# GROUP-THEORETIC CHARACTERIZATIONS OF ALMOST OPEN IMMERSIONS OF CURVES

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**Abstract**

Let $p$ be a prime number and $k$ either a finite field of characteristic $p$ or a generalized sub-$p$-adic field. Let $X_1$ and $X_2$ be hyperbolic curves over $k$. In the present paper, we introduce a kind of morphism between $X_1$ and $X_2$ called an almost open immersion, and give some group-theoretic characterizations for the set of almost open immersions between $X_1$ and $X_2$ via their arithmetic fundamental groups. This result generalizes the Isom-version of Grothendieck's anabelian conjecture for curves over $k$ which has been proven by S. Mochizuki and A. Tamagawa to the case of almost open immersions.

Keywords: hyperbolic curve, fundamental group, anabelian geometry.

Mathematics Subject Classification: Primary 14H30; Secondary 11G20, 14G32.

# Contents



# Introduction

In the present paper, we study the anabelian geometry of curves. Let $p$ be a prime number, $k$ a field, $\overline{k}$ an algebraic closure of $k$, and $G_k$ the absolute Galois group of $k$.

Let $X_i,\ i \in \{1, 2\}$, be a hyperbolic curve of type $(g_{X_i}, n_{X_i})$ over $k$ (i.e., $X_i$ is a smooth, geometrically connected curve over $k$ satisfying $2g_{X_i} + n_{X_i} - 2 > 0$, where $g_{X_i}$ is the genus of the smooth compactification $X_i^{\mathrm{cpt}}$, and $n_{X_i}$ is the cardinality of $(X_i^{\mathrm{cpt}} \setminus X_i)(\overline{k}))$ and $\overline{X}_i$ the curve $X_i \times_k \overline{k}$ over $\overline{k}$. Then we have the following exact sequence of étale fundamental groups:

$$1 \to \pi_1(\overline{X}_i, *) \to \pi_1(X_i, *) \stackrel{\mathrm{pr}_{X_i}}{\to} G_k \to 1,$$

where $*$ is a suitable geometric point.

Let $\mathfrak{Primes}$ be the set of prime numbers, $p \in \Sigma_1 \subseteq \mathfrak{Primes}$ a finite set, and $p \notin \Sigma_2 \subseteq \mathfrak{Primes}$ a finite set. We set

$$\Sigma \in \{\mathfrak{Primes}, \mathfrak{Primes} \setminus \Sigma_1, \text{tame}\} \text{ if } \mathrm{char}(k) = p$$

and

$$\Sigma \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma_2 \text{ if } \mathrm{char}(k) = 0.$$

Write $\Delta_{X_i}$ for $\pi_1^{\Sigma}(\overline{X}_i, *)$, where $\pi_1^{\Sigma}(\overline{X}_i, *)$ denotes the maximal pro-$\Sigma$ quotient of $\pi_1(\overline{X}_i, *)$ if $\Sigma \in \{\mathfrak{Primes}, \mathfrak{Primes} \setminus \Sigma_1, \mathfrak{Primes} \setminus \Sigma_2\}$ and denotes the tame fundamental group of $\overline{X}_i$ if $\Sigma =$ tame. Then the kernel of the natural surjection $\pi_1(\overline{X}_i, *) \twoheadrightarrow \Delta_{X_i}$ is a closed normal subgroup of $\pi_1(X_i, *)$. Moreover, we denote by

$$\Pi_{X_i} \stackrel{\text{def}}{=} \pi_1(X_i, *)/(\mathrm{Ker}(\pi_1(\overline{X}_i, *) \twoheadrightarrow \Delta_{X_i})).$$

Thus, we obtain the following exact sequence of fundamental groups:

$$1 \to \Delta_{X_i} \to \Pi_{X_i} \stackrel{\mathrm{pr}^{\Sigma}_{X_i}}{\to} G_k \to 1.$$

We define

$$\mathrm{Isom}_{\text{pro-gps}}(-,-) \text{ and } \mathrm{Hom}^{\text{open}}_{\text{pro-gps}}(-,-)$$

to be the set of continuous isomorphisms and the set of open continuous homomorphisms of profinite groups between the two profinite groups in parentheses, respectively, and define

$$\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \stackrel{\text{def}}{=} \{\Phi \in \mathrm{Isom}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2}) \mid \mathrm{pr}^{\Sigma}_{X_1} = \mathrm{pr}^{\Sigma}_{X_2} \circ \Phi\},$$

$$\mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \stackrel{\text{def}}{=} \{\Phi \in \mathrm{Hom}^{\text{open}}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2}) \mid \mathrm{pr}^{\Sigma}_{X_1} = \mathrm{pr}^{\Sigma}_{X_2} \circ \Phi\}.$$

Thus, by composing with inner automorphisms, we obtain a natural action of $\Delta_{X_2}$ on $\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ and a natural action of $\Delta_{X_2}$ on $\mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$.

We consider the category $\mathcal{C}_k$ of smooth $k$-curves and dominant $k$-morphisms. If $\mathrm{char}(k) = p$, we denote by $\mathcal{FC}_k$ the localization of $\mathcal{C}_k$ at geometric $k$-Frobenius maps between curves (cf. [S1, Section 3]). The ultimate aim of Grothendieck's anabelian conjectures (or, the Grothendieck conjectures for short) for curves over suitable $k$ is to reconstruct the curves from their fundamental groups. More precisely, these conjectures can be formulated as follows:

**(Isom-version):** *The natural maps*

$$\text{Isom-}\pi_1^{\Sigma} : \mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = 0$ *and*

$$\mathrm{Isom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma} : \mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = p$ *are bijections.*

Suppose that $\mathrm{char}(k) = 0$. If $\Sigma = \mathfrak{Primes}$ and $k$ is a number field, then Isom-version was proved by H. Nakamura (cf. [N1], [N2]) when the genus of $X_i, i \in \{1, 2\}$, is 0, and was proved by A. Tamagawa (cf. [T1]) in the case of arbitrary affine curves. Later, S. Mochizuki (cf. [M2]) generalized their results to the case where $k$ is a generalized sub-$p$-adic field (i.e., a field which can be embedded as a subfield of a finitely generated extension of the quotient field of the ring of Witt vectors with coefficients in an algebraic closed field of $\mathbb{F}_p$), $\Sigma$ is a set which contains $p$, and $X_i, i \in \{1, 2\}$, is an arbitrary hyperbolic curve over $k$.

Suppose that $\mathrm{char}(k) = p$. If $\Sigma \in \{\mathfrak{Primes}, \mathrm{tame}\}$ and $k$ is a finite field, then Isom-version was proved by Tamagawa (cf. [T1]) when $X_i, i \in \{1, 2\}$, is affine, and was proved by Mochizuki (cf. [M4]) when $X_i, i \in \{1, 2\}$, is projective. Recently, M. Saïdi and Tamagawa (cf. [ST1], [ST3]) generalized their results to the case where $p \notin \Sigma$ is a complement of a finite subset of $\mathfrak{Primes}$. On the other hand, J. Stix (cf. [S1], [S2]) proved Isom-version when $\Sigma = \mathrm{tame}$ and $k$ is a field that is finitely generated over $\mathbb{F}_p$.

In fact, by applying $p$-adic Hodge theory, Mochizuki proved a very general version when $k$ is a sub-$p$-adic field (i.e., a field which can be embedded as a subfield of a finitely generated extension of $\mathbb{Q}_p$) as follows (cf. [M1]):

> **(Hom-version of characteristic** 0**):** *Suppose that $k$ is a sub-$p$-adic field. Then natural map*
>
> $$\text{Hom-}\pi_1^{\Sigma} : \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$
>
> *is a bijection.*

Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\text{Isom-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\text{Hom-}\pi_1^{\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}), \end{array}$$

when $\mathrm{char}(k) = 0$. Since all the vertical arrows appeared in the commutative diagrams above are injections, we have that

$$\text{Hom-version of characteristic } 0 \Rightarrow \text{Isom-version of characteristic } 0.$$

On the other hand, since the method used in [M1] can not work well in the case of generalized sub-$p$-adic fields, we do not know whether Hom-version of characteristic 0 above holds or not if $k$ is a generalized sub-$p$-adic field.

Similar like in the case of characteristic 0, we may consider certain Hom-versions of the Grothendieck conjectures for curves in positive characteristic (=**Hom-version of positive characteristic**) which is one of the main open problems in anabelian geometry. Essentially, Hom-version of positive characteristic is a kind of problems concerning the following fundamental anabelian style questions:

- Can we find out all of the continuous homomorphisms contained in $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ which can be induced by morphisms of $X_1$ and $X_2$?

- Can we give a purely group-theoretic characterization of morphisms of curves in terms of $\Pi_{X_1}$ and $\Pi_{X_2}$ as simple as possible?

Moreover, an optimistic expectation for Hom-version of positive characteristic is the following ultimate goal:

> The natural map
>
> $$\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma} : \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$
>
> is a bijection.

Hom-version of positive characteristic is a much more difficult problem than Isom-version of positive characteristic. Since Tamagawa proved the Isom-version of the Grothendieck conjecture for affine curves over finite fields in the 1990s, at the time of writing, except some obvious cases (e.g. $X_1 \to X_2$ is a finite étale morphism, $\Pi_{X_1} \subseteq \Pi_{X_2}$ is an open normal subgroup), no published results concerning the Grothendieck conjecture for curves in positive characteristic for **non-isomorphisms** are known even the following case:

> Suppose that $k$ is a finite field, that $\Sigma = \mathfrak{Primes}$, and that $g_{X_1} = g_{X_2}$ and $n_{X_1} \geq n_{X_2}$. Which elements contained in $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ can be induced by morphisms from $X_1$ to $X_2$? How to give a purely group-theoretic characterization of elements of $\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$ in terms of étale fundamental groups of $X_1$ and $X_2$?

Note that, if $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$, then all of the elements of $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ are surjections, and all of the morphisms between $X_1$ and $X_2$ are compositions of isomorphisms and Frobenius twists. Since $\Pi_{X_i}$, $i \in \{1, 2\}$, is not topologically finitely generated, we do not known whether or not a surjection $\Pi_{X_1} \twoheadrightarrow \Pi_{X_2}$ is an isomorphism in general. Thus, we do not known whether or not the surjection of profinite groups arises from geometry in general.

On the other hand, we would like to mention that Saïdi and Tamagawa obtained a birational version of Hom-version of positive characteristic for function fields of curves over finite fields under certain conditions (cf. [ST2]).

In the present paper, we prove a certain type of Grothendieck's anabelian conjecture for a kind of **non-isomorphisms** called almost open immersions. This result generalizes the Isom-version of the Grothendieck conjecture for curves over either a finite field or a generalized sub-$p$-adic field which has been proven by Mochizuki and Tamagawa.

Before we explain our main theorem of the present paper, let us introduce some notation. Let $f \in \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$ be a separable $k$-morphism. We shall say that $f : X_1 \to X_2$ is separable $\Sigma$-almost open immersion if $f$ is a composition of an open immersion and a **finite** étale morphism such that the Galois group of the Galois closure of the finite étale morphism is a finite quotient of $\Pi_{X_2}$. Suppose that $\mathrm{char}(k) = p$. Let $\phi \in \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$. We shall say that $\phi : X_1 \to X_2$ is a $\Sigma$-almost open immersion if $\phi$ can be represented by the following $k$-morphisms

$$X_1 \cong_k Y(m_1) \leftarrow Y \to Y(m_2) \to X_2$$

such that $Y(m_2) \to X_2$ is a separable $\Sigma$-almost open immersion, where $Y(m_1)$ and $Y(m_2)$ denote the $m_1^{\mathrm{th}}$-Frobenius twist and $m_2^{\mathrm{th}}$-Frobenius twist of $Y$, respectively, and $\cong_k$ is a $k$-isomorphism. Then we define

$$\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$$

if $\mathrm{char}(k) = 0$ and

$$\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$$

if $\mathrm{char}(k) = p$ to be the sets of all the $\Sigma$-almost open immersions between $X_1$ and $X_2$. On the other hand, we introduce a purely group-theoretic condition ($\Sigma$-gnc) concerning genus (cf. cf. Section 1 and Proposition 1.2). We denote by

$$\mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$$

for the elements of $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ satisfying the condition ($\Sigma$-gnc). Then the natural maps $\mathrm{Hom}\text{-}\pi_1^{\Sigma}$ and $\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma}$ induce the following natural maps:

$$\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = 0$ and

$$\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

if $\mathrm{char}(k) = p$ which fit into the following commutative diagrams:

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}\text{-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}}} & \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}\text{-}\pi_1^{\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}} & \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

respectively. Here, all the vertical arrows appeared in the commutative diagrams above are injections. Now, our main theorem of the present paper is as follows (cf. Theorem 4.2 and Theorem 4.3).

**Theorem 0.1.** *Suppose that $k$ is either a finite field of characteristic $p$ or a generalized sub-$p$-adic field. Then the natural maps*

$$\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}_{\mathcal{C}_k}^{\Sigma\text{-al-op-im}}(X_1, X_2) \overset{\sim}{\to} \mathrm{Hom}_{G_k}^{\mathrm{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = 0$ *and*

$$\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}_{\mathcal{FC}_k}^{\Sigma\text{-al-op-im}}(X_1, X_2) \overset{\sim}{\to} \mathrm{Hom}_{G_k}^{\mathrm{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = p$ *are bijections.*

**Remark 0.1.1.** We maintain the notation introduced in Theorem 0.1. Suppose that $g_{X_1} = g_{X_2} \geq 1$ and $n_{X_1} \geq n_{X_2}$. Then all of the morphisms contained in $\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$ (resp. $\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$) are $\Sigma$-almost open immersions. Thus, we obtain a group-theoretic characterization of $\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$ (resp. $\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$) in terms of $\Pi_{X_1}$ and $\Pi_{X_2}$.

Our method of proving Theorem 0.1 is as follows. The main difficult is proving the surjectivity of $\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}}$ and $\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}$. Let $\Phi \in \mathrm{Hom}_{G_k}^{\mathrm{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})$. To verify that the image of $\Phi$ in $\mathrm{Hom}_{G_k}^{\mathrm{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ comes from a morphism of curves, it is easy to see that we may assume that $\Phi$ is a surjection. First, we assume that $\Sigma \neq \mathfrak{Primes}$ when $\mathrm{char}(k) = p$. By using the condition ($\Sigma$-gnc), we prove that the kernel of the surjection $\Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$ induced by $\Phi$ is generated by inertia subgroups of $\Delta_{X_1}$ associated to cups of $X_1$ (cf. Theorem 2.6 (i)). Then we can reduce Theorem 0.1 to the Isom-version of the Grothendieck conjecture for curves over $k$ which has been proven by Mochizuki when $k$ is a generalized sub-$p$-adic field (cf. [M2]), and Saïdi-Tamagawa when $k$ is a finite field (cf. [ST3]).

Next, we assume that $\mathrm{char}(k) = p$ and $\Sigma = \mathfrak{Primes}$. In this case, the proof of Theorem 0.1 is more complicated than the prime-to-$p$ (or tame) case explained above. We introduce a purely group-theoretic condition ($\Sigma$-prc) concerning $p$-rank (cf. Section 1 and Proposition 1.2), and by ($\Sigma$-gnc) and ($\Sigma$-prc), we can also prove that the kernel of the surjection $\Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$ induced by $\Phi$ is generated by inertia subgroups of $\Delta_{X_1}$ associated to cups of $X_1$ (cf. Theorem 2.4 (ii)). By applying the prime-to-$p$ version of Theorem 0.1, then we can reduce Theorem 0.1 to a result concerning Hopfian and weakly Hopfian properties of fundamental groups of curves in positive characteristic (cf. Section 3).

The present paper is organized as follows. In Section 1, we review some well-known facts concerning the Isom-version of the Grothendieck conjecture for curves, introduce two purely group-theoretic conditions ($\Sigma$-gnc) and ($\Sigma$-prc), and give a group-theoretic characterization of the sets of cusps of hyperbolic curves. In Section 2, we study the kernels of surjections of geometric fundamental groups, and prove that the kernels are generated by inertia subgroups under the conditions ($\Sigma$-gnc) and ($\Sigma$-prc). In Section 3, by applying a finiteness theorem concerning étale coverings with restricted ramification of a variety over a finite field obtained by T. Hiranouchi, we study the Hopfian and weakly Hopfian properites of fundamental groups of curves in positive characteristic. In Section 4, by applying the Isom-version of the Grothendieck conjecture for curves and the results obtained in Section 2 and Section 3, we prove our main theorems.

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# 1 Preliminaries

Let $p > 0$ be a prime number, $\mathbb{F}_p$ a finite field of characteristic $p$, and $\overline{\mathbb{F}}_p$ an algebraic closure of $\mathbb{F}_p$. We shall say that a field is generalized sub-$p$-adic if the field may be embedded as a subfield of a finitely generated extension of the quotient field of $W(\overline{\mathbb{F}}_p)$ (i.e., the ring of Witt vectors of $\overline{\mathbb{F}}_p$). Let $k$ be either a **finite field** of characteristic $p$ or a **generalized sub-$p$-adic field** and $\overline{k}$ an algebraic closure of $k$. We shall say that $X_i$, $i \in \{1, 2\}$, is a hyperbolic curve of type $(g_{X_i}, n_{X_i})$ over $k$ if $X_i$ is a smooth, geometrically connected curve over $k$ satisfying $2g_{X_i} + n_{X_i} - 2 > 0$, where $g_{X_i}$ is the genus of the smooth compactification $X_i^{\mathrm{cpt}}$, and $n_{X_i}$ is the cardinality of $(X_i^{\mathrm{cpt}} \setminus X_i)(\overline{k})$. Then we have the following fundamental exact sequence of étale fundamental groups:

$$1 \to \pi_1(\overline{X}_i, *) \to \pi_1(X_i, *) \overset{\mathrm{pr}_{X_i}}{\to} G_k \to 1,$$

where $\overline{X}_i$ denotes the curve $X_i \times_k \overline{k}$, $G_k$ denotes the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of $k$, and $*$ is a suitable geometric point. For simplicity, we omit $*$ and denote by $\pi_1(X_i)$ and $\pi_1(\overline{X}_i)$ the étale fundamental groups $\pi_1(X_i, *)$ and $\pi_1(\overline{X}_i, *)$, respectively.

Let $\mathfrak{Primes}$ be the set of prime numbers, $p \in \Sigma_1 \subseteq \mathfrak{Primes}$ a finite set, and $p \notin \Sigma_2 \subseteq \mathfrak{Primes}$ a finite set. We put

$$\Sigma \in \{\mathfrak{Primes}, \mathfrak{Primes} \setminus \Sigma_1, \mathrm{tame}\} \text{ if } \mathrm{char}(k) = p$$

and

$$\Sigma \overset{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma_2 \text{ if } \mathrm{char}(k) = 0.$$

Write $\Delta_{X_i}$ for $\pi_1^{\Sigma}(\overline{X}_i)$, where $\pi_1^{\Sigma}(\overline{X}_i)$ denotes the maximal pro-$\Sigma$ quotient of $\pi_1(\overline{X}_i)$ if $\Sigma \in \{\mathfrak{Primes}, \mathfrak{Primes} \setminus \Sigma_1, \mathfrak{Primes} \setminus \Sigma_2\}$ and denotes the tame fundamental group of $\overline{X}_i$ if $\Sigma = \mathrm{tame}$. Note that

$$\mathrm{Ker}(\pi_1(\overline{X}_i) \twoheadrightarrow \Delta_{X_i})$$

is also a normal closed subgroup of $\pi_1(X_i)$. Then we denote by

$$\Pi_{X_i} \overset{\mathrm{def}}{=} \pi_1(X_i)/(\mathrm{Ker}(\pi_1(\overline{X}_i) \twoheadrightarrow \Delta_{X_i})).$$

Moreover, we obtain a commutative diagram as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\overline{X}_i) & \longrightarrow & \pi_1(X_i) & \overset{\mathrm{pr}_{X_i}}{\longrightarrow} & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_{X_i} & \longrightarrow & \Pi_{X_i} & \overset{\mathrm{pr}^{\Sigma}_{X_i}}{\longrightarrow} & G_k & \longrightarrow & 1, \end{array}$$

where all the vertical arrows are surjections.

We define

$$\mathrm{Isom}_{\text{pro-gps}}(-,-) \text{ and } \mathrm{Hom}^{\text{open}}_{\text{pro-gps}}(-,-)$$

to be the set of continuous isomorphisms and the set of open continuous homomorphisms of profinite groups between the two profinite groups in parentheses, respectively, and define

$$\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \stackrel{\text{def}}{=} \{\Phi \in \mathrm{Isom}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2}) \mid \mathrm{pr}^{\Sigma}_{X_1} = \mathrm{pr}^{\Sigma}_{X_2} \circ \Phi\},$$

$$\mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \stackrel{\text{def}}{=} \{\Phi \in \mathrm{Hom}^{\text{open}}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2}) \mid \mathrm{pr}^{\Sigma}_{X_1} = \mathrm{pr}^{\Sigma}_{X_2} \circ \Phi\}.$$

Thus, by composing with inner automorphisms, we obtain a natural action of $\Delta_{X_2}$ on $\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ and a natural action of $\Delta_{X_2}$ on $\mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$.

We consider the category $\mathcal{C}_k$ of smooth $k$-curves and dominant $k$-morphisms. If $\mathrm{char}(k) = p$, we denote by $\mathcal{FC}_k$ the localization of $\mathcal{C}_k$ at geometric $k$-Frobenius maps between curves. Then we obtain the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}\text{-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}\text{-}\pi_1^{\Sigma}} & \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}), \end{array}$$

if $\mathrm{char}(k) = 0$ and

$$\begin{array}{ccc} \mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma}} & \mathrm{Hom}^{\text{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \end{array}$$

if $\mathrm{char}(k) = p$, where all the vertical arrows are injections. Moreover, the following Isom-version of the Grothendieck conjecture for hyperbolic curves over $k$ has been known (cf. [M2, Theorem 4.12], [T1, Theorem 0.5 and Theorem 0.6], and [ST3, Theorem 4.22]):

**Theorem 1.1.** *The natural maps*

$$\mathrm{Isom}\text{-}\pi_1^{\Sigma} : \mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = 0$ *and*

$$\mathrm{Isom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma} : \mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = p$ *are bijections.*

Let $\mathcal{F}$ be a geometric object and $\Pi_{\mathcal{F}}$ a profinite group associated to the geometric object $\mathcal{F}$. Given an invariant $\mathrm{Inv}_{\mathcal{F}}$ depending on the isomorphism class of $\mathcal{F}$ (in a certain category), we shall say that $\mathrm{Inv}_{\mathcal{F}}$ **can be reconstructed group-theoretically from**

$\Pi_{\mathcal{F}}$ if $\Pi_{\mathcal{F}_1} \cong \Pi_{\mathcal{F}_2}$ (as profinite groups) implies that $\mathrm{Inv}_{\mathcal{F}_1} = \mathrm{Inv}_{\mathcal{F}_2}$ for two such geometric objects $\mathcal{F}_1$ and $\mathcal{F}_2$. Moreover, suppose that we are given an additional structure $\mathrm{Add}_{\mathcal{F}}$ (e.g., a family of subgroups) on the profinite group $\Pi_{\mathcal{F}}$ depending functorially on $\mathcal{F}$; then we shall say that $\mathrm{Add}_{\mathcal{F}}$ **can be reconstructed group-theoretically from** $\Pi_{\mathcal{F}}$ if all isomorphisms $\Pi_{\mathcal{F}_1} \cong \Pi_{\mathcal{F}_2}$ (as profinite groups) preserve the structures $\mathrm{Add}_{\mathcal{F}_1}$ and $\mathrm{Add}_{\mathcal{F}_2}$.

Let $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_{X_1}, \Delta_{X_2})$. We denote by

$$\Delta_{\Phi} \overset{\mathrm{def}}{=} \mathrm{Im}(\overline{\Phi}) \subseteq \Delta_{X_2}$$

the image of $\overline{\Phi}$. We introduce a condition concerning genus as follows:

> **($\Sigma$-gnc):** For each open subgroup $\overline{H}_2 \subseteq \Delta_{\Phi}$, write $\overline{H}_1$ for the inverse image $\overline{\Phi}^{-1}(\overline{H}_2)$. We denote by $g_{\overline{H}_1}$ and $g_{\overline{H}_2}$ the genera of the curves over $\overline{k}$ corresponding to $\overline{H}_1$ and $\overline{H}_2$, respectively. We shall say that $\overline{\Phi}$ satisfies ($\Sigma$-gnc) if $g_{\overline{H}_1} = g_{\overline{H}_2}$ for each open subgroup $\overline{H}_2 \subseteq \Delta_{\Phi}$.

Let $C$ be a smooth curve over $\overline{k}$ and $C^{\mathrm{cpt}}$ the smooth compactification of $C$. If $\mathrm{char}(k) = p$, we define the $p$-rank of $C$ to be

$$\sigma_C \overset{\mathrm{def}}{=} \dim_{\mathbb{F}_p}(\mathrm{H}^1_{\text{ét}}(C^{\mathrm{cpt}}, \mathbb{F}_p)).$$

Next, we introduce a condition concerning $p$-rank as follows:

> **($\Sigma$-prc):** Suppose that $\mathrm{char}(k) = p$ and $\Sigma \in \{\mathfrak{Primes}, \mathrm{tame}\}$. For each open subgroup $\overline{H}_2 \subseteq \Delta_{\Phi}$, write $\overline{H}_1$ for the inverse image $\overline{\Phi}^{-1}(\overline{H}_2)$. We denote by $\sigma_{\overline{H}_1}$ and $\sigma_{\overline{H}_2}$ the $p$-rank of the curves over $\overline{k}$ corresponding to $\overline{H}_1$ and $\overline{H}_2$, respectively. We shall say that $\overline{\Phi}$ satisfies ($\Sigma$-prc) if $\sigma_{\overline{H}_1} = \sigma_{\overline{H}_2}$ for each open subgroup $\overline{H}_2 \subseteq \Delta_{\Phi}$.

Note that if $\overline{\Phi}$ satisfies ($\Sigma$-gnc) (resp. ($\Sigma$-prc)), then, for each open subgroup $\overline{Q}_2 \subseteq \Delta_{X_2}$, the homomorphism $\overline{\Phi}^{-1}(\overline{Q}_2) \to \overline{Q}_2$ induced by $\overline{\Phi}$ also satisfies ($\Sigma$-gnc) (resp. ($\Sigma$-prc)).

**Proposition 1.2.** *(i) Suppose that* $\mathrm{char}(k) = p$*, and that* $\Sigma$ *is either* $\mathfrak{Primes}$ *or* tame*. Then* ($\Sigma$-gnc) *and* ($\Sigma$-prc) *are group-theoretic properties.*

*(ii) Let* $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$*. Write* $\overline{\Phi} : \Delta_{X_1} \to \Delta_{X_2}$ *for the morphism induced by* $\Phi$*. Then* ($\Sigma$-gnc) *is a group-theoretical property.*

*Proof.* For each $\overline{H}_1 \subseteq \Delta_{\Phi}$ and $\overline{H}_2 \overset{\mathrm{def}}{=} \overline{\Phi}^{-1}(\overline{H}_1)$, we shall write $X_{\overline{H}_i}$, $i \in \{1, 2\}$, for the hyperbolic curve of genus $g_{\overline{H}_i}$ over $\overline{k}$ corresponding to $\overline{H}_i$.

First, we prove (i). Suppose that $\Sigma = \mathrm{tame}$. Then we see immediately that $\sigma_{\overline{H}_i} = \dim_{\mathbb{F}_p}(\overline{H}_i^{\mathrm{ab}} \otimes \mathbb{F}_p)$, where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$. Then ($\Sigma$-prc) is a group-theoretical property. If $\Sigma = \mathfrak{Primes}$, then [T2, Corollary 1.7] implies that $\overline{\Phi}$ satisfies ($\Sigma$-prc) group-theoretically. Moreover, if $\Sigma = \mathfrak{Primes}$ (resp. $\Sigma = \mathrm{tame}$), by [T2, Theorem 1.9] (resp. [T4, Theorem 0.5]), ($\Sigma$-gnc) is a group-theoretical property.

Next, we prove (ii). Suppose that $\mathrm{char}(k) = 0$. To verify the proposition, we may reduce immediately to the case where $k$ is finite over the quotient field of $W(\overline{\mathbb{F}}_p)$. Then the genera $g_{\overline{H}_1}$ and $g_{\overline{H}_2}$ are equal to the dimensions of the weight 0 parts of the Hodge-Tate decompositions of the abelianizations of the maximal pro-$p$ quotients of $\overline{H}_1$ and $\overline{H}_2$ (cf. [Ta, Section 4, Remark]), respectively. Suppose that $\mathrm{char}(k) = p$. Let $\ell$ be a prime number distinct from $p$. Then the genera $g_{\overline{H}_1}$ and $g_{\overline{H}_2}$ are equal to 1/2 the dimensions of the Frobenius weight 1 parts of the abelianizations of the maximal pro-$\ell$ quotients of $\overline{H}_1$ and $\overline{H}_2$, respectively. This completes the proof of the proposition. □

**Remark 1.2.1.** Let $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_{X_1}, \Delta_{X_2})$. The proposition means that we can determine whether $\overline{\Phi}$ satisfies ($\Sigma$-gnc) (resp. ($\Sigma$-prc)) or not group-theoretically from $\Delta_{X_1}$ and $\Delta_{X_2}$.

In the remainder of this section, let $X$ be a hyperbolic curve of type $(g_X, n_X)$ over $\overline{k}$. Write $X^{\mathrm{cpt}}$ for the smooth compactification of $X$ over $\overline{k}$. We define a pointed smooth stable curve

$$X^{\bullet} \stackrel{\mathrm{def}}{=} (X^{\mathrm{cpt}}, D_X \stackrel{\mathrm{def}}{=} X^{\mathrm{cpt}} \setminus X)$$

over $\overline{k}$. Here, $X^{\mathrm{cpt}}$ denotes the underlying curve of $X^{\bullet}$, and $D_X$ denotes the set of marked points of $X^{\bullet}$. By choosing a suitable geometric point, we denote by $\pi_1(X)$ the étale fundamental group of $X$.

Let $K_X$ be the function field of $X$. We define $K_X^{\Sigma}$ to be the maximal pro-$\Sigma$ (resp. the maximal tame if $\Sigma =$ tame) Galois extension of $K_X$ in a fixed separable closure of $K_X$, unramified over $X$ (resp. unramified over $X$, and at most tamely ramified over $D_X$). Then we may identify the maximal pro-$\Sigma$ quotient $\Delta_X$ of $\pi_1(X)$ (resp. the tame fundamental group of $X$ if $\Sigma =$ tame) with $\mathrm{Gal}(K_X^{\Sigma}/K_X)$. We put

$$X^{\bullet,\Sigma} \stackrel{\mathrm{def}}{=} (X^{\Sigma}, D_{X^{\Sigma}}),$$

where $X^{\Sigma}$ denotes the normalization of $X^{\mathrm{cpt}}$ in $K_X^{\Sigma}$, and $D_{X^{\Sigma}}$ denotes the inverse image of $D_X$ in $X^{\Sigma}$. For each $e^{\Sigma} \in D_{X^{\Sigma}}$, we denote by $I_{e^{\Sigma}}$ the inertia subgroup of $\Delta_X$ associated to $e^{\Sigma}$ (i.e., the stabilizer of $e^{\Sigma}$). Let $C_{\Delta_X}$ be a cofinal system of open subgroups of $\Delta_X$. For each $H \in C_{\Delta_X}$, we write $X_H^{\bullet} \stackrel{\mathrm{def}}{=} (X_H, D_{X_H})$ for the smooth pointed stable curve corresponding to $H$ and $e_H \in D_{X_H}$ for the image of $e^{\Sigma}$ in $X_H^{\bullet}$. Write $E_{C_{\Delta_X}}$ for the system $(e_H)_{H \in C_{\Delta_X}}$, and note that $E_{C_{\Delta_X}}$ admits a natural action of $\Delta_X$. Then we may identify the inertia subgroup $I_{e^{\Sigma}}$ associated to $e^{\Sigma}$ with the stabilizer of $E_{C_{\Delta_X}}$.

**Definition 1.3.** Let $f^{\bullet} : Y^{\bullet} \stackrel{\mathrm{def}}{=} (Y, D_Y) \to X^{\bullet}$ be a morphism of smooth pointed stable curves over $k$. We shall say that $f^{\bullet}$ is a Galois tame covering (resp. Galois étale covering) if $f^{\bullet}$ induces a Galois covering of underlying curves which is at most tamely ramified over $D_X$ (resp. $f^{\bullet}$ induces a Galois covering of underlying curves which is étale). Moreover, we put

$$\mathrm{Ram}_{f^{\bullet}} \stackrel{\mathrm{def}}{=} \{e \in D_X \mid f^{\bullet} \text{ is ramified over } e\}.$$

In the remainder of this section, we suppose that $g_X \geq 2$, and that $n_X > 0$. We define

$$(\ell, d, f^{\bullet} : Y^{\bullet} \stackrel{\mathrm{def}}{=} (Y, D_Y) \to X^{\bullet})$$

to be a triple associated to $X$ satisfying the following conditions:

(a) $\ell, d$ are prime numbers distinct from each other and from $p$ such that $\ell \equiv 1 \pmod{d}$; then all $d^{\text{th}}$ roots of unity are contained in $\mathbb{F}_\ell$; Moreover, we assume that $\ell, d \in \Sigma$ if $\Sigma \neq$ tame.

(b) $f^\bullet : Y^\bullet \to X^\bullet$ is a Galois **étale** covering (i.e., the morphism of underlying curves induced by $f^\bullet$ is a Galois étale covering) over $\overline{k}$ whose Galois group is equipped with an isomorphism with $G_d$, where $G_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of $d^{\text{th}}$ roots of unity.

Write $M_{Y^\bullet}^{\text{ét}}$ and $M_{Y^\bullet}$ for $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell)$ and $\mathrm{Hom}(\Delta_Y, \mathbb{F}_\ell)$, respectively, where $\Delta_Y$ denotes the maximal pro-$\Sigma$ quotient of the étale fundamental group of $Y \setminus D_Y$ (resp. the tame fundamental group of $Y \setminus D_Y$ if $\Sigma =$ tame). Note that there is a natural injection

$$M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}$$

induced by the natural surjection $\Delta_Y \twoheadrightarrow \Delta_Y^{\text{ét}}$, where $\Delta_Y^{\text{ét}}$ denotes the étale fundamental group of $Y$. Then we obtain an exact sequence

$$0 \to M_{Y^\bullet}^{\text{ét}} \to M_{Y^\bullet} \to M_{Y^\bullet}^{\text{ra}} \stackrel{\text{def}}{=} \mathrm{coker}(M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}) \to 0$$

with a natural action of $G_d$.

Let

$$M_{Y^\bullet, G_d}^{\text{ra}} \subseteq M_{Y^\bullet}^{\text{ra}}$$

be the subset of elements on which $G_d$ acts via the natural character $G_d \hookrightarrow \mathbb{F}_\ell^\times$ induced by the inclusion $G_d \subseteq \mathbb{F}_\ell$ and

$$U_{Y^\bullet}^* \subseteq M_{Y^\bullet}$$

the subset of elements that map to nonzero elements of $M_{Y^\bullet, G_d}^{\text{ra}}$. For each $\alpha \in U_{Y^\bullet}^*$, write

$$g_\alpha^\bullet : Y_\alpha^\bullet \stackrel{\text{def}}{=} (Y_\alpha, D_{Y_\alpha}) \to Y^\bullet$$

for the Galois tame covering over $\overline{k}$ of degree $\ell$ corresponding to $\alpha$. Then we obtain a map

$$\epsilon : U_{Y^\bullet}^* \to \mathbb{Z}, \ \alpha \mapsto \#D_{Y_\alpha},$$

where $\#(-)$ denotes the cardinality of $(-)$.

We define a subset of $U_{Y^\bullet}^*$ to be

$$U_{Y^\bullet}^{\text{mp}} \stackrel{\text{def}}{=} \{\alpha \in U_{Y^\bullet}^* \mid \#\mathrm{Ram}_{g_\alpha^\bullet} = d\} = \{\alpha \in U_{Y^\bullet}^* \mid \epsilon(\alpha) = \ell(dn_X - d) + d\}.$$

For each $\alpha \in U_{Y^\bullet}^{\text{mp}}$, since the image of $\alpha$ is contained in $M_{Y^\bullet, G_d}^{\text{ra}}$, we obtain that the action of $G_d$ on the set $\mathrm{Ram}_{g_\alpha^\bullet} \subseteq D_{Y^\bullet}$ is transitive. Thus, there exists a unique marked point $e_\alpha$ of $X^\bullet$ such that $f^\bullet(y) = e_\alpha$ for each $y \in \mathrm{Ram}_{g_\alpha^\bullet}$. Moreover, we define a pre-equivalence relation $\sim$ on $U_{Y^\bullet}^{\text{mp}}$ as follows:

> Let $\alpha, \beta \in U_{Y^\bullet}^{\text{mp}}$. Then $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in U_{Y^\bullet}^*$, we have $\lambda\alpha + \mu\beta \in U_{Y^\bullet}^{\text{mp}}$.

On the other hand, for each $e \in D_X$, we define

$$U^{\mathrm{mp}}_{Y^\bullet, e} \stackrel{\mathrm{def}}{=} \{\alpha \in U^{\mathrm{mp}}_{Y^\bullet} \mid g^\bullet_\alpha \text{ is ramified over } (f^\bullet)^{-1}(e)\}.$$

Then, for any two marked points $e, e' \in D_X$ distinct from each other, we have

$$U^{\mathrm{mp}}_{Y^\bullet, e} \cap U^{\mathrm{mp}}_{Y^\bullet, e'} = \emptyset.$$

Moreover, we have

$$U^{\mathrm{mp}}_{Y^\bullet} = \bigcup_{e \in D_X} U^{\mathrm{mp}}_{Y^\bullet, e}.$$

Write $(g_Y, n_Y)$ for the type of $Y^\bullet$. Then the structure of the maximal pro-$\ell$ quotients of tame fundamental groups implies that

$$\Delta^{\mathrm{ab}}_Y \otimes \mathbb{F}_\ell \cong \langle a_1, \dots, a_{g_Y}, b_1, \dots, b_{g_Y}, \{c_{e'}\}_{e' \in D_Y} \mid \prod_{i=1}^{g_Y} [a_i, b_i] \prod_{e' \in D_Y} c_{e'} = 1 \rangle^{\mathrm{ab}} \otimes \mathbb{F}_\ell,$$

where $\{c_{e'}\}_{e' \in D_Y}$ denotes a set of generators of inertia subgroups associated to marked points. Next, let us explain the set $U^{\mathrm{mp}}_{Y^\bullet, e}$ more precisely. Let $\alpha \in U^{\mathrm{mp}}_{Y^\bullet, e}$ and $e'' \in (f^\bullet)^{-1}(e) \subseteq D_Y$. The construction of $U^{\mathrm{mp}}_{Y^\bullet, e}$ implies that $\alpha(c_{e''}) = a$ for some $a \in \mathbb{F}^\times_\ell$, that $\alpha(c_{\tau(e'')}) = \tau a$ for each $\tau \in G_d$, and that $\alpha(c_{e'}) = 0$ for each $e' \in D_Y \setminus (f^\bullet)^{-1}(e)$. Note that

$$\alpha(\prod_{e' \in D_Y} c_{e'}) = a \sum_{\tau \in G_d} \tau = 0$$

holds in $\mathbb{F}_\ell$.

**Proposition 1.4.** *(i) The pre-equivalence relation $\sim$ on $U^{\mathrm{mp}}_{Y^\bullet}$ is an equivalence relation, and there exists a natural bijection*

$$\vartheta : U^{\mathrm{mp}}_{Y^\bullet} / \sim \to D_X.$$

*Moreover, let*

$$(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$$

*be an arbitrary triple associated to $X$. Hence we obtain a resulting set $U^{\mathrm{mp}}_{Y^{\bullet,*}} / \sim$ and a natural bijection*

$$\vartheta^* : U^{\mathrm{mp}}_{Y^{\bullet,*}} / \sim \to D_X.$$

*Then there exists a natural bijection*

$$U^{\mathrm{mp}}_{Y^{\bullet,*}} / \sim \cong U^{\mathrm{mp}}_{Y^\bullet} / \sim$$

*which is compatible with the bijections $\vartheta$ and $\vartheta^*$ (i.e., the set $U^{\mathrm{mp}}_{Y^\bullet} / \sim$ does not depend on the choices of $\ell, d$, and the étale covering $f^\bullet : Y^\bullet \to X^\bullet$).*

*(ii) Write $g_Y$ for the genus of $Y^\bullet$. We have, for each $e \in D_X$,*

$$\# U^{\mathrm{mp}}_{Y^\bullet, e} = \ell^{2g_Y+1} - \ell^{2g_Y} \text{ and } U^{\mathrm{mp}}_{Y^\bullet} = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$

*Proof.* First, let us prove (i). Let $\beta, \gamma \in U^{\mathrm{mp}}_{Y^\bullet}$. If $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\beta + \mu\gamma \neq 0$, we have $\mathrm{Ram}_{g^\bullet_{\lambda\beta+\mu\gamma}} = \mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Thus, $\beta \sim \gamma$. On the other hand, if $\beta \sim \gamma$, we have $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Otherwise, we have $\#\mathrm{Ram}_{g^\bullet_{\beta+\gamma}} = 2d$. Thus, $\beta \sim \gamma$ if and only if $\mathrm{Ram}_{g^\bullet_\beta} = \mathrm{Ram}_{g^\bullet_\gamma}$. Then $\sim$ is an equivalence relation on $U^{\mathrm{mp}}_{Y^\bullet}$.

Let

$$\vartheta : U^{\mathrm{mp}}_{Y^\bullet}/\sim \to D_X$$

be a map defined by $\alpha \mapsto e_\alpha$. We prove that $\vartheta$ is a bijection. It is easy to see that $\vartheta$ is an injection. On the other hand, for each $e \in D_X$, the explanation of $U^{\mathrm{mp}}_{Y^\bullet,e}$ mentioned in front of the proposition implies that we may construct a connected Galois tame covering of $h^\bullet : Z^\bullet \to Y^\bullet$ such that the element corresponding to $h^\bullet$ is contained in $U^{\mathrm{mp}}_{Y^\bullet}$. Then $\vartheta$ is a surjection.

Next, we prove the "moreover" part. First, we suppose that $\ell \neq \ell^*$, and that $d \neq d^*$. Then there exists a natural bijection

$$U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$$

which compatible with the bijections $\vartheta$ and $\vartheta^*$ as follows. Let $\alpha \in U^{\mathrm{mp}}_{Y^\bullet}$ and $\alpha^* \in U^{\mathrm{mp}}_{Y^{\bullet,*}}$. Write $Y^\bullet_\alpha \to Y^\bullet$ and $Y^{\bullet,*}_{\alpha^*} \to Y^{\bullet,*}$ for the tame coverings corresponding to $\alpha$ and $\alpha^*$, respectively. Let us consider

$$Y^\bullet \times_{X^\bullet} Y^{\bullet,*}.$$

Thus, we have a connected Galois tame covering $Y^\bullet \times_{X^\bullet} Y^{\bullet,*} \to X^\bullet$ of degree $dd^*\ell\ell^*$. Then it is easy to check that $\alpha$ and $\alpha^*$ correspond to same marked point if and only if the cardinality of the set of marked points of $Y^\bullet \times_{X^\bullet} Y^{\bullet,*}$ is equal to $dd^*(\ell\ell^*(n_X - 1) + 1)$. In general case, for any two given triples $(\ell, d, f^\bullet : Y^\bullet \to X^\bullet)$ and $(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$, we may choose a triple

$$(\ell^{**}, d^{**}, f^{\bullet,**} : Y^{\bullet,**} \to X^\bullet)$$

associated to $X$ such that $\ell^{**} \neq \ell$, $\ell^{**} \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain a resulting set $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim$ and a natural bijection $\vartheta^{**} : U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \to D_X$. Then we obtain two natural bijections $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$ and $U^{\mathrm{mp}}_{Y^{\bullet,**}}/\sim \cong U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim$. Thus, we have $U^{\mathrm{mp}}_{Y^{\bullet,*}}/\sim \cong U^{\mathrm{mp}}_{Y^\bullet}/\sim$. This completes the proof of (i).

Next, let us prove (ii). Write $E_e \subseteq D_Y$ for the set $(f^\bullet)^{-1}(e)$. Then $U^{\mathrm{mp}}_{Y^\bullet,e}$ can be naturally regarded as a subset of $\mathrm{H}^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell)$ via the natural open immersion $Y \setminus E_e \hookrightarrow Y$. Write $L_e$ for the $\mathbb{F}_\ell$-vector space generated by $U^{\mathrm{mp}}_{Y^\bullet,e}$ in $\mathrm{H}^1_{\text{ét}}(Y \setminus E_e, \mathbb{F}_\ell)$. Then we have

$$U^{\mathrm{mp}}_{Y^\bullet,e} = L_e \setminus \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell).$$

Write $H_e$ for the quotient $L_e/\mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell)$. We have an exact sequence as follows:

$$0 \to \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to L_e \to H_e \to 0.$$

The explanation of $U^{\mathrm{mp}}_{Y^\bullet,e}$ mentioned in front of the proposition implies that

$$\dim_{\mathbb{F}_\ell}(H_e) = 1.$$

On the other hand, since $\dim_{\mathbb{F}_\ell}(\mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell)) = 2g_Y$, we obtain

$$\#U^{\mathrm{mp}}_{Y^\bullet,e} = \ell^{2g_Y+1} - \ell^{2g_Y}.$$

Thus, we have

$$\#U_{Y^\bullet}^{\mathrm{mp}} = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$

This completes the proof of the lemma. □

**Remark 1.4.1.** The proof of the "moreover" part of Proposition 1.4 (i) implies that, if the cardinality of the sets of marked points of smooth pointed stable curves can be reconstructed group-theoretically, then the bijection

$$U_{Y^\bullet,*}^{\mathrm{mp}}/\sim \cong U_{Y^\bullet}^{\mathrm{mp}}/\sim$$

can be determined group-theoretically.

# 2 The kernels of surjections of geometric fundamental groups

We maintain the notation introduced in Section 1. Let $\overline{X}_i^{\mathrm{cpt}}$, $i \in \{1,2\}$, be the smooth compactification of $\overline{X}_i$ over $\overline{k}$. We define a pointed smooth stable curve over $\overline{k}$ to be

$$\overline{X}_i^\bullet \overset{\mathrm{def}}{=} (\overline{X}_i^{\mathrm{cpt}}, D_{\overline{X}_i} \overset{\mathrm{def}}{=} \overline{X}^{\mathrm{cpt}} \setminus \overline{X}_i),\ i \in \{1,2\}.$$

Let $\overline{\Phi} \in \mathrm{Hom}_{\text{pro-gps}}^{\mathrm{open}}(\Delta_{X_1}, \Delta_{X_2})$. In this section, we suppose that $n_{X_2} > 0$, that $\overline{\Phi}$ is a surjective homomorphism, and that $\overline{\Phi}$ satisfies ($\Sigma$-gnc). First, we have the following lemma.

**Lemma 2.1.** *Suppose that $g_{X_2} \geq 2$. For each $i \in \{1,2\}$, we write $\Delta_{X_i}^{\text{ét}}$ for the étale fundamental group of $\overline{X}_i^{\mathrm{cpt}}$ and $\Delta_{X_i}^{\text{ét},p'}$ for the maximal pro-$\Sigma \setminus \{p\}$ quotient of $\Delta_{X_i}$ (resp. maximal prime-to-p quotient of $\Delta_{X_i}$ if $\Sigma =$ tame). Then $\overline{\Phi}$ induces an isomorphism*

$$\Delta_{X_1}^{\text{ét},p'} \overset{\sim}{\to} \Delta_{X_2}^{\text{ét},p'}.$$

*Proof.* Let $N_2 \subseteq \Delta_{X_2}$ be an arbitrary open subgroup such that the Galois covering $X_{N_2} \to \overline{X}_2$ corresponding to $N_2$ is étale, and that $(\#(\Delta_{X_2}/N_2), p) = 1$. Write $N_1 \subseteq \Delta_{X_1}$ for the inverse image $\overline{\Phi}^{-1}(N_2)$. Then the condition ($\Sigma$-gcn) and the Riemann-Hurwitz formula imply that the covering $X_{N_1} \to \overline{X}_1$ corresponding to $N_1$ is étale. Note that $\#(\Delta_{X_1}/N_1) = \#(\Delta_{X_2}/N_2)$ is prime to $p$. Thus, $\overline{\Phi}$ induces a surjection

$$\Delta_{X_1}^{\text{ét},p'} \twoheadrightarrow \Delta_{X_2}^{\text{ét},p'}.$$

On the other hand, since $g_{X_1} = g_{X_2}$, $\Delta_{X_1}^{\text{ét},p'}$ is isomorphic to $\Delta_{X_2}^{\text{ét},p'}$ as abstract profinite groups. Moreover, we have $\Delta_{X_i}^{\text{ét},p'}$, $i \in \{1,2\}$, is topologically finitely generated. Then the surjection $\Delta_{X_1}^{\text{ét},p'} \twoheadrightarrow \Delta_{X_2}^{\text{ét},p'}$ obtained above is an isomorphism (cf. [FJ, Proposition 16.10.6]). This completes the proof of the lemma. □

Suppose that $g_{X_2} \geq 2$. Let

$$(\ell, d, f_2^\bullet : Y_2^\bullet \stackrel{\text{def}}{=} (Y_2, D_{Y_2}) \to \overline{X}_2^\bullet)$$

be a triple associated to $\overline{X}_2$ (cf. Section 1). Lemma 2.1 implies that the triple $(\ell, d, f_2^\bullet : Y_2^\bullet \to \overline{X}_2^\bullet)$ associated to $\overline{X}_2$ induces a triple

$$(\ell, d, f_1^\bullet : Y_1^\bullet \stackrel{\text{def}}{=} (Y_1, D_{Y_1}) \to \overline{X}_1^\bullet)$$

associated to $\overline{X}_1$, where the Galois étale covering $f_1^\bullet$ is induced by $f_2^\bullet$ via the isomorphism $\mathrm{H}^1_{\text{ét}}(\overline{X}_2^{\text{cpt}}, \mathbb{F}_d) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(\overline{X}_1^{\text{cpt}}, \mathbb{F}_d)$ induced by $\Delta_{X_1}^{\text{ét},p'} \twoheadrightarrow \Delta_{X_2}^{\text{ét},p'}$.

Write $\Delta_{Y_1^\bullet} \subseteq \Delta_{X_1}$ and $\Delta_{Y_2^\bullet} \subseteq \Delta_{X_2}$ for open normal subgroups corresponding to $Y_1^\bullet$ and $Y_2^\bullet$, respectively. Write $M_{Y_1^\bullet}$, $M_{Y_1^\bullet}^{\text{ét}}$, $M_{Y_1^\bullet}^{\text{ra}}$, $M_{Y_2^\bullet}$, $M_{Y_2^\bullet}^{\text{ét}}$, and $M_{Y_2^\bullet}^{\text{ra}}$ for $\mathrm{Hom}(\Delta_{Y_1^\bullet}, \mathbb{F}_\ell)$, $\mathrm{H}^1_{\text{ét}}(Y_1, \mathbb{F}_\ell)$, $M_{Y_1^\bullet}/M_{Y_1^\bullet}^{\text{ét}}$, $\mathrm{Hom}(\Delta_{Y_2^\bullet}, \mathbb{F}_\ell)$, $\mathrm{H}^1_{\text{ét}}(Y_2, \mathbb{F}_\ell)$, and $M_{Y_2^\bullet}/M_{Y_2^\bullet}^{\text{ét}}$, respectively. Write $\overline{\Phi}_Y$ for $\overline{\Phi}|_{\Delta_{Y_1}}$. Then $\overline{\Phi}_Y$ induces a homomorphism

$$\overline{\Psi}_{Y,\ell}^{\text{ab}} : M_{Y_2^\bullet} \to M_{Y_1^\bullet}.$$

By replacing $\overline{\Phi}$ by $\overline{\Phi}_Y$, the claim implies that $\overline{\Phi}_Y$ induces the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{Y_1^\bullet}^{\text{ét}} & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & M_{Y_1^\bullet}^{\text{ra}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \overline{\Psi}_{Y,\ell}^{\text{ab}}} & & \uparrow & & \\ 0 & \longrightarrow & M_{Y_2^\bullet}^{\text{ét}} & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & M_{Y_2^\bullet}^{\text{ra}} & \longrightarrow & 0, \end{array}$$

where the vertical arrows on the right-hand side and the middle side are injections, and the vertical arrow on the left-hand side is an isomorphism. Write $U_{Y_1^\bullet}^*$ and $U_{Y_2^\bullet}^*$ for the subsets of $M_{Y_1^\bullet}$ and $M_{Y_2^\bullet}$ defined as in Section 1, respectively. Since the actions of $G_d$ on the exact sequences are compatible with the morphisms appeared in the commutative diagram above, we have

$$\overline{\Psi}_{Y,\ell}^{\text{ab}}(U_{Y_2^\bullet}^*) \subseteq U_{Y_1^\bullet}^*.$$

Let $e_2 \in D_{\overline{X}_2}$, $\alpha_2 \in U_{Y_2^\bullet, e_2}^{\text{mp}}$, and $g_{\alpha_2}^\bullet : Y_{\alpha_2}^\bullet \to Y_2^\bullet$ the Galois tame covering of degree $\ell$ over $\overline{k}$ corresponding to $\alpha_2$. Write

$$g_{\alpha_1}^\bullet : Y_{\alpha_1}^\bullet \to Y_1^\bullet$$

for the tame covering of degree $\ell$ over $\overline{k}$ corresponding to $\alpha_1 \stackrel{\text{def}}{=} \overline{\Psi}_{Y,\ell}^{\text{ab}}(\alpha_2)$. Write $g_{Y_{\alpha_1}}$ and $g_{Y_{\alpha_2}}$ for the genera of $Y_{\alpha_1}^\bullet$ and $Y_{\alpha_2}^\bullet$, respectively. Then the condition ($\Sigma$-gnc) and the Riemann-Hurwitz formula imply that

$$g_{Y_{\alpha_1}} - g_{Y_{\alpha_2}} = \frac{1}{2}(d - \#\mathrm{Ram}_{g_{\alpha_1}^\bullet})(\ell - 1) = 0.$$

Then we have $d = \#\mathrm{Ram}_{g_{\alpha_1}^\bullet}$. This means that $\alpha_1 \in U_{Y_1^\bullet}^{\text{mp}}$. Moreover, there exists $e_1 \in D_{\overline{X}_1}$ such that $\alpha_1 \in U_{Y_1^\bullet, e_1}^{\text{mp}}$.

Let $\alpha_2' \in U^{\mathrm{mp}}_{Y_2^\bullet, e_2}$ distinct from $\alpha_2$. Since, for each $a\alpha_2 + b\alpha_2' \neq 0$, $a, b \in \mathbb{F}_\ell^\times$, $a\alpha_2 + b\alpha_2' \in U^{\mathrm{mp}}_{Y_2^\bullet, e_2}$, we have $\overline{\Psi}^{\mathrm{ab}}_{Y,\ell}(a\alpha_2 + b\alpha_2') \in U^{\mathrm{mp}}_{Y_1^\bullet}$. Moreover, we have $\overline{\Psi}^{\mathrm{ab}}_{Y,\ell}(a\alpha_2 + b\alpha_2') \in U^{\mathrm{mp}}_{Y_1^\bullet, e_1}$. This implies that $\overline{\Psi}^{\mathrm{ab}}_{Y,\ell}(\alpha_2') \in U^{\mathrm{mp}}_{Y_1^\bullet, e_1}$. Thus, we obtain

$$\overline{\Psi}^{\mathrm{ab}}_{Y,\ell}(U^{\mathrm{mp}}_{Y_2^\bullet, e_2}) \subseteq U^{\mathrm{mp}}_{Y_1^\bullet, e_1}.$$

On the other hand, Proposition 1.4 (ii) implies that $\#U^{\mathrm{mp}}_{Y_1^\bullet, e_1} = \#U^{\mathrm{mp}}_{Y_2^\bullet, e_2}$. We have

$$\overline{\Psi}^{\mathrm{ab}}_{Y,\ell}(U^{\mathrm{mp}}_{Y_2^\bullet, e_2}) = U^{\mathrm{mp}}_{Y_1^\bullet, e_1}.$$

Then Proposition 1.4 (i) implies that $\overline{\Psi}^{\mathrm{ab}}_{Y,\ell}$ induces an injection

$$\lambda_{\overline{\Phi}, Y, \ell} : U^{\mathrm{mp}}_{Y_2^\bullet} / \sim \hookrightarrow U^{\mathrm{mp}}_{Y_1^\bullet} / \sim .$$

On the other hand, let $(\ell^*, d^*, f_2^{*,\bullet} : Y_2^{*,\bullet} \stackrel{\mathrm{def}}{=} (Y_2^*, D_{Y_2^*}) \to \overline{X}_2^\bullet)$ be an arbitrary triple associated to $\overline{X}_2$. Then by similar arguments to the arguments given above imply that $\overline{\Psi}^{\mathrm{ab}}_{Y,\ell}$ induces an injection

$$\lambda_{\overline{\Phi}, Y^*, \ell^*} : U^{\mathrm{mp}}_{Y_2^{*,\bullet}} / \sim \hookrightarrow U^{\mathrm{mp}}_{Y_1^{*,\bullet}} / \sim .$$

Since $\overline{\Phi}$ satisfies ($\Sigma$-gnc), we have $n_{\overline{H}_1} = n_{\overline{H}_2}$ for each open normal subgroup $\overline{H}_2 \subseteq \Delta_{X_2}$, where $\overline{H}_1$ denotes the inverse image $\overline{\Phi}^{-1}(\overline{H}_2)$, and $n_{\overline{H}_i}$, $i \in \{1, 2\}$, denotes the cardinality of the marked points of the curves over $\overline{k}$ corresponding to $\overline{H}_i$. Then Remark 1.4.1 implies that the following commutative diagram holds:

$$\begin{array}{ccc} U^{\mathrm{mp}}_{Y_2^{*,\bullet}} / \sim & \xrightarrow{\lambda_{\overline{\Phi}, Y^*, \ell^*}} & U^{\mathrm{mp}}_{Y_1^{*,\bullet}} / \sim \\ \downarrow & & \downarrow \\ U^{\mathrm{mp}}_{Y_2^\bullet} / \sim & \xrightarrow{\lambda_{\overline{\Phi}, Y, \ell}} & U^{\mathrm{mp}}_{Y_1^\bullet} / \sim, \end{array}$$

where the vertical arrows are the bijection constructed in the proof of Proposition 1.4 (i). Moreover, Proposition 1.4 (i) implies that we may identify $D_{\overline{X}_i}$ with $U^{\mathrm{mp}}_{Y_i^\bullet} / \sim$ for $i \in \{1, 2\}$. Then we obtain the following result.

**Lemma 2.2.** *Suppose that $g_{X_2} \geq 2$. Then the surjective $\overline{\Phi}$ induces a map*

$$\lambda_{\overline{\Phi}} : D_{\overline{X}_2} \hookrightarrow D_{\overline{X}_1}.$$

*Moreover, $\lambda_{\overline{\Phi}}$ is an injection.*

Let $P_2 \subseteq \Delta_{X_2}$ be an arbitrary open subgroup and $P_1 \subseteq \Delta_{X_1}$ the inverse image $\overline{\Phi}^{-1}(P_2)$ of $P_2$. Write

$$Z_i^\bullet \stackrel{\mathrm{def}}{=} (Z_i, D_{Z_i}) \ i \in \{1, 2\},$$

for the pointed smooth stable curves of over $\overline{k}$ corresponding to $P_i$. The surjection $\overline{\Phi}$ induces a surjection

$$\overline{\Phi}_Z \stackrel{\text{def}}{=} \overline{\Phi}|_{P_1} : P_1 \twoheadrightarrow P_2.$$

Then Lemma 2.2 implies an injective map

$$\lambda_{\overline{\Phi}_Z} : D_{Z_2} \hookrightarrow D_{Z_1}.$$

On the other hand, $P_i$, $i \in \{1, 2\}$, determines a morphism $f^\bullet_{P_i} : Z^\bullet_i \to \overline{X}^\bullet_i$ of smooth pointed stable curves over $\overline{k}$. Moreover, $f^\bullet_{P_i}$ induces a surjective map of the sets of marked points

$$\gamma_{f_{P_i}} : D_{Z_i} \twoheadrightarrow D_{\overline{X}_i}$$

of $Z^\bullet_i$ and $\overline{X}^\bullet_i$. Furthermore, we have the following lemma.

**Lemma 2.3.** *Suppose that $g_{X_2} \geq 2$. Then the natural diagram*

$$\begin{array}{ccc} D_{Z_2} & \xrightarrow{\lambda_{\overline{\Phi}_Z}} & D_{Z_1} \\ {\scriptstyle \gamma_{f_{P_2}}}\downarrow & & {\scriptstyle \gamma_{f_{P_1}}}\downarrow \\ D_{\overline{X}_2} & \xrightarrow{\lambda_{\overline{\Phi}}} & D_{\overline{X}_1} \end{array}$$

*is commutative.*

*Proof.* Let $e_{Z_2} \in D_{Z_2}$, $e_{Z_1} \stackrel{\text{def}}{=} \lambda_{\overline{\Phi}_Z}(e_{Z_2}) \in D_{Z_1}$, $e_2 \stackrel{\text{def}}{=} \gamma_{f_{P_2}}(e_{Z_2}) \in D_{\overline{X}_2}$, $e_1 \stackrel{\text{def}}{=} (\gamma_{f_{P_1}} \circ \lambda_{\overline{\Phi}_Z})(e_{Z_2}) \in D_{\overline{X}_2}$, and $e'_1 \stackrel{\text{def}}{=} \lambda_{\overline{\Phi}}(e_2) \in D_{\overline{X}_1}$. Let us prove that $e_1 = e'_1$. Write $S_{Z_1}$ and $S_{Z_2}$ for the sets $(\gamma_{f_{P_1}})^{-1}(e'_1)$ and $(\gamma_{f_{P_2}})^{-1}(e_2)$, respectively. Note that $e_{Z_2} \in S_{Z_2}$. To verify $e_1 = e'_1$, it is sufficient to prove that $e_{Z_1} \in S_{Z_1}$

Let $(\ell, d, f^\bullet_2 : Y^\bullet_2 \to \overline{X}^\bullet_2)$ be a triple associated to $\overline{X}_2$ such that $(\ell, \#(\Delta_{X_2}/P_2)) = 1$ and $(d, \#(\Delta_{X_2}/P_2)) = 1$. By Lemma 2.1, we obtain a triple

$$(\ell, d, f^\bullet_1 : Y^\bullet_1 \to \overline{X}^\bullet_1)$$

associated to $\overline{X}_1$ induced by $\overline{\Phi}$ and $(\ell, d, f^\bullet_2 : Y^\bullet_2 \to X^\bullet_2)$. On the other hand, we have a triple

$$(\ell, d, g^\bullet_2 : W^\bullet_2 \stackrel{\text{def}}{=} Y^\bullet_2 \times_{\overline{X}^\bullet_2} Z^\bullet_2 \to Z^\bullet_2)$$

associated to $Z_2$. Again, by By Lemma 2.1, we obtain a triple

$$(\ell, d, g^\bullet_1 : W^\bullet_1 \stackrel{\text{def}}{=} Y^\bullet_1 \times_{\overline{X}^\bullet_1} Z^\bullet_1 \to Z^\bullet_1)$$

associated to $Z_1$ induced by $\overline{\Phi}_Z$ and $(\ell, d, g^\bullet_2 : W^\bullet_2 \to Z^\bullet_2)$.

Let $\alpha_2 \in U^{\text{mp}}_{Y^\bullet_2, e_2}$, where $U^{\text{mp}}_{(-)}$ is defined as in Section 1. Then by similar arguments to the arguments given in the proof of Lemma 2.2 imply that $\alpha_2$ induces an element

$$\alpha_1 \in U^{\text{mp}}_{Y^\bullet_1, e'_1}.$$

Write $Y^\bullet_{\alpha_1}$ and $Y^\bullet_{\alpha_2}$ for the smooth pointed stable curves over $\overline{k}$ corresponding to $\alpha_1$ and $\alpha_2$, respectively. We consider the connected Galois tame covering

$$Y^\bullet_{\alpha_2} \times_{\overline{X}^\bullet_2} Z^\bullet_2 \to W^\bullet_2$$

of degree $\ell$ over $\overline{k}$, and write $\beta_2$ for the element of $U^*_{W^\bullet_2}$ corresponding to this connected Galois tame covering, where $U^*_{(-)}$ is defined as in Section 1. Then we have

$$\beta_2 = \sum_{c_2 \in S_{Z_2}} t_{c_2}\beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in U^{\mathrm{mp}}_{W^\bullet_2, c_2}$. On the other hand, similar arguments to the arguments given in the proof of Lemma 2.2 imply that $\beta_{c_2}$ induced an element $\beta_{\lambda_{\overline{\Phi}_Z}(c_2)} \in U^{\mathrm{mp}}_{W^\bullet_1, \lambda_{\overline{\Phi}_Z}(c_2)}$. Then $\beta_2$ induces an element

$$\beta_1 \overset{\mathrm{def}}{=} \sum_{c_2 \in S_{Z_2} \setminus \{e_{Z_2}\}} t_{c_2}\beta_{\lambda_{\overline{\Phi}_Z}(c_2)} + t_{e_{Z_2}}\beta_{\lambda_{\overline{\Phi}_Z}(e_{Z_2})} \in U^*_{W^\bullet_1}.$$

Note that since $\beta_1$ corresponds to the connected Galois tame covering $Y^\bullet_{\alpha_1} \times_{\overline{X}^\bullet_1} Z^\bullet_1 \to W^\bullet_1$, we have the composition of the connected Galois tame covering $Y^\bullet_{\alpha_1} \times_{\overline{X}^\bullet_1} Z^\bullet_1 \to W^\bullet_1$ and the Galois étale covering $g^\bullet_1 : W^\bullet_1 \to Z^\bullet_1$ is tamely ramified over $S_{Z_2}$. This means that $e_{Z_1} = \lambda_{\overline{\Phi}_Z}(e_{Z_2})$ is contained in $S_{Z_1}$. This completes the proof of the lemma. □

Let $K_{\overline{X}_i}$, $i \in \{1, 2\}$, be the function field of $\overline{X}_i$. We define $K^\Sigma_{\overline{X}_i}$ to be the maximal pro-$\Sigma$ (resp. maximal tame if $\Sigma =$ tame) Galois extension of $K_{\overline{X}_i}$ in a fixed separable closure of $K_{\overline{X}_i}$, unramified over $\overline{X}_i$ (resp. unramified over $\overline{X}_i$, and at most tamely ramified over $D_{\overline{X}_i}$). We put

$$\overline{X}^{\bullet,\Sigma}_i \overset{\mathrm{def}}{=} (\overline{X}^\Sigma_i, D_{\overline{X}^\Sigma_i}),\ i \in \{1, 2\},$$

where $\overline{X}^\Sigma_i$ denotes the normalization of $\overline{X}^{\mathrm{cpt}}_i$ in $K^\Sigma_{\overline{X}_i}$, and $D_{\overline{X}^\Sigma_i}$ denotes the inverse image of $D_X$ in $\overline{X}^\Sigma_i$. We have the following proposition.

**Proposition 2.4.** *Let $e^\Sigma_2 \in D_{\overline{X}^\Sigma_2}$, $e_2 \in D_{\overline{X}_2}$ the image of $e^\Sigma_2$, and $I_{e^\Sigma_2}$ the inertia subgroup of $\Delta_{X_2}$ associated to $e^\Sigma_2$. Then the following statements hold.*

*(i) There exists a point $e^\Sigma_1 \in D_{\overline{X}^\Sigma_1}$ such that $\overline{\Phi}$ induces a surjection*

$$\overline{\Phi}|_{I_{e^\Sigma_1}} : I_{e^\Sigma_1} \twoheadrightarrow I_{e^\Sigma_2},$$

*where $I_{e^\Sigma_1}$ denotes the inertia subgroup associated to $e^\Sigma_1$.*

*(ii) Let $e^\Sigma_{1,1}, e^\Sigma_{1,2} \in D_{\overline{X}^\Sigma_1}$. Write $I_{e^\Sigma_{1,1}}$ and $I_{e^\Sigma_{1,2}}$ for the inertia subgroups of $\Delta_{X_1}$ associated to $e^\Sigma_{1,1}$ and $e^\Sigma_{1,2}$, and write $e_{1,1}$ and $e_{1,2}$ for the images of $e^\Sigma_{1,1}$ and $e^\Sigma_{1,2}$ in $D_{\overline{X}_1}$, respectively. Suppose that $\overline{\Phi}|_{I_{e^\Sigma_{1,1}}} : I_{e^\Sigma_{1,1}} \twoheadrightarrow I_{e^\Sigma_2}$ and $\overline{\Phi}|_{I_{e^\Sigma_{1,2}}} : I_{e^\Sigma_{1,2}} \twoheadrightarrow I_{e^\Sigma_2}$. Then we have $\lambda_{\overline{\Phi}}(e_2) = e_{1,1} = e_{1,2}$.*

*Proof.* By Lemma 2.5 below, we may assume that $g_{X_2} \geq 2$. First, we prove (i). Let $C_{\Delta_{X_2}}$ be a cofinal system of open subgroups of $\Delta_{X_2}$. For each $\overline{H}_2 \in C_{\Delta_{X_2}}$, we write $X^\bullet_{\overline{H}_2} \overset{\text{def}}{=} (X_{\overline{H}_2}, D_{X_{\overline{H}_2}})$ for the smooth pointed stable curve corresponding to $\overline{H}_2$ and $e_{\overline{H}_2} \in D_{X_{\overline{H}_2}}$ for the image of $e_2^\Sigma$ in $X^\bullet_{\overline{H}_2}$. We denote by $E_{C_{\Delta_{X_2}}}$ the inverse system $(e_{\overline{H}_2})_{\overline{H}_2 \in C_{\Delta_{X_2}}}$. Then we may identify the inertia subgroup $I_{e_2^\Sigma}$ associated to $e_2^\Sigma$ with the stabilizer of $E_{C_{\Delta_{X_2}}}$.

On the other hand, write $\overline{H}_1$ for $\overline{\Phi}^{-1}(\overline{H}_2)$, $\overline{H}_2 \in C_{\Delta_{X_2}}$, and $X^\bullet_{\overline{H}_1} \overset{\text{def}}{=} (X_{\overline{H}_1}, D_{X_{\overline{H}_1}})$ for the smooth pointed stable curve corresponding to $\overline{H}_1$. Then Lemma 2.2 implies that, for each $\overline{H}_2 \in C_{\Delta_{X_2}}$, we have an injection

$$\lambda_{\overline{\Phi}|_{\overline{H}_1}} : D_{X_{\overline{H}_2}} \hookrightarrow D_{X_{\overline{H}_1}}.$$

Write $L_{\overline{\Phi}}$ for the kernel of $\overline{\Phi} : \Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$. We denote by $K^{L_{\overline{\Phi}}}_{\overline{X}_1} \subseteq K^{\Sigma}_{\overline{X}_1}$ the subfield corresponding to $L_{\overline{\Phi}}$. We put

$$X^\bullet_{1,L_{\overline{\Phi}}} \overset{\text{def}}{=} (X_{1,L_{\overline{\Phi}}}, D_{X_{1,L_{\overline{\Phi}}}}),$$

where $X_{1,L_{\overline{\Phi}}}$ denotes the normalization of $\overline{X}_1^{\text{cpt}}$ in $K^{L_{\overline{\Phi}}}_{\overline{X}_1}$, $D_{X_{1,L_{\overline{\Phi}}}}$ denotes the inverse image of $D_{\overline{X}_1}$ in $X_{1,L_{\overline{\Phi}}}$. Lemma 2.3 implies that the inverse system $(\lambda_{\overline{\Phi}|_{\overline{H}_1}}(e_{\overline{H}_2}))_{\overline{H}_2 \in C_{\Delta_{X_2}}}$ determines a unique point $e_{1,L_{\overline{\Phi}}} \in D_{X_{1,L_{\overline{\Phi}}}}$. We choose a point $e_1^\Sigma \in D_{\overline{X}_1^\Sigma}$ such that the image of $e_1^\Sigma$ in $D_{X_{1,L_{\overline{\Phi}}}}$ is $e_{1,L_{\overline{\Phi}}}$. Write $I_{e_1^\Sigma}$ for the inertia subgroup of $\Delta_{X_1}$ associated to $e_1^\Sigma$ and $I_{e_{1,L_{\overline{\Phi}}}}$ for the inertia subgroup of $\Delta_{X_1}/L_{\overline{\Phi}}$ associated to $e_{1,L_{\overline{\Phi}}}$. Note that the construction above implies that $I_{e_{1,L_{\overline{\Phi}}}} \cong I_{e_2^\Sigma}$. Then $\overline{\Phi}$ induces a surjection

$$\overline{\Phi}|_{I_{e_1^\Sigma}} : I_{e_1^\Sigma} \twoheadrightarrow I_{e_{1,L_{\overline{\Phi}}}} \overset{\sim}{\to} I_{e_2^\Sigma}.$$

Next, we prove (ii). Write $e_1$ for the image of $e_1^\Sigma$ in $D_{\overline{X}_1}$. Let $(\ell, d, f_2^\bullet : Y_2^\bullet \to \overline{X}_2^\bullet)$ be a triple associated to $\overline{X}_2$ defined as in Section 1. Then $\overline{\Phi}$ induces a triple

$$(\ell, d, f_1^\bullet : Y_1^\bullet \to \overline{X}_1^\bullet)$$

associated to $\overline{X}_1$. Write

$$\Delta_{Y_1} \subseteq \Delta_{X_1} \text{ and } \Delta_{Y_2} \subseteq \Delta_{X_2}$$

for the normal open subgroups corresponding to $Y_1^\bullet$ and $Y_2^\bullet$, respectively. Since $f_2^\bullet$ and $f_1^\bullet$ are étale, we have

$$I_{e_2^\Sigma} \subseteq \Delta_{Y_2},\ I_{e_{1,1}^\Sigma} \subseteq \Delta_{Y_1},\ \text{and } I_{e_{1,2}^\Sigma} \subseteq \Delta_{Y_1}.$$

Let $\alpha_2 \in U^{\text{mp}}_{Y_2^\bullet, e_2}$ be an element such that the composition of the natural injection $I_{e_2^\Sigma} \hookrightarrow \Delta_{Y_2}$ and the morphism $\Delta_{Y_2} \to \mathbb{Z}/\ell\mathbb{Z}$ corresponding to $\alpha_2$ is nontrivial. Then, similar arguments to the arguments given in the proof of Lemma 2.2, we obtain an element $\alpha_1 \in U^{\text{mp}}_{Y_1^\bullet, e_1}$. Moreover, the composition of the natural injection $I_{e_{1,1}^\Sigma} \hookrightarrow \Delta_{Y_1}$ (resp. $I_{e_{1,2}^\Sigma} \hookrightarrow \Delta_{Y_1}$) and the morphism $\Delta_{Y_1} \to \mathbb{Z}/\ell\mathbb{Z}$ corresponding to $\alpha_1$ is nontrivial. This means that $e_1 = e_{1,1} = e_{1,2}$. Moreover, the proof of (i) implies that $\lambda_{\overline{\Phi}}(e_2) = e_1 = e_{1,1} = e_{1,2}$. This completes the proof of the proposition. □

**Lemma 2.5.** *Suppose that Proposition 2.4 holds when $g_{X_2} \geq 2$. Then Proposition 2.4 holds when $g_{X_2} \geq 0$.*

*Proof.* Suppose that $n_{X_2} = 0$. The definition of pointed stable curves implies that $g_{X_2} \geq 2$. Then the lemma is trivial. Suppose that $n_{X_2} \neq 0$. Then $n_{X_1} \neq 0$. We take $\ell', \ell'' \in \mathfrak{Primes}$ distinct from $p$ and from each other. First, by replacing $\overline{X}_1$ and $\overline{X}_2$ by finite étale coverings of $\overline{X}_1$ and $\overline{X}_2$ with degree $\ell''$, respectively, we may assume that $n_{X_1} \geq 3$ and $n_{X_2} \geq 3$. We choose an open normal subgroup $Q_2 \subseteq \Pi_{X_2}$ such that the tame covering

$$g_2^\bullet : X^\bullet_{\overline{Q}_2} \stackrel{\text{def}}{=} (X_{\overline{Q}_2}, D_{X_{\overline{Q}_2}}) \to \overline{X}_2^\bullet$$

over $\overline{k}$ corresponding to $\overline{Q}_2$ is totally ramified over $D_{\overline{X}_2}$, where

$$\overline{Q}_2 \stackrel{\text{def}}{=} Q_2 \cap \Delta_{X_2} = \mathrm{Ker}(\Delta_{X_2} \twoheadrightarrow \mathbb{Z}/\ell'\mathbb{Z}).$$

Write $Q_1$ for $\Phi^{-1}(Q_1)$, $\overline{Q}_1$ for $\overline{\Phi}^{-1}(\overline{Q}_1)$, and

$$g_1^\bullet : X^\bullet_{\overline{Q}_1} \stackrel{\text{def}}{=} (X_{\overline{Q}_1}, D_{X_{\overline{Q}_1}}) \to \overline{X}_1^\bullet$$

for the tame covering over $\overline{k}$ corresponding to $\overline{Q}_1$. Note that the genera of $X^\bullet_{\overline{Q}_1}$ and $X^\bullet_{\overline{Q}_2}$ are $\geq 2$. Write $e_{\overline{Q}_2}$ for the image of $e_2^\Sigma$ in $D_{X_{\overline{Q}_2}}$. By Lemma 2.2, we have $e_{\overline{Q}_1} \stackrel{\text{def}}{=} \lambda_{\overline{\Phi}|_{\overline{Q}_2}}(e_{\overline{Q}_2}) \in D_{\overline{X}_{Q_1}}$. Moreover, by Proposition 2.4 (i), there exists $e_1^\Sigma \in D_{\overline{X}_1^\Sigma}$ such that the inertia subgroup $I_{e_{\overline{Q}_1}}$ of $\overline{Q}_1$ associated to $e_1^\Sigma$ is equal to $I_{e_1^\Sigma} \cap \overline{Q}_1$, and that $\overline{\Phi}|_{I_{e_{\overline{Q}_1}}} : I_{e_{\overline{Q}_1}} \twoheadrightarrow I_{e_{\overline{Q}_2}}$, where $I_{e_1^\Sigma}$ denotes the inertia subgroup of $\Delta_{X_1}$ associated to $e_1^\Sigma$. Since $I_{e_2^\Sigma}$ is commensurable terminal ([M3, Lemma 1.3.7]), we have $\overline{\Phi}(I_{e_1^\Sigma}) \subseteq I_{e_2^\Sigma}$. On the other hand, by applying Proposition 2.4 (ii) to $X^\bullet_{\overline{Q}_1}$ and $X^\bullet_{\overline{Q}_2}$ implies that $g_1^\bullet$ is totally ramified at $e_{\overline{Q}_1}$. Thus, $I_{e_{\overline{Q}_1}} \neq I_{e_1^\Sigma}$. This means that

$$\overline{\Phi}(I_{e_1^\Sigma}) = I_{e_2^\Sigma}.$$

Let $e_{1,1}^\Sigma, e_{1,2}^\Sigma \in D_{\overline{X}_1^\Sigma}$ satisfying Proposition 2.4 (i). Then we have the images of $e_{1,1}^\Sigma$ and $e_{1,2}^\Sigma$ in $D_{X_{\overline{Q}_1}}$ are equal to $e_{\overline{Q}_1}$. Thus, the images of $e_{1,1}^\Sigma$ and $e_{1,2}^\Sigma$ in $D_{X_1}$ are equal. This completes the proof of the lemma. □

We set

$$D_{\overline{X}_1 \setminus \overline{X}_2} \stackrel{\text{def}}{=} D_{\overline{X}_1} \setminus \lambda_{\overline{\Phi}}(D_{\overline{X}_2}).$$

Next, we prove the main theorem of the present section.

**Theorem 2.6.** *Let $L_{\overline{\Phi}}$ be the kernel of $\overline{\Phi} : \Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$. For each $e \in D_{\overline{X}_1 \setminus \overline{X}_2}$ and each $e^\Sigma \in D_{X_1^\Sigma}$ over $e$, we denote by $I_{e^\Sigma}$ the inertia subgroup of $\Delta_{X_2}$ associated to $e^\Sigma$.*

*(i) Suppose that $\Sigma \neq \mathfrak{Primes}$ when* $\mathrm{char}(k) = p$. *Then we have $I_{e^\Sigma} \subseteq L_{\overline{\Phi}}$.*

*(ii) Suppose that* $\mathrm{char}(k) = p$, *that $\Sigma = \mathfrak{Primes}$, and that $\overline{\Phi}$ satisfies* ($\Sigma$-prc). *Then we have $I_{e^\Sigma} \subseteq L_{\overline{\Phi}}$.*

*Proof.* First, let us prove (i). We denote by $I \subseteq \Delta_{X_2}$ the image $\overline{\Phi}(I_{e^\Sigma})$. To verify (i), we may assume that $I$ is not trivial. Then $I$ is a pro-cyclic subgroup of $\Delta_{X_2}$. Let $\overline{H}_2$ be any open subgroup of $\Delta_{X_2}$ and $\overline{H}_1 \overset{\text{def}}{=} \overline{\Phi}^{-1}(\overline{H}_2)$. Write $\overline{H}_1^{\text{ét}}$ and $\overline{H}_2^{\text{ét}}$ for the étale fundamental groups of the smooth compactifications of the curves over $\overline{k}$ corresponding to $\overline{H}_1$ and $\overline{H}_2$, respectively. Then we obtain natural surjections

$$\overline{H}_1 \twoheadrightarrow \overline{H}_1^{\text{ét}} \text{ and } \overline{H}_2 \twoheadrightarrow \overline{H}_2^{\text{ét}}.$$

By Lemma 2.1, $\overline{\Phi}$ induces an isomorphism $\overline{H}_1^{\text{ét},p'} \overset{\sim}{\to} \overline{H}_2^{\text{ét},p'}$. Moreover, since $I_{e^\Sigma} \cap \overline{H}_1$ is contained in the kernel of the surjection $\overline{H}_1 \twoheadrightarrow \overline{H}_1^{\text{ét},p',\text{ab}}$, the natural morphism

$$I \cap \overline{H}_2 \hookrightarrow \overline{H}_2 \twoheadrightarrow \overline{H}_2^{\text{ét},p',\text{ab}}$$

is trivial. Thus, by applying [N3, Lemma 2.1.4], $I$ is contained in an inertia subgroup $J \subseteq \Delta_{X_2}$. By replacing $\overline{X}_2$ and $\overline{X}_1$ by the smooth pointed stable curves corresponding open subgroups $\overline{N}_2 \subseteq \Delta_{X_2}$ and $\overline{\Phi}^{-1}(\overline{N}_2) \subseteq \Delta_{X_1}$, respectively, such that $\overline{N}_2 \cap J = \overline{N}_2 \cap I$, Proposition 2.4 implies that $e \in \lambda_\Phi(D_{\overline{X}_2})$. This contradicts to the assumption $e \in D_{\overline{X}_1 \setminus \overline{X}_2}$. We complete the proof of (i).

Next, let us prove (ii). Let $G$ be an arbitrary finite group and $\delta : \Delta_{X_2} \twoheadrightarrow G$ an arbitrary surjection. To verify (ii), we only need to prove that the image $(\delta \circ \overline{\Phi})(I_{e^\Sigma})$ is trivial. Write $G_p$ for a Sylow $p$-subgroup of $G$. Then we obtain that the index of $G_p$ is prime to $p$. Let $\overline{Q}_2 \overset{\text{def}}{=} \delta^{-1}(G_p)$ and $\overline{Q}_1 \overset{\text{def}}{=} \overline{\Phi}^{-1}(\overline{Q}_2)$. Write $f^\bullet_{\overline{Q}_1} : X^\bullet_{\overline{Q}_1} \overset{\text{def}}{=} (X_{\overline{Q}_1}, D_{X_{\overline{Q}_1}}) \to \overline{X}_1^\bullet$ and $f^\bullet_{\overline{Q}_2} : X^\bullet_{\overline{Q}_2} \overset{\text{def}}{=} (X_{\overline{Q}_2}, D_{X_{\overline{Q}_2}}) \to \overline{X}_2^\bullet$ for the coverings over $\overline{k}$ corresponding to $\overline{Q}_1$ and $\overline{Q}_2$, respectively. Lemma 2.3 implies that

$$\#(f^\bullet_{\overline{Q}_2})^{-1}(e_2) = \#(f^\bullet_{\overline{Q}_1})^{-1}(\lambda_{\overline{\Phi}}(e_2)),\ e_2 \in D_{\overline{X}_2}.$$

Since $\overline{\Phi}$ satisfies ($\Sigma$-gnc), the Riemann-Hurwitz formula implies that

$$\#(f^\bullet_{\overline{Q}_1})^{-1}(e_1) = [G : G_p],\ e_1 \in D_{\overline{X}_1 \setminus \overline{X}_2}.$$

This means that $f^\bullet_{\overline{Q}_1}$ is étale over $e_1 \in D_{\overline{X}_1 \setminus \overline{X}_2}$. Then we have

$$(\delta \circ \overline{\Phi})(I_{e^\Sigma}) \subseteq G_p.$$

Thus, we may assume that $G$ is a $p$-group. Write $\overline{P}_1$ and $\overline{P}_2$ for the kernels of $\delta \circ \overline{\Phi}$ and $\delta$, $X^\bullet_{\overline{P}_1} \overset{\text{def}}{=} (X_{\overline{P}_1}, D_{X_{\overline{P}_1}})$ and $X^\bullet_{\overline{P}_2} \overset{\text{def}}{=} (X_{\overline{P}_2}, D_{X_{\overline{P}_2}})$ for the smooth pointed stable curves over $\overline{k}$ corresponding to $\overline{P}_1$ and $\overline{P}_2$, respectively. For each $e_2 \in D_{X_2}$, Proposition 2.4 (ii) implies that the ramification indexes of points of $D_{X_{\overline{P}_2}}$ over $e_2$ are equal to the ramification indexes of points of $D_{X_{\overline{P}_1}}$ over $\lambda_{\overline{\Phi}}(e_2)$. Since $\overline{\Phi}$ satisfies ($\Sigma$-prc), the Deuring-Shafarevich formula implies that the ramification indexes of points of $D_{X_{\overline{P}_1}}$ over $e$ is 1. This means that $(\delta \circ \overline{\Phi})(I_{e^\Sigma})$ is trivial. We complete the proof of (ii). □

# 3 Hopfian and weakly Hopfian properties

Let $l$ be either a finite field of characteristic $p > 0$ or an algebraically closed field of characteristic $p > 0$. Let $U^{\mathrm{cpt}}$ be a smooth curve over $l$ of genus $g_U$, $\bar{l}$ an algebraic closure of $l$, $G_l$ the absolute Galois group $\mathrm{Gal}(\bar{l}/l)$ of $l$, and $U \subseteq U^{\mathrm{cpt}}$ an open subset. Then we obtain an exact sequence of étale fundamental groups

$$1 \to \pi_1(U \times_l \bar{l}, *) \to \pi_1(U, *) \overset{\mathrm{pr}_U}{\to} G_l \to 1,$$

where $*$ is a suitable geometric point. Write $\Pi_U$ for the étale fundamental group $\pi_1(U, *)$ and $\Delta_U$ for the geometric étale fundamental group $\pi_1(U \times_l \bar{l}, *)$.

**Definition 3.1.** Let $\Pi$ be a profinite group. We shall say that $\Pi$ is **Hopfian**, if every surjection $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi, \Pi)$ is an isomorphism.

**Remark 3.1.1.** Suppose that $U$ is a projective curve, and that $l$ is either a finite field or an algebraically closed field. Since $\Pi_U$ is topologically finitely generated, $\Pi_U$ is Hopfian (cf. [FJ, Proposition 16.10.6]).

**Lemma 3.2.** *Let $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_U, \Pi_U)$ be a surjection. Then $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{G_l}(\Pi_U, \Pi_U)$. In particular, $\Phi$ induces a surjction $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_U, \Delta_U)$.*

*Proof.* The lemma follows from [T1, Proposition 3.3 (iii)]. □

**Definition 3.3.** We shall say that $\Pi_U$ is **weakly Hopfian**, if every surjection $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_U, \Pi_U)$, such that the surjection $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_U, \Delta_U)$ induced by $\Phi$ induces surjections of inertia subgroups and higher ramification subgroups, is an isomorphism.

**Remark 3.3.1.** By the definitions, we see that

$$\text{Hopfian} \Rightarrow \text{weakly Hopfian}.$$

On the other hand, for a given surjection $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_U, \Pi_U)$, we do not know whether or not $\Phi$ induces surjections of inertia subgroups and higher ramification groups. Moreover, we may ask the following question

**Question:** When does "weakly Hopfian ⇒ Hopfian" hold?

For the question in Remark 3.3.1, we have the following proposition.

**Proposition 3.4.** *Let $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_U, \Pi_U)$ be a surjection and $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_U, \Delta_U)$ the surjection induced by $\Phi$. Suppose that $\overline{\Phi}$ satisfies ($\Sigma$-gnc). Then $\overline{\Phi}$ induces surjections of inertia subgroups and higher ramification groups.*

*Proof.* We may assume that $U$ is affine. By Proposition 2.4, we obtain that $\overline{\Phi}$ induces surjections of inertia subgroups. Then we only need to prove that $\overline{\Phi}$ induces surjections of higher ramification groups. Write $\overline{U}^{\mathrm{cpt}}$ and $\overline{U}$ for $U^{\mathrm{cpt}} \times_l \bar{l}$ and $U \times_l \bar{l}$, respectively. Let $e$ be a closed point of $\overline{U}^{\mathrm{cpt}} \setminus \overline{U}$ and $I$ the subgroup of $\Delta_U$ generated by the inertia subgroups associated to the inverse images of $e'$, $e' \in \overline{U}^{\mathrm{cpt}} \setminus (\overline{U} \cup \{e\})$, in the universal covering. Then by applying Proposition 2.4, we have that $\overline{\Phi}$ induces a surjection

$$\overline{\Phi}_{I,e} : \Delta_U/I \twoheadrightarrow \Delta_U/I.$$

Note that $\Delta_U/I$ is the étale fundamental group of $\overline{U}^{\mathrm{cpt}} \setminus (\overline{U}^{\mathrm{cpt}} \setminus (\overline{U} \cup \{e\}))$. Thus, by replacing $\overline{U}^{\mathrm{cpt}}$, $\Delta_U$, and $\overline{\Phi}$ by $\overline{U}^{\mathrm{cpt}} \setminus (\overline{U}^{\mathrm{cpt}} \setminus (\overline{U} \cup \{e\}))$, $\Delta_U/I$, and $\overline{\Phi}_{I,e}$, respectively, we may assume that $\#(\overline{U}^{\mathrm{cpt}} \setminus \overline{U}) = 1$.

Let $I_1$ be an inertia subgroup associated to the unique cusp. Write $I_2$ for $\overline{\Phi}(I_1)$. Note that $I_2$ is also an inertia subgroup of $\Pi_U$. Then $I_1$ and $I_2$ carry the upper filtration $\{I_1^r\}_{r\in\mathbb{R}_{\geq 0}}$ and $\{I_2^r\}_{r\in\mathbb{R}_{\geq 0}}$, respectively. For each finite quotient $I_2 \twoheadrightarrow J$, $\{I_1^r\}_{r\in\mathbb{R}_{\geq 0}}$ and the natural surjection $I_1 \twoheadrightarrow I_2 \twoheadrightarrow J$ (resp. $\{I_2^r\}_{r\in\mathbb{R}_{\geq 0}}$ and the surjection $I_2 \twoheadrightarrow J$) induces an upper filtration $\{J_1^r\}_{r\in\mathbb{R}_{\geq 0}}$ (resp. $\{J_2^r\}_{r\in\mathbb{R}_{\geq 0}}$) on $J$. Thus, we obtain two lower filtrations

$$\{J_{1,s}\}_{s\in\mathbb{R}_{\geq 0}} \text{ and } \{J_{2,s}\}_{s\in\mathbb{R}_{\geq 0}}$$

on $J$ induced by $\{J_1^r\}_{r\in\mathbb{R}_{\geq 0}}$ and $\{J_2^r\}_{r\in\mathbb{R}_{\geq 0}}$, respectively. To verify that $\overline{\Phi}$ induces a surjection $I_1^r \twoheadrightarrow I_2^r$ for each $r \in \mathbb{R}_{\geq 0}$, we only need to prove that $\overline{\Phi}$ induces an isomorphism $J_1^r \xrightarrow{\sim} J_2^r$ for each $r \in \mathbb{R}_{\geq 0}$; moreover, this is equivalent to prove that $\overline{\Phi}$ induces an isomorphism $J_{1,s} \xrightarrow{\sim} J_{2,s}$ for each $r \in \mathbb{R}_{\geq 0}$, or that the Artin character of $J$ determined by the lower filtration $\{J_{1,s}\}_{s\in\mathbb{R}_{\geq 0}}$ is equal to the Artin character of $J$ determined by the lower filtration $\{J_{2,s}\}_{s\in\mathbb{R}_{\geq 0}}$.

Let $\overline{H}_2$ be an arbitrary open normal subgroup of $\Delta_U$, $\overline{H}_1$ the inverse image $\overline{\Phi}^{-1}(\overline{H}_1)$, $G_{\overline{H}}$ the quotient $\Delta_U/\overline{H}_2$, and $J_{\overline{H}}$ the image of $I_2$ in $G_{\overline{H}}$. Write $\overline{U}^{\mathrm{cpt}}_{\overline{H}_1}$ and $\overline{U}^{\mathrm{cpt}}_{\overline{H}_2}$ for the smooth compactifications of the curves corresponding to $\overline{H}_1$ and $\overline{H}_2$, respectively. We denote by $\mathrm{Ar}_{J_{\overline{H},1}} : J_{\overline{H}} \to \mathbb{Z}$ and $\mathrm{Ar}_{J_{\overline{H},2}} : J_{\overline{H}} \to \mathbb{Z}$ the Artin character induced by the natural surjection $I_1 \twoheadrightarrow I_2 \twoheadrightarrow J_{\overline{H}}$ and the surjection $I_2 \twoheadrightarrow J_{\overline{H}}$, respectively. Let $\ell \neq p$ be a prime number. Then the Lefschetz trace formula induces the following formulas for characters (cf. [S, Chapter VI §4 Corollary]):

$$\mathrm{Ind}_{J_{\overline{H}}}^{G_{\overline{H}}}(\mathrm{Ar}_{J_{\overline{H},1}}) = (2-2g_U)r_{G_{\overline{H}}} - 2\cdot u_{G_{\overline{H}}} + h_{\ell,\overline{H}_1}$$

and

$$\mathrm{Ind}_{J_H}^{G_H}(\mathrm{Ar}_{J_{\overline{H},2}}) = (2-2g_U)r_{G_{\overline{H}}} - 2\cdot u_{G_{\overline{H}}} + h_{\ell,\overline{H}_2},$$

where $r_{G_{\overline{H}}}$ denotes the regular representation of $G_{\overline{H}}$, $u_{G_{\overline{H}}}$ denotes the unit representation of $G_{\overline{H}}$, and $h_{\ell,\overline{H}_1}$ (resp. $h_{\ell,\overline{H}_2}$) denotes the character of the $G_{\overline{H}}$-module $\pi_1(\overline{U}^{\mathrm{cpt}}_{\overline{H}_1})^{\mathrm{ab}} \otimes \mathbb{Z}_\ell$ (resp. $\pi_1(\overline{U}^{\mathrm{cpt}}_{\overline{H}_2})^{\mathrm{ab}} \otimes \mathbb{Z}_\ell$) whose $G_{\overline{H}}$-module structure coming from the conjugate action of $\Delta_U$ on $\overline{H}_1$ (resp. $\overline{H}_2$). Since $\overline{\Phi}$ satisfies ($\Sigma$-gnc), Lemma 2.1 implies that $\overline{\Phi}$ induces an isomorphism

$$\pi_1(\overline{U}^{\mathrm{cpt}}_{\overline{H}_1})^{\mathrm{ab}} \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \pi_1(\overline{U}^{\mathrm{cpt}}_{\overline{H}_2})^{\mathrm{ab}} \otimes \mathbb{Z}_\ell$$

as $G_{\overline{H}}$-modules. This implies that $h_{\ell,\overline{H}_1} = h_{\ell,\overline{H}_2}$. Thus, we obtain

$$\mathrm{Ind}_{J_{\overline{H}}}^{G_{\overline{H}}}(\mathrm{Ar}_{J_{\overline{H},1}}) = \mathrm{Ind}_{J_{\overline{H}}}^{G_{\overline{H}}}(\mathrm{Ar}_{J_{\overline{H},2}}).$$

Moreover, similar arguments to the arguments given in the proof of [T2, Theorem 2.7] imply that

$$\mathrm{Ar}_{J_{\overline{H},1}} = \mathrm{Ar}_{J_{\overline{H},2}}.$$

This completes the proof of the proposition. □

In [T3, Section 6], Tamagawa posed two conjectures concerning Hopfian and weakly Hopfian properties of fundamental groups of curves over algebraically closed fields of characteristic $p > 0$ as follows:

**Conjecture 3.5.** *Suppose that $l$ is an algebraically closed field, and that $U$ is affine. Write $\overline{\mathbb{F}}_p$ for the algebraic closure of $\mathbb{F}_p$ in $l$, and $\mathrm{td}(l)$ for the transcendence degree of $l$ over $\overline{\mathbb{F}}_p$. Then $\Pi_U$ is Hopfian if and only if $\mathrm{td}(l) < \aleph_0$, where $\aleph_0$ denotes the countable infinite cardinality.*

**Conjecture 3.6.** *Suppose that $l$ is an algebraically closed field, and that $U$ is affine. Write $\overline{\mathbb{F}}_p$ for the algebraic closure of $\mathbb{F}_p$ in $l$, and $\mathrm{td}(l)$ for the transcendence degree of $l$ over $\overline{\mathbb{F}}_p$. Then $\Pi_U$ is weakly Hopfian if and only if $\mathrm{td}(l) < \aleph_0$, where $\aleph_0$ denotes the countable infinite cardinality.*

Tamagawa proved that, if $\Pi_U$ is either Hopfian or weakly Hopfian, then $\mathrm{td}(l) < \aleph_0$ (i.e., the "only if" parts of Conjecture 3.5 and Conjecture 3.6). Moreover, by the definitions of Hopfian and weakly Hopfian, Conjecture 3.5 implies Conjecture 3.6. For the "if" part of Conjecture 3.5 and Conjecture 3.6, no results are known even when $l = \overline{\mathbb{F}}_p$ and $U = \mathbb{A}^1_{\overline{\mathbb{F}}_p}$. In fact, we don't know any examples of affine curves in positive characteristic whose étale fundamental groups are Hopfian even when $l = \mathbb{F}_p$ and $U = \mathbb{A}^1_{\mathbb{F}_p}$. On the other hand, we can prove that, if $U$ is a curve over a finite field $l$, then $\Pi_U$ is weakly Hopfian (see Proposition 3.8 below).

Let $V$ be an arbitrary separated and connected scheme of finite type over $\mathrm{Spec}\, l$, $V^{\mathrm{cpt}}$ a Nagata compactification of $V$ over $l$, and $B$ an effective Cartier divisor on $V^{\mathrm{cpt}}$ whose support is contained in $V^{\mathrm{cpt}} \setminus V$. We denote by $\Pi_V^B$ the étale fundamental group with restricted ramification bounded by $B$ (cf. [H, Definition 2.4]). Then we have the following proposition.

**Proposition 3.7.** *Suppose that $l$ is a finite field. Then $\Pi_V^B$ is Hopfian.*

*Proof.* Let $\Phi^B \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_V^B, \Pi_V^B)$ be an arbitrary surjection. Let $G$ be an arbitrary finite group and $\delta : \Pi_V^B \twoheadrightarrow G$ an arbitrary surjection. To verify the proposition, we only need to prove that there exists a surjection $\gamma : \Pi_V^B \twoheadrightarrow G$ such that $\delta = \gamma \circ \Phi$. We set

$$S_{\#G} \stackrel{\mathrm{def}}{=} \{H \subseteq \Pi_V^B \mid \#(\Pi_V^B/H) \leq \#G\}.$$

Note that all the étale coverings induced by $H$ is of ramification bounded by $B$ (cf. [H, Definition 2.2]). Moreover, by [H, Theorem 1.2], $S_{\#G}$ is a finite set. This means that $\Pi_V^B$ is small (cf. [H, Definition 3.1]). Then the proposition follows from [FJ, Proposition 16.10.6]. □

**Proposition 3.8.** *Suppose that $l$ is a finite field. Then $\Pi_V$ is weakly Hopfian.*

*Proof.* Let $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_V, \Pi_V)$ be an arbitrary surjection such that the surjection $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_V, \Delta_V)$ induced by $\Phi$ induces surjections of inertia subgroups and higher ramification subgroups. Let $G'$ be an arbitrary finite group, $\delta' : \Pi_V \twoheadrightarrow G'$ a surjection, and $V_{\delta'} \to V$ the Galois étale covering of degree $\#G'$ induced by $\delta'$. To verify the proposition, we only need to prove that there exists a surjection $\gamma' : \Pi_V \twoheadrightarrow G'$ such that $\delta' = \gamma' \circ \Phi$. We see that there exists an effective divisor $B' \overset{\mathrm{def}}{=} \sum_{v' \in V^{\mathrm{cpt}} \setminus V} m_{v'} v'$ on $V^{\mathrm{cpt}}$ which satisfies that the Galois étale covering $V_{\delta'} \to V$ is of ramification bounded by $B'$. Then there exist surjective morphisms $\nu : \Pi_V \twoheadrightarrow \Pi_V^{B'}$ and $\delta_{B'} : \Pi_V^{B'} \twoheadrightarrow G'$ such that $\delta' = \delta_{B'} \circ \nu$, where $\Pi_V^{B'}$ denotes the étale fundamental group with restricted ramification bounded by $B'$.

Moreover, since $\overline{\Phi}$ induces surjections of inertia subgroups and higher ramification subgroups, we obtain a surjection $\Phi^{B'} : \Pi_V^{B'} \twoheadrightarrow \Pi_V^{B'}$ induced by $\Phi$. Then we have the following commutative diagram

$$\begin{array}{ccc} \Pi_V & \xrightarrow{\Phi} & \Pi_V \\ {\scriptstyle \nu}\downarrow & & {\scriptstyle \nu}\downarrow \\ \Pi_V^{B'} & \xrightarrow{\Phi^{B'}} & \Pi_V^{B'} \\ {\scriptstyle \delta_{B'}}\downarrow & & \\ G' & & . \end{array}$$

Since $\Pi_V^{B'}$ is Hopfian (cf. Proposition 3.7), $\Phi^{B'}$ is an isomorphism. Thus, we may define

$$\gamma' \overset{\mathrm{def}}{=} \delta_{B'} \circ (\Phi^{B'})^{-1} \circ \nu.$$

This completes the proof of the proposition. □

The main theorem of the present section is as follows.

**Theorem 3.9.** *Let $\Phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_U, \Pi_U)$ be a surjection. Suppose that the surjection $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_U, \Delta_U)$ induced by $\Phi$ satisfies* ($\Sigma$-gnc). *Then $\Phi$ is an isomorphism.*

*Proof.* The theorem follows from Proposition 3.4 and Proposition 3.8. □

# 4 Group-theoretic characterizations of almost open immersions

We maintain the notation introduced in Section 1 and Section 2.

**Definition 4.1.** Let $f \in \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$ be a separable $k$-morphism. We shall say that $f : X_1 \to X_2$ is **separable $\Sigma$-almost open immersion** if $f$ is a composition of an open immersion and a finite étale morphism such that the Galois group of the Galois closure of the finite étale morphism is a finite quotient of $\Pi_{X_2}$. Note that the open immersion and the finite étale morphism are unique.

Suppose that $\mathrm{char}(k) = p$. Let $\phi \in \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$. We shall say that $\phi : X_1 \to X_2$ is a **$\Sigma$-almost open immersion** if $\phi$ can be represented by the following $k$-morphisms

$$X_1 \cong_k Y(m_1) \leftarrow Y \to Y(m_2) \to X_2$$

such that $Y(m_2) \to X_2$ is a separable $\Sigma$-almost open immersion, where $Y(m_1)$ and $Y(m_2)$ denote the $m_1^{\mathrm{th}}$-Frobenius twist and $m_2^{\mathrm{th}}$-Frobenius twist of $Y$, respectively, and $\cong_k$ is a $k$-isomorphism.

**Remark 4.1.1.** Let $f : X_1 \to X_2$ be a separable morphism over $k$, $K_{X_i}$, $i \in \{1, 2\}$, the function field of $X_i$, and $X_2^{\mathrm{sep}}$ the normalization of $X_2$ in $K_{X_1}$. Then $f$ is a separable $\Sigma$-almost open immersion if and only if the natural finite morphism of $X_2^{\mathrm{sep}} \to X_2$ is étale such that the Galois closure of $X_2^{\mathrm{sep}} \to X_2$ is a finite quotient of $\Pi_{X_2}$, and the natural morphism $X_1 \to X_2^{\mathrm{sep}}$ induced by $f$ is an open immersion. On the other hand, if $X_1$ and $X_2$ are projective, then $f$ is a separable $\Sigma$-almost open immersion if and only if $f$ is a finite étale morphism.

We define
$$\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$$
to be the set of separable $\Sigma$-almost open immersions if $\mathrm{char}(k) = 0$ and
$$\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \subseteq \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$$
if $\mathrm{char}(k) = p$ to be the set of $\Sigma$-almost open immersions. On the other hand, we put
$$\mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \overset{\mathrm{def}}{=} \{\Phi \in \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \mid \overline{\Phi} \text{ satisfies } (\Sigma\text{-gnc})\},$$
where $\overline{\Phi} \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Delta_{X_1}, \Delta_{X_2})$ denotes the morphism induced by $\Phi$. Note that, by Proposition 1.2, $\mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})$ is a purely group-theoretic set. The natural maps
$$\mathrm{Hom}\text{-}\pi_1^{\Sigma} : \mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$
if $\mathrm{char}(k) = 0$ and
$$\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma} : \mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$
if $\mathrm{char}(k) = p$ induce the following natural maps:
$$\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$
and
$$\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$
which fit into the following commutative diagrams:
$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}\text{-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}}} & \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}\text{-}\pi_1^{\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Isom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Isom}\text{-}\pi_1^{\Sigma}} & \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}} & \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2) & \xrightarrow{\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma}} & \mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2}),
\end{array}$$

respectively, where all the vertical arrows are injections. Next, let us start to prove our main theorems.

**Theorem 4.2.** *Suppose that* $\Sigma \neq \mathfrak{Primes}$ *when* $\mathrm{char}(k) = p$. *Then the natural maps*

$$\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{C}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = 0$ *and*

$$\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*if* $\mathrm{char}(k) = p$ *are bijections.*

*Proof.* Frist, let us prove the theorem when $\mathrm{char}(k) = p$. Let us prove that $\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}$ is a surjection. Let

$$\Phi \in \mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2}).$$

To verify the surjectivity, it is sufficient to prove that the image of $\Phi$ in

$$\mathrm{Hom}^{\mathrm{open},\Sigma\text{-gnc}}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

is induced by an almost immersion of $X_1$ and $X_2$. Moreover, $\Phi$ is a composite of an open surjection and an open injection. Since any open injection is induced by a finite étale covering of $X_2$, to verify the surjectivity, we may assume that $\Phi$ is a surjection. Note that ($\Sigma$-gnc) implies that $g \overset{\mathrm{def}}{=} g_{X_1} = g_{X_2}$.

Let $\Sigma_3 \subseteq \mathfrak{Primes}$ be a finite set which contains $\Sigma_1$ (resp. $\Sigma_3 \overset{\mathrm{def}}{=} \{p\}$ if $\Sigma = \mathrm{tame}$) and $\Lambda \overset{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma_3$. Then, for each $i \in \{1, 2\}$, we obtain a surjection

$$\Delta_{X_i} \twoheadrightarrow \Delta^{\Lambda}_{X_i},$$

where $\Delta^{\Lambda}_{X_i}$ denotes the maximal pro-$\Lambda$ quotient of $\Delta_{X_i}$. We denote by

$$\Pi^{\Lambda}_{X_i} \overset{\mathrm{def}}{=} \Pi_{X_i}/(\mathrm{Ker}(\Delta_{X_i} \twoheadrightarrow \Delta^{\Lambda}_{X_i}))$$

for each $i \in \{1, 2\}$. Then the surjection $\Phi$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^{\Lambda}_{X_1} & \longrightarrow & \Pi^{\Lambda}_{X_1} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \overline{\Phi}^{\Lambda}} & & \downarrow{\scriptstyle \Phi^{\Lambda}} & & \| & & \\
1 & \longrightarrow & \Delta^{\Lambda}_{X_2} & \longrightarrow & \Pi^{\Lambda}_{X_2} & \longrightarrow & G_k & \longrightarrow & 1.
\end{array}$$

We define a pointed smooth curve over $\overline{k}$ to be

$$\overline{X}_1^{*,\bullet} \overset{\text{def}}{=} (\overline{X}_1^{\text{cpt}}, D_{\overline{X}_1^*} \overset{\text{def}}{=} \lambda_{\overline{\Phi}}(D_{\overline{X}_2})).$$

Let $X_1^{\text{cpt}}$ be the smooth compactification of $X_1$ over $k$ and $D_{X_1^*}$ the image of $D_{\overline{X}_1^*}$ in $X_1^{\text{cpt}}$. We put

$$X_1^* \overset{\text{def}}{=} X_1^{\text{cpt}} \setminus D_{X_1^*}.$$

Note that $X_1^*$ is a hyperbolic curve of type $(g, n_{X_2})$ over $k$. Write $\Delta_{X_1^*}$ for the maximal pro-$\Sigma$ quotient of $\pi_1(X_1^* \times_k \overline{k})$, $\Delta^{\Lambda}_{X_1^*}$ for the maximal pro-$\Lambda$ quotient of $\Delta_{X_1^*}$, $\Pi_{X_1^*}$ for $\pi_1(X_1^*)/(\text{Ker}(\pi_1(X_1^* \times_k \overline{k}) \twoheadrightarrow \Delta_{X_1^*})$, and $\Pi^{\Lambda}_{X_1^*}$ for $\Pi_{X_1^*}/(\text{Ker}(\Delta_{X_1^*}) \twoheadrightarrow \Delta^{\Lambda}_{X_1^*})$. Since $X_1$ is an open subcurve of $X_1^*$, we have a natural surjection $\Pi_{X_1} \to \Pi_{X_1^*}$.

Write $D_{\overline{X}_1 \setminus \overline{X}_2}$ for $D_{\overline{X}_1} \setminus \lambda_{\overline{\Phi}}(D_{\overline{X}_2})$, and write $E \subseteq \Delta_{X_1}$ for the subgroup generated by the inertia subgroups of $\Delta_{X_1}$ associated the inverse images of the elements of $D_{\overline{X}_1 \setminus \overline{X}_2}$ in $D_{\overline{X}_1^{\Sigma}}$. Then the kernels of $\Pi_{X_1} \to \Pi_{X_1^*}$ and $\Delta_{X_1} \twoheadrightarrow \Delta_{X_1^*}$ are equal to $E$. Moreover, Theorem 2.6 (i) implies that $\Phi$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_1} & \longrightarrow & \Pi_{X_1} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_{X_1^*} & \longrightarrow & \Pi_{X_1^*} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \overline{\Phi}^* \downarrow & & \downarrow \Phi^* & & \| & & \\
1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_{X_2} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array}$$

where all the vertical arrows are surjections. Thus, to verify the surjectivity of $\text{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}$, it is sufficient to prove that the image of $\Phi^*$ in $\text{Hom}^{\text{open}}_{G_k}(\Pi_{X_1^*}, \Pi_{X_2})/\text{Inn}(\Delta_{X_2})$ is induced by an element of $\text{Isom}_{\mathcal{FC}_k}(X_1^*, X_2)$.

On the other hand, $\Phi^*$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^{\Lambda}_{X_1^*} & \longrightarrow & \Pi^{\Lambda}_{X_1^*} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \overline{\Phi}^{*,\Lambda} \downarrow & & \Phi^{*,\Lambda} \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^{\Lambda}_{X_2} & \longrightarrow & \Pi^{\Lambda}_{X_2} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array}$$

where all the vertical arrows are surjections. Since $X_1^*$ and $X_2$ are hyperbolic curves of type $(g, n_{X_2})$, and $p \notin \Lambda$, we obtain that $\overline{\Phi}^{*,\Lambda}$ is an isomorphism. Thus, $\Phi^{*,\Lambda}$ is also an isomorphism. Then Theorem 1.1 implies that the image of $\Phi^{*,\Lambda}$ in $\text{Isom}_{G_k}(\Pi^{\Lambda}_{X_1^*}, \Pi^{\Lambda}_{X_2})/\text{Inn}(\Delta^{\Lambda}_{X_2})$ is induced by an isomorphism of $\text{Isom}_{\mathcal{FC}_k}(X_1^*, X_2)$. Since $\Delta_{X_1}$ and $\Delta_{X_2}$ are topologically finitely generated, the surjection $\overline{\Phi}$ is an isomorphism. Then $\Phi^*$ is an isomorphism. Again, by applying Theorem 1.1, we obtain that the image of $\Phi^*$ in

$$\text{Isom}_{G_k}(\Pi_{X_1^*}, \Pi_{X_2})/\text{Inn}(\Delta_{X_2})$$

is induced by an isomorphism of $\text{Isom}_{\mathcal{FC}_k}(X_1^*, X_2)$.

Next, let us prove $\text{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}$ is an injection. Let

$$\phi_1, \phi_2 \in \text{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2)$$

such that $[\Phi'] \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}(\phi_1) = \mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}(\phi_2)$. We may assume that $\phi_1$ and $\phi_2$ are separable $k$-morphisms. Note that, if $\phi_1$ and $\phi_2$ are finite étale morphisms, then we obtain immediately $\phi_1 = \phi_2$. Since $\phi_1$ and $\phi_2$ are compositions of a unique open immersion and a unique finite étale morphism, to verify the injectivity, we may assume that $\phi_1$ and $\phi_2$ are open immersions. Let $\Phi' \in \mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})$ such that the image of $\Phi$ in $\mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ is $[\Phi']$. Then the kernel of $\Phi'$ is generated by the inertia subgroups associated to the inverse images of the elements of $D_{\overline{X}_1 \setminus \overline{X}_2}$ in $D_{\overline{X}_1^{\Sigma}}$. Then the injectivity follows immediately from [T4, Lemma 5.1] or [M5, Proposition 1.2].

On the other hand, suppose that $\mathrm{char}(k) = 0$. By replacing $\Lambda$ by $\Sigma$, similar arguments to the arguments given in the proof of the case where $k$ is a finite field imply that $\mathrm{Hom}\text{-}\pi_1^{\Sigma\text{-gnc}}$ is a bijection. This completes the proof of the theorem. □

Finally, we treat the case where $\mathrm{char}(k) = p$ and $\Sigma = \mathfrak{Primes}$.

**Theorem 4.3.** *Suppose that* $\mathrm{char}(k) = p$ *and* $\Sigma = \mathfrak{Primes}$*. Then the natural map*

$$\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}} : \mathrm{Hom}_{\mathcal{FC}_k}^{\Sigma\text{-al-op-im}}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

*is a bijection.*

*Proof.* First, let us prove that $\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}$ is a surjection. Let

$$\Phi \in \mathrm{Hom}_{G_k}^{\text{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2}).$$

To verify the theorem, we may assume that $\Phi$ is a surjection. Write $\overline{\Phi} : \Delta_{X_1} \twoheadrightarrow \Delta_{X_2}$ for the surjection induced by $\Phi$. We have the following claim.

**Claim:** The surjection $\overline{\Phi}$ satisfies ($\Sigma$-prc).

Let us prove the claim. For each $\overline{H}_2 \subseteq \Delta_{X_2}$ open normal subgroup, let $H_2 \subseteq \Pi_{X_2}$ be an open normal subgroup that $H_2 \cap \Delta_{X_2} = \overline{H}_2$. Write $G_{k'}$ for the image of $H_2$ in $G_k$, $H_1$ for $\Phi^{-1}(H_2)$, and $X_{H_1}$ and $X_{H_2}$ for the curve corresponding to $H_1$ and $H_2$, respectively. Note that $H_1 \cap \Delta_{X_1} = \overline{\Phi}^{-1}(\overline{H}_2)$, and $\overline{\Phi}|_{H_1} : H_1 \twoheadrightarrow H_2$ satisfies ($\Sigma$-gnc).

Let $\Omega \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{p\}$. Write $\overline{H}_1^{\Omega}$ and $\overline{H}_2^{\Omega}$ for the maximal pro-$\Omega$ quotients of $\overline{H}_1$ and $\overline{H}_2$, respectively. We denote by $\Pi_{X_{H_1}} \stackrel{\text{def}}{=} H_1/\mathrm{Ker}(\overline{H}_1 \twoheadrightarrow H_1^{\Omega})$ and denote by $\Pi_{X_{H_2}} \stackrel{\text{def}}{=} H_2/\mathrm{Ker}(\overline{H}_2 \twoheadrightarrow H_2^{\Omega})$. Then $\Phi$ induces a surjection $\Phi_{X_H} : \Pi_{X_{H_1}} \twoheadrightarrow \Pi_{X_{H_2}}$. Moreover, we have $\Phi_{X_H} \in \mathrm{Hom}_{G_{k'}}^{\text{open},\Omega\text{-gnc}}(\Pi_{X_{H_1}}, \Pi_{X_{H_2}})$. Thus, Theorem 4.2 implies that $X_{H_1}$ is isomorphic to an open subset of $X_{H_2}$ as schemes. Then we obtain that the $p$-rank of $X_{H_1}$ is equal to the $p$-rank of $X_{H_2}$. This completes the proof of the claim.

We define a pointed smooth curve over $\overline{k}$ to be

$$\overline{X}_1^{*,\bullet} \stackrel{\text{def}}{=} (\overline{X}_1^{\text{cpt}}, D_{\overline{X}_1^*} \stackrel{\text{def}}{=} \lambda_{\overline{\Phi}}(D_{\overline{X}_2})).$$

Let $X_1^{\mathrm{cpt}}$ be the smooth compactification of $X_1$ over $k$ and $D_{X_1^*}$ the image of $D_{\overline{X}_1^*}$ in $X_1^{\mathrm{cpt}}$. We set

$$X_1^* \stackrel{\mathrm{def}}{=} X_1^{\mathrm{cpt}} \setminus D_{X_1^*}.$$

Note that $X_1^*$ is a hyperbolic curve of type $(g, n_{X_2})$ over $k$. Write $\Pi_{X_1^*}$ for the étale fundamental group $\pi_1(X_1^*)$ and $\Delta_{X_1^*}$ for the geometric étale fundamental group $\pi_1(X_1^* \times_k \overline{k})$. Since $X_1$ is an open subcurve of $X_1^*$, we have a natural surjection $\Pi_{X_1} \to \Pi_{X_1^*}$.

Write $D_{\overline{X}_1 \setminus \overline{X}_2}$ for $D_{\overline{X}_1} \setminus \lambda_{\overline{\Phi}}(D_{\overline{X}_2})$, and write $E \subseteq \Delta_{X_2}$ for the subgroup generated by the inertia subgroups of $\Delta_{X_2}$ associated the inverse images of the elements of $D_{\overline{X}_1 \setminus \overline{X}_2}$ in $D_{\overline{X}_1^{\Sigma}}$. Then the kernels of $\Pi_{X_1} \to \Pi_{X_1^*}$ and $\Delta_{X_1} \twoheadrightarrow \Delta_{X_1^*}$ are equal to $E$. Moreover, Theorem 2.6 (ii) implies that $\Phi$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{X_1} & \longrightarrow & \Pi_{X_1} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_{X_1^*} & \longrightarrow & \Pi_{X_1^*} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \overline{\Phi}^* \downarrow & & \downarrow \Phi^* & & \| & & \\
1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_{X_2} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array}$$

where all the vertical arrows are surjections. Thus, to verify the surjectivity of $\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}$, it is sufficient to prove that the image of $\Phi^*$ in $\mathrm{Hom}^{\mathrm{open}}_{G_k}(\Pi_{X_1^*}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ is induced by an element of $\mathrm{Isom}_{\mathcal{FC}_k}(X_1^*, X_2)$. Note that $\overline{\Phi}^*$ satisfies ($\Sigma$-gnc).

Let $\Sigma_3 \subseteq \mathfrak{Primes}$ be a finite set which contains $\{p\}$ and $\Lambda \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma_3$. We write $\Delta^{\Lambda}_{X_1^*}$ and $\Delta^{\Lambda}_{X_2}$ for the maximal pro-$\Lambda$ quotients of $\Delta_{X_1^*}$ and $\Delta_{X_2}$, $\Pi^{\Lambda}_{X_1^*}$ and $\Pi^{\Lambda}_{X_2}$ for $\Pi_{X_1^*}/\mathrm{Ker}(\Delta_{X_1^*} \twoheadrightarrow \Delta^{\Lambda}_{X_1^*})$ and $\Pi_{X_2}/\mathrm{Ker}(\Delta_{X_2} \twoheadrightarrow \Delta^{\Lambda}_{X_2})$, respectively. We obtain that $\Phi^*$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^{\Lambda}_{X_1^*} & \longrightarrow & \Pi^{\Lambda}_{X_1^*} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \overline{\Phi}^{*,\Lambda} \downarrow & & \Phi^{*,\Lambda} \downarrow & & \| & & \\
1 & \longrightarrow & \Delta^{\Lambda}_{X_2} & \longrightarrow & \Pi^{\Lambda}_{X_2} & \longrightarrow & G_k & \longrightarrow & 1,
\end{array}$$

where all the vertical arrows are surjections. Then Theorem 4.2 implies that $X_1^*$ is isomorphic to $X_2$ in $\mathcal{FC}_k$. We obtain that $\Pi_{X_1^*}$ is isomorphic to $\Pi_{X_2}$ as abstract profinite groups. Thus, by Theorem 3.9, we obtain that

$$\Phi^* \in \mathrm{Isom}_{G_k}(\Pi_{X_1^*}, \Pi_{X_2}).$$

Moreover, Theorem 1.1 implies that the image of $\Phi^*$ in $\mathrm{Isom}_{G_k}(\Pi_{X_1^*}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ is induced by an isomorphism of $\mathrm{Isom}_{\mathcal{FC}_k}(X_1^*, X_2)$.

Next, let us prove $\mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}$ is an injection. Let

$$\phi_1, \phi_2 \in \mathrm{Hom}^{\Sigma\text{-al-op-im}}_{\mathcal{FC}_k}(X_1, X_2)$$

such that $[\Phi'] \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}(\phi_1) = \mathrm{Hom}_{\mathcal{FC}_k}\text{-}\pi_1^{\Sigma\text{-gnc}}(\phi_2)$. We may assume that $\phi_1$ and $\phi_2$ are separable $k$-morphisms. Note that, if $\phi_1$ and $\phi_2$ are finite étale morphisms, then we obtain immediately $\phi_1 = \phi_2$. Since $\phi_1$ and $\phi_2$ are compositions of a unique open immersion and a unique finite étale morphism, to verify the injectivity, we may assume that $\phi_1$ and $\phi_2$ are open immersions. Let $\Phi' \in \mathrm{Hom}_{G_k}^{\mathrm{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})$ such that the image of $\Phi$ in $\mathrm{Hom}_{G_k}^{\mathrm{open},\Sigma\text{-gnc}}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$ is $[\Phi']$. Then the kernel of $\Phi'$ is generated by the inertia subgroups associated to the inverse images of the elements of $D_{\overline{X}_1 \setminus \overline{X}_2}$ in $D_{\overline{X}_1^{\Sigma}}$. Then the injectivity follows immediately from [T2, Corollary 2.2]. This completes the proof of the theorem. □

**Remark 4.3.1.** Theorem 4.2 and Theorem 4.3 can be regarded as a certain Hom-version of the Grothendieck conjecture for almost open immersion of curves.

**Remark 4.3.2.** Finally, let us come back to Hom-version of positive characteristic. Note that, for any $\phi$ which is either an element of $\mathrm{Hom}_{\mathcal{C}_k}(X_1, X_2)$ or an element of $\mathrm{Hom}_{\mathcal{FC}_k}(X_1, X_2)$, there exists an open sub-curve $U_i \subseteq X_i$, $i \in \{1, 2\}$ such that the restriction of $\phi$ on $U_1$ is an almost open immersion. Write $\Delta_{U_i}$ for the maximal pro-$\Sigma$ quotient of the geometric tame fundamental group $\pi_1^{\mathrm{t}}(U_i \times_k \overline{k})$, $\Pi_{U_i}$ for $\pi^{\mathrm{t}}(U_i)/(\mathrm{Ker}(\pi^{\mathrm{t}}(U_i) \twoheadrightarrow \Delta_{U_i}))$. Let $\Phi$ be an arbitrary element of $\mathrm{Hom}_{G_k}^{\mathrm{open}}(\Pi_{X_1}, \Pi_{X_2})$. If one can develop a suitable theory of anabelian cuspidalizations for surjections (i.e., group-theoretic reconstructions of the fundamental groups of open sub-curves of given curves from the fundamental group of the given curves; moreover, the cases of abelian and pro-$\ell$ cuspidalizations for isomorphisms have already been established by Mochizuki (cf. [M4])), then one may obtain a homomorphism $\Phi^{\mathrm{cusp}} : \Pi_{U_1'} \to \Pi_{U_2'}$ group-theoretically from $\Phi$ such that the condition ($\Sigma$-gnc) is satisfied. Here, $U_i'$, $i \in \{1, 2\}$, is an open sub-curve of $X_i$, and $\Pi_{U_i'}$ is $\pi_1(U_i')/(\mathrm{Ker}(\pi_1(U_i' \times_k \overline{k}) \twoheadrightarrow \Delta_{U_i'}))$, where $\Delta_{U_i'}$ denotes the maximal pro-$\Sigma$ quotient of the geometric étale fundamental group $\pi_1(U_i' \times_k \overline{k})$. Then Hom-version of positive characteristic can be deduced from Theorem 4.2 and Theorem 4.3.

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