}$ for the dual graph of ${}_{\backslash L}X$. It follows from the construction of ${}_{\backslash L}X$ that we have two natural maps

$$v(\Gamma_X) \to v(\Gamma_{{}_{\backslash L}X}), \quad e(\Gamma_X) \setminus L \xrightarrow{\sim} e(\Gamma_{{}_{\backslash L}X})$$

(the latter of which is a bijection); we shall denote this pair of maps by the notation

$$\Gamma_X \to \Gamma_{{}_{\backslash L}X}$$

which we shall refer to as the *contracting morphism* associated to the deformation. Similarly, we can deform the stable curve X along $e(\Gamma_X)\backslash L$ (i.e., by taking "L" to be $e(\Gamma_X)\backslash L$). This yields a new stable curve, which we denote by ${}_L\mathcal{X}$, over S such that the set of nodes $e(\Gamma_{{}_LX})$ of the dual graph of the geometric generic fiber ${}_LX$ of ${}_L\mathcal{X}$ may be naturally identified with L, together with a natural contracting morphism

$$\Gamma_X \to \Gamma_{{}_LX}.$$

Furthermore, we have a classifying morphism ${}_L\phi : S \to \overline{\mathcal{M}}_g$ determined by ${}_L\mathcal{X} \to S$.

On the other hand, we have a log scheme ${}_{\backslash L}S^{\log}$ (resp. ${}_LS^{\log}$) whose underlying scheme is S, and whose log structure is the log structure obtained by pulling back the log structure of $\overline{\mathcal{M}}_g^{\log}$ via ${}_{\backslash L}\phi$ (resp. ${}_L\phi$). Thus, we obtain a stable log curve ${}_{\backslash L}\mathcal{X}^{\log} := \overline{\mathcal{X}}_g^{\log} \times_{\overline{\mathcal{M}}_g^{\log}} {}_{\backslash L}S^{\log}$ over ${}_{\backslash L}S^{\log}$ (resp. ${}_L\mathcal{X}^{\log} := \overline{\mathcal{X}}_g^{\log} \times_{\overline{\mathcal{M}}_g^{\log}} {}_LS^{\log}$ over ${}_LS^{\log}$) whose underlying scheme is ${}_{\backslash L}\mathcal{X}$ (resp. ${}_L\mathcal{X}$). Write

$$\overline{\eta}^{\log}_{{}_{\backslash L}X} := S^{\log}_{{}_{\backslash L}X} \times_S \overline{\eta}, \quad s^{\log}_X := S^{\log}_{{}_{\backslash L}X} \times_S s$$
$$(\text{resp.} \quad \overline{\eta}^{\log}_{{}_LX} := S^{\log}_{{}_LX} \times_S \overline{\eta}, \quad s^{\log}_X := S^{\log}_{{}_LX} \times_S s),$$

where we observe that the log schemes $S^{\log}_{{}_{\backslash L}X} \times_S s$ and $S^{\log}_{{}_LX} \times_S s$ are naturally isomorphic. Thus, we have a natural injection of log fundamental groups as follows:

$$I_{\overline{\eta}^{\log}_{{}_{\backslash L}X}} := \pi_1^\ell(\overline{\eta}^{\log}_{{}_{\backslash L}X}) \cong \bigoplus_{e \in e(\Gamma_{{}_{\backslash L}X})} \mathbb{Z}_\ell(1)_e \hookrightarrow \pi_1^\ell({}_{\backslash L}S^{\log}) \cong I_{s^{\log}_X} := \pi_1^\ell(s^{\log}_X) \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e,$$

(resp.

$$I_{\overline{\eta}^{\log}_{{}_LX}} := \pi_1^\ell(\overline{\eta}^{\log}_{{}_LX}) \cong \bigoplus_{e \in e(\Gamma_{{}_LX})} \mathbb{Z}_\ell(1)_e \hookrightarrow \pi_1^\ell({}_LS^{\log}) \cong I_{s^{\log}_X} := \pi_1^\ell(s^{\log}_X) \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e),$$

where the $\bigoplus_{e \in e(\Gamma_{{}_{\backslash L}X})} \mathbb{Z}_\ell(1)_e$ (resp. $\bigoplus_{e \in e(\Gamma_{{}_LX})} \mathbb{Z}_\ell(1)_e$) maps to the portion of $\bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e$ indexed by $e(\Gamma_{{}_{\backslash L}X})$ (resp. $e(\Gamma_{{}_LX})$).

Write $M_{\backslash_L X}$, $M_{_L X}$ and M_X for the abelianizations of the pro-ℓ admissible fundamental groups of $\backslash_L X$, $_L X$ and X, respectively. By applying the specialization theorem (cf. Proposition 1.1), we obtain a commutative diagram as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^{\mathrm{ver}}_{_L X} & \longrightarrow & M_{_L X} & \longrightarrow & M^{\mathrm{top}}_{_L X} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M^{\mathrm{ver}}_{X} & \longrightarrow & M_{X} & \longrightarrow & M^{\mathrm{top}}_{X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^{\mathrm{ver}}_{\backslash_L X} & \longrightarrow & M_{\backslash_L X} & \longrightarrow & M^{\mathrm{top}}_{\backslash_L X} & \longrightarrow & 0,
\end{array}$$

where the vertical morphisms in the middle (resp. on the right-hand side; on the left-hand side) are the isomorphisms induced by the inverses of the respective specialization isomorphisms (resp. surjective morphisms induced by the respective contracting morphisms; injective). From the commutative diagram above, it follows immediately, by considering the respective actions of $I_{\overline{\eta}^{\log}_{_L X}} \hookrightarrow I_{s^{\log}_X} \hookleftarrow I_{\overline{\eta}^{\log}_{\backslash_L X}}$ on the relevant modules in the above commutative diagram, that we obtain the following commutative diagram of pro-ℓ period matrix morphisms:

$$\begin{array}{ccc}
I_{\overline{\eta}^{\log}_{_L X}} & \xrightarrow{f_{_L X}} & \mathrm{Hom}(M^{\mathrm{top}}_{_L X} \otimes M^{\mathrm{top}}_{_L X}, \mathbb{Z}_\ell(1)) \\
{\scriptstyle _L i}\downarrow & & {\scriptstyle _L j}\downarrow \\
I_{s^{\log}_X} & \xrightarrow{f_X} & \mathrm{Hom}(M^{\mathrm{top}}_{X} \otimes M^{\mathrm{top}}_{X}, \mathbb{Z}_\ell(1)) \\
{\scriptstyle \backslash_L i}\uparrow & & {\scriptstyle \backslash_L j}\uparrow \\
I_{\overline{\eta}^{\log}_{\backslash_L X}} & \xrightarrow{f_{\backslash_L X}} & \mathrm{Hom}(M^{\mathrm{top}}_{\backslash_L X} \otimes M^{\mathrm{top}}_{\backslash_L X}, \mathbb{Z}_\ell(1)).
\end{array}$$

2.2 Degeneration of pro-ℓ period matrices

In this subsection, we study the degeneracy of pro-ℓ period matrices of stable curves. We continue to use the notation of Section 2.1.

Definition 2.5. An element $a = (a_e)_e \in I_{s^{\log}_X} \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ is called *non-degenerate* if $a_e \neq 0$ for each $e \in e(\Gamma_X)$. A non-degenerate element $a = (a_e)_e \in I_{s^{\log}_X} \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$ is called *positive definite* if, for any $e_1, e_2 \in e(\Gamma_X)$, it holds that $a_{e_1}/a_{e_2} \in \mathbb{Q}_{>0} \subset \mathbb{Q}_\ell^\times$.

Remark 2.5.1. Let $S^{\log}_2 \to S^{\log}_1$ be a morphism of log schemes defined at the ending of Section 1.1 and $\pi_1^\ell(S^{\log}_2) \to \pi_1^\ell(S^{\log}_1) \cong \pi_1^\ell(s^{\log}_X)$ the morphism of log étale fundamental groups induced by the morphism $S^{\log}_2 \to S^{\log}_1$. Then the discussion at the ending of Section 1.1 implies that all the elements of the image of $\pi_1^\ell(S^{\log}_2) \to \pi_1^\ell(S^{\log}_1) \cong \pi_1^\ell(s^{\log}_X)$ are positive definite.

Given a *positive definite* element $a = (a_e)_e \in I_{s_X^{\log}} \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$, observe that, for a suitable choice of *generator* $\xi \in \mathbb{Z}_\ell(1)$, it holds that $a_e \in \mathbb{N} \cdot \xi$ for each e. In particular, one verifies immediately that, in the notation of Section 1.1, there exists a morphism $S_2^{\log} \to S_1^{\log}$ such that a is contained in the image of $\pi_1^\ell(S_2^{\log}) \to \pi_1^\ell(S_1^{\log}) \cong \pi_1^\ell(s_X^{\log})$. Thus, the pro-$\ell$ period matrix $f_X(a)$ associated to a is a *positive definite* matrix (cf. [4] Chapter III Corollary 7.3, or, alternatively, the explicit computations given in the proof of [4] Chapter III Theorem 8.3), hence, in particular, *non-degenerate*. The fact that $f_X(a)$ is non-degenerate may also be regarded as a special case of the weight-monodromy conjecture for curves.

If $a \in I_{s_X^{\log}}$ is an *arbitrary* (i.e., not necessarily positive definite) *non-degenerate* element, then $f_X(a)$ will not necessarily be a non-degenerate matrix. It is easy to construct a counterexample (for instance, see [7] Remark 5.9.2).

Definition 2.6. The stable curve X over $s := \operatorname{Spec} k$ will be called a *pro-ℓ period matrix degenerate curve* if the dual graph Γ_X is not a tree (i.e., $r(\Gamma_X) := \operatorname{rank}(\mathrm{H}^1(\Gamma_\mathrm{X}, \mathbb{Z})) \neq 0$), and, moreover, there exists a non-degenerate element $a \in I_{s_X^{\log}}$ such that the pro-ℓ period matrix $f_X(a)$ is degenerate.

Next, we prepare for the proof of our main theorem. We begin by observing that for Question 0.1, we can assume without loss of generality that X is sturdy. More precisely, we have the following lemma.

Lemma 2.7. *Let X be a stable curve over k of type $(g_X, 0)$ and Γ_X the dual graph of X. Then there exist a sturdy stable curve Y over k and a finite morphism $\psi : Y \to X$ over k such that the following two properties hold: (i) the morphism of dual graphs $\Gamma_Y \to \Gamma_X$ induced by ψ is an isomorphism; (ii) the pro-ℓ period matrix morphisms f_Y and f_X fit into the following commutative diagram:*

$$\begin{array}{ccc}
I_{s_Y^{\log}} \cong \bigoplus_{e \in e(\Gamma_Y)} \mathbb{Z}_\ell(1)_e & \xrightarrow{f_Y} & \operatorname{Hom}(M_Y^{\mathrm{top}} \otimes M_Y^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \\
{\scriptstyle \ell}\big\downarrow & & {\scriptstyle \ell}\big\downarrow \\
I_{s_X^{\log}} \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e & \xrightarrow{f_X} & \operatorname{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)),
\end{array}$$

where the vertical arrow on the right-hand side is the multiplication by ℓ relative to the identification of $\operatorname{Hom}(M_Y^{\mathrm{top}} \otimes M_Y^{\mathrm{top}}, \mathbb{Z}_\ell(1))$ *with* $\operatorname{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1))$ *by the isomorphism $M_Y^{\mathrm{top}} \xrightarrow{\sim} M_X^{\mathrm{top}}$ induced by the isomorphism $\Gamma_Y \xrightarrow{\sim} \Gamma_X$ of (i), and the vertical arrow on the left-hand side is the morphism determined by multiplying by ℓ.*

Proof. Let $v \in v(\Gamma_X)$. Then we shall write X_v for the irreducible component of X associated to v, $n_v : X_v^* \to X_v$ for the normalization morphism associated to X_v, and P_v for the set

$$n_v^{-1}(X_v \bigcap \operatorname{Node}(X))$$

which is a subset of the set of closed points of X_v^*. In the following, we shall use the notation $(-)^{\mathrm{cl}}$ to denote the set of closed points of $(-)$. Choose a finite nonempty set

$$Q_v \subset X_v^{*\mathrm{cl}}$$

such that $Q_v \bigcap P_v = \emptyset$, and, moreover, the cardinality of the set $[v] := Q_v \bigcup P_v$ is a positive even number $2m_v$ such that $m_v >> \ell$. Thus, we obtain a pointed smooth curve $(X_v^*, [v])$ of type (g_{X_v}, r_{X_v}), where g_{X_v} denotes the genus of X_v^* and $r_{X_v} = \sharp[v] = 2m_v$. For simplicity, we use the notation $X_{[v]}^\bullet$ to denote the resulting pointed smooth curve.

Recall that the pro-ℓ admissible fundamental group of $X_{[v]}^\bullet$ admits a presentation as follows:

$$\pi_1^{\ell\text{-adm}}(X_{[v]}^\bullet) \cong \langle \{a_s\}_{s=1,\ldots,g_{X_v}}, \{b_t\}_{t=1,\ldots,g_{X_v}}, \{c_i\}_{i=1,\ldots,2m_v} \mid \prod_t [a_t, b_t] \prod_i c_i = 1\rangle^\ell,$$

where $\langle - \rangle^\ell$ denotes the pro-ℓ completion of the group $\langle - \rangle$. We construct a surjective morphism $h_v : \pi_1^{\ell\text{-adm}}(X_{[v]}^\bullet) \to \mathbb{Z}/\ell\mathbb{Z}$ as follows: for $s, t \in \{1, \ldots, g_{X_v}\}$, $h_v(a_s) = h_v(b_t) = 0$; $h_v(c_1) = 1, h_v(c_2) = -1, ..., h_v(c_{2i-1}) = 1, h_v(c_{2i}) = -1, ..., h_v(c_{2m_v-1}) = 1, h_v(c_{2m_v}) = -1$. Thus, we obtain a connected $\mathbb{Z}/\ell\mathbb{Z}$-admissible covering $\psi_v : Y_v^\bullet \to X_{[v]}^\bullet$ that is totally ramified over all the marked points in $[v]$ and étale over $X_v^* \setminus [v]$. We denote the underlying curve of $Y_v^\bullet$ by Y_v.

Write Q_X for the set $\bigcup_{v \in v(\Gamma_X)} Q_v$. Thus, we obtain a pointed stable curve $X^\bullet := (X, Q_X)$ of type (g_X, r_X), where $r_X = \sharp Q_X$. By gluing the $\{Y_v\}_v$ along the set $\bigcup_{v \in v(\Gamma_X)} \psi_v^{-1}(P_v)$ in a fashion that is compatible with the gluing of the $\{X_v\}_v$ that gives rise to X, we obtain a stable curve Y over s. Write Q_Y for the set $\bigcup_{v \in v(\Gamma_X)} \psi_v^{-1}(Q_v)$. Thus, we obtain a new pointed stable curve $Y^\bullet := (Y, Q_Y)$ of type (g_Y, r_Y), where $g_Y := \dim_k \mathrm{H}^1(Y, \mathcal{O}_Y)$ and $r_Y = \sharp Q_Y = \sharp Q_X = r_X$, together with an admissible covering $\psi' : Y^\bullet \to X^\bullet$. It follows from the construction of Y and the Hurwitz formula that Y is sturdy, and, moreover, that the morphism of dual graphs $\Gamma_Y \to \Gamma_X$ induced by ψ' is an isomorphism.

On the other hand, we have a morphism from s to the moduli stack $\overline{\mathcal{M}}_{g_X, r_X}$ (resp. $\overline{\mathcal{M}}_{g_Y, r_Y}$) determined by $X \to s$ (resp. $Y \to s$). By pulling back the log structure of $\overline{\mathcal{X}}_{g_X, r_X}^{\log}$ and $\overline{\mathcal{M}}_{g_X, r_X}^{\log}$ (resp. $\overline{\mathcal{X}}_{g_Y, r_Y}^{\log}$ and $\overline{\mathcal{M}}_{g_Y, r_Y}^{\log}$) to X and s (resp. Y and s), respectively, we obtain a stable log curve $X^{\bullet \log} \to s_X^{\log}$ (resp. $Y^{\bullet \log} \to s_Y^{\log}$). One verifies immediately that the log scheme $s_X^{\log}$ (resp. $s_Y^{\log}$) admits a chart $(\operatorname{Spec} k, \mathbb{N}^r)$ (resp. $(\operatorname{Spec} k, \frac{1}{\ell} \cdot \mathbb{N}^r)$), where $r = \sharp e(\Gamma_X)$ (resp. $r = \sharp e(\Gamma_Y)$). Thus, it follows from [12] Section 3.9 that the admissible covering ψ' determines a commutative diagram as follows:

$$\begin{array}{ccccc}
Y^{\bullet \log} & \longrightarrow & X_\ell^{\bullet \log} := X^{\bullet \log} \times_{s_X^{\log}} s_Y^{\log} & \longrightarrow & X^{\bullet \log} \\
\downarrow & & \downarrow & & \downarrow \\
s_Y^{\log} & = & s_Y^{\log} & \longrightarrow & s_X^{\log},
\end{array}$$

where, for a suitable choice of charts for $s_X^{\log}$ and $s_Y^{\log}$, the morphism of log structures induced by the morphism $s_Y^{\log} \to s_X^{\log}$ may be described as the morphism of log structures induced by the morphism of charts determined by the morphism of monoids $\mathbb{N}^r \to \frac{1}{\ell} \cdot \mathbb{N}^r$ such that $(0, ...0, 1, 0, ..., 0) \longmapsto (0, ..., 0, 1, 0, ..., 0)$, and $Y^{\bullet \log} \to X_\ell^{\bullet \log}$ is the log admissible covering determined by the admissible covering ψ'.

Next, write $M_{X^\bullet}, M_{Y^\bullet}, M_X, M_Y$ for the abelianizations of the pro-ℓ admissible fundamental groups of $X^\bullet, Y^\bullet, X, Y$, respectively. Then we obtain a commutative diagram as follows (cf. Remark 2.3.1):

$$\begin{tikzcd}
0 \arrow[r] & M_{Y^\bullet}^{\mathrm{ver}} \arrow[r] \arrow[d] & M_{Y^\bullet} \arrow[r] \arrow[d, "\psi'_M"'] & M_{Y^\bullet}^{\mathrm{top}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{X^\bullet}^{\mathrm{ver}} \arrow[r] & M_{X^\bullet} \arrow[r] & M_{X^\bullet}^{\mathrm{top}} \arrow[r] & 0,
\end{tikzcd}$$

where ψ'_M denotes the morphism induced by the admissible covering ψ'. By forgetting the marked points in Q_Y and Q_X, we conclude that ψ' determines a finite morphism $\psi : Y \to X$. Moreover, there is a natural surjection $M_{Y^\bullet} \to M_Y$ (resp. $M_{X^\bullet} \to M_X$) whose kernel is $M_{Y^\bullet}^{\mathrm{cusp}}$ (resp. $M_{X^\bullet}^{\mathrm{cusp}}$) (cf. Remark 2.3.1). Note that the image $\psi'_M(M_{Y^\bullet}^{\mathrm{cusp}})$ is contained in $M_{X^\bullet}^{\mathrm{cusp}}$, so we obtain a commutative diagram by passing to quotients as follows:

$$\begin{tikzcd}
0 \arrow[r] & M_{Y}^{\mathrm{ver}} \arrow[r] \arrow[d] & M_{Y} \arrow[r] \arrow[d] & M_{Y}^{\mathrm{top}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_{X}^{\mathrm{ver}} \arrow[r] & M_{X} \arrow[r] & M_{X}^{\mathrm{top}} \arrow[r] & 0.
\end{tikzcd}$$

Moreover, since $\psi : Y \to X$ is totally ramified over all the nodes of X, we obtain that the image of the morphism $M_Y^{\mathrm{edge}} \to M_X^{\mathrm{edge}}$ induced by ψ is $\ell \cdot M_X^{\mathrm{edge}} \subset M_X^{\mathrm{edge}}$. Since this commutative diagram is compatible with the actions of $I_{s_Y^{\log}} := \pi_1^{\ell}(s_Y^{\log}) \to I_{s_X^{\log}} := \pi_1^{\ell}(s_X^{\log})$, the pro-$\ell$ period matrix morphisms associated to X and Y fit into a commutative diagram

$$\begin{tikzcd}
I_{s_Y^{\log}} \cong \bigoplus_{e \in e(\Gamma_Y)} \mathbb{Z}_\ell(1)_e \arrow[r, "f_Y"] \arrow[d, "\ell"'] & \mathrm{Hom}(M_Y^{\mathrm{top}} \otimes M_Y^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \arrow[d, "\ell"'] \\
I_{s_X^{\log}} \cong \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e \arrow[r, "f_X"] & \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)),
\end{tikzcd}$$

where the vertical arrow on the right-hand side is the multiplication by ℓ relative to the identification of $\mathrm{Hom}(M_Y^{\mathrm{top}} \otimes M_Y^{\mathrm{top}}, \mathbb{Z}_\ell(1))$ with $\mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1))$ by the isomorphism $M_Y^{\mathrm{top}} \xrightarrow{\sim} M_X^{\mathrm{top}}$ induced by the isomorphism $\Gamma_Y \xrightarrow{\sim} \Gamma_X$ of (i), and the vertical arrow on the left-hand side is the morphism determined by multiplying by ℓ. This completes the proof of the lemma. $\square$

Definition 2.8. Let X be a stable curve over k and Γ_X the dual graph of X. For any edge $e \in e(\Gamma_X)$, write $v(e)$ for the set of vertices which abut to e. Write

$$e^{\circ}(\Gamma_X) := \left\{ e^{\circ} \in e(\Gamma_X) \mid \sharp v(e^{\circ}) = 1 \right\}$$

for the set of edges which form loops of Γ_X. Note that $\sharp v(e) = 2$ for each $e \in e(\Gamma_X) \backslash e^{\circ}(\Gamma_X)$. We shall refer to the subgraph $\Gamma_X^{\varnothing} := \Gamma_X \setminus e^{\circ}(\Gamma_X)$ as the *maximal untangled subgraph* of Γ_X.

Theorem 2.9. *Let X be a stable curve over k and Γ_X the dual graph of X. Then X is a pro-ℓ period matrix degenerate curve if and only if the maximal untangled subgraph $\Gamma_X^{\varnothing}$ of Γ_X is not a tree (i.e., $r(\Gamma_X^{\varnothing}) := \mathrm{rank}(\mathrm{H}^1(\Gamma_X^{\varnothing}, \mathbb{Z})) \neq 0$).*

Proof. By Lemma 2.7, we can assume that X is sturdy.

If Γ_X is a tree, then by definition, X is not a pro-ℓ period matrix degenerate curve. Hence, we can assume that Γ_X is not a tree.

First, let us prove the "only if" portion of the theorem. Write $L := e^{\circ}(\Gamma_X)$. Let R be a complete discrete valuation ring with residue field k and $\overline{K}$ an algebraic closure of the quotient field K of R. By applying Remark 2.4.2, we can deform the stable curve X along L (resp. $e(\Gamma_X) \setminus L$) so as to obtain a new stable curve ${}_{\setminus L}X$ (resp. ${}_{L}X$) over $\overline{K}$ such that the set of edges $e(\Gamma_{{}_{\setminus L}X})$ (resp. $e(\Gamma_{{}_{L}X})$) of the associated dual graph may be identified with $e(\Gamma_X) \setminus L$ (resp. L).

It is easy to see that the restriction of the contracting morphism $\Gamma_X \to \Gamma_{{}_{\setminus L}X}$ to $\Gamma_X^{\varnothing}$ is an isomorphism. Suppose that $\Gamma_X^{\varnothing}$ is a tree. Thus, the rank of $\Gamma_{{}_{\setminus L}X}$ (i.e., the rank of $\mathrm{H}^1(\Gamma_{{}_{\setminus L}X}, \mathbb{Z})$ as a free $\mathbb{Z}$-module) is 0. By applying Remark 2.4.2, we obtain a commutative diagram of pro-ℓ period matrix morphisms $f_X, f_{{}_{\setminus L}X}, f_{{}_{L}X}$ as follows:

$$\begin{array}{ccc}
I_{\overline{\eta}^{\log}_{{}_{\setminus L}X}} \cong \bigoplus_{e\in e(\Gamma_X)\setminus L} \mathbb{Z}_\ell(1)_e & \xrightarrow{f_{{}_{\setminus L}X}} & 0 \\
\Big\downarrow {\scriptstyle {}_{\setminus L}i} & & \Big\downarrow {\scriptstyle {}_{\setminus L}j} \\
I_{s^{\log}_X} \cong (\bigoplus_{e\in e(\Gamma_X)\setminus L} \mathbb{Z}_\ell(1)_e) \bigoplus (\bigoplus_{e\in L} \mathbb{Z}_\ell(1)_e) & \xrightarrow{f_X} & \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \\
\Big\uparrow {\scriptstyle {}_{L}i} & & \Big\uparrow {\scriptstyle {}_{L}j} \\
I_{\overline{\eta}^{\log}_{{}_{L}X}} \cong \bigoplus_{e\in L} \mathbb{Z}_\ell(1)_e & \xrightarrow{f_{{}_{L}X}} & \mathrm{Hom}(M_{{}_{L}X}^{\mathrm{top}} \otimes M_{{}_{L}X}^{\mathrm{top}}, \mathbb{Z}_\ell(1)),
\end{array}$$

where ${}_{L}j$ is induced by the contracting morphism $\Gamma_X \to \Gamma_{{}_{L}X}$. Moreover, ${}_{L}j$ is an isomorphism. Thus, it follows immediately from this commutative diagram that, by replacing X by ${}_{L}X$, we may assume without loss of generality that $X = {}_{L}X$.

Let $l \in e(\Gamma_X)$. Then we can also deform the stable curve X along $e(\Gamma_X) \setminus \{l\}$. This yields a stable curve ${}_{l}X$ whose set of nodes is $\{l\}$, together with a commutative diagram of pro-ℓ period matrix morphisms $f_{{}_{l}X}$, f_X as follows:

$$\begin{array}{ccc}
I_{\overline{\eta}^{\log}_{{}_{l}X}} \cong \mathbb{Z}_\ell(1)_l & \xrightarrow{f_{{}_{l}X}} & \mathrm{Hom}(M_{{}_{l}X}^{\mathrm{top}} \otimes M_{{}_{l}X}^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \cong \mathbb{Z}_\ell(1) \\
\Big\downarrow {\scriptstyle {}_{l}i} & & \Big\downarrow {\scriptstyle {}_{l}j} \\
I_{s^{\log}_X} \cong (\bigoplus_{e\in e(\Gamma_X)\setminus\{l\}} \mathbb{Z}_\ell(1)_e) \bigoplus \mathbb{Z}_\ell(1)_l) & \xrightarrow{f_X} & \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)).
\end{array}$$

Furthermore, we have $M_X^{\mathrm{top}} \cong \bigoplus_{e\in e(\Gamma_X)} M_{{}_{e}X}^{\mathrm{top}}$. Then for any non-degenerate element $a = (a_e)_e \in \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)_e$, we have a quadratic form

$$h_X := f_X(a) = \sum_{e\in e(\Gamma_X)} h_{{}_{e}X},$$

where we write $h_{{}_{e}X} := {}_{e}j(f_{{}_{e}X}(a_e))$. Since $h_{{}_{e}X}$ restricts to a non-degenerate form on $M_{{}_{e}X}^{\mathrm{top}}$ and to 0 on $\bigoplus_{e'\in e(\Gamma_X)\setminus\{e\}} M_{{}_{e'}X}^{\mathrm{top}}$, it follows that h_X is a non-degenerate quadratic form.

That is to say, X is not a pro-ℓ period matrix degenerate curve. This completes the proof of the "only if" part of the theorem.

Next, let us prove the "if" part of the theorem. Let R be a complete discrete valuation ring with residue field k and $\overline{K}$ an algebraic closure of the quotient field K of R. Since $\Gamma_X^{\emptyset}$ is not a tree, one verifies immediately that there exists an element $l \in e(\Gamma_X^{\emptyset})$ such that l is *not of separating type* (cf. [7], Definition 2.5 (i)). By applying Remark 2.4.2, we can deform the stable curve X along l (resp. $e(\Gamma_X) \setminus \{l\}$) so as to obtain a stable curve ${}_{\setminus l}X$ (resp. ${}_lX$) over $\overline{K}$ such that the set of edges of the associated dual graph may be identified with $e(\Gamma_X) \setminus \{l\}$ (resp. l). One verifies immediately that since l is not of separating type, it follows that l, regarded as an element of $e(\Gamma_{{}_lX})$, is a *loop*, and hence that the rank of $M_{{}_lX}^{\mathrm{top}}$ is 1. Let us consider the pro-ℓ period matrix morphisms of ${}_{\setminus l}X$ and ${}_lX$ with $\mathbb{Q}_\ell$-coefficients. By applying Remark 2.4.2, after tensoring with $\mathbb{Q}_\ell$, we obtain a commutative diagram of pro-ℓ period matrix morphisms of $X, {}_lX$ and ${}_{\setminus l}X$ over $\mathbb{Q}_\ell$ as follows:

$$\begin{array}{ccc}
I_{\overline{\eta}^{\log}_{{}_lX}} \otimes \mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell(1)_l & \xrightarrow{f_{{}_lX}^{\mathbb{Q}_\ell}} & \mathrm{Hom}(M_{{}_lX}^{\mathrm{top}} \otimes M_{{}_lX}^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \\
{}_li^{\mathbb{Q}_\ell}\Big\downarrow & & {}_lj^{\mathbb{Q}_\ell}\Big\downarrow \\
I_{s_X^{\log}} \otimes \mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell(1)_l \bigoplus (\bigoplus_{e \in e(\Gamma_X)\setminus\{l\}} \mathbb{Q}_\ell(1)_e) & \xrightarrow{f_X^{\mathbb{Q}_\ell}} & \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \\
{}_{\setminus l}i^{\mathbb{Q}_\ell}\Big\uparrow & & {}_{\setminus l}j^{\mathbb{Q}_\ell}\Big\uparrow \\
I_{\overline{\eta}^{\log}_{{}_{\setminus l}X}} \otimes \mathbb{Q}_\ell(1) \cong \bigoplus_{e \in e(\Gamma_X)\setminus\{l\}} \mathbb{Q}_\ell(1)_e & \xrightarrow{f_{{}_{\setminus l}X}^{\mathbb{Q}_\ell}} & \mathrm{Hom}(M_{{}_{\setminus l}X}^{\mathrm{top}} \otimes M_{{}_{\setminus l}X}^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,
\end{array}$$

where $f_{{}_lX}^{\mathbb{Q}_\ell}$ (resp. ${}_{\setminus l}j^{\mathbb{Q}_\ell}$) is an isomorphism (resp. the natural isomorphism induced by the isomorphism $M_X^{\mathrm{top}} \xrightarrow{\sim} M_{{}_{\setminus l}X}^{\mathrm{top}}$). By applying the commutative diagram above, for any element $a := (a_l, (a_e)_{e \neq l}) \in \mathbb{Q}_\ell(1)_l \bigoplus (\bigoplus_{e \neq l} \mathbb{Q}_\ell(1)_e)$, we obtain a quadratic form $h_X := f_X^{\mathbb{Q}_\ell}(a)$ on M_X^{top}:

$$h_X = h_{{}_lX}|_{M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}} + h_{{}_{\setminus l}X},$$

where we write $h_{{}_lX}$ (resp. $h_{{}_lX}|_{M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}}$, $h_{{}_{\setminus l}X}$) for the quadratic form $f_{{}_lX}^{\mathbb{Q}_\ell}(a_l)$ (resp. ${}_lj^{\mathbb{Q}_\ell}(f_{{}_lX}^{\mathbb{Q}_\ell}(a_l))$, ${}_{\setminus l}j^{\mathbb{Q}_\ell}(f_{{}_{\setminus l}X}^{\mathbb{Q}_\ell}((a_e)_{e \in e(\Gamma_{{}_{\setminus l}X})}))$) on $M_{{}_lX}^{\mathrm{top}}$ (resp. M_X^{top}, M_X^{top}).

Write p_l for the node of X corresponding to l, X_l for the stable curve obtained from the (sturdy) stable curve X by normalizing at p_l, and Γ_{X_l} for the dual graph of X_l. Note that since l is not of separating type, Γ_{X_l} may be regarded as a connected subgraph of Γ_X whose rank (i.e., the rank of $\mathrm{H}^1(\Gamma_{X_l}, \mathbb{Z})$ as a free $\mathbb{Z}$-module) is $r(\Gamma_X) - 1$. By applying Remark 2.4.1, we have a commutative diagram of pro-ℓ period matrix morphisms of X_l and X over $\mathbb{Q}_\ell$ as follows:

$$\begin{array}{ccc}
\mathbb{Q}_\ell(1)_l \bigoplus(\bigoplus_{e\in e(\Gamma_X)\setminus\{l\}} \mathbb{Q}_\ell(1)_e)) & \xrightarrow{f_X^{\mathbb{Q}_\ell}} & \mathrm{Hom}(M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \\
u_l^{\mathbb{Q}_\ell}\Big\downarrow & & v_l^{\mathbb{Q}_\ell}\Big\downarrow \\
\bigoplus_{e\in e(\Gamma_X)\setminus\{l\}} \mathbb{Q}_\ell(1)_e & \xrightarrow{f_{X_l}^{\mathbb{Q}_\ell}} & \mathrm{Hom}(M_{X_l}^{\mathrm{top}} \otimes M_{X_l}^{\mathrm{top}}, \mathbb{Z}_\ell(1)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\end{array}$$

On the other hand, it follows immediately from the structure of the graphs Γ_X, $\Gamma_{{}_lX}$, and Γ_{X_l} that we have a natural exact sequence as follows:

$$0 \to M_{X_l}^{\mathrm{top}} \to M_X^{\mathrm{top}} \to M_{{}_lX}^{\mathrm{top}} \to 0.$$

Thus, we obtain a quadratic form $h_{X_l} := f_{X_l}^{\mathbb{Q}_\ell}((a_e)_{e\in e(\Gamma_X)\setminus\{l\}})$ on $M_{X_l}^{\mathrm{top}}$ which is equal to the quadratic form given by the restricted forms $h_X|_{M_{X_l}^{\mathrm{top}}\otimes M_{X_l}^{\mathrm{top}}} = h_{\setminus_l X}|_{M_{X_l}^{\mathrm{top}}\otimes M_{X_l}^{\mathrm{top}}}$.

Here, we follow the notational conventions of the discussion preceding Lemma 2.10 below. Write

$$\det(h_X) \in \bigwedge \check{M}_X^{\mathrm{top}} \otimes \bigwedge \check{M}_X^{\mathrm{top}}$$

$$(\text{resp. } \det(h_{\setminus_l X}) \in \bigwedge \check{M}_X^{\mathrm{top}} \otimes \bigwedge \check{M}_X^{\mathrm{top}},$$

$$\det(h_{X_l}) \in \bigwedge \check{M}_{X_l}^{\mathrm{top}} \otimes \bigwedge \check{M}_{X_l}^{\mathrm{top}},$$

$$\det(h_{{}_lX}) \in \bigwedge \check{M}_{{}_lX}^{\mathrm{top}} \otimes \bigwedge \check{M}_{{}_lX}^{\mathrm{top}}),$$

for the determinants associated to the quadratic forms introduced above.

If $\Gamma_{\setminus_l X}$ and Γ_{X_l} are *not trees*, then the rank of M_X^{top} is ≥ 2. Thus, by applying Lemma 2.10 to $h_X = h_{\setminus_l X} + h_{{}_lX}|_{M_X^{\mathrm{top}}\otimes M_X^{\mathrm{top}}}$, we obtain that

$$\det(h_X) = \det(h_{\setminus_l X}) + \det(h_{X_l}) \wedge \det(h_{{}_lX}).$$

Let us take $(a_e)_{e\neq l} \in \bigoplus_{e\neq l} \mathbb{Q}_\ell(1)_e$ to be positive definite and $a_l \in \mathbb{Q}_\ell(1)_l$ to be arbitrary. This implies that the quadratic forms $h_{\setminus_l X}$ and h_{X_l} are positive definite (cf. [4] Chapter III Corollary 7.3). Hence, in particular, $\det(h_{\setminus_l X})$ and $\det(h_{X_l})$ are $\neq 0$ and, moreover, (by definition) independent of the choice of a_l. Thus, since the pro-ℓ period matrix morphism $f_{{}_lX}^{\mathbb{Q}_\ell}$ is an isomorphism, we may modify $a_l \in \mathbb{Q}_\ell(1)_l$ (which determines $\det(h_{{}_lX}) = f_{{}_lX}^{\mathbb{Q}_\ell}(a_l)$) so that

$$f_X^{\mathbb{Q}_\ell}((a_l, (a_e)_{e\neq l})) = \det(h_X) = \det(h_{\setminus_l X}) + \det(h_{X_l}) \wedge \det(h_{{}_lX}) = 0.$$

Finally, by clearing denominators, we conclude that we may choose a non-degenerate element

$$(a_l'', (a_e'')_{e\neq l})) \in \bigoplus_{e\in e(\Gamma_X)} \mathbb{Z}_\ell(1)$$

such that the quadratic form $f_X((a_l'', (a_e'')_{e\neq l}))$ is degenerate. This completes the proof of the theorem in the case under consideration.

If Γ_{X_l} is a *tree*, then $M_{X_l}^{\mathrm{top}}$ is 0, so $M_X^{\mathrm{top}} \cong M_{{}_lX}^{\mathrm{top}} \cong M_{\setminus_l X}^{\mathrm{top}}$ is of rank 1. Then, by applying Lemma 2.10 to $h_X = h_{\setminus_l X} + h_{{}_lX}|_{M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}}$, we obtain that

$$\det(h_X) = \det(h_{\setminus_l X}) + \det(h_{{}_lX}|_{M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}}) \in \check{M}_X^{\mathrm{top}} \otimes \check{M}_X^{\mathrm{top}}.$$

Let us take $(a_e)_{e \neq l} \in \bigoplus_{e \neq l} \mathbb{Q}_\ell(1)_e$ and $a_l \in \mathbb{Q}_\ell(1)_l$ to be positive definite. This implies that $\det(h_{\setminus_l X})$ and $\det(h_{{}_lX}|_{M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}})$ are non-zero (cf. [4] Chapter III Corollary 7.3). Since $\det(h_{\setminus_l X})$ is (by definition) independent of the choice of a_l, we can modify $a_l \in \mathbb{Q}_\ell(1)_l$ (which determines $\det(h_{{}_lX}|_{M_X^{\mathrm{top}} \otimes M_X^{\mathrm{top}}}) = {}_lj^{\mathbb{Q}_\ell}(f_{{}_lX}^{\mathbb{Q}_\ell}(a_l)))$ so that $\det(h_X) = 0$. Finally, by clearing denominators, we conclude that we may choose a non-degenerate element

$$(a''_l, (a''_e)_{e \neq l})) \in \bigoplus_{e \in e(\Gamma_X)} \mathbb{Z}_\ell(1)$$

such that the quadratic form h_X is degenerate.

If $\Gamma_{\setminus_l X}$ is a *tree*, then Γ_X, hence also $\Gamma_X^{\varnothing}$, is a tree. This contradicts our assumption that $\Gamma_X^{\varnothing}$ is not a tree. This completes the proof of the theorem. □

Let W be an n-dimensional vector space over a field k_W and $Q : W \otimes W \to k_W$ a quadratic form on W. Then Q induces a morphism $W \to \check{W}$ from W to the dual space $\check{W} := \mathrm{Hom}(W, k_W)$. Thus, by forming n-th exterior powers, we obtain a natural morphism

$$\det_Q : k_W \to \bigwedge^n \check{W} \otimes \bigwedge^n \check{W}.$$

We use the notation

$$\det(Q) \in \bigwedge^n \check{W} \otimes \bigwedge^n \check{W}$$

to denote $\det_Q(1)$. We have a lemma as follows.

Lemma 2.10. *Let $0 \to V_1 \to V_0 \to V_2 \to 0$ be an exact sequence of vector spaces of finite dimension over a field k_V. Suppose that* $\dim(V_0) =: n \geq 1$ *(resp.* $\dim(V_1) = n-1$, $\dim(V_2) = 1$*). Let $A_0^1, A_0^2 \in \mathrm{Hom}(V_0 \otimes V_0, k_V)$ (resp. $A_1 \in \mathrm{Hom}(V_1 \otimes V_1, k_V)$, $A_2 \in \mathrm{Hom}(V_2 \otimes V_2, k_V)$) be two symmetric quadratic forms on V_0 (resp. a quadratic form on V_1, a quadratic form on V_2). Furthermore, we suppose that the following conditions are satisfied: (i) $A_0^1|_{V_1 \otimes V_1} = A_1$; (ii) $A_0^2 = A_2|_{V_0 \otimes V_0}$ (so $A_0^2|_{V_1 \otimes V_1} = 0$). Let $A_0 := A_0^1 + A_0^2$. Then we have*

$$\det(A_0) = \det(A_0^1) + \det(A_0^2), \qquad \text{if } n = 1;$$

$$\det(A_0) = \det(A_0^1) + \det(A_1) \wedge \det(A_2), \quad \text{if } n \geq 2.$$

Proof. Choose a basis of V_0 that extends a basis of V_1. Then the lemma follows from an elementary matrix computation. □

2.3 Relationship with the weight-monodromy conjecture

In this subsection, we explain the relationship between Theorem 2.9 and the weight-monodromy conjecture for curves.

Let K be a p-adic local field (i.e., a finite extension of $\mathbb{Q}_p$), $\overline{K}$ an algebraic closure of K, R the ring of integers of K, k the residue field of R, $\overline{R}^{\mathrm{unr}}$ the integral closure of R in the maximal unramified extension of K in $\overline{K}$, and $\overline{k}$ the residue field of $\overline{R}^{\mathrm{unr}}$. Let X be a projective hyperbolic curve over K of genus g. Suppose that X admits a stable model $\mathcal{X}_R$ over R. Write $X_{\overline{K}}$ (resp. X_k, $X_{\overline{k}}$) for the geometric generic fiber (resp. special fiber, geometric special fiber) of $\mathcal{X}_R$. Then the reduction curve $X_{\overline{k}} \to \operatorname{Spec} \overline{k}$ determines a classifying morphism $\operatorname{Spec} \overline{k} \to \overline{\mathcal{M}}_g$. Write $s^{\log}_{X_{\overline{k}}}$ for the log scheme whose underlying scheme is $\operatorname{Spec} \overline{k}$ and log structure is the pull-back log structure of $\overline{\mathcal{M}}_g^{\log}$.

Write $M_{X_{\overline{K}}}$ and $M_{X_{\overline{k}}}$ for the respective abelianizations of the pro-ℓ admissible fundamental groups $\pi_1^{\ell\text{-adm}}(X_{\overline{K}})$ and $\pi_1^{\ell\text{-adm}}(X_{\overline{k}})$ (cf. the discussion immediately preceding Proposition 1.3). Note that there is a natural isomorphism $M_{X_{\overline{K}}} \cong M_{X_{\overline{k}}}$ induced by the specialization morphism of the pro-ℓ admissible fundamental groups $\pi_1^{\ell\text{-adm}}(X_{\overline{K}})$ and $\pi_1^{\ell\text{-adm}}(X_{\overline{k}})$ (cf. Proposition 1.1). Recall the natural exact sequence

$$1 \to I_K \to G_K \to G_k \to 1,$$

where I_K, G_K, and G_k denote the inertia group of K determined by $\overline{K}$, the absolute Galois group of K determined by $\overline{K}$, and the absolute Galois group of k determined by $\overline{k}$, respectively. By the ℓ-adic cohomology criterion for stable reduction of curves (cf. [3] Theorem 2.4 and [1] Theorem 7.4.6), the action of the inertia group I_K of G_K on $W := M_{X_{\overline{K}}} \otimes \mathbb{Q}_\ell$ is unipotent. Thus, any lifting to G_K of the Frobenius element $\in G_k$ determines a filtration on W (corresponding to weights ≥ 2, ≥ 1, ≥ 0), which is called the *weight filtration*, and which does not depend on the choice of the lifting, as follows:

$$0 \subseteq W_2 \subseteq W_1 \subseteq W. \qquad (*)$$

Since the action of the inertia group I_K of G_K on W is unipotent, the action of I_K factors through the maximal pro-ℓ quotient of I_K, which we denote by I_K^ℓ. Write

$$\rho^\ell_{I_K} : I_K^\ell \to \mathrm{GL}(W)$$

for the resulting Galois representation. Since the action of I_K^ℓ on W is unipotent, for any generator a of I_K^ℓ, there exists a uniquely determined monodromy operator $N_a : W \to W$ such that $\rho^\ell_{I_K}(a) = \exp(N_a)$. Note that Remark 2.5.1 implies that a induces a positive definite element $\widetilde{a} \in \pi_1^\ell(s^{\log}_{X_{\overline{k}}})$.

For the geometric special fiber $X_{\overline{k}}$, we have the following filtration defined in Section 2.1:

$$0 \subseteq M^{\mathrm{edge}}_{X_{\overline{k}}} \otimes \mathbb{Q}_\ell \subseteq M^{\mathrm{ver}}_{X_{\overline{k}}} \otimes \mathbb{Q}_\ell \subseteq M_{X_{\overline{k}}} \otimes \mathbb{Q}_\ell \cong W. \qquad (**)$$

Since $M^{\mathrm{edge}}_{X_{\overline{k}}}$ is isomorphic to a direct sum of copies of $\mathbb{Z}_\ell(1)$, the weight of $M^{\mathrm{edge}}_{X_{\overline{k}}}$ is equal to 2. Furthermore, by applying Proposition 2.1 and the Weil conjecture for abelian varieties, the weight of $M^{\mathrm{ver}}_{X_{\overline{k}}}/M^{\mathrm{edge}}_{X_{\overline{k}}}$ is equal to 1. Since $M_{X_{\overline{k}}}/M^{\mathrm{ver}}_{X_{\overline{k}}} \cong M^{\mathrm{top}}_{X_{\overline{k}}}$ (cf. the discussion at the beginning of Section 2.1), the weight of $M_{X_{\overline{k}}}/M^{\mathrm{ver}}_{X_{\overline{k}}}$ is 0. Thus, the filtration $(*)$ coincides with the filtration $(**)$. Since any connected étale covering of the geometric special fiber $X_{\overline{k}}$ lifts uniquely to an étale covering of $\mathcal{X}_R \times_{\operatorname{Spec} R} \operatorname{Spec} \overline{R}^{\mathrm{unr}}$ whose domain is a stable curve over $\operatorname{Spec} \overline{R}^{\mathrm{unr}}$, the action of I_K^{ℓ} on $W/W_2 \cong M_{X_{\overline{k}}}/M_{X_{\overline{k}}}^{\mathrm{edge}} \otimes \mathbb{Q}_\ell \cong \operatorname{Hom}(M_{X_{\overline{k}}}^{\mathrm{ver}}, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}_\ell \cong \operatorname{Hom}(W_1, \mathbb{Q}_\ell(1))$ (where the second isomorphism is the isomorphism arising from the Poincaré duality discussed at the beginning of the proof of Proposition 2.2) is trivial, so we have $(\rho_{I_K}^{\ell}(a) - 1)^2 = 0$. Since $\rho_{I_K}^{\ell}(a) - 1$ may be written as the product of N_a with an invertible matrix that commutes with N_a, this implies that $N_a^2 = 0$, $\operatorname{Im}(N_a) \subseteq W_2 \subseteq W_1 \subseteq \operatorname{Ker}(N_a)$. Thus, we obtain a monodromy filtration associated to a as follows (cf. [2] Proposition 1.6.1):

$$0 \subseteq \operatorname{Im}(N_a) \subseteq \operatorname{Ker}(N_a) \subseteq W.$$

Write $\overline{N}_a$ for the isomorphism $W/\operatorname{Ker}(N_a) \xrightarrow{\sim} \operatorname{Im}(N_a)$ induced by N_a. Thus, $\operatorname{rank}(\overline{N}_a) = \dim_{\mathbb{Q}_\ell}(W/\operatorname{Ker}(N_a)) = \dim_{\mathbb{Q}_\ell}(\operatorname{Im}(N_a))) = \operatorname{rank}(f_{X_{\overline{k}}}(\widetilde{a}))$, where $f_{X_{\overline{k}}}(\widetilde{a})$ is the pro-ℓ period matrix associated to $\widetilde{a}$, and

$$\dim_{\mathbb{Q}_\ell}(M_{X_{\overline{k}}}^{\mathrm{top}} \otimes \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell}(W/W_1) = \dim_{\mathbb{Q}_\ell}(W_2),$$

where the equalities follow from the discussion at the beginning of the proof of Proposition 2.2. The weight-monodromy conjecture asserts that the weight filtration coincides with the monodromy filtration associated to a. To prove this assertion, let us first recall that by Faltings-Chai's theory, $f_{X_{\overline{k}}}(\widetilde{a})$ is non-degenerate. Thus, we have $\operatorname{rank}(\overline{N}_a) = \operatorname{rank}(f_X(\widetilde{a})) = \dim_{\mathbb{Q}_\ell}(M_{X_{\overline{k}}}^{\mathrm{top}} \otimes \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell}(W/W_1) = \dim_{\mathbb{Q}_\ell}(W_2)$. These equalities, together with the inclusions $\operatorname{Im}(N_a) \subseteq W_2 \subseteq W_1 \subseteq \operatorname{Ker}(N_a)$, imply that $W_1 = \operatorname{Ker}(N_a)$ and $W_2 = \operatorname{Im}(N_a)$. Thus, the weight-monodromy conjecture for curves holds.

On the other hand, let us consider the action of $\pi_1^{\ell}(s_{X_{\overline{k}}}^{\log})$ on W induced by the homotopy exact sequence of pro-ℓ log étale fundamental groups of stable log curves (cf. Corollary 1.2). Moreover, by the ℓ-adic cohomology criterion for stable reduction, this action is unipotent. For any non-degenerate element b in $\pi_1^{\ell}(s_{X_{\overline{k}}}^{\log})$, by applying similar arguments to the arguments discussed above, we can define a monodromy operator N_b associated to b such that $N_b^2 = 0$, and b acts on W as $\exp(b) = 1 + N_b$; moreover, N_b determines a monodromy filtration. On the other hand, the Frobenius element of G_k determines, by applying similar arguments to the arguments discussed above, a filtration on W, which is called the *weight filtration*, and which, in fact, as can be easily verified, *coincides* with the weight filtration $(*)$ discussed at the beginning of the present subsection. On the other hand, by Theorem 2.9, if the maximal untangled subgraph of the dual graph of $X_{\overline{k}}$ is not a tree, then there exists a non-degenerate element $b \in \pi_1^{\ell}(s_{X_{\overline{k}}}^{\log})$ whose pro-ℓ period matrix is *degenerate*. Thus, we have $\dim_{\mathbb{Q}_\ell}(W/\operatorname{Ker}(N_b)) = \operatorname{rank}(\overline{N}_b) = \operatorname{rank}(f_{X_{\overline{k}}}(b)) < \dim_{\mathbb{Q}_\ell}(M_{X_{\overline{k}}}^{\mathrm{top}} \otimes \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell}(W/W_1)$, which implies that $\operatorname{Ker}(N_b) \neq W_1$. In particular, the weight filtration *does not coincide* with the monodromy filtration defined by b. Put another way, we have shown that Theorem 2.9 implies that if the maximal untangled subgraph of the dual graph of $X_{\overline{k}}$ is not a tree, then there exist non-degenerate elements of $\pi_1^{\ell}(s_{X_{\overline{k}}}^{\log})$ for which *the weight-monodromy conjecture does not hold*. Moreover, we obtain an equivalent form of Theorem 2.9 as follows.

Corollary 2.11. *Let X be a smooth projective hyperbolic curve over a p-adic local field K, $\overline{K}$ an algebraic closure of K, R the ring of integers of K, k the residue field of R, $\overline{R}^{\mathrm{unr}}$ the integral closure of R in the maximal unramified extension of K in $\overline{K}$, and $\overline{k}$ the residue field of $\overline{R}^{\mathrm{unr}}$. Suppose that X admits a stable model $\mathcal{X}_R$ over R. Write X_k for the special fiber of $\mathcal{X}_R$, $X_{\overline{k}}$ for the geometric special fiber of $\mathcal{X}_R$, and $\Gamma_{X_{\overline{k}}}$ for the dual graph of $X_{\overline{k}}$. The geometric special fiber $X_{\overline{k}}$ determines a classifying morphism $\mathrm{Spec}\,\overline{k} \to \overline{\mathcal{M}}_g$, and we shall write $s^{\mathrm{log}}_{X_{\overline{k}}}$ for the log scheme whose underlying scheme is $\mathrm{Spec}\,\overline{k}$, and whose log structure is the pull-back of the log structure of $\overline{\mathcal{M}}_g^{\mathrm{log}}$. Then the weight-monodromy conjecture for X holds for all the non-degenerate elements of $\pi_1^{\ell}(s^{\mathrm{log}}_{X_{\overline{k}}})$ (i.e., the weight filtration on W coincides with the monodromy filtration on W defined by an arbitrary non-degenerate element of $\pi_1^{\ell}(s^{\mathrm{log}}_{X_{\overline{k}}})$) if and only if the maximal untangled subgraph of $\Gamma_{X_{\overline{k}}}$ is a tree.*

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