Maximums of Generalized Hasse-Witt Invariants of Pointed Stable Curves in Positive Characteristic

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Abstract

In the present paper, we study generalized Hasse-Witt invariants of cyclic coverings of curves in positive characteristic. Let $X^\bullet = (X, D_X)$ be a pointed stable curve of topological type (g_X, n_X) over an algebraically closed field of characteristic $p > 0$. We prove that, if $X^\bullet$ is component-generic, then the first generalized Hasse-Witt invariant of each prime-to-p cyclic admissible coverings of $X^\bullet$ attains the maximum under certain assumptions. This result generalizes a result of S. Nakajima concerning the ordinariness of prime-to-p cyclic étale coverings of smooth projective generic curves to the case of (possibly ramified) admissible coverings of (possibly singular) pointed stable curves. Moreover, without any assumptions, we prove that there exists a prime-to-p cyclic admissible covering of $X^\bullet$ such that the first generalized Hasse-Witt invariant of the cyclic admissible covering attains the maximum. This result can be regarded as an analogue of a result of M. Raynaud concerning the new-ordinariness of prime-to-p cyclic étale coverings of arbitrary smooth projective curves in the case of generalized Hasse-Witt invariants of prime-to-p cyclic admissible coverings of arbitrary pointed stable curves. As applications, we obtain a group-theoretical formula for (g_X, n_X), and prove that the field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from surjective open continuous homomorphisms of admissible fundamental groups. Those results generalize A. Tamagawa's results concerning group-theoretical formula for topological types and field structures associated to inertia subgroups of marked points of smooth pointed stable curves to the case of arbitrary pointed stable curves.

Keywords: pointed stable curve, admissible covering, admissible fundamental group, generalized Hasse-Witt invariant, Raynaud-Tamagawa theta divisor, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14G32.

Contents

1 Introduction

In the present paper, we study generalized Hasse-Witt invariants of cyclic coverings of curves in positive characteristic. Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of topological type or type, for short, (g_X, n_X) over an algebraically closed field k, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. Moreover, by choosing a suitable base point of $X^\bullet$, we have the admissible fundamental group

$$\Pi_{X^\bullet}$$

of $X^\bullet$ (cf. Definition 2.2). In particular, if $X^\bullet$ is smooth over k, then $\Pi_{X^\bullet}$ is naturally (outer) isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(X \setminus D_X)$.

Suppose that the characteristic $\mathrm{char}(k)$ of k is 0. Then the structure of $\Pi_{X^\bullet}$ is well-known, and is isomorphic to the profinite completion of the following group (cf. [V, Théorème 2.2 (c)])

$$\langle a_1, \ldots, a_{g_X}, b_1, \ldots, b_{g_X}, c_1, \ldots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle.$$

In particular, $\Pi_{X^\bullet}$ is a free profinite group with $2g_X + n_X - 1$ generators if $n_X > 0$. Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Suppose that $n_{X_i} > 0$, $i \in \{1, 2\}$. Then we see that $\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet}$ if and only if $2g_{X_1} + n_{X_1} - 1 = 2g_{X_2} + n_{X_2} - 1$. Thus, the type (g_X, n_X) cannot be determined group-theoretically from the isomorphism class of $\Pi_{X^\bullet}$.

On the other hand, when $\mathrm{char}(k) = p > 0$, the situation is quite different from that in characteristic 0, and the structure of $\Pi_{X^\bullet}$ is no longer known. In the remainder of the introduction, we assume that $\mathrm{char}(k) = p > 0$. The admissible fundamental group $\Pi_{X^\bullet}$ is very mysterious. In fact, some developments of F. Pop-M. Saïdi, M. Raynaud, A. Tamagawa, and the author (cf. [PS], [R2], [T1], [T2], [T3], [Y1], [Y2], [Y3], [Y5]) showed evidence for very strong *anabelian* phenomena for curves over *algebraically closed fields of characteristic* $p > 0$. In this situation, the Galois group of the base field is trivial, and the étale (or tame) fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field. This kinds of anabelian phenomenon go beyond Grothendieck's anabelian geometry, and show that the admissible (or tame) fundamental group of a pointed stable curve over an algebraically closed field of characteristic p must encode"*moduli*" of the curve. This is the reason that we do not have an explicit description of the admissible (or tame) fundamental group of any pointed stable curve in positive characteristic. Note that since all the admissible coverings (cf. Definition 2.2) in positive characteristic can be lifted to characteristic 0 (cf. [V, Théorème 2.2 (c)]), we obtain that $\Pi_{X^\bullet}$ is topologically finitely generated. Then the isomorphism class of $\Pi_{X^\bullet}$ is determined by the set of finite quotients of $\Pi_{X^\bullet}$ (cf. [FJ, Proposition 16.10.6]).

Furthermore, the theory developed in [T2] and [Y2] implies that the isomorphism class of $X^\bullet$ as a scheme can possibly be determined by not only the isomorphism class of $\Pi_{X^\bullet}$ as a profinite group but also the isomorphism class of the maximal pro-solvable quotient of $\Pi_{X^\bullet}$. Then we may ask the following question:

Which finite solvable groups can appear as a quotient of $\Pi_{X^\bullet}$*?*

Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup and $X_H^\bullet = (X_H, D_{X_H})$ the pointed stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H. We have an important invariant σ_{X_H} associated to $X_H^\bullet$ (or H) which is called *p-rank* (or *Hasse-Witt invariant*, see Definition 2.3). Roughly speaking, σ_{X_H} controls the finite quotients of $\Pi_{X^\bullet}$ which are extensions of the group $\Pi_{X^\bullet}/H$ by p-groups. Since the structures of maximal prime-to-p quotients of admissible fundamental groups have been known, in order to solve the question mentioned above, we need compute the p-rank σ_{X_H} when $\Pi_{X^\bullet}/H$ is abelian. If $\Pi_{X^\bullet}/H$ is a p-group, then σ_{X_H} can be computed by applying the Deuring-Shafarevich formula (cf. [C]). If $\Pi_{X^\bullet}/H$ is not a p-group, the situation of σ_{X_H} is very complicated. The Deuring-Shafarevich formula implies that, to compute σ_{X_H}, we only need to assume that $\Pi_{X^\bullet}/H$ is a prime-to-p group.

First, let us consider the case of generic curves. Suppose that $n_X = 0$, and that $X^\bullet$ is *smooth* over k. If $X^\bullet$ is a curve corresponding to a geometric generic point of the moduli space (i.e., a geometric generic curve), S. Nakajima (cf. [N]) proved that, if $\Pi_{X^\bullet}/H$ is a cyclic group with a prime-to-p order, then σ_{X_H} attains the maximum g_{X_H} (i.e., $X_H^\bullet$ is *ordinary*). Moreover, B. Zhang (cf. [Z]) extended Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is an arbitrary abelian group. Recently, E. Ozman and R. Pries (cf. [OP]) generalized Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is a cyclic group with a prime order distinct from p, and where $X^\bullet$ is a curve corresponding to a geometric point over a generic point of p-rank stratas of the moduli space. Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. In other words, the results of Nakajima, Zhang, and Ozman-Pries say that, for *each* Galois étale covering of $X^\bullet$ with Galois group $\mathbb{Z}/m\mathbb{Z}$, the *generalized Hasse-Witt invariants* (cf. [N]) associated to non-trivial characters of $\mathbb{Z}/m\mathbb{Z}$ attain the maximum $g_X - 1$ except for the eigenspaces associated with eigenvalue 1. The first main result of the present paper is as follows (see Theorem 3.12 for a more precise statement):

Theorem 1.1. *Let $X^\bullet$ be a component-generic pointed stable curve (cf. Section 2.1 for the definition). Then the "first" generalized Hasse-Witt invariant (cf. Section 2.2 for the definition) of each prime-to-p cyclic admissible covering of $X^\bullet$ attains the maximum under certain assumptions.*

If $n_X = 0$ and $X^\bullet$ is smooth over k, then Theorem 1.1 is equivalent to [N, Proposition 4]. Thus, Theorem 1.1 generalizes [N, Proposition 4] to the case of (possibly ramified) admissible coverings of (possibly singular) pointed stable curves. Moreover, by applying this result, we generalize [N, Theorem 2] to the case of tame coverings (cf. Corollary 3.14).

Next, let us consider the general case. If $X^\bullet$ is not geometric generic, σ_{X_H} cannot be computed explicitly in general. Suppose that $X^\bullet$ is *smooth* over k, and that $n_X = 0$. Raynaud (cf. [R1]) developed his theory of theta divisors and proved that, if $\ell >> 0$ is a prime number distinct from p, then there *exists* a Galois étale covering of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the generalized Hasse-Witt invariants associated to non-trivial characters of $\mathbb{Z}/\ell\mathbb{Z}$ attain the maximum $g_X - 1$ except for the eigenspaces associated with eigenvalue 1 (i.e., the étale covering is *new-ordinary*). Moreover, as a consequence, Raynaud obtained that $\Pi_{X^\bullet}$ is not a prime-to-p profinite group. This is the first deep result concerning the global structures of étale fundamental groups of projective curves over algebraically closed fields of characteristic $p > 0$.

Suppose that $X^\bullet$ is *smooth* over k, and that $n_X \geq 0$. The computations of generalized Hasse-invariants of admissible coverings of $X^\bullet$ (i.e., tame coverings of $X \setminus D_X$) are much more difficult than the case where $n_X = 0$. In the remainder of the introduction, let t be an arbitrary positive natural number and $n \stackrel{\mathrm{def}}{=} p^t - 1$. Tamagawa observed that Raynaud's theory of theta divisors can be generalized to the case of tame coverings, and established a tamely ramified version of the theory of Raynaud's theta divisors. By applying the theory of theta divisors, under certain assumptions, Tamagawa proved that, if $n >> 0$ and $n_X > 1$, then the "first" generalized Hasse-Witt invariants of *almost* all of the Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ are equal to g_X. Furthermore, he introduced a kind of group-theoretical invariant $\mathrm{Avr}_p(\Pi_{X^\bullet})$ associated to $\Pi_{X^\bullet}$ (i.e., depends only on the isomorphism class of $\Pi_{X^\bullet}$) called the limit of p-averages (cf. Remark 5.4.1), and proved a highly non-trivial result as follows (cf. [T2, Theorem 0.5]):

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X \geq 2. \end{cases}$$

By applying the formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$, the following *group-theoretical* formula for (g_X, n_X) was essentially obtained by Tamagawa. In particular, we obtain that g_X and n_X are group-theoretical invariants associated to $\Pi_{X^\bullet}$. This result is the main goal of the theory developed in [T2] (cf. [T2, Theorem 0.1]).

Theorem 1.2. *Let Π be an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Suppose that $X^\bullet$ is smooth over k. Then we have (cf. Section 5 for the definitions of group-theoretical invariants b^1_Π, b^2_Π, and c_Π associated to Π)*

$$g_X = \mathrm{Avr}_p(\Pi) + c_\Pi, \ \ n_X = b^1_\Pi - 2\mathrm{Avr}_p(\Pi) - 2c_\Pi - b^2_\Pi + 1.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Remark 1.2.1. Before Tamagawa proved Theorem 1.2, he also obtained an étale fundamental group version formula for (g_X, n_X) in a completely different way (by using wildly ramified coverings) which is much simpler than the case of tame fundamental groups (cf. [T1, §1]). Note that, for any smooth pointed stable curve over an algebraically closed field of positive characteristic, since the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group (cf. [T1, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version.

Remark 1.2.2. The group-theoretical formulas for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ and (g_X, n_X) are key results in the theory of tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (cf. [T2], [Y2]). On the other hand, if $W^\bullet$ is a smooth pointed stable curve of type (g_W, n_W) over an *arithmetic* field (e.g. number field, p-adic field, finite field), then a group-theoretical formula for (g_W, n_W) can be deduced immediately by computing "weight" (e.g. by applying the weight monodromy conjecture or p-adic Hodge theory).

Let us return to the case where $X^\bullet$ is an *arbitrary* pointed stable curve over k. Here our main question of the present paper is the following:

> *Does there exist a group-theoretical formula for (g_X, n_X) when $X^\bullet$ is an arbitrary pointed stable curve over k?*

First, we want to mention that the approach to finding a group-theoretical formula for (g_X, n_X) by applying $\mathrm{Avr}_p(\Pi_{X^\bullet})$ explained above *is difficult to be generalized* to the case where $X^\bullet$ is an arbitrary pointed stable curve. The reason is that the formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ is very complicated in general when $X^\bullet$ is not smooth over k, and $\mathrm{Avr}_p(\Pi_{X^\bullet})$ depends not only on the type (g_X, n_X) but also on *the structure of the dual semi-graph* $\Gamma_{X^\bullet}$ of $X^\bullet$ (cf. [Y4, Theorem 1.3 and Theorem 1.4]).

In the present paper, we solve the problem mentioned above by considering the maximum generalized Hasse-Witt invariants. Let $\overline{\mathbb{F}}_p$ be an arbitrary algebraic closure of $\mathbb{F}_p$, Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups, $\chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}_p^\times)$ such that $\chi \neq 1$, and $\Pi_\chi \subseteq \Pi$ the kernel of χ. The profinite group Π_χ admits a natural action of Π via conjugation. Let

$$\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})[\chi] \overset{\mathrm{def}}{=} \{\pi \in \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \tau \cdot \pi = \chi(\tau)\pi \text{ for all } \tau \in \Pi\},$$

and $\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})) \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})[\chi])$, where $(\tau \cdot \pi)(x) \overset{\mathrm{def}}{=} \pi(\tau^{-1} \cdot x)$ for all $x \in \Pi_\chi$. We put

$$\gamma_\Pi^{\max} \overset{\mathrm{def}}{=} \max\{\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})) \mid \chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}_p^\times) \text{ such that } \chi \neq 1\}.$$

Since the prime number p is a group-theoretical invariant associated to Π (cf. Lemma 5.2 (ii)), we see that $\gamma_\Pi^{\max}$ is also a group-theoretical invariant associated to Π. Then the second main theorem of the present paper is as follows (see also Theorem 5.4):

Theorem 1.3. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Then we have*

$$g_X = b^1_\Pi - \gamma^{\max}_\Pi - 1,\ n_X = 2\gamma^{\max}_\Pi - b^1_\Pi - b^2_\Pi + 3.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Note that $\gamma^{\max}_\Pi$ is equal to the maximum of generalized Hasse-Witt invariants $\gamma^{\max}_{X^\bullet}$ of prime-to-p cyclic admissible coverings (cf. Definition 3.2). Then Theorem 1.3 is an application of the following key observation (see Theorem 4.6 for a more precise statement):

Theorem 1.4. *We maintain the notation introduced above. Then there exist a natural number $m \in \mathbb{N}$ prime to p and a non-trivial Galois admissible covering of $X^\bullet$ over k with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that the "first" generalized Hasse-Witt invariant of the Galois admissible covering attains the maximum*

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Remark 1.4.1. Theorem 1.4 can be regarded as an analogue of [R1, Théorème 4.3.1] in the case of generalized Hasse-Witt invariants of prime-to-p cyclic admissible coverings of pointed stable curves.

Let us explain another application of Theorem 1.4 which is the main motivation of the theory developed in the present paper. Theorem 1.4 is the main tool in the proof of the following important result (cf. [Y5, Theorem 4.11 and Proposition 4.13], Theorem 5.5 and Remark 5.5.1 of the present paper):

> Let $X^\bullet_i$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic $p > 0$, $\Pi_{X^\bullet_i}$ the admissible fundamental group of $X^\bullet_i$, and $I_i \subseteq \Pi_{X^\bullet_i}$ an inertia subgroup associated to a marked point of $X^\bullet_i$. Suppose that $(g, n) \overset{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let
>
> $$\phi : \Pi_{X^\bullet_1} \twoheadrightarrow \Pi_{X^\bullet_2}$$
>
> be an open continuous surjective homomorphism. Then the following statements hold:
>
> (i) $\phi(I_1) \subseteq \Pi_{X^\bullet_2}$ is an inertia subgroup associated to a marked point of $X^\bullet_2$, and that there exists an inertia subgroup $I' \subseteq \Pi_{X^\bullet_1}$ associated to a marked point of $X^\bullet_1$ such that $\phi(I') = I_2$.
>
> (ii) The field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from $\Pi_{X^\bullet_i}$, and that ϕ induces a field isomorphism between the fields associated to I_1 and $\phi(I_1)$ group-theoretically (cf. Theorem 5.5 and Remark 5.5.1).

This result generalizes [T2, Theorem 5.2 and Proposition 5.3] and [Y2, Theorem 5.6 and Proposition 6.1] to the case of arbitrary pointed stable curves. [T2, Theorem 5.2 and Proposition 5.3] and [Y2, Theorem 5.6 and Proposition 6.1] play key roles in the proofs of the weak Isom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic $p > 0$ (cf. [T2, Theorem 0.2]) and the weak Hom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic $p > 0$ ([Y2, Theorem 1.2]), respectively. Moreover, this result is a critical step towards proving the main theorems of [Y5], [Y6]. Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack of type (g, n) over $\mathrm{Spec}\,\mathbb{Z}$ and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, where $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$. Moreover, we may define an equivalence relation $\sim_{fe}$ on $\overline{M}_{g,n}$ (roughly speaking, for any $q_1, q_2 \in \overline{M}_{g,n}$, $q_1 \sim_{fe} q_2$ if the curve corresponding to a geometric point over q_1 is a Frobenius twist of the curve corresponding to a geometric point over q_2). In [Y5], the author introduced a topological space $\overline{\Pi}_{g,n}$ whose points are isomorphism classes (as profinite groups) of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$, which is called *the moduli spaces of admissible fundamental groups of type* (g, n). Moreover, the author showed that there is a natural continuous surjective homomorphism $\pi^{\mathrm{adm}}_{g,n} : \overline{M}_{g,n}/\sim_{fe} \twoheadrightarrow \overline{\Pi}_{g,n}$, and conjectured that $\pi^{\mathrm{adm}}_{g,n}$ is *a homeomorphism*. This means that the moduli spaces of pointed stable curves in positive characteristic can be reconstructed group-theoretically as *topological spaces* from admissible fundamental groups of pointed stable curves in positive characteristic. The main theorems of [Y5], [Y6] are the following: Under certain assumptions, the author prove that $\pi^{\mathrm{adm}}_{g,n}$ is a homeomorphism.

The present paper is organized as follows. In Section 2, we recall some definitions and properties of admissible coverings, admissible fundamental groups, generalized Hasse-Witt invariants, and Raynaud-Tamagawa theta divisors. In Section 3, we study the maximum generalized Hasse-Witt invariants when $X^\bullet$ is a component-generic pointed stable curve. In Section 4, we study the maximum generalized Hasse-Witt invariants when $X^\bullet$ is an arbitrary pointed stable curve. In Section 5, we give some applications to anabelian geometry. In Section 6, we give an appendix in which we prove some results concerning generalized Hasse-Witt invariants of coverings of dual semi-graphs induced by Galois admissible coverings which are used in Section 3.

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2 Preliminaries

2.1 Admissible coverings and admissible fundamental groups

In this subsection, we recall some definitions and results concerning admissible fundamental groups which will be used in the present paper.

Definition 2.1. Let $\mathbb{G} \stackrel{\text{def}}{=} (v(\mathbb{G}), e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})})$ be a semi-graph (cf. [M3, Section 1]). Here, $v(\mathbb{G})$, $e^{\text{op}}(\mathbb{G})$, $e^{\text{cl}}(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of open edges of $\mathbb{G}$, the set of closed edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively. Note that, for each $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$, $e \stackrel{\text{def}}{=} \{b_e^1, b_e^2\}$ is a set of cardinality 2. Then e is a closed edge if $\zeta_e^{\mathbb{G}}(e) \subseteq v(\mathbb{G})$, and e is an open edge if $\zeta_e^{\mathbb{G}}(e) = \{\zeta_e^{\mathbb{G}}(e) \cap v(\mathbb{G}), \{v(\mathbb{G})\}\}$. We denote by $e^{\text{lp}}(\mathbb{G}) \subseteq e^{\text{cl}}(\mathbb{G})$ the subset of closed edges such that $\#\zeta_e^{\mathbb{G}}(e) = 1$ for each $e \in e^{\text{lp}}(\mathbb{G})$ (i.e., a closed edge which abuts to a unique vertex of $\mathbb{G}$), where "lp" means "loop". For each $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$, we denote by $v^{\mathbb{G}}(e) \subseteq v(\mathbb{G})$ the set of vertices of $\mathbb{G}$ to which e abuts. For each $v \in v(\mathbb{G})$, we denote by $e^{\mathbb{G}}(v) \subseteq e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$ the set of edges of $\mathbb{G}$ to which v is abutted. Moreover, we shall say that $\mathbb{G}$ is a tree if the Betti number $\dim_{\mathbb{Q}}(H^1(\mathbb{G}, \mathbb{Q}))$ of $\mathbb{G}$ is equal to 0.

Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve over an algebraically closed field k of characteristic $p > 0$, where X denotes the underlying curve, D_X denotes the set of marked points. Write g_X for the genus of X and n_X for the cardinality $\#D_X$ of D_X. We shall say that the pair (g_X, n_X) is the topological type (or type for short) of $X^\bullet$. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ and $r_X \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\text{op}}(\Gamma_{X^\bullet})$. Note that $X^\bullet$ is allowed to have components with self-intersections in general (i.e., $e^{\text{lp}}(\Gamma_{X^\bullet}) \neq \emptyset$). Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \stackrel{\text{def}}{=} X_v \setminus X_v^{\text{sing}}$, where $(-)^{\text{sing}}$ denotes the singular locus of $(-)$. We define a smooth pointed stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \stackrel{\text{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v)).$$

Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack parameterizing pointed stable curves of type (g, n) over $\operatorname{Spec} \mathbb{Z}$, $\overline{\mathbb{F}}_p$ the algebraically closure of $\mathbb{F}_p$ in k, $\overline{\mathcal{M}}_{g,n} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. Then $X^\bullet \to \operatorname{Spec} k$ determines a morphism $c_X : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X}$ and $\widetilde{X}_v^\bullet \to \operatorname{Spec} k$, $v \in v(\Gamma_{X^\bullet})$, determines a morphism $c_v : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_v,n_v}$. Moreover, we have a clutching morphism of moduli stacks

$$c : \bigtimes_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v} \to \overline{\mathcal{M}}_{g_X,n_X}$$

such that $c \circ (\bigtimes_{v \in v(\Gamma_{X^\bullet})} c_v) = c_X$. We shall say that $X^\bullet$ is a *component-generic* pointed stable curve over k if the image of

$$\operatorname{Spec} k \stackrel{\bigtimes_{v \in v(\Gamma_{X^\bullet})} c_v}{\to} \bigtimes_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v} \to \bigtimes_{v \in v(\Gamma_{X^\bullet})} \overline{M}_{g_v,n_v}$$

is a generic point of $\bigtimes_{v \in v(\Gamma_{X^\bullet})} \overline{M}_{g_v,n_v}$.

Definition 2.2. Let $Y^\bullet = (Y, D_Y)$ be a pointed stable curve over k, $f^\bullet : Y^\bullet \to X^\bullet$ a morphism of pointed stable curves over k, and $f : Y \to X$ the morphism of underlying curves induced by $f^\bullet$.

We shall say $f^\bullet$ a *Galois admissible covering* over k (or Galois admissible covering for short) if the following conditions are satisfied: (i) There exists a finite group $G \subseteq \mathrm{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$. (ii) For each $y \in Y^{\mathrm{sm}} \setminus D_Y$, f is étale at y, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$. (iii) For any $y \in Y^{\mathrm{sing}}$, the image $f(y)$ is contained in X^{sing}, where $(-)^{\mathrm{sing}}$ denotes the set of singular points of $(-)$. (iv) For each $y \in Y^{\mathrm{sing}}$, the local morphism between two nodes induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n,p) = 1$. Moreover, if we write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y, then $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit, and $\#(-)$ denotes the cardinality of $(-)$. (v) The local morphism between two marked points induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[a]] & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m,p) = 1$ (i.e., a tamely ramified extension).

Moreover, we shall say $f^\bullet$ an *admissible covering* if there exists a morphism of pointed stable curves $(f^\bullet)' : (Y^\bullet)' \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ (f^\bullet)' : (Y^\bullet)' \to X^\bullet$ is a Galois admissible covering over k. One can check easily that the definition of admissible covering coincides with the definition of [M1, §3.9 Definition] when the base scheme is k. We shall say an admissible covering $f^\bullet$ *étale* if f is an étale morphism.

Let $Z^\bullet$ be a disjoint union of finitely many pointed stable curves over k. We shall say that a morphism $f_Z^\bullet : Z^\bullet \to X^\bullet$ over k is a *multi-admissible covering* if the restriction of $f_Z^\bullet$ to each connected component of $Z^\bullet$ is admissible. For any category $\mathscr{C}$, we write $\mathrm{Ob}(\mathscr{C})$ for the class of objects of $\mathscr{C}$, and write $\mathrm{Hom}(\mathscr{C})$ for the class of morphisms of $\mathscr{C}$. We denote by

$$\mathrm{Cov}^{\mathrm{adm}}(X^\bullet) \overset{\mathrm{def}}{=} (\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)), \mathrm{Hom}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)))$$

the category which consists of the following data: (i) $\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$ consists of an empty object and all the pairs $(Z^\bullet, f_Z^\bullet : Z^\bullet \to X^\bullet)$, where $Z^\bullet$ is a disjoint union of finitely many pointed stable curves over k, and $f_Z^\bullet$ is a multi-admissible covering over k; (ii) for any $(Z^\bullet, f_Z^\bullet)$, $(Y^\bullet, f_Y^\bullet) \in \mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$, we define

$$\mathrm{Hom}((Z^\bullet, f_Z^\bullet), (Y^\bullet, f_Y^\bullet)) \overset{\mathrm{def}}{=} \{g^\bullet \in \mathrm{Hom}_k(Z^\bullet, Y^\bullet) \mid f_Y^\bullet \circ g^\bullet = f_Z^\bullet\},$$

where $\mathrm{Hom}_k(Z^\bullet, Y^\bullet)$ denotes the set of k-morphisms of pointed stable curves. By applying [M1, §3.11 Proposition] and the theory of Kummer log étale coverings, we may see that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\mathrm{sm}} \setminus D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the *admissible fundamental group* of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by

$$\Pi_{X^\bullet}.$$

The structure of the maximal prime-to-p quotient of $\Pi_{X^\bullet}$ is well-known, and is isomorphic to the prime-to-p completion of the following group (cf. [V, Théorème 2.2 (c)])

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X}[a_i, b_i]\prod_{j=1}^{n_X} c_j = 1\rangle.$$

Write $\Pi^{\text{ét}}_{X^\bullet}$ for the étale fundamental group of the underlying curve X of $X^\bullet$ and $\Pi^{\text{top}}_{X^\bullet}$ for the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$. Note that we have the following natural continuous surjective homomorphisms (for suitable choices of base points)

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \twoheadrightarrow \Pi^{\text{top}}_{X^\bullet}.$$

For each $v \in v(\Gamma_{X^\bullet})$, we denote by

$$\Pi_{\widetilde{X}^\bullet_v}$$

the admissible fundamental group of $\widetilde{X}^\bullet_v$. Then we have a natural (outer) injective homomorphism $\Pi_{\widetilde{X}^\bullet_v} \hookrightarrow \Pi_{X^\bullet}$.

For more details on the theory of admissible coverings and admissible fundamental groups for pointed stable curves, see [M1], [M2].

Remark 2.2.1. Let $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ be the moduli stack parameterizing pointed stable curves of type (g_X, n_X) over $\operatorname{Spec}\mathbb{Z}$ and $\mathcal{M}_{g_X,n_X,\mathbb{Z}}$ the open substack of $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ parameterizing smooth pointed stable curves. Write $\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \setminus \mathcal{M}_{g_X,n_X,\mathbb{Z}} \subset \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ relative to $\operatorname{Spec}\mathbb{Z}$.

The pointed stable curve $X^\bullet$ over k induces a morphism $\operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. Write $s^{\log}_X$ for the log scheme whose underlying scheme is $\operatorname{Spec} k$, and whose log structure is the pulling-back log structure induced by the morphism $\operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. We obtain a natural morphism $s^{\log}_X \to \overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ induced by the morphism $\operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ and a stable log curve $X^{\log} \overset{\text{def}}{=} s^{\log}_X \times_{\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}} \overline{\mathcal{M}}^{\log}_{g_X,n_X+1,\mathbb{Z}}$ over $s^{\log}_X$ whose underlying scheme is X. Let $Y^{\log} \to X^{\log}$ be an arbitrary Kummer log étale covering. One can prove that there exists a Kummer log étale covering $t^{\log}_X \to s^{\log}_X$ such that $Y^{\log} \times_{s^{\log}_X} t^{\log}_X \to X^{\log} \times_{s^{\log}_X} t^{\log}_X$ is a log admissible covering (cf. [M1, §3.5 Definition]) over $t^{\log}_X$. Then the admissible fundamental group of $X^\bullet$ does not depend on the log structure of $X^{\log}$, and [M1, §3.11 Proposition] implies that the admissible fundamental group $\Pi_{X^\bullet}$ of $X^\bullet$ is naturally isomorphic to the geometric log étale fundamental group of $X^{\log}$ (i.e., $\ker(\pi_1(X^{\log}) \to \pi_1(s^{\log}_X))$).

Remark 2.2.2. Suppose that $X^\bullet$ is smooth over k. By the definition of admissible fundamental groups, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X \setminus D_X$.

Definition 2.3. We define the *p-rank* (or *Hasse-Witt invariant*) of $X^\bullet$ to be

$$\sigma_X \stackrel{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(X, \mathbb{F}_p)) = \dim_{\mathbb{F}_p}(\Pi^{\text{ab}}_{X^\bullet} \otimes \mathbb{F}_p),$$

where $(-)^{\text{ab}}$ denotes the abelianization of $(-)$. We shall say that $X^\bullet$ is *ordinary* if $g_X = \sigma_X$.

Remark 2.3.1. One can prove that

$$\sigma_X = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma_{\widetilde{X}_v} + r_X.$$

2.2 Generalized Hasse-Witt invariants of cyclic admissible coverings

In this subsection, we recall some notation concerning generalized Hasse-Witt invariants of cyclic admissible coverings.

We maintain the notation introduced in Section 2.1, and let $X^\bullet = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over k, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Let $\alpha \in \text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$. We denote by $X^\bullet_\alpha = (X_\alpha, D_{X_\alpha}) \to X^\bullet$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition (cf. [S, Section 9])

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha}) = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} \oplus H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}}$. Moreover, we have

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k,$$

where $(-)^{F_{X_\alpha}}$ denotes the subspace of $(-)$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify

$$H_\alpha \stackrel{\text{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$$

with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-vector space H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α. We have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication. We define

$$\gamma_{\alpha,i} \stackrel{\text{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z}.$$

These invariants are called *generalized Hasse-Witt invariants* (cf. [N]) of the cyclic multi-admissible covering $X^\bullet_\alpha \to X^\bullet$. Moreover, we shall say that $\gamma_{\alpha,1}$ is the *first* generalized

Hasse-Witt invariant of the cyclic multi-admissible covering $X^\bullet_\alpha \to X^\bullet$. Note that the decomposition above implies that

$$\dim_k(H_\alpha) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.$$

In particular, if X_α is connected, then $\dim_k(H_\alpha) = \sigma_{X_\alpha}$.

We write $\mathbb{Z}[D_X]$ for the group of divisors whose supports are contained in D_X. Note that $\mathbb{Z}[D_X]$ is a free $\mathbb{Z}$-module with basis D_X. We define

$$c'_n : \mathbb{Z}/n\mathbb{Z}[D_X] \overset{\text{def}}{=} \mathbb{Z}[D_X] \otimes \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z},\ D \bmod n \mapsto \deg(D) \bmod n.$$

Then $\ker(c'_n)$ can be regarded as a subset of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$, where $(\mathbb{Z}/n\mathbb{Z})^\sim$ denotes the set $\{0, 1, \ldots, n-1\}$, and $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$ denotes the subset of $\mathbb{Z}[D_X]$ consisting of the elements whose coefficients are contained in $(\mathbb{Z}/n\mathbb{Z})^\sim$. We denote by $\mathbb{Z}/n\mathbb{Z}[D_X]^0$ the kernel of c'_n and by $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ the subset of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$ corresponding to $\mathbb{Z}/n\mathbb{Z}[D_X]^0$ under the natural bijection $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X] \overset{\sim}{\to} \mathbb{Z}/n\mathbb{Z}[D_X]$. Note that, for each $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, we have $n|\deg(D)$. Then

$$\deg(D) = s(D)n$$

for some integer $s(D)$ such that

$$0 \leq s(D) \leq \begin{cases} 0, & \text{if } n_X \leq 1, \\ n_X - 1, & \text{if } n_X \geq 2. \end{cases}$$

Let $\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}}) \to X^\bullet$ be a universal admissible covering corresponding to $\Pi_{X^\bullet}$. For each $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet})$, write x_e for the marked point corresponding to e, and let $x_{\widehat{e}}$ be a point of the inverse image of x_e in $D_{\widehat{X}}$. Write $I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$ for the inertia subgroup of $x_{\widehat{e}}$. Note that $I_{\widehat{e}}$ is isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Suppose that x_e is contained in X_v. Then we have an injection

$$\phi_{\widehat{e}} : I_{\widehat{e}} \hookrightarrow \Pi^{\mathrm{ab}}_{X^\bullet}$$

induced by the composition of (outer) injective homomorphisms $I_{\widehat{e}} \hookrightarrow \Pi_{\widetilde{X}^\bullet_v} \hookrightarrow \Pi_{X^\bullet}$, where $\Pi_{\widetilde{X}^\bullet_v}$ denotes the admissible fundamental group of $\widetilde{X}^\bullet_v$. Since the image of $\phi_{\widehat{e}}$ depends only on e, we may write I_e for the image $\phi_{\widehat{e}}(I_{\widehat{e}})$. Moreover, the specialization theorem of the maximal prime-to-p quotients of admissible fundamental groups of pointed stable curves (cf. [V, Théorème 2.2 (c)]) implies that, there exists a generator $[s_e]$ of I_e for each $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ such that the following holds

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} [s_e] = 0$$

in $\Pi^{\mathrm{ab}}_{X^\bullet}$.

Definition 2.4. We maintain the notation introduced above.

(i) We put

$$D_\alpha \stackrel{\text{def}}{=} \sum_{e\in e^{\text{op}}(\Gamma_{X^\bullet})} \alpha([s_e])^\sim x_e,\ \alpha \in \text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}),$$

where $\alpha([s_e])^\sim$ denotes the element of $(\mathbb{Z}/n\mathbb{Z})^\sim$ corresponding to $\alpha([s_e])$ via the natural bijection $(\mathbb{Z}/n\mathbb{Z})^\sim \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$. Note that we have $D_\alpha \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$. On the other hand, for each $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, we denote by

$$\text{Rev}^{\text{adm}}_D(X^\bullet) \stackrel{\text{def}}{=} \{\alpha \in \text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \mid D_\alpha = D\}.$$

Moreover, we put

$$\gamma_{(\alpha,D)} \stackrel{\text{def}}{=} \gamma_{\alpha,1}.$$

(ii) Let $t \in \mathbb{N}$ be an arbitrary positive natural number, $n \stackrel{\text{def}}{=} p^t - 1$, and

$$u = \sum_{j=0}^{t-1} u_j p^j,\ u \in \{0, \dots, n\},$$

the p-adic expansion with $u_j \in \{0, \dots, p-1\}$. We identify $\{0, \dots, t-1\}$ with $\mathbb{Z}/t\mathbb{Z}$ naturally. Then $\{0, \dots, t-1\}$ admits an additional structure induced by the natural additional structure of $\mathbb{Z}/t\mathbb{Z}$. We put

$$u^{(i)} \stackrel{\text{def}}{=} \sum_{j=0}^{t-1} u_{i+j} p^j,\ i \in \{0, \dots, t-1\}.$$

Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$. We put

$$D^{(i)} \stackrel{\text{def}}{=} \sum_{x\in D_X} (\text{ord}_x(D))^{(i)} x,\ i \in \{0, \dots, t-1\},$$

which is an effective divisor on X.

2.3 Raynaud-Tamagawa theta divisors

In this subsection, we recall some notation and results concerning theta divisors introduced by Raynaud and Tamagawa (see also [T2, Section 2]).

We maintain the notation introduced in Section 2.2. Moreover, in the present subsection, we suppose that $X^\bullet$ is *smooth* over k. The generalized Hasse-Witt invariants can be also described in terms of line bundles and divisors. Let $m \in \mathbb{N}$ be an arbitrary natural number prime to p. We denote by $\text{Pic}(X)$ the Picard group of X. Consider the following complex of abelian groups:

$$\mathbb{Z}[D_X] \xrightarrow{a_m} \text{Pic}(X) \oplus \mathbb{Z}[D_X] \xrightarrow{b_m} \text{Pic}(X),$$

where $a_m(D) = ([\mathcal{O}_X(-D)], mD)$, $b_m(([\mathcal{L}], D)) = [\mathcal{L}^m \otimes \mathcal{O}_X(D)]$. We denote by

$$\mathscr{P}_{X^\bullet,m} \stackrel{\text{def}}{=} \ker(b_m)/\text{Im}(a_m)$$

the homology group of the complex. Moreover, we have the following exact sequence

$$0 \to \mathrm{Pic}(X)[m] \overset{a'_m}{\to} \mathscr{P}_{X^\bullet,m} \overset{b'_m}{\to} \mathbb{Z}/m\mathbb{Z}[D_X] \overset{c'_m}{\to} \mathbb{Z}/m\mathbb{Z},$$

where $(-)[m]$ means the m-torsion subgroup of $(-)$, and

$$a'_m([\mathcal{L}]) = ([\mathcal{L}], 0) \bmod \mathrm{Im}(a_m),$$

$$b'_m(([\mathcal{L}], D)) \bmod \mathrm{Im}(a_m)) = D \bmod m,$$

$$c'_m(D \bmod m) = \deg(D) \bmod m.$$

We shall define

$$\widetilde{\mathscr{P}}_{X^\bullet,m}$$

to be the inverse image of $(\mathbb{Z}/m\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/m\mathbb{Z})^\sim[D_X] \subseteq \mathbb{Z}[D_X]$ under the projection $\ker(b_m) \to \mathbb{Z}[D_X]$. It is easy to see that $\mathscr{P}_{X^\bullet,m}$ and $\widetilde{\mathscr{P}}_{X^\bullet,m}$ are free $\mathbb{Z}/m\mathbb{Z}$-groups with rank $2g_X + n_X - 1$ if $n_X \neq 0$ and with rank $2g_X$ if $n_X = 0$, and that there is a natural isomorphism $\widetilde{\mathscr{P}}_{X^\bullet,m} \overset{\sim}{\to} \mathscr{P}_{X^\bullet,m}$.

On the other hand, let $\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z})$ and $f^\bullet_\alpha : X^\bullet_\alpha \to X^\bullet$ the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/m\mathbb{Z}$ corresponding to α. Fix a primitive mth root ζ_m, we may identify μ_m with $\mathbb{Z}/m\mathbb{Z}$ via the map $\zeta_m^i \mapsto i$. Then we see that

$$f_{\alpha,*}\mathcal{O}_{X_\alpha} \cong \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathcal{L}_{\alpha,i},$$

where locally $\mathcal{L}_{\alpha,i}$ is the eigenspace of the natural action of i with eigenvalue ζ_m^i. Moreover, we have the following natural isomorphism (cf. [T2, Proposition 3.5]):

$$\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z}) \overset{\sim}{\to} \widetilde{\mathscr{P}}_{X^\bullet,m},\ \alpha \mapsto ([\mathcal{L}_{\alpha,1}], D_\alpha).$$

Then every element of $\widetilde{\mathscr{P}}_{X^\bullet,m}$ induces a Galois multi-admissible covering of $X^\bullet$ over k with Galois group $\mathbb{Z}/m\mathbb{Z}$.

In the remainder of the present paper, we may assume that

$$n \overset{\mathrm{def}}{=} p^t - 1$$

for some positive natural number $t \in \mathbb{N}$ unless indicated otherwise. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$. We fix an isomorphism $\mathcal{L}^n \cong \mathcal{O}_X(-D)$. Note that D is an effective divisor on X. We have the following composition of morphisms of line bundles

$$\mathcal{L} \overset{p^t}{\to} \mathcal{L}^{\otimes p^t} = \mathcal{L}^{\otimes n} \otimes \mathcal{L} \overset{\sim}{\to} \mathcal{O}_X(-D) \otimes \mathcal{L} \hookrightarrow \mathcal{L}.$$

The composite morphism induces a morphism $\phi_{([\mathcal{L}],D)} : H^1(X, \mathcal{L}) \to H^1(X, \mathcal{L})$. We denote by

$$\gamma_{([\mathcal{L}],D)} \overset{\mathrm{def}}{=} \dim_k(\bigcap_{r \geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)})).$$

Write $\alpha_{\mathcal{L}} \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ for the element corresponding to $([\mathcal{L}], D)$ and F_X for the absolute Frobenius morphism on X. Then we see that $\gamma_{\alpha_{\mathcal{L}},1}$ is equal to the dimension over k of the largest subspace of $H^1(X, \mathcal{L})$ on which $F^t_X \stackrel{\mathrm{def}}{=} F_X \circ \cdots \circ F_X$ is a bijection. Moreover, we have

$$\gamma_{\alpha_{\mathcal{L}},1} = \dim_k(H^1(X, \mathcal{L})^{F^t_X} \otimes_{\mathbb{F}_p} k),$$

where $(-)^{F^t_X}$ denotes the subspace of $(-)$ on which F^t_X acts trivially. It is easy to check that

$$H^1(X, \mathcal{L})^{F^t_X} \otimes_{\mathbb{F}_p} k = \bigcap_{r \geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)}).$$

Then we obtain that $\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1}$. Moreover, we observer that $D_{\alpha_{\mathcal{L}}} = D$. Then we obtain that

$$\gamma_{([\mathcal{L}],D)} = \gamma_{(\alpha_{\mathcal{L}},D)} (\stackrel{\mathrm{def}}{=} \gamma_{\alpha_{\mathcal{L}},1}).$$

Lemma 2.5. *We maintain the notation introduced above. Suppose that $X^\bullet$ is smooth over k. Then we have*

$$\gamma_{(\alpha_{\mathcal{L}},D)} \leq \dim_k(H^1(X, \mathcal{L})) = \begin{cases} g_X, & \text{if } ([\mathcal{L}], D) = ([\mathcal{O}_X], 0), \\ g_X - 1, & \text{if } s(D) = 0, \\ g_X + s(D) - 1, & \text{if } s(D) \geq 1. \end{cases}$$

Proof. The first inequality follows from the definition of generalized Hasse-Witt invariants. The Riemann-Roch theorem implies that

$$\dim_k(H^1(X, \mathcal{L})) = g_X - 1 - \deg(\mathcal{L}) + \dim_k(H^0(X, \mathcal{L}))$$
$$= g_X - 1 + \frac{1}{n}\deg(D) + \dim_k(H^0(X, \mathcal{L})) = g_X - 1 + s(D) + \dim_k(H^0(X, \mathcal{L})).$$

This completes the proof of the lemma. □

Next, let us explain the Raynaud-Tamagawa theta divisors. Let F_k be the absolute Frobenius morphism on $\mathrm{Spec}\, k$ and $F_{X/k}$ the relative Frobenius morphism $X \to X_1 \stackrel{\mathrm{def}}{=} X \times_{k,F_k} k$ over k and $F^t_k \stackrel{\mathrm{def}}{=} F_k \circ \cdots \circ F_k$. We define

$$X_t \stackrel{\mathrm{def}}{=} X \times_{k,F^t_k} k,$$

and define a morphism

$$F^t_{X/k} : X \to X_t$$

over k to be $F^t_{X/k} \stackrel{\mathrm{def}}{=} F_{X_{t-1}/k} \circ \cdots \circ F_{X_1/k} \circ F_{X/k}$.

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$ and $\mathcal{L}_t$ the pulling-back of $\mathcal{L}$ by the natural morphism $X_t \to X$. Note that $\mathcal{L}$ and $\mathcal{L}_t$ are line bundles of degree $-s(D)$. We put

$$\mathcal{B}^t_D \stackrel{\mathrm{def}}{=} (F^t_{X/k})_*(\mathcal{O}_X(D))/\mathcal{O}_{X_t}, \ \mathcal{E}_D \stackrel{\mathrm{def}}{=} \mathcal{B}^t_D \otimes \mathcal{L}_t.$$

Write $\mathrm{rk}(\mathcal{E}_D)$ for the rank of $\mathcal{E}_D$. Then we obtain

$$\chi(\mathcal{E}_D) = \deg(\det(\mathcal{E}_D)) - (g_X - 1)\mathrm{rk}(\mathcal{E}_D).$$

Moreover, we have $\chi(\mathcal{E}_D) = 0$ (cf. [T2, Lemma 2.3 (ii)]). In [R1], Raynaud investigated the following property of the vector bundle $\mathcal{E}_D$ on X.

Condition 2.6. We shall say that $\mathcal{E}_D$ satisfies ($\star$) if there exists a line bundle $\mathcal{L}'_t$ of degree 0 on X_t such that

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}'_t)), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}'_t))\}.$$

Let J_{X_t} be the Jacobian variety of X_t, and $\mathcal{L}_{X_t}$ a universal line bundle on $X_t \times J_{X_t}$. Let $\mathrm{pr}_{X_t} : X_t \times J_{X_t} \to X_t$ and $\mathrm{pr}_{J_{X_t}} : X_t \times J_{X_t} \to J_{X_t}$ be the natural projections. We denote by $\mathcal{F}$ the coherent $\mathcal{O}_{X_t}$-module $\mathrm{pr}^*_{X_t}(\mathcal{E}_D) \otimes \mathcal{L}_{X_t}$, and by

$$\chi_{\mathcal{F}} \stackrel{\text{def}}{=} \dim_k(H^0(X_t \times_k k(y), \mathcal{F} \otimes k(y))) - \dim_k(H^1(X_t \times_k k(y), \mathcal{F} \otimes k(y)))$$

for each $y \in J_{X_t}$, where $k(y)$ denotes the residue field of y. Note that since $\mathrm{pr}_{J_{X_t}}$ is flat, $\chi_{\mathcal{F}}$ is independent of $y \in J_{X_t}$. Write $(-\chi_{\mathcal{F}})^+$ for $\max\{0, -\chi_{\mathcal{F}}\}$. We denote by

$$\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$$

the closed subscheme of J_{X_t} defined by the $(-\chi_{\mathcal{F}})^+$-th Fitting ideal

$$\mathrm{Fitt}_{(-\chi_{\mathcal{F}})^+}(R^1(\mathrm{pr}_{J_{X_t}})_*(\mathcal{F})).$$

The definition of $\Theta_{\mathcal{E}_D}$ is independent of the choice of $\mathcal{L}_t$. Moreover, for each line bundle $\mathcal{L}''$ of degree 0 on X_t, we have that $[\mathcal{L}''] \notin \Theta_{\mathcal{E}_D}$ if and only if

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}'')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}''))\},$$

where $[\mathcal{L}'']$ denotes the point of J_{X_t} corresponding to $\mathcal{L}''$ (cf. [T2, Proposition 2.2 (i) (ii)]).

Suppose that $\mathcal{E}_D$ satisfies ($\star$). [R1, Proposition 1.8.1] implies that $\Theta_{\mathcal{E}_D}$ is algebraically equivalent to $\mathrm{rk}(\mathcal{E}_D)\Theta$, where Θ is the classical theta divisor (i.e., the image of $X_t^{g_X-1}$ in J_{X_t}). Then we have the following definition.

Definition 2.7. We shall say $\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$ the *Raynaud-Tamagawa theta divisor* associated to $\mathcal{E}_D$ if $\mathcal{E}_D$ satisfies ($\star$).

Remark 2.7.1. The definition of $\mathcal{E}_D$ implies that the following natural exact sequence holds

$$0 \to \mathcal{L}_t \to (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \to \mathcal{E}_D \to 0.$$

Let $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$. Write $\mathcal{I}_t$ for the pulling-back of $\mathcal{I}$ by the natural morphism $X_t \to X$. we obtain the following exact sequence

$$\ldots \to H^0(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \stackrel{\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}}{\to} H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t)$$
$$\to H^1(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to \ldots.$$

Note that we have that

$$H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}),$$

and that

$$H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{O}_X(D) \otimes (F^t_{X/k})^*(\mathcal{L}_t \otimes \mathcal{I}_t))$$

$$\cong H^1(X, \mathcal{O}_X(D) \otimes (\mathcal{L} \otimes \mathcal{I})^{\otimes p^t}) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}).$$

Moreover, it is easy to see that the homomorphism

$$H^1(X, \mathcal{L} \otimes \mathcal{I}) \to H^1(X, \mathcal{L} \otimes \mathcal{I})$$

induced by $\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}$ coincides with $\phi_{([\mathcal{L} \otimes \mathcal{I}], D)}$. Thus, we obtain that if

$$\gamma_{(([\mathcal{L} \otimes \mathcal{I}], D))} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$$

for some line bundle $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$, then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists (i.e., $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$).

Let N be an arbitrary non-negative integer. We put

$$C(N) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } N = 0, \\ 3^{N-1} N!, & \text{if } N \neq 0. \end{cases}$$

Then we have the following proposition.

Proposition 2.8. *We maintain the notation introduced above. Suppose that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists, and that*

$$n = p^t - 1 > C(g_X) + 1.$$

Then there exists a line bundle $\mathcal{I}$ of degree 0 on X such that $[\mathcal{I}] \neq [\mathcal{O}_X]$, that $[\mathcal{I}^{\otimes n}] = [\mathcal{O}_X]$, and that $\gamma_{(([\mathcal{L} \otimes \mathcal{I}], D))} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$ (i.e., $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$).

Proof. By applying similar arguments to the arguments given in the proof of [T2, Corollary 3.10], the proposition follows immediately from Remark 2.7.1. □

The following fundamental theorem of theta divisors was proved by Raynaud and Tamagawa.

Theorem 2.9. *Suppose that $s(D) \in \{0, 1\}$. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists (i.e., $\mathcal{E}_D$ satisfies $(\star)$).*

Remark 2.9.1. Theorem 2.9 was proved by Raynaud if $s(D) = 0$ (cf. [R1, Théorème 4.1.1]), and by Tamagawa if $s(D) \leq 1$ (cf. [T2, Theorem 2.5]).

We may ask whether or not the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exists in general when $X^\bullet$ is smooth over k.

> *Suppose that $s(D) \geq 2$. Does the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exist?*

Note that since the existence of $\Theta_{\mathcal{E}_D}$ implies that $\mathcal{E}_D$ is a semi-stable bundle, we obtain that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$ (cf. [T2, Lemma 2.15]). Then we may consider the following problem:

> *Suppose that $X^\bullet$ is smooth over k, that $s(D) \geq 2$, and that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$. Does the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exist?*

In fact, the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ does not exist in general. Here, we have an example as follows. Let $X = \mathbb{P}^1_k$, $D_X = \{0, 1, \infty, \lambda\}$, where $w \notin \{0, 1\}$, and

$$D = \sum_{x \in D_X} \frac{p-1}{2} x.$$

Then we have $s(D) = 2$. Let $([\mathcal{L}], D)$ be an arbitrary element of $\widetilde{\mathscr{P}}_{X^\bullet, n}$. We see that $\mathcal{E}_D$ satisfies $(\star)$ if and only if the elliptic curve defined by the equation

$$y^2 = x(x-1)(x-\lambda)$$

is ordinary. Thus, we cannot expect that $\Theta_{\mathcal{E}_D}$ exists in general. On the other hand, we have the following open problem posed by Tamagawa (cf. [T2, Question 2.20]).

Problem 2.10. *Suppose that $X^\bullet$ is a geometric generic pointed stable curve of type (g_X, n_X) over k. Let $([\mathcal{L}], D)$ be an arbitrary element of $\widetilde{\mathscr{P}}_{X^\bullet, n}$. Moreover, suppose that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$. Does the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exist?*

In the next section, we solve Problem 2.10 under the assumption $s(D) = n_X - 1$ (cf. Corollary 3.11 below).

3 Maximum generalized Hasse-Witt invariants of cyclic admissible coverings of component-generic pointed stable curves

In this section, we discuss the maximum generalized Hasse-Witt invariants of cyclic admissible coverings of a component-generic pointed stable curve. We maintain the notation introduced in Section 2.1 and Section 2.2.

Proposition 3.1. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$. Write*

$$f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$$

for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. For each $v \in v(\Gamma_{X^\bullet})$, $f^\bullet$ induces a Galois multi-admissible covering

$$\widetilde{f}^\bullet_v : \widetilde{Y}^\bullet_v \to \widetilde{X}^\bullet_v$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\widetilde{\alpha}_v$ for an element of $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_v}, \mathbb{Z}/n\mathbb{Z})$ induced by $\widetilde{f}^\bullet_v$. Then we have that

$$\gamma_{(\alpha, D)} = \max\{\gamma_{(\alpha', D)} \mid \alpha' \in \mathrm{Rev}_D(X^\bullet),\ \alpha' \neq 0\}$$

$$= \begin{cases} g_X - 1, & \textit{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \textit{if } \mathrm{Supp}(D) \neq \emptyset \end{cases}$$

if and only if for all $v \in v(\Gamma_{X^\bullet})$,

$$\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})} = \begin{cases} g_v, & \textit{if } \widetilde{\alpha}_v = 0, \\ g_v - 1, & \textit{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) = \emptyset, \\ g_v + s(D_{\widetilde{\alpha}_v}) - 1, & \textit{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) \neq \emptyset, \end{cases}$$

where $\mathrm{Supp}(-)$ *denotes the support of* $(-)$.

Proof. We will prove the proposition by induction on the cardinality $\#v(\Gamma_{X^\bullet})$ of $v(\Gamma_{X^\bullet})$. Suppose that $\#v(\Gamma_{X^\bullet}) = 1$ (i.e., X is irreducible). Then we have that $D_{\widetilde{\alpha}_v}|_{U_{X_v}} = D$ and

$$g_v = g_X - \#X^{\mathrm{sing}}.$$

Moreover, since $\widetilde{X}_v^\bullet$ is smooth over k, we write $([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v}) \in \widetilde{\mathscr{P}}_{\widetilde{X}_v^\bullet, n}$ for the pair induced by $\widetilde{\alpha}_v$. First, we suppose that $\#\mathrm{Supp}(D) \leq 1$. Then the structures of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]) implies that f is étale over $\mathrm{Supp}(D)$ when $\#\mathrm{Supp}(D) \leq 1$. Write $\mathscr{N}_X^{\mathrm{ra}} \subseteq X^{\mathrm{sing}}$ for the subset of nodes over which f is ramified and $\mathscr{N}_X^{\mathrm{et}} \subseteq X^{\mathrm{sing}}$ for the subset of nodes over which f is étale. Then we have $s(D_{\widetilde{\alpha}_v}) = s(D) + \#\mathscr{N}_X^{\mathrm{ra}}$.

On the other hand, by Lemma 2.5, we see that

$$\dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v})) = \begin{cases} g_v, & \text{if } \widetilde{\alpha}_v = 0, \\ g_v - 1, & \text{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) = \emptyset, \\ g_v + s(D_{\widetilde{\alpha}_v}) - 1, & \text{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) \neq \emptyset. \end{cases}$$

Write $\Gamma_{Y^\bullet}$ for the dual semi-graph of $Y^\bullet$. The natural $k[\mu_n]$-submodule

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k \subseteq H^1_{\text{ét}}(Y, \mathbb{F}_p) \otimes k$$

admits the following canonical decomposition

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y^\bullet}}(j)$ as the ζ^j-multiplication. Let us compute $\dim_k(M_{\Gamma_{Y^\bullet}}(1))$. By Proposition 6.1 of the appendix of this paper, we have that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \begin{cases} \#\mathscr{N}_X^{\mathrm{et}} - 1, & \text{if } \widetilde{\alpha}_v = 0, \\ \#\mathscr{N}_X^{\mathrm{et}}, & \text{if } \widetilde{\alpha}_v \neq 0. \end{cases}$$

Thus, we obtain that

$$\dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v})) + \dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset. \end{cases}$$

Since $\gamma_{(\alpha, D)} = \gamma_{([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v})} + \dim_k(M_{\Gamma_{Y^\bullet}}(1))$ and $\#X^{\mathrm{sing}} = \#\mathscr{N}_X^{\mathrm{ra}} + \#\mathscr{N}_X^{\mathrm{et}}$, we have that

$$\gamma_{(\alpha, D)} = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset \end{cases}$$

if and only if $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})} = \gamma_{([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v})} = \dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v}))$. This completes the proof of the proposition when $\#v(\Gamma_{X^\bullet}) = 1$.

Suppose $m \overset{\text{def}}{=} \#v(\Gamma_{X^\bullet}) \geq 2$. Let $v_0 \in v(\Gamma_{X^\bullet})$ be a vertex such that $\Gamma_{X^\bullet} \setminus \{v_0, e^{\Gamma_{X^\bullet}}(v_0)\}$ is connected (note that it is easy to see that such v_0 exists). Write X_1 for the topological closure of $X \setminus X_{v_0}$ in X and X_2 for X_{v_0}. Note that X_1 is connected. We define a pointed stable curve

$$X_i^\bullet = (X_i, D_{X_i} \overset{\text{def}}{=} (X_i \cap D_X) \cup (X_1 \cap X_2)),\ i \in \{1, 2\},$$

over k. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f_i^\bullet : Y_i^\bullet \to X_i^\bullet,\ i \in \{1, 2\},$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we denote by

$$\alpha_i \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X_i^\bullet}, \mathbb{Z}/n\mathbb{Z}),\ i \in \{1, 2\},$$

an element induced by $f_i^\bullet$, where $\Pi_{X_i^\bullet}$ denotes the admissible fundamental group of $X_i^\bullet$. Write $\mathscr{N}^{\mathrm{ra}}_{X_1 \cap X_2} \subseteq X_1 \cap X_2$ for the subset of nodes over which f is ramified and $\mathscr{N}^{\mathrm{et}}_{X_1 \cap X_2} \subseteq X_1 \cap X_2$ for the subset of nodes over which f is étale. Note that $\#\mathscr{N}^{\mathrm{ra}}_{X_1 \cap X_2} + \#\mathscr{N}^{\mathrm{et}}_{X_1 \cap X_2} = \#(X_1 \cap X_2)$. By induction, we obtain that, for each $i \in \{1, 2\}$,

$$\gamma_{(\alpha_i, D_{\alpha_i})} \leq \begin{cases} g_{X_i}, & \text{if } \alpha_i = 0, \\ g_{X_i} - 1, & \text{if } \alpha_i \neq 0,\ \mathrm{Supp}(D_{\alpha_i}) = \emptyset, \\ g_{X_i} + s(D_{\alpha_i}) - 1, & \text{if } \alpha_i \neq 0,\ \mathrm{Supp}(D_{\alpha_i}) \neq \emptyset, \end{cases}$$

where g_{X_i} denotes the genus of X_i. Note that the definition of admissible coverings implies that $s(D_{\alpha_1}) + s(D_{\alpha_2}) = s(D) + \#\mathscr{N}^{\mathrm{ra}}_{X_1 \cap X_2}$.

On the other hand, the natural $k[\mu_n]$-modules $H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k$ and $H^1(\Gamma_{Y_i^\bullet}, \mathbb{F}_p) \otimes k$, $i \in \{1, 2\}$, admit the following canonical decomposition

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(j)$$

and

$$H^1(\Gamma_{Y_i^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y_i^\bullet}}(j),\ i \in \{1, 2\},$$

respectively, where $\Gamma_{Y_i^\bullet}$ denotes the dual semi-graph of $Y_i^\bullet$, and $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y^\bullet}}(j)$ and $M_{\Gamma_{Y_i^\bullet}}(j)$, $i \in \{1, 2\}$, as the ζ^j-multiplication, respectively. We put

$$\dim_k(M_{\Gamma_{Y_1 \cap Y_2}}(1)) \overset{\text{def}}{=} \dim_k(M_{\Gamma_{Y^\bullet}}(1)) - \dim_k(M_{\Gamma_{Y_1^\bullet}}(1)) - \dim_k(M_{\Gamma_{Y_2^\bullet}}(1)).$$

Let us compute $\dim_k(M_{\Gamma_{Y_1 \cap Y_2}}(1))$. Note that without loss of generality, we may assume that X_1 and X_2 are non-singular. Then Proposition 6.3 of the appendix of this paper implies that

$$\dim_k(M_{\Gamma_{Y_1 \cap Y_2}}(1)) =$$

$$\begin{cases} \#\mathscr{N}^{\mathrm{et}}_{X_1 \cap X_2} - 1, & \text{if there exists } i \in \{1, 2\} \text{ such that } \alpha_i = 0, \\ \#\mathscr{N}^{\mathrm{et}}_{X_1 \cap X_2}, & \text{if for each } i \in \{1, 2\},\ \alpha_i \neq 0. \end{cases}$$

Thus, we obtain that

$$\gamma_{(\alpha,D)} = \gamma_{(\alpha_1,D_{\alpha_1})} + \gamma_{(\alpha_2,D_{\alpha_2})} + \dim_k(M_{\Gamma_{Y_1\cap Y_2}}(1))$$

$$\leq \begin{cases} g_{X_1} + g_{X_2} + \#(X_1 \cap X_2) - 2 = g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_{X_1} + s(D_{\alpha_1}) + g_{X_2} + s(D_{\alpha_2}) + \#\mathscr{N}^{\mathrm{et}}_{X_1\cap X_2} - 2 = g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset. \end{cases}$$

Thus, we have that

$$\gamma_{(\alpha,D)} = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset \end{cases}$$

if and only if, for each $i \in \{1, 2\}$,

$$\gamma_{(\alpha_i,D_{\alpha_i})} = \begin{cases} g_{X_i}, & \text{if } \alpha_i = 0, \\ g_{X_i} - 1, & \text{if } \alpha_i \neq 0, \ \mathrm{Supp}(D_{\alpha_i}) = \emptyset, \\ g_{X_i} + s(D_{\alpha_i}) - 1, & \text{if } \alpha_i \neq 0, \ \mathrm{Supp}(D_{\alpha_i}) \neq \emptyset. \end{cases}$$

By induction, the proposition follows from the proposition when $\#v(\Gamma_{X^\bullet}) = m - 1$ and $\#v(\Gamma_{X^\bullet}) = 1$. This completes the proof of the proposition. □

Definition 3.2. We put

$$\gamma^{\max}_{X^\bullet} \overset{\mathrm{def}}{=} \max_{t\in\mathbb{N}}\{\gamma_{(\alpha,D_\alpha)} \mid \alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/(p^t - 1)\mathbb{Z}) \text{ and } \alpha \neq 0\}$$

$$= \max_{m\in\mathbb{N} \text{ s.t. } (m,p)=1}\{\gamma_{(\alpha,D_\alpha)} \mid \alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z}) \text{ and } \alpha \neq 0\}.$$

We shall say $\gamma^{\max}_{X^\bullet}$ *the maximum generalized Hasse-Witt invariant of prime-to-p cyclic admissible coverings* of $X^\bullet$.

Remark 3.2.1. Note that Proposition 3.1 implies that

$$\gamma^{\max}_{X^\bullet} \leq \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Lemma 3.3. *Let* $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ *and*

$$\mathrm{ord}_x(D) = \sum_{j=0}^{t-1} d_{x,j}p^j, \ x \in D_X$$

the p-adic expansion. Suppose that $s(D) = n_X - 1$ *(i.e.,* $\deg(D) = (n_X - 1)n$*). Then we have that* $\deg(D^{(i)}) \geq \deg(D)$ *holds for each* $i \in \{0, 1, \ldots, t - 1\}$ *if and only if*

$$\sum_{x\in D_X} d_{x,j} = (n_X - 1)(p - 1), \ j \in \{0, \ldots, t - 1\}.$$

Proof. (a) The "if" part of the lemma is trivial. We only prove the "only if" part of the lemma. Let $d_x \stackrel{\text{def}}{=} \mathrm{ord}_x(D)$, $x \in D_X$. Since $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, \ldots, t-1\}$ and $n|\deg(D^{(i)})$, we have

$$\deg(D^{(i)}) = \sum_{x \in D_X} (\mathrm{ord}_x(D))^{(i)} = \sum_{x \in D_X} d_x^{(i)} = (n_X - 1)n,$$

where $d_x^{(i)} \stackrel{\text{def}}{=} (\mathrm{ord}_x(D))^{(i)}$. Moreover, for each $i \in \{0, \ldots, t-1\}$, we have

$$d_x^{(i+1)} = d_{x,i}p^{t-1} + \frac{d_x^{(i)} - d_{x,i}}{p} = \frac{1}{p}d_x^{(i)} + \frac{p^t - 1}{p}d_{x,i} = \frac{1}{p}d_x^{(i)} + \frac{n}{p}d_{x,i}.$$

Thus, we obtain that

$$\begin{aligned}(n_X - 1)n = \sum_{x \in D_X} d_x^{(i+1)} &= \frac{1}{p}\sum_{x \in D_X} d_x^{(i)} + \frac{n}{p}\sum_{x \in D_X} d_{x,i} \\ &= \frac{1}{p}(n_X - 1)n + \frac{n}{p}\sum_{x \in D_X} d_{x,i}.\end{aligned}$$

This means that

$$\sum_{x \in D_X} d_{x,i} = (n_X - 1)(p - 1), \ i \in \{0, \ldots, t-1\}.$$

We complete the proof of the lemma. □

Remark 3.3.1. Note that there exists $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ such that $s(D) = (n_X - 1)n$ if and only if $n > n_X - 1$. Lemma 3.3 implies that, if $n > n_X - 1$, then there exists $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ such that $\alpha \neq 0$, that $s(D_\alpha) = n_X - 1$, and that

$$\deg(D_\alpha^{(i)}) \geq \deg(D_\alpha), \ i \in \{0, 1, \ldots, t-1\}.$$

Lemma 3.4. *Let* $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ *and*

$$\mathrm{ord}_x(D) = \sum_{j=0}^{t-1} d_{x,j}p^j, \ x \in D_X$$

the p-adic expansion. Suppose that $s(D^{(i)}) = n_X - 1$ *(i.e.,* $\deg(D^{(i)}) = (n_X - 1)n$*) for each* $i \in \{0, 1, \ldots, t-1\}$*. Moreover, we put* $D_X \stackrel{\text{def}}{=} \{x_1, \ldots, x_{n_X}\}$ *and*

$$a_{l,l+1} \stackrel{\text{def}}{=} [\sum_{r=l+1}^{n_X} d_{x_r}], \ b_{l,l+1} \stackrel{\text{def}}{=} [\sum_{r=1}^{l} d_{x_r}], \ l \in \{2, \ldots, n_X - 2\},$$

where $[(-)]$ *denotes the integer which is equal to the image of* $(-)$ *in* $\mathbb{Z}/n\mathbb{Z}$ *when we identify* $\{0, \ldots, n-1\}$ *with* $\mathbb{Z}/n\mathbb{Z}$ *naturally. Then, for each* $i \in \{0, \ldots, t-1\}$*, we have*

$$a_{l,l+1}^{(i)} + b_{l,l+1}^{(i)} = n, \ l \in \{2, \ldots, n_X - 2\},$$

$$d^{(i)}_{x_1} + d^{(i)}_{x_2} + a^{(i)}_{2,3} = 2n,$$
$$b^{(i)}_{n_X-2,n_X-1} + d^{(i)}_{x_{n_X-1}} + d^{(i)}_{x_{n_X}} = 2n,$$
$$b^{(i)}_{l,l+1} + d^{(i)}_{x_{l+1}} + a^{(i)}_{l+1,l+2} = 2n,\ l \in \{2, \dots, n_X - 3\}.$$

Proof. First, let us treat the first equality. Let $l \in \{2, \dots, n_X - 2\}$ and $i \in \{0, \dots, t-1\}$. Since
$$\sum_{r=l+1}^{n_X} d_{x_r} + \sum_{r=1}^{l} d_{x_r} = \deg(D) = (n_X - 1)n,$$
we see that $n|(a_{l,l+1} + b_{l,l+1})$. Note that if $a_{l,l+1} + b_{l,l+1} = 0$, then $\deg(D) < (n_X - 1)n$. Thus, $a_{l,l+1} + b_{l,l+1} \neq 0$. Moreover, since $a_{l,l+1} + b_{l,l+1} \leq n$, we obtain that $a_{l,l+1} + b_{l,l+1} = n$, and that $a^{(i)}_{l,l+1} + b^{(i)}_{l,l+1}$ is divided by n. Moreover, since $a^{(i)}_{l,l+1} < n$ and $b^{(i)}_{l,l+1} < n$, $\deg(D^{(i)}) = (n_X - 1)n$ implies that $a^{(i)}_{l,l+1} + b^{(i)}_{l,l+1} = n$. This completes the proof of the first equality.

Let $i \in \{0, \dots, t-1\}$. We denote by
$$S^{(i)}_1 \stackrel{\text{def}}{=} d^{(i)}_{x_1} + d^{(i)}_{x_2} + a^{(i)}_{2,3},$$
$$S^{(i)}_l \stackrel{\text{def}}{=} b^{(i)}_{l,l+1} + d^{(i)}_{x_{l+1}} + a^{(i)}_{l+1,l+2},\ l \in \{2, \dots, n_X - 3\},$$
$$S^{(i)}_{n_X-2} \stackrel{\text{def}}{=} b^{(i)}_{n_X-2,n_X-1} + d^{(i)}_{x_{n_X-1}} + d^{(i)}_{x_{n_X}}.$$
We have that $S^{(i)}_l \leq 2n,\ \ l \in \{1, \dots, n_X - 2\}$. Moreover, if $i = 1$, the definitions of $a_{l,l+1}$ and $b_{l,l+1}$ imply that
$$S^{(1)}_l = 2n,\ l \in \{1, \dots, n_X - 2\}.$$
Then we have that $S^{(i)}_l$ is divided by n. Since $s(D^{(i)}) = n_X - 1$, the first equality implies that
$$\sum_{l=1}^{n_X-2} S^{(i)}_l = n(n_X - 3) + (n_X - 1)n = 2n(n_X - 2).$$
Since $S^{(i)}_l \leq 2n$, this implies that
$$S^{(i)}_l = 2n,\ l \in \{1, \dots, n_X - 2\},$$
holds for every $i \in \{1, \dots, t-1\}$. This completes the proof of the lemma. □

Lemma 3.5. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$, that $s(D) = n_X - 1$ if $n_X \neq 0$, and that*
$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \dots, t-1\}.$$
Moreover, suppose that $X^\bullet = (X, D_X)$ is a component-generic pointed stable curve over k, that $X^\bullet$ is smooth over k, and that $(g_X, n_X) = (0, 3)$. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists. Moreover, we have
$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X, \mathcal{L}))$$
when $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$.

Proof. This follows immediately from [B, Corollary 6.8]. □

Remark 3.5.1. Note that, if $n_X = 3$, then we have $s(D) \in \{0, 1, 2\}$.

Next, we will generalize Lemma 3.5 to the case where $X^\bullet$ is *an irreducible component-generic pointed stable curve* (cf. Proposition 3.6). Roughly speaking, the idea is the following. By degenerating $X^\bullet$ to a suitable pointed stable curve $X_s^\bullet$ whose set of irreducible components consists of generic pointed stable curves and pointed stable curves of type $(0, 3)$, Lemma 3.5 and Nakajima's result (cf. [N, Proposition 4]) imply that the first generalized Hasse-Witt invariant of the Galois admissible covering of $X_s^\bullet$ induced by the Galois admissible covering of $X^\bullet$ attains the maximum. Then by using specialization homomorphisms of admissible fundamental groups, we obtain that the first generalized Hasse-Witt invariant of the original Galois admissible covering of $X^\bullet$ attains the maximum.

First, let us introduce some degeneration data for $X^\bullet$. Let R be a discrete valuation ring with algebraically closed residue field k_R, K_R the quotient field of R, and $\overline{K}_R$ an algebraic closure of K_R. Suppose that $k \subseteq K_R$. Let

$$\mathcal{X}^\bullet = (\mathcal{X}, D_{\mathcal{X}} \stackrel{\text{def}}{=} \{e_1, \dots, e_{n_X}\})$$

be a pointed stable curve of type (g_X, n_X) over R. We shall write $\mathcal{X}_\eta^\bullet = (\mathcal{X}_\eta, D_{\mathcal{X}_\eta} \stackrel{\text{def}}{=} \{e_{\eta,1}, \dots, e_{\eta,n_X}\})$, $\mathcal{X}_{\overline{\eta}}^\bullet = (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \{e_{\overline{\eta},1}, \dots, e_{\overline{\eta},n_X}\})$, $\mathcal{X}_s^\bullet = (\mathcal{X}_s, D_{\mathcal{X}_s} \stackrel{\text{def}}{=} \{e_{s,1}, \dots, e_{s,n_X}\})$ for the generic fiber $\mathcal{X}^\bullet \times_R K_R$ of $\mathcal{X}^\bullet$, the geometric generic fiber $\mathcal{X}^\bullet \times_R \overline{K}_R$ of $\mathcal{X}^\bullet$, and the special fiber $\mathcal{X}^\bullet \times_R k_R$ of $\mathcal{X}^\bullet$, respectively. Write $\Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}$ and $\Pi_{\mathcal{X}_s^\bullet}$ for the admissible fundamental groups of $\mathcal{X}_{\overline{\eta}}^\bullet$ and $\mathcal{X}_s^\bullet$, respectively. Moreover, we shall say that $X^\bullet$ *admits a (DEG)* if there exists $\mathcal{X}^\bullet$ such that the following conditions hold, where "(DEG)" means "degeneration":

(i) The geometric generic fiber $\mathcal{X}_{\overline{\eta}}^\bullet$ of $\mathcal{X}^\bullet$ is $\overline{K}_R$-isomorphic to $X^\bullet \times_k \overline{K}_R$. Then without loss of generality, we may identify $e_{\overline{\eta},r}$, $r \in \{1, \dots, n_X\}$, with $x_r \times_k \overline{K}_R$ via this isomorphism. Note that since the admissible fundamental groups do not depend on the base fields, $\Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}$ is naturally isomorphic to $\Pi_{X^\bullet}$.

(ii) $\mathcal{X}_s^\bullet$ is a component-generic pointed stable curve over k_R.

(iii) If $n_X \leq 1$ and $\#X^{\text{sing}} = 0$ (i.e., $X^\bullet$ is smooth over k_R), we have $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is isotrivial (i.e., the image of the natural morphism $\operatorname{Spec} R \to \overline{\mathcal{M}}_{g_X,n_X,k_R} \to \overline{M}_{g_X,n_X,k_R}$ determined by $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is a point, where $\overline{\mathcal{M}}_{g_X,n_X,k_R} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{F}_p} k_R$ and $\overline{M}_{g_X,n_X,k_R}$ denotes the coarse moduli space of $\overline{\mathcal{M}}_{g_X,n_X,k_R}$).

(iv) If $(g_X, n_X) = (1, 1)$ and $\#X^{\text{sing}} = 1$, we have that $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is isotrivial.

(v) If $n_X \leq 1$ and $\#X^{\text{sing}} \geq 2$ hold, the underlying curve of $\mathcal{X}_s^\bullet$ is

$$\mathcal{X}_s = (\bigcup_{T \in \mathscr{T}} T) \cup C$$

such that the following conditions hold: (a) $\mathscr{T}$ is a set of singular projective curves of arithmetic genus 1 over k_R such that $\#\mathscr{T} = \#X^{\text{sing}}$. (b) C is an empty set when $n_X = 0$ and $\#X^{\text{sing}} = \#\mathscr{T} = 2$; otherwise, C is a smooth projective curve of genus $g_X - \#\mathscr{T}$

over k_R. (c) If C is empty, we have $\mathscr{T} \overset{\text{def}}{=} \{T_1, T_2\}$ such that $\#(T_1 \cap T_2) = 1$. (d) If C is not empty, we have that $T' \cap T'' \neq \emptyset$ if and only if $T' = T''$ for each $T', T'' \in \mathscr{T}$, that $\#(T \cap C) = 1$ for each $T \in \mathscr{T}$, and that $D_{\mathcal{X}_s} \subseteq C$.

(vi) If $n_X = 2$, the underlying curve of $\mathcal{X}_s^\bullet$ is

$$\mathcal{X}_s = (\bigcup_{T \in \mathscr{T}} T) \cup C \cup P$$

such that the following conditions hold: (a) $\mathscr{T}$ is a set of singular projective curves of arithmetic genus 1 over k_R such that, for each $T, T' \in \mathscr{T}$, $T \cap T' \neq \emptyset$ if and only if $T = T'$, and that $\#\mathscr{T} = \#X^{\text{sing}}$. (b) C is an empty set when $g_X - \#\mathscr{T} = 0$; otherwise, C is a smooth projective curve of genus $g_X - \#\mathscr{T}$ over k_R when $g_X - \#\mathscr{T} \geq 1$. (c) P is k_R-isomorphic to $\mathbb{P}^1_{k_R}$. (d) If C is empty, we have $\#(P \cap T) = 1$ for each $T \in \mathscr{T}$. (e) If C is not empty, we have that $\#(C \cap T) = 1$, that $\#(C \cap P) = 1$, and that $P \cap T = \emptyset$ for each $T \in \mathscr{T}$. (f) $D_{\mathcal{X}_s} \subseteq P$.

(vii) If $n_X \geq 3$, the underlying curve of $\mathcal{X}_s^\bullet$ is

$$\mathcal{X}_s = (\bigcup_{T \in \mathscr{T}} T) \cup C_1 \cup (\bigcup_{v=2}^{n_X - 1} P_v)$$

such that the following conditions hold: (a) $\mathscr{T}$ is a set of singular projective curves of arithmetic genus 1 over k_R such that, for each $T, T' \in \mathscr{T}$, $T \cap T' \neq \emptyset$ if and only if $T = T'$, and that $\#\mathscr{T} = \#X^{\text{sing}}$. (b) C_1 is an empty set when $g_X - \#\mathscr{T} = 0$; otherwise, C_1 is a projective curve of genus $g_X - \#\mathscr{T}$ over k_R when $g_X - \#\mathscr{T} \geq 1$. (c) P_v, $v \in \{2, \ldots, n_X - 1\}$, is k_R-isomorphic to $\mathbb{P}^1_{k_R}$. (d) If C_1 is empty, we have $\#(P_2 \cap T) = 1$ and $P_v \cap T = \emptyset$ for each $T \in \mathscr{T}$ and each $v \in \{3, \ldots, n_X - 1\}$. (e) If C_1 is not empty, we have $\#(C_1 \cap T) = 1$, $\#(C_1 \cap P_2) = 1$, and $P_v \cap T = \emptyset$ for each $T \in \mathscr{T}$ and each $v \in \{2, \ldots, n_X - 1\}$. (f) For each $v \in \{2, \ldots, n_X - 2\}$, $\#(P_v \cap P_{v+1}) = 1$ and $P_v \cap P_{v'} = \emptyset$ when $v' \notin \{v - 1, v, v + 1\}$. (g) If $n_X = 3$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}, e_{s,3}\}$. (h) If $n_X = 4$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}\}$ and $D_{\mathcal{X}_s} \cap P_3 = \{e_{s,3}, e_{s,4}\}$. (i) If $n_X \geq 5$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}\}$, $D_{\mathcal{X}_s} \cap P_{n_X - 1} = \{e_{s,n_X - 1}, e_{s,n_X}\}$, and $D_{\mathcal{X}_s} \cap P_v = \{e_{s,v}\}$, $v \in \{3, \ldots, n_X - 2\}$.

Note that since geometric generic curves admit all degeneration types, we have that $X^\bullet$ admits a (DEG) when $X^\bullet$ is a component-generic pointed stable curve. Moreover, we have the following proposition.

Proposition 3.6. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ and $\alpha \in \text{Rev}_D^{\text{adm}}(X^\bullet)$ such that $\alpha \neq 0$, that $s(D) = n_X - 1$ if $n_X \neq 0$, and that*

$$\deg(D^{(i)}) \geq \deg(D), \ i \in \{0, 1, \ldots, t - 1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X \overset{\text{def}}{=} \{x_1, \ldots, x_{n_X}\})$ is a component-generic pointed stable curve over k, and that $X^\bullet$ is irreducible. Then we have that $\gamma_{(\alpha, D)}$ attains the maximum

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \textit{if } n_X = 0, \\ g_X + n_X - 2, & \textit{if } n_X \neq 0. \end{cases}$$

Proof. Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. We note that, to verify the proposition, we only need to prove the proposition in the case where $Y^\bullet$ is *connected*. Then we may assume that $Y^\bullet$ is connected.

Since $X^\bullet$ is a component-generic pointed stable curve, $X^\bullet$ admits a (DEG). Furthermore, we write $Q_{\overline{\eta}}$ (resp. Q_s) for the effective divisor on $\mathcal{X}_{\overline{\eta}}$ (resp. $\mathcal{X}_s$) induced by D and $\alpha_{\overline{\eta}} \in \mathrm{Rev}^{\mathrm{adm}}_{Q_{\overline{\eta}}}(\mathcal{X}^\bullet_{\overline{\eta}})$ for the element induced by α. Then we have

$$\gamma_{(\alpha,D)} = \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})}.$$

Suppose that $X^\bullet$ satisfies (DEG)-(iii). If $n_X \leq 1$ and $g_X = 1$, then the proposition is trivial. If $n_X \leq 1$ and $g_X \geq 2$, then the proposition follows immediately from [N, Proposition 4] (or [Z, Théorème 3.1]).

Suppose that $X^\bullet$ satisfies (DEG)-(iv). Then we see immediately that $Y^\bullet$ is a pointed stable curve of type $(1, n)$ such that $\#Y^{\mathrm{sing}} = n$ and the underlying curve of $\widetilde{Y}^\bullet_w$, $w \in v(\Gamma_{Y^\bullet})$, is a rational curve, where $\Gamma_{Y^\bullet}$ is the dual semi-graph of $Y^\bullet$. Thus, we obtain that $\gamma_{(\alpha,D)} = 0$.

Suppose that $X^\bullet$ satisfies (DEG)-(vii). Moreover, we suppose that $C_1 \neq \emptyset$, and that $n_X \geq 5$. For each $v \in \{2, \ldots, n_X - 2\}$, we write

$$y_{v,v+1},\ z_{v,v+1}$$

for the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_v \hookrightarrow \mathcal{X}_s$ and the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_{v+1} \hookrightarrow \mathcal{X}_s$, respectively. We define

$$P_2^\bullet = (P_2, D_{P_2} \overset{\mathrm{def}}{=} \{e_{s,1}, e_{s,2}, y_{2,3}\} \cup (C_1 \cap P_2)),$$

$$P^\bullet_{n_X-1} = (P_{n_X-1}, D_{P_{n_X-1}} \overset{\mathrm{def}}{=} \{z_{n_X-2,n_X-1}, e_{s,n_X-1}, e_{s,n_X}\}),$$

and

$$P_v^\bullet = (P_v, D_{P_v} \overset{\mathrm{def}}{=} \{z_{v-1,v}, e_{s,v}, y_{v,v+1}\}),\ v \in \{3, \ldots, n_X - 2\},$$

to be smooth pointed stable curves of types $(0, 4)$, $(0, 3)$, and $(0, 3)$ over k_R, respectively. Moreover, we define

$$C_1^\bullet = (C_1, D_{C_1} \overset{\mathrm{def}}{=} (C_1 \cap P_2) \cup ((\bigcup_{T \in \mathscr{T}} T) \cap C_1))$$

and

$$T^\bullet = (T, D_T \overset{\mathrm{def}}{=} T \cap C_1),\ T \in \mathscr{T},$$

to be a smooth pointed stable curve of type $(g_X - \#\mathscr{T}, 1 + \#\mathscr{T})$ and a singular pointed stable curve of type $(1, 1)$ over k_R, respectively. Note that, Because C_1 is generic, we have $\sigma_{C_1} = g_X$. Let

$$f^\bullet_{\overline{\eta}} \overset{\mathrm{def}}{=} f^\bullet \times_k \overline{K}_R : \mathcal{Y}^\bullet_{\overline{\eta}} = (\mathcal{Y}_{\overline{\eta}}, D_{\mathcal{Y}_{\overline{\eta}}}) \overset{\mathrm{def}}{=} Y^\bullet \times_k \overline{K}_R \to \mathcal{X}^\bullet_{\overline{\eta}}$$

be the Galois admissible covering over $\overline{K}_R$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $f^\bullet$, and $\Pi_{\mathcal{Y}^\bullet_{\overline{\eta}}} \subseteq \Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ the admissible fundamental group of $\mathcal{Y}^\bullet_{\overline{\eta}}$. By the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]), we have

$$sp_R^{p'}: \Pi^{p'}_{\mathcal{X}^\bullet_{\overline{\eta}}} \overset{\sim}{\to} \Pi^{p'}_{\mathcal{X}^\bullet_s},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Then we obtain a normal open subgroup $\Pi^{p'}_{\mathcal{Y}^\bullet_s} \overset{\text{def}}{=} sp_R^{p'}(\Pi^{p'}_{\mathcal{Y}^\bullet_{\overline{\eta}}}) \subseteq \Pi^{p'}_{\mathcal{X}^\bullet_s}$. Write $\Pi_{\mathcal{Y}^\bullet_s} \subseteq \Pi_{\mathcal{X}^\bullet_s}$ for the inverse image of $\Pi^{p'}_{\mathcal{Y}^\bullet_s}$ of the natural surjection $\Pi_{\mathcal{X}^\bullet_s} \twoheadrightarrow \Pi^{p'}_{\mathcal{X}^\bullet_s}$. Then $\Pi_{\mathcal{Y}^\bullet_s}$ determines a Galois admissible covering

$$f_s^\bullet : \mathcal{Y}_s^\bullet = (\mathcal{Y}_s, D_{\mathcal{Y}_s}) \to \mathcal{X}_s^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_s \in \mathrm{Rev}^{\mathrm{adm}}_{Q_s}(\mathcal{X}_s^\bullet)$ for an element induced by $f_s^\bullet$.

The structure of the maximal prime-to-p quotients of admissible fundamental groups implies that f_s is étale over $(\bigcup_{T\in\mathscr{T}} T) \cap C_1$. Then we obtain that f_s is étale over $C_1 \cap P_2$. Thus, f_s is étale over D_{C_1}. Let $Y_v \overset{\text{def}}{=} f_s^{-1}(P_v)$, $v \in \{2, \ldots, n_X - 1\}$. We put

$$Y_v^\bullet \overset{\text{def}}{=} (Y_v, D_{Y_v} \overset{\text{def}}{=} f_s^{-1}(D_{P_v})),\ v \in \{2, \ldots, n_X - 1\}.$$

Then $f_s^\bullet$ induces a Galois multi-admissible covering

$$f_v^\bullet : Y_v^\bullet \to P_v^\bullet,\ v \in \{2, \ldots, n_X - 1\},$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. We maintain the notation introduced in Lemma 3.4 and define the following effective divisors

$$Q_2 \overset{\text{def}}{=} d_{x_1} e_{s,1} + d_{x_2} e_{s,2} + a_{2,3} y_{2,3},$$

$$Q_{n_X-1} \overset{\text{def}}{=} b_{n_X-2,n_X-1} z_{n_X-2,n_X-1} + d_{x_{n_X-1}} e_{s,n_X-1} + d_{x_{n_X}} e_{s,n_X},$$

and

$$Q_v \overset{\text{def}}{=} b_{v-1,v} z_{v-1,v} + d_{x_v} e_{s,v} + a_{v,v+1} y_{v,v+1},\ v \in \{3, \ldots, n_X - 2\},$$

on P_2, P_{n_X-1}, and P_v, $v \in \{3, \ldots, n_X - 2\}$, respectively. Since f_s is étale over $C_1 \cap P_2$, we see that $f_v^\bullet$, $v \in \{2, \ldots, n_X - 1\}$, induces a pair $([\mathcal{L}_v], Q_v) \in \widetilde{\mathscr{P}}_{P_v^\bullet, n}$. Moreover, the $k_R[\mu_n]$-module $H^1_{\text{ét}}(Y_v, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Y_v, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Y_v}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Y_v}(j)$ as the ζ^j-multiplication. Lemma 3.4 implies that $\deg(Q_v^{(i)}) = \deg(Q_v) = 2n$, $i \in \{0, \ldots, t-1\}$. Then Lemma 3.5 implies that

$$\gamma_{([\mathcal{L}_v], Q_v)} = \dim_{k_R}(M_{Y_v}(1)) = \dim_{k_R}(H^1(P_v, \mathcal{L}_v)) = 1.$$

Let $Z_1 \overset{\text{def}}{=} f_s^{-1}(C_1)$ and $\pi_0(Z_1)$ the set of connected components of Z_1. Then $f_s^\bullet$ induces a Galois étale covering (not necessarily connected)

$$f_{C_1}^\bullet : Z_1^\bullet = (Z_1, D_{Z_1} \overset{\text{def}}{=} f_s^{-1}(D_{C_1})) \to C_1^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, $f^\bullet_{C_1}$ induces an element $\alpha_{C_1} \in \mathrm{Rev}^{\mathrm{adm}}_0(C^\bullet_1)$. Suppose that $\#\pi_0(Z_1) \neq n$. Then we have $\alpha_{C_1} \neq 0$. The $k_R[\mu_n]$-module $H^1_{\text{ét}}(Z_1, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Z_1, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Z_1}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Z_1}(j)$ as the ζ^j-multiplication. Because C_1 is generic, [N, Proposition 4] (or [Z, Théorème 3.1]) implies that

$$\gamma_{(\alpha_{C_1},0)} = \dim_{k_R}(M_{Z_1}(1)) = g_X - \#\mathscr{T} - 1 = g_{C_1} - 1,$$

where g_{C_1} denotes the genus of C_1. Suppose that $\#\pi_0(Z_1) = n$. Then we have $\alpha_{C_1} = 0$. Since C_1 is ordinary, we obtain immediately that

$$\gamma_{(\alpha_{C_1},0)} = \sigma(C_1) = g_X - \#\mathscr{T} = g_{C_1}.$$

Let $V_T \overset{\text{def}}{=} f_s^{-1}(T)$, $T \in \mathscr{T}$, and $\widetilde{T}$ the smooth compactification of $U_T \overset{\text{def}}{=} T \setminus T^{\mathrm{sing}}$. Then $f^\bullet_s$ induces a Galois multi-admissible covering

$$f^\bullet_T : V^\bullet_T = (V_T, D_{V_T} \overset{\text{def}}{=} f_s^{-1}(D_T)) \to T^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Since f_s is étale over D_T, we have that the underlying curve of $V^\bullet_{T,w}$, $w \in v(\Gamma_{V^\bullet_T})$, is a rational curve over k_R, where $\Gamma_{V^\bullet_T}$ is the dual semi-graph of $V^\bullet_T$. We put

$$\widetilde{T}^\bullet = (\widetilde{T}, D_{\widetilde{T}} \overset{\text{def}}{=} D_T \cup (\widetilde{T} \setminus U_T)).$$

Then $f^\bullet_T$ induces a Galois multi-admissible covering

$$f^\bullet_{\widetilde{T}} : V^\bullet_{\widetilde{T}} = (V_{\widetilde{T}}, D_{V_{\widetilde{T}}}) \to \widetilde{T}^\bullet$$

over k_R. Write $\alpha_{\widetilde{T}} \in \mathrm{Rev}_0(\widetilde{T}^\bullet)$ for an element induced by $f^\bullet_{\widetilde{T}}$. Then we obtain that $\gamma_{(\alpha_{\widetilde{T}},0)} = 0$. Thus, Proposition 3.1 implies that

$$\gamma_{(\alpha_s, Q_s)} = g_X + n_X - 2.$$

On the other hand, the $k_R[\mu_n]$-modules $H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ and $H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R$ admit the following canonical decompositions

$$H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{\overline{\eta}}}(j)$$

and

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_s}(j),$$

respectively. Moreover, we have an injection as $k_R[\mu_n]$-modules

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R \hookrightarrow H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$$

induced by the specialization map $\Pi_{\mathcal{Y}^\bullet_{\overline{\eta}}} \twoheadrightarrow \Pi_{\mathcal{Y}^\bullet_s}$. Thus, we have

$$g_X + n_X - 2 = \gamma_{(\alpha_s, Q_s)} = \dim_{k_R}(M_{\mathcal{Y}_s}(1)) \leq \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \dim_{k_R}(M_{\mathcal{Y}_{\overline{\eta}}}(1)).$$

Next, we prove that $\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} \leq g_X + n_X - 2$. We write $\widetilde{\mathcal{X}}_{\overline{\eta}}$ for the smooth compactification of $U_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \mathcal{X}_{\overline{\eta}} \setminus \mathcal{X}^{\text{sing}}_{\overline{\eta}}$ and define

$$\widetilde{\mathcal{X}}^\bullet_{\overline{\eta}} = (\widetilde{\mathcal{X}}_{\overline{\eta}}, D_{\widetilde{\mathcal{X}}_{\overline{\eta}}} \stackrel{\text{def}}{=} D_{\mathcal{X}_{\overline{\eta}}} \cup (\widetilde{\mathcal{X}}_{\overline{\eta}} \setminus U_{\mathcal{X}_{\overline{\eta}}}))$$

to be a pointed stable curve of type $(g_{\overline{\eta}}, n_{\overline{\eta}})$ over $\overline{K}_R$, where $g_{\overline{\eta}} = g_X - \#\mathcal{X}^{\text{sing}}_{\overline{\eta}}$ and $n_{\overline{\eta}} = n_X + 2\#\mathcal{X}^{\text{sing}}_{\overline{\eta}}$. Let $\widetilde{\alpha}_{\overline{\eta}} \in \text{Hom}(\Pi^{\text{ab}}_{\widetilde{\mathcal{X}}^\bullet_{\overline{\eta}}}, \mathbb{Z}/n\mathbb{Z})$ be the element induced by α via the natural (outer) injection $\Pi_{\widetilde{\mathcal{X}}^\bullet_{\overline{\eta}}} \hookrightarrow \Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ and $\widetilde{Q}_{\overline{\eta}} \stackrel{\text{def}}{=} D_{\widetilde{\alpha}_{\overline{\eta}}}$ the divisor on $\widetilde{\mathcal{X}}_{\overline{\eta}}$ determined by $\widetilde{\alpha}_{\overline{\eta}}$. Note that $\widetilde{\alpha}_{\overline{\eta}} \in \text{Rev}^{\text{adm}}_{\widetilde{Q}_{\overline{\eta}}}(\widetilde{\mathcal{X}}^\bullet_{\overline{\eta}})$ and $s(\widetilde{Q}_{\overline{\eta}}) = s(Q_{\overline{\eta}}) + \#\mathscr{N}^{\text{ra}}_{\mathcal{X}_{\overline{\eta}}}$, where $\mathscr{N}^{\text{ra}}_{\mathcal{X}_{\overline{\eta}}}$ denotes the set of nodes of $\mathcal{X}_{\overline{\eta}}$ over which $f_{\overline{\eta}}$ is ramified. Then we have

$$\gamma_{(\widetilde{\alpha}_{\overline{\eta}}, \widetilde{Q}_{\overline{\eta}})} \leq g_{\overline{\eta}} + s(\widetilde{Q}_{\overline{\eta}}) - 1.$$

Since $\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \gamma_{(\widetilde{\alpha}_{\overline{\eta}}, \widetilde{Q}_{\overline{\eta}})} + \#\mathcal{X}^{\text{sing}} - \#\mathscr{N}^{\text{ra}}_{\mathcal{X}_{\overline{\eta}}}$, we obtain that $\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} \leq g_X + n_X - 2$. Then we obtain that

$$\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = g_X + n_X - 2.$$

This completes the proof of the proposition when $X^\bullet$ satisfies (DEG)-(vii), $C_1 \neq \emptyset$, and $n_X \geq 5$. By applying similar arguments to the arguments given in the proof above, one can prove the proposition when $X^\bullet$ satisfies (DEG)-(vii) and either $C_1 = \emptyset$ or $n_X \leq 4$ holds.

Moreover, similar arguments to the arguments given in the proof above imply the proposition holds when $X^\bullet$ satisfies either (DEG)-(v) or (DEG)-(vi). We complete the proof of the proposition. □

Next, we will introduce a kind of semi-graphs associated to the sets of marked points of pointed stable curves, which are called *minimal quasi-trees*. Roughly speaking, a minimal quasi-tree is a minimal tree-like sub-semi-graph of the dual semi-graph of a pointed stable curve which contains all open edges. Minimal quasi-trees will play an important role in the remainder of the present paper.

Definition 3.7. (i) Let $Z^\bullet$ be a pointed sable curve over k, $\Gamma_{Z^\bullet}$ the dual semi-graph of $Z^\bullet$ such that $\Gamma_{Z^\bullet} \setminus e^{\text{lp}}(\Gamma_{Z^\bullet})$ is a tree, and $E_Z \subseteq e^{\text{op}}(\Gamma_{Z^\bullet})$ a subset of open edges. Recall that $e^{\text{lp}}(\Gamma_{Z^\bullet})$ is the subset of closed edges of $\Gamma_{Z^\bullet}$ such that every element of $e^{\text{lp}}(\Gamma_{Z^\bullet})$ corresponds to a node which is contained in a unique irreducible component of Z.

We shall say that a subset of vertices $V_Z \subseteq v(\Gamma_{Z^\bullet})$ is *the set of terminal vertices avoiding to* E_Z if the following conditions are satisfied:

$$\#(e^{\Gamma_{Z^\bullet}}(v) \cap (e^{\text{cl}}(\Gamma_{Z^\bullet}) \setminus e^{\text{lp}}(\Gamma_{Z^\bullet}))) = 1,\ Z_v \cap E_Z = \emptyset,\ v \in V_Z,$$

where Z_v denotes the irreducible component corresponding to v.

(ii) Let $W^\bullet = (W, D_W)$ be a pointed stable curve of type (g_W, n_W) over k and $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$.

Let $E \subseteq e^{\mathrm{cl}}(\Gamma_{W^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{W^\bullet})$ be a subset of closed edges such that $\Gamma_{W^\bullet} \setminus (E \cup e^{\mathrm{lp}}(\Gamma_{W^\bullet}))$ is a connected tree. Note that it is easy to see that E exists. Write N for the set of nodes of W corresponding to the closed edges contained in E and $\mathrm{norm}_N : W_N \to W$ for the normalization morphism of the underlying curve W over N. We define a pointed stable curve over k to be

$$W_1^\bullet = (W_1, D_{W_1}) \stackrel{\mathrm{def}}{=} W_N^\bullet = (W_N, D_{W_N} \stackrel{\mathrm{def}}{=} \mathrm{norm}_N^{-1}(D_W \cup N)).$$

Note that $D_W \subseteq D_{W_1}$. Write $\Gamma_{W_1^\bullet}$ for the dual semi-graph of $W_1^\bullet$. Then the construction of $W_1^\bullet$ implies that $\Gamma_{W_1^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{W_1^\bullet})$ is a tree.

Suppose that $n_W \neq 0$. Let $V_1 \subseteq v(\Gamma_{W_1^\bullet})$ be the subset of terminal vertices avoiding to D_W. If $V_1 = \emptyset$, then we put $W_\Gamma^\bullet \stackrel{\mathrm{def}}{=} W_1^\bullet$. If $V_1 \neq \emptyset$, we write W_2 for the topological closure of

$$W_1 \setminus (\bigcup_{v \in V_1} W_v)$$

in W_1. Note that, by the definition of V_1, we have that W_2 is connected, and that $D_W \subseteq W_2$. Then we define a pointed stable curve over k to be

$$W_2^\bullet \stackrel{\mathrm{def}}{=} (W_2, D_{W_2} \stackrel{\mathrm{def}}{=} D_{W_1} \cup ((\bigcup_{v \in V_1} W_v) \cap W_2)).$$

Let $V_2 \subseteq v(\Gamma_{W_2^\bullet})$ be the subset of terminal vertices avoiding to D_W. If $V_2 = \emptyset$, then we put $W_\Gamma^\bullet \stackrel{\mathrm{def}}{=} W_2^\bullet$. By repeating this process, we obtain a pointed stable curve

$$W_\Gamma^\bullet = (W_\Gamma, D_{W_\Gamma})$$

over k such that the subset of terminal vertices avoiding to D_W of $W_\Gamma^\bullet$ is empty. We write Γ for the dual semi-graph of $W_\Gamma^\bullet$. Note that $D_W \subseteq D_{W_\Gamma}$, and that $\Gamma \setminus e^{\mathrm{lp}}(\Gamma)$ is a *tree*.

We shall say that

$$\Gamma_{D_W}$$

is *a minimal quasi-tree associated to* D_W if $\Gamma_{D_W} = \Gamma$ when $n_W \neq 0$, and $\Gamma_{D_W} = \emptyset$ when $n_W = 0$.

(iii) In the remainder of this definition, we suppose that $n_W \neq 0$. We maintain the notation introduced in (ii). The construction of W_Γ implies that there is a natural morphism

$$f_\Gamma : W_\Gamma \to W$$

over k. We shall denote by

$$\phi_\Gamma : \Gamma \to \Gamma_{W^\bullet}$$

the map of dual semi-graphs induced by f_Γ. Let

$$D_{E_1} \subseteq D_{W_\Gamma} \setminus D_W,\ D_{E_2} \subseteq D_{W_\Gamma} \setminus D_W$$

be the subsets of marked points of $W_\Gamma^\bullet$ such that $D_{W_\Gamma} = D_W \cup D_{E_1} \cup D_{E_2}$, that $f_\Gamma(D_{E_1}) \cap N = \emptyset$, and that $N_2 \stackrel{\mathrm{def}}{=} f_\Gamma(D_{E_2}) \subseteq N$. This means that we have the following: (a)

$D_{W_\Gamma} = D_W \cup D_{E_1} \cup D_{E_2}$; (b) $f_\Gamma(D_W) = D_W$; (c) $f_\Gamma(D_{E_1}) \subseteq W^{\mathrm{sing}} \setminus N$; (d) $N_2 \subseteq N$; (e) $f_\Gamma(W_\Gamma^{\mathrm{sing}}) \subseteq W^{\mathrm{sing}}$.

We define a pointed stable curve over k to be

$$W_{\Gamma^{\mathrm{im}}} = (W_{\Gamma^{\mathrm{im}}} \overset{\mathrm{def}}{=} f_\Gamma(W_\Gamma), D_{W_{\Gamma^{\mathrm{im}}}} \overset{\mathrm{def}}{=} f_\Gamma(D_W \cup D_{E_1})).$$

Then we see that the dual semi-graph of $W_{\Gamma^{\mathrm{im}}}$ coincides with the image $\mathrm{Im}(\phi_\Gamma)$. Denote by

$$\Gamma^{\mathrm{im}}$$

the dual semi-graph of $W_{\Gamma^{\mathrm{im}}}$. We shall say that Γ^{im} is *the image of the map* ϕ_Γ. Moreover, we denote by

$$\mathrm{norm}_\Gamma : W_\Gamma \to W_{\Gamma^{\mathrm{im}}}$$

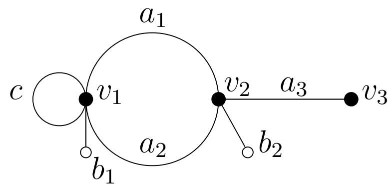
the natural morphism induced by f_Γ which coincides with the normalization morphism of $W_{\Gamma^{\mathrm{im}}}$ over N_2. Then we have that $D_{W_\Gamma} = \mathrm{norm}_\Gamma^{-1}(D_{W_{\Gamma^{\mathrm{im}}}} \cup N_2)$.

Example 3.8. We maintain the notation introduced in Definition 3.7. Let us give an example of minimal quasi-trees associated to the set of marked points.

Let $W^\bullet$ be a pointed stable curve over k and $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$ such that the following conditions hold: (i) $v(\Gamma_{W^\bullet}) \overset{\mathrm{def}}{=} \{v_1, v_2, v_3\}$; (ii) $e^{\mathrm{cl}}(\Gamma_{W^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{W^\bullet}) \overset{\mathrm{def}}{=} \{a_1, a_2, a_3\}$ such that a_1 and a_2 abut to v_1 and v_2, respectively, and that a_3 abuts to v_2 and v_3; (iii) $e^{\mathrm{lp}}(\Gamma_{W^\bullet}) \overset{\mathrm{def}}{=} \{c\}$ and c abuts to v_1; (iv) $e^{\mathrm{op}}(\Gamma_{W^\bullet}) \overset{\mathrm{def}}{=} \{b_1, b_2\}$ such that b_1 and b_2 abut to v_1 and v_2, respectively.

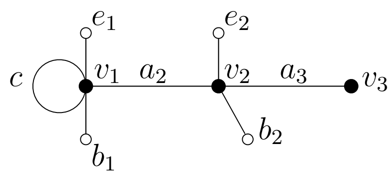
We use the notation "$\bullet$" and "$\circ$" to denote a vertex and an open edge, respectively. Then $\Gamma_{W^\bullet}$ is as follows:

$\Gamma_{W^\bullet}$:



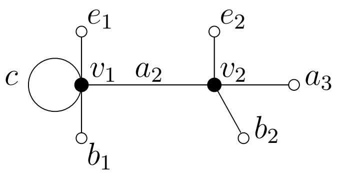
Let $E \overset{\mathrm{def}}{=} \{a_1\}$. Then we see that the dual semi-graph $\Gamma_{W_1^\bullet}$ of $W_1^\bullet$ is as follows:

Γ_{W_1}:



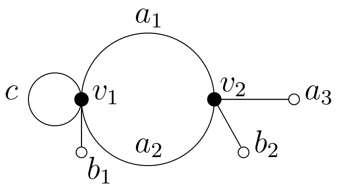
Note that the set of terminal vertices avoiding to $e^{\mathrm{op}}(\Gamma_{W^\bullet})$ of $W_1^\bullet$ is $\{v_3\}$. Then we obtain a minimal quasi-tree $\Gamma_{D_W} \overset{\mathrm{def}}{=} \Gamma$ associated to D_W is as follows:

$\Gamma_{D_W} \overset{\text{def}}{=} \Gamma$:



Moreover, we see that the image Γ^{im} of the map $\phi_\Gamma : \Gamma \to \Gamma^\bullet_W$ is as follows:

Γ^{im}:



Lemma 3.9. *Let $W^\bullet = (W, D_W)$ be a pointed stable curve of type (g_W, n_W) over k. Suppose that $n_W \neq 0$. Then the set of minimal quasi-trees associated to D_W is not empty.*

Proof. The lemma follows immediately from Definition 3.7 (ii). □

Proposition 3.10. *(i) Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$, that $s(D) = n_X - 1$ if $n_X \neq 0$, and that*

$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \dots, t-1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X)$ is a component-generic pointed stable curve over k, and that either $\Gamma_{X^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{X^\bullet})$ is a tree when $n_X = 0$ or $\Gamma_{X^\bullet}$ is a minimal quasi-tree associated to D_X when $n_X \neq 0$. Then we have that $\gamma_{(\alpha,D)}$ attains the maximum

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

(ii) Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ such that $s(D) = n_X - 1$ if $n_X \neq 0$, and that

$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \dots, t-1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X)$ is a component-generic pointed stable curve over k. Then there exists an element $\beta \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\beta \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\beta,D)}$ attains the maximum

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. (i) Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be a Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. To verify (i), we only need to prove (i) in the case where $Y^\bullet$ is *connected*. Then we may assume that $Y^\bullet$ is connected.

Suppose that $n_X = 0$. We see immediately that f is étale over $X^{\mathrm{sing}} \setminus \{x_e\}_{e \in e^{\mathrm{lp}}(\Gamma_{X^\bullet})}$. Then (i) follows from Proposition 3.1 and Proposition 3.6.

Suppose that $n_X > 0$. Let $v \in v(\Gamma_{X^\bullet})$ and $\pi_0(\overline{X \setminus X_v})$ the set of connected components of $\overline{X \setminus X_v}$, where $\overline{X \setminus X_v}$ denotes the topological closure of $X \setminus X_v$ in X. We put

$$D_v \overset{\mathrm{def}}{=} (D_X \cap X_v) \cup \Big(\bigcup_{C \in \pi_0(\overline{X \setminus X_v})} (C \cap X_v) \Big).$$

Note that since we assume that $\Gamma_{X^\bullet}$ is a quasi-tree associated to D_X, we have $\#(C \cap X_v) = 1$ for each $C \in \pi_0(\overline{X \setminus X_v})$. Let $x_C \overset{\mathrm{def}}{=} C \cap X_v$, $C \in \pi_0(\overline{X \setminus X_v})$, be the unique closed point and $Q_v \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_v]^0$ an effective divisor on X_v defined as follows:

$$\mathrm{ord}_x(Q_v) \overset{\mathrm{def}}{=} \mathrm{ord}_x(D),\ x \in D_X \cap X_v,$$

$$\mathrm{ord}_{x_C}(Q_v) \overset{\mathrm{def}}{=} [\sum_{c \in D_X \cap C} \mathrm{ord}_c(D)],\ C \in \pi_0(\overline{X \setminus X_v}),$$

where $[(-)]$ denotes the integer which is equal to the image of $(-)$ in $\mathbb{Z}/n\mathbb{Z}$ when we identify $\{0, \dots, n-1\}$ with $\mathbb{Z}/n\mathbb{Z}$ naturally. By applying similar arguments to the arguments given in the proof of Lemma 3.4, we see that

$$\deg(Q_v) = (\#D_v - 1)n \text{ and } \deg(Q_v^{(i)}) \geq \deg(Q_v),\ i \in \{0, \dots, t-1\}.$$

On the other hand, let

$$X_v^\bullet = (X_v, D_{X_v} \overset{\mathrm{def}}{=} D_v),\ v \in v(\Gamma_{X^\bullet}),$$

be a pointed stable curve of type (g_{X_v}, n_{X_v}) over k. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f_v^\bullet : Y_v^\bullet \to X_v^\bullet,\ v \in v(\Gamma_{X^\bullet}),$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_v \in \mathrm{Rev}_{Q_v}^{\mathrm{adm}}(X_v^\bullet)$ for the element induced by $f_v^\bullet$. If $\alpha_v = 0$, since $X^\bullet$ is component-generic, we have $\gamma_{(\alpha_v, Q_v)} = g_{X_v}$. Then Proposition 3.6 implies that

$$\gamma_{(\alpha_v, Q_v)} = \begin{cases} g_{X_v}, & \text{if } \alpha_v = 0, \\ g_{X_v} - 1, & \text{if } \alpha_v \neq 0,\ \mathrm{Supp}(Q_v) = \emptyset, \\ g_{X_v} + s(Q_v) - 2, & \text{if } \alpha_v \neq 0,\ \mathrm{Supp}(Q_v) \neq \emptyset. \end{cases}$$

Thus, Proposition 3.1 implies that

$$\gamma_{(\alpha, D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

This completes the proof of (i).

(ii) Suppose that $n_X \leq 1$. Then $D = 0$. Let $\beta \in \mathrm{Rev}_0^{\mathrm{adm}}(X^\bullet)$ such that $\beta \neq 0$ and the Galois multi-admissible covering induced by β is étale. By applying [N, Proposition 4] (or [Z, Théoréme 3.1]), we have

$$\gamma_{(\beta, 0)} = \gamma_{X^\bullet}^{\max} = g_X - 1.$$

Then we may assume that $n_X \geq 2$.

Let $\Gamma \stackrel{\text{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X, Γ^{im} the image of the natural morphism $\phi_\Gamma : \Gamma \to \Gamma_{X^\bullet}$, and

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma}),\ X_{\Gamma^{\text{im}}}^\bullet = (X_{\Gamma^{\text{im}}}, D_{X_{\Gamma^{\text{im}}}})$$

the pointed stable curves over k associated to Γ, Γ^{im}, respectively. Note that D is also an effective divisor on $X_{\Gamma^{\text{im}}}$.

Write D_Γ for $\text{norm}_\Gamma^*(D)$ (cf. Definition 3.7 for the definition of norm_Γ). Let $\alpha_\Gamma \in \text{Rev}_{D_\Gamma}^{\text{adm}}(X_\Gamma^\bullet)$ be an arbitrary element such that $\alpha_\Gamma \neq 0$. Then (i) implies that $\gamma_{(\alpha_\Gamma, D_\Gamma)} = \gamma_{X_\Gamma^\bullet}^{\max} = g_{X_\Gamma} + n_X - 2$, where g_{X_Γ} denotes the genus of X_Γ. We denote by

$$g_\Gamma^\bullet : Z_\Gamma^\bullet \to X_\Gamma^\bullet$$

the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_Γ. By gluing $Z_\Gamma^\bullet$ along $g_\Gamma^{-1}(D_{X_\Gamma} \setminus \text{norm}_\Gamma^{-1}(D_{X_{\Gamma^{\text{im}}}}))$ that is compatible with the gluing of $X_\Gamma^\bullet$ that gives rise to $X_{\Gamma^{\text{im}}}^\bullet$, we obtain a pointed stable curve $Z_{\Gamma^{\text{im}}}^\bullet$ over k. Moreover, $g_\Gamma^\bullet$ induces a Galois multi-admissible covering

$$g_{\Gamma^{\text{im}}}^\bullet : Z_{\Gamma^{\text{im}}}^\bullet \to X_{\Gamma^{\text{im}}}^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\Pi_{X_{\Gamma^{\text{im}}}^\bullet}$ for the admissible fundamental group of $X_{\Gamma^{\text{im}}}^\bullet$ and $\alpha_{\Gamma^{\text{im}}}$ for an element of $\text{Hom}(\Pi_{X_{\Gamma^{\text{im}}}^\bullet}^{\text{ab}}, \mathbb{Z}/n\mathbb{Z})$ induced by $g_{\Gamma^{\text{im}}}^\bullet$. We put $D_{\Gamma^{\text{im}}} \stackrel{\text{def}}{=} D_{\alpha_{\Gamma^{\text{im}}}}$. Then Proposition 3.1 implies that

$$\gamma_{(\alpha_{\Gamma^{\text{im}}}, D_{\Gamma^{\text{im}}})} = \gamma_{X_{\Gamma^{\text{im}}}^\bullet}^{\max} = g_{X_{\Gamma^{\text{im}}}} + n_X - 2,$$

where $g_{X_{\Gamma^{\text{im}}}}$ denotes the genus of $X_{\Gamma^{\text{im}}}$.

On the other hand, we write $\pi_0(\overline{X \setminus X_{\Gamma^{\text{im}}}})$ for the set of connected components of $\overline{X \setminus X_{\Gamma^{\text{im}}}}$, where $\overline{X \setminus X_{\Gamma^{\text{im}}}}$ denotes the topological closure of $X \setminus X_{\Gamma^{\text{im}}}$ in X. We define the following pointed stable curve

$$C^\bullet = (C, D_C \stackrel{\text{def}}{=} C \cap X_{\Gamma^{\text{im}}}),\ C \in \pi_0(\overline{X \setminus X_{\Gamma^{\text{im}}}}),$$

over k. Note that since $X^\bullet$ is component-generic, we have that $C^\bullet$ is also component-generic. Then σ_C is equal to the genus of $C^\bullet$.

Let $C \in \pi_0(\overline{X \setminus X_{\Gamma^{\text{im}}}})$. We put

$$Z_C^\bullet \stackrel{\text{def}}{=} \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} C_i^\bullet,$$

where $C_i^\bullet$ is a copy of $C^\bullet$. Then we obtain a Galois multi-admissible covering

$$g_C^\bullet : Z_C^\bullet \to C^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $g_C^\bullet|_{C_i^\bullet}$ is an identity, and the Galois action is $j(C_i) = C_{i+j}$ for each $i, j \in \mathbb{Z}/n\mathbb{Z}$. By gluing $Z_{\Gamma^{\text{im}}}^\bullet$ and

$\{Z^\bullet_C\}_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}$ along $g_\Gamma^{-1}(X_{\Gamma^{\rm im}}\cap(\bigcup_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}C))$ and $\{g_C^{-1}(X_{\Gamma^{\rm im}}\cap C)\}_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}$ in a way that is compatible with the gluing of $\{X^\bullet_{\Gamma^{\rm im}}\}\cup\{C^\bullet\}_{C\in\pi_0(\overline{X\setminus X_{\Gamma^{\rm im}}})}$ that gives rise to $X^\bullet$, we obtain a Galois multi-admissible covering

$$g^\bullet : Z^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we write $\beta \in \mathrm{Rev}^{\rm adm}_D(X^\bullet)$ for an element induced by $g^\bullet$. By applying Proposition 3.1, we see that

$$\gamma_{(\beta,D)} = \gamma^{\max}_{X^\bullet} = g_X + n_X - 2.$$

Then we complete the proof of (ii). □

Remark 3.10.1. Proposition 3.10 (i) does not hold in general. For example, we suppose that $p >> 0$, that $n_X = 0$, and that there exist $v_1, v_2 \in v(\Gamma_{X^\bullet})$ such that $\#(X_{v_1}\cap X_{v_2}) \geq 3$. Then one can construct a Galois admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that (i) of the theorem does not hold.

Indeed, let $X^\bullet$ be a component-generic pointed stable curve of type $(g_X, 0)$ over k with two smooth irreducible components X_1 and X_2 of genus g_1 and g_2, respectively. Moreover, suppose that $X_1 \cap X_2 = \{x_1, x_2, x_3\}$. We put

$$X^\bullet_i = (X_i, D_{X_i} \stackrel{\rm def}{=} X_1 \cap X_2),\ i \in \{1, 2\}.$$

Let $D_1 \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_1}]^0$ such that $s(D_1) = 2$, and that $s(D_1^{(i)}) = 1 < s(D_1) = 2$ for some $i \in \{1, \dots, t-1\}$. We put

$$D_2 \stackrel{\rm def}{=} (n - \mathrm{ord}_{x_1}(D_1))x_1 + (n - \mathrm{ord}_{x_2}(D_1))x_2 + (n - \mathrm{ord}_{x_3}(D_1))x_3.$$

Then $D_2 \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_2}]^0$ is an effective divisor on X_2 with degree n. Let $\alpha_i \in \mathrm{Rev}^{\rm adm}_{D_i}(X^\bullet_i)$, $i \in \{1, 2\}$, and $f^\bullet_i : Y^\bullet_i \to X^\bullet_i$ the Galois multi-admissible covering induced by α_i. Then by gluing $\{X^\bullet_i\}_{i=1,2}$ and $\{Y^\bullet_i\}_{i=1,2}$, we obtain a Galois multi-admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

over k. Write $\alpha \in \mathrm{Rev}^{\rm adm}_0(X^\bullet)$ for the element induced by $f^\bullet$. Then Proposition 3.1 implies that $\gamma_{(\alpha,0)} = g_X - 1 = g_1 + g_2 + 1$ if and only if $\gamma_{(\alpha_1,D_1)} = g_1 + 1$ and $\gamma_{(\alpha_2,D_2)} = g_2$. On the other hand, since $s(D_1^{(i)}) = 1$ for some $i \in \{1, \dots, t-1\}$, we have that $\gamma_{(\alpha_1,D_1)} = g_1$. Then $\gamma_{(\alpha,0)}$ cannot attain the maximum.

Corollary 3.11. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ such that $s(D) = n_X - 1$ if $n_X \neq 0$, and that*

$$\deg(D^{(i)}) \geq \deg(D),\ i \in \{0, 1, \dots, t-1\}.$$

Moreover, suppose that $X^\bullet = (X, D_X \stackrel{\rm def}{=} \{x_1, \dots, x_{n_X}\})$ is a component-generic pointed stable curve over k, and that $X^\bullet$ is smooth over k. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists.

Proof. Since $X^\bullet$ is smooth over k, the corollary follows immediately from Proposition 3.10 and Remark 2.7.1. □

The main result of the present section is as follows.

Theorem 3.12. *Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p and $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$. Let $t \in \mathbb{N}$ be a positive natural number such that $p^t = 1$ in $(\mathbb{Z}/m\mathbb{Z})^{\times}$. Write D' for the divisor $m'D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order m, where $n \overset{\text{def}}{=} p^t - 1$ and $m' \overset{\text{def}}{=} n/m$. Suppose that $X^\bullet = (X, D_X)$ is a component-generic pointed stable curve over k.*

(i) Let $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$ be an arbitrary element such that $\alpha \neq 0$. Suppose that either $\Gamma_{X^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{X^\bullet})$ is a tree when $n_X = 0$ or $\Gamma_{X^\bullet}$ is a minimal quasi-tree associated to D_X when $n_X \neq 0$. Then we have that $\gamma_{(\alpha,D)}$ attains the maximum

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \textit{if } n_X = 0, \\ g_X + n_X - 2, & \textit{if } n_X \neq 0 \end{cases}$$

if and only if

$$s(D) = \begin{cases} 0, & \textit{if } n_X = 0, \\ n_X - 1, & \textit{if } n_X \neq 0 \end{cases}$$

and $\deg((D')^{(i)}) \geq \deg(D'),\ i \in \{0, 1, \ldots, t-1\}$.

(ii) There exists an element $\beta \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$ such that $\beta \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\beta,D)}$ attains the maximum

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \textit{if } n_X = 0, \\ g_X + n_X - 2, & \textit{if } n_X \neq 0 \end{cases}$$

if and only if

$$s(D) = \begin{cases} 0, & \textit{if } n_X = 0, \\ n_X - 1, & \textit{if } n_X \neq 0, \end{cases}$$

and $\deg((D')^{(i)}) \geq \deg(D'),\ i \in \{0, 1, \ldots, t-1\}$.

Proof. (i) Write $\alpha' \in \mathrm{Rev}_{D'}^{\mathrm{adm}}(X^\bullet)$ for the element induced by α. Then we see immediately that $\gamma_{(\alpha,D)} = \gamma_{(\alpha',D')}$. The "only if" part of (i) follows immediately from Proposition 3.1 and [T2, Lemma 2.15]. Moreover, the "if" part of (i) follows immediately from Proposition 3.10 (i).

(ii) Write $\beta' \in \mathrm{Rev}_{D'}^{\mathrm{adm}}(X^\bullet)$ for the element induced by β. Then we see immediately that $\gamma_{(\alpha,D)} = \gamma_{(\beta',D')}$. The "only if" part of (ii) follows immediately from Proposition 3.1 and [T2, Lemma 2.15]. Moreover, the proof of Proposition 3.10 (ii) implies that the "if" part of (ii) holds. $\square$

Definition 3.13. Let $W^\bullet = (W, D_W)$ be a *smooth* pointed stable curve of type (g_W, n_W) over an algebraically closed field of characteristic $p > 0$. Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. We shall say that $W^\bullet$ is (m, n_W)-*ordinary* if for each $\omega \in \mathrm{Rev}_Q^{\mathrm{adm}}(W^\bullet)$ such that $\omega \neq 0$, we have that $\gamma_{(\omega,Q)}$ attains the maximum

$$\gamma_{W^\bullet}^{\max} = \begin{cases} g_W - 1, & \text{if } n_W = 0, \\ g_W + n_W - 2, & \text{if } n_W \neq 0, \end{cases}$$

where Q is an arbitrary divisor on W contained in $(\mathbb{Z}/m\mathbb{Z})^{\sim}[D_W]^0$ satisfying the following conditions: (i) $Q = 0$ if $n_W = 0$, and $\deg(Q) = (n_W - 1)m$ if $n_W \neq 0$. (ii) There exists a positive natural number $d \in \mathbb{N}$ such that $p^d = 1$ in $(\mathbb{Z}/m\mathbb{Z})^{\times}$. (iii) Write Q' for the divisor $m'Q \in (\mathbb{Z}/(p^d-1)\mathbb{Z})^{\sim}[D_W]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/(p^d-1)\mathbb{Z}$ of order m, where $m' \stackrel{\text{def}}{=} (p^d-1)/m$. (iv) $\deg((Q')^{(i)}) \geq \deg(Q')$, $i \in \{0, 1, \ldots, d-1\}$.

Note that, if $n_W = 0$, then the definition of (m, n_W)-ordinary coincides with the definition of m-ordinary defined by Nakajima (cf. [N, §4]).

Corollary 3.14. *Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$ and M_{g_X,n_X} the coarse moduli space of the moduli stack $\mathcal{M}_{g_X,n_X} \stackrel{\text{def}}{=} \mathcal{M}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$.*

(i) Let m be a positive natural number prime to p. We denote by

$$U_{(m,n_X)} \subseteq M_{g_X,n_X}$$

the subset of $\overline{M}_{g_X,n_X}$ consisting of all points which correspond to (m, n_X)-ordinary smooth pointed stable curves. Then

$$U_{(m,n_X)}$$

is a non-empty open subset of M_{g_X,n_X}.

(ii) Suppose that $n_X \leq 1$. We have that

$$M^{\text{cl}}_{g_X,n_X} \cap \Big(\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} U_{(m,n_X)} \Big) = \emptyset,$$

where $M^{\text{cl}}_{g_X,n_X}$ denotes the set of closed points of M_{g_X,n_X}.

(iii) Let $q \in M_{g_X,n_X}$ be an arbitrary point. We denote by Π_q the admissible fundamental group of a smooth pointed stable curve corresponding to a geometric point over q. Note that the isomorphism class of Π_q as a profinite group does not depend on the choices of geometric points over q. Suppose that $n_X \leq 1$. Let $U \subseteq M_{g_X,n_X}$ be an arbitrary non-empty open subset. Then there exist closed points $q_1, q_2 \in U \cap M^{\text{cl}}_{g_X,n_X}$ such that $\Pi_{q_1} \not\cong \Pi_{q_2}$.

Proof. (i) By applying similar arguments to the arguments given in the proof of [N, Theorem 2], (i) follows from Theorem 3.12.

(ii) We maintain the notation introduced in Section 2.2. Suppose that $k = \overline{\mathbb{F}}_p$, and that $X^{\bullet}$ is a smooth pointed stable curve of type (g_X, n_X) over k. To verify (ii), we only need to prove that, if $n_X \leq 1$, $X^{\bullet}$ is not (m, n_X)-ordinary for some positive natural number $m \in \mathbb{N}$ prime to p.

Since $n_X \leq 1$, we have that $(\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0 = \{0\}$. We denote by Θ_X the Raynaud-Tamagawa divisor associated to $\mathcal{B}^1_0$ and by Θ' an arbitrary irreducible component of Θ_X. Write J^1_X for the pulling-back of the Jacobian J_X of X by the Frobenius F_k. If $X^{\bullet}$ is (m, n_X)-ordinary for every positive natural number m prime to p, then a property of Raynaud-Tamagawa theta divisors concerning new-ordinary coverings (e.g. [PS, §1.2 and §1.3]) implies that

$$J^1_X\{p'\} \cap \Theta_X(k) \subseteq \{0_{J^1_X}\},$$

where $0_{J^1_X}$ denotes the zero point of J^1_X, and $J^1_X\{p'\}$ denotes the set of prime-to-p torsion points of $J^1_X(k)$. Since $\dim(\Theta') > 0$, we have

$$\Theta'\{p'\} \stackrel{\text{def}}{=} J^1_X\{p'\} \cap \Theta'(k) \subseteq J^1_X\{p'\} \cap \Theta_X(k)$$

is not dense in Θ'.

On the other hand, since Θ' is defined over $k = \overline{\mathbb{F}}_p$, by applying a result of Anderson-Indik (cf. [T3, §5]), we have that Θ' is a subvariety of a translate of a proper sub-abelian variety of J^1_X. But this contradicts a result of Raynaud (cf. [R2, Proposition 1.2.1]) which says that there exists an irreducible component Θ' of Θ_X such that Θ' is not contained in a translate of a proper sub-abelian variety of J^1_X. This completes the proof of (ii).

(iii) Let q be an arbitrary closed point of U and q^{gen} the generic point of M_{g_X,n_X}. Suppose that (iii) does not hold. Then there exists a closed point $q \in U^{\text{cl}}$ such that $\Pi_{q^{\text{gen}}} \cong \Pi_q$. Then there exist a discrete valuation ring R and a morphism $c_R : \operatorname{Spec} R \to M_{g_X,n_X}$ such that $c_R(\eta_R) = q^{\text{gen}}$ and $c_R(s_R) = q$, where η_R is the generic point of $\operatorname{Spec} R$ and s_R is the closed point of $\operatorname{Spec} R$. By replacing R by a finite extension of R, we have a smooth pointed stable curve

$$\mathcal{X}^\bullet$$

of type (g_X, n_X) over $\operatorname{Spec} R$ determined by c_R. Moreover, we obtain a specialization map

$$sp_R : \Pi_{q^{\text{gen}}} \twoheadrightarrow \Pi_q.$$

Since $\Pi_{q^{\text{gen}}}$ and Π_q are topologically finitely generated, sp_R is an isomorphism.

On the other hand, (ii) implies that there exist a positive integer m prime to p and a connected Galois étale covering $\mathcal{Y} \to \mathcal{X}$ over R with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that the geometric generic fiber of $\mathcal{Y}$ is ordinary and the geometric special fiber of $\mathcal{Y}$ is not ordinary. This means that sp_R is not an isomorphism. We complete the proof of (iii). □

Remark 3.14.1. If $n_X \leq 1$, then Corollary 3.14 (i) was proved by Nakajima (cf. Theorem 2). Then Corollary 3.14 (i) is a generalized version of [N, Theorem 2] to the case of admissible coverings of smooth pointed stable curves. Moreover, Nakajima asked whether or not

$$\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} U_{(m,n_X)}$$

is a non-empty open subset of M_{g_X,n_X} (cf. [N, §4 Remark]). Then Corollary 3.14 (ii) gives an answer of Nakajima's question. Furthermore, we may ask the following question:

Does

$$M^{\text{cl}}_{g_X,n_X} \cap \Big(\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} U_{(m,n_X)}\Big) = \emptyset$$

hold for each non-negative integer n_X?

Remark 3.14.2. Corollary 3.14 (iii) gives an answer of a question of D. Harbater (cf. [H, 4.2]) which was first solved by Pop and Saïdi (cf. [PS, Corollary]). In [PS], Pop and Saïdi proved a result which says that specialization homomorphisms of geometric étale fundamental groups of smooth projective curves in positive characteristic is not an isomorphism under certain assumptions. Then together with a result of C-L. Chai and F. Oort, and a result of J-P. Serre, they obtained Corollary 3.14 (iii).

4 Maximum generalized Hasse-Witt invariants of cyclic admissible coverings of pointed stable curves

In the present section, we discuss the maximum generalized Hasse-Witt invariants of cyclic admissible coverings of an arbitrary pointed stable curve. Let us return to the case where $X^\bullet$ is an arbitrary pointed stable curve over k, and we maintain the notation introduced in Section 2.2. First, We have the following lemma.

Lemma 4.1. *(i) Let $Q \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree* $\deg(Q) = s(Q)n$ *such that* $\mathrm{ord}_x(Q) \leq n$ *for each* $x \in \mathrm{Supp}(Q)$*, $\mathcal{L}_Q$ a line bundle on X such that $\mathcal{L}_Q^{\otimes n} \cong \mathcal{O}_X(-Q)$, and $\mathcal{L}_{Q,t}$ the pulling-back of $\mathcal{L}_Q$ by the natural morphism $X_t \to X$. Suppose that $X^\bullet$ is smooth over k, and that*

$$\#\{x \in X \mid \mathrm{ord}_x(Q) = n\} \geq s(Q) - 1.$$

Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_Q^t \otimes \mathcal{L}_{Q,t}$ exists.

(ii) Let $i \in \{1, 2\}$, $t_i \in \mathbb{Z}_{>0}$, and $n_i \overset{\mathrm{def}}{=} p^{t_i} - 1$. Let $Q_i \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree $\deg(Q_i) = s(Q_i)n_i$ *such that* $\mathrm{ord}_x(Q_i) \leq n_i$ *for each* $x \in \mathrm{Supp}(Q_i)$*, $\mathcal{L}_{Q_i}$ a line bundle on X such that $\mathcal{L}_{Q_i}^{\otimes n_i} \cong \mathcal{O}_X(-Q_i)$, and $\mathcal{L}_{Q_i,t_i}$ the pulling-back of $\mathcal{L}_{Q_i}$ by the natural morphism $X_{t_i} \to X$. Suppose that $s \overset{\mathrm{def}}{=} s(Q_1) = s(Q_2)$. Let $t \overset{\mathrm{def}}{=} t_1 + t_2$, $n \overset{\mathrm{def}}{=} n_1 + p^{t_1} n_2$,*

$$Q \overset{\mathrm{def}}{=} Q_1 + p^{t_1} Q_2 \in \mathbb{Z}[D_X]$$

an effective divisor on X of degree $\deg(Q) = sn$*, $\mathcal{L}_Q$ a line bundle on X such that $\mathcal{L}_Q^{\otimes n} \cong \mathcal{O}_X(-Q)$, and $\mathcal{L}_{Q,t}$ the pulling-back of $\mathcal{L}_Q$ by the natural morphism $X_t \to X$. Suppose that $X^\bullet$ is smooth over k. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_Q^t \otimes \mathcal{L}_{Q,t}$ exists if and only if the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_{Q_i}^{t_i} \otimes \mathcal{L}_{Q_i,t_i}$ exists for each $i \in \{1, 2\}$.*

Proof. See [T2, Corollary 2.6] for (i) and [T2, Lemma 2.12 (ii) and Corollary 2.13] for (ii). □

Lemma 4.1 implies the following proposition.

Proposition 4.2. *Suppose that $X^\bullet$ is irreducible. Then there exist a positive natural number $n \overset{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ on X, and an element $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$ such that $\alpha \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\alpha,D)}$ attains the maximum*

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. We write $\widetilde{X}$ for the smooth compactification of $U_X \overset{\mathrm{def}}{=} X \setminus X^{\mathrm{sing}}$ and define

$$\widetilde{X}^\bullet = (\widetilde{X}, D_{\widetilde{X}} \overset{\mathrm{def}}{=} D_X \cup (\widetilde{X} \setminus U_X))$$

to be a pointed stable curve of type $(g_{\widetilde{X}}, n_{\widetilde{X}})$ over k. Note that $g_{\widetilde{X}} = g_X - \#X^{\mathrm{sing}}$ and $n_{\widetilde{X}} = n_X + 2\#X^{\mathrm{sing}}$. Moreover, we have $(\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{\widetilde{X}}]^0$. By applying Proposition 3.1, to verify the proposition, it is sufficient to prove that there exist a positive natural number $n \stackrel{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective (Weil) divisor $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ of degree $\deg(\widetilde{D}) = (n_X - 1)n$ on $\widetilde{X}$, and an element $\widetilde{\alpha} \in \mathrm{Rev}^{\mathrm{adm}}_{\widetilde{D}}(\widetilde{X}^\bullet)$ such that $\widetilde{\alpha} \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\widetilde{\alpha}, \widetilde{D})}$ attains the maximum

$$\gamma^{\max}_{\widetilde{X}^\bullet} = \begin{cases} g_{\widetilde{X}} - 1, & \text{if } n_X = 0, \\ g_{\widetilde{X}} + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Suppose that $n_X \leq 2$. Then $s(\widetilde{D}) \leq 1$ for each $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. Thus, the proposition follows immediately from Proposition 2.8 and Theorem 2.9.

Suppose that $n_X \geq 3$. Let $D_X \stackrel{\mathrm{def}}{=} \{x_1, \dots, x_{n_X}\}$, $n_i \stackrel{\mathrm{def}}{=} p^{t_i} - 1$ for each $i \in \{1, \dots, n_X - 1\}$ such that $n_i > \max\{C(g_X) + 1, \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet}))\}$, and $0 < a_{i,1}, a_{i,2} < n_i$ for each $i \in \{1, \dots, n_X - 1\}$ such that $a_{i,1} + a_{i,2} = n_i$. We put

$$D_i \stackrel{\mathrm{def}}{=} a_{i,1}x_i + a_{i,2}x_{i+1} + \sum_{x \in D_X \setminus \{x_i, x_{i+1}\}} n_i x, \ i \in \{1, \dots, n_X - 1\},$$

which is an effective divisor on $\widetilde{X}$ with degree $\deg(D_i) = (n_X - 1)n_i$. Moreover, we put

$$\widetilde{D} \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n_X - 1} p^{\sum_{j=0}^{i-1} t_j} D_i$$

and

$$n \stackrel{\mathrm{def}}{=} p^{\sum_{i=0}^{n_X - 1} t_j} - 1 = \sum_{i=1}^{n_X - 1} p^{\sum_{j=0}^{i-1} t_j}(p^{t_i} - 1),$$

where $t_0 \stackrel{\mathrm{def}}{=} 0$. We see immediately that $\deg(\widetilde{D}) = (n_X - 1)n$, and that $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. Let $\mathcal{L}_{\widetilde{D}}$ a line bundle on $\widetilde{X}$ such that $\mathcal{L}_{\widetilde{D}}^{\otimes n} \cong \mathcal{O}_X(-\widetilde{D})$, and $\mathcal{L}_{\widetilde{D},t}$ the pulling-back of $\mathcal{L}_{\widetilde{D}}$ by the natural morphism $\widetilde{X}_t \to \widetilde{X}$. Then Lemma 4.1 implies the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^t_{\widetilde{D}} \otimes \mathcal{L}_{\widetilde{D},t}$ exists. Moreover, Proposition 2.8 implies that there exists a line bundle $\widetilde{\mathcal{I}}$ of degree 0 on $\widetilde{X}$ such that $[\widetilde{\mathcal{I}}] \neq [\mathcal{O}_{\widetilde{X}}]$, that $[\widetilde{\mathcal{I}}^{\otimes n}] = [\mathcal{O}_{\widetilde{X}}]$, and that

$$\gamma_{([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D})} = \begin{cases} g_{\widetilde{X}} - 1, & \text{if } n_X = 0, \\ g_{\widetilde{X}} + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Let $\widetilde{\alpha} \in \mathrm{Rev}^{\mathrm{adm}}_{\widetilde{D}}(\widetilde{X}^\bullet)$ be the element corresponding to the pair $([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D}) \in \widetilde{\mathscr{P}}_{\widetilde{X}^\bullet, n}$. This completes the proof of the proposition. □

Remark 4.2.1. We maintain the notation introduced in the proof of Proposition 4.2. By choosing a suitable $a_{i,1}$ (or $a_{i,2}$) for each $i \in \{1, \dots, n_X - 1\}$, we may obtain that the Galois multi-admissible covering induced by α is connected.

In the remainder of this section, we will generalize Proposition 4.2 to the case where $X^\bullet$ is an arbitrary pointed stable curve over k (cf. Theorem 4.6 below). Let us explain the idea. Let $X^\bullet$ be an arbitrary pointed stable curve over k. For simplicity, we assume that each irreducible component of $X^\bullet$ is non-singular. We maintain the notation introduced in the statement of Proposition 3.1. To verify the generalized version of Proposition 4.2, by applying Proposition 3.1, it is sufficient to construct a prime-to-p cyclic Galois multi-admissible covering $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ for every $v \in v(\Gamma_{X^\bullet})$ such that the following conditions hold: (i) $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})}$ satisfies the conditions mentioned in the statement of Proposition 3.1 (i.e., the Raynaud-Tamagawa theta divisor exists); (ii) $\{\widetilde{f}_v^\bullet\}_{v \in v(\Gamma_{X^\bullet})}$ can be glued together; then we obtain a prime-to-p cyclic Galois multi-admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$; (iii) the divisor $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ associated to $f^\bullet$ such that $s(D) = n_X - 1$ if $n_X \neq 0$.

Suppose that D_X is contained only in a unique irreducible component X_v of X. Then similar arguments to the arguments given in the proof of Proposition 4.2, we may construct a Galois multi-admissible covering $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})}$ satisfies the conditions mentioned in the statement of Proposition 3.1, and that $D_{\widetilde{\alpha}_v} \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $s(D_{\widetilde{\alpha}_v}) = n_X - 1$ if $n_X \neq 0$. On the other hand, for each $w \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, we have a trivial Galois multi-admissible covering

$$\widetilde{f}_w^\bullet : \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} \widetilde{X}_{w,i}^\bullet \to \widetilde{X}_w^\bullet$$

with Galois group $\mathbb{Z}/n\mathbb{Z}$, where $\widetilde{X}_{w,i}^\bullet$, $i \in \mathbb{Z}/n\mathbb{Z}$, is a copy of $\widetilde{X}_w^\bullet$ with a natural action of $\mathbb{Z}/n\mathbb{Z}$. Then by gluing $\{\widetilde{f}^\bullet\}_{v \in v(\Gamma_{X^\bullet})}$, we obtain the desired Galois multi-admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Suppose that there exist $v_1, v_2 \in v(\Gamma_{X^\bullet})$ distinct from each other such that $X_{v_1} \cap D_X \neq \emptyset$ and $X_{v_2} \cap D_X \neq \emptyset$. The constructions of the desired Galois multi-admissible coverings of $X^\bullet$ are very difficult in general. The main difficulty is that we cannot determine the ramifications over nodes of a Galois admissible covering in a unique way when the ramifications over D_X are fixed. For example, Remark 3.10.1 shows that, when $D_X = 0$, there exists a prime-to-p cyclic Galois admissible covering of $X^\bullet$ ramified over every node whose generalized Hasse-Witt invariants cannot attain the maximum.

To overcome this difficulty, we observe that, when $\Gamma_{X^\bullet}$ is a *tree*, the ramifications over nodes of a Galois admissible covering can be determined uniquely if the ramifications over D_X are fixed. Then we may construct the desired Galois multi-admissible covering $f^\bullet$. In order to convince the reader to follow the constructions given in the proof of Lemma 4.5 below, we give the following example for constructing effective divisors $D_{\widetilde{\alpha}_v} \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_{\widetilde{X}_v}]^0$, $v \in v(\Gamma_{X^\bullet})$, and $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$.

Example 4.3. Suppose that $X^\bullet$ is a pointed stable curve of type $(g_X, 3)$ over k, that X has two irreducible components X_1 and X_2, that $X_1 \cap X_2 \stackrel{\text{def}}{=} \{x^-\}$, that $D_X \cap X_1 \stackrel{\text{def}}{=} \{x_{1,1}, x_{1,2}\}$, and that $D_X \cap X_2 \stackrel{\text{def}}{=} \{x_{2,1}\}$. Let $n_0 \stackrel{\text{def}}{=} p^{t_0} - 1 >> 0$, $D_{X_1} \stackrel{\text{def}}{=} \{x_{1,1}, x_{1,2}, x^-\}$, and $D_{X_2} \stackrel{\text{def}}{=} \{x_{2,1}, x^-\}$. We put

$$Q_{1,1} \stackrel{\text{def}}{=} a_1 x_{1,1} + a_2 x_{1,2} + n_0 x^- \in \text{Div}(X_1),$$

$$Q_{1,2} \stackrel{\text{def}}{=} n_0 x_{2,1} \in \mathrm{Div}(X_2),$$

$$Q_1 \stackrel{\text{def}}{=} a_1 x_{1,1} + a_2 x_{1,2} + n_0 x_{2,1} \in \mathrm{Div}(X),$$

where $0 < a_1, a_2 < n_0$ such that $a_1 + a_2 = n_0$. Moreover, we put

$$Q_{2,1} \stackrel{\text{def}}{=} n_0 x_{1,1} + b_1 x_{1,2} + b_2 x^- \in \mathrm{Div}(X_1),$$

$$Q_{2,2} \stackrel{\text{def}}{=} b_1 x^- + b_2 x_{2,1} \in \mathrm{Div}(X_2),$$

$$Q_2 \stackrel{\text{def}}{=} n_0 x_{1,1} + b_1 x_{1,2} + b_2 x_{2,1} \in \mathrm{Div}(X),$$

where $0 < b_1, b_2 < n_0$ such that $b_1 + b_2 = n_0$. Write v_1 and v_2 for the vertices of $v(\Gamma_{X^\bullet})$ corresponding to X_1 and X_2, respectively. We define the following sets of effective divisors

$$\mathrm{Div}_{v_i}^{\text{irr-mp}} \stackrel{\text{def}}{=} \{Q_{1,i}\},\ \mathrm{Div}_{v_i}^{\text{irr-nd}} \stackrel{\text{def}}{=} \{Q_{2,i}\},$$

$$\mathrm{Div}_X^{\text{irr}} \stackrel{\text{def}}{=} \bigsqcup_{i=1,2} (\mathrm{Div}_{v_i}^{\text{irr-mp}} \sqcup \mathrm{Div}_{v_i}^{\text{irr-nd}}),\ \mathrm{Div}_X \stackrel{\text{def}}{=} \{Q_1, Q_2\},$$

where $\sqcup$ means disjoint union. Let $n \stackrel{\text{def}}{=} p^{2t_0} - 1$. Then we obtain

$$D_1 \stackrel{\text{def}}{=} Q_{1,1} + p^{t_0} Q_{2,1} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_1}]^0,$$

$$D_2 \stackrel{\text{def}}{=} Q_{1,2} + p^{t_0} Q_{2,2} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_2}]^0,$$

$$D \stackrel{\text{def}}{=} Q_1 + p^{t_0} Q_2 \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0.$$

Then we have

$$D|_{\{x_{1,1}, x_{1,2}\}} = D_{\widetilde{\alpha}_{v_1}}|_{\{x_{1,1}, x_{1,2}\}} = D_1|_{\{x_{1,1}, x_{1,2}\}}.$$

Moreover, since $\Gamma_{X^\bullet}$ is a tree, we have that

$$D_{\widetilde{\alpha}_{v_1}}|_{x^-} = [\sum_{x \in D_X \setminus \{x_{1,1}, x_{1,2}\}} \mathrm{ord}_x(D)] x^- = \mathrm{ord}_{x_{2,1}}(D) x^- = D_1|_{x^-},$$

where $[(-)]$ denotes the image of $(-)$ in $\mathbb{Z}/n\mathbb{Z}$. Thus, we obtain $D_{\widetilde{\alpha}_{v_1}} = D_1$. By applying similar arguments to the arguments given in the proof above, we obtain $D_{\widetilde{\alpha}_{v_2}} = D_2$. Then D, $D_{\widetilde{\alpha}_{v_1}}$, and $D_{\widetilde{\alpha}_{v_2}}$ satisfy the conditions mentioned in the statement of Proposition 3.1.

In the general case (i.e., $\Gamma_{X^\bullet}$ is not a tree), we take a *minimal quasi-tree* $\Gamma \stackrel{\text{def}}{=} \Gamma_{D_X}$ *associated to* D_X, and construct the desired Galois multi-admissible covering for $X_\Gamma^\bullet$ (this is the motivation of the definition of minimal quasi-trees). Moreover, we have a trivial Galois multi-admissible covering for each irreducible component corresponding to $v \in v(\Gamma_{X^\bullet}) \setminus v(\Gamma)$. Then by gluing the Galois multi-admissible coverings together, we obtain the desired Galois multi-admissible covering of $X^\bullet$.

Definition 4.4. Let $\mathbb{G}$ be a connected semi-graph and $v \in v(\mathbb{G})$ an arbitrary vertex. Moreover, we suppose that $\mathbb{G}$ is a *tree*. For each $v' \in v(\mathbb{G})$, there exists a path $p_{v,v'}$ connecting v and v' in $\mathbb{G}$. We define

$$\mathrm{leng}(p_{v,v'}) \overset{\mathrm{def}}{=} \#\{p_{v,v'} \cap v(\mathbb{G})\} - 1$$

to be the *length* of the path $p_{v,v'}$. Moreover, since $\mathbb{G}$ is a tree, there exists a unique path connecting v and v' whose length is equal to $\min\{\mathrm{leng}(p_{v,v'})\}_{p_{v,v'}}$. We shall write

$$p(\mathbb{G}, v, v')$$

for this unique path connecting v and v' in $\mathbb{G}$, and shall say that $p(\mathbb{G}, v, v')$ is the *minimal path* connecting v and v' in $\mathbb{G}$.

We have the following key lemma of the present section.

Lemma 4.5. *Let $\Gamma \overset{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X,*

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma})$$

the pointed stable curve of type $(g_{X_\Gamma}, n_{X_\Gamma})$ associated to Γ, and $\Pi_{X_\Gamma^\bullet}$ the admissible fundamental group of $X_\Gamma^\bullet$. Suppose that $n_X \geq 2$. Then there exist a positive natural number $n \overset{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_\Gamma}]^0$ on X_Γ, and an element $\alpha_\Gamma \in \mathrm{Rev}^{\mathrm{adm}}_{D_\Gamma}(X_\Gamma^\bullet)$ such that $\alpha_\Gamma \neq 0$, and that the generalized Hasse-Witt invariant

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + n_X - 2.$$

Proof. Since Γ is a minimal quasi-tree associated to D_X, we obtain that $\Gamma' \overset{\mathrm{def}}{=} \Gamma \setminus e^{\mathrm{lp}}(\Gamma)$ is a tree. Then we have $v(\Gamma) = v(\Gamma')$. Note that $D_X \subseteq D_{X_\Gamma}$. Let $v \in v(\Gamma)$ be an arbitrary vertex and $n_0 = p^{t_0} - 1 \in \mathbb{N}$ a positive natural number such that

$$n_0 > \max\{C(g_X) + 1, \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet}))\}.$$

Step 1: Let $v \in v(\Gamma)$. We will construct a family of effective divisors $\mathrm{Div}_v^{\text{irr-mp}}$ on the irreducible component $X_v^\bullet$ associated to the set of marked points $D_X \cap X_v$, and construct a family of effective divisors $\mathrm{Div}_v^{\mathrm{mp}}$ on X_Γ, where "mp" means "marked point" (i.e., the Galois multi-admissible coverings corresponding to the divisors are ramified over marked points).

We put

$$D'_v \overset{\mathrm{def}}{=} D_X \cap X_v,\ m_v \overset{\mathrm{def}}{=} \#D'_v,\ \text{and}\ D'_v \overset{\mathrm{def}}{=} \{x_{v,1}, \dots, x_{v,m_v}\}\ \text{if}\ m_v \neq 0.$$

Moreover, we put

$$D_v \overset{\mathrm{def}}{=} D'_v \cup (X_v \cap (\overline{X_\Gamma \setminus X_v})),$$

where $\overline{X_\Gamma \setminus X_v}$ denotes the topological closure of $X_\Gamma \setminus X_v$ in X_Γ. Note that since $n_X > 0$, we have $\#D_v > 0$. Let $w \in v(\Gamma)$ be an arbitrary vertex distinct from v. Since Γ' is a tree, there exists a unique node

$$x^-_{v,w} \in D_w$$

such that the closed edge of Γ' corresponding to $x^{-}_{v,w}$ is contained in the minimal path $p(\Gamma', v, w)$ connecting v and w in Γ'. On the other hand, we define a set of nodes to be

$$\mathrm{Node}^{+}_{v,w} \overset{\mathrm{def}}{=} \{X_w \cap X_{w'},\ w' \in v(\Gamma) \mid \mathrm{leng}(p(\Gamma', v, w')) = \mathrm{leng}(p(\Gamma', v, w)) + 1\}.$$

Note that $\mathrm{Node}^{+}_{v,w}$ may possibly be an empty set, and that $D_w = \{x^{-}_{v,w}\} \cup \mathrm{Node}^{+}_{v,w} \cup D'_w$.

We define two sets of effective divisors

$$\mathrm{Div}^{\text{irr-mp}}_v,\ \mathrm{Div}^{\text{mp}}_v$$

associated to v on X_v and X_Γ, as follows. Let $i \in \{1, \ldots, m_v - 1\}$ and $0 < a_{v,i,1}, a_{v,i,2} < n_0$ such that $a_{v,i,1} + a_{v,i,2} = n_0$. Suppose that $m_v \leq 1$. Then we put

$$\mathrm{Div}^{\text{irr-mp}}_v \overset{\mathrm{def}}{=} \emptyset,\ \mathrm{Div}^{\text{mp}}_v \overset{\mathrm{def}}{=} \emptyset.$$

Suppose that $m_v \geq 2$. We define

$$Q_{v,v,i} \overset{\mathrm{def}}{=} a_{v,i,1}x_{v,i} + a_{v,i,2}x_{v,i+1} + \sum_{x' \in D'_v \setminus \{x_{v,i}, x_{v,i+1}\}} n_0 x' + \sum_{x \in D_v \setminus D'_v} n_0 x$$

to be an effective divisor on X_v whose support is D_v, and whose degree is equal to $(\#D_v - 1)n_0$. We define

$$Q_{v,w,i} \overset{\mathrm{def}}{=} \sum_{x \in D_w \setminus \{x^{-}_{v,w}\}} n_0 x,\ w \in v(\Gamma) \setminus \{v\},$$

to be an effective divisor on X_w whose support is $D_w \setminus \{x^{-}_{v,w}\}$, and whose degree is equal to $(\#D_w - 1)n_0$. Moreover, we define

$$Q^v_i \overset{\mathrm{def}}{=} a_{v,i,1}x_{v,i} + a_{v,i,2}x_{v,i+1} + \sum_{x \in D_X \setminus \{x_{v,i}, x_{v,i+1}\}} n_0 x,$$

to be an effective divisor on X_Γ whose support is D_X, and whose degree is $(n_X - 1)n_0$. Note that, for each $i, i' \in \{1, \ldots, m_v - 1\}$, we have $Q_{v,w,i} = Q_{v,w,i'}$. Then we put

$$\mathrm{Div}^{\text{irr-mp}}_{v,i} \overset{\mathrm{def}}{=} \bigsqcup_{u \in v(\Gamma)} \{Q_{v,u,i}\},\ \mathrm{Div}^{\text{irr-mp}}_v \overset{\mathrm{def}}{=} \bigsqcup_{i=1}^{m_v - 1} \mathrm{Div}^{\text{irr-mp}}_{v,i},\ \mathrm{Div}^{\text{mp}}_v \overset{\mathrm{def}}{=} \bigsqcup_{i=1}^{m_v - 1} \{Q^v_i\},$$

where $\sqcup$ means disjoint union.

Step 2: Let $v \in v(\Gamma)$. We will construct a family of effective divisors $\mathrm{Div}^{\text{irr-nd}}_v$ on the irreducible component $X^{\bullet}_v$ associated to the set of nodes $D_v \setminus D'_v = X_v \cap (\overline{X_\Gamma \setminus X_v})$, and construct a family of effective divisors $\mathrm{Div}^{\text{nd}}_v$ on X_Γ, where "nd" means "node" (i.e., the Galois multi-admissible coverings corresponding to the divisors are ramified over nodes).

We define two families of effective divisors

$$\mathrm{Div}^{\text{irr-nd}}_v,\ \mathrm{Div}^{\text{nd}}_v$$

associated to v on X_v and X_Γ, respectively, as follows. Let $z \in D_X \setminus D'_v$ and $0 < b_{v,z,1}, b_{v,z,2} < n_0$ such that $b_{v,z,1} + b_{v,z,2} = n_0$. Suppose that $m_v = 0$. Then we put

$$\mathrm{Div}_v^{\text{irr-nd}} \overset{\text{def}}{=} \emptyset,\ \mathrm{Div}_v^{\text{nd}} \overset{\text{def}}{=} \emptyset.$$

Suppose that $m_v \neq 0$. Let w_z be the vertex such that the irreducible component X_{w_z} corresponding to w_z contains z (i.e., $z \in D'_{w_z}$), $p(\Gamma', v, w_z)$ the minimal path connecting v and w_z in Γ', and $w \in v(\Gamma)$ an arbitrary vertex distinct from w_z such that $w \subseteq p(\Gamma', v, w_z)$. Since Γ' is a tree, we have that $\#(\mathrm{Node}^+_{v,w} \cap p(\Gamma', v, w_z)) = 1$. We put

$$x^+_{v,w,w_z} \overset{\text{def}}{=} \mathrm{Node}^+_{v,w} \cap p(\Gamma', v, w_z) \in D_w.$$

We define

$$Q_{v,v,z} \overset{\text{def}}{=} b_{v,z,1} x_{v,m_v} + b_{v,z,2} x^+_{v,v,w_z} + \sum_{x \in D_v \setminus \{x_{v,m_v}, x^+_{v,v,w_z}\}} n_0 x$$

and

$$Q_{v,w_z,z} \overset{\text{def}}{=} b_{v,z,1} x^-_{v,w_z} + b_{v,z,2} z + \sum_{x \in D_{w_z} \setminus \{x_{v,w_z}, z\}} n_0 x$$

to be effective divisors on X_v and X_{w_z} whose supports are D_v and D_{w_z}, and whose degrees are equal to $(\#D_v - 1)n_0$ and $(\#D_{w_z} - 1)n_0$, respectively. Let $w \in v(\Gamma) \setminus \{v, w_z\}$ be an arbitrary vertex such that $w \subseteq p(\Gamma', v, w_z)$. Then we define

$$Q_{v,w,z} \overset{\text{def}}{=} b_{v,z,1} x^-_{v,w} + b_{v,z,2} x^+_{v,w,w_z} + \sum_{x \in D_w \setminus \{x^-_{v,w}, x^+_{v,w,w_z}\}} n_0 x$$

to be an effective divisor on X_w whose support is D_w, and whose degree is equal to $(\#D_w - 1)n_0$. Let $w' \in v(\Gamma)$ be an arbitrary vertex such that $w' \not\subseteq p(\Gamma', v, w_z)$. Then we define

$$Q_{v,w',z} \overset{\text{def}}{=} \sum_{x \in D_{w'} \setminus \{x^-_{v,w'}\}} n_0 x$$

to be an effective divisor on $X_{w'}$ whose support is $D_{w'} \setminus \{x^-_{v,w'}\}$, and whose degree is equal to $(\#D_{w'} - 1)n_0$. Note that, if $w' \not\subseteq p(\Gamma', v, w_z) \cup p(\Gamma', v, w_{z'})$ for $z, z' \in D_X \setminus D'_v$, we have $Q_{v,w',z} = Q_{v,w',z'}$. Moreover, we define

$$Q^v_z \overset{\text{def}}{=} b_{v,z,1} x_{v,m_v} + b_{v,z,2} z + \sum_{x \in D_X \setminus \{x_{v,m_v}, z\}} n_0 x$$

to be an effective divisor on X_Γ whose support is D_X, and whose degree is equal to $(n_X - 1)n_0$. Then we put

$$\mathrm{Div}^{\text{irr-nd}}_{v,z} \overset{\text{def}}{=} \bigsqcup_{u \in v(\Gamma)} \{Q_{v,u,z}\},\ \mathrm{Div}^{\text{irr-nd}}_v \overset{\text{def}}{=} \bigsqcup_{z \in D_X \setminus D'_v} \mathrm{Div}^{\text{irr-nd}}_{v,z},\ \mathrm{Div}^{\text{nd}}_v \overset{\text{def}}{=} \bigsqcup_{z \in D_X \setminus D'_v} \{Q^v_z\}.$$

Step 3: We will construct an effective divisor $P_v \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_v]^0$, $v \in v(\Gamma)$, and $P_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_\Gamma}]^0$ for some n such that the conditions mentioned in Proposition 3.1 hold.

We put

$$\mathrm{Div}_X^{\mathrm{irr}} \stackrel{\mathrm{def}}{=} \bigsqcup_{v \in v(\Gamma)} (\mathrm{Div}_v^{\text{irr-mp}} \sqcup \mathrm{Div}_v^{\text{irr-nd}})$$

and

$$\mathrm{Div}_X \stackrel{\mathrm{def}}{=} \bigsqcup_{v \in v(\Gamma)} (\mathrm{Div}_v^{\mathrm{mp}} \sqcup \mathrm{Div}_v^{\mathrm{nd}}).$$

We denote by $\mathrm{Div}_X^{\mathrm{irr}}(X_u)$, $u \in v(\Gamma)$, the subset of $\mathrm{Div}_X^{\mathrm{irr}}$ whose elements are effective divisors on X_u. Note that $d \stackrel{\mathrm{def}}{=} \#\mathrm{Div}_X^{\mathrm{irr}}(X_{u_1}) = \#\mathrm{Div}_X^{\mathrm{irr}}(X_{u_2}) = \#\mathrm{Div}_X$ for each $u_1, u_2 \in v(\Gamma)$. Moreover, let

$$o_u : \{1, \dots, d\} \xrightarrow{\sim} \mathrm{Div}_X^{\mathrm{irr}}(X_u) \stackrel{\mathrm{def}}{=} \{P_{u,1} \stackrel{\mathrm{def}}{=} o_u(1), \dots, P_{u,d} \stackrel{\mathrm{def}}{=} o_u(d)\},\ u \in v(\Gamma),$$

be a bijective as sets such that, for each $u_1, u_2 \in v(\Gamma)$ and each $j \in \{1, \dots, d\}$, one of the following conditions is satisfied: (i) if $P_{u_1,j} \in \mathrm{Div}_{v,i}^{\text{irr-mp}}$ for some $v \in v(\Gamma)$ and some $i \in \{1, \dots, m_v - 1\}$, then $P_{u_2,j} \in \mathrm{Div}_{v,i}^{\text{irr-mp}}$; (ii) if $P_{u_1,j} \in \mathrm{Div}_{v,z}^{\text{irr-nd}}$ for some $v \in v(\Gamma)$ and some $z \in D_X \setminus D'_v$, then $P_{u_2,j} \in \mathrm{Div}_{v,z}^{\text{irr-nd}}$. Then, by the construction of Div_X, we obtain a bijection

$$o : \{1, \dots, d\} \xrightarrow{\sim} \mathrm{Div}_X \stackrel{\mathrm{def}}{=} \{P_1 \stackrel{\mathrm{def}}{=} o(1), \dots, P_d \stackrel{\mathrm{def}}{=} o(d)\}$$

induced by o_u, $u \in v(\Gamma)$.

Let $t \stackrel{\mathrm{def}}{=} dt_0$ and $n \stackrel{\mathrm{def}}{=} \sum_{j=1}^{d} p^{(j-1)t_0}(p^{t_0} - 1) = p^t - 1$. We define

$$P_u \stackrel{\mathrm{def}}{=} \sum_{j=1}^{d} p^{(j-1)t_0} P_{u,j} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_u]^0,\ u \in v(\Gamma),$$

and

$$P_\Gamma \stackrel{\mathrm{def}}{=} \sum_{j=1}^{d} p^{(j-1)t_0} P_j \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$$

to be effective divisors on X_u and X_Γ, respectively. We see that the support of P_u, $u \in v(\Gamma)$, is D_u, that the support of P_Γ is D_X, that $\deg(P_u) = (\#D_u - 1)n$, and that $\deg(P_\Gamma) = (n_X - 1)n$.

Let $u \in v(\Gamma)$ and $\widetilde{X}_u^\bullet$ the smooth pointed stable curve of type (g_u, n_u) over k defined in Section 2.1. Then P_u can be also regarded as an effective divisor on $\widetilde{X}_u$. By applying similar arguments to the arguments given in the proof of Proposition 4.2, there exists $\widetilde{\alpha}_u \in \mathrm{Rev}_{P_u}^{\mathrm{adm}}(\widetilde{X}_u^\bullet)$ such that

$$\gamma_{(\widetilde{\alpha}_u, P_u)} = g_u + \#D_u - 2.$$

We define

$$X_u^\bullet = (X_u, D_{X_u} \stackrel{\mathrm{def}}{=} D_u)$$

to be a pointed stable curve over k. Then Proposition 3.1 implies that the element $\alpha_u \in \mathrm{Rev}_{P_u}^{\mathrm{adm}}(X_u^\bullet)$ induced by $\widetilde{\alpha}_u$ such that $\gamma_{(\alpha_u, P_u)}$ attains the maximum $g_{X_u} + \#D_u - 2$, where g_{X_u} denotes the genus of X_u. Write

$$f_u^\bullet : Y_u^\bullet \to X_u^\bullet$$

for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_u. By gluing $\{Y_u^\bullet\}_{u\in v(\Gamma)}$ along $\{f_u^{-1}(D_u \setminus D'_u)\}_{u\in v(\Gamma)}$ in a way that is compatible with the gluing of $\{X_u^\bullet\}_{u\in v(\Gamma)}$ that gives rise to $X_\Gamma^\bullet$, we obtain a Galois multi-admissible covering

$$f_\Gamma^\bullet : Y_\Gamma^\bullet \to X_\Gamma^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Note that the construction of $f_\Gamma^\bullet$ implies that f_Γ is étale over $D_{X_\Gamma} \setminus D_X$. We denote by $\alpha_\Gamma \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X_\Gamma^\bullet}, \mathbb{Z}/n\mathbb{Z})$ an element induced by $f_\Gamma^\bullet$.

Let $u \in v(\Gamma)$ and $x \in D'_u$. Then the constructions above implies that $P_\Gamma|_x = P_u|_x$. Moreover, let $x \in D_u \setminus D'_u$, $e_x \in e^{\mathrm{cl}}(\Gamma)$ the closed edge corresponding x, and C_x the connected component of $\Gamma \setminus e_x$ to which e_x is abutted. Write M_x for the subset of marked points of $X_\Gamma^\bullet$ corresponding to the subset of open edges $C_x \cap e^{\mathrm{op}}(\Gamma)$. Then the constructions above imply that

$$P_\Gamma|_x = [\sum_{x'\in M_x} \mathrm{ord}_{x'}(P_\Gamma)]x = \mathrm{ord}_x(P_u)x.$$

This means that $D_{\alpha_\Gamma} = P_\Gamma$. Thus, we put $D_\Gamma \overset{\mathrm{def}}{=} P_\Gamma$. Moreover, Proposition 3.1 implies that

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + n_X - 2.$$

We complete the proof of the lemma. $\square$

Next, we prove the main result of the present paper.

Theorem 4.6. *There exist a positive natural number $n \overset{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ on X, and an element $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$, and that the generalized Hasse-Witt invariant $\gamma_{(\alpha,D)}$ attains the maximum*

$$\gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. Let $t \in \mathbb{N}$ be an arbitrary positive natural number and $n \overset{\mathrm{def}}{=} p^t - 1$ such that

$$n > \max\{C(g_X) + 1, \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet}))\}.$$

First, we suppose that $n_X \leq 1$. We denote by $v(\Gamma_{X^\bullet})^{>0} \subseteq v(\Gamma_{X^\bullet})$ the set of vertices such that $g_v > 0$ for each $v \in v(\Gamma_{X^\bullet})^{>0}$. Suppose that $v(\Gamma_{X^\bullet})^{>0} = \emptyset$. Then $n_X \leq 1$ implies that $\Gamma_{X^\bullet}$ is not a tree. This means that $\Pi^{\mathrm{top}}_{X^\bullet}$ is not trivial. Let $\alpha' : \Pi^{\mathrm{top,ab}}_{X^\bullet} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ be a surjection and

$$\alpha : \Pi^{\mathrm{ab}}_{X^\bullet} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$$

the composite morphism $\Pi^{\mathrm{ab}}_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top,ab}}_{X^\bullet} \overset{\alpha'}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}$. Then the theorem follows from Proposition 3.1.

Suppose that $v(\Gamma_{X^\bullet})^{>0} \neq \emptyset$. Let $v \in v(\Gamma_{X^\bullet})^{>0}$. Then Proposition 2.8 and Theorem 2.9 imply that there exists an element $\widetilde{\alpha}_v \in \mathrm{Rev}^{\mathrm{adm}}_0(\widetilde{X}_v^\bullet)$ such that $\widetilde{\alpha}_v : \Pi^{\mathrm{ab}}_{\widetilde{X}_v^\bullet} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is a surjective, and that

$$\gamma_{(\widetilde{\alpha}_v, 0)} = g_v - 1.$$

Write $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ for the connected Galois étale covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $\widetilde{\alpha}_v$. Let

$$\pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})$$

be the set of connected components of $\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v}$ and $C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})$, where $\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v}$ denotes the topological closure of $X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v$ in X. We define

$$C^\bullet = (C, D_C \stackrel{\text{def}}{=} (C \cap (\bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v)) \cup (D_X \cap C))$$

to be a pointed stable curve over k. Let

$$Y_C^\bullet \stackrel{\text{def}}{=} \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} C_i^\bullet,$$

where $C_i^\bullet$ is a copy of $C^\bullet$. Then we obtain a Galois multi-admissible covering

$$f_C^\bullet : Y_C^\bullet \to C^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $f_C^\bullet|_{C_i^\bullet}$ is an identity, and the Galois action is $j(C_i) = C_{i+j}$ for each $i, j \in \mathbb{Z}/n\mathbb{Z}$. By gluing

$$\{\widetilde{Y}_v^\bullet\}_{v \in v(\Gamma_{X^\bullet})^{>0}} \text{ and } \{Y_C^\bullet\}_{C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})}$$

along $\{D_{\widetilde{X}_v}\}_{v \in v(\Gamma_{X^\bullet})^{>0}}$ and $\{D_C\}_{C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})}$ in a way that is compatible with the gluing of $\{\widetilde{X}_v^\bullet\}_{v \in v(\Gamma_{X^\bullet})^{>0}} \cup \{C^\bullet\}_{C \in \pi_0(\overline{X \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})^{>0}} X_v})}$ that gives rise to $X^\bullet$, we obtain a Galois (étale) admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then theorem follows from Proposition 3.1.

Next, we suppose that $n_X \geq 2$. Let $\Gamma \stackrel{\text{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X, Γ^{im} the image of the natural morphism $\phi_\Gamma : \Gamma \to \Gamma_{X^\bullet}$, and

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma}),\ X_{\Gamma^{\text{im}}}^\bullet = (X_{\Gamma^{\text{im}}}, D_{X_{\Gamma^{\text{im}}}})$$

the pointed stable curves over k associated to Γ, Γ^{im}, respectively.

Lemma 4.5 implies that there exist a natural number $n \stackrel{\text{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \stackrel{\text{def}}{=} D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_\Gamma}]^0$ on X_Γ whose degree is $(n_X - 1)n$, and an element $\alpha_\Gamma \in \text{Rev}_D^{\text{adm}}(X_\Gamma^\bullet)$ such that

$$\gamma_{(\alpha_\Gamma, D)} = g_{X_\Gamma} + n_X - 2,$$

where g_{X_Γ} denotes the genus of X_Γ. We denote by

$$f_\Gamma^\bullet : Z_\Gamma^\bullet \to X_\Gamma^\bullet$$

the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_Γ. Note that f_Γ is étale over $D_{X_\Gamma} \setminus D_X$. By gluing $Z_\Gamma^\bullet$ along $f_\Gamma^{-1}(D_{X_\Gamma} \setminus (D_X \cup \{x_e\}_{e \in \phi_\Gamma^{-1}(e^{\mathrm{op}}(\Gamma^{\mathrm{im}}))}))$ that is compatible with the gluing of $X_\Gamma^\bullet$ that gives rise to $X_{\Gamma^{\mathrm{im}}}^\bullet$, we obtain a pointed stable curve $Z_{\Gamma^{\mathrm{im}}}^\bullet$ over k. Moreover, $f_\Gamma^\bullet$ induces a Galois multi-admissible covering

$$f_{\Gamma^{\mathrm{im}}}^\bullet : Z_{\Gamma^{\mathrm{im}}}^\bullet \to X_{\Gamma^{\mathrm{im}}}^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}$ for the admissible fundamental group of $X_{\Gamma^{\mathrm{im}}}^\bullet$ and $\alpha_{\Gamma^{\mathrm{im}}}$ for an element of $\mathrm{Hom}(\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ induced by $f_{\Gamma^{\mathrm{im}}}^\bullet$. Note that we have $D_{\alpha_{\Gamma^{\mathrm{im}}}} = D$. Then Proposition 3.1 implies that $\gamma_{(\alpha_{\Gamma^{\mathrm{im}}}, D)} = g_{X_{\Gamma^{\mathrm{im}}}} + n_X - 2$, where $g_{X_{\Gamma^{\mathrm{im}}}}$ denotes the genus of $X_{\Gamma^{\mathrm{im}}}$.

On the other hand, we write $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ for the set of connected components of the topological closure $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$ of $X \setminus X_{\Gamma^{\mathrm{im}}}$ in X. We define the following pointed stable curve

$$E^\bullet = (E, D_E \stackrel{\mathrm{def}}{=} E \cap X_{\Gamma^{\mathrm{im}}}),\ E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}),$$

over k. We denote by $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})^{>0}$ the set of curves of $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ whose genus are > 0.

Let $E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})^{>0}$. Similar arguments to the arguments given in the proof of the case where $n_X \leq 1$ and $v(\Gamma_{X^\bullet}) \neq \emptyset$ above imply that there exists a Galois étale covering

$$f_E^\bullet : Z_E^\bullet = (Z_E, D_{Z_E}) \to E^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that

$$\gamma_{(\alpha_E, 0)} = g_E - 1,$$

where g_E denotes the genus of E, and $\alpha_E \in \mathrm{Rev}_0^{\mathrm{adm}}(E^\bullet)$ is an element induced by $f_E^\bullet$.

Let $E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}) \setminus \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})^{>0}$. We put

$$Z_E^\bullet \stackrel{\mathrm{def}}{=} \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} E_i^\bullet,$$

where $E_i^\bullet$ is a copy of $E^\bullet$. Then we obtain a Galois multi-admissible covering

$$f_E^\bullet : Z_E^\bullet \to E^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $f_E^\bullet|_{E_i^\bullet}$ is an identity, and the Galois action is $j(E_i) = E_{i+j}$ for each $i, j \in \mathbb{Z}/n\mathbb{Z}$.

We may glue $Z_{\Gamma^{\mathrm{im}}}^\bullet$ and $\{Z_E^\bullet\}_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$ along $f_\Gamma^{-1}(X_{\Gamma^{\mathrm{im}}} \cap (\bigcup_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})} E))$ and $\{f_E^{-1}(X_{\Gamma^{\mathrm{im}}} \cap E)\}_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$ in a way that is compatible with the gluing of $\{X_{\Gamma^{\mathrm{im}}}^\bullet\} \cup \{E^\bullet\}_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$ that gives rise to $X^\bullet$, then we obtain a Galois multi-admissible covering

$$f^\bullet : Z^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we write

$$\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$$

for an element induced by $f^\bullet$. We see that $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$. By applying Proposition 3.1, we obtain that

$$\gamma_{(\alpha,D)} = g_X + n_X - 2.$$

This completes the proof of the theorem. □

In the remainder of the present section, we will prove a slightly stronger version of Theorem 4.6 when $n_X = 3$. The stronger version will be used in the proof of reconstructions of field structures associated to inertia subgroups (cf. Section 5.2).

Lemma 4.7. *Suppose that $n_X = 3$, and that $D_X \stackrel{\mathrm{def}}{=} \{x_1, x_2, x_3\}$. Let $\Gamma \stackrel{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X,*

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma})$$

the pointed stable curve of type $(g_{X_\Gamma}, n_{X_\Gamma})$ associated to Γ, and $\Pi_{X_\Gamma^\bullet}$ the admissible fundamental group of $X_\Gamma^\bullet$. Let $D_j \in \mathbb{Z}[D_X] \subseteq \mathbb{Z}[D_{X_\Gamma}]$, $j \in \{1,2,3\}$, be an effective divisor on X_Γ such that $\deg(D_j) = 2(p^{t_j} - 1)$, that $\mathrm{ord}_x(D_j) \leq p^{t_j} - 1$ for each $x \in D_X$, and that $\#\{x \in D_X \mid \mathrm{ord}_x(D_j) = p^{t_j} - 1\} \geq 1$. Let $n \stackrel{\mathrm{def}}{=} p^t - 1 \stackrel{\mathrm{def}}{=} p^{t_1+t_2+t_3} - 1$ and

$$D_\Gamma \stackrel{\mathrm{def}}{=} D_1 + p^{t_1} D_2 + p^{t_1+t_2} D_3 \in \mathbb{Z}[D_X] \subseteq \mathbb{Z}[D_{X_\Gamma}]$$

an effective divisor on X_Γ with degree $2n$. Moreover, suppose that $D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_\Gamma}]^0$ (i.e., $\mathrm{ord}_x(D_\Gamma) < n$ for each $x \in D_X$), and that

$$n > \max\{C(g_X) + 1, \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet}))\}.$$

Then there exists an element $\alpha_\Gamma \in \mathrm{Rev}_{D_\Gamma}^{\mathrm{adm}}(X_\Gamma^\bullet)$ such that $\alpha_\Gamma \neq 0$, and that the generalized Hasse-Witt invariant

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + 1.$$

Proof. Since Γ is a minimal quasi-tree associated to D_X, we obtain that $\Gamma' \stackrel{\mathrm{def}}{=} \Gamma \setminus e^{\mathrm{lp}}(\Gamma)$ is a tree. If $D_X \subseteq X_v$ for some $v \in v(\Gamma)$, then the corollary follows from Proposition 4.2. Without loss of generality, we may assume that one of the following cases holds: (i) Let w_1, $w_2 \in v(\Gamma)$ distinct from each other such that $x_1, x_2 \in X_{w_1}$, and $x_3 \in X_{w_2}$. (ii) Let v_1, v_2, $v_3 \in v(\Gamma)$ distinct from each other such that $x_1 \in X_{v_1}$, $x_2 \in X_{v_2}$, and $x_3 \in X_{v_3}$.

We put

$$D'_v \stackrel{\mathrm{def}}{=} D_X \cap X_v, \ D_v \stackrel{\mathrm{def}}{=} D'_v \cup (X_v \cap (\overline{X_\Gamma \setminus X_v})), \ v \in v(\Gamma)$$

where $\overline{X_\Gamma \setminus X_v}$ denotes the topological closure of $X_\Gamma \setminus X_v$ in X_Γ. Next, we construct an effective divisor $P_v \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_v]^0$ for each $v \in v(\Gamma')$.

Suppose that (i) holds. Let $y_{w_1,w_2} \in D_{w_1}$ be the unique node of X_Γ such that the closed edge of Γ' corresponding to y_{w_1,w_2} is contained in the minimal path $p(\Gamma', w_1, w_2)$ and $z_{w_1,w_2} \in D_{w_2}$ the unique node of X_Γ such that the closed edge of Γ' corresponding to z_{w_1,w_2} is contained in the minimal path $p(\Gamma', w_1, w_2)$. On the other hand, let $w \in v(\Gamma) \setminus \{w_1, w_2\}$ be an arbitrary vertex. Note that $n_X = 3$ implies that w is contained in $p(\Gamma', w_1, w_2)$. Since Γ' is a tree, there exist a unique node $x_{w_1,w} \in D_w$ and a unique node $x_{w_2,w} \in D_w$ such that the closed edges of Γ' corresponding to $x_{w_1,w}$ and $x_{w_2,w}$ are contained in $p(\Gamma', w_1, w)$ and $p(\Gamma', w_2, w)$, respectively.

Let $j \in \{1,2,3\}$. We put

$$Q_{w_1,j} \stackrel{\text{def}}{=} \mathrm{ord}_{x_1}(D_j)x_1 + \mathrm{ord}_{x_2}(D_j)x_2 + \mathrm{ord}_{x_3}(D_j)y_{w_1,w_2},$$

$$Q_{w_2,j} \stackrel{\text{def}}{=} [\deg(D_j) - \mathrm{ord}_{x_1}(D_j) - \mathrm{ord}_{x_2}(D_j)]z_{w_1,w_2} + \mathrm{ord}_{x_3}(D_j)x_3,$$

and

$$Q_{w,j} \stackrel{\text{def}}{=} [\deg(D_j) - \mathrm{ord}_{x_1}(D_j) - \mathrm{ord}_{x_2}(D_j)]x_{w_1,w} + \mathrm{ord}_{x_3}(D_j)x_{w_2,w}$$

for all $w \in v(\Gamma) \setminus \{w_1, w_2\}$, where $[(-)]$ denotes the image of $(-)$ in $\mathbb{Z}/(p^{t_j}-1)\mathbb{Z}$. Then $Q_{v,j}$, $v \in v(\Gamma)$, is an effective divisor on X_v whose degree is equal to $(\#D_v - 1)(p^{t_j}-1)$. Moreover, we put

$$P_v \stackrel{\text{def}}{=} Q_{v,1} + p^{t_1}Q_{v,2} + p^{t_1+t_2}Q_{v,3} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_v]^0, \ v \in v(\Gamma).$$

Then P_v is an effective divisor on X_v whose degree is equal to $(\#D_v - 1)n$, and whose support is equal to D_v.

Suppose that (ii) holds. Then one of the following conditions is satisfied: (1) There exist $a, b, c \in \{1,2,3\}$ distinct from each other such that $p(\Gamma', w_a, w_b) \cap p(\Gamma', w_b, w_c) = \emptyset$. (2) For all $a, b, c \in \{1,2,3\}$ distinct from each other, we have that $p(\Gamma', w_a, w_b) \cap p(\Gamma', w_b, w_c) \neq \emptyset$.

Suppose that (1) holds, and that, without loss of generality, we may assume that $a = 1$, $b = 2$, and $c = 3$. Write $y_{v_1,v_3} \in D_{v_1}$ for the unique node of X_Γ such that the closed edge of Γ' corresponding to y_{v_1,v_3} is contained in $p(\Gamma', v_1, v_3)$ and $z_{v_1,v_3} \in D_{v_3}$ for the unique node of X_Γ such that the closed edge of Γ' corresponding to z_{v_1,v_3} is contained in $p(\Gamma', v_1, v_3)$. On the other hand, let $v \in v(\Gamma) \setminus \{v_1, v_3\}$ be an arbitrary vertex. Since $n_X = 3$, we see that $v \in p(\Gamma', v_1, v_3)$, and that either $v \in p(\Gamma', v_1, v_2)$ or $v \in p(\Gamma', v_2, v_3)$ holds. Since Γ' is a tree, there exist a unique node $x_{v_1,v} \in D_v$ and a unique node $x_{v_3,v} \in D_v$ such that the closed edges of Γ' corresponding to $x_{v_1,v}$ and $x_{v_3,v}$ are contained in $p(\Gamma', v_1, v)$ and $p(\Gamma', v, v_3)$, respectively.

Let $j \in \{1,2,3\}$. We put

$$Q_{v_1,j} \stackrel{\text{def}}{=} \mathrm{ord}_{x_1}(D_j)x_1 + [\deg(D_j) - \mathrm{ord}_{x_2}(D_j) - \mathrm{ord}_{x_3}(D_j)]y_{v_1,v_3},$$

$$Q_{v_2,j} \stackrel{\text{def}}{=} \mathrm{ord}_{x_1}(D_j)x_{v_1,v_2} + \mathrm{ord}_{x_2}(D_j)x_{v_2} + \mathrm{ord}_{x_3}(D_j)x_{v_3,v_2}$$

$$Q_{v_3,j} \stackrel{\text{def}}{=} [\deg(D_j) - \mathrm{ord}_{x_1}(D_j) - \mathrm{ord}_{x_2}(D_j)]z_{v_1,v_3} + \mathrm{ord}_{x_3}(D_j)x_3$$

$$Q_{v,j} \stackrel{\text{def}}{=} \mathrm{ord}_{x_1}(D_j)x_{v_1,v} + [\deg(D_j) - \mathrm{ord}_{x_2}(D_j) - \mathrm{ord}_{x_3}(D_j)]x_{v_3,v}$$

for all $v \in (v(\Gamma') \cap p(\Gamma', v_1, v_2)) \setminus \{v_1, v_2\}$, and

$$Q_{v,j} \stackrel{\text{def}}{=} [\deg(D_j) - \mathrm{ord}_{x_1}(D_j) - \mathrm{ord}_{x_2}(D_j)]x_{v_1,v} + \mathrm{ord}_{x_3}(D_j)x_{v_3,v}$$

for all $v \in (v(\Gamma') \cap p(\Gamma', v_2, v_3)) \setminus \{v_2, v_3\}$. Then $Q_{v,j}$, $v \in v(\Gamma)$, is an effective divisor on X_v whose degree is equal to $(\#D_v - 1)(p^{t_j}-1)$. Moreover, we put

$$P_v \stackrel{\text{def}}{=} Q_{v,1} + p^{t_1}Q_{v,2} + p^{t_1+t_2}Q_{v,3} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_v]^0, \ v \in v(\Gamma).$$

Then P_v is an effective divisor on X_v whose degree is equal to $(\#D_v - 1)n$, and whose support is equal to D_v.

Suppose that (2) holds. Then there exists a unique vertex $v_0 \in v(\Gamma')$ such that $\{v_0\} = v(\Gamma') \cap p(\Gamma', v_1, v_2) \cap p(\Gamma', v_2, v_3) \cap p(\Gamma', v_3, v_1)$. Let $v \in v(\Gamma')$. Since $n_X = 3$, we obtain that either $v \in p(\Gamma', v_1, v_0)$ or $v \in p(\Gamma', v_2, v_0)$ or $v \in p(\Gamma', v_3, v_0)$ holds. Let y_{v_i,v_0}, $i \in \{1, 2, 3\}$, be the unique node of X_Γ such that the closed edge of Γ' corresponding to $y_{v_i,v_0} \in D_{v_i}$ is contained in $p(\Gamma', v_i, v_0)$ and $z_{v_i,v_0} \in D_{v_0}$, $i \in \{1, 2, 3\}$, be the unique node of X_Γ such that the closed edge of Γ' corresponding to z_{v_i,v_0} is contained in $p(\Gamma', v_i, v_0)$. Moreover, let $v \in (v(\Gamma') \cap p(\Gamma', v_i, v_0)) \setminus \{v_i, v_0\}$, $i \in \{1, 2, 3\}$. Since Γ' is a tree, there exist a unique node $x_{v_i,v} \in D_v$ and a unique node $x_{v_0,v} \in D_v$ such that the closed edges of Γ' corresponding to $x_{v_i,v}$ and $x_{v_0,v}$ are contained in $p(\Gamma', v_i, v)$ and $p(\Gamma', v_0, v)$, respectively.

Let $j \in \{1, 2, 3\}$. We put

$$Q_{v_1,j} \overset{\text{def}}{=} \text{ord}_{x_1}(D_j)x_1 + [\deg(D_j) - \text{ord}_{x_2}(D_j) - \text{ord}_{x_3}(D_j)]y_{v_1,v_0},$$

$$Q_{v_2,j} \overset{\text{def}}{=} \text{ord}_{x_2}(D_j)x_2 + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_3}(D_j)]y_{v_2,v_0},$$

$$Q_{v_3,j} \overset{\text{def}}{=} \text{ord}_{x_3}(D_j)x_3 + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]y_{v_3,v_0},$$

$$Q_{v_0,j} \overset{\text{def}}{=} \text{ord}_{x_1}(D_j)z_{v_1,v_0} + \text{ord}_{x_2}(D_j)z_{v_2,v_0} + \text{ord}_{x_3}(D_j)z_{v_3,v_0}$$

$$Q_{v,j} \overset{\text{def}}{=} \text{ord}_{x_1}(D_j)x_{v_1,v} + [\deg(D_j) - \text{ord}_{x_2}(D_j) - \text{ord}_{x_3}(D_j)]x_{v_0,v}$$

for all $v \in (v(\Gamma') \cap p(\Gamma', v_1, v_0)) \setminus \{v_1, v_0\}$,

$$Q_{v,j} \overset{\text{def}}{=} \text{ord}_{x_2}(D_j)x_{v_2,v} + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_3}(D_j)]x_{v_0,v}$$

for all $v \in (v(\Gamma') \cap p(\Gamma', v_2, v_0)) \setminus \{v_2, v_0\}$, and

$$Q_{v,j} \overset{\text{def}}{=} \text{ord}_{x_3}(D_j)x_{v_3,v} + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]x_{v_0,v}$$

for all $v \in (v(\Gamma') \cap p(\Gamma', v_3, v_0)) \setminus \{v_3, v_0\}$. Then $Q_{v,j}$, $v \in v(\Gamma)$, is an effective divisor on X_v whose degree is equal to $(\#D_v - 1)(p^{t_j} - 1)$. Moreover, we put

$$P_v \overset{\text{def}}{=} Q_{v,1} + p^{t_1}Q_{v,2} + p^{t_1+t_2}Q_{v,3} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_v]^0, \ v \in v(\Gamma).$$

Then P_v is an effective divisor on X_v whose degree is equal to $(\#D_v - 1)n$, and whose support is equal to D_v.

Let $v \in v(\Gamma)$ and $\widetilde{X}_v^\bullet$ the smooth pointed stable curve of type (g_v, n_v) over k defined in Section 2.1. Then P_v can be also regarded as an effective divisor on $\widetilde{X}_v$. By applying similar arguments to the arguments given in the proof of Proposition 4.2, there exists $\widetilde{\alpha}_v \in \text{Rev}^{\text{adm}}_{P_v}(\widetilde{X}_v^\bullet)$ such that

$$\gamma_{(\widetilde{\alpha}_v, P_v)} = g_v + \#D_v - 2.$$

We define

$$X_v^\bullet = (X_v, D_{X_v} \overset{\text{def}}{=} D_v)$$

to be a pointed stable curve over k. Then Proposition 3.1 implies that the element $\alpha_v \in \mathrm{Rev}^{\mathrm{adm}}_{P_v}(X^\bullet_v)$ induced by $\widetilde{\alpha}_v$ such that $\gamma_{(\alpha_v, P_v)}$ attains the maximum $g_{X_v} + \#D_v - 2$, where g_{X_v} denotes the genus of X_v. Write

$$f^\bullet_v : Y^\bullet_v \to X^\bullet_v$$

for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_v. By gluing $\{Y^\bullet_v\}_{v \in v(\Gamma)}$ along $\{f_v^{-1}(D_v \setminus D'_v)\}_{v \in v(\Gamma)}$ in a way that is compatible with the gluing of $\{X^\bullet_v\}_{v\in v(\Gamma)}$ that gives rise to $X^\bullet_\Gamma$, we obtain a Galois multi-admissible covering

$$f^\bullet_\Gamma : Y^\bullet_\Gamma \to X^\bullet_\Gamma$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Note that the construction of $f^\bullet_\Gamma$ implies that f_Γ is étale over $D_{X_\Gamma} \setminus D_X$. We denote by $\alpha_\Gamma \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet_\Gamma}, \mathbb{Z}/n\mathbb{Z})$ an element induced by $f^\bullet_\Gamma$. By applying similar arguments to the arguments given in the proof of Lemma 4.5 and the construction of D_Γ, we see that

$$\alpha_\Gamma \in \mathrm{Rev}^{\mathrm{adm}}_{D_\Gamma}(X^\bullet_\Gamma).$$

Moreover, Proposition 3.1 implies that

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + 1.$$

We complete the proof of the lemma. □

Theorem 4.8. *Suppose that $n_X = 3$, and that $D_X \overset{\mathrm{def}}{=} \{x_1, x_2, x_3\}$. Let $D_j \in \mathbb{Z}[D_X]$, $j \in \{1, 2, 3\}$, be an effective divisor on X such that $\deg(D_j) = 2(p^{t_j} - 1)$, that $\mathrm{ord}_x(D_j) \leq p^{t_j} - 1$ for each $x \in D_X$, and that $\#\{x \in D_X \mid \mathrm{ord}_x(D_j) = p^{t_j} - 1\} \geq 1$. Let $n = p^t - 1 \overset{\mathrm{def}}{=} p^{t_1+t_2+t_3} - 1$ and*

$$D \overset{\mathrm{def}}{=} D_1 + p^{t_1} D_2 + p^{t_1+t_2} D_3 \in \mathbb{Z}[D_X]$$

an effective divisor on X with degree $2n$. Moreover, suppose that $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, and that

$$n > \max\{C(g_X) + 1, \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet}))\}.$$

Then there exists an element $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha_\Gamma \neq 0$, and that the generalized Hasse-Witt invariant

$$\gamma_{(\alpha, D)} = g_X + 1.$$

Proof. By applying Lemma 4.7 and similar arguments to the arguments given in the proof of Theorem 4.6, we obtain the theorem. □

5 Applications to anabelian geometry

5.1 A group-theoretical formula for topological types of pointed stable curves

In this section, by using Theorem 4.6, we show a group-theoretical formula for the topological type of an arbitrary pointed stable curve over an algebraically closed field of characteristic $p > 0$.

Definition 5.1. (i) Let Δ be an arbitrary profinite group and $m, N \in \mathbb{N}$ positive natural numbers. We define the closed normal subgroup

$$D_N(\Delta)$$

of Δ to be the topological closure of $[\Delta, \Delta]\Delta^N$, where $[\Delta, \Delta]$ denotes the commutator subgroup of Δ. Moreover, we define the closed normal subgroup

$$D_N^{(m)}(\Delta)$$

of Δ inductively by $D_N^{(1)}(\Delta) \stackrel{\text{def}}{=} D_N(\Delta)$ and $D_N^{(i+1)}(\Delta) \stackrel{\text{def}}{=} D_N(D_N^{(i)}(\Delta))$, $i \in \{1, \dots, m-1\}$. Note that $\#(\Delta/D_N^{(m)}(\Delta)) \leq \infty$ when Δ is topologically finitely generated.

(ii) Let ℓ be a prime number and $r, m \in \mathbb{N}$ natural numbers. We denote by

$$F_{r,m}^{\ell}$$

the finite group $\widehat{F}_r/D_\ell^{(m)}(\widehat{F}_r)$, where $\widehat{F}_r$ denotes the free profinite group of rank r.

Let $X^\bullet = (X, D_X)$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. In this section, let

$$\Pi$$

be *an abstract profinite group* which is isomorphic to $\Pi_{X^\bullet}$ as profinite groups. Moreover, we denote by $\pi_A(\Pi)$ the set of finite quotients of Π. We put

$$b_\Pi^1 \stackrel{\text{def}}{=} \max\{r \mid \text{there exists a prime number } \ell \text{ such that } (\mathbb{Z}/\ell\mathbb{Z})^{\oplus r} \in \pi_A(\Pi)\}$$

and

$$b_\Pi^2 \stackrel{\text{def}}{=} \begin{cases} 0, & F_{b_\Pi^1,2}^{\ell} \in \pi_A(\Pi) \text{ for some prime number } \ell, \\ 1, & \text{otherwise.} \end{cases}$$

Note that b_Π^i, $i \in \{1, 2\}$, is a group-theoretical invariant associated to Π (i.e., depends only on the isomorphism class of Π). First, we have the following lemma.

Lemma 5.2. *(i) We have that*

$$b_\Pi^2 = \begin{cases} 1, & \text{if } n_X = 0, \\ 0, & \text{if } n_X \neq 0. \end{cases}$$

and

$$b_\Pi^1 = 2g_X + n_X - 1 + b_\Pi^2.$$

(ii) There exists a unique prime number p_Π such that $(\mathbb{Z}/p_\Pi\mathbb{Z})^{\oplus b_\Pi^1} \notin \pi_A(\Pi)$. In particular, we have $p = p_\Pi$.

Proof. (i) Let $r_\Pi \stackrel{\text{def}}{=} \dim_{\mathbb{F}_\ell}(\Pi^{\text{ab}} \otimes \mathbb{F}_\ell)$, where ℓ an arbitrary prime number $\mathfrak{P} \setminus \{p\}$, and $\mathfrak{P}$ denotes the set of prime numbers. Then the structures of maximal prime-to-p quotients of admissible fundamental groups imply that

$$\Pi^{\text{ab}} \cong \mathbb{Z}_p^{\sigma_{X^\bullet}} \times \prod_{\ell \in \mathfrak{P}\setminus\{p\}} \mathbb{Z}_\ell^{r_\Pi}.$$

Since $X^\bullet$ is a pointed stable curve, we have that

$$\sigma_{X^\bullet} < r_\Pi.$$

This implies that $b^1_\Pi = r_\Pi$. Moreover, we have

$$b^1_\Pi = \begin{cases} 2g_X, & \text{if } n_X = 0, \\ 2g_X + n_X - 1, & \text{if } n_X \neq 0. \end{cases}$$

Suppose that $n_X > 0$. Let $\ell_1 \in \mathfrak{P} \setminus \{p\}$. The specialization theorem of maximal pro-ℓ_1 quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]) implies that the maximal pro-ℓ_1 quotient Π^{ℓ_1} of Π is a free pro-ℓ_1 profinite group of rank b^1_Π. Then we obtain that

$$F^{\ell_1}_{b^1_\Pi, 2} \in \pi_A(\Pi).$$

Thus, we obtain that $b^2_\Pi = 0$ if $n_X > 0$.

Conversely, we assume that $F^{\ell_2}_{b^1_\Pi,2} \in \pi_A(\Pi)$ for some prime number ℓ_2. Then we have $\ell_2 \neq p$. By applying the Schreier index formula, we have the following natural exact sequence

$$1 \to (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus \ell_2^{b^1_\Pi}(b^1_\Pi - 1)+1} \to F^{\ell_2}_{b^1_\Pi,2} \to (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b^1_\Pi} \to 1.$$

Let $\phi : \Pi \twoheadrightarrow F^{\ell_2}_{b^1_\Pi,2}$ be a surjection. We denote by $X^\bullet_{\ell_2}$ the pointed stable curve over k corresponding to the kernel of the natural surjection

$$\Pi_{X^\bullet} \xrightarrow{\sim} \Pi \stackrel{\phi}{\twoheadrightarrow} F^{\ell_2}_{b^1_\Pi,2} \twoheadrightarrow (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b^1_\Pi}$$

and by $\Pi_{\ell_2} \subseteq \Pi$ the kernel of the surjection $\Pi \stackrel{\phi}{\twoheadrightarrow} F^{\ell_2}_{b^1_\Pi,2} \twoheadrightarrow (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b^1_\Pi}$. Then we have

$$(\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus \ell_2^{b^1_\Pi}(b^1_\Pi - 1)+1} \in \pi_A(\Pi_{\ell_2}).$$

Thus, $b^1_{\Pi_{\ell_2}} \geq \ell_2^{b^1_\Pi}(b^1_\Pi - 1) + 1$. If $n_X = 0$, the Riemann-Hurwitz formula implies that

$$g_{X_{\ell_2}} = \ell_2^{b^1_\Pi}(g_X - 1) + 1,$$

where $g_{X_{\ell_2}}$ denotes the genus of $X^\bullet_{\ell_2}$. Then we have

$$b^1_{\Pi_{\ell_2}} = 2(\ell_2^{b^1_\Pi}(g_X - 1) + 1) = \ell_2^{b^1_\Pi}(b^1_\Pi - 2) + 2.$$

On the other hand,

$$\ell_2^{b^1_\Pi}(b^1_\Pi - 2) + 2 < \ell_2^{b^1_\Pi}(b^1_\Pi - 1) + 1.$$

This contradicts the fact that $b^1_{\Pi_{\ell_2}} \geq \ell_2^{b^1_\Pi}(b^1_\Pi - 1) + 1$. Then we obtain that $n_X > 0$ if $b^2_\Pi = 0$. Moreover, we see that

$$b^1_\Pi = 2g_X + n_X - 1 + b^2_\Pi.$$

(ii) This follows immediately from the structure of Π^{ab}. We complete the proof of the lemma. □

Let $\overline{\mathbb{F}}_{p_\Pi}$ be an arbitrary algebraic closure of $\mathbb{F}_{p_\Pi}$. Let $\chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}^\times_{p_\Pi})$. We denote by $\Pi_\chi \subseteq \Pi$ the kernel of χ. The profinite group Π_χ admits a natural action of Π via conjugation. Let

$$\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})[\chi] \stackrel{\mathrm{def}}{=} \{\pi \in \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi} \mid \tau \cdot \pi = \chi(\tau)\pi \text{ for all } \tau \in \Pi\},$$

and $\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})) \stackrel{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_{p_\Pi}}(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})[\chi])$, where $(\tau \cdot \pi)(x) \stackrel{\mathrm{def}}{=} \pi(\tau^{-1} \cdot x)$ for all $x \in \Pi_\chi$. We define a group-theoretical invariant associated to Π as follows:

$$\gamma^{\max}_\Pi \stackrel{\mathrm{def}}{=} \max\{\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z})) \mid \chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}^\times_{p_\Pi}) \text{ and } \chi \neq 1\}.$$

Let $\mu_m \stackrel{\mathrm{def}}{=} \chi(\Pi) \subseteq \overline{\mathbb{F}}^\times_{p_\Pi}$ be the image of χ which is the group of mth roots of unity for some m prime to p_Π, and $X^\bullet_\chi = (X_\chi, D_{X_\chi}) \to X^\bullet$ the Galois multi-admissible covering over k with Galois group μ_m. Then we have a natural Π-equivalent isomorphism

$$H^1_{\text{ét}}(X_\chi, \mathbb{F}_{p_\Pi}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi} \cong \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi}.$$

Moreover, since the actions of Π on $H^1_{\text{ét}}(X_\chi, \mathbb{F}_{p_\Pi}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi}$ and $\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi}$ factor through $\Pi/\Pi_\chi \cong \mu_m$, the isomorphism above is also a μ_m-equivalent. This means that $\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z}))$ is a generalized Hasse-Witt invariant of $\mathbb{Z}/m\mathbb{Z}$-cyclic admissible coverings of $X^\bullet$. Then we have the following lemma.

Lemma 5.3. *Let $\gamma^{\max}_{X^\bullet}$ be the maximum of generalized Hasse-Witt invariant of prime-to-p cyclic admissible coverings of $X^\bullet$ defined in Section 3. Then we have $\gamma^{\max}_\Pi = \gamma^{\max}_{X^\bullet}$. In particular, we have*

$$\gamma^{\max}_\Pi = g_X + n_X - 2 + b^2_\Pi.$$

Proof. The first part of lemma follows the explanation of $\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z}))$ above. The "in particular" part of the lemma follows immediately from Theorem 4.6 and Lemma 5.2 (i). □

We have the following anabelian formula for (g_X, n_X).

Theorem 5.4. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$, and Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Then we have that*

$$g_X = b^1_\Pi - \gamma^{\max}_\Pi - 1,\ n_X = 2\gamma^{\max}_\Pi - b^1_\Pi - b^2_\Pi + 3.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Proof. The theorem follows immediately from Lemma 5.2 and Lemma 5.3. □

Remark 5.4.1. We maintain the notation introduced above. Moreover, suppose that $X^\bullet$ is *smooth* over k. In this remark, we discuss a formula for (g_X, n_X) which was essentially obtained by Tamagawa. Let $n \overset{\text{def}}{=} p^t - 1$ and K_n the kernel of the natural surjection $\Pi \twoheadrightarrow \Pi^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$. In [T2], Tamagawa introduced

$$\text{Avr}_p(\Pi) \overset{\text{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}$$

which is called the limit of p-averages associated to Π. Note that since $p = p_\Pi$ (cf. Lemma 5.2 (ii)), we have that $\text{Avr}_p(\Pi)$ is a group-theoretical invariant associated to Π. Then the main theorem of [T2] (i.e., Tamagawa's p-average theorem, see [T2, Theorem 0.5]) says that

$$\text{Avr}_p(\Pi) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X \geq 2. \end{cases}$$

Let $\ell' \in \mathfrak{P} \setminus \{p_\Pi = p\}$ be an arbitrary prime number distinct from p_Π. Write $\text{Nom}_{\ell'}(\Pi)$ for the set of normal subgroups of Π of index ℓ'. Suppose that $b^2_\Pi = 0$ (i.e., $n_X \neq 0$). Let $\ell >> 0$ be a prime number distinct from p. If $n_X = 1$, then the structures of maximal prime-to-p quotients of admissible fundamental groups imply that every Galois admissible covering of $X^\bullet$ over k is *étale*. Thus, by the Riemann-Hurwitz formula, we obtain that

$$\text{Avr}_p(\Pi(\ell)) - 1 = \ell(\text{Avr}_p(\Pi)).$$

On the other hand, if $n_X \geq 2$, the Riemann-Hurwitz formula implies that

$$\text{Avr}_p(\Pi(\ell)) - 1 = \ell(\text{Avr}_p(\Pi)) - \ell + \frac{1}{2}r(\ell - 1)$$

for some integer number $0 \leq r \leq n_X$. Note that $r(\ell - 1) \neq 2\ell$ when $\ell >> 0$. Then we obtain that

$$\text{Avr}_p(\Pi(\ell)) - 1 = \ell(\text{Avr}_p(\Pi))$$

holds for every $\ell \in \mathfrak{P} \setminus \{p_\Pi\}$ and every $\Pi(\ell) \in \text{Nom}_\ell(\Pi)$ if and only if $n_X = 1$. We define a group-theoretical invariant associated to Π as follows:

$$c_\Pi \overset{\text{def}}{=} \begin{cases} 1, & \text{if } b^2_\Pi = 1, \\ 1, & \text{if } b^2_\Pi = 0,\ \text{Avr}_p(\Pi(\ell)) - 1 = \ell(\text{Avr}_p(\Pi)), \\ & \quad \ell \in \mathfrak{P} \setminus \{p_\Pi\},\ \text{and } \Pi(\ell) \in \text{Nom}_\ell(\Pi), \\ 0, & \text{otherwise.} \end{cases}$$

Then the p-average theorem above implies immediately the following formula

$$g_X = \mathrm{Avr}_p(\Pi) + c_\Pi,\ n_X = b^1_\Pi - 2\mathrm{Avr}_p(\Pi) - 2c_\Pi - b^2_\Pi + 1.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π (cf. [T2, Theorem 0.1]). This result is the main goal of the theory developed in [T2], which plays a key role in the theory of tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (cf. [T2], [Y2]).

On the other hand, the approach to finding a group-theoretical formula for (g_X, n_X) by applying the limit of p-averages associated to Π explained above *is difficult to be generalized* to the case where $X^\bullet$ is an arbitrary (possibly singular) pointed stable curve. The reason is as follows. In [Y4], the author generalized Tamagawa's p-average theorem to the case of pointed stable curves (cf. [Y4, Theorem 1.3 and Theorem 1.4]). The generalized formula concerning the limit of p-averages associated to Π is very complicated in general when $X^\bullet$ is not smooth over k, and $\mathrm{Avr}_p(\Pi)$ depends not only on the topological type (g_X, n_X) but also on the structure of dual semi-graph $\Gamma_{X^\bullet}$.

5.2 Field structures associated to inertia subgroups

In this subsection, we will prove that the field structures associated to inertia subgroups of marked points of arbitrary pointed stable curves can be reconstructed group-theoretically from admissible fundamental groups, and that a surjective open continuous homomorphism of admissible fundamental groups induces a field isomorphism of the fields associated to inertia subgroups of marked points.

Let $i \in \{1, 2\}$, and let $X_i^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Let $\widehat{X}_i^\bullet = (\widehat{X}_i, D_{\widehat{X}_i})$ be the universal admissible covering of $X_i^\bullet$ corresponding to $\Pi_{X_i^\bullet}$, $\Gamma_{\widehat{X}_i^\bullet}$ the dual semi-graph of $\widehat{X}_i^\bullet$, $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X_i^\bullet})$, and x_{e_i} the closed point of X_i corresponding to e_i. This means that $\mathrm{Aut}(\widehat{X}_i^\bullet / X_i^\bullet) = \Pi_{X_i^\bullet}$. Let $\widehat{e}_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{\widehat{X}_i^\bullet})$ be an edge over e_i. We denote by

$$I_{\widehat{e}_i} \subseteq \Pi_{X_i^\bullet}$$

the stabilizer subgroup of $\widehat{e}_i$. We see that $I_{\widehat{e}}$ is (outer) isomorphic to an inertia subgroup associated to the closed point of X corresponding to e_i. Then $I_{\widehat{e}_i} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal pro-prime-to-p quotient of $(-)$.

Write $\overline{\mathbb{F}}_{p,i}$ for the algebraic closure of $\mathbb{F}_p$ in k_i. We put

$$\mathbb{F}_{\widehat{e}_i} \overset{\mathrm{def}}{=} (I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widehat{e}_i}\},$$

where $\{*_{\widehat{e}_i}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_i^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with

$$\bigcup_{(p,m)=1} \mu_m(k_i).$$

Moreover, $\mathbb{F}_{\widehat{e}_i}$ can be identified with $\overline{\mathbb{F}}_{p,i}$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widehat{e}_i}\otimes_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})_i^{p'}$, and whose zero element is $*_{\widehat{e}_i}$. We have the following important result.

Theorem 5.5. *We maintain the notation introduced above. Let $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ be an arbitrary surjective open continuous homomorphism of admissible fundamental groups. Suppose that $\phi(I_{\widehat{e}_1}) = I_{\widehat{e}_2}$, and that $n_X = 3$. Then there exists a group-theoretical algorithm whose input data are $\Pi_{X_i^\bullet}$ and $I_{\widehat{e}_i}$, and whose output datum is $\mathbb{F}_{\widehat{e}_i}$. Moreover, ϕ induces a field isomorphism*

$$\phi^{\mathrm{fd}}_{\widehat{e}_1,\widehat{e}_2} : \mathbb{F}_{\widehat{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widehat{e}_2},$$

where "fd" means "field".

Proof. Let $t \in \mathbb{Z}_{>0}$. We denote by $\mathbb{F}_{p^t,\widetilde{e}_i}$ the unique subfield of $\mathbb{F}_{\widetilde{e}_i}$ whose cardinality is equal to p^t. On the other hand, we fix a finite field $\mathbb{F}_{p^t}$ of cardinality p^t and an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ which contains $\mathbb{F}_{p^t}$. Note that the field structure of $\mathbb{F}_{p^t,\widehat{e}_i}$ is equivalent to a subset

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^t,\widehat{e}_i}, \mathbb{F}_{p^t}) \subseteq \mathrm{Hom}_{\mathrm{group}}(\mathbb{F}^\times_{p^t,\widehat{e}_i}, \mathbb{F}^\times_{p^t}).$$

Then in order to prove the first part of the theorem, it is sufficient to prove that there exists a group-theoretical algorithm whose input datum are $\Pi_{X_i^\bullet}$ and $I_{\widehat{e}_i}$, and whose output datum is the subset $\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^t,\widehat{e}_i}, \mathbb{F}_{p^t})$ for t in a cofinal subset of $\mathbb{N}$ with respect to division.

We put $n \overset{\mathrm{def}}{=} p^t - 1$. Let

$$\chi_i \in \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}^\times_{p^t}).$$

Write H_{χ_i} for the kernel of $\Pi_{X_i^\bullet} \twoheadrightarrow \Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{\chi_i} \mathbb{F}^\times_{p^t}$, M_{χ_i} for $H^{\mathrm{ab}}_{\chi_i} \otimes \mathbb{F}_p$, and $X^\bullet_{H_{\chi_i}} = (X_{H_{\chi_i}}, D_{X_{H_{\chi_i}}})$ for the smooth pointed stable curve over k_i induced by H_{χ_i}. Note that M_{χ_i} admits a natural action of $\Pi_{X_i^\bullet}$ via conjugation. Moreover, this action factors through $\Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}$. We define

$$M_{\chi_i}[\chi_i] \overset{\mathrm{def}}{=} \{a \in M_{\chi_i} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma \cdot a = \chi_i(\sigma)a \text{ for all } \sigma \in \Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}\}$$

and $\gamma_{\chi_i}(M_{\chi_i}) \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(M_{\chi_i}[\chi_i])$. Note that $\gamma_{\chi_i}(M_{\chi_i})$ is equal to a generalized Hasse-Witt invariant of a cyclic multi-admissible covering of $X_i^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$, and that Lemma 2.5 implies that $\gamma_{\chi_i}(M_{\chi_i}) \leq g_X + 1$ if $n_X = 3$. Moreover, we define two maps

$$\mathrm{Res}_{i,t} : \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}^\times_{p^t}) \to \mathrm{Hom}_{\mathrm{groups}}(\mathbb{F}^\times_{p^t,\widehat{e}_i}, \mathbb{F}^\times_{p^t})$$

and

$$\Gamma_{i,t} : \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}^\times_{p^t}) \to \mathbb{Z}_{\geq 0},\ \chi_i \mapsto \gamma_{\chi_i}(M_{\chi_i}),$$

where the map $\mathrm{Res}_{i,t}$ is the restriction with respect to the natural inclusion

$$\mathbb{F}^\times_{p^t,\widehat{e}_i} = I_{\widehat{e}_i} \otimes \mathbb{Z}/n\mathbb{Z} \hookrightarrow \Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}.$$

Let m_0 be the product of all prime numbers $\leq p-2$ if $p \neq 2, 3$ and $m_0 = 1$ if $p = 2, 3$. Let t_0 be the order of p in the multiplicative group $(\mathbb{Z}/m_0\mathbb{Z})^\times$. We have the following claim (cf. [T2, Claim 5.4] for the case where $X_i^\bullet$ is smooth over k_i):

Claim: There exists a constant $C(g_X)$ which only depends on g_X such that, for each $t > \max\{\log_p(C(g_X)+1), \log_p(\#e^{\mathrm{cl}}(\Gamma_{X_i^\bullet}) + \#e^{\mathrm{op}}(\Gamma_{X_i^\bullet}))\}$ divisible by t_0, we have

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^t,\widehat{e}_i}, \mathbb{F}_{p^t}) = \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(\mathbb{F}^\times_{p^t,\widehat{e}_i}, \mathbb{F}^\times_{p^t}) \setminus \mathrm{Res}_{i,t}(\Gamma^{-1}_{i,t}(\{g_X+1\})),\ i \in \{1,2\},$$

where $\mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(-,-)$ denotes the set of surjections whose elements are contained in $\mathrm{Hom}_{\mathrm{groups}}(-,-)$.

Let us prove the claim. By applying similar arguments to the arguments given in the proof of [T2, Claim 5.4], the claim is equivalent to the following:

Let $m \in (\mathbb{Z}/n\mathbb{Z})^\sim$. Then the following statements are equivalent:

(i) We have $m \in \{p^r \mid r = 0, \dots, t-1\}$.

(ii) We have that $(m,n) = 1$, and that, there does not exist $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X_i^\bullet)$ such that $\mathrm{ord}_{x_{e_i}}(D) = m$ and $\gamma_{(\alpha,D)} = g_X + 1$.

First, we prove (i) $\Rightarrow$ (ii). If $s(D) = 1$, then Lemma 2.5 implies that $\gamma_{(\alpha,D)} \leq g_X$. Thus, we may assume that $s(D) = 2$. We put

$$D_{X_i} \stackrel{\mathrm{def}}{=} \{x_{i,1} \stackrel{\mathrm{def}}{=} x_{e_i}, x_{i,2}, x_{i,3}\}.$$

Then [T2, Proposition 2.21 (iv-a)] implies that either $\mathrm{ord}_{x_{i,1}}(D) = n$ or $\mathrm{ord}_{x_{i,3}}(D) = n$ holds, which is impossible as $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$. Next, we prove (ii) $\Rightarrow$ (i). Suppose that $m \notin \{p^r \mid r = 0, \dots, t-1\}$, and that $(m,n) = 1$. Since t is divisible by t_0, n is divisible by all prime numbers $\leq p-2$. Then the assumption $(m,n) = 1$ implies that $m \notin \{ap^r \mid a = 0, \dots, p-2, r = 0, \dots, t-1\}$. Then [T2, Proposition 2.21 (iv-c)] and Theorem 4.8 imply that there exist $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X_i^\bullet)$ such that $\mathrm{ord}_{x_{e_i}}(D) = m$ and $\gamma_{(\alpha,D)} = g_X + 1$. This completes the proof of the claim. Then we complete the proof of the first part of the theorem.

Next, we prove the "moreover" part of the theorem. Let $\kappa_2 \in \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_2^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}^\times_{p^t})$. Then ϕ induces a character

$$\kappa_1 \in \mathrm{Hom}_{\mathrm{groups}}(\Pi^{\mathrm{ab}}_{X_1^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}^\times_{p^r}).$$

Moreover, the surjection $\phi|_{H_{\kappa_1}}$ induces a surjection

$$M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2].$$

Suppose that $\kappa_2 \in \Gamma^{-1}_{2,r}(\{g_X+1\})$. The surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$ implies that $\gamma_{\kappa_1}(M_{\kappa_1}) = g_X + 1$. This means that $\kappa_1 \in \Gamma^{-1}_{1,r}(\{g_X+1\})$. On the other hand, the isomorphism $\phi|_{I_{\widehat{e}_1}} : I_{\widehat{e}_1} \xrightarrow{\sim} I_{\widehat{e}_2}$ induces an injection

$$\mathrm{Res}_{2,r}(\Gamma^{-1}_{2,r}(\{g_X+1\})) \hookrightarrow \mathrm{Res}_{1,r}(\Gamma^{-1}_{1,r}(\{g_X+1\})).$$

Since $\#\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widehat{e}_1}, \mathbb{F}_{p^r}) = \#\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widehat{e}_2}, \mathbb{F}_{p^r})$, we obtain that $\phi|_{I_{\widetilde{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widehat{e}_2}, \mathbb{F}_{p^r}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widehat{e}_1}, \mathbb{F}_{p^r}).$$

Thus, $\phi|_{I_{\widehat{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{\widehat{e}_2}, \overline{\mathbb{F}}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{\widehat{e}_1}, \overline{\mathbb{F}}_p).$$

If we choose $\overline{\mathbb{F}}_p = \mathbb{F}_{\widehat{e}_2}$, then the image of $\mathrm{id}_{\mathbb{F}_{\widehat{e}_2}}$ via the bijection above induces an isomorphism

$$\phi^{\mathrm{fd}}_{\widehat{e}_1, \widehat{e}_2} : \mathbb{F}_{\widehat{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widehat{e}_2}$$

as fields. This completes the proof of the theorem. □

Remark 5.5.1. We maintain the notation introduced above. In fact, by applying Theorem 4.6, we can prove that $I_{\widehat{e}_i}$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$, and that $\phi(I_{\widehat{e}_1})$ is always a stabilizer of an edge of $\Gamma_{\widehat{X}_2^\bullet}$ over an open edge of $\Gamma_{X_2^\bullet}$ (cf. [Y5, Theorem 4.11]). Moreover, by applying similar arguments to the arguments given in the proof of [Y2, Proposition 6.1], to reconstruct the field structures, we may assume that $n_X = 3$. Thus, Theorem 5.5 implies that the field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from surjective open continuous homomorphisms of admissible fundamental groups (cf. [Y5, Proposition 4.13]).

6 Appendix: generalized Hasse-Witt invariants of coverings of dual semi-graphs

In this appendix, we prove two propositions concerning generalized Hasse-Witt invariants of map of dual semi-graphs induced by a cyclic prime-to-p admissible coverings, which were used in the proof of Proposition 3.1.

We maintain the notation introduced in Section 2.1 and Section 2.2. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$. In the present appendix, let n be an arbitrary positive natural number prime to p, and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Let

$$f^\bullet : Y^\bullet \to X^\bullet$$

be a Galois admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, $\mathscr{N}_X^{\mathrm{et}}$ the set of nodes of X over which f is étale, and

$$f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$$

the map of dual semi-graphs of $Y^\bullet$ and $X^\bullet$ induced by $f^\bullet$ with natural action $\mathbb{Z}/n\mathbb{Z}$ on $\Gamma_{Y^\bullet}$, where "sg" means "semi-graph". Note that $Y^\bullet$ is connected.

We denote by $M_{\Gamma_{X^\bullet}}$ and $M_{\Gamma_{Y^\bullet}}$ the first singular cohomology groups $H^1(\Gamma_{X^\bullet}, \mathbb{F}_p) \otimes k$ and $H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k$, respectively. Then $M_{\Gamma_{Y^\bullet}}$ is a $k[\mu_n]$-module and admits the following canonical decomposition

$$M_{\Gamma_{Y^\bullet}} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y^\bullet}}(j)$ as the ζ^j-multiplication.

Let $l \subseteq \Gamma_{Y^\bullet}$ be a loop. We shall write

$$\alpha_l \in M_{\Gamma_{Y^\bullet}}$$

for the vector corresponding to l. For each $j \in \mathbb{Z}/n\mathbb{Z}$, we denote by $j \cdot l$ the natural action of j on l, and by $j \cdot \alpha_l$ the vector $\alpha_{j \cdot l}$. Note that $j'' \cdot (j' \cdot \alpha_l) = (j'' + j') \cdot \alpha_l$ for all $j', j'' \in \mathbb{Z}/n\mathbb{Z}$.

Moreover, we shall say that l is *a minimal loop of* $\Gamma_{Y^\bullet}$ if, for any loop $l' \subseteq l \subseteq \Gamma_{Y^\bullet}$ such that l and l' are homotopic (i.e., $\alpha_l = \alpha_{l'}$), then $l = l'$.

Proposition 6.1. *We maintain the notation introduced above. Moreover, suppose that* $v(\Gamma_{X^\bullet}) = \{v_X\}$. *Then we have that*

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \begin{cases} \#\mathscr{N}_X^{\mathrm{et}} - 1, & \text{if } \#v(\Gamma_{Y^\bullet}) = n\#v(\Gamma_{X^\bullet}), \\ \#\mathscr{N}_X^{\mathrm{et}}, & \text{if } \#v(\Gamma_{Y^\bullet}) \neq n\#v(\Gamma_{X^\bullet}). \end{cases}$$

Proof. Since $\#v(\Gamma_{X^\bullet}) = 1$, we have that the vector space $M_{\Gamma_{X^\bullet}}$ is spanned by the vectors corresponding to the closed edges of $\Gamma_{X^\bullet}$, and that $\dim_k(M_{\Gamma_{X^\bullet}}) = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) = \#e^{\mathrm{lp}}(\Gamma_{X^\bullet})$. Write $E^{\mathrm{lp}} \subseteq e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ for the subset of closed edges such that $(f^{\mathrm{sg}})^{-1}(E^{\mathrm{lp}}) \subseteq e^{\mathrm{lp}}(\Gamma_{Y^\bullet})$ and $E^{\mathrm{tr}} \overset{\mathrm{def}}{=} e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \setminus E^{\mathrm{lp}}$, where "tr" means "tree". Note that for every $e_X \in E^{\mathrm{tr}}$, every closed edge $e_Y \in (f^{\mathrm{sg}})^{-1}(e_X)$ abuts exactly to two different vertices of $\Gamma_{Y^\bullet}$ (i.e., $\#v^{\Gamma_{Y^\bullet}}(e_Y) = 2$). Moreover, we write $E^{\mathrm{lp,et}}$ (resp. $E^{\mathrm{tr,et}}$) for the subset of E^{lp} (resp. E^{tr}) such that f is étale over the node corresponding $e \in E^{\mathrm{lp,et}}$ (resp. $e \in E^{\mathrm{tr,et}}$).

Let $e_X \in E^{\mathrm{lp}}$, $e_Y \in (f^{\mathrm{sg}})^{-1}(e_X)$, $D_{e_Y} \subseteq \mathbb{Z}/n\mathbb{Z}$ the decomposition group of e_Y, and $1 \leq m_{e_Y} \leq n$ the order of D_{e_Y}. Note that e_Y is a minimal loop of $\Gamma_{Y^\bullet}$, and that D_{e_Y} does not depend on the choices of e_Y (i.e., $D_{e'_Y} = D_{e''_Y}$ for all $e'_Y, e''_Y \in (f^{\mathrm{sg}})^{-1}(e_X)$). Write n_{e_Y} for n/m_{e_Y} and M_{e_X} for the subspace of $M_{\Gamma_{Y^\bullet}}$ spanned by $\{\alpha_{e_Y}\}_{e_Y \in (f^{\mathrm{sg}})^{-1}(e_X)}$. Then we see that M_{e_X} is a $k[(\mathbb{Z}/n\mathbb{Z})/D_{e_Y}]$-module, that

$$M_{e_X} \subseteq \bigoplus_{0 \leq i \leq n_{e_Y}-1} M_{\Gamma_{Y^\bullet}}(im_{e_Y}),$$

and that $\dim_k(M_{e_X}) = n_{e_Y}$. Write $M_{e_X}(1) \subseteq M_{e_X}$ for the subspace on which $\zeta \in \mu_n$ acts as the ζ-multiplication. Thus, we have that $\dim_k(M_{e_X} \cap M_{\Gamma_{Y^\bullet}}(1)) = 1$ if and only if $m_{e_Y} = 1$ (i.e., f is étale over the node of X corresponding to e_X).

Suppose that $E^{\mathrm{tr}} = \emptyset$. We have that $\#v(\Gamma_{Y^\bullet}) \neq n\#v(\Gamma_{X^\bullet})$, and that

$$M_{\Gamma_{Y^\bullet}}(1) = \bigoplus_{e_X \in E^{\mathrm{lp,et}}} M_{e_X}(1).$$

Then we obtain that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#E^{\mathrm{lp,et}} = \#\mathscr{N}_X^{\mathrm{et}}$$

when $E^{\mathrm{tr}} = \emptyset$ and $\#v(\Gamma_{Y^\bullet}) \neq n\#v(\Gamma_{X^\bullet})$.

Suppose that $\#v(\Gamma_{Y^\bullet}) = n\#v(\Gamma_{X^\bullet})$. Then we have that $f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ is a Galois topological covering with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then we see that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(i)) = r_X - 1 = \#\mathscr{N}_X^{\mathrm{et}} - 1$$

for all $i \in \{1, \ldots, n-1\}$ and $\dim_k(M_{\Gamma_{Y^\bullet}}(0)) = r_X$, where $r_X = \#e^{\mathrm{cl}}(X^\bullet) - \#v(\Gamma_{X^\bullet}) + 1 = \#e^{\mathrm{cl}}(X^\bullet) = \#\mathscr{N}_X^{\mathrm{et}}$ denotes the Betti number of $\Gamma_{X^\bullet}$. Indeed, since this is a topological question, to see this, we may assume that $X^\bullet$ is an ordinary pointed stable curve whose normalization of underlying curve is a rational curve. Then $Y^\bullet$ is also an ordinary pointed stable curve such that the normalizations of irreducible components are rational curves. Then there exists a line bundle $\mathcal{L}$ on X whose degree is 0, and whose order is n, such that

$$f_*\mathcal{O}_Y \cong \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \mathcal{L}^{\otimes i}.$$

Then we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(i)) = \dim_k(H^1(X, \mathcal{L})) = g_X - 1 = r_X - 1$$

for all $i \in \{1, \ldots, n-1\}$ and $\dim_k(M_{\Gamma_{Y^\bullet}}(0)) = \dim_k(H^1(X, \mathcal{O}_X)) = g_X$. Thus, we obtain that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#\mathscr{N}_X^{\mathrm{et}} - 1$$

when $E^{\mathrm{tr}} \neq \emptyset$ and $\#v(\Gamma_{Y^\bullet}) = n\#v(\Gamma_{X^\bullet})$.

Suppose that $\#v(\Gamma_{Y^\bullet}) \neq n\#v(\Gamma_{X^\bullet})$, and that $E^{\mathrm{tr}} \neq \emptyset$. Then we have $n \neq 2$. Let $v_Y \in (f^{\mathrm{sg}})^{-1}(v_X)$, D_{v_Y} the decomposition group of v_Y which does not depend on the choices of v_Y (i.e., $D_{v'_Y} = D_{v''_Y}$ for all $v'_Y, v''_Y \in (f^{\mathrm{sg}})^{-1}(v_X)$), $1 < m_{v_Y} \overset{\mathrm{def}}{=} \#D_{v_Y}$, and $n_{v_Y} \overset{\mathrm{def}}{=} n/m_{v_Y}$. We put

$$(f^{\mathrm{sg}})^{-1}(v_X) \overset{\mathrm{def}}{=} \{v_{Y,0}, \ldots, v_{Y,n_{v_Y}-1}\},$$

which admits a natural action of $r \in \mathbb{Z}/n\mathbb{Z}$ such that $r \cdot v_{Y,0} = v_{Y,\overline{r}}$, where $\overline{r}$ denotes the image of $\mathbb{Z}/n\mathbb{Z} \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})/D_{v_Y} \overset{\sim}{\to} \mathbb{Z}/n_{v_Y}\mathbb{Z}$. Moreover, without loss of generality, we may assume that $Y_{v_{Y,i}} \cap Y_{v_{Y,i+1}} \neq \emptyset$ for all $i \in \{0, \ldots, n_{v_Y} - 2\}$ and $Y_{v_{Y,n_Y-1}} \cap Y_{v_{Y,0}} \neq \emptyset$; otherwise, $Y_{v_{Y,j'}} \cap Y_{v_{Y,j''}} = \emptyset$ for all $j', j'' \in \{0, \ldots, n_{v_Y} - 1\}$.

Let

$$T_{Y,e_X} \overset{\mathrm{def}}{=} \{e_{Y,0}, \ldots, e_{Y,m_{v_Y}-1}\} \subseteq (f^{\mathrm{sg}})^{-1}(e_X) \subseteq e^{\mathrm{cl}}(\Gamma_{Y^\bullet}),\ e_X \in E^{\mathrm{tr,et}},$$

the subset of closed edges such that $v^{\Gamma_{Y^\bullet}}(e_{Y,j'}) = v^{\Gamma_{Y^\bullet}}(e_{Y,j''}) = \{v_{Y,0}, v_{Y,1}\}$ for all $j', j'' \in \{1, \ldots, m_{v_Y} - 1\}$. Then

$$l_{e_X,i} \overset{\mathrm{def}}{=} v_{Y,0}e_{Y,i}v_{Y,1}e_{Y,i+1}v_{Y,0},\ i \in \{0, \ldots, m_{v_Y} - 2\},$$

can be regarded as a minimal loop of $\Gamma_{Y^\bullet}$. Moreover, the set of vectors

$$\{j \cdot \alpha_{l_{e_X,i}}\}_{i \in \{0,\ldots,m_{v_Y}-2\}, j \in \{0,\ldots,n_{v_Y}-1\}, e_X \in E^{\mathrm{tr,et}}} \subseteq M_{\Gamma_{Y^\bullet}}$$

is to be linearly independent. We denote by $M_{T_{Y,e_X}} \subseteq M_{\Gamma_{Y^\bullet}}$, $e_X \in E^{\mathrm{tr,et}}$, the subspace spanned by

$$\{j \cdot \alpha_{l_{e_X,i}}\}_{i \in \{0,\ldots,m_{v_Y}-2\}, j \in \{0,\ldots,n_{v_Y}-1\}}.$$

Then we see that $M_{T_{Y,e_X}}$, $e_X \in E^{\mathrm{tr,et}}$, is a $k[\mu_n]$-module. Let $r = a + bn_{v_Y} \in \mathbb{Z}/n\mathbb{Z}$, where $0 \leq a \leq n_{v_Y} - 1$ and $0 \leq b \leq m_{v_Y} - 1$. Then we have that

$$r \cdot \alpha_{l_{e_X},0} = a \cdot \alpha_{l_{e_X},b}$$

if $0 \leq a \leq n_{v_Y} - 1$ and $0 \leq b \leq m_{v_Y} - 2$, and that

$$r \cdot \alpha_{l_{e_X},0} = (a + (m_{v_Y} - 1)n_{v_Y}) \cdot \alpha_{l_{e_X},0} = -\sum_{i=0}^{m_{v_Y}-2} a \cdot \alpha_{l_{e_X},i}$$

if $0 \leq a \leq n_{v_Y} - 1$ and $b = m_{v_Y} - 1$ In particular, we have

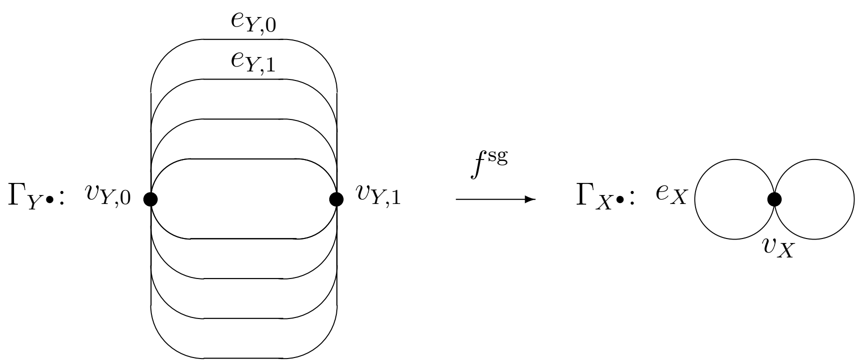
$$1 \cdot ((n_{v_Y} - 1) \cdot \alpha_{l_{e_Y,m_{v_Y}-2}}) = (0 + (m_{v_Y} - 1)n_{v_Y}) \cdot \alpha_{l_{e_X},0}$$

$$= -\sum_{i=0}^{m_{v_Y}-2} 0 \cdot \alpha_{l_{e_X},i} = -\sum_{i=0}^{m_{v_Y}-2} \alpha_{l_{e_X},i}.$$

Then we see that the eigenspace of $M_{T_{Y,e_X}}$ associated with eigenvalue ζ is an one dimensional subspace. Thus, we obtain that

$$\dim_k(M_{T_{Y,e_X}}(1)) = 1,$$

where $M_{T_{Y,e_X}}(1)$ denotes the subspace on which $\zeta \in \mu_n$ acts as the ζ-multiplication. For example, if $n = 4$, $m_{v_Y} = 2$ and $\#E^{\mathrm{tr,et}} = 2$, we have the following, where $v_{Y,1} = 1 \cdot v_{Y,0}$.



Let $m \overset{\mathrm{def}}{=} \#E^{\mathrm{tr,et}}$ and put $E^{\mathrm{tr,et}} \overset{\mathrm{def}}{=} \{e_{X,1}, \ldots, e_{X,m}\}$. Let $e_Y^i \in T_{Y,e_{X,i}}$, $i \in \{1, \ldots, m\}$, be an arbitrary element. Then

$$l_i \overset{\mathrm{def}}{=} v_{Y,0} e_Y^i v_{Y,1} e_Y^{i+1} v_{Y,0}, \ i \in \{1, \ldots, m-1\}$$

can be regarded as a minimal loop of $\Gamma_{Y^\bullet}$. Let

$$\alpha_i \overset{\mathrm{def}}{=} \sum_{j=0}^{m_{v_Y}-1} jn_{v_Y} \cdot \alpha_{l_i}.$$

Note that the decomposition group of α_i is D_{v_Y}. Then the set of vector

$$\{j \cdot \alpha_i\}_{i \in \{1,\ldots,m-1\}, j \in \{0,\ldots,n_{v_Y}-1\}} \subseteq M_{\Gamma_{Y^\bullet}}$$

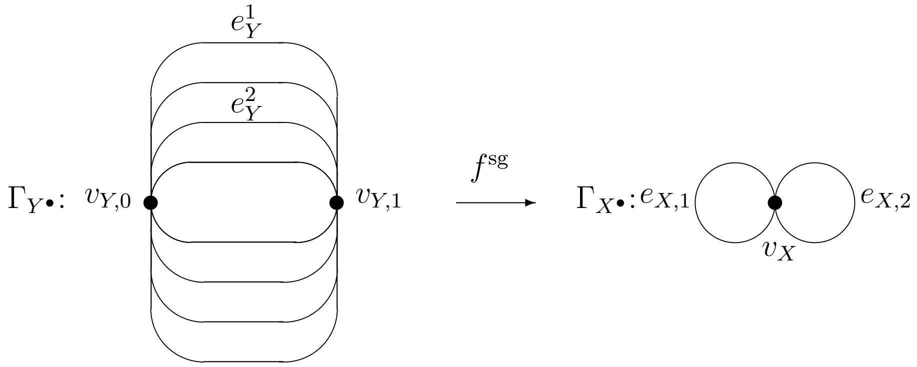
is to be linearly independent. We denote by $M_i \subseteq M_{\Gamma_{Y^\bullet}}$, $i \in \{1, \ldots, m-1\}$, the subspace spanned by $\{j \cdot \alpha_i\}_{j \in \{0,\ldots,n_{v_Y}-1\}}$. Then M_i is a $k[(\mathbb{Z}/n\mathbb{Z})/D_{v_Y}]$-module. Thus, we obtain that

$$M_i \subseteq \bigoplus_{0 \leq j < n_{v_Y}-1} M_{\Gamma_{Y^\bullet}}(jm_{v_Y}),\ i \in \{1, \ldots, m-1\}.$$

Moreover, we see that the set of vectors

$$\{j \cdot \alpha_i\}_{i \in \{1,\ldots,m-1\}, j \in \{0,\ldots,n_{v_Y}-1\}} \cup \{j \cdot \alpha_{l_{e_X,i}}\}_{i \in \{0,\ldots,m_{v_Y}-2\}, j \in \{0,\ldots,n_{v_Y}-1\}, e_X \in E^{\mathrm{tr,et}}} \subseteq M_{\Gamma_{Y^\bullet}}$$

is to be linearly independent. For example, if $n = 4$, $m_{v_Y} = 2$ and $\#E^{\mathrm{tr,et}} = 2$, we have the following, where $v_{Y,1} = 1 \cdot v_{Y,0}$.



Let $e_Y \in T_{Y,e_X}$ for some $e_X \in E^{\mathrm{tr,et}}$. Then we see that $j \cdot e_Y$, $j \in \{0, \ldots, n_{v_Y} - 2\}$, abuts to $v_{Y,j}$ and $v_{Y,1+j}$, and that $(n_{v_Y} - 1) \cdot e_Y$ abuts to v_{Y,n_Y-1} and $v_{Y,0}$. Thus, we have that

$$l_Y \stackrel{\mathrm{def}}{=} v_{Y,0}(0 \cdot e_Y)v_{Y,1} \cdots v_{Y,n_Y-1}((n_{v_Y} - 1) \cdot e_Y)v_{Y,0}$$

can be regarded as a minimal loop of $\Gamma_{Y^\bullet}$. We put

$$\alpha_Y \stackrel{\mathrm{def}}{=} \sum_{j=0}^{m_{v_Y}-1} jn_{v_Y} \cdot \alpha_{l_Y} \in M_{\Gamma_{Y^\bullet}}.$$

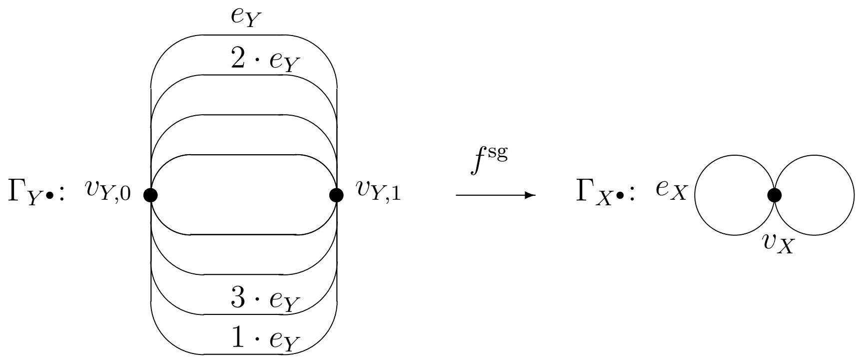
We see that the decomposition group of α_Y is $\mathbb{Z}/n\mathbb{Z}$. Indeed, note that α_Y corresponds to the loop

$$\pi \stackrel{\mathrm{def}}{=} l_Y(n_{v_Y} \cdot l_Y) \cdots ((m_{v_Y} - 1)n_{v_Y} \cdot l_Y).$$

Let e be an arbitrary closed edge contained in π. Then there exists $r \in \mathbb{Z}/n_{v_Y}\mathbb{Z}$ such that e abuts to $v_{Y,r}$ and $v_{Y,r+1}$. Thus, e can be regraded as an oriented edge induced by the oriented loop π. The starting of e is $v_{Y,r}$ (resp. $v_{Y,n_{v_Y}-1}$), and the ending of e is $v_{Y,r+1}$ if $0 \leq r \leq n_Y - 2$ (resp. $v_{Y,1}$ if $r = n_Y - 1$). Consider the action of $1 \in \mathbb{Z}/n\mathbb{Z}$ on e. We see that $1 \cdot e$ is an oriented edge contained in π. Moreover, the starting of $1 \cdot e$ is $v_{Y,r+1}$, and the ending of $1 \cdot e$ is $v_{Y,r+2}$, where $r+1, r+2 \in \mathbb{Z}/n_{v_Y}\mathbb{Z}$. This means that $1 \cdot \pi = \pi$. Thus, the stabilizer of α_Y is $\mathbb{Z}/n\mathbb{Z}$. Write $M_Y \subseteq M_{\Gamma_{Y^\bullet}}$ for the subspace spanned by α_Y. Then we have that $M_Y \subseteq M_{\Gamma_{Y^\bullet}}(0)$, and that

$$\{j \cdot \alpha_i\}_{i \in \{1,\ldots,m-1\}, j \in \{0,\ldots,n_{v_Y}-1\}} \cup \{j \cdot \alpha_{l_{e_X,i}}\}_{i \in \{1,\ldots,m_{v_Y}-1\}, j \in \{0,\ldots,n_{v_Y}-1\}, e_X \in E^{\mathrm{tr,et}}} \cup \{\alpha_Y\}$$

is to be linearly independent. For example, if $n = 4$, $m_{v_Y} = 2$ and $\#E^{\mathrm{tr,et}} = 2$, we have the following, where $v_{Y,1} = 1 \cdot v_{Y,0}$.



Let $l \subseteq \Gamma_{Y^\bullet}$ be an arbitrary minimal loop of $\Gamma_{Y^\bullet}$. Let $V_l \subseteq M_{\Gamma_{Y^\bullet}}$ be the subspace spanned by $\{j \cdot v_l\}_{j \in \mathbb{Z}/n\mathbb{Z}}$. Then we have that

$$V_l \subseteq (\bigoplus_{e_X \in E^{\mathrm{tr,et}}} M_{T_{Y,e_X}}) \oplus (\bigoplus_{e_X \in E^{\mathrm{lp,et}}} M_{e_X}) \oplus (\bigoplus_{i=1}^{m} M_i) \oplus M_Y$$

if there exists a closed edge contained in l whose decomposition group is trivial, and that

$$V_l \subseteq \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(iq)$$

for some $1 < q \leq n$ if all the decomposition groups of closed edges contained in l are not trivial. Thus, we obtain

$$M_{\Gamma_{Y^\bullet}}(1) = (\bigoplus_{e_X \in E^{\mathrm{tr,et}}} M_{T_{Y,e_X}}(1)) \oplus (\bigoplus_{e_X \in E^{\mathrm{lp,et}}} M_{e_X}(1)).$$

Thus, we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#E^{\mathrm{tr,et}} + \#E^{\mathrm{lp,et}} = \#\mathscr{N}_X^{\mathrm{et}}$$

when $E^{\mathrm{tr}} \neq \emptyset$ and $\#v(\Gamma_{Y^\bullet}) \neq n\#v(\Gamma_{X^\bullet})$. This completes the proof of the proposition. □

Lemma 6.2. *We maintain the notation introduced above. Suppose that the set of irreducible components of $X^\bullet$ is $\{X_1, X_2\}$, and that X_1 and X_2 are non-singular. Write v_1 and v_2 for the vertices of $\Gamma_{X^\bullet}$ corresponding to X_1 and X_2, respectively. Let $f_i^\bullet \overset{\mathrm{def}}{=} \widetilde{f}_{v_i}^\bullet : Y_i^\bullet \overset{\mathrm{def}}{=} \widetilde{Y}_{v_i}^\bullet \to X_i^\bullet \overset{\mathrm{def}}{=} \widetilde{X}_{v_i}^\bullet$, $i \in \{1, 2\}$, be the Galois multi-admissible covering over k induced by $f^\bullet$. Then the following statements hold:*

(i) Suppose that there exists $i \in \{1, 2\}$ such that $\#v(\Gamma_{Y_i^\bullet}) = n\#v(\Gamma_{X_i^\bullet})$. Then we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#\mathscr{N}_X^{\mathrm{et}} - 1.$$

(ii) Suppose that $\#(X_1 \cap X_2) = 1$ (i.e., $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) = 1$). Then we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) =$$

$$\begin{cases} \#\mathscr{N}_X^{\mathrm{et}} - 1, & \textit{if there exists } i \in \{1,2\} \textit{ such that } \#v(\Gamma_{Y_i^\bullet}) = n\#v(\Gamma_{X_i^\bullet}), \\ \#\mathscr{N}_X^{\mathrm{et}}, & \textit{if for each } i \in \{1,2\},\ \#v(\Gamma_{Y_i^\bullet}) \neq n\#v(\Gamma_{X_i^\bullet}). \end{cases}$$

Proof. (i) Suppose that either $\#v(\Gamma_{Y_1^\bullet}) = n\#v(\Gamma_{X_1^\bullet})$ and $\#v(\Gamma_{Y_2^\bullet}) = \#v(\Gamma_{X_2^\bullet})$ or $\#v(\Gamma_{Y_1^\bullet}) = \#v(\Gamma_{X_1^\bullet})$ and $\#v(\Gamma_{Y_1^\bullet}) = n\#v(\Gamma_{X_1^\bullet})$ holds. Then either $f_1^\bullet$ or $f_2^\bullet$ is a trivial Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then we see that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = r_X - r_{X_1} - r_{X_2} = r_X - 1 = \#(X_1 \cap X_2) - 1 = \#\mathscr{N}_X^{\text{et}} - 1,$$

where r_{X_i}, $i \in \{1,2\}$, denotes the Betti number of the dual semi-graph of $X_i^\bullet$.

Suppose that $\#v(\Gamma_{Y_1^\bullet}) = n\#v(\Gamma_{X_1^\bullet})$ and $\#v(\Gamma_{Y_2^\bullet}) = n\#v(\Gamma_{X_2^\bullet})$. Then f^{sg} is a Galois topological covering of $\Gamma_{X^\bullet}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then similar arguments to the arguments given in the proof of Proposition 6.1 in the case of $\#v(\Gamma_{Y^\bullet}) = n\#v(\Gamma_{X^\bullet})$ imply that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#\mathscr{N}_X^{\text{et}} - 1.$$

Thus, we obtain that $\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \mathscr{N}_X^{\text{et}} - 1$ when there exists $i \in \{1,2\}$ such that $\#v(\Gamma_{Y_i^\bullet}) = n\#v(\Gamma_{X_i^\bullet})$. This completes the proof of (i).

(ii) Suppose that there exists $i \in \{1,2\}$ such that $\#v(\Gamma_{Y_i^\bullet}) = n\#v(\Gamma_{X_i^\bullet})$. Then (ii) follows immediately from (i).

Suppose that $\#v(\Gamma_{Y_i^\bullet}) \neq n\#v(\Gamma_{X_i^\bullet})$ for all $i \in \{1,2\}$. Write v_1 and v_2 for the vertices of $\Gamma_{X^\bullet}$ corresponding to X_1 and X_2, respectively. Let $w_1 \in (f^{\text{sg}})^{-1}(v_1)$ and $w_2 \in (f^{\text{sg}})^{-1}(v_2)$ be arbitrary vertices of $\Gamma_{Y^\bullet}$. We denote by $D_1 \subseteq \mathbb{Z}/n\mathbb{Z}$ and $D_2 \subseteq \mathbb{Z}/n\mathbb{Z}$ the decomposition groups of w_1 and w_2, respectively, which do not depend on the choices of w_1 and w_2. Since $\#(X_1 \cap X_2) = 1$, Galois topological coverings of $\Gamma_{X^\bullet}$ with cyclic Galois groups do not exist. This means that either $D_1 = \mathbb{Z}/n\mathbb{Z}$ or $D_2 = \mathbb{Z}/n\mathbb{Z}$ holds. Without loss of generality, we may assume that $D_1 = \mathbb{Z}/n\mathbb{Z} \supseteq D_2$. We put $m_2 \overset{\text{def}}{=} \#D_2$ and $n_2 \overset{\text{def}}{=} n/m_2$.

Let e_X be the unique closed edge of $\Gamma_{X^\bullet}$, $(f^{\text{sg}})^{-1}(v_1) \overset{\text{def}}{=} \{w_1\}$, $w_{2,0} \in (f^{\text{sg}})^{-1}(v_2)$, and $e_{Y,0} \in (f^{\text{sg}})^{-1}(e_X)$ such that $e_{Y,0}$ abuts to w_1 and $w_{2,0}$. Let $D_{e_{Y,0}}$ be the decomposition group of $e_{Y,0}$, $m_{e_{Y,0}} = \#D_{e_{Y,0}}$ and $n_{e_{Y,0}} \overset{\text{def}}{=} n/m_{e_{Y,0}}$. Then we see that

$$M_{\Gamma_{Y^\bullet}} = \bigoplus_{0 \leq i \leq n_{e_{Y,0}}-1} M_{\Gamma_{Y^\bullet}}(im_{e_{Y,0}}).$$

Then we obtain that $\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = 0$ if $\#\mathscr{N}_X^{\text{et}} = 0$.

Suppose that $\#\mathscr{N}_X^{\text{et}} = 1$. Note that $w_{2,0}$ admits a natural action of $\mathbb{Z}/n\mathbb{Z}$ such that the stabilizer is D_2. Moreover, $e_{Y,0}$ admits a natural action of $\mathbb{Z}/n\mathbb{Z}$, and $rn_2 \cdot e_{Y,0}$, $r \in \{0, \ldots, m_2 - 1\}$, abuts to w_1 and $w_{2,0}$.

Then

$$l_{e_X} \overset{\text{def}}{=} w_1 e_{Y,0} w_{2,0} (n_2 \cdot e_{Y,0}) w_1$$

can be regarded as a minimal loop of $\Gamma_{Y^\bullet}$. Note that

$$\{j \cdot \alpha_{l_{e_X}}\}_{j \in \{0, \ldots, (m_2-1)n_2-1\}}$$

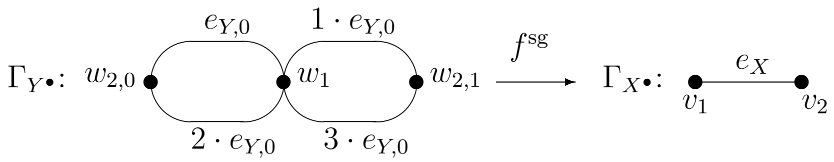
is a basis of $M_{\Gamma_{Y^\bullet}}$. Moreover, the action of $\mathbb{Z}/n\mathbb{Z}$ on l_{e_X} implies that

$$((m_2-1)n_2 + i) \cdot \alpha_{l_{e_X}} = -\sum_{j=0}^{m_2-1} (jn_2 + i) \cdot \alpha_{l_{e_X}}, \ i \in \{0, \ldots, n_2 - 1\}.$$

In particular, we have that

$$1 \cdot (((m_2 - 1)n_2 - 1) \cdot \alpha_{l_{e_X}})) = ((m_2 - 1)n_2) \cdot \alpha_{l_{e_X}} = -\sum_{j=0}^{m_2-1} (jn_2) \cdot \alpha_{l_{e_X}}.$$

For example, if $n = 4$ and $m_{v_Y} = 2$, we have the following



Then we see that the eigenspace of $M_{\Gamma_{Y^\bullet}}$ associated with eigenvalue ζ is an one dimensional subspace. Thus, we obtain that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = 1 = \#\mathscr{N}_X^{\mathrm{et}}.$$

This completes the proof of (ii). $\square$

Proposition 6.3. *We maintain the notation introduced in Lemma 6.2. Suppose that the set of irreducible components of $X^\bullet$ is $\{X_1, X_2\}$, that X_1 and X_2 are non-singular. Then we have*

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) =$$

$$\begin{cases} \#\mathscr{N}_X^{\mathrm{et}} - 1, & \textit{if there exists } i \in \{1,2\} \textit{ such that } \#v(\Gamma_{Y_i^\bullet}) = n\#v(\Gamma_{X_i^\bullet}), \\ \#\mathscr{N}_X^{\mathrm{et}}, & \textit{if for each } i \in \{1,2\},\ \#v(\Gamma_{Y_i^\bullet}) \neq n\#v(\Gamma_{X_i^\bullet}). \end{cases}$$

Proof. Suppose that there exists $i \in \{1,2\}$ such that $\#v(\Gamma_{Y_i^\bullet}) = n\#v(\Gamma_{X_i^\bullet})$. Then the proposition follows from Lemma 6.2 (i). To verify the proposition, we may assume that $\#v(\Gamma_{Y_i^\bullet}) \neq n\#v(\Gamma_{X_i^\bullet})$ for all $i \in \{1,2\}$. Moreover, if $\#(X_1 \cap X_2) = 1$, then the proposition follows from Lemma 6.2 (ii). Thus, we may assume that $\#(X_1 \cap X_2) \geq 2$.

First, let us construct two Galois admissible coverings associated to $f^\bullet : Y^\bullet \to X^\bullet$. Let R be a complete discrete valuation ring with residue field k, K the quotient field of R, $\overline{K}$ an algebraic closure of K, $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ an arbitrary closed edge, and x_e the node of X corresponding to e. By deforming X along x_e, we obtain a pointed stable curve $\mathcal{X}$ over R whose special fiber is $X^\bullet$, and whose generic fiber $X_K^\bullet$ is an irreducible pointed stable curve over K such that $\#e^{\mathrm{cl}}(\Gamma_{X_K^\bullet}) = \#e^{\mathrm{lp}}(\Gamma_{X_K^\bullet}) = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - 1$, where $\Gamma_{X_K^\bullet}$ denotes the dual semi-graph of $X_K^\bullet$. Moreover, since the specialization homomorphism of admissible fundamental groups of generic fiber of special fiber is a surjection, by replacing R by a finite extension of R, $f^\bullet$ can be lifted to a finite morphism

$$f_K^\bullet : Y_K^\bullet \to X_K^\bullet$$

over K such that

$$f_{\backslash e}^\bullet \overset{\mathrm{def}}{=} f_K^\bullet \times_K \overline{K} : Y_{\backslash e}^\bullet \overset{\mathrm{def}}{=} Y_K^\bullet \times_K \overline{K} \to X_{\backslash e}^\bullet \overset{\mathrm{def}}{=} X_K^\bullet \times_K \overline{K}$$

is a Galois admissible covering over $\overline{K}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. We write $\Gamma_{Y^\bullet_{\backslash e}}$ for the dual semi-graph of $Y^\bullet_{\backslash e}$ and denote by

$$M_{\Gamma_{Y^\bullet_{\backslash e}}} \overset{\text{def}}{=} H^1(\Gamma_{Y^\bullet_{\backslash e}}, \mathbb{F}_p) \otimes k.$$

Then $M_{\Gamma_{Y^\bullet_{\backslash e}}}$ is a $k[\mu_n]$-module and admits the following canonical decomposition

$$M_{\Gamma_{Y^\bullet_{\backslash e}}} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet_{\backslash e}}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y^\bullet_{\backslash e}}}(j)$ as the ζ^j-multiplication.

On the other hand, let $\mathrm{norm}_e : X_e \to X$ be the normalization morphism of X over the nodes corresponding the closed edges contained in $e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \setminus \{e\}$. Then we obtain a pointed stable curve

$$X^\bullet_e = (X_e, D_{X_e} \overset{\text{def}}{=} \{\mathrm{norm}^{-1}(x_{e'})\}_{e' \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \setminus \{e\}})$$

over k. Note that X_e has two non-singular irreducible components X_1 and X_2, and that $X_1 \cap X_2 = \{x_e\}$ in X_e. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f^\bullet_e : Y^\bullet_e \to X^\bullet_e$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. We write $\Gamma_{Y^\bullet_e}$ for the dual semi-graph of $Y^\bullet_e$ and denote by

$$M_{\Gamma_{Y^\bullet_e}} \overset{\text{def}}{=} H^1(\Gamma_{Y^\bullet_e}, \mathbb{F}_p) \otimes k.$$

Then $M_{\Gamma_{Y^\bullet_e}}$ is a $k[\mu_n]$-module and admits the following canonical decomposition

$$M_{\Gamma_{Y^\bullet_e}} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet_e}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y^\bullet_e}}(j)$ as the ζ^j-multiplication.

Write $\mathscr{N}^{\mathrm{et}}_{X_{\backslash e}}$ and $\mathscr{N}^{\mathrm{et}}_{X_e}$ for the sets of nodes of $X_{\backslash e}$ and X_e, respectively, over which $f_{\backslash e}$ and f_e are étale. Then we see immediately that

$$\#\mathscr{N}^{\mathrm{et}}_{X_{\backslash e}} + \#\mathscr{N}^{\mathrm{et}}_{X_e} = \#\mathscr{N}^{\mathrm{et}}_X.$$

The constructions of $\Gamma_{Y^\bullet_{\backslash e}}$ and $\Gamma_{Y^\bullet_e}$ imply that

$$M_{\Gamma_{Y^\bullet}} = M_{\Gamma_{Y^\bullet_{\backslash e}}} \oplus M_{\Gamma_{Y^\bullet_e}}$$

as $k[\mu_n]$-modules. Then we obtain

$$M_{\Gamma_{Y^\bullet}}(1) = M_{\Gamma_{Y^\bullet_{\backslash e}}}(1) \oplus M_{\Gamma_{Y^\bullet_e}}(1).$$

Then Proposition 6.1 and Lemma 6.2 imply that

$$\begin{aligned} \dim_k(M_{\Gamma_{Y^\bullet}}(1)) &= \dim_k(M_{\Gamma_{Y^\bullet_{\backslash e}}}(1)) + \dim_k(M_{\Gamma_{Y^\bullet_e}}(1)) \\ &= \#\mathscr{N}^{\mathrm{et}}_{X_{\backslash e}} + \#\mathscr{N}^{\mathrm{et}}_{X_e} = \#\mathscr{N}^{\mathrm{et}}_X. \end{aligned}$$

This completes the proof of the proposition. □

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