GENERALIZED HASSE-WITT INVARIANTS FOR COVERINGS WITH PRESCRIBED RAMIFICATIONS

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Abstract. Let $X^\bullet = (X, D_X)$ be a pointed stable curve over an algebraically closed field of characteristic $p > 0$ and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. In the present paper, we prove that the generalized Hasse-Witt invariants of prime-to-p cyclic admissible coverings of $X^\bullet$ with certain prescribed ramifications can attain maximum. As an application, we prove that the field structures associated to inertia subgroups of marked points of $X^\bullet$ can be reconstructed group-theoretically from certain finite quotients of $\Pi_{X^\bullet}$.

Keywords: pointed stable curve, admissible covering, admissible fundamental group, generalized Hasse-Witt invariant, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14G32.

Contents

1. Introduction

Let $X^\bullet = (X, D_X)$ be a pointed stable curve of (topological) type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, where X denotes the underlying curve with genus g_X, and D_X denotes the (finite) set of marked points with cardinality $n_X \overset{\text{def}}{=} \#(D_X)$. By choosing a suitable base point of $X^\bullet$, we have the admissible fundamental group $\Pi_{X^\bullet}$ (2.2.2) of $X^\bullet$. The admissible fundamental groups of pointed stable curves are natural generalizations of the tame fundamental groups of smooth pointed stable curves.

In particular, $\Pi_{X^\bullet}$ is isomorphic to the tame fundamental group of $X^\bullet$ if $X^\bullet$ is smooth over k.

1.1. **Fundamental groups of curves in positive characteristic.**

1.1.1. Write $\Pi^{p'}_{X^\bullet}$ for the maximal prime-to-p quotient of $\Pi_{X^\bullet}$. Then $\Pi^{p'}_{X^\bullet}$ can be determined by (g_X, n_X), and it is isomorphic to the prime-to-p completion of the topological fundamental group of a Riemann surface of type (g_X, n_X) (2.2.2). However, the full admissible fundamental group $\Pi_{X^\bullet}$ is very mysterious, and its structure is no longer known. In fact, since the 1990s, some developments of F. Pop-M. Saïdi ([PS]), M. Raynaud ([R2]), A. Tamagawa ([T1], [T2], [T3]), and the author of the present paper ([Y1], [Y4]) showed that there exist *anabelian phenomena* for curves over algebraically closed fields of characteristic p. This means that the isomorphism class of $X^\bullet$ as a scheme can be completely determined by the isomorphism class of $\Pi_{X^\bullet}$ as a profinite group. Furthermore, by the theory developed in [T2] and [Y1] (e.g. see [Y1, Remark 1.2.2]), we can expect that the maximal pro-solvable quotient $\Pi^{\mathrm{sol}}_{X^\bullet}$ of $\Pi_{X^\bullet}$ is sufficiently to determine the isomorphism class of $X^\bullet$ as a scheme. Moreover, since $\Pi_{X^\bullet}$ is topologically finitely generated, the isomorphism class of $\Pi_{X^\bullet}$ is completely determined by the set of finite quotients of $\Pi_{X^\bullet}$ ([FJ, Proposition 16.10.6]). Then to understand the anabelian phenomena of curves in positive characteristic, we may ask the following question: *Which finite solvable groups can appear as quotients of* $\Pi_{X^\bullet}$*?*

1.1.2. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup and $X^\bullet_H = (X_H, D_{X_H})$ the pointed stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H. We have an important invariant σ_{X_H} associated to $X^\bullet_H$ (or H) which is called *p-rank* (or *Hasse-Witt invariant*). Roughly speaking, σ_{X_H} controls the finite quotients of $\Pi_{X^\bullet}$ which are extensions of the group $\Pi_{X^\bullet}/H$ by p-groups.

Since the structures of maximal prime-to-p quotients of admissible fundamental groups are known, to find all the solvable quotients of $\Pi_{X^\bullet}$, we need to compute the p-rank σ_{X_H} when $\Pi_{X^\bullet}/H$ is *abelian*. If $\Pi_{X^\bullet}/H$ is a p-group, then σ_{X_H} can be computed by using the Deuring-Shafarevich formula ([C], [Su]). If $\Pi_{X^\bullet}/H$ is not a p-group, the situation of σ_{X_H} is very complicated. Moreover, the Deuring-Shafarevich formula implies that, to compute σ_{X_H}, we only need to assume that $\Pi_{X^\bullet}/H$ is a prime-to-p group (i.e., the order of $\Pi_{X^\bullet}/H$ is prime to p). In this situation, we have the so-called *generalized Hasse-Witt invariants* associated to (prime-to-p) cyclic admissible coverings of $X^\bullet$ (2.3.2) which are defined as the dimensions of canonical decomposition of $H^{p,\mathrm{ab}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p (\overset{\sim}{\to} H^1_{\text{ét}}(X_H, \overline{\mathbb{F}}_p)^\vee)$ under the natural actions of $\Pi_{X^\bullet}/H$, and which are refined invariants of p-rank, where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$, and $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$.

1.2. **The first main result.**

1.2.1. *Previous results of Raynaud, Tamagawa, and the author of the present paper.* We fix some notation. Let n be a positive natural number prime to p, and let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ and $D_{f^\bullet}$ the ramification divisor induced by $f^\bullet$ (Definition 2.4 (i)). Note that $D_{f^\bullet}$ is an effective divisor on X whose support is contained in D_X, and whose degree $\deg(D_{f^\bullet})$ is divisible by n such that $\deg(D_{f^\bullet}) = 0$ if $n_X = 0$ and $0 \leq \deg(D_{f^\bullet}) \leq (n_X - 1)n$ if $n_X \neq 0$. We put $s(D_{f^\bullet}) \overset{\mathrm{def}}{=} \deg(D_{f^\bullet})/n$.

Suppose that $X^\bullet$ is *smooth* over k, and that $n_X = 0$ (i.e., $D_X = \emptyset$). Raynaud ([R1]) developed his theory of theta divisors and proved that, if $n >> 0$ is a natural number prime to p, then there *exists* a Galois *étale* covering $f^\bullet$ of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ (i.e., $s(D_{f^\bullet}) = 0$) whose "first" generalized Hasse-Witt invariant (2.3.2) is as large as possible, namely equal to $g_X - 1$ ([R1, Théorème 4.3.1]). Moreover, as a consequence, Raynaud obtained that $\Pi_{X^\bullet}$ is not a prime-to-p profinite group. This is the first deep result concerning the global structures of étale fundamental groups of curves over algebraically closed fields of characteristic p.

Suppose that $X^\bullet$ is *smooth* over k, and that $n_X \geq 0$. The computations of generalized Hasse-invariants of admissible coverings of $X^\bullet$ (i.e., tame coverings of $X^\bullet$) are much more difficult than the case of $n_X = 0$. Tamagawa observed that Raynaud's theory of theta divisors can be generalized to the case of tame coverings, and established a tamely ramified version of the theory of Raynaud's theta divisors. By applying the theory of theta divisors, Tamagawa ([T2]) proved that, if $n_X \geq 2$ and $n >> 0$ is a natural number prime to p, then there *exists* a Galois admissible covering (i.e., Galois tame covering) $f^\bullet$ of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that $\deg(D_{f^\bullet}) = n$ (i.e., $s(D_{f^\bullet}) = 1$), and that the "first" generalized Hasse-Witt invariant of $f^\bullet$ is as large as possible, namely equal to g_X. Note that since all abelian tame coverings of $X^\bullet$ are étale if $n_X \leq 1$, the calculations of generalized Hasse-Witt invariants can be deduced from Raynaud's result mentioned above if $n_X \leq 1$. As an application, Tamagawa obtained that the type (g_X, n_X) can be reconstructed group-theoretically from the tame fundamental group $\Pi_{X^\bullet}$ when $X^\bullet$ is smooth over k (i.e., an anabelian formula for (g_X, n_X), see [T2, Theorem 0.1]) which is the most critical step in his proof of Grothendieck's anabelian conjecture for curves over algebraically closed fields of characteristic p ([T2, Theorem 0.2]).

Suppose that $X^\bullet$ is an *arbitrary* pointed stable curve (i.e., possibly singular) over k. In [Y3], the author of the present paper consider the case of $s(D_{f^\bullet}) = n_X - 1$, and by using the theory of Raynaud-Tamagawa theta divisors, we proved ([Y3, Theorem 1.2]) that, if $n_X \neq 0$ and $n >> 0$ is a natural number prime to p, then there *exists* a Galois admissible covering $f^\bullet$ of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that $\deg(D_{f^\bullet}) = (n_X - 1)n$, and that the "first" generalized Hasse-Witt invariant of $f^\bullet$ is as large as possible, namely equal to $g_X + s(D_{f^\bullet}) - 1 = g_X + n_X - 2$. As an application, we obtained that the type (g_X, n_X) can be reconstructed group-theoretically from the admissible fundamental group $\Pi_{X^\bullet}$ when $X^\bullet$ is an arbitrary pointed stable curve over k ([Y3, Theorem 1.3]). On the other hand, this result is one of main tools to establish the theory of *moduli spaces of admissible fundamental groups* by the author in [Y5] (a theory which gives a general framework for describing the anabelian phenomena of curves over algebraically closed fields of characteristic p).

1.2.2. In the present paper, we study genrealized Hasse-Witt invariants of prime-to-p cyclic admissible coverings with certain *prescribed ramifications*. Let $m \in \mathbb{N}$ be an arbitrary natural number. We denote by $(\mathbb{Z}/m\mathbb{Z})^\sim \overset{\text{def}}{=} \{0, \dots, m-1\}$. Then there is a natural bijection (as sets) $\mathbb{Z}/m\mathbb{Z} \overset{\sim}{\to} (\mathbb{Z}/m\mathbb{Z})^\sim$. The first main result of the present paper is as follows (see Theorem 3.5 for a more precise statement):

Theorem 1.1. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over k and $D \in \mathbb{Z}[D_X]$ a given effective divisor with degree $\deg(D) = (n_X - 1)n$ satisfying certain given conditions introduced in Condition 3.3. Write $\overline{D}$ for the effective divisor on X induced by D via the natural map $\mathbb{Z}[D_X] \to \mathbb{Z}/n\mathbb{Z}[D_X] \overset{\text{def}}{=} \mathbb{Z}[D_X] \otimes \mathbb{Z}/n\mathbb{Z} \overset{\sim}{\to} (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$, where the second arrow induced by the above bijection $\mathbb{Z}/m\mathbb{Z} \overset{\sim}{\to} (\mathbb{Z}/m\mathbb{Z})^\sim$.*

Suppose that $n >> 0$ is a natural number prime to p. Then there exists a Galois admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that $D_{f^\bullet} = \overline{D}$, and that the "first" generalized Hasse-Witt invariant of $f^\bullet$ is as large as possible, namely equal to $g_X + s(\overline{D}) - 1$.

Theorem 1.1 implies the following corollary (see Corollary 3.6 for a more precise statement):

Corollary 1.2. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over k and $s \in \{0, \dots, n_X - 1\}$ an integer. Suppose that $n >> 0$ is a natural number prime to p. Then there exists an effective divisor D with degree $\deg(D) = (n_X - 1)n$ such that the following hold:*

- *D satisfies certain given conditions introduced in Condition 3.3.*
- *$\deg(\overline{D}) = sn$ (i.e., $s(\overline{D}) = s$), where $\overline{D}$ is the effective divisor on X as defined in the statement of Theorem 1.1.*
- *There exists a Galois admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that $D_{f^\bullet} = \overline{D}$, and that the "first" generalized Hasse-Witt invariant of $f^\bullet$ is as large as possible, namely equal to $g_X + s - 1$.*

In particular, we obtain the results proved by Raynaud and Tamagawa mentioned in the second and the third paragraphs of 1.2.1 if $s \in \{0, 1\}$ and $X^\bullet$ is smooth over k, and obtain the result proved by the author of the present paper mentioned in the fourth paragraph of 1.2.1 if $s = n_X - 1$.

1.3. **The second main result.** Let us explain the second main result of the present paper that motivated the theory developed in the present paper.

1.3.1. We fix some notation. Let $\Gamma_{X^\bullet}$ be the dual semi-graph of $X^\bullet$, and $\Gamma_{\widehat{X}^\bullet}$ the dual semi-graph of the universal admissible covering of $X^\bullet$ corresponding to $\Pi_{X^\bullet}$ (2.3.4). Moreover, we shall suppose $n_X > 0$. Let e be an open edge of $\Gamma_{X^\bullet}$ (i.e., an edge corresponding to a marked point of D_X, see 2.2.1) and $\widehat{e}$ an arbitrary element of the inverse images of e of the natural surjection $\Gamma_{\widehat{X}^\bullet} \twoheadrightarrow \Gamma_{X^\bullet}$ (2.3.4). Since the dual semi-graph $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of $\Pi_{X^\bullet}$, we put $I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$ the stabilizer subgroup of $\widehat{e}$.

Write $x_e \in D_X$ for the marked point of $X^\bullet$ corresponding to e. Then the general theory of admissible fundamental groups implies $I_{\widehat{e}} \xrightarrow{\sim} \mathrm{Gal}(\widehat{K}^{\mathrm{t}}_{X,x_e}/\widehat{K}_{X,x_e}) \xrightarrow{\sim} \widehat{\mathbb{Z}}(1)^{p'}$, where $\widehat{K}_{X,x_e}$ denotes the quotient field of the completion of the local ring $\mathcal{O}_{X,x_e}$, and $\widehat{K}^{\mathrm{t}}_{X,x_e}$ denotes the maximal tamely ramified extension of $\widehat{K}_{X,x_e}$.

Suppose that $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$ in k. Then we have the following (see 4.2.3 for a more precise explanation): The set

$$\mathbb{F}_{\widehat{e}} \overset{\text{def}}{=} (I_{\widehat{e}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}) \sqcup \{*_{\widehat{e}}\}$$

can be identified with $\overline{\mathbb{F}}_p$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widehat{e}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}$, and whose zero element is $*_{\widehat{e}}$. where $\{*_{\widehat{e}}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with $(\mathbb{Q}/\mathbb{Z})^{p'}(1) \overset{\text{def}}{=} \bigcup_{(p,m)=1} \mu_m(k)$. Moreover, the set

$$\mathbb{F}_{\widehat{e},p^t} \overset{\text{def}}{=} I_{\widehat{e}} \otimes \mathbb{Z}/(p^t - 1)\mathbb{Z} \sqcup \{*_{\widehat{e}}\} \subseteq \mathbb{F}_{\widehat{e}}$$

admits a structure of field induced by $\mathbb{F}_{\widehat{e}}(\xrightarrow{\sim} \overline{\mathbb{F}}_p)$ which is isomorphic to the subfield of $\overline{\mathbb{F}}_p$ with cardinality p^t.

1.3.2. *Reconstructions of field structures in anabelian geometry.* Tamagawa ([T2, Proposition 5.3]) proved that the field structure of $\mathbb{F}_{\widehat{e}}$ defined above *can be reconstructed group-theoretically* from the admissible fundamental group (=tame fundamental group) $\Pi_{X^\bullet}$ if $X^\bullet$ is *smooth* over k. Namely, there exists a group-theoretical algorithm whose input datum is $\Pi_{X^\bullet}$, and whose output datum is the field $\mathbb{F}_{\widehat{e}}$. Furthermore, the author of the present paper extended Tamagawa's result to the case where $X^\bullet$ is a (possibly singular) pointed stable curve over k (see [Y3, Theorem 6.4] and [Y5, Theorem 4.13]). These results play an important role in the theory of anabelian geometry of curves over algebraically closed fields of characteristic p, and are key steps to prove the Grothendieck's anabelian conjecture for certain curves over algebraically closed fields of characteristic p ([T2, Theorem 0.2]) and the homeomorphism conjecture for 1-dimensional moduli spaces ([Y5, Theorem 0.1]).

On the other hand, motivated by the theory of moduli spaces of admissible fundamental groups, the author of the present paper observed ([Y6]) that the anabelian phenomena for curves over algebraically closed fields of characteristic p can be understood by using not only full tame fundamental groups but also certain *finite quotients* of them. More precisely, we obtained a "*finite version*" of Grothendieck's anabelian conjecture for certain curves over algebraically closed fields of characteristic p which is a strong generalization of Tamagawa's result [T2, Theorem 0.2] (namely, the isomorphism classes of curves as schemes can be completely determined by certain *finite quotients* of their tame fundamental groups, see [Y6, Corollary 1.4]). One of main steps in the proof of [Y6, Corollary 1.4] is a "finite version" of [T2, Proposition 5.3] which says that the field structure of $\mathbb{F}_{\widehat{e},p^t}$ can be reconstructed group-theoretically from certain *finite quotients* of $\Pi_{X^\bullet}$ if $X^\bullet$ is smooth over k and $t >> 0$ (see [Y6, Proposition 5.2]). Note that [Y6, Proposition 5.2] implies [T2, Proposition 5.3].

1.3.3. By applying Theorem 1.1, we obtain the second main result of the present paper which generalizes [Y6, Proposition 5.2] to the case of (possibly singular) pointed stable curves (see Theorem 4.2 for a more precise statement):

Theorem 1.3. *We maintain the notation introduced in 1.3.1. Then the field structure of $\mathbb{F}_{\widehat{e},p^t}$ can be reconstructed group-theoretically from certain finite quotients of $\Pi_{X^\bullet}$ if $t >> 0$.*

In [Y7], we will use Theorem 1.3 to study the topological properties of the moduli spaces of admissible fundamental groups and prove a "finite version" of Grothendieck's anabelian conjecture for certain (possibly singular) pointed stable curves over algebraically closed fields of characteristic p. On the other hand, see Remark 4.2.1 for some further explanations about the applications of Theorem 1.3 to anabelian geometry.

1.4. **Structure of the present paper.** The present paper is organized as follows. In §2, we recall some notation and results concerning pointed stable curves and generalized Hasse-Witt invariants. In §3 and §4, we prove the first main result and the second main result, respectively.

1.5. **Acknowledgements.** The author was supported by JSPS Grant-in-Aid for Young Scientists Grant Numbers 20K14283.

2. Preliminaries

In the present section, we recall some notation and results concerning semi-graphs, pointed stable curves, admissible fundamental groups, and generalized Hasse-Witt invariants of cyclic admissible coverings.

2.1. Semi-graphs.

2.1.1. Let $\mathbf{G} \overset{\text{def}}{=} (v(\mathbf{G}), e(\mathbf{G}), \zeta^{\mathbf{G}} : e(\mathbf{G}) \to v(\mathbf{G}) \cup \{v(\mathbf{G})\})$ be a semi-graph. Here, $v(\mathbf{G})$, $e(\mathbf{G})$, and $\zeta^{\mathbf{G}}$ denote the set of vertices of $\mathbf{G}$, the set of edges of $\mathbf{G}$, and the set of coincidence maps of $\mathbf{G}$, respectively. Note that $\{v(\mathbf{G})\}$ is a set with exactly one element.

Let $e \in e(\mathbf{G})$ be an edge. Then $e \overset{\text{def}}{=} \{b_e^1, b_e^2\}$ is a set of cardinality 2. The set $e(\mathbf{G})$ consists of closed edges and open edges defined as follows: If e is a closed edge, then the coincidence map $\zeta^{\mathbf{G}}$ is a map from e to the set of vertices to which e abuts. If e is an open edge, then the coincidence map $\zeta^{\mathbf{G}}$ is a map from e to the set which consists of the vertex to which e abuts and the set $\{v(\mathbf{G})\}$ (i.e., either $\zeta^{\mathbf{G}}(b_e^1)$ or $\zeta^{\mathbf{G}}(b_e^2)$ is not contained in $v(\mathbf{G})$).

We denote by $e^{\mathrm{op}}(\mathbf{G}) \subseteq e(\mathbf{G})$ the set of open edges of $\mathbf{G}$ and $e^{\mathrm{cl}}(\mathbf{G}) \subseteq e(\mathbf{G})$ the set of closed edges of $\mathbf{G}$. Note that we have $e(\mathbf{G}) = e^{\mathrm{op}}(\mathbf{G}) \cup e^{\mathrm{cl}}(\mathbf{G})$. Moreover, we denote by $e^{\mathrm{lp}}(\mathbf{G}) \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\mathbf{G}) \mid \#(\zeta^{\mathbf{G}}(e)) = 1\}$ (i.e., a closed edge which abuts to a unique vertex of $\mathbf{G}$), where "lp" means "loop". For each $e \in e(\mathbf{G})$, we denote by $v^{\mathbf{G}}(e) \subseteq v(\mathbf{G})$ the set of vertices of $\mathbf{G}$ to which e abuts. For each $v \in v(\mathbf{G})$, we denote by $e^{\mathbf{G}}(v) \subseteq e(\mathbf{G})$ the set of edges of $\mathbf{G}$ to which v is abutted.

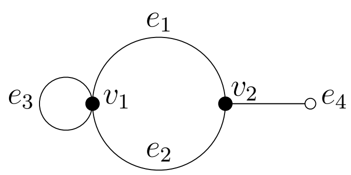
We shall say $\mathbf{G}$ *connected* if $\mathbf{G}$ is connected as a topological space whose topology is induced by the topology of $\mathbb{R}^2$, where $\mathbb{R}$ denotes the real number field. Denote by $r_{\mathbf{G}} \overset{\mathrm{def}}{=} \dim_{\mathbb{Q}}(H^1(\mathbf{G}, \mathbb{R}))$ the Betti number of $\mathbf{G}$. Moreover, we shall call $\mathbf{G}$ a *tree* if $r_{\mathbf{G}} = 0$.

Remark. The motivations of the above notation concerning semi-graphs arise from the dual semi-graphs of pointed stable curves (see 2.2.1 below).

Example 2.1. Let us give an example of semi-graph to explain the above notation. We use the notation "$\bullet$" and "$\circ$ with a line segment" to denote a vertex and an open edge, respectively.

Let $\mathbf{G}$ be a semi-graph as follows:

$\mathbf{G}$:



Then we see $v(\mathbf{G}) = \{v_1, v_2\}$, $e^{\mathrm{cl}}(\mathbf{G}) = \{e_1, e_2, e_3\}$, $e^{\mathrm{op}}(\mathbf{G}) = \{e_4\}$, $\zeta^{\mathbf{G}}(e_1) = \{v_1, v_2\}$, $\zeta^{\mathbf{G}}(e_2) = \{v_1, v_2\}$, $\zeta^{\mathbf{G}}(e_3) = \{v_1\}$, and $\zeta^{\mathbf{G}}(e_4) = \{v_2, \{v(\mathbf{G})\}\}$. Moreover, we have $e^{\mathrm{lp}}(\mathbf{G}) = \{e_3\}$, $v^{\mathbf{G}}(e_1) = \{v_1, v_2\}$, $v^{\mathbf{G}}(e_2) = \{v_1, v_2\}$, $v^{\mathbf{G}}(e_3) = \{v_1\}$, $v^{\mathbf{G}}(e_4) = \{v_2\}$, $e^{\mathbf{G}}(v_1) = \{e_1, e_2, e_3\}$, and $e^{\mathbf{G}}(v_2) = \{e_1, e_2, e_4\}$.

2.1.2. Let $\mathbf{G}'$ be a connected semi-graph. We shall say $\mathbf{G}'$ a *sub-semi-graph* of $\mathbf{G}$ if either $\mathbf{G}' = \{e\}$ for some $e \in e(\mathbf{G})$ or the following conditions hold:

(i) $v(\mathbf{G}') \neq \emptyset$ and $v(\mathbf{G}') \subseteq v(\mathbf{G})$.
(ii) $e^{\mathrm{cl}}(\mathbf{G}') \subseteq e^{\mathrm{cl}}(\mathbf{G})$ is the subset of closed edges of $\mathbf{G}$ such that $v(e) \subseteq v(\mathbf{G}')$.
(iii) $e^{\mathrm{op}}(\mathbf{G}') \subseteq e(\mathbf{G}) \setminus e^{\mathrm{cl}}(\mathbf{G}')$ is the subset of edges of $\mathbf{G}$ such that $\#(v^{\mathbf{G}}(e) \cap v(\mathbf{G}')) = 1$.

Note that the definition of $\mathbf{G}'$ implies that $\mathbf{G}'$ can be completely determined by $v(\mathbf{G}')$ if $v(\mathbf{G}') \neq \emptyset$.

The conditions (ii), (iii) imply that, if $e \in e^{\mathrm{lp}}(\mathbf{G})$ is a loop and $v^{\mathbf{G}}(e) \subseteq v(\mathbf{G}')$, then $e \in e^{\mathrm{cl}}(\mathbf{G}')$. If $\mathbf{G}' = \{e\}$ for some $e \in e(\mathbf{G})$, we will use e to denote $\mathbf{G}'$. Moreover, there exists a natural injection $\mathbf{G}' \hookrightarrow \mathbf{G}$, and $\mathbf{G}'$ can be regarded as *a topological subspace* of $\mathbf{G}$ via this injection.

Suppose that $\mathbf{G}'$ is a sub-semi-graph of $\mathbf{G}$ such that $v(\mathbf{G}') \neq \emptyset$. Let $L \subseteq e^{\mathrm{cl}}(\mathbf{G}')$ be a *subset of closed edges* of $\mathbf{G}'$ such that $\mathbf{G}' \setminus L$ (i.e., removing L from $\mathbf{G}'$) is *connected*. For any $e \overset{\mathrm{def}}{=} \{b_e^1, b_e^2\} \in L$, we put $e^i \overset{\mathrm{def}}{=} \{b_{e^i}^1, b_{e^i}^2\}$, $i \in \{1, 2\}$, and shall call e^i the i-edge associated to e. We shall say that $\mathbf{G}'_L$ is *the semi-graph associated to* $\mathbf{G}'$ *and* L if the following conditions hold:

(i) $v(\mathbf{G}'_L) \overset{\text{def}}{=} v(\mathbf{G}')$.
(ii) $e^{\text{op}}(\mathbf{G}'_L) \overset{\text{def}}{=} e^{\text{op}}(\mathbf{G}') \cup \{e^1, e^2\}_{e\in L}$ such that $\zeta^{\mathbf{G}'_L}(e) = \zeta^{\mathbf{G}'}(e)$ if $e \in e^{\text{op}}(\mathbf{G}')$, that $\zeta^{\mathbf{G}'_L}(e^1) \overset{\text{def}}{=} \{\zeta^{\mathbf{G}'}(b^1_e), \{v(\mathbf{G}'_L)\}\}$ if e^1 is the 1-edge associated to $e \in L$, and that $\zeta^{\mathbf{G}'_L}(e^2) \overset{\text{def}}{=} \{\zeta^{\mathbf{G}'}(b^2_e), \{v(\mathbf{G}'_L)\}\}$ if e^2 is the 2-edge associated to $e \in L$.
(iii) $e^{\text{cl}}(\mathbf{G}'_L) \overset{\text{def}}{=} e^{\text{cl}}(\mathbf{G}') \setminus L$ such that $\zeta^{\mathbf{G}'_L}(e) \overset{\text{def}}{=} \zeta^{\mathbf{G}'}(e)$ for all $e \in e^{\text{cl}}(\mathbf{G}') \setminus L$.

Then we have a natural map of semi-graphs

$$\delta_{(\mathbf{G}',L)} : \mathbf{G}'_L \to \mathbf{G}'$$

which is defined as follows:

- $\delta_{(\mathbf{G}',L)}(v) = v$ for $v \in v(\mathbf{G}'_L)$.
- $\delta_{(\mathbf{G}',L)}(e) = e$ for $e \in e(\mathbf{G}'_L) \setminus \{e^1, e^2\}_{e\in L}$.
- $\delta_{(\mathbf{G}',L)}(e^i) = e$, $i \in \{1, 2\}$, for i-edge associated to $e \in L$.

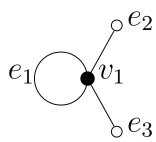
Moreover, we put $\delta_{\mathbf{G}'_L} : \mathbf{G}'_L \overset{\delta_{(\mathbf{G}',L)}}{\to} \mathbf{G}' \hookrightarrow \mathbf{G}$ the composition of maps of semi-graphs. Note that $\delta_{\mathbf{G}'_L}|_{\mathbf{G}'_L \setminus \{e^1,e^2\}_{e\in L}}$ is an injection.

Remark. The motivations of the above notation concerning semi-graphs arise from the dual semi-graphs of pointed stable sub-curves (see 2.2.3 below).

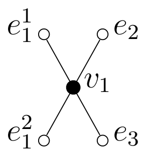
Example 2.2. We give some examples of semi-graphs to explain the above notation. We use the notation "$\bullet$" and "$\circ$" to denote a vertex and an open edge, respectively.

Let $\mathbf{G}$ be a semi-graph constructed in Example 2.1, and let $\mathbf{G}'$ be a sub-semi-graph of $\mathbf{G}$ such that $v(\mathbf{G}') = \{v_1\}$, and $L \overset{\text{def}}{=} \{e_1\} \subseteq e^{\text{cl}}(\mathbf{G}')$ a subset of edges of $\mathbf{G}'$ and $\{e^1_1, e^2_1\}$ the set of 1-edge and 2-edge associated to e_1. Then we have the following:

$\mathbf{G}'$:



$\mathbf{G}'_L$:



2.2. **Pointed stable curves and their admissible fundamental groups.**

2.2.1. Let p be a prime number, and let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve over an algebraically closed field k of characteristic p, where X denotes the underlying curve, D_X denotes a finite set of marked points satisfying [K, Definition 1.1 (iv)]. Write g_X for the *arithmetic* genus (or genus for short) of X and n_X

for the cardinality $\#(D_X)$ of D_X. We call the pair (g_X, n_X) the *topological type* (or type for short) of $X^\bullet$.

Recall that the *dual semi-graph* $\Gamma_{X^\bullet} \overset{\mathrm{def}}{=} (v(\Gamma_{X^\bullet}), e(\Gamma_{X^\bullet}), \zeta^{\Gamma_{X^\bullet}})$ of $X^\bullet$ is a semi-graph associated to $X^\bullet$ defined as follows: (i) $v(\Gamma_{X^\bullet})$ is the set of irreducible components of X. (ii) $e^{\mathrm{op}}(\Gamma_{X^\bullet})$ is the set of marked points D_X. (iii) $e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ is the set of singular points (or nodes) X^{sing} of X. (iv) $\zeta^{\Gamma_{X^\bullet}}(e)$, $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$, consists of the set $\{v(\Gamma_{X^\bullet})\}$ and the unique irreducible component containing e. (v) $\zeta^{\Gamma_{X^\bullet}}(e)$, $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, consists of the irreducible components containing e.

2.2.2. By choosing a base point $x \in X \setminus (X^{\mathrm{sing}} \cup D_X)$, we have the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ of $X^\bullet$ (see [Y3, §2.1.5] for definitions of (Galois) admissible coverings and (Galois) multi-admissible coverings, and [Y5, §1.2] for a definition of admissible fundamental groups). Since we only focus on the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ in the present paper, for simplicity of notation, we omit the base point and denote by

$$\Pi_{X^\bullet}$$

the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$. Note that, by the definition of admissible coverings ([Y3, §2.1.5]), the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X^\bullet$ when $X^\bullet$ is smooth over k. Moreover, the structure of the maximal prime-to-p quotient of $\Pi_{X^\bullet}$ is well-known, and is isomorphic to the prime-to-p completion of the following group

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X}[a_i, b_i]\prod_{j=1}^{n_X} c_j = 1\rangle.$$

We denote by $\Pi_{X^\bullet}^{\text{ét}}$ the étale fundamental group of the underlying curve X of $X^\bullet$. We have the following natural continuous surjective homomorphisms (for suitable choices of base points)

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}}.$$

2.2.3. We define pointed stable curves associated to various semi-graphs introduced in 2.1.2. Let $\Gamma \subseteq \Gamma_{X^\bullet}$ be a sub-semi-graph (2.1.2). Write X_Γ for the semi-stable subcurve of X (i.e., a closed subscheme of X which is a semi-stable curve) whose irreducible components are the irreducible components corresponding to the vertices of Γ, and whose nodes are the nodes corresponding to the closed edges of Γ. Moreover, write D_{X_Γ} for the set of closed points $X_\Gamma \cap \{x_e\}_{e \in e^{\mathrm{op}}(\Gamma) \subseteq e(\Gamma_{X^\bullet})}$, where $x_e \in X$ denotes the closed point corresponding to $e \in e(\Gamma_{X^\bullet})$. We define a pointed stable curve of type (g_Γ, n_Γ) over k to be

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma}).$$

Note that the dual semi-graph of $X_\Gamma^\bullet$ is naturally isomorphic to Γ. We shall call $X_\Gamma^\bullet$ *the pointed stable curve of type* (g_Γ, n_Γ) *associated to* Γ, and denote by $\Pi_{X_\Gamma^\bullet}$ the admissible fundamental group of $X_\Gamma^\bullet$.

Let $\Gamma \subseteq \Gamma_{X^\bullet}$ be a sub-semi-graph and $L \subseteq e^{\mathrm{cl}}(\Gamma)$ such that $\Gamma \setminus L$ is connected. Let Γ_L be the semi-graph associated to Γ and L (2.1.2), and $\mathrm{Node}_L(X_\Gamma) \subseteq X_\Gamma^{\mathrm{sing}}$ the set of nodes of X_Γ corresponding to L. Write $\mathrm{nor}_L : X_{\Gamma_L} \to X_\Gamma$ for the normalization of X_Γ at $\mathrm{Node}_L(X_\Gamma)$, and put $D_{X_{\Gamma_L}} \overset{\mathrm{def}}{=} \mathrm{nor}_L^{-1}(D_{X_\Gamma} \cup \mathrm{Node}_L(X_\Gamma))$. We define a pointed stable curve of type $(g_{\Gamma_L}, n_{\Gamma_L})$ to be

$$X_{\Gamma_L}^\bullet = (X_{\Gamma_L}, D_{\Gamma_L}).$$

Note that the dual semi-graph of $X_{\Gamma_L}^\bullet$ is naturally isomorphic to Γ_L. We shall call $X_{\Gamma_L}^\bullet$ *the pointed stable curve of type* $(g_{\Gamma_L}, n_{\Gamma_L})$ *associated to* Γ_L. We denote by $\Pi_{X_{\Gamma_L}^\bullet}$ the admissible fundamental group of $X^\bullet_{\Gamma_L}$. Moreover, we have the following natural outer injections (i.e., up to inner automorphism of $\Pi_{X^\bullet}$)

$$\Pi_{X^\bullet_{\Gamma_L}} \hookrightarrow \Pi_{X^\bullet_{\Gamma}} \hookrightarrow \Pi_{X^\bullet}.$$

Let $v \in v(\Gamma_{X^\bullet})$ and $\Gamma_v \subseteq \Gamma_{X^\bullet}$ the sub-semi-graph such that $v(\Gamma_v) = \{v\}$. Let $e^{\mathrm{lp}}(\Gamma_v)$ be the set of loops of Γ_v (2.1.1). Note that in this situation, we have $e^{\mathrm{lp}}(\Gamma_v) = e^{\mathrm{cl}}(\Gamma_v)$. Write X_v for the irreducible component corresponding to v and $\mathrm{nor}_v : \widetilde{X}_v \to X_v$ for the normalization of X_v. We put $D_{\widetilde{X}_v} \overset{\mathrm{def}}{=} \mathrm{nor}_v^{-1}\big((D_X \cap X_v) \cup (X^{\mathrm{sing}} \cap X_v)\big)$. Then we have $\widetilde{X}_v = X_{(\Gamma_v)_{e^{\mathrm{lp}}(\Gamma_v)}}$ and $D_{\widetilde{X}_v} = D_{X_{(\Gamma_v)_{e^{\mathrm{lp}}(\Gamma_v)}}}$. Moreover, we shall call

$$\widetilde{X}^\bullet_v \overset{\mathrm{def}}{=} (\widetilde{X}_v, D_{\widetilde{X}_v}) = X^\bullet_{(\Gamma_v)_{e^{\mathrm{lp}}(\Gamma_v)}}$$

the smooth pointed stable curve of type $(g_v, n_v) \overset{\mathrm{def}}{=} (g_{(\Gamma_v)_{e^{\mathrm{lp}}(\Gamma_v)}}, n_{(\Gamma_v)_{e^{\mathrm{lp}}(\Gamma_v)}})$ *associated to* v. We denote by $\Pi_{\widetilde{X}^\bullet_v}$ the admissible fundamental group of $\widetilde{X}^\bullet_v$. Suppose $\Gamma_v \subseteq \Gamma$. Then we have the following natural outer injections

$$\Pi_{\widetilde{X}^\bullet_v} \hookrightarrow \Pi_{X^\bullet_{\Gamma_v}} \hookrightarrow \Pi_{X^\bullet_{\Gamma}} \hookrightarrow \Pi_{X^\bullet}.$$

Example 2.3. Suppose that the dual semi-graph $\Gamma_{X^\bullet}$ is equal to the semi-graph constructed in Example 2.1. Then we have that $\Gamma_{v_1} = \Gamma_{X^\bullet_{\Gamma_{v_1}}}$, $\Gamma_{\widetilde{X}^\bullet_{v_1}} = \Gamma_{X^\bullet_{(\Gamma_{v_1})_{e^{\mathrm{lp}}(\Gamma_{v_1})}}}$ are equal to the semi-graphs $\mathbf{G}'$, $\mathbf{G}'_L$, constructed in Example 2.2, respectively.

2.3. Generalized Hasse-Witt invariants.

2.3.1. *Notation and Settings.* We maintain the notation and the settings introduced in 2.2.1 and 2.2.2.

2.3.2. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. By fixing a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Let $\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$. We denote by $X^\bullet_\alpha = (X_\alpha, D_{X_\alpha}) \to X^\bullet$ the Galois multi-admissible covering ([Y3, §2.1.5]) with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α.

We put $H_\alpha \overset{\mathrm{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$. Then H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α, moreover, it admits the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication. We shall call

$$\gamma_{\alpha,i} \overset{\mathrm{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z},$$

a *generalized Hasse-Witt invariant* (see [B], [N], [T2] for the case of étale or tame coverings of smooth pointed stable curves) of the cyclic multi-admissible covering $X^\bullet_\alpha \to X^\bullet$. In particular, we shall call $\gamma_{\alpha,1}$ the *first* generalized Hasse-Witt invariant of the cyclic multi-admissible covering $X^\bullet_\alpha \to X^\bullet$.

2.3.3. Write $\mathbb{Z}[D_X]$ for the group of divisors whose supports are contained in D_X. Note that $\mathbb{Z}[D_X]$ is a free $\mathbb{Z}$-module with basis D_X. We put

$$\mathbb{Z}/n\mathbb{Z}[D_X] \overset{\mathrm{def}}{=} \mathbb{Z}[D_X] \otimes \mathbb{Z}/n\mathbb{Z},$$

$$c'_n : \mathbb{Z}/n\mathbb{Z}[D_X] \to \mathbb{Z}/n\mathbb{Z},\ D \bmod n \mapsto \deg(D) \bmod n.$$

Write $(\mathbb{Z}/n\mathbb{Z})^{\sim}$ for the set $\{0, 1, \ldots, n-1\}$ and $(\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]$ for the subset of $\mathbb{Z}[D_X]$ consisting of the elements whose coefficients are contained in $(\mathbb{Z}/n\mathbb{Z})^{\sim}$. Then we have a natural bijection $\iota_n : (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X] \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}[D_X]$. We put

$$(\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0 \stackrel{\text{def}}{=} \iota_n^{-1}(\ker(c'_n)).$$

Note that we have $n|\deg(D)$ for all $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. Moreover, we put

$$s(D) \stackrel{\text{def}}{=} \frac{\deg(D)}{n} \in \mathbb{Z}_{\geq 0}.$$

Since every $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ can be regarded as a ramification divisor associated to some cyclic admissible covering, the structure of the maximal prime-to-p quotient of $\Pi_{X^\bullet}$ (2.2.2) implies the following:

$$0 \leq s(D) \leq \begin{cases} 0, & \text{if } n_X \leq 1, \\ n_X - 1, & \text{if } n_X \geq 2. \end{cases}$$

2.3.4. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup and $X^\bullet_H = (X_H, D_{X_H})$ the pointed stable curve over k corresponding to H. We put

$$\widehat{X} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} X_H,\ D_{\widehat{X}} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} D_{X_H},\ \Gamma_{\widehat{X}^\bullet} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} \Gamma_{X^\bullet_H}.$$

We call $\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}})$ the universal admissible covering of $X^\bullet$ corresponding to $\Pi_{X^\bullet}$, and $\Gamma_{\widehat{X}^\bullet}$ the dual semi-graph of $\widehat{X}^\bullet$. Note that $\mathrm{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of $\Pi_{X^\bullet}$.

Let $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. Write $\widehat{e} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet})$ for an open edge over e (i.e., the image of $\widehat{e}$ of the natural surjection $D_{\widehat{X}} \twoheadrightarrow D_X$ is e) and $x_e \in D_X$ for the marked point corresponding to e. We denote by $I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$ the stabilizer subgroup of $\widehat{e}$. The definition of admissible coverings ([Y3, §2.1.5]) implies that $I_{\widehat{e}}$ is (outer) isomorphic to the Galois group $\mathrm{Gal}(\widehat{K}^{\mathrm{t}}_{x_e}/\widehat{K}_{x_e}) \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $\widehat{K}_{x_e}$ denotes the quotient field of $\widehat{\mathcal{O}}_{X,x_e}$, $\widehat{K}^{\mathrm{t}}_{x_e}$ denotes a maximal tamely ramified extension of $\widehat{K}_{x_e}$, and $\widehat{\mathbb{Z}}(1)^{p'}$ denotes the maximal prime-to-p quotient of $\widehat{\mathbb{Z}}(1)$. Then we have an injection $\phi_{\widehat{e}} : I_{\widehat{e}} \hookrightarrow \Pi^{\mathrm{ab}}_{X^\bullet}$. Since the image of $\phi_{\widehat{e}}$ depends only on e, we may write I_e for the image $\phi_{\widehat{e}}(I_{\widehat{e}})$. Moreover, the structures of maximal prime-to-p quotients of admissible fundamental groups of pointed stable curves (2.2.2) imply that the following holds: There exists a generator s_e of I_e for each $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ such that

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} s_e = 0$$

in $\Pi^{\mathrm{ab}}_{X^\bullet}$. In the remainder of the present paper, we *fix* a set of generators $\{s_e\}_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})}$ of I_e satisfying the above condition. Then we have the following definitions:

Definition 2.4. (i) For $\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$, we put

$$D_\alpha \stackrel{\text{def}}{=} \sum_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} \alpha(s_e)^{\sim} x_e,$$

where $\alpha(s_e)^{\sim}$ denotes the element of $(\mathbb{Z}/n\mathbb{Z})^{\sim}$ corresponding to $\alpha(s_e)$ via the natural bijection $(\mathbb{Z}/n\mathbb{Z})^{\sim} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$. Note that we have $D_\alpha \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. On the other hand, for each $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$, we put

$$\mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet) \stackrel{\text{def}}{=} \{\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \mid D_\alpha = D\},$$

$$\gamma_{(\alpha,D)} \stackrel{\text{def}}{=} \gamma_{\alpha,1} \text{ (2.3.2)}.$$

(ii) Let $t \in \mathbb{N}$ be an arbitrary positive natural number, and $n \stackrel{\text{def}}{=} p^t - 1$. For $u \in \{0, \dots, n\}$, we write

$$u = \sum_{r=0}^{t-1} u_r p^r$$

for the p-adic expansion with $u_r \in \{0, \dots, p-1\}$. We identify $\{0, \dots, t-1\}$ with $\mathbb{Z}/t\mathbb{Z}$ naturally. Then $\{0, \dots, t-1\}$ admits an additional structure induced by the natural additional structure of $\mathbb{Z}/t\mathbb{Z}$. We put

$$u^{(i)} \stackrel{\text{def}}{=} \sum_{r=0}^{t-1} u_{i+r} p^r, \ i \in \{0, \dots, t-1\}.$$

Let $D \in \mathbb{Z}[D_X]$ be an effective divisor on X such that $\mathrm{ord}_x(D) \leq n$ for all $x \in D_X$ and $n | \deg(D)$. For $i \in \{0, \dots, t-1\}$, we put

$$D^{(i)} \stackrel{\text{def}}{=} \sum_{x \in D_X} \left(\mathrm{ord}_x(D)\right)^{(i)} x \in \mathbb{Z}[D_X].$$

We shall call D a *Frobenius stable* effective divisor on X if

$$\deg(D) = \deg(D^{(i)})$$

holds for each $i \in \{0, \dots, t-1\}$.

2.4. **Generalized Hasse-Witt invariants via line bundles.** The generalized Hasse-Witt invariants can be also described in terms of line bundles and divisors.

2.4.1. *Notation and Settings.* We maintain the notation and the settings introduced in 2.2.1 and 2.2.2.

2.4.2. Let $n \in \mathbb{N}$ be an arbitrary natural number prime to p. We denote by $\mathrm{Pic}(X)$ the Picard group of X. Consider the following complex of abelian groups:

$$\mathbb{Z}[D_X] \xrightarrow{a_n} \mathrm{Pic}(X) \oplus \mathbb{Z}[D_X] \xrightarrow{b_n} \mathrm{Pic}(X),$$

where $a_n(D) = ([\mathcal{O}_X(-D)], nD)$, $b_n(([\mathcal{L}], D)) = [\mathcal{L}^n \otimes \mathcal{O}_X(D)]$. We denote by

$$\mathscr{P}_{X^\bullet, n} \stackrel{\text{def}}{=} \ker(b_n)/\mathrm{Im}(a_n)$$

the homology group of the complex. Moreover, we have the following exact sequence

$$0 \to \mathrm{Pic}(X)[n] \xrightarrow{a'_n} \mathscr{P}_{X^\bullet, n} \xrightarrow{b'_n} \mathbb{Z}/n\mathbb{Z}[D_X] \xrightarrow{c'_n} \mathbb{Z}/n\mathbb{Z},$$

where $\mathrm{Pic}(X)[n]$ denotes the n-torsion subgroup of $\mathrm{Pic}(X)$, and

$$a'_n([\mathcal{L}]) = ([\mathcal{L}], 0) \bmod \mathrm{Im}(a_n), \ b'_n(([\mathcal{L}], D)) \bmod \mathrm{Im}(a_n)) = D \bmod n,$$

$$c'_n(D \bmod n) = \deg(D) \bmod n.$$

We shall define

$$\widetilde{\mathscr{P}}_{X^\bullet, n} \subseteq \ker(b_n) \subseteq \mathrm{Pic}(X) \oplus \mathbb{Z}[D_X]$$

to be the inverse image of $(\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$(2.3.3) $\subseteq (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X] \subseteq \mathbb{Z}[D_X]$ under the projection $\ker(b_n) \to \mathbb{Z}[D_X]$. It is easy to see that $\mathscr{P}_{X^\bullet, n}$ and $\widetilde{\mathscr{P}}_{X^\bullet, n}$ are free $\mathbb{Z}/n\mathbb{Z}$-modules with rank $2g_X + n_X - 1$ if $n_X \neq 0$ and with rank $2g_X$ if $n_X = 0$, and that there is a natural isomorphism $\widetilde{\mathscr{P}}_{X^\bullet, n} \xrightarrow{\sim} \mathscr{P}_{X^\bullet, n}$.

On the other hand, let $\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ and $f^\bullet_\alpha : X^\bullet_\alpha \to X^\bullet$ the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. By fixing a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Then we see

$$f_{\alpha,*}\mathcal{O}_{X_\alpha} \cong \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \mathcal{L}_{\alpha,i},$$

where locally $\mathcal{L}_{\alpha,i}$ is the eigenspace of the natural action of i with eigenvalue ζ^i. Moreover, by similar arguments to the arguments given in [T2, Proposition 3.5], we have the following isomorphism:

$$\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \overset{\sim}{\to} \widetilde{\mathscr{P}}_{X^\bullet,n},\ \ \alpha \mapsto ([\mathcal{L}_{\alpha,1}], D_\alpha).$$

Then every element of $\widetilde{\mathscr{P}}_{X^\bullet,n}$ induces a Galois multi-admissible covering of $X^\bullet$ over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

2.4.3. In 2.4.3, we suppose $n \overset{\mathrm{def}}{=} p^t - 1$ for some positive natural number $t \in \mathbb{N}$. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$. We fix an isomorphism $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(-D)$. Note that D is an effective divisor on X. We have the following composition of morphisms of line bundles

$$\mathcal{L} \overset{p^t}{\to} \mathcal{L}^{\otimes p^t} = \mathcal{L}^{\otimes n} \otimes \mathcal{L} \overset{\sim}{\to} \mathcal{O}_X(-D) \otimes \mathcal{L} \hookrightarrow \mathcal{L}.$$

This composite morphism induces a homomorphism $\phi_{([\mathcal{L}],D)} : H^1(X, \mathcal{L}) \to H^1(X, \mathcal{L})$. We denote by

$$\gamma_{([\mathcal{L}],D)} \overset{\mathrm{def}}{=} \dim_k\big(\bigcap_{r\geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)})\big),$$

and write $\alpha_{\mathcal{L}} \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ for the element corresponding to $([\mathcal{L}], D)$ via the isomorphism $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \overset{\sim}{\to} \widetilde{\mathscr{P}}_{X^\bullet,n}$. Then we have the following lemma:

Lemma 2.5. *We maintain the notation and the settings introduced above. Then the following statements hold:*

(i) We have $\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1}$ (2.3.2). Moreover, since $D_{\alpha_{\mathcal{L}}} = D$, we have

$$\gamma_{([\mathcal{L}],D)} = \gamma_{(\alpha_{\mathcal{L}},D)}\ (\overset{\mathrm{def}}{=} \gamma_{\alpha_{\mathcal{L}},1}).$$

(ii) We have

$$\gamma_{(\alpha_{\mathcal{L}},D)} \leq \dim_k(H^1(X, \mathcal{L})) = \begin{cases} g_X, & \text{if } ([\mathcal{L}], D) = ([\mathcal{O}_X], 0), \\ g_X - 1, & \text{if } s(D) = 0,\ [\mathcal{L}] \neq [\mathcal{O}_X], \\ g_X + s(D) - 1, & \text{if } s(D) \geq 1, \end{cases}$$

where $s(D)$ is the natural number defined in 2.3.3.

Proof. See [Y3, Lemma 2.6 and Lemma 2.7]. □

We shall say that *the generalized Hasse-Witt invariant $\gamma_{(\alpha_{\mathcal{L}},D)}$ can attain maximum* if

$$\gamma_{(\alpha_{\mathcal{L}},D)} = \dim_k(H^1(X, \mathcal{L}))$$

holds.

2.5. **Raynaud-Tamagawa theta divisors.** We recall the theory of Raynaud-Tamagawa theta divisors which was introduced by Raynaud in the case of étale coverings ([R1]), and which was generalized by Tamagawa in the case of tame coverings ([T2]).

2.5.1. *Notation and Settings.* We maintain the notation and the settings introduced in 2.4.1. Moreover, we suppose that $X^\bullet$ is *smooth* over k.

2.5.2. Let F_k be the absolute Frobenius morphism on Spec k, $F_{X/k}$ the relative Frobenius morphism $X \to X_1 \overset{\text{def}}{=} X \times_{k,F_k} k$ over k, and $F_k^t \overset{\text{def}}{=} F_k \circ \cdots \circ F_k$. We put $X_t \overset{\text{def}}{=} X \times_{k,F_k^t} k$, and define a morphism $F_{X/k}^t : X \to X_t$ over k to be $F_{X/k}^t \overset{\text{def}}{=} F_{X_{t-1}/k} \circ \cdots \circ F_{X_1/k} \circ F_{X/k}$.

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$, and let $\mathcal{L}_t$ be the pulling back of $\mathcal{L}$ by the natural morphism $X_t \to X$. Note that $\mathcal{L}$ and $\mathcal{L}_t$ are line bundles of degree $-s(D)$ (2.3.3). We put $\mathcal{B}_D^t \overset{\text{def}}{=} (F_{X/k}^t)_*\big(\mathcal{O}_X(D)\big)/\mathcal{O}_{X_t}$ and

$$\mathcal{E}_D \overset{\text{def}}{=} \mathcal{B}_D^t \otimes \mathcal{L}_t.$$

Let J_{X_t} be the Jacobian variety of X_t and $\mathcal{L}_{X_t}$ a universal line bundle on $X_t \times J_{X_t}$. Let $\mathrm{pr}_{X_t} : X_t \times J_{X_t} \to X_t$ and $\mathrm{pr}_{J_{X_t}} : X_t \times J_{X_t} \to J_{X_t}$ be the natural projections. We denote by $\mathcal{F}$ the coherent $\mathcal{O}_{X_t}$-module $\mathrm{pr}_{X_t}^*(\mathcal{E}_D) \otimes \mathcal{L}_{X_t}$, and by

$$\chi_{\mathcal{F}} \overset{\text{def}}{=} \dim_k(H^0(X_t \times_k k(y), \mathcal{F} \otimes k(y))) - \dim_k(H^1(X_t \times_k k(y), \mathcal{F} \otimes k(y)))$$

for each $y \in J_{X_t}$, where $k(y)$ denotes the residue field of y. Note that since $\mathrm{pr}_{J_{X_t}}$ is flat, $\chi_{\mathcal{F}}$ is independent of $y \in J_{X_t}$. Write $(-\chi_{\mathcal{F}})^+$ for $\max\{0, -\chi_{\mathcal{F}}\}$. We denote by $\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$ the closed subscheme of J_{X_t} defined by the $(-\chi_{\mathcal{F}})^+$th Fitting ideal $\mathrm{Fitt}_{(-\chi_{\mathcal{F}})^+}\big(R^1(\mathrm{pr}_{J_{X_t}})_*(\mathcal{F})\big)$. The definition of $\Theta_{\mathcal{E}_D}$ is independent of the choice of $\mathcal{L}_t$. Moreover, we have $\mathrm{codim}(\Theta_{\mathcal{E}_D}) \leq 1$.

We shall call

$$\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$$

the *Raynaud-Tamagawa theta divisor* associated to $\mathcal{E}_D$ if there exists a line bundle $\mathcal{L}_t'$ of degree 0 on X_t such that

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}_t')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}_t'))\}.$$

The following fundamental theorem of theta divisors was proved by Raynaud when $s(D) = 0$ ([R1, Théorème 4.1.1]), and by Tamagawa when $s(D) \leq 1$ ([T2, Theorem 2.5]).

Theorem 2.6. *Suppose that $s(D) \in \{0, 1\}$ (2.3.3). Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists.*

2.5.3. Let N be an arbitrary non-negative integer. We put

$$C(N) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } N = 0, \\ 3^{N-1}N!, & \text{if } N \neq 0. \end{cases}$$

Then we have the following proposition.

Proposition 2.7. *We maintain the notation introduced above. Suppose that*

$$n \overset{\text{def}}{=} p^t - 1 > C(g_X) + 1,$$

and that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists. Then there exists a line bundle $\mathcal{I}$ of degree 0 on X such that $[\mathcal{I}] \neq [\mathcal{O}_X]$, that $[\mathcal{I}^{\otimes n}] = [\mathcal{O}_X]$, and that

$$\gamma_{([\mathcal{L}\otimes\mathcal{I}],D)} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I})) = \begin{cases} g_X, & \text{if } ([\mathcal{L}], D) = ([\mathcal{O}_X], 0), \\ g_X - 1, & \text{if } s(D) = 0,\ [\mathcal{L}] \neq [\mathcal{O}_X], \\ g_X + s(D) - 1, & \text{if } s(D) \geq 1. \end{cases}$$

Namely, the first generalized Hasse-Witt invariant (2.3.2) of the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to $([\mathcal{L} \otimes \mathcal{I}], D)$ (2.4.2) can attain maximum.

Proof. See [Y3, Proposition 2.10]. □

3. Maximum generalized Hasse-Witt invariants with prescribed ramifications

In this section, we prove that the generalized Hasse-Witt invariants of cyclic admissible coverings with certain prescribed ramifications can attain maximum. The main result of the present section is Theorem 3.5.

3.1. **Minimal quasi-trees.** In [Y3], we introduced a kind of semi-graph which we call "a minimal quasi-tree", and which plays an important role for studying maximum generalized Hasse-Witt invariants of cyclic admissible coverings. Roughly speaking, the key point is that we can completely control the ramifications of admissible coverings at *nodes* of pointed stable curves by using minimal quasi-trees. For the convenience of readers, we recall the definition of minimal quasi-trees and give some examples.

3.1.1. Let $W^\bullet$ be a pointed stable curve of type (g_W, n_W) over an algebraically closed field l and $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$. We have the following:

Definition 3.1. Let Γ' be a sub-semi-graph (2.1.2) of $\Gamma_{W^\bullet}$ and $L \subseteq e^{\mathrm{cl}}(\Gamma') \setminus e^{\mathrm{lp}}(\Gamma')$ (see 2.1.1 for $e^{\mathrm{lp}}(\Gamma')$). We shall call the semi-graph Γ'_L associated to Γ' and L (2.1.2) a *quasi-tree associated to* D_W if the following conditions are satisfied:

- $\Gamma'_L \setminus e^{\mathrm{lp}}(\Gamma'_L)$ is a tree (i.e., the Betti number is 0, see 2.1.1).
- $e^{\mathrm{op}}(\Gamma_{W^\bullet})$ is contained in $e^{\mathrm{op}}(\Gamma'_L)$.

Moreover, we shall call a semi-graph

$$\Gamma_{D_W}$$

a *minimal quasi-tree associated to* D_W if either $\Gamma_{D_W} = \emptyset$ when $n_W = 0$ or the following conditions are satisfied when $n_W \neq 0$:

- Γ_{D_W} is a quasi-tree associated to D_W.
- Suppose that Γ'' is a quasi-tree associated to D_W such that $\Gamma'' \subseteq \Gamma_{D_W}$. Then we have $\Gamma'' = \Gamma_{D_W}$.

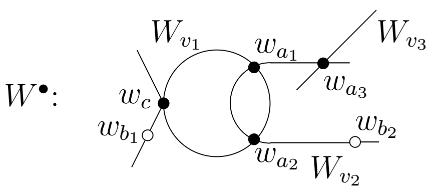
Note that by the definition of Γ_{D_W}, we have that $\Gamma_{D_W} \setminus e^{\mathrm{lp}}(\Gamma_{D_W})$ is a tree.

In particular, when $e^{\mathrm{lp}}(\Gamma_{W^\bullet}) = \emptyset$, minimal quasi-trees are very simple. Namely, Γ_{D_W} is a *minimal tree-like semi-graph* contained in $\Gamma_{W^\bullet}$ such that Γ_{D_W} *contains all of the open edges* of $\Gamma_{W^\bullet}$.

Remark 3.1.1. For any pointed stable curves, minimal quasi-tree associated to the sets of marked points always *exist* (see [Y3, §4.4.5]).

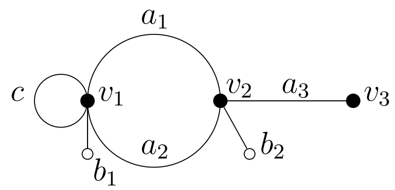
3.1.2. We give an example concerning minimal quasi-trees.

Example 3.2. (a) Let $W^\bullet$ be a pointed stable curve over k such that the following conditions hold: (i) The set of irreducible components of W is $\{W_{v_1}, W_{v_2}, W_{v_3}\}$; (ii) $D_W = \{w_{b_1}, w_{b_2}\}$; (iii) The set of nodes is $\{w_c, w_{a_1}, w_{a_2}, w_{a_3}\}$; (iv) W_{v_1} is a singular curve with the unique node w_c; (v) $w_{b_1} \in W_{v_1}$ and $w_{b_2} \in W_{v_2}$; (vi) $w_{a_1}, w_{a_2} \in W_{v_1} \cap W_{v_2}$; (vii) $w_{a_3} \in W_{v_2} \cap W_{v_3}$. We use the notation "$\bullet$" and "$\circ$" to denote a node and a marked point, respectively. Then $W^\bullet$ is as follows:



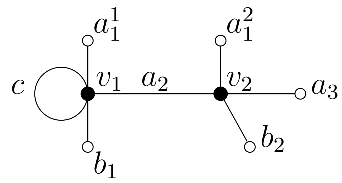
The dual semi-graph $\Gamma_{W^\bullet}$ of $W^\bullet$ such that the following conditions hold: (i) $v(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{v_1, v_2, v_3\}$; (ii) $e^{\text{cl}}(\Gamma_{W^\bullet}) \setminus e^{\text{lp}}(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{a_1, a_2, a_3\}$ such that a_1 and a_2 abut to v_1 and v_2, respectively, and that a_3 abuts to v_2 and v_3; (iii) $e^{\text{lp}}(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{c\}$ and c abuts to v_1; (iv) $e^{\text{op}}(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{b_1, b_2\}$ such that b_1 and b_2 abut to v_1 and v_2, respectively. We use the notation " $\bullet$ " and" $\circ$ with a line segment" to denote a vertex and an open edge, respectively. Then $\Gamma_{W^\bullet}$ is as follows:

$\Gamma_{W^\bullet}$:



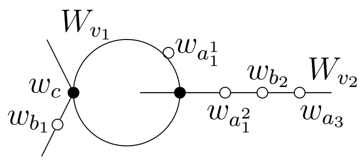
(b) We obtain a minimal quasi-tree $\Gamma_{D_W} \overset{\text{def}}{=} \Gamma$ associated to D_W is as follows:

$\Gamma_{D_W} \overset{\text{def}}{=} \Gamma$:



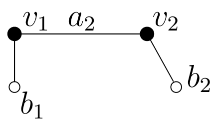
On the other hand, the pointed stable curve $W_\Gamma^\bullet$ associated to Γ (2.2.3) is as follows:

$W_\Gamma^\bullet$:



(c) Next, we give an example $\Gamma' \subsetneq \Gamma_{W^\bullet}$, which is a tree containing all open edges of $\Gamma_{W^\bullet}$, and which is *not* a (minimal) quasi-tree associated to D_W

Γ':



If Γ'' is a quasi-tree, then by the definition of quasi-trees (Definition 3.1), Γ' is equal to a semi-graph $\Gamma''_{L''}$ (2.1.2) associated to a sub-semi-graph Γ'' of $\Gamma_{W^\bullet}$ and a subset of closed edges $L'' \subseteq e^{\text{cl}}(\Gamma'') \setminus e^{\text{lp}}(\Gamma'')$. Thus, the definition of $\Gamma''_{L''}$ implies $e^{\Gamma''_{L''}}(v_1) = \{c, b_1, a_1^1, a_2\}$. This means that Γ' is not a quasi-tree.

3.2. **Maximum generalized Hasse-Witt invariants.**

3.2.1. *Notation and Settings.* We maintain the notation introduced in 2.2.1 and 2.2.2. Let Γ_{D_X} be a minimal quasi-tree associated to D_X. Then by definition, there exist a sub-semi-graph Γ' of $\Gamma_{X^\bullet}$ and a subset of closed edges $L \subseteq e^{\text{cl}}(\Gamma') \setminus e^{\text{lp}}(\Gamma')$ such that

$\Gamma_{D_X} = \Gamma'_L$. For simplicity, we denote by $\Gamma \overset{\text{def}}{=} \Gamma_{D_X}$. Moreover, let $X^\bullet_\Gamma = (X_\Gamma, D_{X_\Gamma})$ be the pointed stable curve of type (g_Γ, n_Γ) over k associated to $\Gamma(\overset{\text{def}}{=} \Gamma'_L)$ (2.2.3). Note that we have $e^{\text{op}}(\Gamma_{X^\bullet}) \subseteq e^{\text{op}}(\Gamma)$ (i.e., $D_X \subseteq D_{X_\Gamma}$).

Let $t \in \mathbb{N}$ be a positive natural number, $n \overset{\text{def}}{=} p^t - 1$, and $D \in \mathbb{Z}[D_X] \subseteq \mathbb{Z}[D_{X_\Gamma}]$ an effective divisor on X with degree $(n_X - 1)n$ such that $\text{ord}_x(D) \leq n$ for all $x \in D_X$. Write (see 2.3.3 for $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$)

$$\overline{D} \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$$

for the image of D via the composition of maps $\mathbb{Z}[D_X] \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}[D_X] \overset{\iota_n^{-1}}{\xrightarrow{\sim}} (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$, where the second arrow is the map defined in 2.3.3. Moreover, since $\#\big(\text{Supp}(D)\big) \in \{n_X - 1, n_X\}$, we have (see 2.3.3 for $s(\overline{D})$)

$$s(\overline{D}) = \begin{cases} \#\Big(\text{Supp}(D) \setminus \{x \in D_X \mid \text{ord}_x(D) = n\}\Big) - 1, & \text{if } \#\big(\text{Supp}(D)\big) = n_X, \\ 0, & \text{if } \#\big(\text{Supp}(D)\big) = n_X - 1. \end{cases}$$

Note that we have $D = \overline{D}$ and $s(D) = s(\overline{D}) = n_X - 1$ if $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$.

3.2.2. We maintain the notation and the settings introduced in 3.2.1. We introduce a condition concerning the effective divisor D which plays a central role in the remainder of the present paper.

Condition 3.3. Suppose $n_X > 0$. There exist a positive natural number $m \in \mathbb{N}$ and a set of effective divisors $\{D_j \in \mathbb{Z}[D_X]\}_{j \in \{1,\ldots,m\}}$ on X such that the following conditions are satisfied:

- (i) $\deg(D_j) = (n_X - 1)(p^{t_j} - 1)$, where $t_j \in \mathbb{N}$ is a positive natural number.
- (ii) $\text{ord}_x(D_j) \leq p^{t_j} - 1$ for all $x \in D_X$.
- (iii) $\#\big(\{x \in D_X \mid \text{ord}_x(D_j) = p^{t_j} - 1\}\big) \geq n_X - 2$.
- (iv) $D \overset{\text{def}}{=} D_1 + p^{t_1} D_2 + p^{t_1 + t_2} D_3 + \cdots + p^{\sum_{j=1}^{m-1} t_j} D_m$.

Note that if D satisfies Condition 3.3, we see that $t = \sum_{j=1}^m t_j$, and that D_j, $j \in \{1, \ldots, m\}$, is Frobenius stable (see Definition 2.4 (ii)). Then D and $\overline{D}$ are also Frobenius stable.

3.2.3. Firstly, we have the following lemma:

Lemma 3.4. *We maintain the notation and the settings introduced in 3.2.1. Suppose that $s(\overline{D}) \geq 1$, that $n > \max\{C(g_X) + 1, \#(X^{\text{sing}}) + n_X\}$ (see 2.5.3 for $C(g_X)$), and that D satisfies Condition 3.3. Then there exists an element $\alpha_\Gamma \in \text{Rev}^{\text{adm}}_{\overline{D}}(X^\bullet_\Gamma) \setminus \{0\}$ (Definition 2.4 (i)) such that*

$$\gamma_{(\alpha_\Gamma, \overline{D})} = g_\Gamma + s(\overline{D}) - 1.$$

Proof. We divide the proof of the lemma into the following parts:

Constructions of ramification divisors for irreducible components. Firstly, we construct explicitly ramification divisors on irreducible components of $X^\bullet_\Gamma$ induced by the divisor D on X_Γ.

Let $v \in v(\Gamma)$ be an arbitrary vertex of Γ, X_v the irreducible component of X_Γ corresponding to v, and $\pi_0(v)$ the set of connected components of $\overline{\{X_\Gamma \setminus X_v\}}$, where $\overline{\{X_\Gamma \setminus X_v\}}$ denotes the topological closure of $X_\Gamma \setminus X_v$ in X_Γ. We denote by $D_{X_v} \overset{\text{def}}{=} (D_{X_\Gamma} \cap X_v) \cup \big(\bigcup_{C \in \pi_0(v)} (C \cap X_v)\big)$ and put $X^\bullet_v = (X_v, D_{X_v})$. Then $X^\bullet_v$ is a pointed stable curve of type

(g_{X_v}, n_{X_v}) over k. Note that $X_v^\bullet = \widetilde{X}_v^\bullet$ if X_v is smooth over k, where $\widetilde{X}_v^\bullet$ denotes the smooth pointed stable curve associated to v (see 2.2.3).

Let $C \in \pi_0(v)$ be an arbitrary connected component. Since Γ is a minimal quasi-tree associated to D_X, we have $\#(C \cap X_v) = 1$. Then we shall put

$$x_C \overset{\text{def}}{=} C \cap X_v,\ C \in \pi_0(v).$$

Note that $C \cap \overline{\{X_\Gamma \setminus C\}} = C \cap X_v = \{x_C\}$. We denote by $D_C \overset{\text{def}}{=} (D_{X_\Gamma} \cap C) \cup \{x_C\}$. Then $C^\bullet = (C, D_C)$ is a pointed stable curve of type (g_C, n_C) over k. We put

$$D'_{X_v} \overset{\text{def}}{=} D_{X_v} \setminus (D_{X_\Gamma} \setminus D_X) = \Big(\bigcup_{C \in \pi_0(v)} \{x_C\}\Big) \cup (D_X \cap X_v),\ n'_{X_v} \overset{\text{def}}{=} \#(D'_{X_v}),$$

$$D'_C \overset{\text{def}}{=} D_C \setminus (D_{X_\Gamma} \setminus D_X) = \{x_C\} \cup (D_X \cap C),\ n'_C \overset{\text{def}}{=} \#(D'_C).$$

We see immediately

$$n'_{X_v} = n_X + 2\#(\pi_0(v)) - \sum_{C \in \pi_0(v)} n'_C.$$

Let $j \in \{1, \dots, m\}$ and $\{D_j\}_{j \in \{1,\dots,m\}}$ the set of effective divisors introduced in Condition 3.3. We put

$$d^v_{x_C,j} \overset{\text{def}}{=} \begin{cases} p^{t_j} - 1, & \text{if } d_{x,j} = p^{t_j} - 1 \text{ for all } x \in D_X \cap C, \\ \left[\sum_{x \in D_X \cap C} \mathrm{ord}_x(D_j)\right], & \text{otherwise,} \end{cases}$$

where $[(-)]$ denotes the image of $(-)$ of the natural surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/(p^{t_j}-1)\mathbb{Z}$. Moreover, we put

$$D_{v,j} \overset{\text{def}}{=} \sum_{C \in \pi_0(v)} d^v_{x_C,j} x_C + \sum_{x \in D_X \cap X_v} \mathrm{ord}_x(D_j)x \in \mathbb{Z}[D'_{X_v}],$$

$$D_v \overset{\text{def}}{=} D_{v,1} + p^{t_1} D_{v,2} + p^{t_1+t_2} D_{v,3} + \dots + p^{\sum_{j=1}^{m-1}} D_{v,m} \in \mathbb{Z}[D'_{X_v}].$$

Note that $\#\big(\{x \in D_X \mid \mathrm{ord}_x(D_j) = p^{t_j} - 1\}\big) \geq n_X - 2$ (i.e., Condition 3.3 (iii)) implies

$$\#\big(\{x \in D'_{X_v} \mid \mathrm{ord}_x(D_{v,j}) = p^{t_j} - 1\}\big) \geq n'_{X_v} - 2.$$

Calculations of degrees of ramification divisors. Next, we calculate the degrees of $D_{v,j}$, $j \in \{1, \dots, m\}$, and D_v. Let $C \in \pi_0(v)$ and $j \in \{1, \dots, m\}$. We shall put

$$Q_{C,j} \overset{\text{def}}{=} (p^{t_j} - 1 - d^v_{x_C,j})x_C + \sum_{x \in D_X \cap C} \mathrm{ord}_x(D_j)x \in \mathbb{Z}[D'_C],$$

$$Q_C \overset{\text{def}}{=} Q_{C,1} + p^{t_1} Q_{C,2} + p^{t_1+t_2} Q_{C,3} + \dots + p^{\sum_{j=1}^{m-1}} Q_{C,m} \in \mathbb{Z}[D'_C].$$

Moreover, $\deg(D_j) = (n_X - 1)(p^{t_j} - 1)$ (i.e., Condition 3.3 (i)) and the definition of $d^v_{x_C,j}$ imply

$$\sum_{x \in D_X \setminus D'_C} \mathrm{ord}_x(D_j) + d^v_{x_C,j} + (p^{t_j} - 1 - d^v_{x_C,j}) + \sum_{x \in D_X \cap C} \mathrm{ord}_x(D_j) = n_X(p^{t_j} - 1),$$

$$(p^{t_j} - 1 - d^v_{x_C,j}) + \sum_{x \in D_X \cap C} \mathrm{ord}_x(D_j) \leq (n'_C - 1)(p^{t_j} - 1),$$

$$\sum_{x \in D_X \setminus D'_C} \mathrm{ord}_x(D_j) + d^v_{x_C,j} \leq (n_X - n'_C + 1)(p^{t_j} - 1).$$

Then we have $\deg(Q_{C,j}) = (n'_C - 1)(p^{t_j} - 1)$.

On the other hand, we see

$$(n_X + \#(\pi_0(v)) - 1)(p^{t_j} - 1) = \deg(D_{v,j}) + \sum_{C \in \pi_0(v)} \deg(Q_{C,j})$$

$$= \deg(D_{v,j}) + \Big(\sum_{C\in\pi_0(v)} n'_C - \#(\pi_0(v))\Big)(p^{t_j}-1).$$

Then we obtain

$$\begin{aligned}\deg(D_{v,j}) &= (n_X + 2\#(\pi_0(v)) - \sum_{C\in\pi_0(v)} n'_C - 1)(p^{t_j}-1)\\ &= (n'_{X_v}-1)(p^{t_j}-1),\end{aligned}$$

and

$$\deg(D_v) = (n'_{X_v}-1)n.$$

Constructions of Galois multi-admissible coverings for $X_v^\bullet$. Next, we construct Galois multi-admissible coverings for irreducible components with ramifications divisors constructing above.

Let $v \in v(\Gamma)$ and $j \in \{1,\dots,m\}$. Write $\overline{D}_v \in (\mathbb{Z}/n\mathbb{Z})^\sim[D'_{X_v}]^0$ for the image of D_v via the composition of maps $\mathbb{Z}[D'_{X_v}] \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}[D'_{X_v}] \overset{\iota_n^{-1}}{\overset{\sim}{\to}} (\mathbb{Z}/n\mathbb{Z})^\sim[D'_{X_v}]$, where the second arrow is the map defined in 2.3.3. We denote by

$$B_{v,j} \overset{\text{def}}{=} D_{v,j}|_{\mathrm{Supp}(\overline{D}_v)}.$$

Then the constructions of D_v and $D_{v,j}$ imply the following conditions are satisfied:

- $\deg(B_{v,j}) = s(\overline{D}_v)(p^{t_j}-1)$.
- $\mathrm{ord}_x(B_{v,j}) \leq p^{t_j}-1$ for all $x \in \mathrm{Supp}(\overline{D}_v)$.
- $\#\big(\{x \in \mathrm{Supp}(\overline{D}_v) \mid \mathrm{ord}_x(B_{v,j}) = p^{t_j}-1\}\big) \geq s(\overline{D}_v)-1$.
- $\overline{D}_v = B_{v,1} + p^{t_1}B_{v,2} + \cdots + p^{\sum_{j=1}^{m-1}}B_{v,m}$

Let $v \in v(\Gamma)$ and $\widetilde{X}_v^\bullet$ the smooth pointed stable curve of type (g_v, n_v) over k. Write $\mathrm{nom}_v : \widetilde{X}_v \to X_v$ for the normalization morphism. We use the notation $\widetilde{D}_v$ to denote $\mathrm{nom}^*(\overline{D}_v)$. Note that $\widetilde{D}_v$ and $\overline{D}_v$ are equal via the isomorphism $\widetilde{X}_v \setminus \mathrm{nom}_v^{-1}(X_v^{\mathrm{sing}}) \overset{\sim}{\to} X_v \setminus X_v^{\mathrm{sing}}$. In particular, we have $s(\widetilde{D}_v) = s(\overline{D}_v)$. Let $\mathcal{L}_{\widetilde{D}_v}$ be a line bundle on $\widetilde{X}_v$ such that $\mathcal{L}_{\widetilde{D}_v}^{\otimes n} \cong \mathcal{O}_{\widetilde{X}_v}(-\widetilde{D}_v)$. Write $\mathcal{L}_{\widetilde{D}_v,t}$ for the pulling back line bundle of $\mathcal{L}_{\widetilde{D}_v}$ by the natural morphism $\widetilde{X}_{v,t} \times_{k,F_k^t} k \to \widetilde{X}_v$, where F_k denotes the absolute Frobenius morphism on $\mathrm{Spec}\, k$. Then by applying [T2, Corollary 2.6, Lemma 2.12 (ii), and Corollary 3.13] for $\widetilde{D}_v$ and $B_{v,j}$, $j \in \{1,\dots,m\}$, the Raynaud-Tamagawa theta divisor associated to the vector bundle (2.5.2)

$$\mathcal{B}^t_{\widetilde{D}_v} \otimes \mathcal{L}_{\widetilde{D}_v,t}$$

exists. Moreover, by Proposition 2.7, there exists $\widetilde{\alpha}_v \in \mathrm{Rev}^{\mathrm{adm}}_{\widetilde{D}_v}(\widetilde{X}_v^\bullet)$ such that $\gamma_{(\widetilde{\alpha}_v,\widetilde{D}_v)} = g_v + s(\overline{D}_v) - 1$.

We put $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$, $v \in v(\Gamma)$, the Galois multi-admissible covering over k induced by $\widetilde{\alpha}_v$ whose Galois group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Then $\widetilde{f}_v^\bullet$ induces a Galois multi-admissible covering

$$f_v^\bullet : Y_v^\bullet \to X_v^\bullet,\ v \in v(\Gamma),$$

over k whose Galois group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Let $\Pi_{X_v^\bullet}, \Pi_{\widetilde{X}_v^\bullet}$ be the admissible fundamental groups of $X_v^\bullet, \widetilde{X}_v^\bullet$, respectively, and $\alpha_v \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X_v^\bullet}, \mathbb{Z}/n\mathbb{Z})$ an element induced by $f_v^\bullet$ satisfying the composition of the homomorphisms $\Pi^{\mathrm{ab}}_{\widetilde{X}_v^\bullet} \to \Pi^{\mathrm{ab}}_{X_v^\bullet} \overset{\alpha_v}{\to} \mathbb{Z}/n\mathbb{Z}$ is equal to $\widetilde{\alpha}_v$ (see 2.2.3 for the first arrow). Then we see $\alpha_v \in \mathrm{Rev}^{\mathrm{adm}}_{\overline{D}_v}(X_v^\bullet) \setminus \{0\}$. By [Y3, Theorem 3.9], we obtain

$$\gamma_{(\alpha_v,\overline{D}_v)} = g_{X_v} + s(\overline{D}_v) - 1.$$

Namely, the generalized Hasse-Witt invariant $\gamma_{(\alpha_v,\overline{D}_v)}$ can attain maximum.

Constructions of Galois multi-admissible coverings for $X^\bullet_\Gamma$. Next, we prove that the Galois multi-admissible covering $f^\bullet_v : Y^\bullet_v \to X^\bullet_v$, $v \in v(\Gamma)$, constructed above can be glued as a Galois multi-admissible covering of $X^\bullet_\Gamma$.

Let $v_1, v_2 \in v(\Gamma)$ be vertices of Γ distinct from each other such that $X_{v_1} \cap X_{v_2}$ is not empty. Since Γ is a minimal quasi-tree associated to D_X, we have $\#(X_{v_1} \cap X_{v_2}) = 1$. Let $C_1 \in \pi_0(v_1)$ and $C_2 \in \pi_0(v_2)$ be the connected components such that $X_{v_2} \subseteq C_1$ and $X_{v_1} \subseteq C_2$. Then we have $x_{1,2} \overset{\text{def}}{=} x_{C_1} = x_{C_2} = X_{v_1} \cap X_{v_2} = C_1 \cap C_2$. Note that $C_1 \cup C_2 = X_\Gamma$. The definitions of $d^{v_1}_{x_{C_1},j}, d^{v_2}_{x_{C_2},j}$ imply

$$d^{v_1}_{x_{C_1},j} + d^{v_2}_{x_{C_2},j} = \begin{cases} d^{v_1}_{x_{C_1},j} = p^{t_j} - 1, & \text{if } d_{x,j} = p^{t_j} - 1 \text{ for all } x \in D_X \cap C_1, \\ d^{v_2}_{x_{C_2},j} = p^{t_j} - 1, & \text{if } d_{x,j} = p^{t_j} - 1 \text{ for all } x \in D_X \cap C_2, \end{cases}$$

otherwise,

$$d^{v_1}_{x_{C_1},j} + d^{v_2}_{x_{C_2},j} = \Big[\sum_{x \in D_X \cap C_1} \mathrm{ord}_x(D_j)\Big] + \Big[\sum_{x \in D_X \cap C_2} \mathrm{ord}_x(D_j)\Big].$$

Since $\sum_{x \in D_X \cap C_1} \mathrm{ord}_x(D_j) + \sum_{x \in D_X \cap C_2} \mathrm{ord}_x(D_j) = \deg(D_j)$ is divided by $p^{t_j} - 1$, we have

$$\Big[\sum_{x \in D_X \cap C_1} \mathrm{ord}_x(D_j)\Big] + \Big[\sum_{x \in D_X \cap C_2} \mathrm{ord}_x(D_j)\Big] = p^{t_j} - 1.$$

Then we obtain

$$d^{v_1}_{x_{C_1},j} + d^{v_2}_{x_{C_2},j} = p^{t_j} - 1.$$

On the other hand, $\deg(D_{v_i}) = (n'_X - 1)n$ and $\mathrm{ord}_x(D_{v_i}) \leq n$ for all $x \in D'_{X_{v_i}}$ imply

$$0 < \mathrm{ord}_{x_{1,2}}(D_{v_i}) = d^{v_i}_{x_{C_i},1} + p^{t_1} d^{v_i}_{x_{C_i},2} + \cdots + p^{\sum_{j=1}^{m-1} t_j} d^{v_i}_{x_{C_i},m} \leq n$$

for all $i \in \{1, 2\}$. Then we obtain $\mathrm{ord}_{x_{1,2}}(D_{v_1}) + \mathrm{ord}_{x_{1,2}}(D_{v_2}) = n$. Moreover, we see

$$\mathrm{ord}_{x_{1,2}}(D_{v_1}) + \mathrm{ord}_{x_{1,2}}(D_{v_2}) = \begin{cases} \mathrm{ord}_{x_{1,2}}(D_{v_2}) = n, & \text{if } \mathrm{Supp}(\overline{D}) \subseteq C_1, \\ \mathrm{ord}_{x_{1,2}}(D_{v_1}) = n, & \text{if } \mathrm{Supp}(\overline{D}) \subseteq C_2. \end{cases}$$

Thus, we have

$$\mathrm{ord}_{x_{1,2}}(\overline{D}_{v_1}) + \mathrm{ord}_{x_{1,2}}(\overline{D}_{v_2}) = \begin{cases} 0, & \text{if either } \mathrm{Supp}(\overline{D}) \subseteq C_1 \text{ or } \mathrm{Supp}(\overline{D}) \subseteq C_2, \\ n, & \text{otherwise.} \end{cases}$$

This means that we may glue $\{f^\bullet_v\}_{v \in v(\Gamma)}$ along $\{f_v^{-1}\big(D'_{X_v} \setminus (D'_{X_v} \cap D_X)\big)\}_{v \in v(\Gamma)}$ in a way that is compatible with the actions of $\mathbb{Z}/n\mathbb{Z}$ and the gluing of $\{X^\bullet_v\}_{v \in v(\Gamma)}$ that gives rise to $X^\bullet_\Gamma$. Then we obtain a Galois multi-admissible covering

$$f^\bullet_\Gamma : Y^\bullet_\Gamma = (Y_\Gamma, D_{Y_\Gamma}) \to X^\bullet_\Gamma$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Note that the construction of $f^\bullet_\Gamma$ implies that f_Γ is *étale* over $D_{X_\Gamma} \setminus D_X$, where $f_\Gamma : Y_\Gamma \to X_\Gamma$ denotes the morphism of underlying curves induced by $f^\bullet_\Gamma$.

Let $\Pi_{X^\bullet_v}$, $v \in v(\Gamma)$, be the admissible fundamental group of $X^\bullet_v$ and $\alpha_\Gamma \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet_\Gamma}, \mathbb{Z}/n\mathbb{Z})$ an element induced by $f^\bullet_\Gamma$ satisfying the composition of the homomorphisms $\Pi^{\mathrm{ab}}_{X^\bullet_v} \to \Pi^{\mathrm{ab}}_{X^\bullet_\Gamma} \xrightarrow{\alpha_\Gamma} \mathbb{Z}/n\mathbb{Z}$ is equal to α_v for all $v \in v(\Gamma)$ (see 2.2.3 for the first arrow). Then we see $\alpha_\Gamma \in \mathrm{Rev}^{\mathrm{adm}}_{\overline{D}}(X^\bullet_\Gamma) \setminus \{0\}$. By [Y3, Theorem 3.9], we obtain

$$\gamma_{(\alpha_\Gamma,\overline{D})} = g_{X_\Gamma} + s(\overline{D}) - 1.$$

This completes the proof of the lemma.

□

3.2.4. Now, we can prove the first main result of the present paper:

Theorem 3.5. *Let $X^\bullet = (X, D_X)$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, and $n \overset{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$ a natural number satisfying $n > \max\{C(g_X)+1, \#(X^{\mathrm{sing}})+n_X\}$ (see 2.5.3 for $C(g_X)$). Let $D \in \mathbb{Z}[D_X]$ be an effective divisor on X such that $\mathrm{ord}_x(D) \leq n$ for all $x \in D_X$, and that*

$$\overline{D} \overset{\mathrm{def}}{=} \begin{cases} 0, & \text{if } n_X = 0,\\ \text{an effective divisor with degree } (n_X-1)n \text{ satisfies Condition 3.3}, & \text{if } n_X \neq 0. \end{cases}$$

Then there exists an element $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_{\overline{D}}(X^\bullet) \setminus \{0\}$ (see 3.2.1 for $\overline{D}$, and see Definition 2.4 (i) for $\mathrm{Rev}^{\mathrm{adm}}_{\overline{D}}(X^\bullet)$) such that the generalized Hasse-Witt invariant $\gamma_{(\alpha,\overline{D})}$ can attain maximum. Namely, the following holds:

$$\gamma_{(\alpha,\overline{D})} = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(\overline{D}) = \emptyset,\\ g_X + s(\overline{D}) - 1, & \text{if } \mathrm{Supp}(\overline{D}) \neq \emptyset. \end{cases}$$

Proof. Suppose $\mathrm{Supp}(\overline{D}) = \emptyset$. Then we have $\overline{D} = 0$. Thus, the theorem follows immediately from Theorem 2.6 (i.e., Raynaud's result for zero divisor), Proposition 2.7, and [Y3, Theorem 3.9].

Suppose $\mathrm{Supp}(\overline{D}) \neq \emptyset$ (note that this implies $n_X \geq 2$). Let $\Gamma \overset{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X. There is a natural map of semi-graphs $\delta_\Gamma : \Gamma \to \Gamma_{X^\bullet}$ defined in 2.1.2. Write Γ^{im} for the image of δ_Γ. Note that Γ^{im} is a sub-semi-graph (2.1.2) of $\Gamma_{X^\bullet}$. We put

$$X^\bullet_\Gamma = (X_\Gamma, D_{X_\Gamma}),\ X^\bullet_{\Gamma^{\mathrm{im}}} = (X_{\Gamma^{\mathrm{im}}}, D_{X_{\Gamma^{\mathrm{im}}}})$$

the pointed stable curves of types (g_Γ, n_Γ), $(g_{\Gamma^{\mathrm{im}}}, n_{\Gamma^{\mathrm{im}}})$ over k corresponding to Γ, Γ^{im} (2.2.3), respectively, and $\Pi_{X^\bullet_\Gamma}$, $\Pi_{X^\bullet_{\Gamma^{\mathrm{im}}}}$ the admissible fundamental groups of $X^\bullet_\Gamma$, $X^\bullet_{\Gamma^{\mathrm{im}}}$, respectively.

By Lemma 3.4, there exists an element $\alpha_\Gamma \in \mathrm{Rev}^{\mathrm{adm}}_{\overline{D}}(X^\bullet_\Gamma) \setminus \{0\}$ such that

$$\gamma_{(\alpha_\Gamma,\overline{D})} = g_\Gamma + s(\overline{D}) - 1$$

holds. Write $f^\bullet_\Gamma : Y^\bullet_\Gamma = (Y_\Gamma, D_{Y_\Gamma}) \to X^\bullet_\Gamma$ for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_Γ. Note that the construction of $f^\bullet_\Gamma$ given in the proof of Lemma 3.4 (see the penultimate paragraph of the proof of Lemma 3.4) implies that the morphism $f_\Gamma : Y_\Gamma \to X_\Gamma$ of underlying curves is étale at

$$f_\Gamma^{-1}\Big(D_{X_\Gamma} \setminus (D_X \cup \{x_e\}_{e \in \delta_\Gamma^{-1}\left(e^{\mathrm{op}}(\Gamma^{\mathrm{im}})\right)})\Big).$$

By gluing $Y^\bullet_\Gamma$ along $f_\Gamma^{-1}\Big(D_{X_\Gamma} \setminus (D_X \cup \{x_e\}_{e \in \delta_\Gamma^{-1}\left(e^{\mathrm{op}}(\Gamma^{\mathrm{im}})\right)})\Big)$ in a way that is compatible with the actions of $\mathbb{Z}/n\mathbb{Z}$ and the gluing of $X^\bullet_\Gamma$ that gives rise to $X^\bullet_{\Gamma^{\mathrm{im}}}$, we obtain a pointed stable curve $Y^\bullet_{\Gamma^{\mathrm{im}}}$ over k. Moreover, $f^\bullet_\Gamma$ induces a Galois multi-admissible covering $f^\bullet_{\Gamma^{\mathrm{im}}} : Y^\bullet_{\Gamma^{\mathrm{im}}} \to X^\bullet_{\Gamma^{\mathrm{im}}}$ over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_{\Gamma^{\mathrm{im}}}$ for an element of $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet_{\Gamma^{\mathrm{im}}}}, \mathbb{Z}/n\mathbb{Z})$ induced by $f^\bullet_{\Gamma^{\mathrm{im}}}$ such that the composition of the homomorphisms $\Pi^{\mathrm{ab}}_{X^\bullet_\Gamma} \to \Pi^{\mathrm{ab}}_{X^\bullet_{\Gamma^{\mathrm{im}}}} \overset{\alpha_{\Gamma^{\mathrm{im}}}}{\to} \mathbb{Z}/n\mathbb{Z}$ is equal to α_Γ. Note that we have $D_{\alpha_{\Gamma^{\mathrm{im}}}} = \overline{D}$, where $D_{\alpha_{\Gamma^{\mathrm{im}}}}$ denotes the effective divisor on $X_{\Gamma^{\mathrm{im}}}$ determined by $\alpha_{\Gamma^{\mathrm{im}}}$ via the bijection $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet_{\Gamma^{\mathrm{im}}}}, \mathbb{Z}/n\mathbb{Z}) \overset{\sim}{\to} \widetilde{\mathscr{P}}_{X^\bullet_{\Gamma^{\mathrm{im}}}, n}$ (see 2.4.2). Then [Y3, Theorem 3.9] implies

$$\gamma_{(\alpha_{\Gamma^{\mathrm{im}}},\overline{D})} = g_{\Gamma^{\mathrm{im}}} + s(\overline{D}) - 1.$$

Write $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ for the set of connected components of the topological closure $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$ of $X \setminus X_{\Gamma^{\mathrm{im}}}$ in X. We define the following pointed stable curve

$$E^\bullet = (E, D_E \stackrel{\mathrm{def}}{=} E \cap X_{\Gamma^{\mathrm{im}}}),\ E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}),$$

over k. By Theorem 2.6 (i.e., Raynaud's result for zero divisor), Proposition 2.7, and [Y3, Theorem 3.9], there exists a Galois *étale* covering $f_E^\bullet : Y_E^\bullet = (Y_E, D_{Y_E}) \to E^\bullet$ over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that the following holds:

$$\gamma_{(\alpha_E,0)} = \begin{cases} g_E, & \text{if } g_E = 0, \\ g_E - 1, & \text{if } g_E \neq 0, \end{cases}$$

where g_E denotes the genus of E, and $\alpha_E \in \mathrm{Rev}_0^{\mathrm{adm}}(E^\bullet)$ is an element induced by $f_E^\bullet$ (note that $\alpha_E = 0$ if $g_E = 0$).

Since f_Γ and f_E are étale at

$$f_\Gamma^{-1}(X_{\Gamma^{\mathrm{im}}} \cap (\bigcup_{E \in \pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})} E)),\ f_E^{-1}(X_{\Gamma^{\mathrm{im}}} \cap E),$$

respectively, we can glue $Y_{\Gamma^{\mathrm{im}}}^\bullet$ and $\{Y_E^\bullet\}_{E \in \pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})}$ along

$$f_\Gamma^{-1}(X_{\Gamma^{\mathrm{im}}} \cap (\bigcup_{E \in \pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})} E)) \text{ and } \{f_E^{-1}(X_{\Gamma^{\mathrm{im}}} \cap E)\}_{E \in \pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})}$$

in a way that is compatible with the actions of $\mathbb{Z}/n\mathbb{Z}$ and the gluing of $\{X_{\Gamma^{\mathrm{im}}}^\bullet\} \cup \{E^\bullet\}_{E \in \pi_0(\overline{X\setminus X_{\Gamma^{\mathrm{im}}}})}$ that gives rise to $X^\bullet$. Then we obtain a Galois multi-admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Let $\Pi_{X^\bullet}$, $\Pi_{E^\bullet}$ be the admissible fundamental groups of $X^\bullet$, $E^\bullet$, $E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$, respectively. Write $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ for an element induced by $f^\bullet$ such that the compositions of the homomorphisms $\Pi^{\mathrm{ab}}_{X^\bullet_{\Gamma^{\mathrm{im}}}} \to \Pi^{\mathrm{ab}}_{X^\bullet} \stackrel{\alpha}{\to} \mathbb{Z}/n\mathbb{Z}$, $\Pi^{\mathrm{ab}}_{E^\bullet} \to \Pi^{\mathrm{ab}}_{X^\bullet} \stackrel{\alpha}{\to} \mathbb{Z}/n\mathbb{Z}$, $E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$, are equal to $\alpha_{\Gamma^{\mathrm{im}}}$ and α_E, respectively. We see $\alpha \in \mathrm{Rev}_{\overline{D}}^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$. By applying [Y3, Theorem 3.9], we obtain

$$\gamma_{(\alpha,\overline{D})} = g_X + s(\overline{D}) - 1.$$

This completes the proof of the theorem.

□

3.2.5. Let $X^\bullet = (X, D_X)$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. We put (see also [Y3, Definition 3.10 (i)])

$$\gamma_{X^\bullet}^{\max} \stackrel{\mathrm{def}}{=} \max_{d \in \mathbb{N} \text{ s.t. } (d,p)=1} \{\gamma_{(\alpha, D_\alpha)} \mid \alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/d\mathbb{Z}) \setminus \{0\}\}.$$

Note that Lemma 2.5 (ii) implies

$$\gamma_{X^\bullet}^{\max} \leq \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X + n_X - 2, & \text{if } n_X \geq 2. \end{cases}$$

In [Y3], we proved the following significant result concerning existence of maximum generalized Hasse-Witt invariants of cyclic admissible coverings:

([Y3, Theorem 5.4]): We maintain the notation introduced above. Then there exist a natural number $n = p^t - 1$, an effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$, and an element $\mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$ such that the following holds:

$$\gamma_{(\alpha,D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X + n_X - 2, & \text{if } n_X \geq 2. \end{cases}$$

This result is one of main results of [Y3], and it is an important tool to study admissible fundamental groups of pointed stable curves in positive characteristic and the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (e.g. [Y3], [Y5], [Y6]). On the other hand, we have the following:

Corollary 3.6. *Let $X^\bullet = (X, D_X)$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, and $n \overset{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$ a natural number satisfying $n > \max\{C(g_X)+1, \#(X^{\mathrm{sing}}) + n_X\}$ (see 2.5.3 for $C(g_X)$). Let $s \in \{0, \dots, n_X - 1\}$ be an integer. Then there exists an effective divisor $D \in \mathbb{Z}[D_X]$on X such that $\mathrm{ord}_x(D) \leq n$ for all $x \in D_X$, and that the following conditions are satisfied:*

- *We have*

$$D \overset{\mathrm{def}}{=} \begin{cases} 0, & \textit{if } n_X = 0, \\ \textit{an effective divisor with degree } (n_X - 1)n \textit{ satisfies Condition 3.3}, & \textit{if } n_X \neq 0 \end{cases}$$

  such that $s(\overline{D}) = s$ (see 3.2.1 for $\overline{D}$).
- *There exists an element $\alpha \in \mathrm{Rev}_{\overline{D}}^{\mathrm{adm}}(X^\bullet)\setminus\{0\}$ (see Definition 2.4 (i) for $\mathrm{Rev}_{\overline{D}}^{\mathrm{adm}}(X^\bullet)$) such that the generalized Hasse-Witt invariant $\gamma_{(\alpha,\overline{D})}$ can attain maximum. Namely, the following holds:*

$$\gamma_{(\alpha,\overline{D})} = \begin{cases} g_X - 1, & \textit{if } \mathrm{Supp}(\overline{D}) = \emptyset, \\ g_X + s(\overline{D}) - 1, & \textit{if } \mathrm{Supp}(\overline{D}) \neq \emptyset. \end{cases}$$

In particular, we obtain [Y3, Theorem 5.4] *if $s \overset{\mathrm{def}}{=} n_X - 1$.*

Proof. If $s = 0$, the corollary follows immediately from Theorem 3.5. Then we may suppose $s \geq 1$. We put $m \overset{\mathrm{def}}{=} s$ and $t_j \in \mathbb{N}$, $j \in \{1, \dots, m\}$, a positive natural number satisfying $n \overset{\mathrm{def}}{=} p^{\sum_{j=1}^m t_j} - 1 > \max\{C(g_X) + 1, \#(X^{\mathrm{sing}}) + n_X\}$.

Let $D_X \overset{\mathrm{def}}{=} \{x_1, \dots, x_{n_X}\}$. Then we put

$$D_j \overset{\mathrm{def}}{=} x_j + (p^{t_j} - 2)x_{j+1} + \sum_{x \in D_X \setminus \{x_j, x_{j+1}\}} (p^{t_j} - 1)x \in \mathbb{Z}[D_X], \; j \in \{1, \dots, m \overset{\mathrm{def}}{=} s\}.$$

Moreover, we put

$$D \overset{\mathrm{def}}{=} D_1 + p^{t_1} D_2 + p^{t_1+t_2} D_3 + \cdots + p^{\sum_{j=1}^{m-1} t_j} D_m \in \mathbb{Z}[D_X].$$

We see immediately that D satisfies Condition 3.3, and that $\overline{D} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ is an effective divisor on X whose support is equal to $\{x_1, \dots, x_{s+1}\}$, and whose degree is equal to sn (i.e., $s(\overline{D}) = s$). Then the corollary follows from Theorem 3.5. This completes the proof of the corollary. □

4. Reconstructions of field structures via finite groups

In this section, by applying Theorem 3.5, we prove that the field structures associated to inertia subgroups can be reconstructed group-theoretically from certain finite quotients of admissible fundamental groups. The main result of the present section is Theorem 4.2.

4.1. A lemma for constructing effective divisors.

4.1.1. *Notation and Settings.* We maintain the notation and the settings introduced in 2.2.1. Moreover, suppose $n_X > 0$.

Let $t \in \mathbb{N}$ be a positive natural number, $n \stackrel{\text{def}}{=} p^t - 1$, and $D \in \mathbb{Z}[D_X]$ an effective divisor on X with degree $(n_X - 1)n$ such that $\mathrm{ord}_x(D) \leq n$ for all $x \in D_X$. Suppose the following holds:

- D is Frobenius stable (Definition 2.4 (ii)). Namely, $\deg(D) = \deg(D^{(i)}) = (n_X - 1)n$ for all $i \in \{0, \ldots, t-1\}$.

Note that the above condition is a necessary condition for the existence of the Raynaud-Tamagawa theta divisor associated to the vector bundle $\mathcal{E}_D$ defined in 2.5.2 (see [T2, Lemma 2.15]).

We denote by $d_x^{(i)} \stackrel{\text{def}}{=} \mathrm{ord}_x(D^{(i)})$ (see Definition 2.4 (ii)), $x \in D_X$, and put

$$d_x^{(i)} = \sum_{r=0}^{t-1} d_{x,r}^{(i)} p^r$$

the p-adic expansion of $d_x^{(i)}$. In particular, $D = D^{(0)}$ if $i = 0$. We shall write d_x, $d_{x,r}$ for $d_x^{(0)}$, $d_{x,r}^{(0)}$, respectively.

On the other hand, let $u \in \mathbb{N}$ be an arbitrary positive natural number. We put

$$S_u \stackrel{\text{def}}{=} \{0, \ldots, u-1\}.$$

4.1.2. We have the following lemma:

Lemma 4.1. *We maintain the notation and the settings introduced in 4.1.1. Let $x_1 \in D_X$. Then the following statements hold:*

(i) We have

$$\sum_{x \in D_X} d_{x,r}^{(i)} = (n_X - 1)(p-1)$$

for each $i \in \{0, \ldots, t-1\}$ and each $r \in \{0, \ldots, t-1\}$.

(ii) If $d_{x_1} \in p^{S_t} \stackrel{\text{def}}{=} \{p^b \mid b \in S_t\}$, then we have that

$$\#\big(\{d_x = n \mid x \in D_X\}\big) \geq n_X - 2,$$

and that D satisfies Condition 3.3.

(iii) If $d_{x_1} \in S_{p-1}p^{S_t} \stackrel{\text{def}}{=} \{ap^b \mid a \in S_{p-1},\ b \in S_t\}$, then

$$\#\big(\{d_x = n \mid x \in D_X\}\big) \geq n_X - 2$$

holds if and only if D satisfies Condition 3.3.

(iv) Suppose

$$d_{x_1} \notin p^{S_t} \cup S_{p-1}p^{S_t} = \begin{cases} S_{p-1}p^{S_t}, & \text{if } p \neq 2, \\ S_p p^{S_t}, & \text{if } p = 2. \end{cases}$$

We divide the set $S_t \stackrel{\text{def}}{=} \{0, \ldots, t-1\}$ into the following parts

$$S_t^{\neq 0, p-1} \stackrel{\text{def}}{=} \{r \in S_t \mid d_{x_1,r} \neq 0, p-1\},$$

$$S_t^{=0} \stackrel{\text{def}}{=} \{r \in S_t \mid d_{x_1,r} = 0\},$$

$$S_t^{=p-1} \stackrel{\text{def}}{=} \{r \in S_t \mid d_{x_1,r} = p-1\}.$$

Note that the above assumption implies $\#(S_t^{\neq 0,p-1} \cup S_t^{=p-1}) \geq 2$. *Then there exists a set* $\{d'_x \in S_n\}_{x\in D_X\setminus\{x_1\}}$ *of natural numbers such that the effective divisor*

$$D' \stackrel{\text{def}}{=} d_{x_1}x_1 + \sum_{x\in D_X\setminus\{x_1\}} d'_x x \in \mathbb{Z}[D_X]$$

on X *satisfies Condition 3.3 (in particular,* D' *and* $\overline{D}'$ *are Frobenius stable), and that*

$$s(\overline{D}') = \begin{cases} \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}), & \text{if } \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}) + 1 \leq n_X, \\ n_X - 1, & \text{if } \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}) + 1 > n_X, \end{cases}$$

where $\overline{D}' \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ *denotes the image of* D' *of the composition of maps* $\mathbb{Z}[D_X] \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}[D_X] \overset{\iota_n^{-1}}{\xrightarrow{\sim}} (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]$ *(see 2.3.3 for* ι_n*). Note that we have* $2 \leq s(\overline{D}') \leq n_X - 1$ *if* $n_X \geq 3$.

Proof. (i) This is [Y2, Lemma 3.2].

(ii) If $d_{x_1} = p^b \in p^{S_t}$, then (i) implies that

$$\sum_{x\in D_X\setminus\{x_1\}} d_{x,r} = \begin{cases} (n_X-1)(p-1), & \text{if } r \neq b, \\ (n_X-1)(p-1)-1, & \text{if } r = b, \end{cases}$$

holds for all $r \in \{0,\dots,t-1\}$. If $r \in \{0,\dots,t-1\}\setminus\{b\}$, we see immediately $d_{x,r} = p-1$ for each $x \in D_X \setminus \{x_1\}$. If $r = b$, then there exists $x' \in D_X$ such that $d_{x',b} = p-2$ and $d_{x,b} = p-1$ for all $x \in D_X \setminus \{x_1, x'\}$. We put

$$Q_r \stackrel{\text{def}}{=} \sum_{x\in D_X\setminus\{x_1\}} (p-1)x \in \mathbb{Z}[D_X],\ r \in \{0,\dots,t-1\}\setminus\{b\},$$

$$Q_b \stackrel{\text{def}}{=} x_1 + (p-2)x' + \sum_{x\in D_X\setminus\{x_1,x'\}} (p-1)x \in \mathbb{Z}[D_X].$$

If we put $m \stackrel{\text{def}}{=} t$ and $D_i \stackrel{\text{def}}{=} Q_{i-1}$, $i \in \{1,\dots,t\}$, then we see $D = D_1 + pD_2 + \cdots + p^{t-1}D_t$ satisfying Condition 3.3.

(iii) If $d_{x_1} = ap^b \in S_{p-1}p^{S_t}$, then (i) implies that

$$\sum_{x\in D_X\setminus\{x_1\}} d_{x,r} = \begin{cases} (n_X-1)(p-1), & \text{if } r \neq b, \\ (n_X-1)(p-1)-a, & \text{if } r = b, \end{cases}$$

holds for each $r \in \{0,\dots,t-1\}$. If $r \in \{0,\dots,t-1\}\setminus\{b\}$, we see immediately $d_{x,r} = p-1$ for all $x \in D_X \setminus \{x_1\}$.

Suppose $\#\big(\{d_x = n \mid x \in D_X\}\big) \geq n_X - 2$. If $r = b$, then there exists $x' \in D_X$ such that $d_{x',b} = p-1-a$ and $d_{x,b} = p-1$ for all $x \in D_X \setminus \{x_1, x'\}$. We put

$$Q_r \stackrel{\text{def}}{=} \sum_{x\in D_X\setminus\{x_1\}} (p-1)x \in \mathbb{Z}[D_X],\ r \in \{0,\dots,t-1\}\setminus\{b\},$$

$$Q_b \stackrel{\text{def}}{=} ax_1 + (p-1-a)x' + \sum_{x\in D_X\setminus\{x_1,x'\}} (p-1)x \in \mathbb{Z}[D_X].$$

If we put $m \stackrel{\text{def}}{=} t$ and $D_i \stackrel{\text{def}}{=} Q_{i-1}$, $i \in \{1,\dots,t\}$, then we see $D = D_1 + pD_2 + \cdots + p^{t-1}D_t$ satisfying Condition 3.3.

On the other hand, suppose that D satisfies Condition 3.3. We maintain the notation introduced in Condition 3.3. Note that there exists a unique natural number $s \in \{1, \dots, m\}$ such that

$$\sum_{j=1}^{s} t_j \leq b < \sum_{j=1}^{s+1} t_j.$$

We put $b' \stackrel{\text{def}}{=} b - \sum_{j=1}^{s} t_j < t_{s+1}$. Then we have

$$D_j = \sum_{x \in D_X \setminus \{x_1\}} (p^{t_j} - 1)x$$

for all $j \in \{1, \dots, m\} \setminus \{s+1\}$, and

$$D_{s+1} = ap^{b'}x_1 + (p-1-a)p^{b'}x' + \Big(\sum_{r \in \{0, \dots t_{s+1}-1\} \setminus \{b'\}} (p-1)p^r\Big)x' + \sum_{x \in D_X \setminus \{x_1, x'\}} (p^{t_{s+1}} - 1)x$$

for some $x' \in D_X \setminus \{x_1\}$ if $j = s+1$. Then we see immediately $d_x = n$ for all $D_X \setminus \{x_1, x'\}$. This completes the proof of (ii).

(iv) Suppose

$$d_{x_1} = \sum_{r=0}^{t-1} d_{x_1, r} p^r \notin p^{S_t} \cup S_{p-1} p^{S_t}.$$

Then we have $S_t^{\neq 0, p-1} \cup S_t^{=p-1} \neq \emptyset$. We put

$$d'_{x'[r], r} \stackrel{\text{def}}{=} p - 1 - d_{x_1, r}$$

for some $x'[r] \in D_X \setminus \{x_1\}$ if $r \in S_t^{\neq 0, p-1}$,

$$d'_{x, r} \stackrel{\text{def}}{=} p - 1$$

for all $x \in D_X \setminus \{x_1\}$ if $r \in S_t^{=0}$, and

$$d'_{x''[r], r} \stackrel{\text{def}}{=} p - 2, \ d'_{x'''[r], r} \stackrel{\text{def}}{=} 1$$

for some $x''[r], x'''[r] \in D_X \setminus \{x_1\}$ if $r \in S_t^{=p-1}$. Denote by

$$D'_X \stackrel{\text{def}}{=} \Big(\bigcup_{r \in S_t^{\neq 0, p-1}} \{x'[r]\}\Big) \cup \Big(\bigcup_{r \in S_t^{=p-1}} \{x''[r], x'''[r]\}\Big).$$

By taking suitably chosen marked points $x'[r], x''[r], x'''[r] \in D_X \setminus \{x_1\}$, we have

$$\#(D'_X) = \begin{cases} \#(S_t^{\neq 0, p-1}) + 2\#(S_t^{=p-1}), & \text{if } \#(S_t^{\neq 0, p-1}) + 2\#(S_t^{=p-1}) + 1 \leq n_X, \\ n_X - 1, & \text{if } \#(S_t^{\neq 0, p-1}) + 2\#(S_t^{=p-1}) + 1 > n_X, \end{cases}$$

Furthermore, we put

$$Q'_r \stackrel{\text{def}}{=} d_{x_1, r} + \sum_{x \in D_X \setminus \{x_1\}} d'_{x, r} x \in \mathbb{Z}[D_X], \ r \in \{0, \dots, t-1\},$$

$$d'_x \stackrel{\text{def}}{=} \sum_{r=0}^{t-1} d'_{x, r} p^r \in S_n.$$

Then we see immediately that

$$D' \stackrel{\text{def}}{=} D'_1 + pD'_2 + \cdots + p^{t-1} D'_t = d_{x_1} x_1 + \sum_{x \in D_X \setminus \{x_1\}} d'_x x \in \mathbb{Z}[D_X]$$

satisfies Condition 3.3 if we put $m \stackrel{\text{def}}{=} t$ and $D'_i \stackrel{\text{def}}{=} Q'_{i-1}$, $i \in \{1\ldots,t\}$. Note that the construction of Q'_r, $r \in \{0,\ldots,t-1\}$, implies $\deg(D') = (n_X-1)n$ and

$$\#\big(\{x \in D_X \mid \mathrm{ord}_x(D') = n\}\big) =$$

$$\begin{cases} n_X - \#(S_t^{\neq 0,p-1}) - 2\#(S_t^{=p-1}) - 1, & \text{if } \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}) + 1 \leq n_X, \\ 0, & \text{if } \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}) + 1 > n_X. \end{cases}$$

Thus, we obtain (see the second paragraph of 3.2.1)

$$s(\overline{D}') = \begin{cases} \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}), & \text{if } \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}) + 1 \leq n_X, \\ n_X - 1, & \text{if } \#(S_t^{\neq 0,p-1}) + 2\#(S_t^{=p-1}) + 1 > n_X. \end{cases}$$

This completes the proof of the lemma. □

4.2. **Reconstructions of field structures.**

4.2.1. *Notation and Settings.* Let $i \in \{1,2\}$, and let $X_i^\bullet = (X_i, D_{X_i})$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Moreover, we suppose $n_X > 0$.

4.2.2. Recall that $\widehat{X}_i^\bullet$ is the universal admissible covering of $X_i^\bullet$ corresponding to $\Pi_{X_i^\bullet}$ (2.3.4), and that $\Gamma_{\widehat{X}_i^\bullet}$ is the dual semi-graph of $\widehat{X}_i^\bullet$. We put

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet}) \stackrel{\text{def}}{=} \{I_{\widehat{e}_i}\}_{\widehat{e}_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet})},$$

where $I_{\widehat{e}_i} \subseteq \Pi_{X_i^\bullet}$ denotes the stabilizer subgroup of $\widehat{e}_i$ (2.3.4), and "op" means "open edge".

Let $N_i \subseteq \Pi_{X_i^\bullet}$ be an arbitrary open *normal* subgroup of $\Pi_{X_i^\bullet}$, $Q_{N_i} \stackrel{\text{def}}{=} \Pi_{X_i^\bullet}/N_i$, $X_{N_i}^\bullet = (X_{N_i}, D_{X_{N_i}})$ the pointed stable curve of type (g_{N_i}, n_{N_i}) corresponding to N_i, and $\Gamma_{X_{N_i}^\bullet}$ the dual semi-graph of $X_{N_i}^\bullet$. Write $f_{N_i}^\bullet : X_{N_i}^\bullet \to X_i^\bullet$ for the Galois admissible covering over k_i with Galois group Q_{N_i} corresponding to $N_i \hookrightarrow \Pi_{X_i^\bullet}$. The set of open edges $e^{\mathrm{op}}(\Gamma_{X_{N_i}^\bullet})$ admits an action of Q_{N_i} induced by the Galois admissible covering $f_{N_i}^\bullet$. Let $e_{N_i} \in e^{\mathrm{op}}(\Gamma_{X_{N_i}^\bullet})$ be an open edge of $\Gamma_{X_{N_i}^\bullet}$. We denote by $I_{e_{N_i}} \subseteq Q_{N_i}$ the stabilizer subgroup of e_{N_i}. Moreover, we put

$$\mathrm{Edg}^{\mathrm{op}}(Q_{N_i}) \stackrel{\text{def}}{=} \{I_{e_{N_i}}\}_{e_{N_i} \in e^{\mathrm{op}}(\Gamma_{X_{N_i}^\bullet})}.$$

Let $\widehat{e}'_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet})$ and $e'_{N_i} \in e^{\mathrm{op}}(\Gamma_{X_{N_i}^\bullet})$ the image of $\widehat{e}'_i$ of the natural surjection $\Gamma_{\widehat{X}_i^\bullet} \twoheadrightarrow \Gamma_{X_{N_i}^\bullet}$. Then the image of $I_{\widehat{e}'_i}$ of the natural surjection $\Pi_{X_i^\bullet} \twoheadrightarrow Q_{N_i}$ is equal to $I_{e'_{N_i}}$. Thus, the surjection $\Pi_{X_i^\bullet} \twoheadrightarrow Q_{N_i}$ induces a surjection

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{N_i}),\ I_{\widehat{e}'_i} \mapsto I_{e'_{N_i}}.$$

4.2.3. *Field structures associated to inertia subgroups.* Let $\widehat{e}_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet})$. We put

$$\mathbb{F}_{\widehat{e}_i} \stackrel{\text{def}}{=} (I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widehat{e}_i}\},$$

where $\{*_{\widehat{e}_i}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_i^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with $(\mathbb{Q}/\mathbb{Z})_i^{p'}(1) \stackrel{\text{def}}{=} \bigcup_{(p,m)=1} \mu_m(k_i)$. Moreover, let $a_{\widehat{e}_i}$ be a generator of $I_{\widehat{e}_i}$. Then we have a natural bijection

$$I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'},\ a_{\widehat{e}_i} \otimes 1 \mapsto 1 \otimes 1.$$

Let $\overline{\mathbb{F}}_{p,j}$ be the algebraic closure of $\mathbb{F}_p$ in k_i. Thus, we obtain the following bijections

$$I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_i^{p'}(1) \xrightarrow{\sim} \overline{\mathbb{F}}_{p,i}^{\times}.$$

This means that $\mathbb{F}_{\widehat{e}_i}$ can be identified with $\overline{\mathbb{F}}_{p,i}$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}$, and whose zero element is $*_{\widehat{e}_i}$.

Let Δ be a profinite group and $b \in \mathbb{N}$ a positive natural number. We denote by $D_b(\Delta) \subseteq \Delta$ the topological closure of $[\Delta, \Delta]\Delta^b$, where $[\Delta, \Delta]$ denotes the commutator subgroup of Δ. Let $t \in \mathbb{N}$ be a positive natural number and $n \overset{\text{def}}{=} p^t - 1$. Let $\Pi^{\text{ét}}_{X_i^\bullet}$ be the étale fundamental group of the underlying curve X and $A_i \subseteq \Pi_{X_i^\bullet}$ the inverse image of $D_n(\Pi^{\text{ét}}_{X_i^\bullet})$ of the natural surjection $\Pi_{X_i^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X_i^\bullet}$ (2.2.2). We shall put

$$O_i \overset{\text{def}}{=} \begin{cases} D_n(A_i), & \text{if } n_X < 3, \\ D_n(\Pi_{X_i^\bullet}), & \text{if } n_X \geq 3. \end{cases}$$

Note that the structures of maximal prime-to-p quotients of admissible fundamental groups (2.2.2) imply $I \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ for all $I \in \mathrm{Edg}^{\mathrm{op}}(Q_{O_i})$, where $Q_{O_i} \overset{\text{def}}{=} \Pi_{X_i^\bullet}/O_i$.

Let $f^\bullet_{O_i} : X^\bullet_{O_i} = (X_{O_i}, D_{X_{O_i}}) \to X_i^\bullet$ be the Galois admissible covering over k_i corresponding to $O_i \hookrightarrow \Pi_{X_i^\bullet}$, $\Gamma_{X^\bullet_{O_i}}$ the dual semi-graph of $X^\bullet_{O_i}$, and $e_{O_i} \in e^{\mathrm{op}}(\Gamma_{X^\bullet_{O_i}})$ the image of $\widehat{e}_i$ of the natural surjection $\Gamma_{\widehat{X}_i^\bullet} \twoheadrightarrow \Gamma_{X^\bullet_{O_i}}$. Write $I_{e_{O_i}} \in \mathrm{Edg}^{\mathrm{op}}(Q_{O_i})$ for the stabilizer subgroup of e_{O_i}. Then the image of $I_{\widehat{e}_i}$ of the natural surjection $\Pi_{X_i^\bullet} \twoheadrightarrow Q_{O_i}$ is equal to $I_{e_{O_i}}$. Moreover, write $a_{e_{O_i}}$ for the image of $a_{\widehat{e}_i}$ of the surjection $I_{\widehat{e}_i} \twoheadrightarrow I_{e_{O_i}}$. Since $I_{e_{O_i}} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\sim} \mu_n(k_i) \hookrightarrow \overline{\mathbb{F}}_{p,i}^{\times}$, where the first arrow is determined by $a_{e_{O_i}} \mapsto 1$, the set

$$\mathbb{F}_{e_{O_i},t} \overset{\text{def}}{=} I_{e_{O_i}} \sqcup \{*_{\widehat{e}_i}\} \subseteq \mathbb{F}_{\widehat{e}_i}$$

admits a structure of field induced by $\mathbb{F}_{\widehat{e}_i}(\xrightarrow{\sim} \overline{\mathbb{F}}_{p,i})$ which is isomorphic to the subfield of $\overline{\mathbb{F}}_{p,i}$ with cardinality p^t.

4.2.4. Now, we can state the second main result of the present paper:

Theorem 4.2. *We maintain the notation and the settings introduced in 4.2.1. Let* $n \overset{\text{def}}{=} p^t - 1 \in \mathbb{N}$ *be a positive natural number satisfying the following condition:*

- *Let m_0 be the product of all prime numbers $\leq p-2$ if $p > 3$ and $m_0 \overset{\text{def}}{=} \{1\}$ if $p \in \{2, 3\}$. We put t_0 the order of p in the multiplicative group $(\mathbb{Z}/m_0\mathbb{Z})^{\times}$. Then we have*

$$n \overset{\text{def}}{=} p^t - 1 > \max\{C(g_X) + 1, \#(X^{\text{sing}}) + n_X, 2\}, \ (p^{t_0} - 1)|n.$$

Let $H_i \subseteq \Pi_{X_i^\bullet}$ be an open normal subgroup satisfying $H_i \subseteq D_p(O_i)$, where $O_i \subseteq \Pi_{X_i^\bullet}$ denotes the open subgroup defined in 4.2.3. Then the following statements hold:

(i) The field structure of $\mathbb{F}_{e_{O_i},t}$ (4.2.3) can be reconstructed group-theoretically from $Q_{H_i} \overset{\text{def}}{=} \Pi_{X_i^\bullet}/H_i$, Q_{O_i}, and $\mathrm{Edg}^{\mathrm{op}}(Q_{O_i})$. Namely, there exists a group-theoretical algorithm whose input data are Q_{H_i}, Q_{O_i}, and $\mathrm{Edg}^{\mathrm{op}}(Q_{O_i})$, and whose output datum is $\mathbb{F}_{e_{O_i},t}$ as a field.

(ii) Let $\phi : Q_{H_1} \twoheadrightarrow Q_{H_2}$ be a surjection. Suppose that the following conditions are satisfied:

- *ϕ fits into the following commutative diagram*

$$\begin{tikzcd} Q_{H_1} \arrow[r, "\phi"] \arrow[d] & Q_{H_2} \arrow[d] \\ \Pi_{X_1^\bullet}/D^{(1)}(O_1) \arrow[r, "\psi"] \arrow[d] & \Pi_{X_2^\bullet}/D^{(1)}(O_2) \arrow[d] \\ Q_{O_1} \arrow[r, "\rho"] & Q_{O_2} \end{tikzcd}$$

such that ρ is an isomorphism.
- *ρ induces a bijection $\rho^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(Q_{O_1}) \xrightarrow{\sim} \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$, $I \mapsto \rho(I)$, such that $\rho(I_{e_{O_1}}) = I_{e_{O_2}}$.*

Then the isomorphism $\rho_{e_{O_1},e_{O_2}} \overset{\mathrm{def}}{=} \rho|_{I_{e_{O_1}}} : I_{e_{O_1}} \xrightarrow{\sim} I_{e_{O_2}}$ induces a field isomorphism

$$\rho^{\mathrm{fd}}_{e_{O_1},e_{O_2}} : \mathbb{F}_{e_{O_1},t} \xrightarrow{\sim} \mathbb{F}_{e_{O_2},t},$$

where "fd" means "field".

Proof. Suppose $n_X < 3$. Let $O'_i \overset{\mathrm{def}}{=} D_n(A_i)$ and $f^\bullet_{O'_i} : X^\bullet_{O'_i} = (X_{O'_i}, D_{X_{O'_i}}) \to X^\bullet_i$ the Galois admissible covering over k_i corresponding to $O'_i \hookrightarrow \Pi_{X_i^\bullet}$. The definition of A_i implies that $f^\bullet_{O'_i}$ is étale (i.e., the morphism of underlying curves induced by $f^\bullet_{O'_i}$ is étale). Then $I_{e_{O_i}}$ is contained in $O'_i/D_n(O'_i)$. Moreover, we have $n_{O'_i} \overset{\mathrm{def}}{=} \#(D_{X_{O'_i}}) \geq 3$ since $n > 2$. Thus, by replacing $X^\bullet_i$, $\Pi_{X_i^\bullet}$, and Q_{O_i} by $X^\bullet_{O'_i}$, O'_i, and $O'_i/D_n(O'_i)$, respectively, to verify the theorem, it is sufficient to assume $n_X \geq 3$.

From now on, we suppose $n_X \geq 3$. Note that since $O_i \overset{\mathrm{def}}{=} D_n(\Pi_{X_i^\bullet})$ when $n_X \geq 2$ (4.2.3), we have the following isomorphism

$$Q_{O_i} \xrightarrow{\sim} \langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X}[a_i, b_i]\prod_{j=1}^{n_X} c_j = 1\rangle^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}.$$

Let $e' \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$ and $e'_{O_i} \in f^{\mathrm{sg},-1}_{O_i}(e') \subseteq e^{\mathrm{op}}(\Gamma_{X^\bullet_{O_i}})$, where $f^{\mathrm{sg}}_{O_i} : \Gamma_{X^\bullet_{O_i}} \to \Gamma_{X_i^\bullet}$ denotes the map of dual semi-graphs induced by $f^\bullet_{O_i}$ introduced in 4.2.3. We see that $I_{e'_{O_i}}$ does not depend on the choices of $e'_{O_i} \in f^{\mathrm{sg},-1}_{O_i}(e')$. Thus, we may denote by $I_{e'} \overset{\mathrm{def}}{=} I_{e'_{O_i}}$, and we have

$$\mathrm{Edg}^{\mathrm{op}}(Q_{O_i}) = \{I_{e'}\}_{e' \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})},$$

$$\mathrm{Edg}^{\mathrm{op}}(Q_{O_i}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_i^\bullet}),\ I_{e'} \mapsto e'.$$

Moreover, there exists a generator s_e of I_e for each $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ such that

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} s_e = 0$$

in Q_{O_i}, and $s_e = a_{e_{O_i}}$ if $e_{O_i} \in f^{\mathrm{sg},-1}_{O_i}(e)$ (see 4.2.3 for $a_{e_{O_i}}$).

For $\alpha_i \in \mathrm{Hom}_{\mathrm{gp}}(Q_{O_i}, \mathbb{F}^\times_{p^t}) = \mathrm{Hom}_{\mathrm{gp}}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{F}^\times_{p^t})$ ("gp" means "group"), we put

$$D_{\alpha_i} \overset{\mathrm{def}}{=} \sum_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} \alpha_i(s_e)^{\sim} x_e \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$$

where $x_e \in D_X$ denotes the marked point corresponding to e, and $\alpha_i(s_e)^\sim$ denotes the element of $(\mathbb{Z}/n\mathbb{Z})^\sim$ corresponding to $\alpha_i(s_e)$ via the natural bijection $(\mathbb{Z}/n\mathbb{Z})^\sim \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ defined in 2.3.3. We shall put

$$\mathrm{Hom}^{\mathrm{fs}}_{\mathrm{gp}}(Q_{O_i}, \mathbb{F}^\times_{p^t}) \overset{\mathrm{def}}{=} \{\alpha_i \in \mathrm{Hom}_{\mathrm{gp}}(Q_{O_i}, \mathbb{F}^\times_{p^t}) \mid D_{\alpha_i} \text{ is Frobenius stable (see Definition 2.4)}\}.$$

Note that the above constructions imply that $\mathrm{Hom}^{\mathrm{fs}}_{\mathrm{gp}}(Q_{O_i}, \mathbb{F}^\times_{p^t})$ can be reconstructed group-theoretically from Q_{O_i} and $\mathrm{Edg}^{\mathrm{op}}(Q_{O_i})$.

(i) Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$ and $\mathbb{F}_{p^t} \subseteq \overline{\mathbb{F}}_p$ the subfield with cardinality p^t. The field structure of $\mathbb{F}_{x_{O_i},t}$ is equivalent to the subset

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_i},t}, \mathbb{F}_{p^t}) \subseteq \mathrm{Hom}_{\mathrm{gp}}(\mathbb{F}^\times_{x_{O_i},t}, \mathbb{F}^\times_{p^t}),$$

where "fd" means "field". Then in order to prove (i), it is sufficient to prove that the set $\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_i},t}, \mathbb{F}_{p^t})$ can be reconstructed group-theoretically from Q_{H_i}, Q_{O_i}, and $\mathrm{Edg}^{\mathrm{op}}(Q_{O_i})$.

Let $\chi_i \in \mathrm{Hom}_{\mathrm{gp}}(Q_{O_i}, \mathbb{F}^\times_{p^t})$. We put

$$H_{\chi_i} \overset{\mathrm{def}}{=} \ker(Q_{H_i} \twoheadrightarrow Q_{O_i} \xrightarrow{\chi_i} \mathbb{F}^\times_{p^t}),\ M_{\chi_i} \overset{\mathrm{def}}{=} H^{\mathrm{ab}}_{\chi_i} \otimes \mathbb{F}_p.$$

Then M_{χ_i} admits a natural action of Q_{O_i} via the conjugation action. Since we assume $H_i \subseteq D_p(O_i)$, we see $M_{\chi_i} = \left(\ker(\Pi_{X_i^\bullet} \twoheadrightarrow Q_{O_i} \xrightarrow{\chi_i} \mathbb{F}^\times_{p^t})\right)^{\mathrm{ab}} \otimes \mathbb{F}_p$. Denote by

$$M_{\chi_i}[\chi_i] \overset{\mathrm{def}}{=} \{a \in M_{\chi_i} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma \cdot a = \chi_i(\sigma)a \text{ for all } \sigma \in Q_{O_i}\},$$

$$\gamma_{\chi_i}(M_{\chi_i}) \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(M_{\chi_i}[\chi_i]).$$

The integer $\gamma_{\chi_i}(M_{\chi_i})$ is a generalized Hasse-Witt invariant of the cyclic admissible covering of $X_i^\bullet$ corresponding to $\ker(\Pi_{X_i^\bullet} \twoheadrightarrow Q_{O_i} \xrightarrow{\chi_i} \mathbb{F}^\times_{p^t}) \hookrightarrow \Pi_{X_i^\bullet}$. Note that we have $\gamma_{\chi_i}(M_{\chi_i}) \leq g_X + s(D_{\chi_i}) - 1$ if D_{χ_i} is not zero (see Lemma 2.5). We define two maps

$$\mathrm{Res}^{\mathrm{fs}}_{i,t} : \mathrm{Hom}^{\mathrm{fs}}_{\mathrm{gp}}(Q_{O_i}, \mathbb{F}^\times_{p^t}) \to \mathrm{Hom}_{\mathrm{gp}}(\mathbb{F}^\times_{x_{O_i},t}, \mathbb{F}^\times_{p^t}),$$

$$\Gamma^{\mathrm{fs}}_{i,t} : \mathrm{Hom}^{\mathrm{fs}}_{\mathrm{gp}}(Q_{O_i}, \mathbb{F}^\times_{p^t}) \to \mathbb{Z}_{\geq 0},\ \chi_i \mapsto \gamma_{\chi_i}(M_{\chi_i}),$$

where the map $\mathrm{Res}^{\mathrm{fs}}_{i,t}$ is the restriction with respect to the natural inclusion $\mathbb{F}^\times_{x_{O_i},t} = I_{x_{O_i}} \hookrightarrow Q_{O_i}$. It is easy to see that $\mathrm{Res}^{\mathrm{fs}}_{i,t}$ is a surjection. We put $\mathcal{H} \overset{\mathrm{def}}{=} \{g_X + 1, g_X + 2, \dots, g_X + n_X - 2\}$. Then (i) follows from the following claim:

Claim. We have

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{e_{O_i},t}, \mathbb{F}_{p^t}) = \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{gp}}(\mathbb{F}^\times_{e_{O_i},t}, \mathbb{F}^\times_{p^t}) \setminus \mathrm{Res}^{\mathrm{fs}}_{i,t}\left((\Gamma^{\mathrm{fs}}_{i,t})^{-1}(\mathcal{H})\right),$$

where $\mathrm{Hom}^{\mathrm{surj}}_{\mathrm{gp}}(-,-)$ denotes the subset of $\mathrm{Hom}_{\mathrm{gp}}(-,-)$ whose elements are surjections.

Proof of Claim. Fix a primitive nth root $\zeta \in \mathbb{F}_{p^t}$, we may identify $\mathbb{F}^\times_{p^t}$ with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta \mapsto 1$, and identify $I_{e_{O_i}} = \mathbb{F}^\times_{e_{O_i},t}$ with $\mathbb{Z}/n\mathbb{Z}$ via the map $a_{e_{O_i}} \mapsto 1$ (4.2.3). By considering the Frobenius element of $\mathrm{Gal}(\mathbb{F}_{p^t}/\mathbb{F}_p)$, we see that a homomorphism $\omega \in \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{gp}}(\mathbb{F}^\times_{e_{O_i},t}, \mathbb{F}^\times_{p^t}) = \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{gp}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is a field isomorphism contained in $\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{e_{O_i},t}, \mathbb{F}_{p^t})$ if and only if $\omega(1) \in p^{S_t} \overset{\mathrm{def}}{=} \{1, p, \dots, p^{t-1}\}$.

Let $f^\bullet_{O_i} : X^\bullet_{O_i} = (X_{O_i}, D_{X_{O_i}}) \to X_i^\bullet$ be the Galois admissible covering over k_i corresponding to $O_i \hookrightarrow \Pi_{X_i^\bullet}$. Write $x_{O_i} \in D_{X_{O_i}}$ for the marked point corresponding to e_{O_i} and $x_{i,1} \in D_{X_i}$ for $f_{O_i}(x_{O_i})$, where f_{O_i} denotes the morphism of underlying curves induced by $f^\bullet_{O_i}$. Then the claim is equivalent to the following statement:

$\omega(1) \in p^{S_t}$ if and only if $(\omega(1), n) = 1$, and there does not exist an effective divisor $D'_i \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_i}]^0$ such that D'_i is Frobenius stable, that $\mathrm{ord}_{x_{i,1}}(D'_i) = \omega(1)$, and that $\gamma_{(\alpha, D'_i)} = g_X + s(D'_i) - 1 \in \mathcal{H}$ for some $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_{D'_i}(X^\bullet_i) \setminus \{0\}$.

Firstly, we treat the "only if" part of the above statement. Suppose that there exists an effective divisor $D'_i \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_i}]^0$ such that D'_i is Frobenius stable, that $\mathrm{ord}_{x_{i,1}}(D'_i) = \omega(1)$, and that $\gamma_{(\alpha, D'_i)} = g_X + s(D'_i) - 1 \in \mathcal{H}$ for some $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_{D'_i}(X^\bullet_i) \setminus \{0\}$. Since D'_i is Frobenius stable, Lemma 4.1 (ii) implies $s(D'_i) = 1$. This means $\gamma_{(\alpha', D'_i)} \leq g_X \notin \mathcal{H}$ for all $\alpha' \in \mathrm{Rev}^{\mathrm{adm}}_{D'_i}(X^\bullet_i) \setminus \{0\}$. This contradicts $\gamma_{(\alpha, D'_i)} = g_X + s(D'_i) - 1 \in \mathcal{H}$.

Next, to verify the "if" part of the above statement, suppose $\omega(1) \notin p^{S_t}$. Since $\omega(1)$ is prime to n, the assumption $m_0|(p^{t_0} - 1)|n$ implies that $\omega(1) \notin S_{p-1}p^{S_t} \overset{\mathrm{def}}{=} \{ap^b \mid a = 0, \dots, p-2, b = 0, \dots, t-1\}$. Then Lemma 4.1 (iv) implies that there exists $\overline{D}'_i \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_i}]^0$ on X_i such that Condition 3.3 is satisfied, and that $2 \leq s(\overline{D}'_i) \leq n_X - 1$ holds since we assume $n_X \geq 3$. Moreover, since $n \overset{\mathrm{def}}{=} p^t - 1 > \max\{C(g_X) + 1, \#(X^{\mathrm{sing}}) + n_X\}$, Theorem 3.5 implies that $\gamma_{(\alpha, \overline{D}'_i)} \in \mathcal{H}$ for some $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_{\overline{D}'_i}(X^\bullet_i) \setminus \{0\}$. This contradicts our assumptions. Then we obtain $\omega(1) \in p^{S_t}$. This completes the proof of the claim.

(ii) Let $\kappa_2 \in \mathrm{Hom}^{\mathrm{fs}}_{\mathrm{gp}}(Q_{O_2}, \mathbb{F}^\times_{p^t})$. Then we obtain a character

$$\kappa_1 \in \mathrm{Hom}^{\mathrm{fs}}_{\mathrm{gp}}(Q_{O_1}, \mathbb{F}^\times_{p^t})$$

induced by $\rho : Q_{O_1} \overset{\sim}{\to} Q_{O_2}$ and $\rho^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(Q_{O_1}) \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$. Moreover, the surjection $\phi|_{H_{\kappa_1}} : H_{\kappa_1} \twoheadrightarrow H_{\kappa_2}$ induces a surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$. Suppose $\kappa_2 \in (\Gamma^{\mathrm{fs}}_{2,r})^{-1}(\mathcal{H})$. The surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$ implies $\gamma_{\kappa_1}(M_{\kappa_1}) \geq \gamma_{\kappa_2}(M_{\kappa_2})$. Namely, we have $\kappa_1 \in (\Gamma^{\mathrm{fs}}_{1,t})^{-1}(\mathcal{H})$. Thus, the isomorphism $\rho_{e_{O_1}, e_{O_2}} : I_{e_{O_1}} \overset{\sim}{\to} I_{e_{O_2}}$ induces an injection

$$\mathrm{Res}^{\mathrm{fs}}_{2,t}((\Gamma^{\mathrm{fs}}_{2,t})^{-1}(\mathcal{H})) \hookrightarrow \mathrm{Res}^{\mathrm{fs}}_{1,t}((\Gamma^{\mathrm{fs}}_{1,t})^{-1}(\mathcal{H})).$$

Since $\#(\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{e_{O_1},t}, \mathbb{F}_{p^t})) = \#(\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{e_{O_2},t}, \mathbb{F}_{p^t}))$, we obtain that $\rho_{e_{O_1}, e_{O_2}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{e_{O_2},t}, \mathbb{F}_{p^t}) \overset{\sim}{\to} \mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{e_{O_1},t}, \mathbb{F}_{p^t}).$$

If we choose $\mathbb{F}_{p^t} = \mathbb{F}_{e_{O_2},t}$, then the image of $\mathrm{id}_{\mathbb{F}_{e_{O_2},t}}$ via the above bijection induces a field isomorphism

$$\rho^{\mathrm{fd}}_{e_{O_1}, e_{O_2}} : \mathbb{F}_{e_{O_1},t} \overset{\sim}{\to} \mathbb{F}_{e_{O_2},t}.$$

This completes the proof of (ii) of the theorem. □

Remark 4.2.1. In the statement of Theorem 4.2 (i) and (ii), we assume that the following group-theoretical data are given:

- The set of inertia subgroups $\mathrm{Edg}^{\mathrm{op}}(Q_{O_i})$ of open edges of the dual semi-graph of $X^\bullet_{O_i}$ (=the set of inertia subgroups of marked points of $X^\bullet_{O_i}$).
- The surjection $\phi : Q_{H_1} \twoheadrightarrow Q_{H_2}$ induces a commutative diagram

$$\begin{array}{ccc} \Pi_{X^\bullet_1}/D^{(1)}(O_1) & \xrightarrow{\psi} & \Pi_{X^\bullet_2}/D^{(1)}(O_2) \\ \downarrow & & \downarrow \\ Q_{O_1} & \xrightarrow{\rho} & Q_{O_2} \end{array}$$

  such that ρ is an isomorphism, that ρ induces a bijection $\rho^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(Q_{O_1}) \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$, $I \mapsto \rho(I)$, and that $\rho(I_{e_{O_1}}) = I_{e_{O_2}}$.

As we mentioned in §1.3 of the introduction of the present paper, reconstructions of field structures associated to inertia subgroups play a critical role in the theory of anabelian geometry of curves over algebraically closed fields of characteristic p. However, in general, the above group-theoretical data *do not hold* for an arbitrary open subgroup $H_i \subseteq \Pi_{X_i^\bullet}$ such that $H_i \subseteq D_p(O_i)$. This means that we *cannot* apply directly Theorem 4.2 to anabelian geometry.

To overcome the difficulties, in [Y7], the author of the present paper introduced the so-called "*quasi-anabelian pairs*" (for various combinatorial data of dual semi-graphs) associated to admissible fundamental groups (see [Y6, §4] for the case of the tame fundamental groups of *smooth* pointed stable curves). More precisely, for the case of open edges, we have the following:

Quasi-anabelian pairs for open-edge-like subgroups. Let $A \subseteq B \subseteq \Pi_{X_2^\bullet}$ be open characteristic subgroups of $\Pi_{X_2^\bullet}$. The pair of finite quotients

$$(Q_A \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/A, Q_B \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/B)$$

is called a *quasi-anabelian pair for open-edge-like subgroups associated to* $\Pi_{X_2^\bullet}$ if, for *any* surjection $\alpha : \Pi_{X_1^\bullet} \twoheadrightarrow Q_A$, the composition of surjections $\beta : \Pi_{X_1^\bullet} \stackrel{\alpha}{\twoheadrightarrow} Q_A \twoheadrightarrow Q_B$ induces a surjection

$$\beta^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_B),$$

where $Q_A \twoheadrightarrow Q_B$ denotes the natural surjection induced by $A \subseteq B$.

In [Y7], we established a general method for constructing explicitly quasi-anabelian pairs associated to admissible fundamental groups of arbitrary pointed stable curves (see [Y6, §4] for the case of the tame fundamental groups of *smooth* pointed stable curves).

Once a quasi-anabelian pair can be explicitly constructed, we can construct a quasi-anabelian pair (Q_A, Q_B) for open-edge-like subgroups associated to $\Pi_{X_2^\bullet}$ such that $B \subseteq D_p(O_1)$ holds. We put $H_2 \stackrel{\mathrm{def}}{=} B$ and $H_1 \stackrel{\mathrm{def}}{=} \ker(\beta)$. Then we see that β induces a surjection $\phi : Q_{H_1} \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/H_1 \twoheadrightarrow Q_{H_2}$. Moreover, $\beta^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_B)$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Edg}^{\mathrm{op}}(Q_{H_1}) & \xrightarrow{\phi^{\mathrm{op}}} & \mathrm{Edg}^{\mathrm{op}}(Q_{H_2}) \\ \downarrow & & \downarrow \\ \mathrm{Edg}^{\mathrm{op}}(Q_{O_1}) & \xrightarrow{\rho^{\mathrm{op}}} & \mathrm{Edg}^{\mathrm{op}}(Q_{O_2}), \end{array}$$

where the horizontal arrows are surjections, and the vertical arrows are surjections induced by the natural surjections $Q_{H_1} \twoheadrightarrow Q_{O_1}$ and $Q_{H_2} \twoheadrightarrow Q_{O_2}$. Then we see immediately that the group-theoretical data mentioned in the first paragraph of the remark are satisfied. Namely, we have the following strong version of Theorem 4.2:

> We maintain the notation and the settings introduced in 4.2.1. Let $n \stackrel{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$ be a positive natural number satisfying the following condition:
> - Let m_0 be the product of all prime numbers $\leq p-2$ if $p > 3$ and $m_0 \stackrel{\mathrm{def}}{=} \{1\}$ if $p \in \{2, 3\}$. We put t_0 the order of p in the multiplicative group $(\mathbb{Z}/m_0\mathbb{Z})^\times$. Then we have
>
> $$n \stackrel{\mathrm{def}}{=} p^t - 1 > \max\{C(g_X) + 1, \#(X^{\mathrm{sing}}) + n_X, 2\},\ (p^{t_0} - 1)|n.$$
>
> Let $N_2 \subseteq H_2 \subseteq \Pi_{X^\bullet}$ are open characteristic subgroups such that (Q_{N_2}, Q_{H_2}) is a quasi-anabelian pair for open-edge-like subgroups associated to $\Pi_{X_2^\bullet}$,

and that $H_2 \subseteq D_p(O_2)$, where $O_2 \subseteq \Pi_{X_2^\bullet}$ denotes the open subgroup defined in 4.2.3. Let $\alpha : \Pi_{X_1^\bullet} \twoheadrightarrow Q_{N_2}$ be an arbitrary surjection and $\beta : \Pi_{X_1^\bullet} \overset{\alpha}{\twoheadrightarrow} Q_{N_2} \twoheadrightarrow Q_{H_2}$ the composition of surjections. We put $H_1 \overset{\text{def}}{=} \ker(\beta)$. Then the following statements hold:

(i) We have that H_1 is contained in $D_p(O_1)$, and that the field structure of $\mathbb{F}_{e_{O_i},t}$ can be reconstructed group-theoretically from Q_{H_i} and Q_{O_i}, $i \in \{1, 2\}$, where $O_1 \subseteq \Pi_{X_1^\bullet}$ denotes the open subgroup defined in 4.2.3.

(ii) Let $\phi : Q_{H_1} \twoheadrightarrow Q_{H_2}$ be the surjection induced by β. Then the isomorphism $\rho_{e_{O_1},e_{O_2}} \overset{\text{def}}{=} \rho|_{I_{e_{O_1}}} : I_{e_{O_1}} \overset{\sim}{\to} I_{e_{O_2}}$ induces a field isomorphism $\rho^{\text{fd}}_{e_{O_1},e_{O_2}} : \mathbb{F}_{e_{O_1},t} \overset{\sim}{\to} \mathbb{F}_{e_{O_2},t}$.

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