MAXIMUM GENERALIZED HASSE-WITT INVARIANTS AND THEIR APPLICATIONS TO ANABELIAN GEOMETRY

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ABSTRACT. Let $X^\bullet = (X, D_X)$ be a pointed stable curve of topological type (g_X, n_X) over an algebraically closed field of characteristic $p > 0$.

Under certain assumptions, we prove that, if $X^\bullet$ is *component-generic*, then the first generalized Hasse-Witt invariant of *every* prime-to-p cyclic admissible covering of $X^\bullet$ attains maximum. This result generalizes a result of S. Nakajima concerning the ordinariness of prime-to-p cyclic étale coverings of smooth projective generic curves to the case of (possibly ramified) admissible coverings of (possibly singular) pointed stable curves.

Moreover, we prove that, if $X^\bullet$ is an *arbitrary* pointed stable curve, then there *exists* a prime-to-p cyclic admissible covering of $X^\bullet$ whose first generalized Hasse-Witt invariant attains maximum. This result generalizes a result of M. Raynaud concerning the new-ordinariness of prime-to-p cyclic étale coverings of smooth projective curves to the case of (possibly ramified) admissible coverings of (possibly singular) pointed stable curves.

As applications, we obtain *an anabelian formula for* (g_X, n_X), and prove that the field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from open continuous homomorphisms of admissible fundamental groups. Those results generalize A. Tamagawa's results concerning an anabelian formula for topological types and reconstructions of field structures associated to inertia subgroups of marked points of smooth pointed stable curves to the case of *arbitrary* pointed stable curves.

Keywords: pointed stable curve, admissible covering, generalized Hasse-Witt invariant, Raynaud-Tamagawa theta divisor, admissible fundamental group, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14G32.

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1. INTRODUCTION

Let $X^\bullet = (X, D_X)$ be a pointed stable curve over an algebraically closed field k, where X denotes the underlying curve, and D_X denotes the (finite) set of marked points. Write g_X for the arithmetic genus of X and n_X for the cardinality $\#(D_X)$ of D_X. We call (g_X, n_X) the topological type (or type for short) of $X^\bullet$. By choosing a suitable base point of $X^\bullet$, we have the admissible fundamental group $\Pi_{X^\bullet}$ of $X^\bullet$. The admissible fundamental groups of pointed stable curves are natural generalizations of the tame fundamental groups of smooth pointed stable curves. In particular, $\Pi_{X^\bullet}$ is isomorphic to the tame fundamental group of $X^\bullet$ if $X^\bullet$ is smooth over k.

Suppose that the characteristic $\mathrm{char}(k)$ of k is 0 (resp. $p > 0$). Then the structure of $\Pi_{X^\bullet}$ (resp. the maximal prime-to-p quotient $\Pi_{X^\bullet}^{p'}$ of $\Pi_{X^\bullet}$) is well-known, and it is isomorphic to the profinite completion (resp. prime-to-p completion) of the topological fundamental group of a Riemann surface of type (g_X, n_X) (2.1.6). In particular, $\Pi_{X^\bullet}$ (resp. $\Pi_{X^\bullet}^{p'}$) is a free profinite group with $2g_X + n_X - 1$ generators if $n_X > 0$. We see that the type (g_X, n_X) cannot be determined group-theoretically from the isomorphism class (as a profinite group) of $\Pi_{X^\bullet}$ if $\mathrm{char}(k) = 0$ (resp. $\Pi_{X^\bullet}^{p'}$ if $\mathrm{char}(k) = p$).

1.1. Fundamental groups of curves in positive characteristic.

1.1.1.   If $\mathrm{char}(k) = p > 0$, the situation is quite different from that in characteristic 0, and the structure of $\Pi_{X^\bullet}$ is no longer known. In the remainder of the introduction, we assume $\mathrm{char}(k) = p > 0$. In the case of positive characteristic, the admissible fundamental group $\Pi_{X^\bullet}$ is very mysterious. After M. Raynaud and D. Harbater proved Abhyankar's

conjecture, Harbater asked whether or not the geometric information of a curve over in positive characteristic can be carried out from its geometric fundamental groups ([H]). Some developments of F. Pop-M. Saïdi ([PS]), M. Raynaud ([R2]), A. Tamagawa ([T1], [T2], [T3]), and the author of the present paper ([Y1], [Y4]) showed evidence for very strong *anabelian* phenomena for curves over *algebraically closed fields of characteristic* $p > 0$. In this situation, the Galois group of the base field is trivial, and the arithmetic fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field. These kinds of anabelian phenomena go beyond Grothendieck's anabelian geometry (since no Galois actions), and show that the admissible (or tame) fundamental group of a pointed stable curve over an algebraically closed field of characteristic p must encode "*moduli*" of the curve. This is the reason why we do not have an explicit description of the admissible (or tame) fundamental group of any pointed stable curve in positive characteristic.

1.1.2. Furthermore, the theory developed in [T2] and [Y3] implies that the isomorphism class of $X^\bullet$ (as a scheme) can possibly be determined by not only the isomorphism class of $\Pi_{X^\bullet}$ but also the isomorphism class of the maximal pro-solvable quotient of $\Pi_{X^\bullet}$. On the other hand, since all the admissible coverings (see Definition 2.3) of $X^\bullet$ can be lifted to characteristic 0 ([V, Théorème 2.2 (c)]), we obtain that $\Pi_{X^\bullet}$ is topologically finitely generated. This implies that the isomorphism class of $\Pi_{X^\bullet}$ is determined by the set of finite quotients of $\Pi_{X^\bullet}$ ([FJ, Proposition 16.10.6]). Then to understand the anabelian phenomena of curves in positive characteristic, we may ask the following question: *Which finite solvable groups can appear as quotients of $\Pi_{X^\bullet}$?*

1.1.3. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup and $X_H^\bullet = (X_H, D_{X_H})$ the pointed stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H. We have an important invariant σ_{X_H} associated to $X_H^\bullet$ (or H) which is called *p-rank* (or *Hasse-Witt invariant*, see 2.2.2). Roughly speaking, σ_{X_H} controls the finite quotients of $\Pi_{X^\bullet}$ which are extensions of the group $\Pi_{X^\bullet}/H$ by p-groups.

Since the structures of maximal prime-to-p quotients of admissible fundamental groups are known, to find all the solvable quotients of $\Pi_{X^\bullet}$, we need to compute the p-rank σ_{X_H} when $\Pi_{X^\bullet}/H$ is *abelian*. If $\Pi_{X^\bullet}/H$ is a p-group, then σ_{X_H} can be computed by using the Deuring-Shafarevich formula ([C], [Su]). If $\Pi_{X^\bullet}/H$ is not a p-group, the situation of σ_{X_H} is very complicated. Moreover, the Deuring-Shafarevich formula implies that, to compute σ_{X_H}, we only need to assume that $\Pi_{X^\bullet}/H$ is a prime-to-p group (i.e., the order of $\Pi_{X^\bullet}/H$ is prime to p).

1.2. Generalized Hasse-Witt invariants for generic curves.

1.2.1. *Nakajima, Zhang, and Ozman-Pries' results.* Firstly, let us consider the case of generic curves. Suppose that $n_X = 0$, and that $X^\bullet$ is *smooth* over k. If $X^\bullet$ is a curve corresponding to a geometric generic point of the moduli space (i.e., a geometric generic curve), S. Nakajima ([N, Proposition 4]) proved that, if $\Pi_{X^\bullet}/H$ is a prime-to-p cyclic group, then σ_{X_H} attains the maximum g_{X_H} (i.e., $X_H^\bullet$ is *ordinary*). Moreover, B. Zhang ([Z]) generalized Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is an arbitrary abelian group. Recently, E. Ozman and R. Pries ([OP]) generalized Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is a cyclic group with a prime order distinct from p, and where $X^\bullet$ is a curve corresponding to a geometric point over a generic point of p-rank stratum of the moduli space.

Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. In other words, the results of Nakajima, Zhang, and Ozman-Pries say that the *"first" generalized Hasse-Witt invariant* (see [N] or 2.2.4 of the present paper) of *every* Galois *étale* covering of $X^\bullet$ with Galois group $\mathbb{Z}/m\mathbb{Z}$ attains the maximum $g_X - 1$.

1.2.2. Let $\mathcal{M}_{g,0,\mathbb{Z}}$ be the moduli stack over $\operatorname{Spec} \mathbb{Z}$ parameterizing smooth pointed stable curves of type $(g, 0)$ and $M_{g,0,k}$ the coarse moduli space of $\mathcal{M}_{g,0,\mathbb{Z}} \times_{\mathbb{Z}} k$. Let $q \in M_{g,0,k}$, $X_q^\bullet$ a smooth pointed stable curve determined by a geometric point over q (i.e., a morphism $\operatorname{Spec} k' \to M_{g,0,k}$ whose image is q, where k' is an algebraically closed field containing k), and $m \in \mathbb{N}$ an arbitrary positive natural number prime to p . We denote by $U_m \subseteq M_{g,0,k}$ the subset of points of $M_{g,0,k}$ such that for $q \in U_m$, the "first" generalized Hasse-Witt invariant of every Galois étale covering of $X_q^\bullet$ with Galois group $\mathbb{Z}/m\mathbb{Z}$ attains the maximum. By applying [N, Proposition 4], Nakajima proved that U_m is a non-empty Zariski *open* subset of $M_{g,0,k}$ ([N, Theorem 2]). Note that we have $U_m \neq M_{g,0,k}$ in general (see 4.6.3).

1.2.3. The first main result of the present paper is as follows (see Theorem 4.13 for a more precise statement):

Theorem 1.1. *Let $X^\bullet$ be a component-generic pointed stable curve (see 2.1.2 for the definition) over an algebraically closed field k of characteristic $p > 0$. Then the "first" generalized Hasse-Witt invariant of every prime-to-p cyclic admissible covering of $X^\bullet$ attains maximum under certain assumptions.*

If $n_X = 0$ and $X^\bullet$ is smooth over k, then Theorem 1.1 is equivalent to [N, Proposition 4]. Thus, Theorem 1.1 generalizes Nakajima's result to the case of (possibly ramified) admissible coverings of (possibly singular) pointed stable curves. Moreover, by applying Theorem 1.1, we generalize [N, Theorem 2] to the case of tame coverings (see Proposition 4.15 (i)).

1.3. **Generalized Hasse-Witt invariants for arbitrary curves.** Next, let us consider the case where $X^\bullet$ is an *arbitrary* pointed stable curve. Let m be a positive natural number prime to p, and let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois admissible covering over k with Galois group $\mathbb{Z}/m\mathbb{Z}$ and D the ramification divisor associated to $f^\bullet$. Note that the degree $\deg(D)$ of D is divisible by m, and that $0 \leq \deg(D) \leq (n_X - 1)m$ if $n_X \neq 0$.

1.3.1. *Raynaud and Tamagawa's results.* If $X^\bullet$ is not generic, the computation of σ_{X_H} is very difficult in general. Suppose that $X^\bullet$ is *smooth* over k, and that $n_X = 0$. Raynaud ([R1]) developed his theory of theta divisors and proved that, if $m >> 0$ is a natural number prime to p, then there *exists* a Galois *étale* covering $f^\bullet$ of $X^\bullet$ with Galois group $\mathbb{Z}/m\mathbb{Z}$ whose "first" generalized Hasse-Witt invariant attains the maximum $g_X - 1$ ([R1, Théorème 4.3.1]). Moreover, as a consequence, Raynaud obtained that $\Pi_{X^\bullet}$ is not a prime-to-p profinite group. This is the first deep result concerning the global structures of étale fundamental groups of curves over algebraically closed fields of characteristic $p > 0$.

Suppose that $X^\bullet$ is *smooth* over k, and that $n_X \geq 0$. The computations of generalized Hasse-invariants of admissible coverings of $X^\bullet$ (i.e., tame coverings of $X^\bullet$) are much more difficult than the case of $n_X = 0$. Tamagawa observed that Raynaud's theory of theta divisors can be generalized to the case of tame coverings, and he established a tamely ramified version of the theory of Raynaud's theta divisors. By applying the theory of theta divisors, Tamagawa proved that, if $n_X \neq 0$ and $m >> 0$, then there *exists* a Galois

admissible covering (i.e., Galois tame covering) $f^\bullet$ of $X^\bullet$ with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that $\deg(D) = m$, and that the "first" generalized Hasse-Witt invariants of $f^\bullet$ is as large as possible, namely equal to g_X.

1.3.2. In the present paper, we study maximum generalized Hasse-Witt invariants for *arbitrary* (possibly singular) pointed stable curves. The second main theorem of the present paper is as follows (see Theorem 5.4 for a more precise statement):

Theorem 1.2. *We maintain the notation introduced above. Let $X^\bullet$ be an arbitrary pointed stable curve over an algebraically closed field k of characteristic $p > 0$. Suppose that $m >> 0$. Then there exists a Galois admissible covering $f^\bullet$ of $X^\bullet$ with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that $\deg(D) = (n_X - 1)m$ if $n_X \neq 0$, and that the "first" generalized Hasse-Witt invariant (see 2.2.4 for the definition) attains the maximum (see Definition 3.10 for $\gamma_{X^\bullet}^{\max}$)*

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Remark 1.2.1. If $n_X = 3$, we prove a strong version of Theorem 1.2 (see Theorem 5.7) which is the key tool for reconstructing field structures associated to marked points from $\Pi_{X^\bullet}$ under certain conditions (see Theorem 6.4 and Remark 6.4.1).

If $n_X = 0$ and $X^\bullet$ is smooth over k, then Theorem 1.2 is equivalent to [R1, Théorème 4.3.1]. Thus, Theorem 1.2 generalizes Raynaud's result to the case of (possibly ramified) admissible coverings of arbitrary (possibly singular) pointed stable curves. On the other hand, Theorem 1.2 can be regarded as an analogue of Tamagawa's result in the case of arbitrary (possibly singular) pointed stable curves when $\deg(D)$ attains maximum, namely equal to $(n_X - 1)m$.

1.4. **An anabelian formula for topological types.**
As we mentioned above, the generalized Hasse-Witt invariants can help us to analyze the structures of admissible fundamental groups in positive characteristic, and moreover, to understand the anabelian phenomena of curves in positive characteristic.

1.4.1. By applying the result explained in 1.3.1, Tamagawa obtained a group-theoretical formula for the type (g_X, n_X) by using the tame fundamental group $\Pi_{X^\bullet}$ when $X^\bullet$ is *smooth* over k (see 6.1.6). In particular, g_X and n_X are group-theoretical invariants associated to $\Pi_{X^\bullet}$ ([T2, Theorem 0.1]). This result is the main goal of [T2], which plays a key role in the theory of tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (e.g. [T2], [Y3]).

On the other hand, we mention that Tamagawa's method for finding a group-theoretical formula for topological types *is difficult to generalize* to the case of arbitrary pointed stable curves (see 6.1.7). If $X^\bullet$ is an *arbitrary* pointed stable curve over k, we ask the following question:

> *Does there exist a group-theoretical formula for (g_X, n_X) when $X^\bullet$ is an arbitrary pointed stable curve over k?*

This problem can be completely solved by applying Theorem 1.2 as explained below.

1.4.2.   Let $\overline{\mathbb{F}}_p$ be an arbitrary algebraic closure of $\mathbb{F}_p$, Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups, $\chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}_p^\times)$ such that $\chi \neq 1$, and $\Pi_\chi \subseteq \Pi$ the kernel of χ. The profinite group Π_χ admits a natural action of Π via the conjugation action. We put

$$H_{\chi,p} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p,$$

$$N_\chi \overset{\mathrm{def}}{=} \{\pi \in H_{\chi,p} \mid \tau \cdot \pi = \chi(\tau)\pi \text{ for all } \tau \in \Pi\},$$

$$\gamma_{N_\chi} \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(N_\chi),$$

where $(\tau \cdot \pi)(x) \overset{\mathrm{def}}{=} \pi(\tau^{-1} \cdot x)$ for all $x \in \Pi_\chi$. Moreover, we put

$$\gamma_\Pi^{\max} \overset{\mathrm{def}}{=} \max\{\gamma_{N_\chi} \mid \chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}_p^\times) \text{ such that } \chi \neq 1\}.$$

Since the prime number p is a group-theoretical invariant associated to Π (Lemma 6.1 (ii)), we see that $\gamma_\Pi^{\max}$ is also a group-theoretical invariant associated to Π. Moreover, we have $\gamma_\Pi^{\max} = \gamma_{X^\bullet}^{\max}$ (Lemma 6.2). Then we obtain the following formula (see also Theorem 6.3):

Theorem 1.3. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Then we have (see 6.1.3 for the definitions of group-theoretical invariants b_Π^1 and b_Π^2 associated to Π)*

$$g_X = b_\Pi^1 - \gamma_\Pi^{\max} - 1, \ \ n_X = 2\gamma_\Pi^{\max} - b_\Pi^1 - b_\Pi^2 + 3.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Remark 1.3.1. If $W^\bullet$ is a pointed stable curve of type (g_W, n_W) over an *arithmetic* field (e.g. a number field, a p-adic field, a finite field), then a group-theoretical formula for (g_W, n_W) can be deduced immediately by computing "weight" (e.g. by applying the weight monodromy conjecture or p-adic Hodge theory, see [Y2, Proposition 1.2]).

1.5. **Some further applications to anabelian geometry.** Let us explain some further applications of Theorem 1.2 (and Theorem 5.7) that motivated the theory developed in the present paper.

Theorem 1.2 and Theorem 5.7 are the main ingredients in the proof a critical step towards proving the main theorems of [Y7], [Y8]. We explain briefly as follows. Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack over $\mathrm{Spec}\,\mathbb{Z}$ parameterizing pointed stable curves of type (g, n) and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$. Moreover, we may define an equivalence relation $\sim_{fe}$ on $\overline{M}_{g,n}$ induced by Frobenius actions (roughly speaking, for any $q_1, q_2 \in \overline{M}_{g,n}$, $q_1 \sim_{fe} q_2$ if the curve corresponding to a geometric point over q_1 is a Frobenius twist of the curve corresponding to a geometric point over q_2). In [Y7], the author introduced a topological space $\overline{\Pi}_{g,n}$ whose points are isomorphism classes (as profinite groups) of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$, which we call *the moduli space of admissible fundamental groups of type (g, n)*. Moreover, the author proved that there is a natural continuous surjective homomorphism

$$\pi_{g,n}^{\mathrm{adm}} : \overline{M}_{g,n}/ \sim_{fe} \twoheadrightarrow \overline{\Pi}_{g,n},$$

and formulated the so-called *Homeomorphism Conjecture* which says that $\pi_{g,n}^{\mathrm{adm}}$ is a homeomorphism. The Homeomorphism Conjecture shows a new kind of anabelian phenomenon which *cannot be explained by using Grothendieck's original anabelian philosophy*, and which means that the moduli spaces of pointed stable curves in positive characteristic *can be reconstructed group-theoretically as topological spaces* from admissible fundamental groups of curves (see [Y7, Introduction]). Furthermore, the main theorems of [Y7], [Y8] say that the Homeomorphism Conjecture holds when $\dim(\overline{M}_{g,n}) \leq 1$ (i.e., $(g, n) = (0, 3)$, $(0, 4)$, $(1, 1)$).

1.6. **Structure of the present paper.** The present paper is organized as follows.

In Section 2, we recall some definitions and properties of admissible coverings, admissible fundamental groups, generalized Hasse-Witt invariants, and Raynaud-Tamagawa theta divisors.

In Section 3, we study the relation of generalized Hasse-Witt invariants between a pointed stable curve and the smooth pointed stable curves associated to its irreducible components. Moreover, we introduce maximum generalized Hasse-Witt invariants and prove our main results in the case of $n_X \leq 1$.

In Section 4, we study maximum generalized Hasse-Witt invariants when $X^\bullet$ is a component-generic pointed stable curve and prove the first main result.

In Section 5, we study maximum generalized Hasse-Witt invariants when $X^\bullet$ is an arbitrary pointed stable curve and prove the second main result.

In Section 6, by applying the results obtained in previous sections, we prove an anabelian formula for types of arbitrary pointed stable curves, and prove that the field structures associated to inertia subgroups of marked points can be group-theoretically reconstructed from admissible fundamental groups.

2. Preliminaries

2.1. **Admissible coverings and admissible fundamental groups.** In this subsection, we recall some definitions and results concerning admissible coverings and admissible fundamental groups.

2.1.1. We recall some notation concerning semi-graphs ([M, Section 1]). Let

$$\mathbf{G} \stackrel{\mathrm{def}}{=} (v(\mathbf{G}), e(\mathbf{G}), \zeta^{\mathbf{G}})$$

be a semi-graph. Here, $v(\mathbf{G})$, $e(\mathbf{G})$, and $\zeta^{\mathbf{G}}$ denote the set of vertices of $\mathbf{G}$, the set of edges of $\mathbf{G}$, and the coincidence map of $\mathbf{G}$, respectively.

Let $e \in e(\mathbf{G})$ be an edge. Then $e \stackrel{\mathrm{def}}{=} \{b_e^1, b_e^2\}$ is a set of cardinality 2 for each $e \in e(\mathbf{G})$. The set $e(\mathbf{G})$ consists of closed edges and open edges. If e is a closed edge, then the coincidence map $\zeta^{\mathbf{G}}(e)$ is the set of vertices to which e abuts. If e is an open edge, then $\zeta^{\mathbf{G}}(e)$ is the set which consists of the unique vertex to which e abuts and the one element set $\{v(\mathbf{G})\}$ (i.e., either $\zeta^{\mathbf{G}}(b_e^1)$ or $\zeta^{\mathbf{G}}(b_e^2)$ is not contained in $v(\mathbf{G})$).

We shall write $e^{\mathrm{op}}(\mathbf{G}) \subseteq e(\mathbf{G})$ for the set of open edges of $\mathbf{G}$ and $e^{\mathrm{cl}}(\mathbf{G}) \subseteq e(\mathbf{G})$ for the set of closed edges of $\mathbf{G}$. Note that we have

$$e(\mathbf{G}) = e^{\mathrm{op}}(\mathbf{G}) \cup e^{\mathrm{cl}}(\mathbf{G}).$$

Moreover, we denote by $e^{\mathrm{lp}}(\mathbf{G}) \subseteq e^{\mathrm{cl}}(\mathbf{G})$ the subset of closed edges such that $\#(\zeta_e^{\mathbf{G}}(e)) = 1$ for each $e \in e^{\mathrm{lp}}(\mathbf{G})$ (i.e., a closed edge which abuts to a unique vertex of $\mathbf{G}$), where "lp" means "loop". For each $e \in e(\mathbf{G})$, we denote by $v^{\mathbf{G}}(e) \stackrel{\mathrm{def}}{=} \zeta^{\mathbf{G}}(e) \cap v(\mathbf{G}) \subseteq v(\mathbf{G})$ the set of vertices of $\mathbf{G}$ to which e abuts. For each $v \in v(\mathbf{G})$, we denote by $e^{\mathbf{G}}(v) \subseteq e(\mathbf{G})$ the set of edges of $\mathbf{G}$ to which v is abutted.

We shall say that $\mathbf{G}$ is a *tree* if the Betti number $\dim_{\mathbb{Q}}(H^1(\mathbf{G}, \mathbb{Q}))$ of $\mathbf{G}$ is equal to 0, where $\mathbb{Q}$ denotes the rational number field.

Example 2.1. Let us give an example of semi-graph to explain the above definitions. We use the notation "•" and "○ with a line segment" to denote a vertex and an open edge, respectively.

Let $\mathbf{G}$ be a semi-graph as follows:
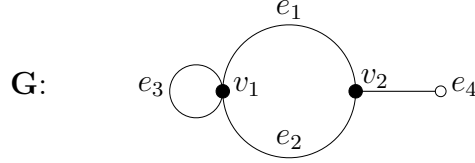
Then we see that $v(\mathbf{G}) = \{v_1, v_2\}$, $e^{\mathrm{cl}}(\mathbf{G}) = \{e_1, e_2, e_3\}$, $e^{\mathrm{op}}(\mathbf{G}) = \{e_4\}$, $\zeta^{\mathbf{G}}(e_1) = \{v_1, v_2\}$, $\zeta^{\mathbf{G}}(e_2) = \{v_1, v_2\}$, $\zeta^{\mathbf{G}}(e_3) = \{v_1\}$, and $\zeta^{\mathbf{G}}(e_4) = \{v_2, \{v(\mathbf{G})\}\}$. Moreover, we have $e^{\mathrm{lp}}(\mathbf{G}) = \{e_3\}$, $v^{\mathbf{G}}(e_1) = \{v_1, v_2\}$, $v^{\mathbf{G}}(e_2) = \{v_1, v_2\}$, $v^{\mathbf{G}}(e_3) = \{v_1\}$, $v^{\mathbf{G}}(e_4) = \{v_2\}$, $e^{\mathbf{G}}(v_1) = \{e_1, e_2, e_3\}$, and $e^{\mathbf{G}}(v_2) = \{e_1, e_2, e_4\}$.

2.1.2.   Let p be a prime number, and let

$$X^{\bullet} = (X, D_X)$$

be a pointed stable curve over an algebraically closed field k of characteristic p, where X denotes the underlying curve, D_X denotes a finite set of marked points satisfying [K, Definition 1.1 (iv)]. Write g_X for the *arithmetic* genus (or genus for short) of X and n_X for the cardinality $\#(D_X)$ of D_X. We call the pair (g_X, n_X) the *topological type* (or type for short) of $X^{\bullet}$.

Recall that the *dual semi-graph* $\Gamma_{X^{\bullet}} \stackrel{\mathrm{def}}{=} (v(\Gamma_{X^{\bullet}}), e(\Gamma_{X^{\bullet}}), \zeta^{\Gamma_{X^{\bullet}}})$ of $X^{\bullet}$ is a semi-graph associated to $X^{\bullet}$ defined as follows: (i) $v(\Gamma_{X^{\bullet}})$ is the set of irreducible components of X; (ii) $e^{\mathrm{op}}(\Gamma_{X^{\bullet}})$ is the set of marked points D_X; (iii) $e^{\mathrm{cl}}(\Gamma_{X^{\bullet}})$ is the set of nodes of X; (iv) $\zeta^{\Gamma_{X^{\bullet}}}(e)$, $e \in e^{\mathrm{op}}(\Gamma_{X^{\bullet}})$, consists of the unique irreducible component containing e and the set $\{v(\Gamma_{X^{\bullet}})\}$; (v) $\zeta^{\Gamma_{X^{\bullet}}}(e)$, $e \in e^{\mathrm{cl}}(\Gamma_{X^{\bullet}})$, consists of the irreducible components containing e.

Moreover, we write $r_X \stackrel{\mathrm{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^{\bullet}}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^{\bullet}}$.

Example 2.2. We give an example to explain dual semi-graphs of pointed stable curves. Let $X^{\bullet}$ be a pointed stable curve over k whose irreducible components are X_{v_1} and X_{v_2},

whose node is x_{e_1}, and whose marked point is $x_{e_2} \in X_{v_2}$. We use the notation "$\bullet$" and "$\circ$" to denote a node and a marked point, respectively. Then $X^\bullet$ is as follows:

$$X^\bullet:$$
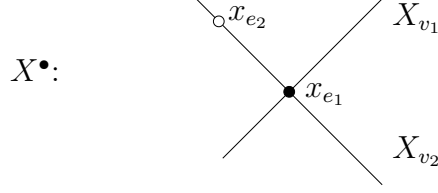
We write v_1 and v_2 for the vertices of $\Gamma_{X^\bullet}$ corresponding to X_{v_1} and X_{v_2}, respectively, e_1 for the closed edge corresponding to x_{e_1}, and e_2 for the open edge corresponding to x_{e_2}. Moreover, we use the notation "$\bullet$" and "$\circ$ with a line segment" to denote a vertex and an open edge, respectively. Then the dual semi-graph $\Gamma_{X^\bullet}$ of $X^\bullet$ is as follows:

$$\Gamma_{X^\bullet}: \qquad v_1 \;\bullet\!\!\overset{e_1}{\rule{2cm}{0.4pt}}\!\!\bullet\; v_2 \rule{1.2cm}{0.4pt}\circ\, e_2$$

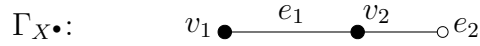2.1.3. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, x_e for the node of X corresponding to e if $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, and x_e for the marked point of X corresponding to e if $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. Note that $X^\bullet$ is allowed to have components with self-intersections in general (i.e., $e^{\mathrm{lp}}(\Gamma_{X^\bullet}) \neq \emptyset$). Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \overset{\mathrm{def}}{=} X_v \setminus X_v^{\mathrm{sing}}$, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$. We call

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \overset{\mathrm{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v))$$

the smooth pointed stable curve of type (g_v, n_v) over k associated to v (or the smooth pointed stable curve associated to v for short). Note that $\widetilde{X}_v$ is the normalization of X_v.

2.1.4. Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack parameterizing pointed stable curves of type (g,n) over $\mathrm{Spec}\,\mathbb{Z}$, $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$ in k, $\overline{\mathcal{M}}_{g,n} \overset{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. Then $X^\bullet \to \mathrm{Spec}\,k$ determines a morphism $c_X : \mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_X,n_X}$ and $\widetilde{X}_v^\bullet \to \mathrm{Spec}\,k$, $v \in v(\Gamma_{X^\bullet})$, determines a morphism $c_v : \mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_v,n_v}$. Moreover, we have a clutching morphism of moduli stacks ([K, Definition 3.8])

$$c : \prod_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v} \to \overline{\mathcal{M}}_{g_X,n_X}$$

such that $c \circ (\prod_{v \in v(\Gamma_{X^\bullet})} c_v) = c_X$. We shall say that $X^\bullet$ is a *component-generic* pointed stable curve over k if the image of

$$\prod_{v \in v(\Gamma_{X^\bullet})} c_v : \mathrm{Spec}\,k \to \prod_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v}$$

is a generic point in $\prod_{v \in v(\Gamma_{X^\bullet})} \overline{M}_{g_v,n_v}$.

2.1.5. We recall the definition of admissible coverings of pointed stable curves. Let $Y^\bullet = (Y, D_Y)$ be a pointed stable curve over k and $\Gamma_{Y^\bullet}$ the dual semi-graph of $Y^\bullet$. Let

$$f^\bullet : Y^\bullet \to X^\bullet$$

be a *surjective, generically étale, finite* morphism of pointed stable curves over k such that $f(y)$ is a smooth (resp. singular) point of X if y is a smooth (resp. singular) point of Y. Write $f : Y \to X$ for the morphism of underlying curves induced by $f^\bullet$ and $f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by $f^\bullet$. Let $v \in v(\Gamma_{X^\bullet})$ and $w \in (f^{\mathrm{sg}})^{-1}(v) \subseteq v(\Gamma_{Y^\bullet})$. Then $f^\bullet$ induces a morphism of smooth pointed stable curves

$$\widetilde{f}^\bullet_{w,v} : \widetilde{Y}^\bullet_w \to \widetilde{X}^\bullet_v$$

associated to w and v (2.1.3).

Definition 2.3. We shall say that $f^\bullet : Y^\bullet \to X^\bullet$ is a *Galois admissible covering* over k with Galois group G if the following conditions are satisfied: (i) There exists a finite group $G \subseteq \mathrm{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$; (ii) $\widetilde{f}^\bullet_{w,v}$ is a tame covering over k for each $v \in v(\Gamma_{X^\bullet})$ and each $w \in (f^{\mathrm{sg}})^{-1}(v)$; (iii) For each $y \in Y^{\mathrm{sing}}$, we write $D_y \subseteq G$ for the decomposition group of y and τ a generator of D_y. Then the local morphism between singular points induced by f is

$$\begin{array}{ccc}
\widehat{\mathcal{O}}_{X,f(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\
u & \mapsto & s^{\#(D_y)} \\
v & \mapsto & t^{\#(D_y)},
\end{array}$$

and $\tau(s) = \zeta_{\#(D_y)} s$ and $\tau(t) = \zeta_{\#(D_y)}^{-1} t$, where $\zeta_{\#(D_y)}$ is a primitive $\#(D_y)$th root of unity.

Moreover, we shall say that $f^\bullet$ is an *admissible covering* if there exists a morphism of pointed stable curves $h^\bullet : W^\bullet \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ h^\bullet : W^\bullet \to X^\bullet$ is a Galois admissible covering over k.

Let $Z^\bullet$ be a disjoint union of finitely many pointed stable curves over k. We shall say that a morphism $f^\bullet_Z : Z^\bullet \to X^\bullet$ over k is a *multi-admissible covering* if the restriction of $f^\bullet_Z$ to each connected component of $Z^\bullet$ is admissible. Moreover, we shall say that $f^\bullet_Z$ is *étale* if the underlying morphism of curves f_Z induced by $f^\bullet_Z$ is an étale morphism.

2.1.6. By choosing a base point $x \in X \setminus X^{\mathrm{sing}}$, we have the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ of $X^\bullet$ (see [Y7, 1.2.1] for the definition of admissible fundamental groups). Since we only focus on the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ in the present paper, for simplicity of notation, we omit the base point and denote by

$$\Pi_{X^\bullet}$$

the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$. Note that, by the definition of admissible coverings, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X^\bullet$ when $X^\bullet$ is smooth over k. The structure of the maximal prime-to-p quotient of $\Pi_{X^\bullet}$ is well-known, and is isomorphic to the prime-to-p completion of the following group ([V, Théorème 2.2 (c)])

$$\langle a_1, \ldots, a_{g_X}, b_1, \ldots, b_{g_X}, c_1, \ldots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle.$$

We denote by $\Pi_{X^\bullet}^{\text{ét}}$ and $\Pi_{X^\bullet}^{\text{top}}$ the étale fundamental group of the underlying curve X of $X^\bullet$ and the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, respectively. We have the following natural continuous surjective homomorphisms (for suitable choices of base points)

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \Pi_{X^\bullet}^{\text{top}}.$$

Moreover, for each $v \in v(\Gamma_{X^\bullet})$, we denote by

$$\Pi_{\widetilde{X}_v^\bullet}$$

the admissible fundamental group of $\widetilde{X}_v^\bullet$ (i.e., the tame fundamental group of the smooth pointed stable curve associated to v). Then we have a natural (outer) injective homomorphism $\Pi_{\widetilde{X}_v^\bullet} \hookrightarrow \Pi_{X^\bullet}$.

2.2. Generalized Hasse-Witt invariants.

In this subsection, we recall some notation concerning generalized Hasse-Witt invariants of cyclic admissible coverings of arbitrary pointed stable curves. On the other hand, in the case of *smooth* pointed stable curves, the generalized Hasse-Witt invariants of cyclic tame coverings have been studied by I. Bouw ([B, Section 2]) and Tamagawa ([T2, Section 3]).

2.2.1. **Settings.** We maintain the notation introduced in 2.1.2. Moreover, let $\Pi_{X^\bullet}$ be the admissible fundamental group of $X^\bullet$.

2.2.2. We define the *p-rank* (or *Hasse-Witt invariant*) of $X^\bullet$ to be

$$\sigma_X \overset{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(X, \mathbb{F}_p)) = \dim_{\mathbb{F}_p}(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{F}_p),$$

where $\Pi_{X^\bullet}^{\text{ab}}$ denotes the abelianization of $\Pi_{X^\bullet}$. We shall say that $X^\bullet$ is *ordinary* if $g_X = \sigma_X$. Moreover, we have the following:

$$\sigma_X = r_X + \sum_{v \in v(\Gamma_{X^\bullet})} \sigma_{\widetilde{X}_v},$$

where r_X denotes the Betti number of $\Gamma_{X^\bullet}$ (2.1.2).

2.2.3. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Let $\alpha \in \text{Hom}(\Pi_{X^\bullet}^{\text{ab}}, \mathbb{Z}/n\mathbb{Z})$. We denote by $X_\alpha^\bullet = (X_\alpha, D_{X_\alpha}) \to X^\bullet$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition ([S1, Section 9])

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha}) = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} \oplus H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}}$. Moreover, we have $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k$, where $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}}$ denotes the subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify

$$H_\alpha \overset{\text{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$$

with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-linear space H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α. Then we have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication.

2.2.4.   We call

$$\gamma_{\alpha,i} \overset{\mathrm{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z},$$

a *generalized Hasse-Witt invariant* (see [B], [N], [T2] for the case of étale or tame coverings of smooth pointed stable curves) of the cyclic multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. In particular, we call

$$\gamma_{\alpha,1}$$

the *first* generalized Hasse-Witt invariant of the cyclic multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. Note that the above decomposition implies that

$$\dim_k(H_\alpha) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.$$

In particular, if X_α is connected, then $\dim_k(H_\alpha) = \sigma_{X_\alpha}$.

2.2.5.   We write $\mathbb{Z}[D_X]$ for the group of divisors whose supports are contained in D_X. Note that $\mathbb{Z}[D_X]$ is a free $\mathbb{Z}$-module with basis D_X. We put

$$\mathbb{Z}/n\mathbb{Z}[D_X] \overset{\mathrm{def}}{=} \mathbb{Z}[D_X] \otimes \mathbb{Z}/n\mathbb{Z},$$

$$c_n' : \mathbb{Z}/n\mathbb{Z}[D_X] \to \mathbb{Z}/n\mathbb{Z},\ D \bmod n \mapsto \deg(D) \bmod n.$$

Write $(\mathbb{Z}/n\mathbb{Z})^\sim$ for the set $\{0, 1, \dots, n-1\}$ and $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$ for the subset of $\mathbb{Z}[D_X]$ consisting of the elements whose coefficients are contained in $(\mathbb{Z}/n\mathbb{Z})^\sim$. Then we have a natural bijection $\iota_n : (\mathbb{Z}/n\mathbb{Z})^\sim[D_X] \overset{\sim}{\to} \mathbb{Z}/n\mathbb{Z}[D_X]$.

We put

$$(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \overset{\mathrm{def}}{=} \iota_n^{-1}(\ker(c_n')).$$

Note that we have $n|\deg(D)$ for all $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$. Moreover, we put

$$s(D) \overset{\mathrm{def}}{=} \frac{\deg(D)}{n} \in \mathbb{Z}_{\geq 0}.$$

Since every $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ can be regarded as a ramification divisor associated to some cyclic admissible covering, the structure of the maximal prime-to-p quotient of $\Pi_{X^\bullet}$ (2.1.6) implies the following:

$$0 \leq s(D) \leq \left\{ \begin{array}{ll} 0, & \text{if } n_X \leq 1, \\ n_X - 1, & \text{if } n_X \geq 2. \end{array} \right.$$

2.2.6. We put

$$\widehat{X} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} X_H, \ D_{\widehat{X}} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} D_{X_H}, \ \Gamma_{\widehat{X}^\bullet} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} \Gamma_{X_H^\bullet}.$$

We call

$$\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}})$$

the universal admissible covering of $X^\bullet$ corresponding to $\Pi_{X^\bullet}$, and $\Gamma_{\widehat{X}^\bullet}$ the dual semi-graph of $\widehat{X}^\bullet$. Note that $\text{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of $\Pi_{X^\bullet}$.

Let $\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}}) \to X^\bullet$ be a universal admissible covering corresponding to $\Pi_{X^\bullet}$. For every $e \in e^{\text{op}}(\Gamma_{X^\bullet})$, write $\widehat{e} \in e^{\text{op}}(\Gamma_{\widehat{X}^\bullet})$ for an open edge over e and x_e for the marked point corresponding to e.

We denote by $I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$ the stabilizer of $\widehat{e}$. The definition of admissible coverings (2.1.5) implies that $I_{\widehat{e}}$ is isomorphic to the Galois group $\text{Gal}(K_{x_e}^{\text{t}}/K_{x_e}) \cong \widehat{\mathbb{Z}}(1)^{p'}$, where K_{x_e} denotes the quotient field of $\mathcal{O}_{X,x_e}$, $K_{x_e}^{\text{t}}$ denotes a maximal tamely ramified extension, and $\widehat{\mathbb{Z}}(1)^{p'}$ denotes the maximal prime-to-p quotient of $\widehat{\mathbb{Z}}(1)$. Suppose that x_e is contained in X_v. Then we have an injection

$$\phi_{\widehat{e}} : I_{\widehat{e}} \hookrightarrow \Pi_{X^\bullet}^{\text{ab}}$$

which factors through $I_{\widehat{e}} \hookrightarrow \Pi_{\widetilde{X}_v^\bullet}^{\text{ab}}$ induced by the composition of (outer) injective homomorphisms $I_{\widehat{e}} \hookrightarrow \Pi_{\widetilde{X}_v^\bullet} \hookrightarrow \Pi_{X^\bullet}$, where $\Pi_{\widetilde{X}_v^\bullet}$ denotes the admissible fundamental group of the smooth pointed stable curve $\widetilde{X}_v^\bullet$ associated to v (2.1.3). Since the image of $\phi_{\widehat{e}}$ depends only on e, we may write I_e for the image $\phi_{\widehat{e}}(I_{\widehat{e}})$. Moreover, the structure of maximal prime-to-p quotients of admissible fundamental groups of pointed stable curves (2.1.6) implies that the following holds: There exists a generator $[s_e]$ of I_e for each $e \in e^{\text{op}}(\Gamma_{X^\bullet})$ such that

$$\sum_{e \in e^{\text{op}}(\Gamma_{X^\bullet})} [s_e] = 0$$

in $\Pi_{X^\bullet}^{\text{ab}}$. In the remainder of the present paper, we *fix* a set of generators $\{[s_e]\}_{e \in e^{\text{op}}(\Gamma_{X^\bullet})}$ of I_e satisfying the above condition.

Definition 2.4. We maintain the notation introduced above.
   (i) For $\alpha \in \text{Hom}(\Pi_{X^\bullet}^{\text{ab}}, \mathbb{Z}/n\mathbb{Z})$, we put

$$D_\alpha \stackrel{\text{def}}{=} \sum_{e \in e^{\text{op}}(\Gamma_{X^\bullet})} \alpha([s_e])^\sim x_e,$$

where $\alpha([s_e])^\sim$ denotes the element of $(\mathbb{Z}/n\mathbb{Z})^\sim$ corresponding to $\alpha([s_e])$ via the natural bijection $(\mathbb{Z}/n\mathbb{Z})^\sim \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$. Note that we have $D_\alpha \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$. On the other hand, for each $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, we put

$$\text{Rev}_D^{\text{adm}}(X^\bullet) \stackrel{\text{def}}{=} \{\alpha \in \text{Hom}(\Pi_{X^\bullet}^{\text{ab}}, \mathbb{Z}/n\mathbb{Z}) \mid D_\alpha = D\}.$$

Moreover, we put

(1) $$\gamma_{(\alpha,D)} \stackrel{\text{def}}{=} \gamma_{\alpha,1} \ (2.2.4).$$

(ii) Let $t \in \mathbb{N}$ be an arbitrary positive natural number, and $n \overset{\text{def}}{=} p^t - 1$. For $u \in \{0, \ldots, n\}$, we write

$$u = \sum_{j=0}^{t-1} u_j p^j$$

for the p-adic expansion with $u_j \in \{0, \ldots, p-1\}$. We identify $\{0, \ldots, t-1\}$ with $\mathbb{Z}/t\mathbb{Z}$ naturally. Then $\{0, \ldots, t-1\}$ admits an additional structure induced by the natural additional structure of $\mathbb{Z}/t\mathbb{Z}$. We put

$$u^{(i)} \overset{\text{def}}{=} \sum_{j=0}^{t-1} u_{i+j} p^j, \ i \in \{0, \ldots, t-1\}.$$

Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_X]^0$. For $i \in \{0, \ldots, t-1\}$, we put

$$D^{(i)} \overset{\text{def}}{=} \sum_{x \in D_X} \big(\mathrm{ord}_x(D)\big)^{(i)} x$$

which is an effective divisor on X.

Roughly speaking, $D^{(i)}$ is the ramification divisor associated to a suitable Frobenius action of a Galois admissible covering whose ramification divisor is D (see the "in particular" part of Lemma 2.5 below for the relationship between the generalized Hasse-Witt invariants associated to D and $D^{(i)}$).

Lemma 2.5. *We maintain the notation introduced above. Let $l \in \{1, \ldots, n-1\}$. We put*

$$D(l) \overset{\text{def}}{=} lD - \Big(\sum_{x \in D_X} n \cdot \Big[\frac{l \cdot \mathrm{ord}_x(D)}{n} \Big] x \Big),$$

where $[(-)]$ denotes the largest integer $\leq (-)$. Then the following holds (see 2.2.4 for the definition of $\gamma_{\alpha,l}$)

$$\gamma_{\alpha,l} = \gamma_{l\alpha,1} = \gamma_{(l\alpha, D(l))}.$$

In particular, if $n \overset{\text{def}}{=} p^t - 1$, we have $\gamma_{(\alpha,D)} = \gamma_{\alpha,1} = \gamma_{\alpha,p^{t-i}} = \gamma_{p^{t-i}\alpha,1} = \gamma_{(p^{t-i}\alpha, D(p^{t-i}))} = \gamma_{(p^{t-i}\alpha, D^{(i)})}, \ i \in \{0, \ldots, t-1\}.$

2.3. Generalized Hasse-Witt invariants via line bundles.
The generalized Hasse-Witt invariants can be also described in terms of line bundles and divisors.

2.3.1. **Settings.** We maintain the settings introduced in 2.2.1. Moreover, in the present subsection, we suppose that $X^\bullet$ is *smooth* over k.

2.3.2. Let $n \in \mathbb{N}$ be an arbitrary natural number prime to p. We denote by $\mathrm{Pic}(X)$ the Picard group of X. Consider the following complex of abelian groups:

$$\mathbb{Z}[D_X] \overset{a_n}{\to} \mathrm{Pic}(X) \oplus \mathbb{Z}[D_X] \overset{b_n}{\to} \mathrm{Pic}(X),$$

where $a_n(D) = ([\mathcal{O}_X(-D)], nD)$, $b_n(([\mathcal{L}], D)) = [\mathcal{L}^n \otimes \mathcal{O}_X(D)]$. We denote by

$$\mathscr{P}_{X^\bullet, n} \overset{\text{def}}{=} \ker(b_n)/\mathrm{Im}(a_n)$$

the homology group of the complex. Moreover, we have the following exact sequence

$$0 \to \mathrm{Pic}(X)[n] \overset{a'_n}{\to} \mathscr{P}_{X^\bullet, n} \overset{b'_n}{\to} \mathbb{Z}/n\mathbb{Z}[D_X] \overset{c'_n}{\to} \mathbb{Z}/n\mathbb{Z},$$

where $\mathrm{Pic}(X)[n]$ denotes the n-torsion subgroup of $\mathrm{Pic}(X)$, and

$$
\begin{aligned}
a'_n([\mathcal{L}]) &= ([\mathcal{L}], 0) \bmod \mathrm{Im}(a_n), \\
b'_n(([\mathcal{L}], D)) \bmod \mathrm{Im}(a_n)) &= D \bmod n, \\
c'_n(D \bmod n) &= \deg(D) \bmod n.
\end{aligned}
$$

We shall define

$$
\widetilde{\mathscr{P}}_{X^\bullet, n} \subseteq \ker(b_n) \subseteq \mathrm{Pic}(X) \oplus \mathbb{Z}[D_X]
$$

to be the inverse image of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_X] \subseteq \mathbb{Z}[D_X]$ under the projection $\ker(b_n) \to \mathbb{Z}[D_X]$. It is easy to see that $\mathscr{P}_{X^\bullet, n}$ and $\widetilde{\mathscr{P}}_{X^\bullet, n}$ are free $\mathbb{Z}/n\mathbb{Z}$-modules with rank $2g_X + n_X - 1$ if $n_X \neq 0$ and with rank $2g_X$ if $n_X = 0$, and that there is a natural isomorphism $\widetilde{\mathscr{P}}_{X^\bullet, n} \xrightarrow{\sim} \mathscr{P}_{X^\bullet, n}$.

On the other hand, let $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ and $f_\alpha^\bullet : X_\alpha^\bullet \to X^\bullet$ the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta^i \mapsto i$. Then we see

$$
f_{\alpha,*}\mathcal{O}_{X_\alpha} \cong \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \mathcal{L}_{\alpha,i},
$$

where locally $\mathcal{L}_{\alpha,i}$ is the eigenspace of the natural action of i with eigenvalue ζ^i. Moreover, we have the following natural isomorphism ([T2, Proposition 3.5]):

$$
\mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathscr{P}}_{X^\bullet, n}, \ \alpha \mapsto ([\mathcal{L}_{\alpha,1}], D_\alpha).
$$

Then every element of $\widetilde{\mathscr{P}}_{X^\bullet, n}$ induces a Galois multi-admissible covering of $X^\bullet$ over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

2.3.3. **Assumptions.** In the remainder of the present paper, *we may assume that*

$$
n \stackrel{\mathrm{def}}{=} p^t - 1
$$

for some positive natural number $t \in \mathbb{N}$ unless indicated otherwise.

2.3.4. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet, n}$. We fix an isomorphism $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(-D)$. Note that D is an effective divisor on X. We have the following composition of morphisms of line bundles

$$
\mathcal{L} \xrightarrow{p^t} \mathcal{L}^{\otimes p^t} = \mathcal{L}^{\otimes n} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X(-D) \otimes \mathcal{L} \hookrightarrow \mathcal{L}.
$$

This composite morphism induces a homomorphism $\phi_{([\mathcal{L}], D)} : H^1(X, \mathcal{L}) \to H^1(X, \mathcal{L})$. We denote by

$$
\gamma_{([\mathcal{L}], D)} \stackrel{\mathrm{def}}{=} \dim_k \Big(\bigcap_{r \geq 1} \mathrm{Im}(\phi_{([\mathcal{L}], D)}^r) \Big).
$$

Write $\alpha_{\mathcal{L}} \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ for the element corresponding to $([\mathcal{L}], D)$ and F_X for the absolute Frobenius morphism on X. Then we see that $\gamma_{\alpha_{\mathcal{L}}, 1}$ (2.2.4) is equal to the dimension over k of the largest subspace of $H^1(X, \mathcal{L})$ on which $F_X^t \stackrel{\mathrm{def}}{=} F_X \circ \cdots \circ F_X$ is a bijection. Moreover, we have

$$
\gamma_{\alpha_{\mathcal{L}}, 1} = \dim_k (H^1(X, \mathcal{L})^{F_X^t} \otimes_{\mathbb{F}_p} k),
$$

where $H^1(X, \mathcal{L})^{F_X^t}$ denotes the subspace of $H^1(X, \mathcal{L})$ on which F_X^t acts trivially. Then we have the following lemmas.

Lemma 2.6. *We maintain the notation introduced above. Then we have* $\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1}$. *Moreover, since* $D_{\alpha_{\mathcal{L}}} = D$, *we have*

$$\gamma_{([\mathcal{L}],D)} = \gamma_{(\alpha_{\mathcal{L}},D)} \ (\stackrel{\text{def}}{=} \gamma_{\alpha_{\mathcal{L}},1}).$$

Proof. The lemma follows immediately from $H^1(X,\mathcal{L})^{F_X^t} \otimes_{\mathbb{F}_p} k = \bigcap_{r \geq 1} \text{Im}(\phi^r_{([\mathcal{L}],D)})$. $\square$

Lemma 2.7. *We maintain the notation introduced above. Suppose that* $X^\bullet$ *is smooth over* k. *Then we have*

$$\gamma_{(\alpha_{\mathcal{L}},D)} \leq \dim_k(H^1(X,\mathcal{L})) = \begin{cases} g_X, & \text{if } ([\mathcal{L}],D) = ([\mathcal{O}_X],0), \\ g_X - 1, & \text{if } s(D) = 0, \ [\mathcal{L}] \neq [\mathcal{O}_X], \\ g_X + s(D) - 1, & \text{if } s(D) \geq 1, \end{cases}$$

where $s(D)$ *is the integer defined in* 2.2.5.

Proof. The first inequality follows from the definition of generalized Hasse-Witt invariants. On the other hand, the Riemann-Roch theorem implies that

$$\dim_k(H^1(X,\mathcal{L})) = g_X - 1 - \deg(\mathcal{L}) + \dim_k(H^0(X,\mathcal{L}))$$

$$= g_X - 1 + \frac{1}{n}\deg(D) + \dim_k(H^0(X,\mathcal{L})) = g_X - 1 + s(D) + \dim_k(H^0(X,\mathcal{L})).$$

This completes the proof of the lemma. $\square$

2.4. **Raynaud-Tamagawa theta divisors.**

In this subsection, we recall the theory of theta divisors which was introduced by Raynaud in the case of étale coverings ([R1]), and which was generalized by Tamagawa in the case of tame coverings ([T2]).

2.4.1. **Settings.**

We maintain the settings introduced in 2.3.1.

2.4.2.

Let F_k be the absolute Frobenius morphism on $\text{Spec}\,k$, $F_{X/k}$ the relative Frobenius morphism $X \to X_1 \stackrel{\text{def}}{=} X \times_{k,F_k} k$ over k, and $F_k^t \stackrel{\text{def}}{=} F_k \circ \cdots \circ F_k$. We put $X_t \stackrel{\text{def}}{=} X \times_{k,F_k^t} k$, and define a morphism

$$F_{X/k}^t : X \to X_t$$

over k to be $F_{X/k}^t \stackrel{\text{def}}{=} F_{X_{t-1}/k} \circ \cdots \circ F_{X_1/k} \circ F_{X/k}$.

Let $([\mathcal{L}],D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$, and let $\mathcal{L}_t$ be the pulling back of $\mathcal{L}$ by the natural morphism $X_t \to X$. Note that $\mathcal{L}$ and $\mathcal{L}_t$ are line bundles of degree $-s(D)$ (2.2.5). We put

$$\mathcal{B}_D^t \stackrel{\text{def}}{=} (F_{X/k}^t)_*(\mathcal{O}_X(D))/\mathcal{O}_{X_t}, \ \mathcal{E}_D \stackrel{\text{def}}{=} \mathcal{B}_D^t \otimes \mathcal{L}_t.$$

Write $\text{rk}(\mathcal{E}_D)$ for the rank of $\mathcal{E}_D$. Then we obtain

$$\chi(\mathcal{E}_D) = \deg(\det(\mathcal{E}_D)) - (g_X - 1)\text{rk}(\mathcal{E}_D).$$

Moreover, we have $\chi(\mathcal{E}_D) = 0$ ([T2, Lemma 2.3 (ii)]).

2.4.3.   Let J_{X_t} be the Jacobian variety of X_t and $\mathcal{L}_{X_t}$ a universal line bundle on $X_t \times J_{X_t}$. Let $\mathrm{pr}_{X_t} : X_t \times J_{X_t} \to X_t$ and $\mathrm{pr}_{J_{X_t}} : X_t \times J_{X_t} \to J_{X_t}$ be the natural projections. We denote by $\mathcal{F}$ the coherent $\mathcal{O}_{X_t}$-module $\mathrm{pr}_{X_t}^*(\mathcal{E}_D) \otimes \mathcal{L}_{X_t}$, and by

$$\chi_{\mathcal{F}} \overset{\mathrm{def}}{=} \dim_k(H^0(X_t \times_k k(y), \mathcal{F} \otimes k(y))) - \dim_k(H^1(X_t \times_k k(y), \mathcal{F} \otimes k(y)))$$

for each $y \in J_{X_t}$, where $k(y)$ denotes the residue field of y. Note that since $\mathrm{pr}_{J_{X_t}}$ is flat, $\chi_{\mathcal{F}}$ is independent of $y \in J_{X_t}$. Write $(-\chi_{\mathcal{F}})^+$ for $\max\{0, -\chi_{\mathcal{F}}\}$. We denote by

$$\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$$

the closed subscheme of J_{X_t} defined by the $(-\chi_{\mathcal{F}})^+$th Fitting ideal $\mathrm{Fitt}_{(-\chi_{\mathcal{F}})^+}\left(R^1(\mathrm{pr}_{J_{X_t}})_*(\mathcal{F})\right)$. The definition of $\Theta_{\mathcal{E}_D}$ is independent of the choice of $\mathcal{L}_t$. Moreover, we have $\mathrm{codim}(\Theta_{\mathcal{E}_D}) \leq 1$.

2.4.4.   In [R1], Raynaud investigated the following property of the vector bundle $\mathcal{E}_D$ on X.

Condition 2.8. We shall say that $\mathcal{E}_D$ satisfies $(\star)$ if there exists a line bundle $\mathcal{L}_t'$ of degree 0 on X_t such that

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}_t')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}_t'))\}.$$

Moreover, [T2, Proposition 2.2 (i) (ii)] implies that $[\mathcal{L}'] \notin \Theta_{\mathcal{E}_D}$ if and only if $\mathcal{E}_D$ satisfies $(\star)$ for $\mathcal{L}'$, where $[\mathcal{L}']$ denotes the point of J_{X_t} corresponding to $\mathcal{L}'$. Namely, $\Theta_{\mathcal{E}_D}$ is a *divisor* of J_{X_t} when $\mathcal{E}_D$ satisfies $(\star)$. Then we have the following definition:

Definition 2.9. We shall say that $\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$ is the *Raynaud-Tamagawa theta divisor* associated to $\mathcal{E}_D$ if $\mathcal{E}_D$ satisfies $(\star)$.

Remark 2.9.1. Suppose that $\mathcal{E}_D$ satisfies $(\star)$ (i.e., Condition 2.8). [R1, Proposition 1.8.1] implies that $\Theta_{\mathcal{E}_D}$ is algebraically equivalent to $\mathrm{rk}(\mathcal{E}_D)\Theta$, where Θ is the classical theta divisor (i.e., the image of $X_t^{g_X-1}$ in J_{X_t}).

Remark 2.9.2. Note that (see [Y6, Lemma 2.8]) if

$$\gamma_{([\mathcal{L} \otimes \mathcal{I}], D)} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$$

for some $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$, then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists (i.e., $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$).

2.4.5.   Let N be an arbitrary non-negative integer. We define a constant concerning N as following:

$$C(N) \overset{\mathrm{def}}{=} \begin{cases} 0, & \text{if } N = 0, \\ 3^{N-1}N!, & \text{if } N \neq 0. \end{cases}$$

Then we have the following proposition.

Proposition 2.10. *We maintain the notation introduced above. Suppose that*

$$n = p^t - 1 > C(g_X) + 1,$$

and that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists. Then there exists a line bundle $\mathcal{I}$ of degree 0 on X such that $[\mathcal{I}] \neq [\mathcal{O}_X]$, that $[\mathcal{I}^{\otimes n}] = [\mathcal{O}_X]$, and that $\gamma_{(([\mathcal{L} \otimes \mathcal{I}], D))} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$ (i.e., $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$).

Proof. By applying similar arguments to the arguments given in the proof of [T2, Corollary 3.10], the proposition follows from Remark 2.9.2.   □

Namely, if $n >> 0$ and the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists, then there exists a line bundle $\mathcal{I}$ of degree 0 with order n such that the following hold:

- $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$
- the first generalized Hasse-Witt invariant (2.2.4) of the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to $([\mathcal{L} \otimes \mathcal{I}], D)$ (2.3.2) is as large as possible.

2.4.6.   The following fundamental theorem of theta divisors was proved by Raynaud when $s(D) = 0$ ([R1, Théorème 4.1.1]), and by Tamagawa when $s(D) \leq 1$ ([T2, Theorem 2.5]).

Theorem 2.11. *Suppose that $s(D) \in \{0, 1\}$ (2.2.5). Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists.*

We may ask whether or not the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exists for an arbitrary $s(D)$ when $X^\bullet$ is smooth over k. However, the following example shows that the Raynaud-Tamagawa theta divisor *does not exist* in general.

Example 2.12. Let $p > 2$, $n = p - 1$, $X = \mathbb{P}_k^1$, $D_X = \{0, 1, \infty, \lambda\}$ such that $\lambda \in k \setminus \{0, 1\}$, and

$$D = \sum_{x \in D_X} \frac{n}{2} \cdot x$$

an effective divisor on X. Then we have $s(D) = 2$. Let $([\mathcal{L}], D)$ be an arbitrary element of $\widetilde{\mathscr{P}}_{X^\bullet, n}$. We see that $\mathcal{E}_D$ satisfies $(\star)$ if and only if the elliptic curve defined by the equation

$$y^2 = x(x - 1)(x - \lambda)$$

is ordinary. Thus, we cannot expect that $\Theta_{\mathcal{E}_D}$ exists in general.

3. Maximum generalized Hasse-Witt invariants

In this section, we introduce the maximum generalized Hasse-Witt invariants for cyclic prime-to-p admissible coverings. The main result of the present section is Theorem 3.9 which says that the first generalized Hasse-Witt invariant of a prime-to-p cyclic admissible covering attains maximum *if and only if* the first generalized Hasse-Witt invariants of the induced admissible coverings of irreducible components attain maximum.

By the formula mentioned in 2.2.2 and the definition of generalized Hasse-Witt invariants (2.2.3), to prove Theorem 3.9 for a cyclic prime-to-p admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ of degree n, the most difficult part is to compute the following:

- The generalized Hasse-Witt invariants arising from the covering of dual semi-graphs

$$f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$$

  induced by $f^\bullet$ (i.e., the dimensions of irreducible representations

$$\mathbb{Z}/n\mathbb{Z} \to \mathrm{GL}(H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k)$$

  induced by the natural action of $\mathbb{Z}/n\mathbb{Z}$ on $\Gamma_{Y^\bullet}$).

Note that the computations concerning generalized Hasse-Witt invariants of coverings of dual semi-graphs are not easy since f^{sg} *is not a topological covering* in general (i.e., $f^{\bullet}$ is ramified over singular points and marked points of $X^{\bullet}$).

In Section 3.1, we will compute generalized Hasse-Witt invariants of coverings of dual semi-graphs and prove two technical propositions (Proposition 3.5 and Proposition 3.8). In Section 3.2, we will use the propositions to obtain Theorem 3.9.

3.1. Generalized Hasse-Witt invariants for coverings of dual semi-graphs. The readers who would like to start the proofs of Theorem 3.9 quickly may skip this subsection, after glancing at the statements of Proposition 3.5 and Proposition 3.8.

3.1.1. **Settings.** We maintain the notation introduced in 2.1.2. Moreover, in this subsection, let n be an *arbitrary positive natural number prime to p* and $\mu_n \subseteq k^{\times}$ the group of nth roots of unity. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Let

$$f^{\bullet} : Y^{\bullet} \to X^{\bullet}$$

be a Galois admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, f the underlying morphism of $f^{\bullet}$, $\mathscr{N}_X^{\mathrm{et}}$ the set of nodes of X over which f is étale, and

$$f^{\mathrm{sg}} : \Gamma_{Y^{\bullet}} \to \Gamma_{X^{\bullet}}$$

the map of dual semi-graphs of $Y^{\bullet}$ and $X^{\bullet}$ induced by $f^{\bullet}$, where "sg" means "semi-graph". Note that $Y^{\bullet}$ is connected.

We put $M_{Y^{\bullet}} \overset{\mathrm{def}}{=} H^1_{\mathrm{ét}}(Y, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$. Then $M_{Y^{\bullet}}$ is a $k[\mu_n]$-module and admits the following canonical decomposition

$$M_{Y^{\bullet}} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Y^{\bullet}}(j),$$

where $\zeta \in \mu_n$ (or $1 \in \mathbb{Z}/n\mathbb{Z}$) acts on $M_{Y^{\bullet}}(j)$ as the ζ^j-multiplication. Note that $\dim_k(M_{Y^{\bullet}}(j))$, $j \in \mathbb{Z}/n\mathbb{Z}$, is a generalized Hasse-Witt invariant of the Galois admissible covering $f^{\bullet}$ (2.2.4).

3.1.2. The dual semi-graph $\Gamma_{Y^{\bullet}}$ admits a natural action of $\mathbb{Z}/n\mathbb{Z}$ induced by the Galois admissible covering $f^{\bullet}$. We write $M_{\Gamma_{X^{\bullet}}}$, $M_{\Gamma_{Y^{\bullet}}}$, and $M^{\vee}_{\Gamma_{Y^{\bullet}}}$ for $H^1(\Gamma_{X^{\bullet}}, \mathbb{F}_p) \otimes k$, $H^1(\Gamma_{Y^{\bullet}}, \mathbb{F}_p) \otimes k$, and the dual vector space $\pi_1^{\mathrm{top}}(\Gamma_{Y^{\bullet}})^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{F}_p \otimes_{\mathbb{F}_p} k$ of $M_{\Gamma_{Y^{\bullet}}}$, respectively. Moreover, we *fix* a basis $\{\lambda_q\}_q$ for $M_{\Gamma_{Y^{\bullet}}}$. Then we have a dual basis $\{\lambda_q^{\vee}\}_q$ for $M^{\vee}_{\Gamma_{Y^{\bullet}}}$.

Let $l \subseteq \Gamma_{Y^{\bullet}}$ be a loop and $\alpha_l^{\vee} \overset{\mathrm{def}}{=} \sum_q a_q \lambda_q^{\vee}$, $a_q \in k$, the vector of $M^{\vee}_{\Gamma_{Y^{\bullet}}}$ corresponding to l. We shall call

$$\alpha_l \overset{\mathrm{def}}{=} \sum_q a_q \lambda_q \in M_{\Gamma_{Y^{\bullet}}}$$

the vector of $M_{\Gamma_{Y^{\bullet}}}$ corresponding to l.

Moreover, we shall say that l is *a minimal loop of* $\Gamma_{Y^{\bullet}}$ if, for any loop $l' \subseteq l \subseteq \Gamma_{Y^{\bullet}}$ such that l and l' are homotopic, then $l = l'$.

3.1.3. Let $r \in \mathbb{Z}/n\mathbb{Z}$. We denote by $r \cdot l$ and $r \cdot \alpha_l^\vee$ the natural actions of r on l and $\alpha_l^\vee$, respectively. Note that we have $r \cdot \alpha_l^\vee = \alpha_{r \cdot l}^\vee$.

On the other hand, we denote by $r * \alpha_l$ the action of $r \in \mathbb{Z}/n\mathbb{Z}$ on $\alpha_l \in M_{\Gamma_{Y^\bullet}}$ induced by the action of $\mathbb{Z}/n\mathbb{Z}$ on $M_{\Gamma_{Y^\bullet}}^\vee$. Then we have $(-r) * \alpha_l = \alpha_{r \cdot l}$. For convenience, we put

$$r \cdot \alpha_l \stackrel{\text{def}}{=} (-r) * \alpha_l.$$

Then we have $r \cdot \alpha_l = \alpha_{r \cdot l}$ and $r'' \cdot (r' \cdot \alpha_l) = (r' + r'') \cdot \alpha_l$ for all $r', r'' \in \mathbb{Z}/n\mathbb{Z}$. Moreover, $M_{\Gamma_{Y^\bullet}}$ is a $k[\mu_n]$-module and admits the following canonical decomposition

$$M_{\Gamma_{Y^\bullet}} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(j),$$

where $\zeta \in \mu_n$ (or $1 \in \mathbb{Z}/n\mathbb{Z}$) acts on $M_{\Gamma_{Y^\bullet}}(j)$ as the ζ^j-multiplication.

3.1.4. Firstly, we have the following lemma.

Lemma 3.1. *We maintain the notation introduced above. Then the following statements hold:*

(i) Suppose that $f^\bullet : X^\bullet \to Y^\bullet$ is a Galois étale covering (i.e., f is étale) with Galois group $\mathbb{Z}/n\mathbb{Z}$, and that $X^\bullet$ and $Y^\bullet$ are ordinary (2.2.2). Then we have

$$\dim_k(M_{Y^\bullet}(j)) = \begin{cases} g_X, & \text{if } j = 0, \\ g_X - 1, & \text{if } j \in \{1, \ldots, n-1\}, \end{cases}$$

(ii) Suppose that $f^{\text{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ is a Galois topological covering with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(j)) = \begin{cases} r_X, & \text{if } j = 0, \\ r_X - 1, & \text{if } j \in \{1, \ldots, n-1\}, \end{cases}$$

where r_X denotes the Betti number of $\Gamma_{X^\bullet}$ (2.1.2).

Proof. (i) Since the underlying morphism f is étale, there exists a line bundle $\mathcal{L}$ on X whose degree is 0, and whose order is n, such that

$$f_* \mathcal{O}_Y \cong \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} \mathcal{L}^{\otimes j}.$$

Then we have $\dim_k(M_{Y^\bullet}(j)) = \dim_k(H^1(X, \mathcal{L})) = g_X - 1$ for all $j \in \{1, \ldots, n-1\}$ and $\dim_k(M_{Y^\bullet}(0)) = \dim_k(H^1(X, \mathcal{O}_X)) = g_X$. This completes the proof of (i).

(ii) Since this is a topological question, to verify (ii), by adding certain marked points, we may assume that $X^\bullet$ is an *ordinary* pointed stable curve such that the normalization of every irreducible component is *isomorphic to* $\mathbb{P}_k^1$. Then $Y^\bullet$ is also an *ordinary* pointed stable curve such that the normalizations of irreducible components are isomorphic to $\mathbb{P}_k^1$. Then (ii) follows immediately from (i). This completes the proof of the lemma. □

3.1.5. By using admissible coverings, we compute the dimensions of irreducible representations of a certain representation of $\mathbb{Z}/n\mathbb{Z}$. We have the following linear algebra result which follows immediately from Lemma 3.4 below (i.e., by replacing V, $\alpha_{a,b}$ by $M_{\Gamma_{Y^\bullet}}$, $\alpha_{l_{a,b}}$, respectively).

Proposition 3.2. *Let $s, t, n \in \mathbb{N}$ be positive natural numbers prime to p such that $n = st$, and let V_b, $b \in \{1, \ldots, t\}$, be a k-linear space of dimension $s - 1$ and $\{\alpha_{1,b}, \ldots, \alpha_{s-1,b}\}$ a basis for V_b. We put*

$$V \stackrel{\text{def}}{=} \bigoplus_{b=1}^{t} V_b,$$

and let $1 \in \mathbb{Z}/n\mathbb{Z}$. We define an action of $\mathbb{Z}/n\mathbb{Z}$ on V as follows: $1 \cdot \alpha_{a,b} = \alpha_{a,b+1}$ if $1 \leq a \leq s - 1$ and $1 \leq b \leq t - 1$, $1 \cdot \alpha_{a,t} = \alpha_{a+1,1}$ if $1 \leq a \leq s - 2$, and

$$1 \cdot \alpha_{s-1,t} = -\sum_{a=1}^{s-1} \alpha_{a,1}.$$

Then we have the following:

$$\dim_k(V(j)) = \begin{cases} 0, & \text{if } j \in \{0, s, 2s, \ldots, (t-1)s\}, \\ 1, & \text{otherwise,} \end{cases}$$

where $V(j) \subseteq V$ denotes the k-linear subspace on which $1 \in \mathbb{Z}/n\mathbb{Z}$ acts as the ζ^j-multiplication.

We give an algebraic geometric proof of the above proposition. We maintain the notation introduced above. Suppose that $f^{\bullet} : Y^{\bullet} \to X^{\bullet}$ is a Galois *étale* covering with Galois group $\mathbb{Z}/n\mathbb{Z}$, that the set of irreducible components of $X^{\bullet}$ is $\{X_1, X_2\}$, and that X_1, X_2 are non-singular. Write v_1 and v_2 for the vertices of $\Gamma_{X^{\bullet}}$ corresponding to X_1 and X_2, respectively. Let $f_i^{\bullet} \stackrel{\text{def}}{=} \tilde{f}_{v_i}^{\bullet} : Y_i^{\bullet} \stackrel{\text{def}}{=} \tilde{Y}_{v_i}^{\bullet} \to X_i^{\bullet} \stackrel{\text{def}}{=} \tilde{X}_{v_i}^{\bullet}$, $i \in \{1, 2\}$, be the Galois multi-admissible covering over k induced by $f^{\bullet}$ (see 2.1.3 for $\tilde{X}_{v_i}^{\bullet}$). Moreover, we suppose that the following conditions are satisfied:

- $X_1^{\bullet}$ and $X_2^{\bullet}$ are ordinary pointed stable curves of type $(1, 1)$. This implies that the connected components of Y_i are non-singular curves of genus (resp. p-rank) 1.
- Y_1 is connected and $\#(\pi_0(Y_2)) = t$, where $\pi_0(Y_2)$ denotes the set of connected components of Y_2. Then the decomposition group of a connected component of Y_2 is a subgroup of $\mathbb{Z}/n\mathbb{Z}$ with order $s \stackrel{\text{def}}{=} n/t$.

Note that the structures of maximal prime-to-p quotients of admissible fundamental groups (2.1.6) imply that the above Galois admissible covering exists.

Write $w_1 \in v(\Gamma_{Y^{\bullet}})$ for the vertex corresponding to Y_1 and $w_{2,b} \in v(\Gamma_{Y^{\bullet}})$, $b \in \{1, \ldots, t\}$ for the vertex corresponding to a connected component of Y_2. Then we see $v(\Gamma_{Y^{\bullet}}) = \{w_1, w_{2,1}, \ldots, w_{2,t}\}$. Moreover, we write $\{e_{1,b}, \ldots, e_{s,b}\}$ for the set of closed edges of $\Gamma_{Y^{\bullet}}$ connecting w_1 and $w_{2,b}$. We define a minimal loop as following (for instance, see Example 3.3 below):

$$l_{a,b} \stackrel{\text{def}}{=} w_1 e_{a,b} w_{2,b} e_{a+1b} w_1, \quad a \in \{1, \ldots, s-1\}, \ b \in \{1, \ldots, t\}.$$

Then we have $1 \cdot \alpha_{l_{a,b}} = \alpha_{l_{a,b+1}}$ if $1 \leq b \leq t - 1$, and $1 \cdot \alpha_{l_{a,t}} = \alpha_{l_{a+1,1}}$ if $1 \leq a \leq s - 2$, and

$$1 \cdot \alpha_{l_{s-1,t}} = -\sum_{a=1}^{s-1} \alpha_{l_{a,1}}.$$

Note that the set of vectors $\{\alpha_{l_{a,b}}\}_{a \in \{1, \ldots, s-1\}, b \in \{1, \ldots, t\}}$ is linearly independent and is a basis for $M_{\Gamma_{Y^{\bullet}}}$.

Example 3.3. We maintain the notation introduced above. If $n = 4$ and $b = 2$, we have the following

$$\Gamma_{Y^\bullet}: \quad w_{2,1} \quad \overset{e_{1,1} \quad e_{1,2}}{\underset{e_{2,1} \quad e_{2,2}}{\bullet \quad w_1 \quad \bullet}} \quad w_{2,2} \quad \xrightarrow{\; f^{\mathrm{sg}} \;} \quad \Gamma_{X^\bullet}: \quad \underset{v_1}{\bullet} \!\!\!\!\!\!\!\!\!\!\! \underset{v_2}{\bullet}$$
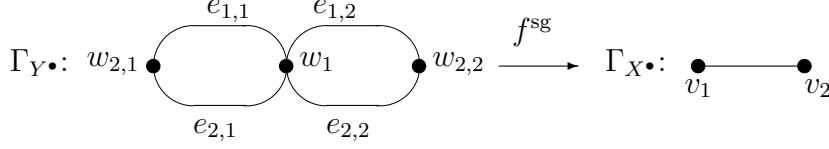
By replacing V, $\alpha_{a,b}$ by $M_{\Gamma_{Y^\bullet}}$, $\alpha_{l_{a,b}}$, respectively, to verify Proposition 3.2, it is sufficient to prove the following lemma:

Lemma 3.4. *We maintain the notation introduced above. Then we have the following:*

$$\dim_k(M_{\Gamma_{Y^\bullet}}(j)) = \begin{cases} 0, & \text{if } j \in \{0, s, 2s, \dots, (t-1)s\}, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. We put

$$M_1 \overset{\mathrm{def}}{=} H^1_{\text{ét}}(Y_1, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k, \quad M_2 \overset{\mathrm{def}}{=} \bigoplus_{b=1}^{t} H^1_{\text{ét}}(Y_{w_{2,b}}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k.$$

Then since all the connected components of Y_1 and Y_2 are non-singular curves of genus (resp. p-rank) 1, we obtain

$$\dim_k(M_1(j)) = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\dim_k(M_2(j)) = \begin{cases} 1, & \text{if } j \in \{0, s, 2s, \dots, (t-1)s\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $M_1(j) \overset{\mathrm{def}}{=} M_{\Gamma_{Y^\bullet}}(j) \cap M_1$ and $M_2(j) \overset{\mathrm{def}}{=} M_{\Gamma_{Y^\bullet}}(j) \cap M_2$. On the other hand, Lemma 3.1 (i) implies

$$\dim_k(M_{Y^\bullet}(j)) = \dim_k(M_1(1)) + \dim_k(M_2(j)) + \dim_k(M_{\Gamma_{Y^\bullet}}(j))$$

$$= \begin{cases} 2, & \text{if } j = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then we complete the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.1.6. Next, we calculate generalized Hasse-Witt invariants of dual semi-graphs when $X^\bullet$ is irreducible.

Proposition 3.5. *We maintain the notation introduced above. We maintain the notation introduced above. Recall that $\mathscr{N}_X^{\text{ét}} \subseteq X^{\text{sing}}$ (3.1.1) is the subset of nodes over which f is étale. Suppose that $v(\Gamma_{X^\bullet}) = \{v_X\}$. Then we have*

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \begin{cases} \#(\mathscr{N}_X^{\text{ét}}) - 1, & \text{if } \#(v(\Gamma_{Y^\bullet})) = n, \\ \#(\mathscr{N}_X^{\text{ét}}), & \text{if } \#(v(\Gamma_{Y^\bullet})) \neq n. \end{cases}$$

Proof. We compute $\dim_k(M_{\Gamma_{Y^\bullet}}(1))$ by analyzing the minimal loops of $\Gamma_{Y^\bullet}$. Let $l \subseteq \Gamma_{Y^\bullet}$ be an arbitrary minimal loop of $\Gamma_{Y^\bullet}$ and $V_l \subseteq M_{\Gamma_{Y^\bullet}}$ the subspace spanned by $\{j \cdot \alpha_l\}_{j \in \mathbb{Z}/n\mathbb{Z}}$. Firstly, we note that

$$V_l \subseteq \bigoplus_{0 \leq j \leq n/m, \ m|n} M_{\Gamma_{Y^\bullet}}(jm)$$

for some natural number $m > 1$ if all the decomposition groups of the elements of $e^{\mathrm{cl}}(\Gamma_{Y^\bullet}) \cap l$ are *not trivial* (i.e., there exists a closed edge e contained in l such that f is *étale* at the node of Y corresponding to e). Then to compute $\dim_k(M_{\Gamma_{Y^\bullet}}(1))$, it is sufficient to consider the subspaces of $M_{\Gamma_{Y^\bullet}}$ which are generalized by the corresponding vectors (3.1.2) of the minimal loops of $\Gamma_{Y^\bullet}$ such that the decomposition groups of the closed edges contained in the minimal loops are trivial (see M_{e_X}, $M_{T_{Y,e_X}}$, M_i, and M_π defined below).

We prove the proposition by dividing into various cases.

Case 1: Suppose $\#(v(\Gamma_{Y^\bullet})) = n$. Then we have that $f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ is a Galois topological covering with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then Lemma 3.1 (ii) implies

$$\dim_k(M_{\Gamma_{Y^\bullet}}(i)) = r_X - 1 = \#(e^{\mathrm{cl}}(X^\bullet)) - \#(v(\Gamma_{X^\bullet})) + 1 - 1 = \#(e^{\mathrm{cl}}(X^\bullet)) - 1 = \#(\mathcal{N}_X^{\mathrm{et}}) - 1$$

for all $i \in \{1, \ldots, n-1\}$ and $\dim_k(M_{\Gamma_{Y^\bullet}}(0)) = r_X$. Thus, we obtain the proposition when $\#(v(\Gamma_{Y^\bullet})) = n$.

Case 2: Suppose $\#(v(\Gamma_{Y^\bullet})) \neq n$. Since $\#(v(\Gamma_{X^\bullet})) = 1$, we have that the linear space $M_{\Gamma_{X^\bullet}}$ is spanned by the vectors corresponding to the closed edges of $\Gamma_{X^\bullet}$, and that $\dim_k(M_{\Gamma_{X^\bullet}}) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet})) = \#(e^{\mathrm{lp}}(\Gamma_{X^\bullet}))$. Write $E^{\mathrm{lp}} \subseteq e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ for the subset of closed edges such that $(f^{\mathrm{sg}})^{-1}(E^{\mathrm{lp}}) \subseteq e^{\mathrm{lp}}(\Gamma_{Y^\bullet})$ and $E^{\mathrm{tr}} \overset{\mathrm{def}}{=} e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \setminus E^{\mathrm{lp}}$, where "tr" means "tree". Note that for every $e_X \in E^{\mathrm{tr}}$, every closed edge $e_Y \in (f^{\mathrm{sg}})^{-1}(e_X)$ abuts exactly to two different vertices of $\Gamma_{Y^\bullet}$. Moreover, we write $E^{\mathrm{lp,et}}$ (resp. $E^{\mathrm{tr,et}}$) for the subset of E^{lp} (resp. E^{tr}) such that f is étale over all the nodes corresponding to the elements of $E^{\mathrm{lp,et}}$ (resp. $E^{\mathrm{tr,et}}$).

> *The subspace M_{e_X}:* Roughly speaking, this kind of subspace is generated by the corresponding vectors of the following minimal loops: Let l be a minimal loop of $\Gamma_{Y^\bullet}$; Then $l \cap e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$ is an element of $e^{\mathrm{lp}}(\Gamma_{Y^\bullet})$.

Let $e_X \in E^{\mathrm{lp}}$, $e_Y \in (f^{\mathrm{sg}})^{-1}(e_X)$, and let $D_{e_Y} \subseteq \mathbb{Z}/n\mathbb{Z}$ be the decomposition group of e_Y and m_{e_Y} the order of D_{e_Y}. Note that e_Y is a minimal loop of $\Gamma_{Y^\bullet}$, and that D_{e_Y} does not depend on the choice of e_Y (i.e., $D_{e_Y'} = D_{e_Y''}$ for all $e_Y', e_Y'' \in (f^{\mathrm{sg}})^{-1}(e_X)$). Write n_{e_Y} for n/m_{e_Y} and M_{e_X} for the subspace of $M_{\Gamma_{Y^\bullet}}$ spanned by $\{\alpha_{e_Y}\}_{e_Y \in (f^{\mathrm{sg}})^{-1}(e_X)}$. Then we see that M_{e_X} is a $k[(\mathbb{Z}/n\mathbb{Z})/D_{e_Y}]$-module, that

$$M_{e_X} \subseteq \bigoplus_{0 \leq j \leq n_{e_Y}-1} M_{\Gamma_{Y^\bullet}}(jm_{e_Y}),$$

and that $\dim_k(M_{e_X}) = n_{e_Y}$. Write $M_{e_X}(1) \subseteq M_{e_X}$ for the subspace on which $\zeta \in \mu_n$ acts as the ζ-multiplication. Thus, we obtain $\dim_k(M_{e_X} \cap M_{\Gamma_{Y^\bullet}}(1)) = 1$ if and only if $m_{e_Y} = 1$ (i.e., f is étale over the node of X corresponding to e_X).

Case 2A: Suppose $E^{\mathrm{tr}} = \emptyset$. We have

$$M_{\Gamma_{Y^\bullet}}(1) = \bigoplus_{e_X \in E^{\mathrm{lp,et}}} M_{e_X}(1).$$

Then we obtain

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#(E^{\mathrm{lp,et}}) = \#(\mathscr{N}_X^{\mathrm{et}})$$

if $E^{\mathrm{tr}} = \emptyset$ and $\#(v(\Gamma_{Y^\bullet})) \neq n$.

Case 2B: Suppose $E^{\mathrm{tr}} \neq \emptyset$. Then we have $n \neq 2$. Let $v_Y \in (f^{\mathrm{sg}})^{-1}(v_X)$, and let D_{v_Y} be the decomposition group of v_Y which does not depend on the choice of v_Y (i.e., $D_{v'_Y} = D_{v''_Y}$ for all $v'_Y, v''_Y \in (f^{\mathrm{sg}})^{-1}(v_X)$), $1 < m_{v_Y} \overset{\mathrm{def}}{=} \#(D_{v_Y})$, and $n_{v_Y} \overset{\mathrm{def}}{=} n/m_{v_Y}$. We put

$$(f^{\mathrm{sg}})^{-1}(v_X) \overset{\mathrm{def}}{=} \{v_{Y,0}, \dots, v_{Y,n_{v_Y}-1}\},$$

which admits a natural action of $r \in \mathbb{Z}/n\mathbb{Z}$ such that $r \cdot v_{Y,0} = v_{Y,\bar{r}}$, where $\bar{r}$ denotes the image of $\mathbb{Z}/n\mathbb{Z} \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})/D_{v_Y} \overset{\sim}{\to} \mathbb{Z}/n_{v_Y}\mathbb{Z}$. Moreover, without loss of generality, we may assume $Y_{v_{Y,i}} \cap Y_{v_{Y,i+1}} \neq \emptyset$ for all $i \in \{0, \dots, n_{v_Y} - 2\}$, $Y_{v_{Y,n_Y-1}} \cap Y_{v_{Y,0}} \neq \emptyset$, and $Y_{v_{Y,j'}} \cap Y_{v_{Y,j''}} = \emptyset$ for all $j', j'' \in \{0, \dots, n_{v_Y} - 1\}$.

> *The subspace $M_{T_{Y,e_X}}$:* Roughly speaking, this kind of subspace is generated by the corresponding vectors of the minimal loops constructed by the following minimal loops via Galois actions: Let $l_{e_X,i}$ be a minimal loop of $\Gamma_{Y^\bullet}$; Then $l_{e_X,i} \cap e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$ is *stable* under the actions of decomposition groups of vertices $l_{e_X,i} \cap v(\Gamma_{Y^\bullet})$, and the images of $l_{e_X,i} \cap e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$ in $\Gamma_{X^\bullet}$ is an element of $E^{\mathrm{tr,et}}$.

We put

$$T_{Y,e_X} \overset{\mathrm{def}}{=} \{e_{Y,0}, \dots, e_{Y,m_{v_Y}-1}\} \subseteq (f^{\mathrm{sg}})^{-1}(e_X) \subseteq e^{\mathrm{cl}}(\Gamma_{Y^\bullet}), \ e_X \in E^{\mathrm{tr,et}},$$

the subset of closed edges such that $v^{\Gamma_{Y^\bullet}}(e_{Y,i}) = \{v_{Y,0}, v_{Y,1}\}$ for all $i \in \{0, \dots, m_{v_Y} - 1\}$ (see 2.1.1 for $v^{\Gamma_{Y^\bullet}}(e_{Y,i})$). Then

$$l_{e_X,i} \overset{\mathrm{def}}{=} v_{Y,0} e_{Y,i} v_{Y,1} e_{Y,i+1} v_{Y,0}, \ i \in \{0, \dots, m_{v_Y} - 2\},$$

can be regarded as a minimal loop of $\Gamma_{Y^\bullet}$ (for instance, see Example 3.6 (a) below). Moreover, the set of vectors

$$\{j \cdot \alpha_{l_{e_X,i}}\}_{i \in \{0,\dots,m_{v_Y}-2\}, j \in \{0,\dots,n_{v_Y}-1\}, e_X \in E^{\mathrm{tr,et}}} \subseteq M_{\Gamma_{Y^\bullet}}$$

is linearly independent. We denote by $M_{T_{Y,e_X}} \subseteq M_{\Gamma_{Y^\bullet}}$, $e_X \in E^{\mathrm{tr,et}}$, the subspace spanned by

$$\{j \cdot \alpha_{l_{e_X,i}}\}_{i \in \{0,\dots,m_{v_Y}-2\}, j \in \{0,\dots,n_{v_Y}-1\}}.$$

Then we see that $M_{T_{Y,e_X}}$, $e_X \in E^{\mathrm{tr,et}}$, is a $k[\mu_n]$-module. Moreover, we have

$$1 \cdot (j \cdot \alpha_{l_{e_X,i}}) = (j+1) \cdot \alpha_{l_{e_X,i}}$$

if $0 \leq j \leq n_{v_Y} - 2$ and $0 \leq i \leq m_{v_Y} - 2$,

$$1 \cdot ((n_{v_Y} - 1) \cdot \alpha_{l_{e_X,i}}) = \alpha_{l_{e_X,i+1}}$$

if $0 \leq i \leq m_{v_Y} - 3$, and

$$1 \cdot \left((n_{v_Y} - 1) \cdot \alpha_{l_{e_Y}, m_{v_Y} - 2} \right) = - \sum_{i=0}^{m_{v_Y} - 2} \alpha_{l_{e_X}, i}.$$

Then Proposition 3.2 implies

$$\dim_k(M_{T_{Y, e_X}}(1)) = 1,$$

where $M_{T_{Y, e_X}}(1)$ denotes the subspace on which $\zeta \in \mu_n$ acts as the ζ-multiplication.

> *The subspace M_i:* Roughly speaking, this kind of subspace is generated by the corresponding vectors of the minimal loops constructed by the following minimal loops via Galois actions: Let l_i be a minimal loop of $\Gamma_{Y\bullet}$; Then the images of $l_i \cap e^{\mathrm{cl}}(\Gamma_{Y\bullet})$ in $\Gamma_{X\bullet}$ is two different elements of $E^{\mathrm{tr, et}}$.

Let $m \overset{\mathrm{def}}{=} \#(E^{\mathrm{tr, et}})$ and $E^{\mathrm{tr, et}} \overset{\mathrm{def}}{=} \{e_{X, 1}, \ldots, e_{X, m}\}$. Let $e_Y^i \in T_{Y, e_{X, i}}$, $i \in \{1, \ldots, m\}$, be an arbitrary element. Note that $v^{\Gamma_{Y\bullet}}(e_Y^i) = \{v_{Y, 0}, v_{Y, 1}\}$. Then

$$l_i \overset{\mathrm{def}}{=} v_{Y, 0} e_Y^i v_{Y, 1} e_Y^{i+1} v_{Y, 0}, \ i \in \{1, \ldots, m-1\},$$

can be regarded as a minimal loop of $\Gamma_{Y\bullet}$ (for instance, see Example 3.6 (b) below). Note that $e_Y^{i+1} \in T_{Y, e_{X, i+1}}$. Let

$$\alpha_i \overset{\mathrm{def}}{=} \sum_{j=0}^{m_{v_Y} - 1} j n_{v_Y} \cdot \alpha_{l_i}.$$

Note that the decomposition group of α_i is D_{v_Y}. Then the set of vectors

$$\{j \cdot \alpha_i\}_{i \in \{1, \ldots, m-1\}, j \in \{0, \ldots, n_{v_Y} - 1\}} \subseteq M_{\Gamma_{Y\bullet}}$$

is linearly independent. We denote by $M_i \subseteq M_{\Gamma_{Y\bullet}}$, $i \in \{1, \ldots, m-1\}$, the subspace spanned by $\{j \cdot \alpha_i\}_{j \in \{0, \ldots, n_{v_Y} - 1\}}$. Then M_i is a $k[(\mathbb{Z}/n\mathbb{Z})/D_{v_Y}]$-module. Thus, we obtain that

$$M_i \subseteq \bigoplus_{0 \leq j < n_{v_Y} - 1} M_{\Gamma_{Y\bullet}}(j m_{v_Y}), \ i \in \{1, \ldots, m-1\}.$$

Moreover, we see that the set of vectors

$$\{j \cdot \alpha_i\}_{i \in \{1, \ldots, m-1\}, j \in \{0, \ldots, n_{v_Y} - 1\}} \cup \{j \cdot \alpha_{l_{e_X}, i}\}_{i \in \{0, \ldots, m_{v_Y} - 2\}, j \in \{0, \ldots, n_{v_Y} - 1\}, e_X \in E^{\mathrm{tr, et}}} \subseteq M_{\Gamma_{Y\bullet}}$$

is linearly independent.

> *The subspace M_π:* Roughly speaking, this kind of subspace is generated by the corresponding vectors of the minimal loops constructed by the following minimal loops via Galois actions: Let l_Y be a minimal loop of $\Gamma_{Y\bullet}$; Then $l_Y \cap e^{\mathrm{cl}}(\Gamma_{Y\bullet})$ is *not stable* under the actions of decomposition groups of vertices $l_Y \cap v(\Gamma_{Y\bullet})$, and the images of $l_Y \cap e^{\mathrm{cl}}(\Gamma_{Y\bullet})$ in $\Gamma_{X\bullet}$ is an element of $E^{\mathrm{tr, et}}$.

Let $e_Y \in T_{Y, e_X}$ for some $e_X \in E^{\mathrm{tr}, \mathrm{et}}$. Then we see that $j \cdot e_Y$, $j \in \{0, \ldots, n_{v_Y} - 2\}$, abuts to $v_{Y,j}$ and $v_{Y,j+1}$ (i.e., $v^{\Gamma_{Y\bullet}}(j \cdot e_Y) = \{v_{Y,j}, v_{Y,j+1}\}$), and that $(n_{v_Y} - 1) \cdot e_Y$ abuts to $v_{Y, n_Y - 1}$ and $v_{Y,0}$ (i.e., $v^{\Gamma_{Y\bullet}}((n_{v_Y} - 1) \cdot e_Y) = \{v_{Y, n_{v_Y} - 1}, v_{Y,0}\}$). Thus, we have that

$$l_Y \overset{\mathrm{def}}{=} v_{Y,0}(0 \cdot e_Y)v_{Y,1} \cdots v_{Y, n_Y - 1}((n_{v_Y} - 1) \cdot e_Y)v_{Y,0}$$

can be regarded as a minimal loop of $\Gamma_{Y\bullet}$ (for instance, see Example 3.6 (c) below). Note that $0 \cdot e_Y = e_Y$. We put

$$\alpha_\pi \overset{\mathrm{def}}{=} \sum_{j=0}^{m_{v_Y} - 1} j n_{v_Y} \cdot \alpha_{l_Y} \in M_{\Gamma_{Y\bullet}}.$$

Let us prove that the decomposition group of α_π is $\mathbb{Z}/n\mathbb{Z}$. Note that α_π corresponds to the loop

$$\pi \overset{\mathrm{def}}{=} l_Y(n_{v_Y} \cdot l_Y) \cdots ((m_{v_Y} - 1)n_{v_Y} \cdot l_Y).$$

Let e be an arbitrary closed edge which is contained in π. Then there exists $r \in \mathbb{Z}/n_{v_Y}\mathbb{Z}$ such that e abuts to $v_{Y,r}$ and $v_{Y,r+1}$. Thus, e can be regraded as an oriented edge induced by the oriented loop π as follows:

- If $0 \le r \le n_Y - 2$, the starting of e is $v_{Y,r}$, and the ending of e is $v_{Y,r+1}$.
- If $r = n_Y - 1$, the starting of e is $v_{Y, n_{v_Y} - 1}$, and the ending of e is $v_{Y,1}$.

Consider the action of $1 \in \mathbb{Z}/n\mathbb{Z}$ on e. We see that $1 \cdot e$ is an oriented edge which is contained in π. Moreover, the starting of $1 \cdot e$ is $v_{Y,r+1}$, and the ending of $1 \cdot e$ is $v_{Y,r+2}$, where $r+1, r+2 \in \mathbb{Z}/n_{v_Y}\mathbb{Z}$. Namely, we have $1 \cdot \pi = \pi$. Thus, the stabilizer of α_π is $\mathbb{Z}/n\mathbb{Z}$. Write $M_\pi \subseteq M_{\Gamma_{Y\bullet}}$ for the subspace spanned by α_π. Then we have that

$$M_\pi \subseteq M_{\Gamma_{Y\bullet}}(0),$$

and that

$$\{j \cdot \alpha_i\}_{i \in \{1, \ldots, m-1\}, j \in \{0, \ldots, n_{v_Y} - 1\}} \cup \{j \cdot \alpha_{l_{e_X}, i}\}_{i \in \{1, \ldots, m_{v_Y} - 1\}, j \in \{0, \ldots, n_{v_Y} - 1\}, e_X \in E^{\mathrm{tr}, \mathrm{et}}} \cup \{\alpha_\pi\}$$

is linearly independent.

Let $l \subseteq \Gamma_{Y\bullet}$ be an arbitrary minimal loop of $\Gamma_{Y\bullet}$ and $V_l \subseteq M_{\Gamma_{Y\bullet}}$ the subspace spanned by $\{j \cdot \alpha_l\}_{j \in \mathbb{Z}/n\mathbb{Z}}$. Then if $\dim_k(V_l(1)) \neq 0$, we have

$$V_l(1) \subseteq \Big(\bigoplus_{e_X \in E^{\mathrm{tr}, \mathrm{et}}} M_{T_{Y, e_X}}\Big) \oplus \Big(\bigoplus_{e_X \in E^{\mathrm{lp}, \mathrm{et}}} M_{e_X}\Big) \oplus \Big(\bigoplus_{i=1}^{m} M_i\Big) \oplus M_\pi.$$

Thus, we obtain

$$M_{\Gamma_{Y\bullet}}(1) = \Big(\bigoplus_{e_X \in E^{\mathrm{tr}, \mathrm{et}}} M_{T_{Y, e_X}}(1)\Big) \oplus \Big(\bigoplus_{e_X \in E^{\mathrm{lp}, \mathrm{et}}} M_{e_X}(1)\Big).$$

This implies

$$\dim_k(M_{\Gamma_{Y\bullet}}(1)) = \#(E^{\mathrm{tr}, \mathrm{et}}) + \#(E^{\mathrm{lp}, \mathrm{et}}) = \#(\mathscr{N}_X^{\mathrm{et}})$$

if $E^{\mathrm{tr}} \neq \emptyset$ and $\#(v(\Gamma_{Y\bullet})) \neq n$. We complete the proof of the proposition. $\square$

Example 3.6. We maintain the notation introduced in the proof of Proposition 3.5. Suppose $n = 4$, $m_{v_Y} = 2$ and $\#(E^{\mathrm{tr}, \mathrm{et}}) = 2$.

(a) The case of $M_{T_{Y, e_X}}$: We have $l_{e_X, 0} \overset{\mathrm{def}}{=} v_{Y,0}e_{Y,0}v_{Y,1}e_{Y,1}v_{Y,0}$, where f^{sg} is as follows:
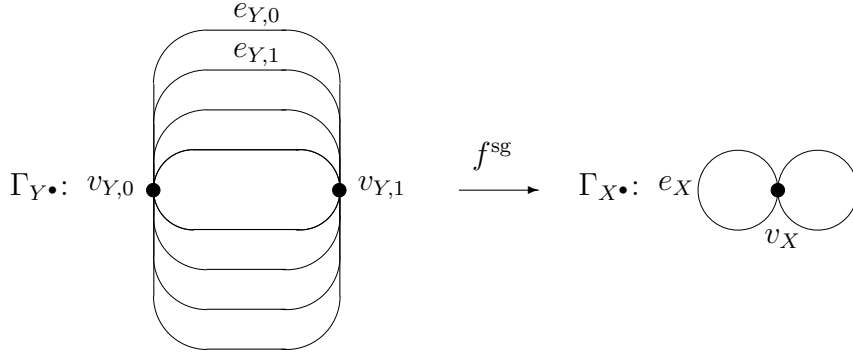
Here, $v_{Y,1} = 1 \cdot v_{Y,0}$.

(b) The case of M_i: We have $l_1 \overset{\text{def}}{=} v_{Y,0} e_Y^1 v_{Y,1} e_Y^2 v_{Y,0}$, where f^{sg} is as follows:

Here, $v_{Y,1} = 1 \cdot v_{Y,0}$.

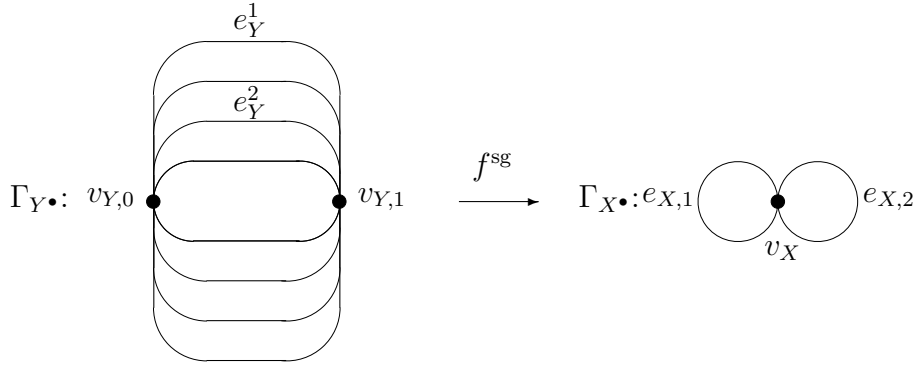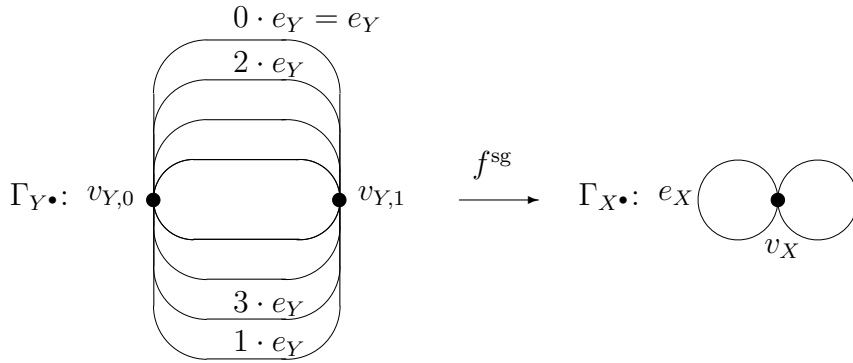(c) The case of M_π: We have $l_Y \overset{\text{def}}{=} v_{Y,0} (0 \cdot e_Y) v_{Y,1} (1 \cdot e_Y) v_{Y,0}$, where f^{sg} is as follows:

Here, $v_{Y,1} = 1 \cdot v_{Y,0}$.

3.1.7. We calculate generalized Hasse-Witt invariants of dual semi-graphs when $X^\bullet$ is not irreducible.

Lemma 3.7. *We maintain the notation introduced above. Suppose that the set of irreducible components of $X^\bullet$ is $\{X_1, X_2\}$, and that X_1, X_2 are non-singular. Write v_1 and v_2 for the vertices of $\Gamma_{X^\bullet}$ corresponding to X_1 and X_2, respectively. Let $f_i^\bullet \overset{\text{def}}{=} \widetilde{f}_{v_i}^\bullet : Y_i^\bullet \overset{\text{def}}{=} \widetilde{Y}_{v_i}^\bullet \to X_i^\bullet \overset{\text{def}}{=} \widetilde{X}_{v_i}^\bullet$, $i \in \{1, 2\}$, be the Galois multi-admissible covering over k induced by $f^\bullet$. Then the following statements hold:*

(i) Suppose that there exists $i \in \{1, 2\}$ such that $\#(v(\Gamma_{Y_i^\bullet})) = n$. Then we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#(\mathscr{N}_X^{\mathrm{et}}) - 1.$$

(ii) Suppose that $\#(X_1 \cap X_2) = 1$ (i.e., $\#(e^{\mathrm{cl}}(\Gamma_{X^\bullet})) = 1$) and $\#(v(\Gamma_{Y_i^\bullet})) \neq n$ for each $i \in \{1, 2\}$. Then we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \#(\mathscr{N}_X^{\mathrm{et}}).$$

Proof. (i) Suppose that either $\#(v(\Gamma_{Y_1^\bullet})) = 1$ or $\#(v(\Gamma_{Y_2^\bullet})) = 1$ holds. Then either $f_1^\bullet$ or $f_2^\bullet$ is a trivial Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then we see

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = r_X - r_{X_1} - r_{X_2} = r_X = \#(X_1 \cap X_2) - 1 = \#(\mathscr{N}_X^{\mathrm{et}}) - 1,$$

where r_{X_i}, $i \in \{1, 2\}$, denotes the Betti number of the dual semi-graph of $X_i^\bullet$.

Suppose $\#(v(\Gamma_{Y_1^\bullet})) = \#(v(\Gamma_{Y_2^\bullet})) = n$. Then f^{sg} is a Galois topological covering of $\Gamma_{X^\bullet}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then Lemma 3.1 (ii) implies

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = r_X - 1 = \#(\mathscr{N}_X^{\mathrm{et}}) - 1.$$

This completes the proof of (i).

(ii) Let $w_1 \in (f^{\mathrm{sg}})^{-1}(v_1)$ and $w_2 \in (f^{\mathrm{sg}})^{-1}(v_2)$ be arbitrary vertices of $\Gamma_{Y^\bullet}$. We denote by $D_1 \subseteq \mathbb{Z}/n\mathbb{Z}$ and $D_2 \subseteq \mathbb{Z}/n\mathbb{Z}$ the decomposition groups of w_1 and w_2, respectively, which do not depend on the choices of w_1 and w_2. Since $\#(X_1 \cap X_2) = 1$ (i.e., $\Gamma_{X^\bullet}$ is a tree), the Galois topological coverings of $\Gamma_{X^\bullet}$ with cyclic Galois groups do not exist. Namely, either $D_1 = \mathbb{Z}/n\mathbb{Z}$ or $D_2 = \mathbb{Z}/n\mathbb{Z}$ holds. Without loss of generality, we may assume that $D_1 = \mathbb{Z}/n\mathbb{Z} \supseteq D_2$. We put $m_2 \stackrel{\mathrm{def}}{=} \#(D_2)$ and $n_2 \stackrel{\mathrm{def}}{=} n/m_2$.

Let e_X be the unique closed edge of $\Gamma_{X^\bullet}$, $(f^{\mathrm{sg}})^{-1}(v_1) \stackrel{\mathrm{def}}{=} \{w_1\}$, $w_{2,0} \in (f^{\mathrm{sg}})^{-1}(v_2)$, and $e_{Y,0} \in (f^{\mathrm{sg}})^{-1}(e_X)$ such that $e_{Y,0}$ abuts to w_1 and $w_{2,0}$ (i.e., $v^{\Gamma_{Y^\bullet}}(e_{Y,0}) = \{w_1, w_{2,0}\}$). Let $D_{e_{Y,0}}$ be the decomposition group of $e_{Y,0}$, $m_{e_{Y,0}} = \#(D_{e_{Y,0}})$ and $n_{e_{Y,0}} \stackrel{\mathrm{def}}{=} n/m_{e_{Y,0}}$. Then we see that

$$M_{\Gamma_{Y^\bullet}} = \bigoplus_{0 \leq j \leq n_{e_{Y,0}} - 1} M_{\Gamma_{Y^\bullet}}(jm_{e_{Y,0}}).$$

Then we obtain that $\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = 0$ if $\#(\mathscr{N}_X^{\mathrm{et}}) = 0$.

Suppose that $\#(\mathscr{N}_X^{\mathrm{et}}) = 1$. Then $f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ is equal to the covering of dual semi-graphs induced by the Galois admissible covering constructed in 3.1.5. Thus, Lemma 3.4 implies $\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = 1 = \#(\mathscr{N}_X^{\mathrm{et}})$. This completes the proof of the lemma. $\square$

Proposition 3.8. *We maintain the notation introduced above. Suppose that the set of irreducible components of $X^\bullet$ is $\{X_1, X_2\}$, and that X_1, X_2 are non-singular. Then we have*

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) =$$

$$\begin{cases} \#(\mathscr{N}_X^{\mathrm{et}}) - 1, & \text{if there exists } i \in \{1, 2\} \text{ such that } \#(v(\Gamma_{Y_i^\bullet})) = n, \\ \#(\mathscr{N}_X^{\mathrm{et}}), & \text{if for each } i \in \{1, 2\}, \ \#(v(\Gamma_{Y_i^\bullet})) \neq n. \end{cases}$$

Proof. Suppose that there exists $i \in \{1, 2\}$ such that $\#(v(\Gamma_{Y_i^\bullet})) = n$. Then the proposition follows from Lemma 3.7 (i). To verify the proposition, we may assume that $\#(v(\Gamma_{Y_i^\bullet})) \neq n$ for all $i \in \{1, 2\}$. Moreover, if $\#(X_1 \cap X_2) = 1$, then the proposition follows from Lemma 3.7 (ii). Thus, we may assume that $\#(X_1 \cap X_2) \geq 2$.

First, let us construct two Galois admissible coverings associated to $f^\bullet : Y^\bullet \to X^\bullet$. Let R be a complete discrete valuation ring with residue field k, K the quotient field of

R, $\overline{K}$ an algebraic closure of K, $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ an arbitrary closed edge, and x_e the node of X corresponding to e. By deforming X along x_e, we obtain a pointed stable curve $\mathcal{X}$ over R whose special fiber is $X^\bullet$, and whose generic fiber $X_K^\bullet$ is an irreducible pointed stable curve over K such that $\#(e^{\mathrm{cl}}(\Gamma_{X_K^\bullet})) = \#(e^{\mathrm{lp}}(\Gamma_{X_K^\bullet})) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet})) - 1$, where $\Gamma_{X_K^\bullet}$ denotes the dual semi-graph of $X_K^\bullet$. Moreover, since the specialization homomorphism of admissible fundamental groups is a surjection, by replacing R by a finite extension of R, $f^\bullet$ can be lifted to a finite morphism $f_K^\bullet : Y_K^\bullet \to X_K^\bullet$ over K such that

$$f_{\backslash e}^\bullet \overset{\mathrm{def}}{=} f_K^\bullet \times_K \overline{K} : Y_{\backslash e}^\bullet \overset{\mathrm{def}}{=} Y_K^\bullet \times_K \overline{K} \to X_{\backslash e}^\bullet \overset{\mathrm{def}}{=} X_K^\bullet \times_K \overline{K}$$

is a Galois admissible covering over $\overline{K}$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Note that $X_{\backslash e}^\bullet$ is irreducible. We write $\Gamma_{Y_{\backslash e}^\bullet}$ for the dual semi-graph of $Y_{\backslash e}^\bullet$ and denote by $M_{\Gamma_{Y_{\backslash e}^\bullet}} \overset{\mathrm{def}}{=} H^1(\Gamma_{Y_{\backslash e}^\bullet}, \mathbb{F}_p) \otimes k$. Then $M_{\Gamma_{Y_{\backslash e}^\bullet}}$ is a $k[\mu_n]$-module and admits the following canonical decomposition

$$M_{\Gamma_{Y_{\backslash e}^\bullet}} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y_{\backslash e}^\bullet}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y_{\backslash e}^\bullet}}(j)$ as the ζ^j-multiplication.

On the other hand, let $\mathrm{norm}_e : X_e \to X$ be the normalization morphism of X over the nodes corresponding to the closed edges which are contained in $e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \setminus \{e\}$. Then we obtain a pointed stable curve

$$X_e^\bullet = (X_e, D_{X_e} \overset{\mathrm{def}}{=} \mathrm{norm}_e^{-1}(D_X) \cup \{\mathrm{norm}^{-1}(x_{e'})\}_{e' \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \setminus \{e\}})$$

over k. Note that X_e has two non-singular irreducible components X_1 and X_2, and that $X_1 \cap X_2 = \{x_e\}$ in X_e. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f_e^\bullet : Y_e^\bullet \to X_e^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. We write $\Gamma_{Y_e^\bullet}$ for the dual semi-graph of $Y_e^\bullet$ and denote by $M_{\Gamma_{Y_e^\bullet}} \overset{\mathrm{def}}{=} H^1(\Gamma_{Y_e^\bullet}, \mathbb{F}_p) \otimes k$. Then $M_{\Gamma_{Y_e^\bullet}}$ is a $k[\mu_n]$-module and admits the following canonical decomposition

$$M_{\Gamma_{Y_e^\bullet}} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y_e^\bullet}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y_e^\bullet}}(j)$ as the ζ^j-multiplication.

Write $\mathscr{N}_{X_{\backslash e}}^{\mathrm{et}}$ and $\mathscr{N}_{X_e}^{\mathrm{et}}$ for the sets of nodes of $X_{\backslash e}$ and X_e over which $f_{\backslash e}$ and f_e are étale, respectively. Then we see immediately $\#(\mathscr{N}_{X_{\backslash e}}^{\mathrm{et}}) + \#(\mathscr{N}_{X_e}^{\mathrm{et}}) = \#(\mathscr{N}_X^{\mathrm{et}})$. The constructions of $\Gamma_{Y_{\backslash e}^\bullet}$ and $\Gamma_{Y_e^\bullet}$ imply $M_{\Gamma_{Y^\bullet}} = M_{\Gamma_{Y_{\backslash e}^\bullet}} \oplus M_{\Gamma_{Y_e^\bullet}}$ as $k[\mu_n]$-modules. Then we obtain

$$M_{\Gamma_{Y^\bullet}}(1) = M_{\Gamma_{Y_{\backslash e}^\bullet}}(1) \oplus M_{\Gamma_{Y_e^\bullet}}(1).$$

On the other hand, Proposition 3.5 and Lemma 3.7 imply that

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \dim_k(M_{\Gamma_{Y_{\backslash e}^\bullet}}(1)) + \dim_k(M_{\Gamma_{Y_e^\bullet}}(1))$$

$$= \#(\mathscr{N}_{X_{\backslash e}}^{\mathrm{et}}) + \#(\mathscr{N}_{X_e}^{\mathrm{et}}) = \#(\mathscr{N}_X^{\mathrm{et}}).$$

This completes the proof of the proposition. $\qquad\square$

3.2. **Generalized Hasse-Witt invariants of curves and their irreducible components.**

3.2.1. **Settings.** We maintain the notation introduced in 2.1.2 and the assumption introduced in 2.3.3 (i.e., $n \stackrel{\text{def}}{=} p^t - 1$).

3.2.2. The main result of the present section is as follows:

Theorem 3.9. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5) and $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$ (Definition 2.4 (i)). Let $\Pi_{X^\bullet}$ be the admissible fundamental group of $X^\bullet$ and*

$$f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$$

the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. For each $v \in v(\Gamma_{X^\bullet})$, $f^\bullet$ induces a Galois multi-admissible covering for the smooth pointed stable curve of type (g_v, n_v) associated to v (2.1.3)

$$\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Let $\widetilde{\alpha}_v \in \mathrm{Hom}(\Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ be the homomorphism $\Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}} \xrightarrow{\alpha} \mathbb{Z}/n\mathbb{Z}$, where $\Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}}$ is the natural homomorphism induced by the natural (outer) injection $\Pi_{\widetilde{X}_v^\bullet} \hookrightarrow \Pi_{X^\bullet}$. Then we have (see Definition 2.4 (i)-(1) for $\gamma_{(\alpha,D)}$ and 2.2.5 for $s(D)$)

$$\gamma_{(\alpha,D)} = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset, \end{cases}$$

if and only if, for each $v \in v(\Gamma_{X^\bullet})$,

$$\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})} = \begin{cases} g_v, & \text{if } \widetilde{\alpha}_v = 0, \\ g_v - 1, & \text{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) = \emptyset, \\ g_v + s(D_{\widetilde{\alpha}_v}) - 1, & \text{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) \neq \emptyset, \end{cases}$$

where $\mathrm{Supp}(-)$ denotes the support of $(-)$.

Proof. We prove the theorem by induction on the cardinality $\#(v(\Gamma_{X^\bullet}))$ of $v(\Gamma_{X^\bullet})$. Suppose that $\#(v(\Gamma_{X^\bullet})) = 1$ (i.e., X is irreducible). Then we have $D_{\widetilde{\alpha}_v}|_{U_{X_v}} = D$ and

$$g_v = g_X - \#(X^{\mathrm{sing}}),$$

where U_{X_v} is the open subset of X_v defined in 2.1.3. Moreover, since $\widetilde{X}_v^\bullet$ is smooth over k, we write $([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v}) \in \widetilde{\mathscr{P}}_{\widetilde{X}_v^\bullet, n}$ for the pair induced by $\widetilde{\alpha}_v$ (2.3.2). Write $\mathscr{N}_X^{\mathrm{ra}} \subseteq X^{\mathrm{sing}}$ for the subset of nodes over which f is ramified and $\mathscr{N}_X^{\mathrm{et}} \subseteq X^{\mathrm{sing}}$ for the subset of nodes over which f is étale. Then we have $s(D_{\widetilde{\alpha}_v}) = s(D) + \#(\mathscr{N}_X^{\mathrm{ra}})$.

On the other hand, by Lemma 2.7, we have

$$\dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v})) = \begin{cases} g_v, & \text{if } \widetilde{\alpha}_v = 0, \\ g_v - 1, & \text{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) = \emptyset, \\ g_v + s(D_{\widetilde{\alpha}_v}) - 1, & \text{if } \widetilde{\alpha}_v \neq 0, \ \mathrm{Supp}(D_{\widetilde{\alpha}_v}) \neq \emptyset. \end{cases}$$

Write $\Gamma_{Y^\bullet}$ for the dual semi-graph of $Y^\bullet$. The natural $k[\mu_n]$-submodule

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k \subseteq H^1_{\text{ét}}(Y, \mathbb{F}_p) \otimes k$$

admits the following canonical decomposition

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y^\bullet}}(j)$ as the ζ^j-multiplication. By Proposition 3.5, we have

$$\dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \begin{cases} \#(\mathscr{N}_X^{\mathrm{et}}) - 1, & \text{if } \widetilde{\alpha}_v = 0, \\ \#(\mathscr{N}_X^{\mathrm{et}}), & \text{if } \widetilde{\alpha}_v \neq 0. \end{cases}$$

Thus, we obtain

$$\dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v})) + \dim_k(M_{\Gamma_{Y^\bullet}}(1)) = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset. \end{cases}$$

Since $\gamma_{(\alpha,D)} = \gamma_{([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v})} + \dim_k(M_{\Gamma_{Y^\bullet}}(1))$ and $\#(X^{\mathrm{sing}}) = \#(\mathscr{N}_X^{\mathrm{ra}}) + \#(\mathscr{N}_X^{\mathrm{et}})$, we have

$$\gamma_{(\alpha,D)} = \begin{cases} g_X - 1, & \text{if } \mathrm{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \mathrm{Supp}(D) \neq \emptyset \end{cases}$$

if and only if $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})} = \gamma_{([\mathcal{L}_{\widetilde{\alpha}_v}], D_{\widetilde{\alpha}_v})} = \dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\widetilde{\alpha}_v}))$. This completes the proof of the proposition when $\#(v(\Gamma_{X^\bullet})) = 1$.

Suppose $m \overset{\mathrm{def}}{=} \#(v(\Gamma_{X^\bullet})) \geq 2$. Let $v_0 \in v(\Gamma_{X^\bullet})$ be a vertex such that $\Gamma_{X^\bullet} \setminus \{v_0, e^{\Gamma_{X^\bullet}}(v_0)\}$ (see 2.1.1 for $e^{\Gamma_{X^\bullet}}(v_0)$) is connected (note that v_0 always exists). Write X_1 for the topological closure of $X \setminus X_{v_0}$ in X and X_2 for X_{v_0}. Note that X_1 is connected. We define a pointed stable curve

$$X_i^\bullet = (X_i, D_{X_i} \overset{\mathrm{def}}{=} (X_i \cap D_X) \cup (X_1 \cap X_2)), \ i \in \{1, 2\},$$

of type (g_{X_i}, n_{X_i}) over k. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f_i^\bullet : Y_i^\bullet \to X_i^\bullet, \ i \in \{1, 2\},$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we denote by $\alpha_i \in \mathrm{Hom}(\Pi_{X_i^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$, $i \in \{1, 2\}$, the composition of the natural homomorphisms $\Pi_{X_i^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}} \overset{\alpha}{\to} \mathbb{Z}/n\mathbb{Z}$, where $\Pi_{X_i^\bullet}$ denotes the admissible fundamental group of $X_i^\bullet$.

Write $\mathscr{N}_{X_1 \cap X_2}^{\mathrm{ra}} \subseteq X_1 \cap X_2$ for the subset of nodes over which f is ramified and $\mathscr{N}_{X_1 \cap X_2}^{\mathrm{et}} \subseteq X_1 \cap X_2$ for the subset of nodes over which f is étale. Then we have $\#(\mathscr{N}_{X_1 \cap X_2}^{\mathrm{ra}}) + \#(\mathscr{N}_{X_1 \cap X_2}^{\mathrm{et}}) = \#(X_1 \cap X_2)$. Moreover, we have

$$\gamma_{(\alpha_i, D_{\alpha_i})} \leq \begin{cases} g_{X_i}, & \text{if } \alpha_i = 0, \\ g_{X_i} - 1, & \text{if } \alpha_i \neq 0, \ \mathrm{Supp}(D_{\alpha_i}) = \emptyset, \\ g_{X_i} + s(D_{\alpha_i}) - 1, & \text{if } \alpha_i \neq 0, \ \mathrm{Supp}(D_{\alpha_i}) \neq \emptyset, \end{cases}$$

for all $i \in \{1, 2\}$. Note that the definition of admissible coverings implies that $s(D_{\alpha_1}) + s(D_{\alpha_2}) = s(D) + \#(\mathscr{N}_{X_1 \cap X_2}^{\mathrm{ra}})$.

On the other hand, the natural $k[\mu_n]$-modules $H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k$ and $H^1(\Gamma_{Y_i^\bullet}, \mathbb{F}_p) \otimes k$, $i \in \{1, 2\}$, admit the following canonical decomposition

$$H^1(\Gamma_{Y^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y^\bullet}}(j),$$

$$H^1(\Gamma_{Y_i^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{Y_i^\bullet}}(j), \ i \in \{1, 2\},$$

respectively, where $\Gamma_{Y_i^\bullet}$ denotes the dual semi-graph of $Y_i^\bullet$, and $\zeta \in \mu_n$ acts on $M_{\Gamma_{Y^\bullet}}(j)$ and $M_{\Gamma_{Y_i^\bullet}}(j)$, $i \in \{1, 2\}$, as the ζ^j-multiplication, respectively. We put

$$\dim_k(M_{\Gamma_{Y_1 \cap Y_2}}(1)) \overset{\text{def}}{=} \dim_k(M_{\Gamma_{Y^\bullet}}(1)) - \dim_k(M_{\Gamma_{Y_1^\bullet}}(1)) - \dim_k(M_{\Gamma_{Y_2^\bullet}}(1)).$$

Let us compute $\dim_k(M_{\Gamma_{Y_1 \cap Y_2}}(1))$. Without loss of generality, we may assume that X_1 and X_2 are non-singular. Then Proposition 3.8 implies that the following holds:

$$\dim_k(M_{\Gamma_{Y_1 \cap Y_2}}(1)) =$$

$$\begin{cases} \#(\mathscr{N}^{\text{et}}_{X_1 \cap X_2}) - 1, & \text{if there exists } i \in \{1, 2\} \text{ such that } \alpha_i = 0, \\ \#(\mathscr{N}^{\text{et}}_{X_1 \cap X_2}), & \text{if for each } i \in \{1, 2\}, \ \alpha_i \neq 0. \end{cases}$$

Since $\alpha \neq 0$, we obtain

$$\gamma_{(\alpha, D)} = \gamma_{(\alpha_1, D_{\alpha_1})} + \gamma_{(\alpha_2, D_{\alpha_2})} + \dim_k(M_{\Gamma_{Y_1 \cap Y_2}}(1))$$

$$\leq g_{X_1} + s(D_{\alpha_1}) + g_{X_2} + s(D_{\alpha_2}) + \#(\mathscr{N}^{\text{et}}_{X_1 \cap X_2}) - 2 = g_X + s(D) - 1$$

$$= \begin{cases} g_X - 1, & \text{if } \text{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \text{Supp}(D) \neq \emptyset. \end{cases}$$

Thus, we have

$$\gamma_{(\alpha, D)} = \begin{cases} g_X - 1, & \text{if } \text{Supp}(D) = \emptyset, \\ g_X + s(D) - 1, & \text{if } \text{Supp}(D) \neq \emptyset \end{cases}$$

if and only if

$$\gamma_{(\alpha_i, D_{\alpha_i})} = \begin{cases} g_{X_i}, & \text{if } \alpha_i = 0, \\ g_{X_i} - 1, & \text{if } \alpha_i \neq 0, \ \text{Supp}(D_{\alpha_i}) = \emptyset, \\ g_{X_i} + s(D_{\alpha_i}) - 1, & \text{if } \alpha_i \neq 0, \ \text{Supp}(D_{\alpha_i}) \neq \emptyset, \end{cases}$$

for all $i \in \{1, 2\}$. By induction, the theorem follows from the theorem when $\#(v(\Gamma_{X^\bullet})) = m - 1$ and $\#(v(\Gamma_{X^\bullet})) = 1$. This completes the proof of the theorem. $\square$

3.2.3. We define maximum generalized Hasse-Witt invariants as follows:

Definition 3.10. (i) We put

$$\gamma_{X^\bullet}^{\max} \overset{\text{def}}{=} \max_{t \in \mathbb{N}} \{\gamma_{(\alpha, D_\alpha)} \mid \alpha \in \text{Hom}(\Pi_{X^\bullet}^{\text{ab}}, \mathbb{Z}/(p^t - 1)\mathbb{Z}) \text{ and } \alpha \neq 0\}$$

$$= \max_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} \{\gamma_{(\alpha, D_\alpha)} \mid \alpha \in \text{Hom}(\Pi_{X^\bullet}^{\text{ab}}, \mathbb{Z}/m\mathbb{Z}) \text{ and } \alpha \neq 0\},$$

and call $\gamma_{X^\bullet}^{\max}$ the maximum generalized Hasse-Witt invariant of prime-to-p cyclic admissible coverings of $X^\bullet$. Note that Theorem 3.9 implies that

$$\gamma_{X^\bullet}^{\max} \leq \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

(ii) Let m be a natural number prime to p and $\alpha \in \text{Hom}(\Pi_{X^\bullet}^{\text{ab}}, \mathbb{Z}/m\mathbb{Z})$. We shall say that $\gamma_{(\alpha, D_\alpha)}$ *attains maximum* if

$$\gamma_{(\alpha, D_\alpha)} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

We shall say that $\gamma_{X^\bullet}^{\max}$ *attains maximum* if there exist a prime-to-p natural number m' and an element $\alpha' \in \text{Hom}(\Pi_{X^\bullet}^{\text{ab}}, \mathbb{Z}/m'\mathbb{Z})$ such that $\gamma_{(\alpha', D_{\alpha'})}$ attains maximum.

3.3. **An easy case of main results.**

3.3.1. **Settings.** We maintain the notation introduced in 2.1.2.

3.3.2. By applying Nakajima and Raynaud's results concerning ordinariness of cyclic étale coverings, we have the following result.

Proposition 3.11. *(i) Let $n \in \mathbb{N}$ be an arbitrary natural number prime to p, $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$, and $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$. Suppose that $X^\bullet$ is a smooth component-generic pointed stable curve over k, and that $n_X \leq 1$. Then $\gamma_{(\alpha,D)}$ attains maximum. Namely, the following holds:*

$$\gamma_{(\alpha,D)} = g_X - 1.$$

(ii) Let $X^\bullet$ be an arbitrary pointed stable curve over k. Suppose that $n_X \leq 1$. Then $\gamma_{X^\bullet}^{\max}$ attains maximum. Namely, the following holds:

$$\gamma_{X^\bullet}^{\max} = g_X - 1.$$

Proof. Since $n_X \leq 1$, the structures of maximal prime-to-p quotients of admissible fundamental groups (2.1.6) imply $D = 0$. Namely, all of the prime-to-p cyclic Galois admissible coverings of $X^\bullet$ are étale over D_X (in particular, are étale if $X^\bullet$ is smooth over k).

(i) This follows immediately from [N, Proposition 4].

(ii) Let $v \in v(\Gamma_{X^\bullet})$, and let $\widetilde{X}_v^\bullet$ be the smooth pointed stable curve of type (g_v, n_v) over k associated to v (2.1.3). We denote by

$$\mathscr{V} \overset{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid g_v > 0\}.$$

Suppose that $\mathscr{V} = \emptyset$. Then $n_X \leq 1$ implies that $\Gamma_{X^\bullet}$ is not a tree. Namely, $\Pi_{X^\bullet}^{\mathrm{top}}$ is not trivial. Let $\alpha' : \Pi_{X^\bullet}^{\mathrm{top,ab}} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ be a surjection and $\alpha : \Pi_{X^\bullet}^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ the composite of the homomorphisms $\Pi_{X^\bullet}^{\mathrm{ab}} \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{top,ab}} \overset{\alpha'}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}$. Then the theorem follows from Lemma 3.1 (ii).

Suppose that $\mathscr{V} \neq \emptyset$. Let $v \in \mathscr{V}$. Then Proposition 2.10 and Theorem 2.11 imply that there exists an element $\widetilde{\alpha}_v \in \mathrm{Rev}_0^{\mathrm{adm}}(\widetilde{X}_v^\bullet)$ such that $\widetilde{\alpha}_v : \Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ is a surjection, and that

$$\gamma_{(\widetilde{\alpha}_v,0)} = g_v - 1.$$

Write $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ for the connected Galois étale covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $\widetilde{\alpha}_v$.

Let $\mathscr{C}$ be the set of connected components of $\overline{X \setminus \bigcup_{v \in \mathscr{V}} X_v}$ and $C \in \mathscr{C}$, where $\overline{X \setminus \bigcup_{v \in \mathscr{V}} X_v}$ denotes the topological closure of $X \setminus \bigcup_{v \in \mathscr{V}} X_v$ in X. We define

$$C^\bullet = (C, D_C \overset{\mathrm{def}}{=} (C \cap \bigcup_{v \in \mathscr{V}} X_v) \cup (D_X \cap C))$$

to be a pointed stable curve over k. Note that the normalization of each irreducible component of C is isomorphic to $\mathbb{P}_k^1$. We put

$$Y_C^\bullet \overset{\mathrm{def}}{=} \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} C_i^\bullet,$$

where $C_i^\bullet$ is a copy of $C^\bullet$. Then we obtain a Galois multi-admissible covering $f_C^\bullet : Y_C^\bullet \to C^\bullet$ over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $f_C^\bullet|_{C_i^\bullet}$ is an identity, and the Galois action is $j(C_i) = C_{i+j}$ for all $i, j \in \mathbb{Z}/n\mathbb{Z}$. By gluing $\{\widetilde{Y}_v^\bullet\}_{v \in \mathscr{V}}$ and $\{Y_C^\bullet\}_{C \in \mathscr{C}}$

along $\{D_{\widetilde{X}_v}\}_{v \in \mathscr{V}}$ and $\{D_C\}_{C \in \mathscr{C}}$ in a way that is compatible with the gluing of $\{\widetilde{X}_v^\bullet\}_{v \in \mathscr{V}} \cup \{C^\bullet\}_{C \in \mathscr{C}}$ that gives rise to $X^\bullet$, we obtain a Galois (étale) admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Write $\alpha \in \mathrm{Rev}_0^{\mathrm{adm}}(X^\bullet)$ for an element induced by $f^\bullet$ such that the composition of the natural homomorphisms $\Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}} \xrightarrow{\alpha} \mathbb{Z}/n\mathbb{Z}$ is equal to $\widetilde{\alpha}_v$ constructed above if $v \in \mathscr{V}$, and is equal to $\widetilde{\alpha}_v \overset{\mathrm{def}}{=} 0$ if $v \notin \mathscr{V}$. This means that the following holds:

$$\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})} = \begin{cases} g_v - 1, & \text{if } v \in \mathscr{V}, \\ g_v = 0, & \text{if } v \notin \mathscr{V}. \end{cases}$$

Then (ii) follows from Theorem 3.9. $\qquad\square$

3.3.3. If $X^\bullet$ is an arbitrary *component-generic pointed stable curve* (i.e., reducible and the dual semi-graph is not a tree) over k, we see that Proposition 3.11 (i) *does not hold in general*. For instance, we have the following example:

Example 3.12. Let $X^\bullet$ be a component-generic pointed stable curve of type $(g_X, 0)$ over k with two smooth irreducible components X_1 and X_2 of genus g_1 and g_2, respectively. Moreover, suppose that $X_1 \cap X_2 = \{x_1, x_2, x_3\}$. We put

$$X_j^\bullet = (X_j, D_{X_j} \overset{\mathrm{def}}{=} X_1 \cap X_2), \ j \in \{1, 2\}.$$

Let $n \overset{\mathrm{def}}{=} p^t - 1$, and let $D_1 \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_{X_1}]^0$ be an effective divisor on X_1 such that $s(D_1) = 2$, and that $s(D_1^{(i)}) = 1 < s(D_1) = 2$ for some $i \in \{1, \ldots, t-1\}$ (see Definition 2.4 (ii) for $D_1^{(i)}$). We put

$$D_2 \overset{\mathrm{def}}{=} (n - \mathrm{ord}_{x_1}(D_1))x_1 + (n - \mathrm{ord}_{x_2}(D_1))x_2 + (n - \mathrm{ord}_{x_3}(D_1))x_3.$$

Then $D_2 \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_{X_2}]^0$ is an effective divisor on X_2 with degree n (i.e., $s(D_2) = 1$).

Let $\alpha_j \in \mathrm{Rev}_{D_j}^{\mathrm{adm}}(X_j^\bullet)$, $j \in \{1, 2\}$, and let $f_j^\bullet : Y_j^\bullet \to X_j^\bullet$ be the Galois multi-admissible covering induced by α_j. Then by gluing $\{X_j^\bullet\}_{j=1,2}$ and $\{Y_j^\bullet\}_{j=1,2}$, we obtain a Galois multi-admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

over k. Write $\alpha \in \mathrm{Rev}_0^{\mathrm{adm}}(X^\bullet)$ for the element induced by $f^\bullet$. Then Theorem 3.9 implies that $\gamma_{(\alpha, 0)} = g_X - 1 = g_1 + g_2 + 1$ if and only if $\gamma_{(\alpha_1, D_1)} = g_1 + 1$ and $\gamma_{(\alpha_2, D_2)} = g_2$. On the other hand, since $s(D_1^{(i)}) = 1$ for some $i \in \{1, \ldots, t-1\}$, we have that $\gamma_{(\alpha_1, D_1)} = g_1$. Then we have

$$\gamma_{(\alpha, 0)} \neq g_X - 1.$$

Namely, $\gamma_{(\alpha, 0)}$ does not attain maximum.

On the other hand, if $X^\bullet$ *is not a component-generic pointed stable curve*, then Proposition 3.11 (i) does not hold in general. In fact, if k is an algebraic closure of $\mathbb{F}_p$, then Proposition 3.11 (i) *does not hold for an arbitrary pointed stable curve* over k (see Proposition 4.15 (ii)).

3.3.4. The main goals of the next two sections are to generalize Proposition 3.11 to the case of (possibly singular) pointed stable curves of an arbitrary type (g_X, n_X). More precisely, we will generalize Proposition 3.11 as follows:

The first main theorem: a generalization of Proposition 3.11 (i). Let n be a natural number prime to p and $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. In order for $\gamma_{(\alpha',D)}$ to attain the maximum for some $\alpha' \in \mathrm{Rev}_D^{\mathrm{adm}}(X^{\bullet})$, it is necessary for D to satisfy a certain condition, which we call being *Frobenius stable* (see [Y6, Definition 3.3] or 4.1.2 below). For a Frobenius stable divisor D, we will prove that $\gamma_{(\alpha,D)}$ attains maximum for *all* $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^{\bullet})$ when $X^{\bullet}$ is a certain component-generic pointed stable curve (Theorem 4.13 (i)). Moreover, we will prove that $\gamma_{(\beta,D)}$ attains maximum for *some* $\beta \in \mathrm{Rev}_D^{\mathrm{adm}}(X^{\bullet})$ when $X^{\bullet}$ is an *arbitrary* component-generic pointed stable curve (Theorem 4.13 (ii)).

The second main theorem: a generalization of Proposition 3.11 (ii). We will prove that $\gamma_{X^{\bullet}}^{\max}$ attains maximum for *arbitrary* pointed stable curve of type (g_X, n_X) over k (Theorem 5.4).

4. Maximum generalized Hasse-Witt invariants for generic curves

In the present section, we discuss the maximum generalized Hasse-Witt invariants of cyclic admissible coverings for generic curves. The main result of this section is Theorem 4.13.

4.1. **Idea.** We briefly explain the idea of our proof of Theorem 4.13.

4.1.1. **Settings.** We maintain the notation introduced in 2.1.2. Moreover, we suppose that $X^{\bullet}$ is a *component-generic* pointed stable curve over k (2.1.4).

4.1.2. Let n be a natural number prime to p, $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5), $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^{\bullet})$ (Definition 2.4 (i)), and $f_{\alpha}^{\bullet} : X_{\alpha}^{\bullet} \to X^{\bullet}$ the Galois multi-admissible covering corresponding to α. Note that there exists a natural number $t \in \mathbb{N}$ such that $p^t = 1$ in $(\mathbb{Z}/n\mathbb{Z})^{\times}$. Write m' for $(p^t - 1)/n$. To compute $\gamma_{(\alpha,D)}$, by replacing α and D by the composition of homomorphisms $\Pi_{X^{\bullet}}^{\mathrm{ab}} \xrightarrow{\alpha} \mathbb{Z}/n\mathbb{Z} \xrightarrow{m'} \mathbb{Z}/(p^t-1)\mathbb{Z}$ and $m'D$, we may assume that $n \overset{\mathrm{def}}{=} p^t - 1$.

A necessary condition: The generalized Hasse-Witt invariant $\gamma_{(\alpha,D)}$ attains maximum (Definition 3.10 (ii)) if the following holds (see Definition 2.4 (ii) for $D^{(i)}$):

$$\deg(D^{(i)}) = \deg(D) = (n_X - 1)n, \ i \in \{0, \ldots, t-1\}.$$

The above condition concerning D is a special case of *Frobenius stable* ([Y6, Definition 3.3]) when $\deg(D) = (n_X - 1)n$. Then to verify our main result of this section, we may assume that D satisfies the above condition.

4.1.3. *Irreducible case.* Suppose that $X^{\bullet}$ is irreducible. Moreover, by Proposition 3.11, we may assume $n_X \geq 2$. First, if $(g_X, n_X) = (0, 3)$, then the first main result follows from a result of I. Bouw (Lemma 4.4). Next, for the case of an arbitrary (g_X, n_X), since $X^{\bullet}$ is component-generic, we introduce certain degeneration data (i.e., degenerations of $X^{\bullet}$) concerning $X^{\bullet}$ (see 4.3.2) such that, for instance, the irreducible components of the degeneration data are either type of $(0, 3)$ or $(1, 0)$. Then by applying Lemma 4.4, Proposition 3.11 (i), and Theorem 3.9, we see that the first generalized Hasse-Witt invariants (2.2.4) of Galois multi-admissible coverings of the degenerations of $X^{\bullet}$ induced by $f_{\alpha}^{\bullet}$ attain maximum. Moreover, the specialization homomorphisms of admissible fundamental groups implies that $\gamma_{(\alpha,D)}$ attains maximum. This completes the proof of Theorem 4.13 (i) when $X^{\bullet}$ is irreducible (Proposition 4.6).

4.1.4. *General case.* We have seen that $\gamma_{(\alpha,D)}$ does not attain maximum for all $\alpha \in \text{Rev}_D^{\text{adm}}(X^\bullet)$ in general if $X^\bullet$ is an arbitrary component-generic pointed stable curve (e.g. Example 3.12). To avoid such situation, we introduce a kind of semi-graph associated to the sets of marked points of pointed stable curves, which we call *a minimal quasi-tree associated to D_X* (4.4). Roughly speaking, a minimal quasi-tree is a minimal tree-like semi-graph contained in the dual semi-graph of a pointed stable curve which contains all open edges.

Firstly, we see that, if the dual semi-graph $\Gamma_{X^\bullet}$ is a minimal quasi-tree, then the ramifications over nodes of a Galois multi-admissible covering can be completely determined by the ramifications over the marked points D_X. To explain this observation, let us see the following example.

Example 4.1. Let $X^\bullet$ be a pointed stable curve of type $(g_X, 2)$ over k with two smooth irreducible components X_1 and X_2. Moreover, we assume that $X_1 \cap D_X = \{x_1\}$, $X_2 \cap D_X = \{x_2\}$, and $X_1 \cap X_2 = X^{\text{sing}} = \{x\}$. Let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ and $D = ax_1 + bx_2 \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ the ramification divisor associated to $f^\bullet$. Note that we have $a + b = 0 \mod n$. Write $f_i^\bullet : Y_i^\bullet \to X_i^\bullet \overset{\text{def}}{=} (X_i, D_{X_i} \overset{\text{def}}{=} \{x_i, x\})$, $i \in \{1, 2\}$, for the Galois multi-admissible covering induced by $f^\bullet$ and $D_i \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_i}]^0$ for the ramification divisor associated to $f_i^\bullet$. Note that $\text{ord}_{x_1}(D_1) = a$ and $\text{ord}_{x_2}(D_2) = b$. Moreover, since $\text{ord}_{x_1}(D_1) + \text{ord}_x(D_1) = 0 \mod n$ and $\text{ord}_x(D_2) + \text{ord}_{x_2}(D_2) = 0 \mod n$, we obtain

$$\text{ord}_x(D_1) = b, \ \text{ord}_x(D_2) = a.$$

Namely, we have $D_1 = ax_1 + bx$, $D_2 = ax + bx_2$. This means that the ramification of $f^\bullet$ over the node x can be determined completely by the ramification of $f^\bullet$.

Secondly, by applying Theorem 3.9, we can reduce Theorem 4.13 (i) to the case where $X^\bullet$ is irreducible. This completes Theorem 4.13 (i) (Proposition 4.12). Moreover, by gluing certain Galois multi-admissible coverings, Theorem 4.13 (ii) follows from Theorem 3.9, Proposition 3.11 (i), and Theorem 4.13 (i).

4.2. **A necessary condition.**

4.2.1. **Settings.** We maintain the notation introduced in 2.1.2 and the assumption introduced in 2.3.3 (i.e., $n \overset{\text{def}}{=} p^t - 1$). Moreover, let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$, and let

$$\text{ord}_x(D) = \sum_{j=0}^{t-1} d_{x,j} p^j, \ x \in D_X$$

be the *p*-adic expansion. For any $x \in D_X$, we put $d_x \overset{\text{def}}{=} \text{ord}_x(D)$ and $d_x^{(i)} \overset{\text{def}}{=} (\text{ord}_x(D))^{(i)}$ (Definition 2.4 (ii)).

4.2.2. First, we have the following necessary condition that the first generalized Hasse-Witt invariants attain maximum.

Lemma 4.2. *We maintain the notation introduced above. Suppose that $n_X \geq 2$. Then the following statements hold:*

(i) Suppose that $s(D) = n_X - 1$. Then $\deg(D^{(i)}) \geq \deg(D)$ for all $i \in \{0, 1, \ldots, t-1\}$ if and only if D is Frobenius stable. Moreover, D is Frobenius stable if and only if

$$\sum_{x \in D_X} d_{x,j} = (n_X - 1)(p-1), \ j \in \{0, \ldots, t-1\}.$$

(ii) Suppose that there exists an element $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$ such that $\gamma_{(\alpha,D)}$ attains maximum. Then we have D is a Frobenius stable divisor with degree $(n_X - 1)n$.

(iii) Suppose that $n > n_X - 1$. Then there exists $D' \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_X]^0$ such that $s(D') = n_X - 1$, and that D' is Frobenius stable.

Proof. (i) Since $\deg(D^{(i)}) \geq \deg(D) = (n_X - 1)n$ is equivalent to $\deg(D^{(i)}) = \deg(D) = (n_X - 1)n$, (i) follows immediately from [Y6, Lemma 3.2] and the definition of Frobenius stable.

(ii) This follows from Definition 3.10 (ii) and Lemma 2.5.

(iii) Note that there exists $D'' \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_X]^0$ such that $s(D'') = (n_X - 1)n$ if $n > n_X - 1$. Then (iii) follows from (i). We complete the proof of the lemma. $\square$

4.2.3. The following lemma will be used in the constructions of Galois admissible coverings of degenerations of $X^\bullet$.

4.2.4. The following lemma will be used in the constructions of Galois admissible coverings of degenerations of $X^\bullet$.

Lemma 4.3. *We maintain the notation introduced above. Suppose that $n_X \geq 3$, and that $\deg(D^{(i)}) = \deg(D) = (n_X - 1)n$ for all $i \in \{0, 1, \ldots, t-1\}$. Moreover, we put $D_X \overset{\mathrm{def}}{=} \{x_1, \ldots, x_{n_X}\}$ and*

$$a_{l,l+1} \overset{\mathrm{def}}{=} [\sum_{r=l+1}^{n_X} d_{x_r}],$$

$$b_{l,l+1} \overset{\mathrm{def}}{=} [\sum_{r=1}^{l} d_{x_r}], \ l \in \{2, \ldots, n_X - 2\},$$

where $[(-)]$ denotes the integer which is equal to the image of $(-)$ in $\mathbb{Z}/n\mathbb{Z}$ when we identify $\{0, \ldots, n-1\}$ with $\mathbb{Z}/n\mathbb{Z}$ naturally. Then, for each $i \in \{0, \ldots, t-1\}$, we have

$$a_{l,l+1}^{(i)} + b_{l,l+1}^{(i)} = n, \ l \in \{2, \ldots, n_X - 2\},$$
$$d_{x_1}^{(i)} + d_{x_2}^{(i)} + a_{2,3}^{(i)} = 2n,$$
$$b_{n_X-2,n_X-1}^{(i)} + d_{x_{n_X-1}}^{(i)} + d_{x_{n_X}}^{(i)} = 2n,$$
$$b_{l,l+1}^{(i)} + d_{x_{l+1}}^{(i)} + a_{l+1,l+2}^{(i)} = 2n, \ l \in \{2, \ldots, n_X - 3\}.$$

Proof. First, let us treat the first equality. Let $l \in \{2, \ldots, n_X - 2\}$ and $i \in \{0, \ldots, t-1\}$. Since

$$\sum_{r=l+1}^{n_X} d_{x_r} + \sum_{r=1}^{l} d_{x_r} = \deg(D) = (n_X - 1)n,$$

we see $n | (a_{l,l+1} + b_{l,l+1})$. Note that if $a_{l,l+1} + b_{l,l+1} = 0$, then $\deg(D) < (n_X - 1)n$. Thus, $a_{l,l+1} + b_{l,l+1} \neq 0$. Moreover, since $a_{l,l+1} + b_{l,l+1} \leq n$, we obtain that $a_{l,l+1} + b_{l,l+1} = n$, and

that $a_{l,l+1}^{(i)} + b_{l,l+1}^{(i)}$ is divided by n. On the other hand, since $0 < a_{l,l+1}^{(i)}, b_{l,l+1}^{(i)} < n$, we have $a_{l,l+1}^{(i)} + b_{l,l+1}^{(i)} = n$. This completes the proof of the first equality.

Let $i \in \{0, \ldots, t-1\}$. We denote by

$$
\begin{aligned}
S_1^{(i)} &\overset{\text{def}}{=} d_{x_1}^{(i)} + d_{x_2}^{(i)} + a_{2,3}^{(i)}, \\
S_l^{(i)} &\overset{\text{def}}{=} b_{l,l+1}^{(i)} + d_{x_{l+1}}^{(i)} + a_{l+1,l+2}^{(i)}, \ l \in \{2, \ldots, n_X - 3\}, \\
S_{n_X-2}^{(i)} &\overset{\text{def}}{=} b_{n_X-2,n_X-1}^{(i)} + d_{x_{n_X-1}}^{(i)} + d_{x_{n_X}}^{(i)}.
\end{aligned}
$$

We have $S_l^{(i)} \leq 2n, \ l \in \{1, \ldots, n_X - 2\}$. Moreover, if $i = 0$, the definitions of $a_{l,l+1}$ and $b_{l,l+1}$ imply

$$S_l^{(0)} = 2n, \ l \in \{1, \ldots, n_X - 2\}.$$

Then we have that $S_l^{(i)}$ is divided by n for all $i \in \{0, \ldots, t-1\}$. Since $s(D^{(i)}) = n_X - 1$, the first equality implies that

$$\sum_{l=1}^{n_X-2} S_l^{(i)} = n(n_X - 3) + (n_X - 1)n = 2n(n_X - 2).$$

On the other hand, since $S_l^{(i)} \leq 2n$, we have

$$S_l^{(i)} = 2n, \ l \in \{1, \ldots, n_X - 2\}, \ i \in \{0, \ldots, t-1\}.$$

This completes the proof of the lemma. $\qquad\square$

4.2.5. The following lemma follows immediately from [B, Corollary 6.8].

Lemma 4.4. *We maintain the notation introduced above. Suppose that $X^\bullet = (X, D_X)$ is a smooth component-generic pointed stable curve of type $(g_X, n_X) = (0, 3)$ over k. Moreover, we suppose that D is a Frobenius stable divisor with degree $2n$ (i.e., $\deg(D^{(i)}) = \deg(D) = (n_X - 1)n = 2n$ for all $i \in \{0, 1, \ldots, t-1\}$). Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists (see Definition 2.9). Moreover, we have (see 2.3.4 for $\gamma_{([\mathcal{L}],D)}$)*

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X, \mathcal{L})), \ ([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet, n}.$$

Remark 4.4.1. Note that, if $n_X = 3$, then we have $s(D) \in \{0, 1, 2\}$.

4.3. Degenerations. In this subsection, we introduce certain degenerations of $X^\bullet$ and prove the first main result in the case of *irreducible* component-generic pointed stable curves (Proposition 4.6).

4.3.1. Settings. We maintain the notation introduced in 2.1.2 and the assumption introduced in 2.3.3 (i.e., $n \overset{\text{def}}{=} p^t - 1$). Moreover, we assume that $X^\bullet = (X, D_X \overset{\text{def}}{=} \{x_1, \ldots, x_{n_X}\})$ is an *irreducible* component-generic pointed stable curve over k.

4.3.2. *Degeneration data.* We introduce some degeneration data for $X^\bullet$. Let R be a discrete valuation ring with algebraically closed residue field k_R, K_R the quotient field of R, and $\overline{K}_R$ an algebraic closure of K_R. Suppose that $k \subseteq K_R$. Let $\mathcal{X}^\bullet = (\mathcal{X}, D_\mathcal{X} \stackrel{\text{def}}{=} \{e_1, \ldots, e_{n_X}\})$ be a pointed stable curve of type (g_X, n_X) over R. We put

$$
\begin{aligned}
\mathcal{X}_\eta^\bullet &= (\mathcal{X}_\eta, D_{\mathcal{X}_\eta} \stackrel{\text{def}}{=} \{e_{\eta,1}, \ldots, e_{\eta,n_X}\}) \stackrel{\text{def}}{=} \mathcal{X}^\bullet \times_R K_R, \\
\mathcal{X}_{\overline{\eta}}^\bullet &= (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \{e_{\overline{\eta},1}, \ldots, e_{\overline{\eta},n_X}\}) \stackrel{\text{def}}{=} \mathcal{X}^\bullet \times_R \overline{K}_R, \\
\mathcal{X}_s^\bullet &= (\mathcal{X}_s, D_{\mathcal{X}_s} \stackrel{\text{def}}{=} \{e_{s,1}, \ldots, e_{s,n_X}\}) \stackrel{\text{def}}{=} \mathcal{X}^\bullet \times_R k_R.
\end{aligned}
$$

We shall say that $X^\bullet$ *admits a (DEG)* if $(g_X, n_X) \neq (1,1)$ and there exists $\mathcal{X}^\bullet$ such that the following conditions hold, where "(DEG)" means "degeneration":

(i) We have that $\mathcal{X}_{\overline{\eta}}^\bullet$ is $\overline{K}_R$-isomorphic to $X^\bullet \times_k \overline{K}_R$, and that $\mathcal{X}_s^\bullet$ is a *component-generic* pointed stable curve over k_R. Then without loss of generality, we may identify $e_{\overline{\eta},r}$, $r \in \{1, \ldots, n_X\}$, with $x_r \times_k \overline{K}_R$ via this isomorphism.

(ii) Let $\mathscr{T}$ be a set with cardinality $\#(\mathscr{T}) = \#(X^{\text{sing}})$ which consists of irreducible singular projective semi-stable curves of genus 1 (i.e., the normalizations are $\mathbb{P}^1_{k_R}$). Let $\mathscr{P}$ be a set which consists of smooth semi-stable curves of genus 0 (i.e., a set of $\mathbb{P}^1_{k_R}$). Let C_1 be either an empty set or a smooth semi-stable curve of genus $g_X - \#(X^{\text{sing}})$. We have the set of irreducible components of $\mathcal{X}_s$ is

$$
\{T\}_{T \in \mathscr{T}} \cup \{C_1\} \cup \{P\}_{P \in \mathscr{P}}.
$$

Moreover, one of the following conditions is satisfied (see Example 4.5 below for examples of (a) (b) (c)):

(a) Suppose that $n_X \leq 1$ and $\#(X^{\text{sing}}) \geq 1$. Then the following conditions hold:
- $C_1 = \emptyset$ when $n_X = 0$ and $\#(X^{\text{sing}}) = 2$; otherwise, $C_1 \neq \emptyset$.
- When $C_1 = \emptyset$, we have $\mathscr{T} \stackrel{\text{def}}{=} \{T_1, T_2\}$ such that $\#(T_1 \cap T_2) = 1$.
- When $C_1 \neq \emptyset$, we have that $T' \cap T'' \neq \emptyset$ if and only if $T' = T''$ for all $T', T'' \in \mathscr{T}$, that $\#(T \cap C_1) = 1$ for all $T \in \mathscr{T}$, and that $D_{\mathcal{X}_s} \subseteq C_1$.
- $\mathscr{P} = \emptyset$.

(b) Suppose that $n_X = 2$. Then the following conditions hold:
- $T' \cap T'' \neq \emptyset$ if and only if $T' = T''$ for all $T', T'' \in \mathscr{T}$.
- $C_1 = \emptyset$ when $g_X - \#(X^{\text{sing}}) = 0$; otherwise, $C_1 \neq \emptyset$ when $g_X - \#(X^{\text{sing}}) \geq 1$.
- $\mathscr{P} \stackrel{\text{def}}{=} \{P\}$ such that $D_{\mathcal{X}_s} \subseteq P$.
- When $C_1 = \emptyset$, we have $\#(P \cap T) = 1$ for all $T \in \mathscr{T}$.
- When $C_1 \neq \emptyset$, we have that $\#(C_1 \cap T) = 1$, that $\#(C_1 \cap P) = 1$, and that $P \cap T = \emptyset$ for all $T \in \mathscr{T}$.

(c) Suppose that $n_X \geq 3$. Then the following conditions hold:
- The first and the second conditions of (b) hold.
- $\mathscr{P} \stackrel{\text{def}}{=} \{P_v\}_{v \in \{2, \ldots, n_X - 1\}}$ such that $D_{\mathcal{X}_s} \subseteq \bigcup_{v \in \{2, \ldots, n_X - 1\}} P_v$.
- When $C_1 = \emptyset$, we have $\#(P_2 \cap T) = 1$ and $P_v \cap T = \emptyset$ for all $T \in \mathscr{T}$ and all $v \neq 2$.
- When $C_1 \neq \emptyset$, we have $\#(C_1 \cap T) = 1$, $\#(C_1 \cap P_2) = 1$, $C_1 \cap P_v = \emptyset$, and $P_v \cap T = \emptyset$ for all $T \in \mathscr{T}$ and all $v \neq 2$.

- If $n_X \geq 4$, for each $v \in \{2, \ldots, n_X - 2\}$, we have $\#(P_v \cap P_{v+1}) = 1$ and $P_v \cap P_{v'} = \emptyset$ when $v' \notin \{v-1, v, v+1\}$.
- If $n_X = 3$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}, e_{s,3}\}$.
- If $n_X = 4$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}\}$ and $D_{\mathcal{X}_s} \cap P_3 = \{e_{s,3}, e_{s,4}\}$.
- If $n_X \geq 5$, we have $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,1}, e_{s,2}\}$, $D_{\mathcal{X}_s} \cap P_{n_X-1} = \{e_{s,n_X-1}, e_{s,n_X}\}$, and $D_{\mathcal{X}_s} \cap P_v = \{e_{s,v}\}$, $v \in \{3, \ldots, n_X - 2\}$.

Note that since generic curves admit all degeneration types, we have that $X^\bullet$ admits a (DEG) when $X^\bullet$ is a component-generic pointed stable curve of type $(g_X, n_X) \neq (1, 1)$.

4.3.3.   Next, we give some examples to explain the degeneration data introduced above. For simplicity, we assume that $\#(X^{\mathrm{sing}}) = 2$, $C_1 \neq \emptyset$, and $n_X \neq 0$.

Example 4.5. We use the notation "$\bullet$" and "$\circ$ with a line segment" to denote a vertex and an open edge, respectively. Moreover, we use $v_{(-)}$ to denote the vertex corresponding to the irreducible component $(-)$.
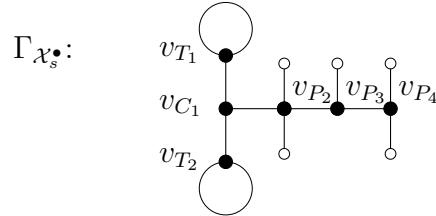
(a) If $n_X = 1$, then the dual semi-graph $\Gamma_{\mathcal{X}_s^\bullet}$ of $\mathcal{X}_s^\bullet$ is as follows (see 4.3.2 (ii)-(a)):
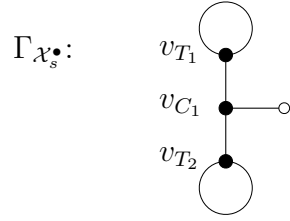
(b) If $n_X = 2$, then the dual semi-graph $\Gamma_{\mathcal{X}_s^\bullet}$ of $\mathcal{X}_s^\bullet$ is as follows (see 4.3.2 (ii)-(b)):
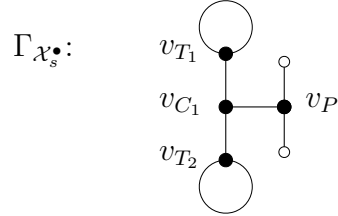
(c) If $n_X = 5$, then the dual semi-graph $\Gamma_{\mathcal{X}_s^\bullet}$ of $\mathcal{X}_s^\bullet$ is as follows (see 4.3.2 (ii)-(c)):

4.3.4.   Proposition 4.6 below was obtained by the author of the present paper when $X^\bullet$ is *smooth* over k (see [Y6, Proposition 3.4]), and its proof is similar to the proof of [Y6, Proposition 3.4]. On the other hand, the degenerations appeared in the proof of Proposition 4.6 are different from that of the proof of [Y6, Proposition 3.4]. Then for convenience of the reader, we give the proof here.

Proposition 4.6. *Let* $X^\bullet = (X, D_X \stackrel{\text{def}}{=} \{x_1, \dots, x_{n_X}\})$ *be an irreducible component-generic pointed stable curve over* k, $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, *and* $\alpha \in \text{Rev}_D^{\text{adm}}(X^\bullet) \setminus \{0\}$. *Suppose that* $\deg(D) = (n_X - 1)n$ *(i.e.,* $s(D) = n_X - 1$*) if* $n_X \neq 0$*, and that* D *is Frobenius stable (i.e.,* $\deg(D^{(i)}) = \deg(D) = (n_X - 1)n$ *for all* $i \in \{0, 1, \dots, t-1\}$*). Then* $\gamma_{(\alpha,D)}$ *attains maximum. Namely, the following holds:*

$$\gamma_{(\alpha,D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. Firstly, we note that, to verify the proposition, we may assume that $Y^\bullet$ is connected.

Suppose that $X^\bullet$ is smooth over k, and that $n_X \leq 1$. Then the proposition follows immediately from Proposition 3.11 (i). Thus, to verify the proposition, it is sufficient to assume that one of the following conditions holds: (1) $\#(X^{\text{sing}}) \geq 1$ and $n_X \leq 1$; (2) $n_X \geq 2$.

Suppose that $X^\bullet$ is a singular curve of type $(1, 1)$ (i.e., $\#(X^{\text{sing}}) = 1$). Since $f^\bullet$ is étale, the proposition follows immediately from Lemma 3.1 (i). Thus, to verify the proposition, we may assume $(g_X, n_X) \neq (1, 1)$. Now, we can use the degeneration data introduced in 4.3.2.

Since $X^\bullet$ is a component-generic pointed stable curve, $X^\bullet$ admits a (DEG). We maintain the notation introduced in 4.3.2. Furthermore, we write $Q_{\overline{\eta}}$ (resp. Q_s) for the effective divisor on $\mathcal{X}_{\overline{\eta}}$ (resp. $\mathcal{X}_s$) induced by D and $\alpha_{\overline{\eta}} \in \text{Rev}_{Q_{\overline{\eta}}}^{\text{adm}}(\mathcal{X}_{\overline{\eta}}^\bullet)$ for the element induced by α. Then we have $\gamma_{(\alpha,D)} = \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})}$. Write $\Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}$ and $\Pi_{\mathcal{X}_s^\bullet}$ for the admissible fundamental groups of $\mathcal{X}_{\overline{\eta}}^\bullet$ and $\mathcal{X}_s^\bullet$, respectively. Then we have a specialization surjective homomorphism

$$sp_R : \Pi_{\mathcal{X}_{\overline{\eta}}^\bullet} \twoheadrightarrow \Pi_{\mathcal{X}_s^\bullet}.$$

We suppose that $X^\bullet$ satisfies (DEG)-(ii)-(c) (4.3.2). Moreover, we suppose that $C_1 \neq \emptyset$ and $n_X \geq 5$ (see Example 4.5 (c)).

Step 1: We define certain smooth pointed stable curves associated to irreducible components of $\mathcal{X}_s^\bullet$.

We write $y_{v,v+1}$, $z_{v,v+1}$, $v \in \{2, \dots, n_X - 2\}$, for the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_v \hookrightarrow \mathcal{X}_s$ and the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_{v+1} \hookrightarrow \mathcal{X}_s$, respectively. We define

$$\begin{aligned}
P_2^\bullet &= (P_2, D_{P_2} \stackrel{\text{def}}{=} \{e_{s,1}, e_{s,2}, y_{2,3}\} \cup (C_1 \cap P_2)), \\
P_{n_X-1}^\bullet &= (P_{n_X-1}, D_{P_{n_X-1}} \stackrel{\text{def}}{=} \{z_{n_X-2,n_X-1}, e_{s,n_X-1}, e_{s,n_X}\}), \\
P_v^\bullet &= (P_v, D_{P_v} \stackrel{\text{def}}{=} \{z_{v-1,v}, e_{s,v}, y_{v,v+1}\}), \; v \in \{3, \dots, n_X - 2\},
\end{aligned}$$

to be smooth pointed stable curves of types $(0, 4)$, $(0, 3)$, and $(0, 3)$ over k_R, respectively. Moreover, we define

$$C_1^\bullet = (C_1, D_{C_1} \stackrel{\text{def}}{=} (C_1 \cap P_2) \cup ((\bigcup_{T \in \mathscr{T}} T) \cap C_1)),$$

$$T^\bullet = (T, D_T \stackrel{\text{def}}{=} T \cap C_1), \; T \in \mathscr{T},$$

to be a smooth pointed stable curve of type $(g_X - \#(X^{\mathrm{sing}}), 1 + \#(X^{\mathrm{sing}}))$ and a singular pointed stable curve of type $(1,1)$ over k_R, respectively. Note that since C_1 is generic, we have $\sigma_{C_1} = g_X - \#(X^{\mathrm{sing}})$ (i.e., $C_1^\bullet$ is ordinary (2.2.2)).

Step 2: We construct a Galois admissible covering of $\mathcal{X}_s^\bullet$ by using specialization isomorphisms of maximal prime-to-p quotients of admissible fundamental groups.

Let $f_{\overline{\eta}}^\bullet \stackrel{\mathrm{def}}{=} f^\bullet \times_k \overline{K}_R : \mathcal{Y}_{\overline{\eta}}^\bullet = (\mathcal{Y}_{\overline{\eta}}, D_{\mathcal{Y}_{\overline{\eta}}}) \stackrel{\mathrm{def}}{=} Y^\bullet \times_k \overline{K}_R \to \mathcal{X}_{\overline{\eta}}^\bullet$ be the Galois admissible covering over $\overline{K}_R$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $f^\bullet$, and $\Pi_{\mathcal{Y}_{\overline{\eta}}^\bullet} \subseteq \Pi_{\mathcal{X}_{\overline{\eta}}^\bullet} \cong \Pi_{X^\bullet}$ the admissible fundamental group of $\mathcal{Y}_{\overline{\eta}}^\bullet$. By the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups ([V, Théorème 2.2 (c)]), we have

$$sp_R^{p'} : \Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}^{p'} \stackrel{\sim}{\to} \Pi_{\mathcal{X}_s^\bullet}^{p'}.$$

Then we obtain a normal open subgroup $\Pi_{\mathcal{Y}_s^\bullet}^{p'} \stackrel{\mathrm{def}}{=} sp_R^{p'}(\Pi_{\mathcal{Y}_{\overline{\eta}}^\bullet}^{p'}) \subseteq \Pi_{\mathcal{X}_s^\bullet}^{p'}$. Write $\Pi_{\mathcal{Y}_s^\bullet} \subseteq \Pi_{\mathcal{X}_s^\bullet}$ for the inverse image of $\Pi_{\mathcal{Y}_s^\bullet}^{p'}$ of the natural surjection $\Pi_{\mathcal{X}_s^\bullet} \twoheadrightarrow \Pi_{\mathcal{X}_s^\bullet}^{p'}$. Then $\Pi_{\mathcal{Y}_s^\bullet}$ determines a Galois admissible covering

$$f_s^\bullet : \mathcal{Y}_s^\bullet = (\mathcal{Y}_s, D_{\mathcal{Y}_s}) \to \mathcal{X}_s^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. We denote by $\alpha_s \in \mathrm{Rev}_{Q_s}^{\mathrm{adm}}(\mathcal{X}_s^\bullet)$ the element induced by the composition homomorphism $\Pi_{\mathcal{X}_s^\bullet}^{p',\mathrm{ab}} \stackrel{(sp_R^{p',\mathrm{ab}})^{-1}}{\to} \Pi_{\mathcal{X}_{\overline{\eta}}^\bullet}^{p',\mathrm{ab}} \stackrel{\alpha_{\overline{\eta}}}{\to} \mathbb{Z}/n\mathbb{Z}$.

Step 3: We compute the generalized Hasse-Witt invariant $\gamma_{(\alpha_s, D_{\alpha_s})}$ by applying Theorem 3.9, Lemma 4.3, and Lemma 4.4.

The structure of $\Pi_{\mathcal{X}_s^\bullet}^{p'}$ (2.1.6) implies that f_s is étale over $(\bigcup_{T \in \mathscr{T}} T) \cap C_1$. Then we obtain that f_s is étale over $C_1 \cap P_2$. Thus, f_s is étale over D_{C_1}. Let $Y_v \stackrel{\mathrm{def}}{=} f_s^{-1}(P_v)$, $v \in \{2, \ldots, n_X - 1\}$. We put

$$Y_v^\bullet \stackrel{\mathrm{def}}{=} (Y_v, D_{Y_v} \stackrel{\mathrm{def}}{=} f_s^{-1}(D_{P_v})), \ v \in \{2, \ldots, n_X - 1\}.$$

Then $f_s^\bullet$ induces a Galois multi-admissible covering

$$f_v^\bullet : Y_v^\bullet \to P_v^\bullet, \ v \in \{2, \ldots, n_X - 1\},$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. We maintain the notation introduced in Lemma 4.3. Then we see that the ramification divisor on P_v, $v \in \{2, \ldots, n_X - 1\}$, induced by $f_v^\bullet$ and α_s is as follows:

$$Q_2 \stackrel{\mathrm{def}}{=} d_{x_1} e_{s,1} + d_{x_2} e_{s,2} + a_{2,3} y_{2,3},$$
$$Q_{n_X-1} \stackrel{\mathrm{def}}{=} b_{n_X-2, n_X-1} z_{n_X-2, n_X-1} + d_{x_{n_X-1}} e_{s, n_X-1} + d_{x_{n_X}} e_{s, n_X},$$
$$Q_v \stackrel{\mathrm{def}}{=} b_{v-1,v} z_{v-1,v} + d_{x_v} e_{s,v} + a_{v,v+1} y_{v,v+1}, \ v \in \{3, \ldots, n_X - 2\}.$$

Since f_s is étale over $C_1 \cap P_2$, we see that $f_v^\bullet$, $v \in \{2, \ldots, n_X - 1\}$, induces a pair $([\mathcal{L}_v], Q_v) \in \widetilde{\mathscr{P}}_{P_v^\bullet, n}$. Moreover, the $k_R[\mu_n]$-module $H^1_{\mathrm{ét}}(Y_v, \mathbb{F}_p) \otimes k_R$ admits the following

canonical decomposition

$$H^1_{\text{ét}}(Y_v, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Y_v}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Y_v}(j)$ as the ζ^j-multiplication. On the other hand, Lemma 4.3 implies $\deg(Q_v^{(i)}) = \deg(Q_v) = 2n$, $i \in \{0, \ldots, t-1\}$. Then by applying Lemma 4.4, we have

$$\gamma_{([\mathcal{L}_v], Q_v)} = \dim_{k_R}(M_{Y_v}(1)) = \dim_{k_R}(H^1(P_v, \mathcal{L}_v)) = 1.$$

Let $Z_1 \stackrel{\text{def}}{=} f_s^{-1}(C_1)$. Then $f_s^\bullet$ induces a Galois étale covering (not necessarily connected)

$$f_{C_1}^\bullet : Z_1^\bullet = (Z_1, D_{Z_1} \stackrel{\text{def}}{=} f_s^{-1}(D_{C_1})) \to C_1^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_{C_1} \in \text{Rev}_0^{\text{adm}}(C_1^\bullet)$ for the element induced by $f_{C_1}^\bullet$ and α_s. The $k_R[\mu_n]$-module $H^1_{\text{ét}}(Z_1, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Z_1, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Z_1}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Z_1}(j)$ as the ζ^j-multiplication. Then Proposition 3.11 (i) implies

$$\gamma_{(\alpha_{C_1}, 0)} = \begin{cases} g_X - \#(X^{\text{sing}}), & \text{if } \alpha_{C_1} = 0, \\ g_X - \#(X^{\text{sing}}) - 1, & \text{if } \alpha_{C_1} \neq 0. \end{cases}$$

Let $V_T \stackrel{\text{def}}{=} f_s^{-1}(T)$, $T \in \mathscr{T}$, and $\widetilde{T}$ the smooth compactification of $U_T \stackrel{\text{def}}{=} T \setminus T^{\text{sing}}$. Then $f_s^\bullet$ induces a Galois multi-admissible covering

$$f_T^\bullet : V_T^\bullet = (V_T, D_{V_T} \stackrel{\text{def}}{=} f_s^{-1}(D_T)) \to T^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then $f_T^\bullet$ induces a Galois multi-admissible covering

$$f_{\widetilde{T}}^\bullet : V_{\widetilde{T}}^\bullet = (V_{\widetilde{T}}, D_{V_{\widetilde{T}}}) \to \widetilde{T}^\bullet$$

over k_R. Write $\alpha_{\widetilde{T}} \in \text{Rev}_0(\widetilde{T}^\bullet)$ for the element induced by $f_{\widetilde{T}}^\bullet$ and α_s. By using the proposition of the case where $X^\bullet$ is a singular curve of type $(g_X, n_X) = (1, 1)$, we obtain that $\gamma_{(\alpha_{\widetilde{T}}, 0)} = 0$. Thus, Theorem 3.9 implies that

$$\gamma_{(\alpha_s, Q_s)} = g_X + n_X - 2.$$

Step 4: We compute $\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})}$ by using specialization surjective homomorphisms of admissible fundamental groups.

The $k_R[\mu_n]$-modules $H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ and $H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R$ admit the following canonical decompositions

$$H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{\overline{\eta}}}(j),$$

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_s}(j),$$

respectively. Moreover, we have an injection as $k_R[\mu_n]$-modules

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R \hookrightarrow H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$$

induced by the specialization map $\Pi_{\mathcal{Y}_{\overline{\eta}}^{\bullet}} \twoheadrightarrow \Pi_{\mathcal{Y}_s^{\bullet}}$. Thus, we have

$$g_X + n_X - 2 = \gamma_{(\alpha_s, Q_s)} = \dim_{k_R}(M_{\mathcal{Y}_s}(1))$$

$$\leq \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \dim_{k_R}(M_{\mathcal{Y}_{\overline{\eta}}}(1)) \leq g_X + n_X - 2.$$

This means $\gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = g_X + n_X - 2$. We complete the proof of the proposition when $X^{\bullet}$ satisfies (DEG)-(ii)-(c), $C_1 \neq \emptyset$, and $n_X \geq 5$.

By applying similar arguments to the arguments given in the proof above, one can prove the proposition when $X^{\bullet}$ satisfies (DEG)-(ii)-(c) and either $C_1 = \emptyset$ or $n_X \leq 4$ holds. Moreover, similar arguments to the arguments given in the proof above imply that the proposition holds when $X^{\bullet}$ satisfies either (DEG)-(ii)-(a) or (DEG)-(ii)-(b). We complete the proof of the proposition. $\qquad\square$

Remark 4.6.1. Suppose that $X^{\bullet}$ is a *smooth* component-generic pointed stable curve over k. By applying [Y6, Proposition 3.4], we obtain a *necessary and sufficient* condition for ordinariness (i.e., the genus and the p-rank of coverings are equal, see 2.2.2) of prime-to-p cyclic admissible coverings of $X^{\bullet}$ (see [Y6, Theorem 1.2]). On the other hand, in the theory of anabelian geometry, we expect that the geometric information of $X^{\bullet}$ (e.g. the type (g_X, n_X), the dual semi-graph $\Gamma_{X^{\bullet}}$, the isomorphism class of $X^{\bullet}$) can be reconstructed group-theoretically from its fundamental group. From this point of view, the generalized Hasse-Witt invariants of coverings are much better than the p-rank of coverings since they contain the information concerning the type (g_X, n_X) of $X^{\bullet}$. This is the reason why we focus on the maximum generalized Hasse-Witt invariants in the present paper.

In the remainder of this section, under certain assumptions, we generalize Proposition 4.6 to the case where $X^{\bullet}$ is not necessary to be irreducible.

4.4. **Minimal quasi-trees.** In this subsection, we introduce the so-called *minimal quasi-trees* which play an important role in the remainder of the present paper.

4.4.1. **Settings.** Let $W^{\bullet} = (W, D_W)$ be a pointed stable curve of type (g_W, n_W) over k and $\Gamma_{W^{\bullet}}$ the dual semi-graph of $W^{\bullet}$.

4.4.2. Before we give the definition of minimal quasi-trees, we introduce some notation concerning sub-semi-graphs. Let $\mathbf{G}'$ be a connected semi-graph. We shall call $\mathbf{G}'$ a *sub-semi-graph* of $\mathbf{G}$ if the following conditions hold:

(i) $v(\mathbf{G}') \neq \emptyset$ and $v(\mathbf{G}') \subseteq v(\mathbf{G})$.
(ii) $e^{\mathrm{cl}}(\mathbf{G}') \subseteq e^{\mathrm{cl}}(\mathbf{G})$ is the subset of closed edges such that $v(e) \subseteq v(\mathbf{G}')$.
(iii) $e^{\mathrm{op}}(\mathbf{G}') \subseteq (e^{\mathrm{cl}}(\mathbf{G}) \cup e^{\mathrm{op}}(\mathbf{G})) \setminus e^{\mathrm{cl}}(\mathbf{G}')$ is the subset of edges such that $\#(v(e) \cap v(\mathbf{G}')) = 1$.

Note that the definition of $\mathbf{G}'$ implies that $\mathbf{G}'$ can be completely determined by $v(\mathbf{G}')$ if $v(\mathbf{G}') \neq \emptyset$. The conditions (ii), (iii) imply that, if $e \in e^{\mathrm{lp}}(\mathbf{G})$ is a loop and $v(e) \subseteq v(\mathbf{G}')$, then $e \in e^{\mathrm{cl}}(\mathbf{G}')$.

Suppose that $\mathbf{G}'$ is a sub-semi-graph of $\mathbf{G}$ such that $v(\mathbf{G}') \neq \emptyset$. Let $L \subseteq e^{\mathrm{cl}}(\mathbf{G}')$ be a *subset of closed edges* of $\mathbf{G}'$ such that $\mathbf{G}' \setminus L$ (i.e., removing L from $\mathbf{G}'$) is *connected*. For any $e \overset{\text{def}}{=} \{b_e^1, b_e^2\} \in L$, we put $e^i \overset{\text{def}}{=} \{b_{e^i}^1, b_{e^i}^2\}$, $i \in \{1, 2\}$, and shall call e^i the i-edge associated to e. We shall call $\mathbf{G}'_L$ *the semi-graph associated to* $\mathbf{G}'$ *and* L if the following conditions hold:

(i) $v(\mathbf{G}'_L) \overset{\mathrm{def}}{=} v(\mathbf{G})$.

(ii) $e^{\mathrm{op}}(\mathbf{G}'_L) \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\mathbf{G}') \cup \{e^1, e^2\}_{e \in L}$ such that $\zeta^{\mathbf{G}'_L}(e) = \zeta^{\mathbf{G}'}(e)$ if $e \in e^{\mathrm{op}}(\mathbf{G}')$, that $\zeta^{\mathbf{G}'_L}(e^1) \overset{\mathrm{def}}{=} \{\zeta^{\mathbf{G}'}(b_e^1), \{v(\mathbf{G}'_L)\}\}$ if e^1 is the 1-edge associated to $e \in L$, and that $\zeta^{\mathbf{G}'_L}(e^2) \overset{\mathrm{def}}{=} \{\zeta^{\mathbf{G}'}(b_e^2), \{v(\mathbf{G}'_L)\}\}$ if e^2 is the 2-edge associated to $e \in L$.

(iii) $e^{\mathrm{cl}}(\mathbf{G}'_L) \overset{\mathrm{def}}{=} e^{\mathrm{cl}}(\mathbf{G}') \setminus L$ such that $\zeta^{\mathbf{G}'_L}(e) \overset{\mathrm{def}}{=} \zeta^{\mathbf{G}'}(e)$ if $e \in e^{\mathrm{cl}}(\mathbf{G}') \setminus L$.

Example 4.7. We give some examples of semi-graphs to explain the above notation. We use the notation "$\bullet$" and "$\circ$" to denote a vertex and an open edge, respectively.

Let $\mathbf{G}$ be a semi-graph and $\mathbf{G}'$ a sub-semi-graph of $\mathbf{G}$ whose set of vertices is $\{v_1, v_2\}$ as follows:
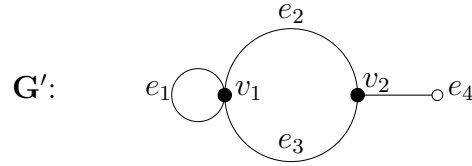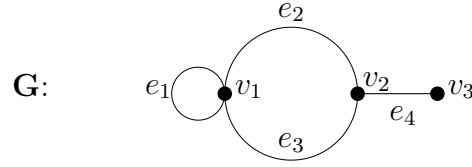
Moreover, let $L \overset{\mathrm{def}}{=} \{e_2\} \subseteq e^{\mathrm{cl}}(\mathbf{G}')$ be a subset of edges of $\mathbf{G}'$ and $\{e_2^1, e_2^2\}$ the set of 1-edge and 2-edge associated to e_2. Then we have the following:
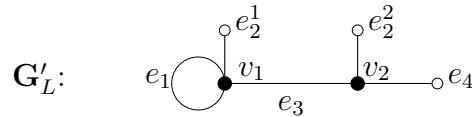
4.4.3. *Definition of minimal quasi-trees associated to D_W.*

Definition 4.8. We maintain the notation introduced above. Let Γ' be a sub-semi-graph (4.4.2) of $\Gamma_{W\bullet}$ and $L \subseteq e^{\mathrm{cl}}(\Gamma') \setminus e^{\mathrm{lp}}(\Gamma')$ (see 2.1.1 for $e^{\mathrm{lp}}(\Gamma')$). We shall call the semi-graph Γ'_L associated to Γ' and L (4.4.2) a *quasi-tree associated to D_W* if the following conditions are satisfied:

- $\Gamma'_L \setminus e^{\mathrm{lp}}(\Gamma'_L)$ is a tree.
- $e^{\mathrm{op}}(\Gamma_{W\bullet})$ is contained in $e^{\mathrm{op}}(\Gamma'_L)$.

Moreover, we shall call a semi-graph

$$\Gamma_{D_W}$$

a *minimal quasi-tree associated to* D_W if either $\Gamma_{D_W} = \emptyset$ when $n_W = 0$ or the following conditions are satisfied when $n_W \neq 0$:

- Γ_{D_W} is a quasi-tree associated to D_W.
- Suppose that Γ'' is a quasi-tree associated to D_W such that $\Gamma'' \subseteq \Gamma_{D_W}$. Then we have $\Gamma'' = \Gamma_{D_W}$.

Note that by the definition of Γ_{D_W}, we have that $\Gamma_{D_W} \setminus e^{\mathrm{lp}}(\Gamma_{D_W})$ is a tree.

In particular, when $e^{\mathrm{lp}}(\Gamma_{W^\bullet}) = \emptyset$, minimal quasi-trees are very simple. Namely, a minimal quasi-tree Γ_{D_W} is a *minimal tree-like semi-graph* contained in $\Gamma_{W^\bullet}$ such that Γ_{D_W} *contains all of the open edges* of $\Gamma_{W^\bullet}$.

We will see that there exists a minimal quasi-tree associated to D_W for an arbitrary pointed stable curve $W^\bullet$ (see 4.4.5 below).

4.4.4. Before we start to explain the constructions of minimal quasi-trees defined above, we introduce some notation. Let $Z^\bullet$ be a pointed stable curve over k, $\Gamma_{Z^\bullet}$ the dual semi-graph of $Z^\bullet$ such that $\Gamma_{Z^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{Z^\bullet})$ is a *tree*, and that $E_Z \subseteq e^{\mathrm{op}}(\Gamma_{Z^\bullet})$ is a subset of open edges.

Firstly, we introduce a subset of vertices $V_Z \subseteq v(\Gamma_{Z^\bullet})$ which consists of terminal vertices (i.e., the vertex v such that $\#(Z_v \cap (\bigcup_{v' \in v(\Gamma_{Z^\bullet}) \setminus \{v\}} Z_{v'})) \leq 1$, where Z_v denotes the irreducible component corresponding to v) of $\Gamma_{Z^\bullet}$ such that for each $v \in V_Z$, v is not abutted to $e \in E_Z$. Namely, we put (see 2.1.1 for $e^{\Gamma_{Z^\bullet}}(v)$)
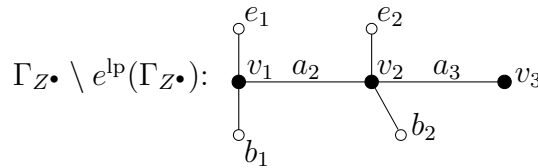
$$V_Z \overset{\mathrm{def}}{=} \{v \in v(\Gamma_{Z^\bullet}) \mid \#(e^{\Gamma_{Z^\bullet}}(v) \cap (e^{\mathrm{cl}}(\Gamma_{Z^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{Z^\bullet}))) \leq 1, \ e^{\Gamma_{Z^\bullet}}(v) \cap E_Z = \emptyset\} \subseteq v(\Gamma_{Z^\bullet}).$$

Note that $\#(e^{\Gamma_{Z^\bullet}}(v) \cap (e^{\mathrm{cl}}(\Gamma_{Z^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{Z^\bullet}))) = 0 < 1$ if and only if $Z^\bullet$ is irreducible. We call V_Z *the set of terminal vertices avoiding to* E_Z.

Example 4.9. We give an example of V_Z. Suppose that the dual semi-graph $\Gamma_{Z^\bullet}$ is as follows:
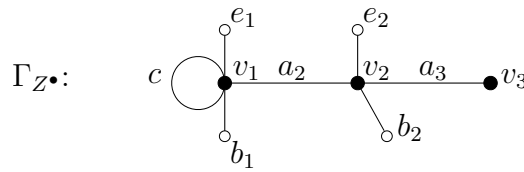
We see that $e^{\mathrm{lp}}(\Gamma_{Z^\bullet}) = \{c\}$, and that the set of terminal vertices is $\{v_1, v_3\}$. Then $\Gamma_{Z^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{Z^\bullet})$ is as follows, which is a tree:

Let $E_Z \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma_{Z^\bullet})$. Since $e^{\Gamma_{Z^\bullet}}(v_1) = \{e_1, b_1\}$ and $e^{\Gamma_{Z^\bullet}}(v_2) = \{e_2, b_2\}$, we have $V_Z = \{v_3\}$.

4.4.5. *Constructions of minimal quasi-trees associated to D_W and their associated pointed stable curves.* Next, we define a minimal quasi-tree associated to D_W. Let $E \subseteq e^{\mathrm{cl}}(\Gamma_{W^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{W^\bullet})$ be a (possibly empty) subset of closed edges such that $\Gamma_{W^\bullet} \setminus (E \cup e^{\mathrm{lp}}(\Gamma_{W^\bullet}))$ is a connected tree. Note that it is easy to see that E exists.

Write N for the set of nodes of W corresponding to the closed edges which are contained in E, and write $\mathrm{norm}_N : W_N \to W$ for the normalization morphism of the underlying curve W over N. We define a pointed stable curve over k to be

$$W_1^\bullet = (W_1, D_{W_1}) \overset{\mathrm{def}}{=} W_N^\bullet = (W_N, D_{W_N} \overset{\mathrm{def}}{=} \mathrm{norm}_N^{-1}(D_W \cup N)).$$

Note that the above construction implies $D_W \subseteq D_{W_1}$ (i.e., $e^{\mathrm{op}}(\Gamma_{W^\bullet}) \subseteq e^{\mathrm{op}}(\Gamma_{W_1^\bullet})$). Write $\Gamma_{W_1^\bullet}$ for the dual semi-graph of $W_1^\bullet$. Then the construction of $W_1^\bullet$ implies that $\Gamma_{W_1^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{W_1^\bullet})$ is a *tree*.

Suppose that $n_W \neq 0$. Then for all $i \in \mathbb{N}$, if $W_i^\bullet$ has already been defined, we may define $W_{i+1}^\bullet$ as follows: Let $V_i \subseteq v(\Gamma_{W_i^\bullet})$ be the subset of terminal vertices avoiding to $e^{\mathrm{op}}(\Gamma_{W^\bullet}) \subseteq e^{\mathrm{op}}(\Gamma_{W_i^\bullet})$. If $V_i = \emptyset$, then we put $W_{i+1}^\bullet \overset{\mathrm{def}}{=} W_i^\bullet$. If $V_i \neq \emptyset$, we write W_{i+1} for the topological closure of

$$W_i \setminus (\bigcup_{v \in V_i} W_v)$$

in W_i. Note that the definition of V_i implies that W_{i+1} is connected, and that D_W is contained in W_{i+1}. Then we define a pointed stable curve over k to be

$$W_{i+1}^\bullet \overset{\mathrm{def}}{=} (W_{i+1}, D_{W_{i+1}} \overset{\mathrm{def}}{=} (D_{W_i} \cap W_{i+1}) \cup ((\bigcup_{v \in V_i} W_v) \cap W_{i+1}))).$$

Note that we have $e^{\mathrm{op}}(\Gamma_{W^\bullet}) \subseteq e^{\mathrm{op}}(\Gamma_{W_{i+1}^\bullet})$.

Let i_0 be the minimal natural number such that $V_{i_0} = \emptyset$. Note that the above construction implies that $W_{i_0}^\bullet = W_j^\bullet$ for all $j \geq i_0$. We put

$$W_\Gamma^\bullet = (W_\Gamma, D_{W_\Gamma}) \overset{\mathrm{def}}{=} W_{i_0}^\bullet,$$

$$\Gamma \overset{\mathrm{def}}{=} \Gamma_{W_{i_0}^\bullet}.$$

Then $\Gamma_{D_W} \overset{\mathrm{def}}{=} \Gamma$ is a *minimal quasi-tree associated to D_W*. We shall call $W_\Gamma^\bullet$ the pointed stable curve associated to Γ.

Note that we have that $D_W \subseteq D_{W_\Gamma}$, and that $\Gamma \setminus e^{\mathrm{lp}}(\Gamma)$ is a *tree*. Moreover, we see $v(\Gamma) \subseteq v(\Gamma_{W^\bullet})$ and $e(\Gamma) \subseteq e(\Gamma_{W^\bullet})$, where $e(-)$ denotes the set of edges of $(-)$ (2.1.1). Note that an open edge of Γ is *not* an open edge of $\Gamma_{W^\bullet}$ in general, and that a closed edge of Γ is a closed edge of $\Gamma_{W^\bullet}$.

Moreover, the construction of Γ *depends* on the choice of E (i.e., a subset of closed edges of $e^{\mathrm{cl}}(\Gamma_{W^\bullet}) \setminus e^{\mathrm{lp}}(\Gamma_{W^\bullet})$ such that $\Gamma_{W^\bullet} \setminus (E \cup e^{\mathrm{lp}}(\Gamma_{W^\bullet}))$ is a tree).

4.4.6. We maintain the notation introduced in 4.4.5. Let us give some examples to explain the above constructions.

Example 4.10. (a) Let $W^\bullet$ be a pointed stable curve over k such that the following conditions hold: (i) The set of irreducible components of W is $\{W_{v_1}, W_{v_2}, W_{v_3}\}$; (ii) $D_W = \{w_{b_1}, w_{b_2}\}$; (iii) The set of nodes is $\{w_c, w_{a_1}, w_{a_2}, w_{a_3}\}$; (iv) W_{v_1} is a singular curve with the unique node w_c; (v) $w_{b_1} \in W_{v_1}$ and $w_{b_2} \in W_{v_2}$; (vi) $w_{a_1}, w_{a_2} \in W_{v_1} \cap W_{v_2}$;

(vii) $w_{a_3} \in W_{v_2} \cap W_{v_3}$. We use the notation "•" and "∘" to denote a node and a marked point, respectively. Then $W^\bullet$ is as follows:
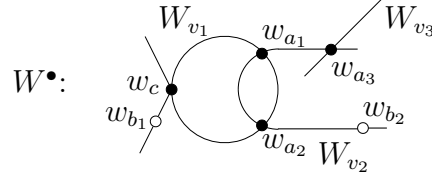
The dual semi-graph $\Gamma_{W^\bullet}$ of $W^\bullet$ such that the following conditions hold: (i) $v(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{v_1, v_2, v_3\}$; (ii) $e^{\text{cl}}(\Gamma_{W^\bullet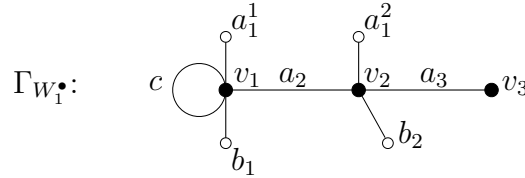}) \setminus e^{\text{lp}}(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{a_1, a_2, a_3\}$ such that a_1 and a_2 abut to v_1 and v_2, respectively, and that a_3 abuts to v_2 and v_3; (iii) $e^{\text{lp}}(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{c\}$ and c abuts to v_1; (iv) $e^{\text{op}}(\Gamma_{W^\bullet}) \overset{\text{def}}{=} \{b_1, b_2\}$ such that b_1 and b_2 abut to v_1 and v_2, respectively. We use the notation "•" and "∘ with a line segment" to denote a vertex and an open edge, respectively. Then $\Gamma_{W^\bullet}$ is as follows:
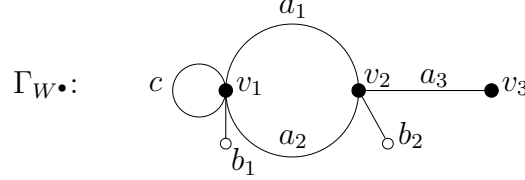
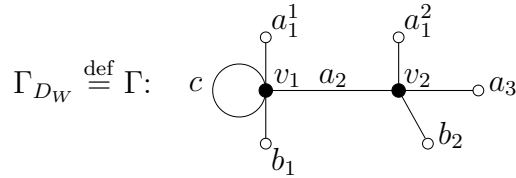(b) Let $E \overset{\text{def}}{=} \{a_1\}$. Then we see that the dual semi-graph $\Gamma_{W_1^\bullet}$ of $W_1^\bullet$ is as follows:
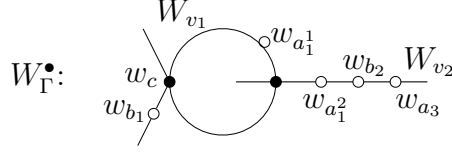

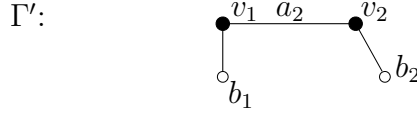Note that the set of terminal vertices avoiding to $e^{\text{op}}(\Gamma_{W^\bullet})$ of $W_1^\bullet$ is $\{v_3\}$.

(c) We obtain a minimal quasi-tree $\Gamma_{D_W} \overset{\text{def}}{=} \Gamma$ associated to D_W is as follows:

On the other hand, $W_\Gamma^\bullet$ is as follows:

(d) Next, we give an example $\Gamma' \subseteq \Gamma_{W^\bullet}$, which is a tree containing all open edges of $\Gamma_{W^\bullet}$, and which is *not* a (minimal) quasi-tree associated to D_W

If Γ'' is a quasi-tree, then by the definition of quasi-trees (Definition 4.8), Γ' is equal to a semi-graph $\Gamma''_{L''}$ (4.4.2) associated to a sub-semi-graph Γ'' of $\Gamma_{W^\bullet}$ and a subset of closed edges $L'' \subseteq e^{\mathrm{cl}}(\Gamma'') \setminus e^{\mathrm{lp}}(\Gamma'')$. Thus, the definition of $\Gamma''_{L''}$ implies that $e^{\Gamma''_{L''}}(v_1) = \{c, b_1, a_1^1, a_2\}$. This means that Γ' is not a quasi-tree.

4.4.7. We maintain the notation introduced in 4.4.5. Suppose that $n_W \neq 0$, and that $\Gamma = \Gamma_{D_W}$ is a minimal quasi-tree associated to D_W.

The construction of W_Γ implies that there is a natural morphism

$$f_\Gamma : W_\Gamma \to W$$

over k. We denote by

$$\phi_\Gamma : \Gamma \to \Gamma_{W^\bullet}$$

the map of dual semi-graphs induced by f_Γ. Note that the construction of W_Γ implies that f_Γ induces an injection of sets of irreducible components $f_\Gamma^{\mathrm{irr}} : \mathrm{Irr}(W_\Gamma) \hookrightarrow \mathrm{Irr}(W)$ (or equivalently, ϕ_Γ induces an injection of sets of vertices $v(\Gamma) \hookrightarrow v(\Gamma_{W^\bullet})$). By the construction of Γ, we see that the set of marked points D_{W_Γ} of $W_\Gamma^\bullet$ can be divided into the following three parts:

- The set of marked points D_W of $W^\bullet$ (then $f_\Gamma(D_{W_\Gamma} \setminus D_W) \subseteq W^{\mathrm{sing}}$).
- Let $w \in D_{W_\Gamma} \setminus D_W$ and W_w the unique irreducible component of W_Γ containing w. Since $f_\Gamma(w) \in W^{\mathrm{sing}}$, we have $f_\Gamma(w) \in f_\Gamma(W_w) \cap W'_w$, where $W'_w \in \mathrm{Irr}(W)$ is the unique irreducible component containing $f_\Gamma(w)$ if $f_\Gamma(w) \in f_\Gamma(W_w)^{\mathrm{sing}}$ (i.e., $W'_w = f_\Gamma(W_w)$), and is the unique irreducible component such that $f_\Gamma(W_w) \neq W'_w$ if $f_\Gamma(w) \in f_\Gamma(W_w) \setminus f_\Gamma(W_w)^{\mathrm{sing}}$. Then we put

$$D_{E_1} \stackrel{\mathrm{def}}{=} \{w \in D_{W_\Gamma} \setminus D_W \mid W'_w \notin \mathrm{Im}(f_\Gamma^{\mathrm{irr}})\}.$$

- We put

$$D_{E_2} \stackrel{\mathrm{def}}{=} \{w \in D_{W_\Gamma} \setminus D_W \mid W'_w \in \mathrm{Im}(f_\Gamma^{\mathrm{irr}})\}.$$

Note that we have $D_{W_\Gamma} = D_W \sqcup D_{E_1} \sqcup D_{E_2}$, where $\sqcup$ means disjoint union.

Note that $f_\Gamma : W_\Gamma \setminus D_{E_2} \xrightarrow{\sim} f_\Gamma(W_\Gamma)$. Namely, the semi-stable sub-curve $f_\Gamma(W_\Gamma) \subseteq W$ is glued by W_Γ along D_{E_2} (or W_Γ is the normalization of $f_\Gamma(W_\Gamma)$ at $f_\Gamma(D_{E_2})$), and $f_\Gamma|_{W_\Gamma \setminus D_{E_2}}$ is an isomorphism.

We define (for instance, see Example 4.11 below)

$$W^\bullet_{\Gamma^{\mathrm{im}}} = (W_{\Gamma^{\mathrm{im}}} \overset{\mathrm{def}}{=} f_\Gamma(W_\Gamma), D_{W_{\Gamma^{\mathrm{im}}}} \overset{\mathrm{def}}{=} f_\Gamma(D_W \cup D_{E_1}))$$

to be a pointed stable curve over k. Then we see that the dual semi-graph of $W^\bullet_{\Gamma^{\mathrm{im}}}$ coincides with the image $\mathrm{Im}(\phi_\Gamma)$. We call

$$\Gamma^{\mathrm{im}} \overset{\mathrm{def}}{=} \Gamma_{W^\bullet_{\Gamma^{\mathrm{im}}}}$$

the image of the map ϕ_Γ (for instance, see Example 4.11 below). Moreover, we denote by

$$\mathrm{norm}_\Gamma : W_\Gamma \to W_{\Gamma^{\mathrm{im}}}$$

the natural morphism induced by f_Γ which coincides with the normalization morphism of $W_{\Gamma^{\mathrm{im}}}$ over the set of nodes $f_\Gamma(D_{E_2})$.

4.4.8.   We maintain the notation introduced in Example 4.10 and give an example of Γ^{im}.

Example 4.11. We see that the set of open edges of $W^\bullet_\Gamma$ corresponding to D_{E_1} is $\{a_3\}$ (Example 4.10 (c)), and that the set of open edges of $W^\bullet_\Gamma$ corresponding to D_{E_2} is $\{e_1, e_2\}$ (Example 4.10 (c)). Then the image Γ^{im} of the map $\phi_\Gamma : \Gamma \to \Gamma^\bullet_W$ is as follows:
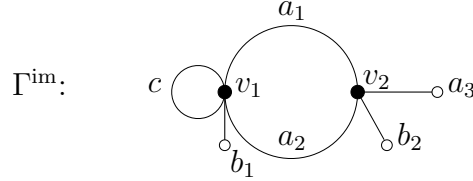
On the other hand, $W^\bullet_{\Gamma^{\mathrm{im}}}$ is as follows:
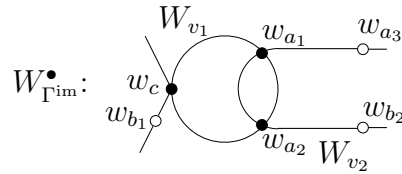
Note that $D_{E_1} = \{w_{a_3}\}$ and $D_{E_2} = \{w_{e_1}, w_{e_2}\}$, where w_{a_3}, w_{e_i}, $i \in \{1,2\}$, denote the marked points of $W^\bullet_\Gamma$ corresponding a_3, e_i, respectively. We have $f_\Gamma(W_{w_{a_3}}) = W_{v_2}$, $f_\Gamma(W_{w_{e_1}}) = W_{v_1}$, $f_\Gamma(W_{w_{e_2}}) = W_{v_2}$, $W'_{w_{e_1}} = W_{v_2}$, and $W'_{w_{e_2}} = W_{v_1}$.

4.5. **The first main theorem.** We are going to prove the first main theorem of the present paper.

4.5.1. **Settings.** We maintain the notation introduced in 2.1.2 and the assumption introduced in 2.3.3 (i.e., $n \overset{\mathrm{def}}{=} p^t - 1$). Moreover, we assume that $X^\bullet$ is a component-generic pointed stable curve over k.

4.5.2. As we have mentioned before (Example 3.12), Proposition 3.11 (i) does not hold in general. The key is that the ramifications over nodes cannot be completely determined by the ramifications over marked points if the dual semi-graphs *are not tree-like*. Then by Proposition 4.6, we see that, to generalize Proposition 3.11 (i) to the case of (possibly reducible and possibly singular) pointed stable curves, we need a *tree-like structure* concerning $\Gamma_{X^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{X^\bullet})$. Note that, if $n_X \neq 0$, the ramifications over marked points can completely determine the ramifications over *nodes contained in minimal quasi-trees*. Then we introduce the following conditions: $\Gamma_{X^\bullet}$ is a tree if $n_X = 0$, and $\Gamma_{X^\bullet}$ is a minimal quasi-tree associated to D_X if $n_X \neq 0$. Moreover, we have the following proposition.

Proposition 4.12. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ be an effective divisor on X such that $s(D) = n_X - 1$ if $n_X \neq 0$, and that D is Frobenius stable (i.e., $\deg(D^{(i)}) = \deg(D)$, $i \in \{0, 1, \ldots, t-1\}$). Then the following statements hold:*

(i) Suppose that one of the following conditions is satisfied:

- $n_X = 0$ *and* $\Gamma_{X^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{X^\bullet})$ *is a tree.*
- $n_X \neq 0$ *and* $\Gamma_{X^\bullet} = \Gamma_{D_X}$ *is a minimal quasi-tree associated to D_X (see Definition 4.8).*

Then $\gamma_{(\alpha,D)}$ attains maximum for all $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$. Namely, the following holds:

$$\gamma_{(\alpha,D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

(ii) There exists an element $\beta \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$ such that $\gamma_{(\beta,D)}$ attains maximum. Namely, the following holds:

$$\gamma_{(\beta,D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. (i) Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be a Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. To verify (i), we may assume that $Y^\bullet$ is connected.

Suppose that $n_X = 0$. Since $\Gamma_{X^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{X^\bullet})$ is a tree, we see immediately that f is étale. Then (i) follows from Theorem 3.9 and Proposition 4.6.

Suppose that $n_X \neq 0$. Let $v \in v(\Gamma_{X^\bullet})$ and $\pi_0(\overline{X \setminus X_v})$ the set of connected components of $\overline{X \setminus X_v}$, where $\overline{X \setminus X_v}$ denotes the topological closure of $X \setminus X_v$ in X. We put

$$D_v \overset{\text{def}}{=} (D_X \cap X_v) \cup \Big(\bigcup_{C \in \pi_0(\overline{X \setminus X_v})} (C \cap X_v) \Big).$$

Let $X_v^\bullet = (X_v, D_{X_v} \overset{\text{def}}{=} D_v)$, $v \in v(\Gamma_{X^\bullet})$, be a pointed stable curve of type (g_{X_v}, n_{X_v}) over k. Then $f^\bullet$ induces a Galois multi-admissible covering

$$f_v^\bullet : Y_v^\bullet \to X_v^\bullet, \ v \in v(\Gamma_{X^\bullet}),$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Note that since $\Gamma_{X^\bullet}$ is a minimal quasi-tree associated to D_X, we have that $C \cap X_v = \{x_C\}$ is a closed point of X for all $C \in \pi_0(\overline{X \setminus X_v})$. We put

$$Q_v \overset{\text{def}}{=} \sum_{x \in D_X \cap X_v} \mathrm{ord}_x(Q_v) x + \sum_{C \in \pi_0(\overline{X \setminus X_v})} \mathrm{ord}_{x_C}(Q_v) x_C \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_v]^0, \ v \in v(\Gamma_{X^\bullet}),$$

the ramification divisor on X_v induced by f_v. Since $\Gamma_{X^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{X^\bullet})$ is a tree, then Q_v satisfies the following:

$$\mathrm{ord}_x(Q_v) \overset{\mathrm{def}}{=} \mathrm{ord}_x(D), \ x \in D_X \cap X_v,$$

$$\mathrm{ord}_{x_C}(Q_v) \overset{\mathrm{def}}{=} [\sum_{c \in D_X \cap C} \mathrm{ord}_c(D)], \ C \in \pi_0(\overline{X \setminus X_v}),$$

where $[(-)]$ denotes the integer which is equal to the image of $(-)$ in $\mathbb{Z}/n\mathbb{Z}$ when we identify $\{0, \ldots, n-1\}$ with $\mathbb{Z}/n\mathbb{Z}$ naturally. By applying similar arguments to the arguments given in the proof of Lemma 4.3, we have

$$\deg(Q_v) = (\#(D_v) - 1)n \text{ and } \deg(Q_v^{(i)}) = \deg(Q_v), \ i \in \{0, \ldots, t-1\}.$$

Let $\Pi_{X_v^\bullet}$, $v \in v(\Gamma_{X^\bullet})$, be the admissible fundamental group of $X_v^\bullet$. Write $\alpha_v \in \mathrm{Rev}_{Q_v}^{\mathrm{adm}}(X_v^\bullet)$ for the composition of the natural homomorphisms $\Pi_{X_v^\bullet} \to \Pi_{X^\bullet}^{\mathrm{ab}} \overset{\alpha}{\to} \mathbb{Z}/n\mathbb{Z}$. Note that $\alpha_v \neq 0$ for all $v \in v(\Gamma_{X^\bullet})$. Then Proposition 4.6 implies

$$\gamma_{(\alpha_v, Q_v)} = \begin{cases} g_{X_v} - 1, & \text{if } \mathrm{Supp}(Q_v) = \emptyset, \\ g_{X_v} + s(Q_v) - 2, & \text{if } \mathrm{Supp}(Q_v) \neq \emptyset. \end{cases}$$

Thus, Theorem 3.9 implies that

$$\gamma_{(\alpha, D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

This completes the proof of (i).

(ii) Suppose that $n_X \leq 1$. (ii) follows from Proposition 3.11 (ii). Then to verify (ii), we may assume that $n_X \geq 2$.

Let $\Gamma \overset{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X, Γ^{im} the image of the natural morphism $\phi_\Gamma : \Gamma \to \Gamma_{X^\bullet}$, and

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma}), \ X_{\Gamma^{\mathrm{im}}}^\bullet = (X_{\Gamma^{\mathrm{im}}}, D_{X_{\Gamma^{\mathrm{im}}}})$$

the pointed stable curves over k associated to Γ, Γ^{im}, respectively (4.4.5, 4.4.7). Note that D is also an effective divisor on $X_{\Gamma^{\mathrm{im}}}$.

Write D_Γ for the pulling back divisor $\mathrm{norm}_\Gamma^*(D)$ (see 4.4.7 for the definition of norm_Γ). Let $\alpha_\Gamma \in \mathrm{Rev}_{D_\Gamma}^{\mathrm{adm}}(X_\Gamma^\bullet)$ be an arbitrary element such that $\alpha_\Gamma \neq 0$. Then similar arguments to the arguments given in (i) imply $\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + n_X - 2$, where g_{X_Γ} denotes the genus of X_Γ. We denote by

$$g_\Gamma^\bullet : Z_\Gamma^\bullet \to X_\Gamma^\bullet$$

the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_Γ. By gluing $Z_\Gamma^\bullet$ along $g_\Gamma^{-1}(D_{X_\Gamma} \setminus \mathrm{norm}_\Gamma^{-1}(D_{X_{\Gamma^{\mathrm{im}}}}))$ in a way that is compatible with the gluing of $X_\Gamma^\bullet$ that gives rise to $X_{\Gamma^{\mathrm{im}}}^\bullet$, we obtain a pointed stable curve $Z_{\Gamma^{\mathrm{im}}}^\bullet$ over k. Moreover, $g_\Gamma^\bullet$ induces a Galois multi-admissible covering

$$g_{\Gamma^{\mathrm{im}}}^\bullet : Z_{\Gamma^{\mathrm{im}}}^\bullet \to X_{\Gamma^{\mathrm{im}}}^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Let $\Pi_{X_\Gamma^\bullet}$, $\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}$ be the admissible fundamental groups of $X_\Gamma^\bullet$, $X_{\Gamma^{\mathrm{im}}}^\bullet$, respectively. Write $\alpha_{\Gamma^{\mathrm{im}}}$ for an element of $\mathrm{Hom}(\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ induced by

$g_{\Gamma^{\mathrm{im}}}^{\bullet}$ such that the composition of the natural homomorphisms $\Pi_{X_{\Gamma}^{\bullet}}^{\mathrm{ab}} \to \Pi_{X_{\Gamma^{\mathrm{im}}}^{\bullet}}^{\mathrm{ab}} \overset{\alpha_{\Gamma^{\mathrm{im}}}}{\to} \mathbb{Z}/n\mathbb{Z}$ is equal to α_{Γ}. We put $D_{\Gamma^{\mathrm{im}}} \overset{\mathrm{def}}{=} D_{\alpha_{\Gamma^{\mathrm{im}}}}$. Then Theorem 3.9 implies that

$$\gamma_{(\alpha_{\Gamma^{\mathrm{im}}}, D_{\Gamma^{\mathrm{im}}})} = g_{X_{\Gamma^{\mathrm{im}}}} + n_X - 2,$$

where $g_{X_{\Gamma^{\mathrm{im}}}}$ denotes the genus of $X_{\Gamma^{\mathrm{im}}}$.

On the other hand, we write $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ for the set of connected components of $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$, where $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$ denotes the topological closure of $X \setminus X_{\Gamma^{\mathrm{im}}}$ in X. We define the following pointed stable curve

$$C^{\bullet} = (C, D_C \overset{\mathrm{def}}{=} C \cap X_{\Gamma^{\mathrm{im}}}), \ C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}),$$

over k. Note that since $X^{\bullet}$ is component-generic, we have that $C^{\bullet}$ is also component-generic. Then the p-rank σ_C is equal to the genus of $C^{\bullet}$.

Let $C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$. We put

$$Z_C^{\bullet} \overset{\mathrm{def}}{=} \bigsqcup_{i \in \mathbb{Z}/n\mathbb{Z}} C_i^{\bullet},$$

where $C_i^{\bullet}$ is a copy of $C^{\bullet}$, and $\sqcup$ means disjoint union. Then we obtain a Galois multi-admissible covering

$$g_C^{\bullet} : Z_C^{\bullet} \to C^{\bullet}$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$, where the restriction morphism $g_C^{\bullet}|_{C_i^{\bullet}}$ is an identity, and the Galois action is $j(C_i) = C_{i+j}$ for all $i, j \in \mathbb{Z}/n\mathbb{Z}$. By gluing $Z_{\Gamma^{\mathrm{im}}}^{\bullet}$ and $\{Z_C^{\bullet}\}_{C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$ along

$$g_{\Gamma^{\mathrm{im}}}^{-1}(X_{\Gamma^{\mathrm{im}}} \cap (\bigcup_{C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})} C)) \text{ and } \{g_C^{-1}(X_{\Gamma^{\mathrm{im}}} \cap C)\}_{C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$$

in a way that is compatible with the gluing of $\{X_{\Gamma^{\mathrm{im}}}^{\bullet}\} \cup \{C^{\bullet}\}_{C \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$ that gives rise to $X^{\bullet}$, we obtain a Galois multi-admissible covering

$$g^{\bullet} : Z^{\bullet} \to X^{\bullet}$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Let $\Pi_{X^{\bullet}}$ be the admissible fundamental group of $X^{\bullet}$. Moreover, we write $\beta \in \mathrm{Rev}_D^{\mathrm{adm}}(X^{\bullet})$ for an element induced by $g^{\bullet}$ such that the composition of the natural homomorphisms $\Pi_{X_{\Gamma^{\mathrm{im}}}^{\bullet}}^{\mathrm{ab}} \to \Pi_{X^{\bullet}}^{\mathrm{ab}} \overset{\beta}{\to} \mathbb{Z}/n\mathbb{Z}$ is equal to $\alpha_{\Gamma^{\mathrm{im}}}$. By applying Theorem 3.9, we see

$$\gamma_{(\beta, D)} = \gamma_{X^{\bullet}}^{\max} = g_X + n_X - 2.$$

We complete the proof of (ii). $\qquad\square$

4.5.3. Now, the main result of the present section is as follows.

Theorem 4.13. *Let $X^{\bullet} = (X, D_X)$ be a component-generic pointed stable curve over k. Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p and $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5). Let $t \in \mathbb{N}$ be a positive natural number such that $p^t = 1$ in $(\mathbb{Z}/m\mathbb{Z})^{\times}$. Write n for $p^t - 1$, m' for n/m, and D' for the divisor $m'D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order m. Then the following statements hold:*

(a) *We have (see 2.2.5 and Definition 2.4 for $(D')^{(i)}$)*

$$s(D) = \begin{cases} 0, & \text{if } n_X = 0, \\ n_X - 1, & \text{if } n_X \neq 0 \end{cases}$$

and D' is Frobenius stable (i.e., $\deg((D')^{(i)}) = \deg(D')$, $i \in \{0, 1, \ldots, t - 1\}$).

(b) *There exists an element $\beta \in \operatorname{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$ (Definition 2.4 (i)) such that $\gamma_{(\beta,D)}$ attains maximum (Definition 3.10 (ii)). Namely, the following holds (see Definition 3.10 (i) for $\gamma_{X^\bullet}^{\max}$):*

$$\gamma_{(\beta,D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Moreover, suppose that one of the following conditions is satisfied:

- $n_X = 0$ *and* $\Gamma_{X^\bullet} \setminus e^{\mathrm{lp}}(\Gamma_{X^\bullet})$ *is a tree.*
- $n_X \neq 0$ *and* $\Gamma_{X^\bullet} = \Gamma_{D_W}$ *is a minimal quasi-tree associated to D_X (4.4.5).*

Then (a) and (b) are equivalent to the following statement:

(c) $\gamma_{(\alpha,D)}$ *attains maximum for all $\alpha \in \operatorname{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$. Namely, the following holds:*

$$\gamma_{(\alpha,D)} = \gamma_{X^\bullet}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. "(b)⇒(a)" follows from Lemma 4.2. We prove "(a)⇒(b)". Let $\Gamma \overset{\text{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X, Γ^{im} the image of the natural morphism $\phi_\Gamma : \Gamma \to \Gamma_{X^\bullet}$, $X_{\Gamma^{\mathrm{im}}}^\bullet = (X_{\Gamma^{\mathrm{im}}}, D_{X_{\Gamma^{\mathrm{im}}}})$ the pointed stable curve over k associated to Γ^{im} (4.4.7), and $\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}$ the admissible fundamental group of $X_{\Gamma^{\mathrm{im}}}^\bullet$. Note that D is also an effective divisor on $X_{\Gamma^{\mathrm{im}}}$. Let $\beta_{\Gamma^{\mathrm{im}}}$ be an arbitrary element of $\operatorname{Rev}_D^{\mathrm{adm}}(X_{\Gamma^{\mathrm{im}}}^\bullet) \setminus \{0\}$. Write $\beta'_{\Gamma^{\mathrm{im}}} \in \operatorname{Rev}_{D'}^{\mathrm{adm}}(X_{\Gamma^{\mathrm{im}}}^\bullet)$ for the element induced by $\beta_{\Gamma^{\mathrm{im}}}$.

Let $\Pi_{X^\bullet}$ be the admissible fundamental group of $X^\bullet$. Proposition 4.12 (ii) implies that there exists $\beta' \in \operatorname{Rev}_{D'}^{\mathrm{adm}}(X^\bullet)$ such that the composition of the natural homomorphisms $\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet} \overset{\beta'}{\to} \mathbb{Z}/n\mathbb{Z}$ is equal to $\beta'_{\Gamma^{\mathrm{im}}}$, and that $\gamma_{(\beta',D')}$ attains maximum. Moreover, the construction of β' given in the proof of Proposition 4.12 (ii) (i.e., the Galois multi-admissible covering of $\widetilde{X}_v^\bullet$, $v \in v(\Gamma_{X^\bullet}) \setminus v(\Gamma^{\mathrm{im}})$, induced by β' is trivial) implies that β' can be induced by an element of $\beta \in \operatorname{Rev}_D^{\mathrm{adm}}(X^\bullet)$. Then $\gamma_{(\beta,D)} = \gamma_{(\beta',D')}$ attains maximum. This completes the proof of "(b)⇔(a)".

Next, we prove the "moreover" part of the theorem. Write $\alpha' \in \operatorname{Rev}_{D'}^{\mathrm{adm}}(X^\bullet)$ for the element induced by α. Then we have $\gamma_{(\alpha,D)} = \gamma_{(\alpha',D')}$. "(c)⇒(a)" follows from Lemma 4.2. Moreover, "(a)⇒(c)" follows immediately from Proposition 4.12 (i). This completes the proof of the theorem. $\qquad\square$

4.6. (m, n_X)-**ordinary curves.**

4.6.1.   We maintain the notation introduced in 2.1.2. Moreover, let $\overline{\mathcal{M}}_{g_X, n_X}$ be the moduli stack parameterizing pointed stable curves of type (g_X, n_X) (2.1.4) and $\mathcal{M}_{g_X, n_X} \subseteq \overline{\mathcal{M}}_{g_X, n_X}$ the open substack parameterizing smooth pointed stable curves of type (g_X, n_X). We denote by $\overline{M}_{g_X, n_X}$, M_{g_X, n_X} the coarse moduli spaces of $\overline{\mathcal{M}}_{g_X, n_X}$, $\mathcal{M}_{g_X, n_X}$, respectively.

4.6.2. Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p, and let $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$ be an effective divisor on X such that $\deg(D) = (n_X - 1)m$ if $n_X \neq 0$.

Definition 4.14. We shall say that $X^{\bullet}$ is (m, n_X)-*ordinary* if, for *all* $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^{\bullet}) \setminus \{0\}$, we have

$$\gamma_{(\alpha, D)} = \gamma_{X^{\bullet}}^{\max} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Note that, if $n_X = 0$ and $X^{\bullet}$ is non-singular, then the definition of (m, n_X)-ordinary coincides with the definition of *m-ordinary* introduced by Nakajima ([N, §4]).

On the other hand, let $t \in \mathbb{N}$ be a positive natural number such that $p^t = 1$ in $(\mathbb{Z}/m\mathbb{Z})^{\times}$. Write n for $p^t - 1$, m' for n/m, and D' for the divisor $m'D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order m. If $n_X \neq 0$ and $X^{\bullet}$ is (m, n_X)-ordinary, then Lemma 4.2 (ii) implies that D' is Frobenius stable. Namely, we have

$$\deg((D')^{(i)}) = \deg(D') = (n_X - 1)n, \ i \in \{0, 1, \ldots, t-1\}.$$

4.6.3. We denote by

$$\overline{U}_{(m, n_X)} \stackrel{\text{def}}{=} \{q \in \overline{M}_{g_X, n_X} \mid \text{curves corresponding to } q \text{ is } (m, n_X)\text{-ordinary}\} \subseteq \overline{M}_{g_X, n_X}.$$

Moreover, we put $U_{(m, n_X)} \stackrel{\text{def}}{=} \overline{U}_{(m, n_X)} \cap M_{g_X, n_X}$ (i.e., the set of points corresponding to *smooth* (m, n_X)-ordinary curves). Then we have the following result.

Proposition 4.15. *The set* $U_{(m, n_X)}$ *is a non-empty open subset of* M_{g_X, n_X}.

Proof. By applying similar arguments to the arguments given in the proof of [N, Theorem 2], the proposition follows from Theorem 4.13 (i). $\qquad\square$

If $n_X \leq 1$, then Proposition 4.15 was proved by Nakajima ([N, Theorem 2]). Then Proposition 4.15 is a generalized version of Nakajima's result to the case of admissible coverings of smooth pointed stable curves. Moreover, Nakajima ([N, §4 Remark]) asked whether or not

$$\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} U_{(m, n_X)}$$

is a non-empty open subset of M_{g_X, n_X}. By applying the theory of Raynaud-Tamagawa theta divisors and [R2, Proposition 1.2.1], we see that the following holds:

Suppose that $n_X \leq 1$. Then we have

$$M_{g_X, n_X}^{\mathrm{cl}} \cap \Big(\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} U_{(m, n_X)} \Big) = \emptyset,$$

where $M_{g_X, n_X}^{\mathrm{cl}}$ denotes the set of closed points of M_{g_X, n_X}.

This gives a *negative* answer of Nakajima's question. On the other hand, we may ask the following question:

Problem 4.16. *(i) Suppose that* $X^{\bullet}$ *is a component-generic pointed stable curve over* k. *Can we find a necessary and sufficient condition that* $X^{\bullet}$ *is* (m, n_X)-*ordinary for all* $m \in \mathbb{N}$ *prime to* p?

(ii) Does

$$M^{\mathrm{cl}}_{g_X, n_X} \cap \left(\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} U_{(m, n_X)} \right) = \emptyset$$

hold for an arbitrary non-negative integer n_X? Moreover, does

$$\overline{M}^{\mathrm{cl}}_{g_X, n_X} \cap \left(\bigcap_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} \overline{U}_{(m, n_X)} \right) = \emptyset$$

hold for an arbitrary non-negative integer n_X?

5. Maximum generalized Hasse-Witt invariants for arbitrary curves

In the present section, we discuss the maximum generalized Hasse-Witt invariants of cyclic admissible coverings for arbitrary pointed stable curves. The main result of this section is Theorem 5.4.

5.1. **Idea.** We briefly explain the idea of our proof of Theorem 5.4.

5.1.1. **Settings.** We maintain the notation introduced in 2.1.2.

5.1.2. *An easy case.* Firstly, let us prove an easy case (i.e., $X^\bullet$ is irreducible) of the main result of the present section.

Proposition 5.1. *Suppose that $X^\bullet$ is irreducible. Then there exist a positive natural number $n \stackrel{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ (2.2.5) on X of degree $(n_X - 1)n$ if $n_X \neq 0$ (resp. degree 0 if $n_X = 0$), and an element $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet) \setminus \{0\}$ (Definition 2.4 (i)) such that $\gamma_{(\alpha, D)}$ attains maximum (Definition 3.10 (ii)). Namely, the following holds:*

$$\gamma_{(\alpha, D)} = \gamma^{\max}_{X^\bullet} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. Since $X^\bullet$ is irreducible, we write $\widetilde{X}^\bullet$ for the smooth pointed stable curve of type $(g_{\widetilde{X}}, n_{\widetilde{X}})$ over k associated to the unique vertex of the dual semi-graph $\Gamma_{X^\bullet}$ of $X^\bullet$ (2.1.3). Note that we have

$$g_{\widetilde{X}} = g_X - \#(X^{\mathrm{sing}}), \quad n_{\widetilde{X}} = n_X + 2\#(X^{\mathrm{sing}}).$$

Moreover, $D_X \subseteq D_{\widetilde{X}}$ implies $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_{\widetilde{X}}]^0$.

By applying Theorem 3.9, to verify the proposition, it is sufficient to prove that there exist a positive natural number $n \stackrel{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ of degree $(n_X - 1)n$ if $n_X \neq 0$ (resp. degree 0 if $n_X = 0$) on $\widetilde{X}$, and an element $\widetilde{\alpha} \in \mathrm{Rev}^{\mathrm{adm}}_{\widetilde{D}}(\widetilde{X}^\bullet) \setminus \{0\}$ such that the following holds:

$$\gamma_{(\widetilde{\alpha}, \widetilde{D})} = \begin{cases} g_{\widetilde{X}} - 1, & \text{if } n_X = 0, \\ g_{\widetilde{X}} + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Suppose that $n_X \leq 1$. Then the proposition follows immediately from Proposition 3.11 (ii).

Suppose that $n_X \geq 2$. Let $D_X \stackrel{\mathrm{def}}{=} \{x_1, \ldots, x_{n_X}\}$ and $n_i \stackrel{\mathrm{def}}{=} p^{t_i} - 1$, $i \in \{1, \ldots, n_X - 1\}$, such that the following conditions are satisfied (see 2.4.5 for $C(g_X)$):

- $n_i > \max\{C(g_X) + 1, \#(e(\Gamma_{X^\bullet}))\}$.
- $0 < a_{i,1}, a_{i,2} < n_i$ and $a_{i,1} + a_{i,2} = n_i$ for all $i \in \{1, \ldots, n_X - 1\}$.

Furthermore, we put

$$D_i \overset{\text{def}}{=} a_{i,1}x_i + a_{i,2}x_{i+1} + \sum_{x \in D_X \setminus \{x_i, x_{i+1}\}} n_i x, \ i \in \{1, \dots, n_X - 1\},$$

which is an effective divisor on $\widetilde{X}$ of degree $(n_X - 1)n_i$. Moreover, we put

$$\widetilde{D} \overset{\text{def}}{=} \sum_{i=1}^{n_X - 1} p^{\sum_{j=0}^{i-1} t_j} D_i,$$

$$n \overset{\text{def}}{=} p^{\sum_{i=0}^{n_X - 1} t_j} - 1 = \sum_{i=1}^{n_X - 1} p^{\sum_{j=0}^{i-1} t_j}(p^{t_i} - 1),$$

where $t_0 \overset{\text{def}}{=} 0$. We see that $\widetilde{D}$ is an effective divisor on $\widetilde{X}$ of degree $(n_X - 1)n$ such that $\widetilde{D} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$.

Let $\mathcal{L}_{\widetilde{D}}$ be a line bundle on $\widetilde{X}$ such that $\mathcal{L}_{\widetilde{D}}^{\otimes n} \cong \mathcal{O}_X(-\widetilde{D})$, and $\mathcal{L}_{\widetilde{D},t}$ the pulling back of $\mathcal{L}_{\widetilde{D}}$ by the natural morphism $\widetilde{X}_t \to \widetilde{X}$ defined in 2.4.2. Then by applying [T2, Corollary 2.6, Lemma 2.12 (ii), and Corollary 2.13], we see that the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_{\widetilde{D}}^t \otimes \mathcal{L}_{\widetilde{D},t}$ exists (Definition 2.9). Moreover, Proposition 2.10 implies that there exists a line bundle $\widetilde{\mathcal{I}}$ of degree 0 on $\widetilde{X}$ such that $[\widetilde{\mathcal{I}}] \neq [\mathcal{O}_{\widetilde{X}}]$, that $[\widetilde{\mathcal{I}}^{\otimes n}] = [\mathcal{O}_{\widetilde{X}}]$, and that (see 2.3.4 for $\gamma_{([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D})}$)

$$\gamma_{([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D})} = \begin{cases} g_{\widetilde{X}} - 1, & \text{if } n_X = 0, \\ g_{\widetilde{X}} + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Let $\widetilde{\alpha} \in \text{Rev}_{\widetilde{D}}^{\text{adm}}(\widetilde{X}^\bullet)$ be the element corresponding to the pair $([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D}) \in \widetilde{\mathscr{P}}_{\widetilde{X}^\bullet, n}$ (2.3.2). Then we have $\gamma_{(\widetilde{\alpha}, \widetilde{D})} = \gamma_{([\mathcal{L}_{\widetilde{D}} \otimes \widetilde{\mathcal{I}}], \widetilde{D})}$. This completes the proof of the proposition. $\qquad\square$

5.1.3. *Strategy of the proof of Theorem 5.4.* In the remainder of this section, we will generalize Proposition 5.1 to the case where $X^\bullet$ is an arbitrary pointed stable curve over k (i.e., Theorem 5.4 below).

Let $X^\bullet$ be an arbitrary pointed stable curve over k. For simplicity, we assume that every irreducible component of $X^\bullet$ is non-singular. Moreover, by Proposition 3.11 (ii), we assume $n_X \geq 2$. We maintain the notation introduced in the statement of Theorem 3.9. To verify Theorem 5.4, by applying Theorem 3.9, it is sufficient to construct a prime-to-p cyclic Galois multi-admissible covering $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ for every $v \in v(\Gamma_{X^\bullet})$ such that the following conditions are satisfied:

- $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})}$ satisfies the conditions mentioned in the statement of Theorem 3.9.
- The ramification divisor $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ associated to $f^\bullet$ is such that $s(D) = n_X - 1$ (2.2.5).
- $\{\widetilde{f}_v^\bullet\}_{v \in v(\Gamma_{X^\bullet})}$ can be glued together. This means that $D_{\widetilde{\alpha}_v}$, $v \in v(\Gamma_{X^\bullet})$, is completely determined by D such that the following *gluing condition* holds (e.g. see Example 5.2 Goal below):

  Let x be a node of X contained in the intersection $X_{v_1} \cap X_{v_2}$ of irreducible components X_{v_1} and X_{v_2}. Then we have

  $$\text{ord}_x(D_{\widetilde{\alpha}_{v_1}}) + \text{ord}_x(D_{\widetilde{\alpha}_{v_2}}) = 0 \bmod n.$$

Thus, by the definition of admissible coverings (Definition 2.3), we obtain a prime-to-p cyclic Galois multi-admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$.

If $X^\bullet$ is *not* irreducible, the constructions of desired Galois multi-admissible coverings of $X^\bullet$ are very difficult in general.

The main difficulty. We *cannot determine the ramifications over nodes* of a Galois admissible covering in a unique way when the ramifications over D_X are fixed. Then we have the following:

(i) Let $f^\bullet$ be a Galois admissible covering whose ramification divisor satisfies the third condition mentioned above. Since the Raynaud-Tamagawa theta divisor concerning $D_{\widetilde{\alpha}_v}$, $v \in v(\Gamma_{X^\bullet})$, *does not exist* in general, $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})}$ does not satisfy the first condition mentioned above.

(ii) Even though we can construct $\widetilde{f}_v^\bullet$ for every $v \in v(\Gamma_{X^\bullet})$ (by applying Proposition 5.1) such that $\gamma_{(\widetilde{\alpha}_v, D_{\widetilde{\alpha}_v})}$ satisfies the conditions mentioned in the statement of Theorem 3.9, $\{\widetilde{f}_v^\bullet\}_{v \in v(\Gamma_{X^\bullet})}$ *cannot be glued together (as admissible coverings) in general.*

To overcome this difficulty, we observe that, when $\Gamma_{X^\bullet}$ is a *minimal quasi-tree* (Definition 4.8), the ramifications over nodes of a Galois admissible covering can be uniquely determined if the ramifications over D_X are fixed. Then for every $v \in v(\Gamma_{X^\bullet})$, we may construct certain effective divisors on $\widetilde{X}_v$ for every marked point of $D_X \cap \widetilde{X}_v$ and every node of $X^{\mathrm{sing}} \cap \widetilde{X}_v$. Furthermore, by similar arguments to the arguments given in the proof of Proposition 5.1, we may construct an effective divisor on $\widetilde{X}_v$ such that the Raynaud-Tamagawa theta divisor concerning this effective divisor exists. Then we can obtain Galois multi-admissible coverings for all $v \in v(\Gamma_{X^\bullet})$ whose first generalized Hasse-Witt invariants attain maximum. On the other hand, since the ramifications over nodes of a Galois admissible covering can be uniquely determined by the ramifications of marked points when $\Gamma_{X^\bullet}$ is a minimal quasi-tree, then we also obtain an effective divisor on X whose restriction on $\widetilde{X}_v$, $v \in v(\Gamma_{X^\bullet})$, is the effective divisor constructed above. Then we may construct a desired Galois multi-admissible covering $f^\bullet$ (e.g. see Example 5.2 below).

In the general case (i.e., $\Gamma_{X^\bullet}$ is not a tree), we take a *minimal quasi-tree* $\Gamma \overset{\mathrm{def}}{=} \Gamma_{D_X}$ *associated to* D_X (Definition 4.8) and the pointed stable curve $X_\Gamma^\bullet$ (4.4.7) associated to Γ. Then we may construct a desired Galois multi-admissible covering for $X_\Gamma^\bullet$ (see Lemma 5.3). In fact, this is the motivation of the definition of minimal quasi-trees. Moreover, Raynaud's theorem (i.e., [R1, Théorème 4.1.1] or Theorem 2.11) implies that we may construct a Galois (étale) multi-admissible covering for each irreducible component corresponding to $w \in v(\Gamma_{X^\bullet}) \setminus v(\Gamma)$ whose first generalized Hasse-Witt invariant (2.2.4) attains maximum. Then by gluing the Galois multi-admissible coverings together, we obtain a desired Galois multi-admissible covering of $X^\bullet$.

5.2. **A key lemma.**

5.2.1. **Settings.** We maintain the notation introduced in 2.1.2.

5.2.2. In order to convince the reader to follow the constructions given in the proof of Lemma 5.3 below, we give an example for constructing effective divisors $D_{\widetilde{\alpha}_v} \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_{\widetilde{X}_v}]^0$, $v \in v(\Gamma_{X^\bullet})$, and $D \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_X]^0$ such that $\{D_{\widetilde{\alpha}_v}\}_{v \in v(\Gamma_{X^\bullet})}$ satisfies the gluing condition.

Example 5.2. Let $X^\bullet$ be a pointed stable curve of type $(g_X, 3)$ over k. Suppose that X has two non-singular irreducible components X_1 and X_2, that $X_1 \cap X_2 \overset{\text{def}}{=} \{x^-\}$, that $D_X \cap X_1 \overset{\text{def}}{=} \{x_{1,1}, x_{1,2}\}$, and that $D_X \cap X_2 \overset{\text{def}}{=} \{x_{2,1}\}$. Let $n_0 \overset{\text{def}}{=} p^{t_0} - 1 >> 0$, $D_{X_1} \overset{\text{def}}{=} \{x_{1,1}, x_{1,2}, x^-\}$, $D_{X_2} \overset{\text{def}}{=} \{x_{2,1}, x^-\}$, $X_1^\bullet = (X_1, D_{X_1})$ a pointed stable curve of type $(g_{X_1}, 3)$, and $X_2^\bullet = (X_2, D_{X_2})$ a pointed stable curve of type $(g_{X_2}, 2)$. Note that $g_X = g_{X_1} + g_{X_2}$. Then we have the following:

$X^\bullet$:
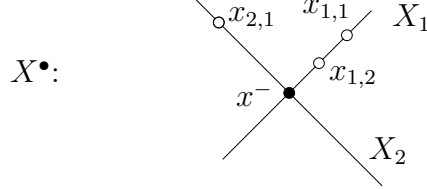
Goal: We will construct effective divisors $D_1 \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_1}]^0$, $D_2 \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_2}]^0$, and $D_3 \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ on X_1, X_2, and X, respectively, for some $n = p^t - 1$ such that the Raynaud-Tamagawa theta divisors associated to D_1 and D_2 exist, and that $D|_{x_{1,1}} = D_1|_{x_{1,1}}$, $D|_{x_{1,2}} = D_1|_{x_{1,2}}$, $D|_{x_{2,1}} = D_2|_{x_{2,1}}$, and the gluing condition (as an admissible covering) $D_1|_{x^-} + D_2|_{x^-} = 0 \bmod n$ hold.

Step 1: We construct effective divisors of degree $2n_0$ (resp. n_0) on X_1 (resp. X_2) such that the Raynaud-Tamagawa theta divisors concerning the divisors exist.

We put

$$
\begin{aligned}
Q_{1,1} &\overset{\text{def}}{=} a_1 x_{1,1} + a_2 x_{1,2} + n_0 x^- \in \text{Div}(X_1), \\
Q_{1,2} &\overset{\text{def}}{=} n_0 x_{2,1} \in \text{Div}(X_2), \\
Q_1 &\overset{\text{def}}{=} a_1 x_{1,1} + a_2 x_{1,2} + n_0 x_{2,1} \in \text{Div}(X),
\end{aligned}
$$

where $0 < a_1, a_2 < n_0$ are such that $a_1 + a_2 = n_0$. Moreover, we put

$$
\begin{aligned}
Q_{2,1} &\overset{\text{def}}{=} n_0 x_{1,1} + b_1 x_{1,2} + b_2 x^- \in \text{Div}(X_1), \\
Q_{2,2} &\overset{\text{def}}{=} b_1 x^- + b_2 x_{2,1} \in \text{Div}(X_2), \\
Q_2 &\overset{\text{def}}{=} n_0 x_{1,1} + b_1 x_{1,2} + b_2 x_{2,1} \in \text{Div}(X),
\end{aligned}
$$

where $0 < b_1, b_2 < n_0$ are such that $b_1 + b_2 = n_0$.

Since $n_0 >> 0$, by applying [T2, Corollary 2.6, Lemma 2.12 (ii), and Corollary 2.13], the Raynaud-Tamagawa theta divisors concerning Q_1, Q_2, $Q_{1,1}$, $Q_{1,2}$, $Q_{2,1}$, $Q_{2,2}$ exist, respectively.

Note that since $\Gamma_{X^\bullet}$ is a *tree*, $Q_{1,1}$ and $Q_{1,2}$ (resp. $Q_{2,1}$ and $Q_{2,2}$) can be completely determined by Q_1 (resp. Q_2). On the other hand, at present, we *cannot* construct Galois multi-admissible coverings whose ramification divisors are the above divisors since the degrees of Q_1, $Q_{1,1}$, $Q_{1,2}$ (resp. Q_2, $Q_{2,1}$, $Q_{2,2}$) *are not multiples of n* (i.e., they *are not contained in* $(\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_1}]^0$ (resp. $(\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_2}]^0$)).

Step 2: By using the effective divisors constructed in Step 1, we construct a global effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ on X and $D_i \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_i}]^0$, $i \in \{1, 2\}$, on X_i such that D_i is completely determined by D.

Write v_1 and v_2 for the vertices of $v(\Gamma_{X^\bullet})$ corresponding to X_1 and X_2, respectively. We define the following sets of effective divisors

$$\mathrm{Div}_{v_i}^{\mathrm{irr\text{-}mp}} \overset{\mathrm{def}}{=} \{Q_{1,i}\}, \ \mathrm{Div}_{v_i}^{\mathrm{irr\text{-}nd}} \overset{\mathrm{def}}{=} \{Q_{2,i}\},$$

$$\mathrm{RT}_X^{\mathrm{irr}} \overset{\mathrm{def}}{=} \bigsqcup_{i=1,2} (\mathrm{Div}_{v_i}^{\mathrm{irr\text{-}mp}} \sqcup \mathrm{Div}_{v_i}^{\mathrm{irr\text{-}nd}}), \ \mathrm{RT}_X \overset{\mathrm{def}}{=} \{Q_1, Q_2\},$$

where $\sqcup$ means disjoint union, and "RT" means "Raynaud-Tamagawa" since we use the divisors to construct certain divisors D_1, D_2, and D whose associated Raynaud-Tamagawa divisors exist (see Step 3 below), respectively. Let $n \overset{\mathrm{def}}{=} p^{t_0} - 1 + p^{t_0}(p^{t_0} - 1) = p^{2t_0} - 1$. We put

$$\begin{aligned} D_1 &\overset{\mathrm{def}}{=} Q_{1,1} + p^{t_0}Q_{2,1} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_1}]^0, \\ D_2 &\overset{\mathrm{def}}{=} Q_{1,2} + p^{t_0}Q_{2,2} \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_{X_2}]^0, \\ D &\overset{\mathrm{def}}{=} Q_1 + p^{t_0}Q_2 \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0. \end{aligned}$$

Then we have

$$D|_{\{x_{1,1}, x_{1,2}\}} = D_1|_{\{x_{1,1}, x_{1,2}\}}, \ D|_{\{x_{2,1}\}} = D_2|_{\{x_{2,1}\}}.$$

Note that, it is easy to check that D_1, D_2, and D are Frobenius stable.

Step 3: We construct a desired Galois multi-admissible covering of $X^\bullet$.

By applying [T2, Corollary 2.6, Lemma 2.12 (ii), and Corollary 2.13], the Raynaud-Tamagawa theta divisors concerning D_1 and D_2 exist. By Proposition 2.10, if $t_0 \gg 0$, then there exists $\alpha_{v_i} \in \mathrm{Rev}_{D_i}^{\mathrm{adm}}(X_i^\bullet) \setminus \{0\}$, $i \in \{1, 2\}$, such that $\gamma_{(\alpha_{v_1}, D_1)} = g_{X_1} + 1$ and $\gamma_{(\alpha_{v_1}, D_1)} = g_{X_2}$. Moreover, since $\Gamma_{X^\bullet}$ is a tree, $\mathrm{ord}_{x^-}(D_1)$ and $\mathrm{ord}_{x^-}(D_2)$ can be completely determined by D. Namely, we have

$$D_1|_{x^-} = \Big[\sum_{x \in D_X \setminus \{x_{1,1}, x_{1,2}\}} \mathrm{ord}_x(D) \Big] x^- = \mathrm{ord}_{x_{2,1}}(D)x^- = (n_0 + p^{t_0}b_2)x^-,$$

$$D_2|_{x^-} = \mathrm{ord}_{x^-}(D_2)x^- = \Big[\sum_{x \in D_X \setminus \{x_{2,1}\}} \mathrm{ord}_x(D) \Big] x^- = (n - \mathrm{ord}_{x_{2,1}}(D_2))x^- = (n - n_0 - p^{t_0}b_2)x^-,$$

where $[(-)]$ denotes the image of $(-)$ in $\mathbb{Z}/n\mathbb{Z}$. Namely, the Galois multi-admissible coverings induced by α_{v_1} and α_{v_2} *can be glued* (as admissible coverings) since $\mathrm{ord}_{x^-}(D_1) + \mathrm{ord}_{x^-}(D_2) = 0 \bmod n$. Then we obtain a desired Galois multi-admissible covering of $X^\bullet$.

5.2.3. Let $\mathbf{G}$ be a connected semi-graph and $v \in v(\mathbf{G})$ an arbitrary vertex. Moreover, we suppose that $\mathbf{G}$ is a *tree*. For each $v' \in v(\mathbf{G})$, there exists a path $\delta_{v,v'}$ connecting v and v' in $\mathbf{G}$. We shall call a path $\delta(\mathbf{G}, v, v')$ *minimal* if we have $\delta_{v,v'} = \delta(\mathbf{G}, v, v')$ for every path $\delta_{v,v'} \subseteq \delta(\mathbf{G}, v, v')$ connecting v and v'. Moreover, since $\mathbf{G}$ is a tree, $\delta(\mathbf{G}, v, v')$ is the unique minimal path connecting v and v'. On the other hand, we put

$$\mathrm{leng}(\delta(\mathbf{G}, v, v')) \overset{\mathrm{def}}{=} \#\{\delta(\mathbf{G}, v, v') \cap v(\mathbf{G})\} - 1$$

the *length* of the minimal path $\delta(\mathbf{G}, v, v')$.

5.2.4. Now, we are going to prove the key lemma of the present section.

Lemma 5.3. *Let* $\Gamma \overset{\text{def}}{=} \Gamma_{D_X}$ *be a minimal quasi-tree associated to* D_X, $X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma})$ *the pointed stable curve of type* $(g_{X_\Gamma}, n_{X_\Gamma})$ *associated to* Γ *(4.4.5), and* $\Pi_{X_\Gamma^\bullet}$ *the admissible fundamental group of* $X_\Gamma^\bullet$. *Suppose that* $n_X \geq 2$. *Then there exist a positive natural number* $n \overset{\text{def}}{=} p^t - 1 \in \mathbb{N}$, *an effective divisor* $D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_\Gamma}]^0$ *on* X_Γ, *and an element* $\alpha_\Gamma \in \text{Rev}_{D_\Gamma}^{\text{adm}}(X_\Gamma^\bullet) \setminus \{0\}$ *such that the following holds:*

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + n_X - 2.$$

Proof. Since Γ is a minimal quasi-tree associated to D_X, we obtain that $\Gamma' \overset{\text{def}}{=} \Gamma \setminus e^{\text{lp}}(\Gamma)$ is a tree. Then we have $v(\Gamma) = v(\Gamma')$. Note that $D_X \subseteq D_{X_\Gamma}$. Let $v \in v(\Gamma)$ be an arbitrary vertex and $n_0 = p^{t_0} - 1 \in \mathbb{N}$ a positive natural number satisfying (see 2.4.5 for $C(g_X)$)

$$n_0 > \max\{C(g_X) + 1, \#(e(\Gamma_{X^\bullet}))\}.$$

Outline of the proof: Before we start to prove the lemma, we give a brief outline of the proof. The proof is divided into three steps. In Step 1, we construct certain effective divisors associated to marked points and nodes on irreducible components of X. In Step 2, by using the effective divisors constructed in Step 1, we construct certain effective divisors on irreducible components of X and an effective divisor on X. In Step 3, we prove the existence of the Raynaud-Tamagawa theta divisors associated to the effective divisors on irreducible components constructed in Step 2. Moreover, we prove that the effective divisors satisfy "gluing conditions" (as admissible coverings). Then we obtain a desired Galois multi-admissible covering of $X^\bullet$.

Step 1 (mp): Let $v \in v(\Gamma)$. We construct a family of effective divisors $\text{Div}_v^{\text{irr-mp}}$ on the irreducible component $X_v^\bullet$ associated to the set of marked points $D_X \cap X_v$, and construct a family of effective divisors Div_v^{mp} on X_Γ, where "mp" means "marked point".

We put

$$D_v' \overset{\text{def}}{=} D_X \cap X_v, \ m_v \overset{\text{def}}{=} \#(D_v'), \ \text{and} \ D_v' \overset{\text{def}}{=} \{x_{v,1}, \dots, x_{v,m_v}\} \ \text{if} \ m_v \neq 0.$$

Note that $D_v' \neq D_{X_\Gamma} \cap X_v$ in general. Moreover, we put

$$D_v \overset{\text{def}}{=} D_v' \cup (X_v \cap (\overline{X_\Gamma \setminus X_v})),$$

where $\overline{X_\Gamma \setminus X_v}$ denotes the topological closure of $X_\Gamma \setminus X_v$ in X_Γ. Note that since $n_X > 0$, we have $\#(D_v) > 0$. Let $w \in v(\Gamma) \setminus \{v\}$ be an arbitrary vertex distinct from v. Since Γ' is a tree, there exists a unique node

$$x_{v,w}^- \in D_w \subseteq X_w$$

such that the closed edge of Γ' corresponding to $x_{v,w}^-$ is contained in the minimal path $\delta(\Gamma', v, w)$ (i.e., the minimal path connecting v and w in Γ' defined in 5.2.3). On the other hand, we define a set of nodes to be

$$\text{Node}_{v,w}^+ \overset{\text{def}}{=} \{X_w \cap X_{w'}, \ w' \in v(\Gamma) \mid \text{leng}(\delta(\Gamma', v, w')) = \text{leng}(\delta(\Gamma', v, w)) + 1\}.$$

Note that $\mathrm{Node}_{v,w}^+$ may possibly be an empty set, and that $D_w = \{x_{v,w}^-\} \cup \mathrm{Node}_{v,w}^+ \cup D_w'$.
    We define two sets of effective divisors

$$\mathrm{Div}_v^{\mathrm{irr\text{-}mp}}, \ \mathrm{Div}_v^{\mathrm{mp}}$$

associated to v on X_v and X_Γ, as follows. Let $i \in \{1, \ldots, m_v - 1\}$ and $0 < a_{v,i,1}, a_{v,i,2} < n_0$ such that $a_{v,i,1} + a_{v,i,2} = n_0$. Suppose that $m_v \leq 1$. Then we put

$$\mathrm{Div}_v^{\mathrm{irr\text{-}mp}} \overset{\mathrm{def}}{=} \emptyset, \ \mathrm{Div}_v^{\mathrm{mp}} \overset{\mathrm{def}}{=} \emptyset.$$

Suppose that $m_v \geq 2$. We define an effective divisor

$$Q_{v,v,i} \overset{\mathrm{def}}{=} a_{v,i,1} x_{v,i} + a_{v,i,2} x_{v,i+1} + \sum_{x' \in D_v' \setminus \{x_{v,i}, x_{v,i+1}\}} n_0 x' + \sum_{x \in D_v \setminus D_v'} n_0 x, \ i \in \{1, \ldots, m_v - 1\},$$

on X_v whose support is D_v, and whose degree is equal to $(\#(D_v) - 1)n_0$. We define an effective divisor

$$Q_{v,w,i} \overset{\mathrm{def}}{=} \sum_{x \in D_w \setminus \{x_{v,w}^-\}} n_0 x, \ w \in v(\Gamma) \setminus \{v\},$$

on X_w whose support is $D_w \setminus \{x_{v,w}^-\}$, and whose degree is equal to $(\#(D_w) - 1)n_0$.
    Moreover, we define

$$Q_i^v \overset{\mathrm{def}}{=} a_{v,i,1} x_{v,i} + a_{v,i,2} x_{v,i+1} + \sum_{x \in D_X \setminus \{x_{v,i}, x_{v,i+1}\}} n_0 x,$$

to be an effective divisor on X_Γ whose support is D_X, and whose degree is $(n_X - 1)n_0$. Note that $Q_i^v|_{D_u'} = Q_{v,u,i}|_{D_u'}$ for all $u \in v(\Gamma)$, and that $Q_{v,w,i} = Q_{v,w,i'}$ for all $i, i' \in \{1, \ldots, m_v - 1\}$.
    We put

$$\mathrm{Div}_{v,i}^{\mathrm{irr\text{-}mp}} \overset{\mathrm{def}}{=} \bigsqcup_{u \in v(\Gamma)} \{Q_{v,u,i}\}, \ \mathrm{Div}_v^{\mathrm{irr\text{-}mp}} \overset{\mathrm{def}}{=} \bigsqcup_{i=1}^{m_v - 1} \mathrm{Div}_{v,i}^{\mathrm{irr\text{-}mp}}, \ \mathrm{Div}_v^{\mathrm{mp}} \overset{\mathrm{def}}{=} \bigsqcup_{i=1}^{m_v - 1} \{Q_i^v\},$$

where $\sqcup$ means disjoint union.

Step 1 (nd): Let $v \in v(\Gamma)$. We construct a family of effective divisors $\mathrm{Div}_v^{\mathrm{irr\text{-}nd}}$ on the irreducible component $X_v^\bullet$ associated to the set of nodes $D_v \setminus D_v' = X_v \cap (\overline{X_\Gamma \setminus X_v})$, and construct a family of effective divisors $\mathrm{Div}_v^{\mathrm{nd}}$ on X_Γ, where "nd" means "node".

    We define two families of effective divisors

$$\mathrm{Div}_v^{\mathrm{irr\text{-}nd}}, \ \mathrm{Div}_v^{\mathrm{nd}}$$

associated to v on X_v and X_Γ, respectively, as follows. Let $z \in D_X \setminus D_v'$ and $0 < b_{v,z,1}, b_{v,z,2} < n_0$ such that $b_{v,z,1} + b_{v,z,2} = n_0$. Suppose that $m_v = 0$. Then we put

$$\mathrm{Div}_v^{\mathrm{irr\text{-}nd}} \overset{\mathrm{def}}{=} \emptyset, \ \mathrm{Div}_v^{\mathrm{nd}} \overset{\mathrm{def}}{=} \emptyset.$$

Suppose that $m_v \neq 0$. Let w_z be the vertex such that the irreducible component X_{w_z} corresponding to w_z contains z (i.e., $z \in D_{w_z}' \overset{\mathrm{def}}{=} D_X \cap X_{w_z}$). Note that $w_z \neq v$. Moreover, let $\delta(\Gamma', v, w_z)$ be the minimal path connecting v and w_z in Γ' and $w \in v(\Gamma)$ an arbitrary

vertex distinct from w_z such that $w \subseteq \delta(\Gamma', v, w_z)$. Since Γ' is a tree, we have $\#(\mathrm{Node}^+_{v,w} \cap \delta(\Gamma', v, w_z)) = 1$. Then we put

$$x^+_{v,w,w_z} \overset{\mathrm{def}}{=} \mathrm{Node}^+_{v,w} \cap \delta(\Gamma', v, w_z) \in D_w \subseteq X_w.$$

For v and w_z, we define

$$Q_{v,v,z} \overset{\mathrm{def}}{=} b_{v,z,1} x_{v,m_v} + b_{v,z,2} x^+_{v,v,w_z} + \sum_{x \in D_v \setminus \{x_{v,m_v}, x^+_{v,v,w_z}\}} n_0 x,$$

$$Q_{v,w_z,z} \overset{\mathrm{def}}{=} b_{v,z,1} x^-_{v,w_z} + b_{v,z,2} z + \sum_{x \in D_{w_z} \setminus \{x_{v,w_z}, z\}} n_0 x$$

to be effective divisors on X_v and X_{w_z} whose supports are D_v and D_{w_z}, and whose degrees are equal to $(\#(D_v) - 1)n_0$ and $(\#(D_{w_z}) - 1)n_0$, respectively.

Let $w \in v(\Gamma) \setminus \{v, w_z\}$ be an arbitrary vertex such that $w \subseteq \delta(\Gamma', v, w_z)$. Then we define

$$Q_{v,w,z} \overset{\mathrm{def}}{=} b_{v,z,1} x^-_{v,w} + b_{v,z,2} x^+_{v,w,w_z} + \sum_{x \in D_w \setminus \{x^-_{v,w}, x^+_{v,w,w_z}\}} n_0 x$$

to be an effective divisor on X_w whose support is D_w, and whose degree is equal to $(\#(D_w) - 1)n_0$.

Let $w' \in v(\Gamma)$ be an arbitrary vertex such that $w' \not\subseteq \delta(\Gamma', v, w_z)$. Then we define

$$Q_{v,w',z} \overset{\mathrm{def}}{=} \sum_{x \in D_{w'} \setminus \{x^-_{v,w'}\}} n_0 x$$

to be an effective divisor on $X_{w'}$ whose support is $D_{w'} \setminus \{x^-_{v,w'}\}$, and whose degree is equal to $(\#(D_{w'}) - 1)n_0$. Note that, if $w'' \not\subseteq \delta(\Gamma', v, w_z) \cup \delta(\Gamma', v, w_{z'})$ for $z, z' \in D_X \setminus D'_v$, we have $Q_{v,w'',z} = Q_{v,w'',z'}$.

Moreover, we define

$$Q^v_z \overset{\mathrm{def}}{=} b_{v,z,1} x_{v,m_v} + b_{v,z,2} z + \sum_{x \in D_X \setminus \{x_{v,m_v}, z\}} n_0 x$$

to be an effective divisor on X_Γ whose support is D_X, and whose degree is equal to $(n_X - 1)n_0$. Note that $Q^v_z|_{D'_u} = Q_{v,u,z}|_{D'_u}$ for all $u \in v(\Gamma)$.

We put

$$\mathrm{Div}^{\mathrm{irr\text{-}nd}}_{v,z} \overset{\mathrm{def}}{=} \bigsqcup_{u \in v(\Gamma)} \{Q_{v,u,z}\}, \ \mathrm{Div}^{\mathrm{irr\text{-}nd}}_v \overset{\mathrm{def}}{=} \bigsqcup_{z \in D_X \setminus D'_v} \mathrm{Div}^{\mathrm{irr\text{-}nd}}_{v,z}, \ \mathrm{Div}^{\mathrm{nd}}_v \overset{\mathrm{def}}{=} \bigsqcup_{z \in D_X \setminus D'_v} \{Q^v_z\}.$$

Step 2: We construct an effective divisor $P_u \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_u]^0$, $v \in v(\Gamma)$, on X_u and an effective divisor $P_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_{X_\Gamma}]^0$ on X_Γ for some n.

We put

$$\mathrm{RT}^{\mathrm{irr}}_X \overset{\mathrm{def}}{=} \bigsqcup_{v \in v(\Gamma)} (\mathrm{Div}^{\mathrm{irr\text{-}mp}}_v \sqcup \mathrm{Div}^{\mathrm{irr\text{-}nd}}_v),$$

$$\mathrm{RT}_X \overset{\mathrm{def}}{=} \bigsqcup_{v \in v(\Gamma)} (\mathrm{Div}^{\mathrm{mp}}_v \sqcup \mathrm{Div}^{\mathrm{nd}}_v).$$

We denote by $\mathrm{RT}_X^{\mathrm{irr}}(X_u) \subseteq \mathrm{RT}_X^{\mathrm{irr}}$, $u \in v(\Gamma)$, the subset whose elements are effective divisors on X_u. Note that the above constructions imply $d \overset{\mathrm{def}}{=} \#(\mathrm{RT}_X^{\mathrm{irr}}(X_{u_1})) = \#(\mathrm{RT}_X^{\mathrm{irr}}(X_{u_2})) = \#(\mathrm{RT}_X)$ for all $u_1, u_2 \in v(\Gamma)$. Moreover, let

$$o_u : \{1, \ldots, d\} \overset{\sim}{\to} \mathrm{RT}_X^{\mathrm{irr}}(X_u) \overset{\mathrm{def}}{=} \{P_{u,1} \overset{\mathrm{def}}{=} o_u(1), \ldots, P_{u,d} \overset{\mathrm{def}}{=} o_u(d)\}, \ u \in v(\Gamma),$$

be a bijection (as sets) such that, for all $u_1, u_2 \in v(\Gamma)$ and all $j \in \{1, \ldots, d\}$, the following conditions are satisfied:

- if $P_{u_1,j} \in \mathrm{Div}_{v,i}^{\mathrm{irr\text{-}mp}}$ for some $v \in v(\Gamma)$ and some $i \in \{1, \ldots, m_v - 1\}$, then $P_{u_2,j} \in \mathrm{Div}_{v,i}^{\mathrm{irr\text{-}mp}}$.
- If $P_{u_1,j} \in \mathrm{Div}_{v,z}^{\mathrm{irr\text{-}nd}}$ for some $v \in v(\Gamma)$ and some $z \in D_X \setminus D_v'$, then $P_{u_2,j} \in \mathrm{Div}_{v,z}^{\mathrm{irr\text{-}nd}}$.

Then, by the construction of RT_X, we obtain a bijection

$$o : \{1, \ldots, d\} \overset{\sim}{\to} \mathrm{RT}_X \overset{\mathrm{def}}{=} \{P_1 \overset{\mathrm{def}}{=} o(1), \ldots, P_d \overset{\mathrm{def}}{=} o(d)\}$$

induced by o_u, $u \in v(\Gamma)$.

Let $t \overset{\mathrm{def}}{=} dt_0$ and $n \overset{\mathrm{def}}{=} \sum_{j=1}^{d} p^{(j-1)t_0}(p^{t_0} - 1) = p^t - 1$. We define

$$P_u \overset{\mathrm{def}}{=} \sum_{j=1}^{d} p^{(j-1)t_0} P_{u,j} \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_u]^0, \ u \in v(\Gamma),$$

$$P_\Gamma \overset{\mathrm{def}}{=} \sum_{j=1}^{d} p^{(j-1)t_0} P_j \in (\mathbb{Z}/n\mathbb{Z})^\sim [D_X]^0$$

to be effective divisors of degrees $\deg(P_u) = (\#(D_u) - 1)n$ and $\deg(P_\Gamma) = (n_X - 1)n$ on X_u and X_Γ, respectively. We see that the support of P_u, $u \in v(\Gamma)$, is D_u, and that the support of P_Γ is D_X.

Let $u \in v(\Gamma)$ and $x' \in D_u'$. Then the above constructions imply $P_\Gamma|_{x'} = P_u|_{x'}$. Moreover, let $x \in D_u \setminus D_u'$. Since $\Gamma \setminus \{e^{\mathrm{lp}}(\Gamma)\}$ is a tree, $X_\Gamma \setminus \{x\}$ has two connected components C_1, C_2. Let C_2 be the connected component such that $X_u \setminus \{x\}$ is not contained in C_2. We denote by $C_x \overset{\mathrm{def}}{=} \overline{\{C_2\}}$ the topological closure of C_2 in X_Γ and $M_x \overset{\mathrm{def}}{=} C_x \cap D_X$. Then the above constructions imply

$$P_u|_x = \mathrm{ord}_x(P_u)x = [\sum_{x' \in M_x} \mathrm{ord}_{x'}(P_\Gamma)]x,$$

where $[(-)]$ denotes the image of $(-)$ in $\mathbb{Z}/n\mathbb{Z}$.

Step 3: We construct a desired Galois multi-admissible covering of $X^\bullet$.

Let $u \in v(\Gamma)$ and $\widetilde{X}_u^\bullet$ the smooth pointed stable curve of type (g_u, n_u) over k (2.1.3). Write $\mathrm{norm}_u : \widetilde{X}_u \to X_u$ for the normalization morphism. Then we obtain the pulling back divisor $\mathrm{norm}_u^*(P_u)$ on $\widetilde{X}_u$. Note that since $\mathrm{Supp}(P_u)$ is contained in the smooth locus of X_u, we have $\mathrm{norm}_u^*(P_u) = P_u$. Let $\mathcal{L}_{P_u}$ be a line bundle on $\widetilde{X}_u$ such that $\mathcal{L}_{P_u}^{\otimes n} \cong \mathcal{O}_{\widetilde{X}_u}(-P_u)$, and $\mathcal{L}_{P_u,t}$ the pulling back of $\mathcal{L}_{P_u}$ by the natural morphism $\widetilde{X}_{u,t} \to \widetilde{X}$ defined in 2.4.2. Then by applying [T2, Corollary 2.6, Lemma 2.12 (ii), and Corollary 2.13], we see that the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_{P_u}^t \otimes \mathcal{L}_{P_u,t}$ exists (Definition 2.9).

Moreover, Proposition 2.10 implies that there exists $\widetilde{\alpha}_u \in \mathrm{Rev}^{\mathrm{adm}}_{P_u}(\widetilde{X}^{\bullet}_u)$ such that

$$\gamma_{(\widetilde{\alpha}_u, P_u)} = g_u + \#(D_u) - 2.$$

Then Theorem 3.9 implies that the element $\alpha_u \in \mathrm{Rev}^{\mathrm{adm}}_{P_u}(X^{\bullet}_u)$ induced by $\widetilde{\alpha}_u$ has the following property:

$$\gamma_{(\alpha_u, P_u)} = g_{X_u} + \#(D_u) - 2,$$

where g_{X_u} denotes the genus of X_u. Write

$$f^{\bullet}_u : Y^{\bullet}_u \to X^{\bullet}_u$$

for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_u.

Let $u' \in v(\Gamma) \setminus \{u\}$ such that $X_u \cap X_{u'} \neq \emptyset$. We denote by $x_{u,u'} \stackrel{\mathrm{def}}{=} X_u \cap X_{u'}$ the unique node. Then the above constructions imply

$$0 < \mathrm{ord}_{x_{u,u'}}(P_u), \mathrm{ord}_{x_{u,u'}}(P_{u'}) < n, \ \mathrm{ord}_{x_{u,u'}}(P_u) + \mathrm{ord}_{x_{u,u'}}(P_{u'}) = n.$$

This means that we may glue (as admissible coverings) $\{Y^{\bullet}_u\}_{u \in v(\Gamma)}$ along $\{f^{-1}_u(D_u \setminus D'_u)\}_{u \in v(\Gamma)}$ in a way that is compatible with the gluing of $\{X^{\bullet}_u\}_{u \in v(\Gamma)}$ that gives rise to $X^{\bullet}_{\Gamma}$. Then we obtain a Galois multi-admissible covering

$$f^{\bullet}_{\Gamma} : Y^{\bullet}_{\Gamma} \to X^{\bullet}_{\Gamma}$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Note that the construction of $f^{\bullet}_{\Gamma}$ implies that f_{Γ} is étale over $D_{X_{\Gamma}} \setminus D_X$.

Let $\Pi_{X^{\bullet}_u}$, $u \in v(\Gamma)$, be the admissible fundamental group of $X^{\bullet}_u$. We denote by $\alpha_{\Gamma} \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^{\bullet}_{\Gamma}}, \mathbb{Z}/n\mathbb{Z})$ an element induced by $f^{\bullet}_{\Gamma}$ such that the composition of the natural homomorphisms $\Pi^{\mathrm{ab}}_{X^{\bullet}_u} \to \Pi^{\mathrm{ab}}_{X^{\bullet}_{\Gamma}} \stackrel{\alpha_{\Gamma}}{\to} \mathbb{Z}/n\mathbb{Z}$ is equal to α_u for all $u \in v(\Gamma)$. We put $D_{\Gamma} \stackrel{\mathrm{def}}{=} P_{\Gamma}$. Then we see $\alpha_{\Gamma} \in \mathrm{Rev}^{\mathrm{adm}}_{D_{\Gamma}}(X^{\bullet}_{\Gamma}) \setminus \{0\}$. Theorem 3.9 implies

$$\gamma_{(\alpha_{\Gamma}, D_{\Gamma})} = g_{X_{\Gamma}} + s(P_{\Gamma}) - 1 = g_{X_{\Gamma}} + n_X - 2.$$

We complete the proof of the lemma. $\square$

5.3. The second main theorem.
Now, we prove the second main result of the present paper.

Theorem 5.4. *Let $X^{\bullet}$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$. Then there exist a positive natural number $n \stackrel{\mathrm{def}}{=} p^t - 1 \in \mathbb{N}$, an effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5) on X of degree $(n_X - 1)n$ if $n_X \neq 0$ (resp. degree 0 if $n_X = 0$), and an element $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^{\bullet}) \setminus \{0\}$ (Definition 2.4 (i)) such that $\gamma_{(\alpha, D)}$ attains maximum (Definition 3.10 (ii)). Namely, the following holds (see Definition 3.10 (i) for $\gamma^{\mathrm{max}}_{X^{\bullet}}$):*

$$\gamma_{(\alpha, D)} = \gamma^{\mathrm{max}}_{X^{\bullet}} = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

Proof. Suppose that $n_X \leq 1$. Then the theorem follows from Proposition 3.11 (ii). To verify the theorem, we may assume $n_X \geq 2$.

Let $\Gamma \stackrel{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X, Γ^{im} the image of the natural morphism $\phi_{\Gamma} : \Gamma \to \Gamma_{X^{\bullet}}$, $X^{\bullet}_{\Gamma} = (X_{\Gamma}, D_{X_{\Gamma}})$, $X^{\bullet}_{\Gamma^{\mathrm{im}}} = (X_{\Gamma^{\mathrm{im}}}, D_{X_{\Gamma^{\mathrm{im}}}})$ the pointed stable curves over k of types $(g_{X_{\Gamma}}, n_{X_{\Gamma}})$, $(g_{X_{\Gamma^{\mathrm{im}}}}, n_{X_{\Gamma^{\mathrm{im}}}})$ associated to Γ, Γ^{im}, respectively (see

4.4.5, 4.4.7 for the definitions of Γ^{im}, ϕ_Γ, $X_\Gamma^\bullet$, and $X_{\Gamma^{\mathrm{im}}}^\bullet$), and $\Pi_{X_\Gamma^\bullet}$, $\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}$ the admissible fundamental groups of $X_\Gamma^\bullet$, $X_{\Gamma^{\mathrm{im}}}^\bullet$, respectively.

Lemma 5.3 implies that there exist a natural number

$$n \stackrel{\text{def}}{=} p^t - 1 > \max\{C(g_X) + 1, \#(e(\Gamma_{X^\bullet}))\},$$

an effective divisor $D \stackrel{\text{def}}{=} D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_\Gamma}]^0$ on X_Γ of degree $(n_X - 1)n$, and an element $\alpha_\Gamma \in \mathrm{Rev}_D^{\mathrm{adm}}(X_\Gamma^\bullet) \setminus \{0\}$ such that the following holds:

$$\gamma_{(\alpha_\Gamma, D)} = g_{X_\Gamma} + n_X - 2.$$

We denote by $f_\Gamma^\bullet : Z_\Gamma^\bullet \to X_\Gamma^\bullet$ the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_Γ. Note that f_Γ is étale over $D_{X_\Gamma} \setminus D_X$. By gluing $Z_\Gamma^\bullet$ along

$$f_\Gamma^{-1}(D_{X_\Gamma} \setminus (D_X \cup \{x_e\}_{e \in \phi_\Gamma^{-1}(e^{\mathrm{op}}(\Gamma^{\mathrm{im}}))}))$$

in a way that is compatible with the gluing of $X_\Gamma^\bullet$ that gives rise to $X_{\Gamma^{\mathrm{im}}}^\bullet$, we obtain a pointed stable curve $Z_{\Gamma^{\mathrm{im}}}^\bullet$ over k. Moreover, $f_\Gamma^\bullet$ induces a Galois multi-admissible covering

$$f_{\Gamma^{\mathrm{im}}}^\bullet : Z_{\Gamma^{\mathrm{im}}}^\bullet \to X_{\Gamma^{\mathrm{im}}}^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_{\Gamma^{\mathrm{im}}}$ for an element of $\mathrm{Hom}(\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ induced by $f_{\Gamma^{\mathrm{im}}}^\bullet$ such that the composition of the natural homomorphisms $\Pi_{X_\Gamma^\bullet}^{\mathrm{ab}} \to \Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}^{\mathrm{ab}} \stackrel{\alpha_{\Gamma^{\mathrm{im}}}}{\to} \mathbb{Z}/n\mathbb{Z}$ is equal to α_Γ. Note that we have $D_{\alpha_{\Gamma^{\mathrm{im}}}} = D$. Then Theorem 3.9 implies

$$\gamma_{(\alpha_{\Gamma^{\mathrm{im}}}, D)} = g_{X_{\Gamma^{\mathrm{im}}}} + n_X - 2.$$

On the other hand, we write $\pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$ for the set of connected components of the topological closure $\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}$ of $X \setminus X_{\Gamma^{\mathrm{im}}}$ in X. We define the following pointed stable curve

$$E^\bullet = (E, D_E \stackrel{\text{def}}{=} E \cap X_{\Gamma^{\mathrm{im}}}), \ E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}}),$$

over k. Proposition 3.11 (ii) implies that there exists a Galois *étale* covering

$$f_E^\bullet : Z_E^\bullet = (Z_E, D_{Z_E}) \to E^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that the following holds:

$$\gamma_{(\alpha_E, 0)} = \begin{cases} g_E, & \text{if } g_E = 0, \\ g_X - 1, & \text{if } g_E \neq 0, \end{cases}$$

where g_E denotes the genus of E, and $\alpha_E \in \mathrm{Rev}_0^{\mathrm{adm}}(E^\bullet)$ is an element induced by $f_E^\bullet$.

We may glue $Z_{\Gamma^{\mathrm{im}}}^\bullet$ and $\{Z_E^\bullet\}_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$ along

$$f_\Gamma^{-1}(X_{\Gamma^{\mathrm{im}}} \cap (\bigcup_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})} E)) \text{ and } \{f_E^{-1}(X_{\Gamma^{\mathrm{im}}} \cap E)\}_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$$

in a way that is compatible with the gluing of $\{X_{\Gamma^{\mathrm{im}}}^\bullet\} \cup \{E^\bullet\}_{E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})}$ that gives rise to $X^\bullet$, then we obtain a Galois multi-admissible covering

$$f^\bullet : Z^\bullet \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

Let $\Pi_{X^\bullet}$, $\Pi_{E^\bullet}$ be the admissible fundamental groups of $X^\bullet$, $E^\bullet$, $E \in \pi_0(\overline{X \setminus X_{\Gamma^{\mathrm{im}}}})$, respectively. We write $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ for an element induced by $f^\bullet$ such that the compositions of the natural homomorphisms $\Pi_{X_{\Gamma^{\mathrm{im}}}^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}} \stackrel{\alpha}{\to} \mathbb{Z}/n\mathbb{Z}$, $\Pi_{E^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}} \stackrel{\alpha}{\to}$

$\mathbb{Z}/n\mathbb{Z}$, $E \in \pi_0(\overline{X \setminus X_{\Gamma^{im}}})$, are equal to $\alpha_{\Gamma^{im}}$ and α_E, respectively. We see $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$. By applying Theorem 3.9, we obtain

$$\gamma_{(\alpha,D)} = g_X + n_X - 2.$$

This completes the proof of the theorem. □

5.4. **A stronger version when $n_X = 3$.** In the remainder of the present section, we prove a slightly stronger version of Theorem 5.4 when $n_X = 3$ (i.e., we prove Theorem 5.4 for a certain fixed effective divisor $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$). The stronger version will be used in the proof of reconstructions of field structures associated to inertia subgroups (cf. Section 6.2). The main result of this subsection is Theorem 5.7.

5.4.1. **Settings.** We maintain the notation introduced in 2.1.2. Moreover, *we suppose that $n_X = 3$.*

5.4.2. We generalize Lemma 5.3 to the case of certain fixed ramification divisors when $n_X = 3$. Let $D_X \overset{\mathrm{def}}{=} \{x_1, x_2, x_3\}$, and let $\Gamma \overset{\mathrm{def}}{=} \Gamma_{D_X}$ be a minimal quasi-tree associated to D_X,

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma})$$

the pointed stable curve of type $(g_{X_\Gamma}, n_{X_\Gamma})$ associated to Γ, and $\Pi_{X_\Gamma^\bullet}$ the admissible fundamental group of $X_\Gamma^\bullet$. Let $D_j \in \mathbb{Z}[D_X] \subseteq \mathbb{Z}[D_{X_\Gamma}]$, $j \in \{1, 2, 3\}$, be an effective divisor on X_Γ such that the following conditions are satisfied:

- $\deg(D_j) = 2(p^{t_j} - 1)$.
- $\mathrm{ord}_x(D_j) \leq p^{t_j} - 1$ for each $x \in D_X$.
- $\#\{x \in D_X \mid \mathrm{ord}_x(D_j) = p^{t_j} - 1\} \geq 1$.

Let $n \overset{\mathrm{def}}{=} p^t - 1 \overset{\mathrm{def}}{=} p^{t_1+t_2+t_3} - 1$, and let

$$D_\Gamma \overset{\mathrm{def}}{=} D_1 + p^{t_1}D_2 + p^{t_1+t_2}D_3 \in \mathbb{Z}[D_X] \subseteq \mathbb{Z}[D_{X_\Gamma}]$$

be an effective divisor on X_Γ with degree $2n$. Then we have the following lemma.

Lemma 5.5. *We maintain the notation introduced above. Suppose that $D_\Gamma \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_{X_\Gamma}]^0$ (i.e., $\mathrm{ord}_x(D_\Gamma) < n$ for all $x \in D_X$), and that (see 2.4.5 for $C(g_X)$)*

$$n > \max\{C(g_X) + 1, \#(e(\Gamma_{X^\bullet}))\}.$$

Then there exists an element $\alpha_\Gamma \in \mathrm{Rev}_{D_\Gamma}^{\mathrm{adm}}(X_\Gamma^\bullet) \setminus \{0\}$ such that the following holds:

$$\gamma_{(\alpha_\Gamma, D_\Gamma)} = g_{X_\Gamma} + 1.$$

Proof. Since Γ is a minimal quasi-tree associated to D_X, we obtain that $\Gamma' \overset{\mathrm{def}}{=} \Gamma \setminus e^{\mathrm{lp}}(\Gamma)$ is a tree. If $D_X \subseteq X_v$ for some $v \in v(\Gamma)$, then the lemma follows from Proposition 5.1. Without loss of generality, we may assume that one of the following conditions holds:

    (i) Let $w_1, w_2 \in v(\Gamma)$ be vertices distinct from each other such that x_1, $x_2 \in X_{w_1}$, and $x_3 \in X_{w_2}$ (see Example 5.6 (i) below).
    (ii) Let $v_1, v_2, v_3 \in v(\Gamma)$ be vertices distinct from each other such that $x_1 \in X_{v_1}$, $x_2 \in X_{v_2}$, and $x_3 \in X_{v_3}$ (see Example 5.6 (ii) below).

We put

$$D'_v \overset{\text{def}}{=} D_X \cap X_v, \ D_v \overset{\text{def}}{=} D'_v \cup (X_v \cap (\overline{X_\Gamma \setminus X_v})), \ v \in v(\Gamma)$$

where $\overline{X_\Gamma \setminus X_v}$ denotes the topological closure of $X_\Gamma \setminus X_v$ in X_Γ. Next, we construct an effective divisor $P_v \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_v]^0$ on X_v for all $v \in v(\Gamma)$.

Suppose that (i) holds. Let $y_{w_1,w_2} \in D_{w_1}$ be the unique node of X_Γ such that the closed edge of Γ' corresponding to y_{w_1,w_2} is contained in the minimal path $\delta(\Gamma', w_1, w_2)$ (5.2.3) and $z_{w_1,w_2} \in D_{w_2}$ the unique node of X_Γ such that the closed edge of Γ' corresponding to z_{w_1,w_2} is contained in the minimal path $\delta(\Gamma', w_1, w_2)$. On the other hand, let $w \in v(\Gamma) \setminus \{w_1, w_2\}$ be an arbitrary vertex (possibly empty). Note that $n_X = 3$ implies that w is contained in $\delta(\Gamma', w_1, w_2)$. Since Γ' is a tree, there exist a unique node $x_{w_1,w} \in D_w$ and a unique node $x_{w_2,w} \in D_w$ such that the closed edges of Γ' corresponding to $x_{w_1,w}$ and $x_{w_2,w}$ are contained in $\delta(\Gamma', w_1, w)$ and $\delta(\Gamma', w_2, w)$, respectively.

Let $j \in \{1, 2, 3\}$. We put

$$Q_{w_1,j} \overset{\text{def}}{=} \text{ord}_{x_1}(D_j)x_1 + \text{ord}_{x_2}(D_j)x_2 + \text{ord}_{x_3}(D_j)y_{w_1,w_2},$$

$$Q_{w_2,j} \overset{\text{def}}{=} [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]z_{w_1,w_2} + \text{ord}_{x_3}(D_j)x_3,$$

$$Q_{w,j} \overset{\text{def}}{=} [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]x_{w_1,w} + \text{ord}_{x_3}(D_j)x_{w_2,w}, \ w \in v(\Gamma) \setminus \{w_1, w_2\},$$

where $[(-)]$ denotes the image of $(-)$ in $\mathbb{Z}/(p^{t_j} - 1)\mathbb{Z}$. Then $Q_{v,j}$, $v \in v(\Gamma)$, is an effective divisor on X_v whose degree is equal to $(\#(D_v) - 1)(p^{t_j} - 1)$. Moreover, we put

$$P_v \overset{\text{def}}{=} Q_{v,1} + p^{t_1} Q_{v,2} + p^{t_1+t_2} Q_{v,3} \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_v]^0, \ v \in v(\Gamma).$$

Then P_v is an effective divisor on X_v whose degree is equal to $(\#(D_v) - 1)n$, and whose support is equal to D_v.

Suppose that (ii) holds. Then one of the following conditions is satisfied:

> (1) There exist $a, b, c \in \{1, 2, 3\}$ distinct from each other such that $\delta(\Gamma', v_a, v_b) \cap \delta(\Gamma', v_b, v_c) \cap e^{\text{cl}}(\Gamma) = \emptyset$ (i.e., $\delta(\Gamma', v_a, v_c) = \delta(\Gamma', v_a, v_b) \cup \delta(\Gamma', v_b, v_c)$) (see Example 5.6 (ii)-(1) below).
> (2) For all $a, b, c \in \{1, 2, 3\}$ distinct from each other, we have that $\delta(\Gamma', v_a, v_b) \cap \delta(\Gamma', v_b, v_c) \cap e^{\text{cl}}(\Gamma) \neq \emptyset$ (see Example 5.6 (ii)-(2) below).

Suppose that (1) holds. Without loss of generality, we may assume that $a = 1$, $b = 2$, and $c = 3$. Note that $\delta(\Gamma', v_1, v_3) = \delta(\Gamma', v_1, v_2) \cup \delta(\Gamma', v_2, v_3)$. Write $y_{v_1,v_3} \in D_{v_1}$ for the unique node of X_Γ such that the closed edge of Γ' corresponding to y_{v_1,v_3} is contained in $\delta(\Gamma', v_1, v_3)$ and $z_{v_1,v_3} \in D_{v_3}$ for the unique node of X_Γ such that the closed edge of Γ' corresponding to z_{v_1,v_3} is contained in $\delta(\Gamma', v_1, v_3)$. On the other hand, let $v \in v(\Gamma) \setminus \{v_1, v_3\}$ be an arbitrary vertex. Since $n_X = 3$, we see that $v \in \delta(\Gamma', v_1, v_3)$, and that either $v \in \delta(\Gamma', v_1, v_2)$ or $v \in \delta(\Gamma', v_2, v_3)$ holds. Since Γ' is a tree, there exist a unique node $x_{v_1,v} \in D_v$ and a unique node $x_{v_3,v} \in D_v$ such that the closed edges of Γ' corresponding to $x_{v_1,v}$ and $x_{v_3,v}$ are contained in $\delta(\Gamma', v_1, v)$ and $\delta(\Gamma', v, v_3)$, respectively.

Let $j \in \{1, 2, 3\}$. We put

$$Q_{v_1,j} \overset{\text{def}}{=} \text{ord}_{x_1}(D_j)x_1 + [\deg(D_j) - \text{ord}_{x_2}(D_j) - \text{ord}_{x_3}(D_j)]y_{v_1,v_3},$$

$$Q_{v_2,j} \overset{\text{def}}{=} \text{ord}_{x_1}(D_j)x_{v_1,v_2} + \text{ord}_{x_2}(D_j)x_2 + \text{ord}_{x_3}(D_j)x_{v_3,v_2},$$

$$Q_{v_3,j} \overset{\text{def}}{=} [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]z_{v_1,v_3} + \text{ord}_{x_3}(D_j)x_3.$$

Morever, we put

$$Q_{v,j} \stackrel{\text{def}}{=} \text{ord}_{x_1}(D_j)x_{v_1,v} + [\deg(D_j) - \text{ord}_{x_2}(D_j) - \text{ord}_{x_3}(D_j)]x_{v_3,v}$$

for all $v \in (v(\Gamma) \cap \delta(\Gamma', v_1, v_2)) \setminus \{v_1, v_2\}$, and

$$Q_{v,j} \stackrel{\text{def}}{=} [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]x_{v_1,v} + \text{ord}_{x_3}(D_j)x_{v_3,v}$$

for all $v \in (v(\Gamma) \cap \delta(\Gamma', v_2, v_3)) \setminus \{v_2, v_3\}$. Then $Q_{v,j}$, $v \in v(\Gamma)$, is an effective divisor on X_v whose degree is equal to $(\#(D_v) - 1)(p^{t_j} - 1)$. Moreover, we put

$$P_v \stackrel{\text{def}}{=} Q_{v,1} + p^{t_1}Q_{v,2} + p^{t_1+t_2}Q_{v,3} \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_v]^0, \ v \in v(\Gamma).$$

Then P_v is an effective divisor on X_v whose degree is equal to $(\#(D_v) - 1)n$, and whose support is equal to D_v.

Suppose that (2) holds. Then we have

$$\{v_0\} = v(\Gamma) \cap \delta(\Gamma', v_1, v_2) \cap \delta(\Gamma', v_2, v_3) \cap \delta(\Gamma', v_3, v_1).$$

Let $v \in v(\Gamma)$. Since $n_X = 3$, we obtain that either $v \in \delta(\Gamma', v_1, v_0)$ or $v \in \delta(\Gamma', v_2, v_0)$ or $v \in \delta(\Gamma', v_3, v_0)$ holds. Let $y_{v_i,v_0} \in D_{v_i}$, $i \in \{1, 2, 3\}$, be the unique node of X_Γ such that the closed edge of Γ' corresponding to y_{v_i,v_0} is contained in $\delta(\Gamma', v_i, v_0)$ and $z_{v_i,v_0} \in D_{v_0}$, $i \in \{1, 2, 3\}$, be the unique node of X_Γ such that the closed edge of Γ' corresponding to z_{v_i,v_0} is contained in $\delta(\Gamma', v_i, v_0)$. Moreover, let $v \in (v(\Gamma) \cap \delta(\Gamma', v_i, v_0)) \setminus \{v_i, v_0\}$, $i \in \{1, 2, 3\}$. Since Γ' is a tree, there exist a unique node $x_{v_i,v} \in D_v$ and a unique node $x_{v_0,v} \in D_v$ such that the closed edges of Γ' corresponding to $x_{v_i,v}$ and $x_{v_0,v}$ are contained in $\delta(\Gamma', v_i, v)$ and $\delta(\Gamma', v_0, v)$, respectively.

Let $j \in \{1, 2, 3\}$. We put

$$\begin{aligned}
Q_{v_1,j} &\stackrel{\text{def}}{=} \text{ord}_{x_1}(D_j)x_1 + [\deg(D_j) - \text{ord}_{x_2}(D_j) - \text{ord}_{x_3}(D_j)]y_{v_1,v_0}, \\
Q_{v_2,j} &\stackrel{\text{def}}{=} \text{ord}_{x_2}(D_j)x_2 + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_3}(D_j)]y_{v_2,v_0}, \\
Q_{v_3,j} &\stackrel{\text{def}}{=} \text{ord}_{x_3}(D_j)x_3 + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]y_{v_3,v_0}, \\
Q_{v_0,j} &\stackrel{\text{def}}{=} \text{ord}_{x_1}(D_j)z_{v_1,v_0} + \text{ord}_{x_2}(D_j)z_{v_2,v_0} + \text{ord}_{x_3}(D_j)z_{v_3,v_0}.
\end{aligned}$$

Moreover, we put

$$Q_{v,j} \stackrel{\text{def}}{=} \text{ord}_{x_1}(D_j)x_{v_1,v} + [\deg(D_j) - \text{ord}_{x_2}(D_j) - \text{ord}_{x_3}(D_j)]x_{v_0,v}$$

for all $v \in (v(\Gamma) \cap \delta(\Gamma', v_1, v_0)) \setminus \{v_1, v_0\}$,

$$Q_{v,j} \stackrel{\text{def}}{=} \text{ord}_{x_2}(D_j)x_{v_2,v} + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_3}(D_j)]x_{v_0,v}$$

for all $v \in (v(\Gamma) \cap \delta(\Gamma', v_2, v_0)) \setminus \{v_2, v_0\}$, and

$$Q_{v,j} \stackrel{\text{def}}{=} \text{ord}_{x_3}(D_j)x_{v_3,v} + [\deg(D_j) - \text{ord}_{x_1}(D_j) - \text{ord}_{x_2}(D_j)]x_{v_0,v}$$

for all $v \in (v(\Gamma) \cap \delta(\Gamma', v_3, v_0)) \setminus \{v_3, v_0\}$. Then $Q_{v,j}$, $v \in v(\Gamma)$, is an effective divisor on X_v whose degree is equal to $(\#(D_v) - 1)(p^{t_j} - 1)$. Moreover, we put

$$P_v \stackrel{\text{def}}{=} Q_{v,1} + p^{t_1}Q_{v,2} + p^{t_1+t_2}Q_{v,3} \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_v]^0, \ v \in v(\Gamma).$$

Then P_v is an effective divisor on X_v whose degree is equal to $(\#(D_v) - 1)n$, and whose support is equal to D_v.

Let $v \in v(\Gamma)$, and let $\widetilde{X}_v^\bullet$ be the smooth pointed stable curve of type (g_v, n_v) over k associated to v (2.1.3). Since D_X is contained in the smooth locus of X_v, P_v can be also

regarded as an effective divisor on $\widetilde{X}_v$. By applying similar arguments to the arguments given in the proof of Proposition 5.1, there exists $\widetilde{\alpha}_v \in \mathrm{Rev}^{\mathrm{adm}}_{P_v}(\widetilde{X}^{\bullet}_v)$ such that

$$\gamma_{(\widetilde{\alpha}_v, P_v)} = g_v + \#(D_v) - 2.$$

Then Theorem 3.9 implies that the element $\alpha_v \in \mathrm{Rev}^{\mathrm{adm}}_{P_v}(X^{\bullet}_v)$ induced by $\widetilde{\alpha}_v$ such that the following holds:

$$\gamma_{(\alpha_v, P_v)} = g_{X_v} + \#(D_v) - 2,$$

where g_{X_v} denotes the genus of X_v. Write $f^{\bullet}_v : Y^{\bullet}_v \to X^{\bullet}_v$ for the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α_v.

By applying similar arguments to the arguments given in Step 3 of the proof of Lemma 5.3, we may glue $\{Y^{\bullet}_v\}_{v \in v(\Gamma)}$ along $\{f_v^{-1}(D_v \setminus D'_v)\}_{v \in v(\Gamma)}$ (as admissible coverings) in a way that is compatible with the gluing of $\{X^{\bullet}_v\}_{v \in v(\Gamma)}$ that gives rise to X_{Γ}. Then we obtain a Galois multi-admissible covering

$$f^{\bullet}_{\Gamma} : Y^{\bullet}_{\Gamma} \to X^{\bullet}_{\Gamma}$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Note that the construction of $f^{\bullet}_{\Gamma}$ implies that f_{Γ} is étale over $D_{X_{\Gamma}} \setminus D_X$. Moreover, there exists an element $\alpha_{\Gamma} \in \mathrm{Rev}^{\mathrm{adm}}_{D_{\Gamma}}(X^{\bullet}_{\Gamma}) \setminus \{0\}$ induced by $f^{\bullet}_{\Gamma}$ such that the following holds:

$$\gamma_{(\alpha_{\Gamma}, D_{\Gamma})} = g_{X_{\Gamma}} + 1.$$

We complete the proof of the lemma.   □

Example 5.6. We give some examples to explain the conditions considered in the proof of Lemma 5.5. We maintain the notation introduced in the proof of Lemma 5.5. Moreover, we suppose that $\Gamma_{X^{\bullet}} = \Gamma = \Gamma'$ is a tree.
   (i) Suppose that $v(\Gamma) = \{w_1, w_2\}$. Then we have
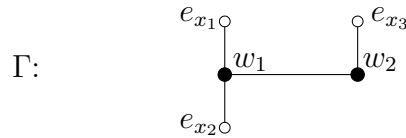
(ii)-(1) Suppose that $v(\Gamma) = \{v_1, v_2, v_3\}$. Then we have
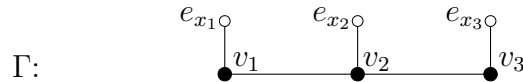
(ii)-(2) Suppose that $v(\Gamma) = \{v_0, v_1, v_2, v_3\}$. Then we have
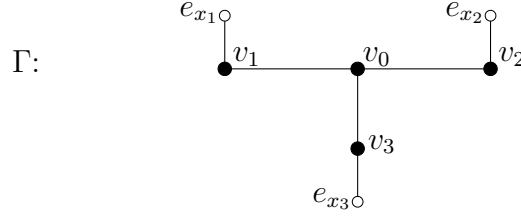
5.4.3. We generalize Theorem 5.4 to the case of certain fixed ramification divisors as follows:

Theorem 5.7. *Let $X^\bullet$ be an arbitrary pointed stable curve of type $(g_X, 3)$ over an algebraically closed field k of characteristic $p > 0$. Let $D_X \overset{\text{def}}{=} \{x_1, x_2, x_3\}$, and let $D_j \in \mathbb{Z}[D_X]$, $j \in \{1, 2, 3\}$, be an effective divisor on X such that the following conditions are satisfied:*
- $\deg(D_j) = 2(p^{t_j} - 1)$.
- $\mathrm{ord}_x(D_j) \le p^{t_j} - 1$ *for each* $x \in D_X$.
- $\#\{x \in D_X \mid \mathrm{ord}_x(D_j) = p^{t_j} - 1\} \ge 1$.

Let $n = p^t - 1 \overset{\text{def}}{=} p^{t_1 + t_2 + t_3} - 1$, and let $D \overset{\text{def}}{=} D_1 + p^{t_1} D_2 + p^{t_1 + t_2} D_3 \in \mathbb{Z}[D_X]$ be an effective divisor on X with degree $2n$. Moreover, suppose that $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ (2.2.5), and that

$$n > \max\{C(g_X) + 1, \#(e(\Gamma_{X^\bullet}))\}.$$

Then there exists an element $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet) \setminus \{0\}$ (Definition 2.4 (i)) such that the following holds:

$$\gamma_{(\alpha, D)} = g_X + 1.$$

Proof. By applying Lemma 5.5 and similar arguments to the arguments given in the proof of Theorem 5.4, we obtain the theorem. $\square$

6. Applications to anabelian geometry

In this section, we give some applications of results obtained in previous sections. The main results of the present section are Theorem 6.3 and Theorem 6.4.

6.1. **An anabelian formula for topological types.** In this subsection, by using Theorem 5.4, we prove a group-theoretical formula for the topological type of an arbitrary pointed stable curve over an algebraically closed field of characteristic $p > 0$.

6.1.1. **Settings.** We maintain the notation introduced in 2.1.2. Moreover, let $\Pi_{X^\bullet}$ be the admissible fundamental group of $X^\bullet$.

6.1.2. Let Δ be an arbitrary profinite group, and let $m \in \mathbb{N}$ be a positive natural number and ℓ a prime number. We denote by $D_\ell(\Delta) \subseteq \Delta$ the topological closure of $[\Delta, \Delta]\Delta^\ell$, where $[\Delta, \Delta]$ denotes the commutator subgroup of Δ, and Δ^ℓ denotes the maximal pro-ℓ quotient of Δ. Moreover, we define the closed normal subgroup $D_\ell^{(m)}(\Delta)$ of Δ inductively by $D_\ell^{(1)}(\Delta) \overset{\text{def}}{=} D_\ell(\Delta)$ and $D_\ell^{(i+1)}(\Delta) \overset{\text{def}}{=} D_\ell(D_\ell^{(i)}(\Delta))$, $i \in \{1, \ldots, m-1\}$. Note that $\#(\Delta/D_\ell^{(m)}(\Delta)) \le \infty$ when Δ is topologically finitely generated. On the other hand, we denote by $F_{r,m}^\ell$ the finite group $\widehat{F}_r/D_\ell^{(m)}(\widehat{F}_r)$, where $\widehat{F}_r$ denotes the free profinite group of rank r.

6.1.3. Let Π be *an abstract profinite group* which is isomorphic to $\Pi_{X^\bullet}$ as profinite groups. Moreover, we denote by $\pi_A(\Pi)$ the set of finite quotients of Π. We put

$$b_\Pi^1 \overset{\text{def}}{=} \max\{r \mid \text{there exists a prime number } \ell \text{ such that } (\mathbb{Z}/\ell\mathbb{Z})^{\oplus r} \in \pi_A(\Pi)\},$$

$$b_\Pi^2 \overset{\text{def}}{=} \begin{cases} 0, & F_{b_\Pi^1,2}^\ell \in \pi_A(\Pi) \text{ for some prime number } \ell, \\ 1, & \text{otherwise.} \end{cases}$$

Note that b_Π^i, $i \in \{1, 2\}$, is a group-theoretical invariant associated to Π (i.e., depends only on the isomorphism class of Π). Firstly, we have the following lemma.

Lemma 6.1. *(i) We have*

$$b_\Pi^1 = 2g_X + n_X - 1 + b_\Pi^2,$$

$$b_\Pi^2 = \begin{cases} 1, & \text{if } n_X = 0, \\ 0, & \text{if } n_X \neq 0. \end{cases}$$

(ii) There exists a unique prime number p_Π such that $(\mathbb{Z}/p_\Pi\mathbb{Z})^{\oplus b_\Pi^1} \notin \pi_A(\Pi)$. In particular, we have $p = p_\Pi$.

Proof. (i) We put $r_\Pi \overset{\text{def}}{=} \dim_{\mathbb{F}_\ell}(\Pi^{\text{ab}} \otimes \mathbb{F}_\ell)$, where ℓ is an arbitrary prime number $\mathfrak{P} \setminus \{p\}$, and $\mathfrak{P}$ denotes the set of prime numbers. Then the structures of maximal prime-to-p quotients of admissible fundamental groups (2.1.6) imply that (see 2.2.2 for σ_X)

$$\Pi^{\text{ab}} \cong \mathbb{Z}_p^{\oplus \sigma_X} \times \prod_{\ell \in \mathfrak{P} \setminus \{p\}} \mathbb{Z}_\ell^{\oplus r_\Pi}.$$

Since $X^\bullet$ is a pointed stable curve, we have that $\sigma_X < r_\Pi$. This implies that $b_\Pi^1 = r_\Pi$. Moreover, we have

$$b_\Pi^1 = \begin{cases} 2g_X, & \text{if } n_X = 0, \\ 2g_X + n_X - 1, & \text{if } n_X \neq 0. \end{cases}$$

Suppose that $n_X > 0$. Let $\ell_1 \in \mathfrak{P} \setminus \{p\}$. The specialization theorem of maximal pro-ℓ_1 quotients of admissible fundamental groups ([V, Théorème 2.2 (c)]) implies that the maximal pro-ℓ_1 quotient Π^{ℓ_1} of Π is a free pro-ℓ_1 profinite group of rank b_Π^1. Then we have $F_{b_\Pi^1,2}^{\ell_1} \in \pi_A(\Pi)$. Thus, we obtain $b_\Pi^2 = 0$ if $n_X > 0$.

Conversely, we assume that $F_{b_\Pi^1,2}^{\ell_2} \in \pi_A(\Pi)$ for some prime number ℓ_2. Then we have $\ell_2 \neq p$. The Schreier index formula ([S2, Chapter I §3 3.4 Corollary]) implies the following natural exact sequence

$$1 \to (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus(\ell_2^{b_\Pi^1}(b_\Pi^1-1)+1)} \to F_{b_\Pi^1,2}^{\ell_2} \to (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b_\Pi^1} \to 1.$$

Let $\phi : \Pi \twoheadrightarrow F_{b_\Pi^1,2}^{\ell_2}$ be a surjection. We denote by $X_{\ell_2}^\bullet$ the pointed stable curve over k corresponding to the kernel of the natural surjection $\Pi_{X^\bullet} \overset{\sim}{\to} \Pi \overset{\phi}{\twoheadrightarrow} F_{b_\Pi^1,2}^{\ell_2} \twoheadrightarrow (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b_\Pi^1}$ and by $\Pi_{\ell_2} \subseteq \Pi$ the kernel of the surjection $\Pi \overset{\phi}{\twoheadrightarrow} F_{b_\Pi^1,2}^{\ell_2} \twoheadrightarrow (\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus b_\Pi^1}$. Then we have $(\mathbb{Z}/\ell_2\mathbb{Z})^{\oplus(\ell_2^{b_\Pi^1}(b_\Pi^1-1)+1)} \in \pi_A(\Pi_{\ell_2})$. This implies $b_{\Pi_{\ell_2}}^1 \geq \ell_2^{b_\Pi^1}(b_\Pi^1 - 1) + 1$. If $n_X = 0$, the Riemann-Hurwitz formula implies $g_{X_{\ell_2}} = \ell_2^{b_\Pi^1}(g_X - 1) + 1$, where $g_{X_{\ell_2}}$ denotes the genus

of $X_{\ell_2}^\bullet$. Then we have $b_{\Pi_{\ell_2}}^1 = 2(\ell_2^{b_\Pi^1}(g_X - 1) + 1) = \ell_2^{b_\Pi^1}(b_\Pi^1 - 2) + 2$. On the other hand, $\ell_2^{b_\Pi^1}(b_\Pi^1 - 2) + 2 < \ell_2^{b_\Pi^1}(b_\Pi^1 - 1) + 1$. This contradicts the fact that $b_{\Pi_{\ell_2}}^1 \geq \ell_2^{b_\Pi^1}(b_\Pi^1 - 1) + 1$. Then we obtain $n_X > 0$ if $b_\Pi^2 = 0$. Moreover, we see

$$b_\Pi^1 = 2g_X + n_X - 1 + b_\Pi^2.$$

(ii) This follows immediately from the structure of Π^{ab}. This completes the proof of the lemma. □

6.1.4.   Let $\overline{\mathbb{F}}_{p_\Pi}$ be an arbitrary algebraic closure of $\mathbb{F}_{p_\Pi}$. Let $\chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}_{p_\Pi}^\times)$. We denote by $\Pi_\chi \subseteq \Pi$ the kernel of χ. The profinite group Π_χ admits a natural action of Π via the conjugation action. We put

$$H_{\chi,p_\Pi} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p_\Pi\mathbb{Z}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi},$$

$$N_\chi \overset{\mathrm{def}}{=} \{\pi \in H_{\chi,p_\Pi} \mid \tau \cdot \pi = \chi(\tau)\pi \text{ for all } \tau \in \Pi\},$$
$$\gamma_{N_\chi} \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_{p_\Pi}}(N_\chi),$$

where $(\tau \cdot \pi)(x) \overset{\mathrm{def}}{=} \pi(\tau^{-1} \cdot x)$ for all $x \in \Pi_\chi$. We define a group-theoretical invariant associated to Π as follows:

$$\gamma_\Pi^{\max} \overset{\mathrm{def}}{=} \max\{\gamma_{N_\chi} \mid \chi \in \mathrm{Hom}(\Pi, \overline{\mathbb{F}}_{p_\Pi}^\times) \text{ and } \chi \neq 1\}.$$

Let $\mu_m \overset{\mathrm{def}}{=} \chi(\Pi) \subseteq \overline{\mathbb{F}}_{p_\Pi}^\times$ be the image of χ which is the group of mth roots of unity for some m prime to p_Π, and $X_\chi^\bullet = (X_\chi, D_{X_\chi}) \to X^\bullet$ the Galois multi-admissible covering over k with Galois group μ_m induced by χ. Then we have a natural Π-equivariant isomorphism

$$H_{\text{ét}}^1(X_\chi, \mathbb{F}_{p_\Pi}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi} \cong H_{\chi,p_\Pi}.$$

Moreover, since the actions of Π on $H_{\text{ét}}^1(X_\chi, \mathbb{F}_{p_\Pi}) \otimes_{\mathbb{F}_{p_\Pi}} \overline{\mathbb{F}}_{p_\Pi}$ and H_{χ,p_Π} factor through $\Pi/\Pi_\chi \cong \mu_m$, the isomorphism above is also μ_m-equivariant. Namely, γ_{N_χ} is a generalized Hasse-Witt invariant of a $\mathbb{Z}/m\mathbb{Z}$-cyclic admissible covering of $X^\bullet$ (2.2.4). Then we have the following result.

Lemma 6.2. *We maintain the notation introduced above. Then we have $\gamma_\Pi^{\max} = \gamma_{X^\bullet}^{\max}$ (see Definition 3.10 (i) for $\gamma_\Pi^{\max}$). In particular, we have*

$$\gamma_\Pi^{\max} = g_X + n_X - 2 + b_\Pi^2.$$

Proof. The first part of the lemma follows from the above explanation concerning γ_{N_χ}. The "in particular" part of the lemma follows immediately from Theorem 5.4 and Lemma 6.1 (i). □

6.1.5.   We have the following anabelian formula for (g_X, n_X).

Theorem 6.3. *Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$, and Π an abstract profinite group such that $\Pi \cong \Pi_{X^\bullet}$ as profinite groups. Then we have*

$$g_X = b_\Pi^1 - \gamma_\Pi^{\max} - 1,$$
$$n_X = 2\gamma_\Pi^{\max} - b_\Pi^1 - b_\Pi^2 + 3.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π.

Proof. The theorem follows immediately from Lemma 6.1 and Lemma 6.2.   □

6.1.6. We maintain the notation introduced above. Moreover, suppose that $X^\bullet$ is *smooth* over k. In the remainder of this subsection, we discuss a formula for (g_X, n_X) which was essentially obtained by Tamagawa. Let $n \overset{\text{def}}{=} p^t - 1$, and let K_n be the kernel of the natural surjection $\Pi \twoheadrightarrow \Pi^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$. In [T2], Tamagawa introduced

$$\mathrm{Avr}_p(\Pi) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}$$

which is called the limit of p-averages associated to Π. Note that since $p = p_\Pi$ (Lemma 6.1 (ii)), we have that $\mathrm{Avr}_p(\Pi)$ is a group-theoretical invariant associated to Π. By applying Theorem 2.11, Tamagawa proved the following result (i.e., Tamagawa's p-average theorem, see [T2, Theorem 0.5]):

$$\mathrm{Avr}_p(\Pi) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X \geq 2. \end{cases}$$

Let $\ell >> 0$ be an arbitrary prime number distinct from p_Π. Write $\mathrm{Nom}_\ell(\Pi)$ for the set of normal subgroups of Π of index ℓ. Suppose that $b_\Pi^2 = 0$ (i.e., $n_X \neq 0$). It is well-known that

$$\mathrm{Avr}_p(\Pi(\ell)) - 1 = \ell(\mathrm{Avr}_p(\Pi))$$

for every $\ell \in \mathfrak{P} \setminus \{p_\Pi\}$ and every $\Pi(\ell) \in \mathrm{Nom}_\ell(\Pi)$ if and only if $n_X = 1$. We may define a group-theoretical invariant associated to Π as follows:

$$c_\Pi \overset{\text{def}}{=} \begin{cases} 1, & \text{if } b_\Pi^2 = 1, \\ 1, & \text{if } b_\Pi^2 = 0, \ \mathrm{Avr}_p(\Pi(\ell)) - 1 = \ell(\mathrm{Avr}_p(\Pi)) \text{ for} \\ & \quad \text{all } \ell \in \mathfrak{P} \setminus \{p_\Pi\} \text{ and all } \Pi(\ell) \in \mathrm{Nom}_\ell(\Pi), \\ 0, & \text{otherwise.} \end{cases}$$

Then Tamagawa's p-average theorem implies immediately the following formula:

$$g_X = \mathrm{Avr}_p(\Pi) + c_\Pi, \ n_X = b_\Pi^1 - 2\mathrm{Avr}_p(\Pi) - 2c_\Pi - b_\Pi^2 + 1.$$

In particular, g_X and n_X are group-theoretical invariants associated to Π when $X^\bullet$ is smooth over k.

Before Tamagawa proved the above result, he also obtained an étale fundamental group version formula for (g_X, n_X) in a completely different way (by using wildly ramified coverings) which is much simpler than the case of tame fundamental groups (see [T1, §1]). Note that, for any smooth pointed stable curve over an algebraically closed field of positive characteristic, since the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group ([T1, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version. On the other hand, tame fundamental groups are much better than étale fundamental groups if we study anabelian geometry in positive characteristic from the point of view of moduli spaces (e.g. [T3], [Y3]).

6.1.7. The approach to finding a group-theoretical formula for (g_X, n_X) by applying the limit of p-averages associated to Π explained above *is difficult to generalize* to the case where $X^\bullet$ is an arbitrary (possibly singular) pointed stable curve. The reason is as follows. In [Y5], the author generalized Tamagawa's p-average theorem to the case of pointed stable curves ([Y5, Theorem 1.3 and Theorem 1.4]). The generalized formula

concerning the limit of p-averages associated to Π is very complicated in general when $X^\bullet$ is not smooth over k, and $\mathrm{Avr}_p(\Pi)$ depends not only on the topological type (g_X, n_X) but also on the structure of dual semi-graph $\Gamma_{X^\bullet}$.

6.2. Field structures associated to inertia subgroups of marked points.

In this subsection, we prove that the field structures associated to inertia subgroups of marked points of arbitrary pointed stable curves can be reconstructed group-theoretically from admissible fundamental groups and inertia subgroups of marked points, and that a surjective open continuous homomorphism of admissible fundamental groups induces a field isomorphism of the fields associated to inertia subgroups of marked points.

6.2.1. **Settings.** Let $i \in \{1, 2\}$, and let $X_i^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Let $\widehat{X}_i^\bullet = (\widehat{X}_i, D_{\widehat{X}_i})$ be the universal admissible covering of $X_i^\bullet$ corresponding to $\Pi_{X_i^\bullet}$ (2.2.6), $\Gamma_{\widehat{X}_i^\bullet}$ the dual semi-graph of $\widehat{X}_i^\bullet$, $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$ an open edge, and x_{e_i} the closed point of X_i corresponding to e_i. Note that we have $\mathrm{Aut}(\widehat{X}_i^\bullet / X_i^\bullet) = \Pi_{X_i^\bullet}$.

Let $\widehat{e}_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet})$ be an open edge over e_i. We denote by

$$I_{\widehat{e}_i} \cong \widehat{\mathbb{Z}}(1)^{p'} \subseteq \Pi_{X_i^\bullet}$$

the stabilizer subgroup of $\widehat{e}_i$.

6.2.2. Write $\overline{\mathbb{F}}_{p,i}$ for the algebraic closure of $\mathbb{F}_p$ in k_i. We put

$$\mathbb{F}_{\widehat{e}_i} \overset{\mathrm{def}}{=} (I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widehat{e}_i}\},$$

where $\{*_{\widehat{e}_i}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_i^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with

$$\bigcup_{(p,m)=1} \mu_m(k_i).$$

Moreover, $\mathbb{F}_{\widehat{e}_i}$ can be identified with $\overline{\mathbb{F}}_{p,i}$ as sets, hence, admits *a structure of field*, whose multiplicative group is $I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}$, and whose zero element is $*_{\widehat{e}_i}$. We have the following important result.

Theorem 6.4. *We maintain the notation introduced above. Let $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ be an arbitrary surjective open continuous homomorphism of admissible fundamental groups. Suppose that $\phi(I_{\widehat{e}_1}) = I_{\widehat{e}_2}$, and that $n_X = 3$. Then there exists a group-theoretical algorithm whose input data are $\Pi_{X_i^\bullet}$ and $I_{\widehat{e}_i}$, and whose output datum is $\mathbb{F}_{\widehat{e}_i}$ (as a field). Moreover, ϕ induces a field isomorphism*

$$\phi_{\widehat{e}_1, \widehat{e}_2}^{\mathrm{fd}} : \mathbb{F}_{\widehat{e}_1} \overset{\sim}{\to} \mathbb{F}_{\widehat{e}_2},$$

where "fd" means "field".

Proof. Let $t \in \mathbb{Z}_{>0}$. We denote by $\mathbb{F}_{p^t, \widehat{e}_i}$ the unique subfield of $\mathbb{F}_{\widehat{e}_i}$ whose cardinality is equal to p^t. On the other hand, we fix a finite field $\mathbb{F}_{p^t}$ of cardinality p^t and an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ containing $\mathbb{F}_{p^t}$. Note that the field structure of $\mathbb{F}_{p^t, \widehat{e}_i}$ is equivalent to a subset

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{p^t, \widehat{e}_i}, \mathbb{F}_{p^t}) \subseteq \mathrm{Hom}_{\mathrm{gp}}(\mathbb{F}_{p^t, \widehat{e}_i}^\times, \mathbb{F}_{p^t}^\times),$$

where "gp" means "group", and "fd" means "field". Then in order to prove the first part of the theorem, it is sufficient to prove that there exists a group-theoretical algorithm whose input data are $\Pi_{X_i^\bullet}$ and $I_{\widehat{e}_i}$, and whose output datum is the subset $\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{p^t,\widehat{e}_i}, \mathbb{F}_{p^t})$ for t in a cofinal subset of $\mathbb{N}$ with respect to division.

Let $n \stackrel{\mathrm{def}}{=} p^t - 1$ and $\chi_i \in \mathrm{Hom}_{\mathrm{gp}}(\Pi_{X_i^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}_{p^t}^\times)$. Write H_{χ_i} for the kernel of $\Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{\chi_i} \mathbb{F}_{p^t}^\times$, M_{χ_i} for $H_{\chi_i}^{\mathrm{ab}} \otimes \mathbb{F}_p$, and $X_{H_{\chi_i}}^\bullet = (X_{H_{\chi_i}}, D_{X_{H_{\chi_i}}})$ for the pointed stable curve over k_i induced by H_{χ_i}. Note that M_{χ_i} admits a natural action of $\Pi_{X_i^\bullet}$ via the conjugation action. Moreover, this action factors through $\Pi_{X_i^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$. We put

$$M_{\chi_i}[\chi_i] \stackrel{\mathrm{def}}{=} \{a \in M_{\chi_i} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma \cdot a = \chi_i(\sigma)a \text{ for all } \sigma \in \Pi_{X_i^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}\},$$

$$\gamma_{\chi_i}(M_{\chi_i}) \stackrel{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(M_{\chi_i}[\chi_i]).$$

Note that $\gamma_{\chi_i}(M_{\chi_i})$ is a generalized Hasse-Witt invariant of a cyclic multi-admissible covering of $X_i^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$, and that Lemma 2.7 implies that $\gamma_{\chi_i}(M_{\chi_i}) \leq g_X + 1$ if $n_X = 3$. Moreover, we define two maps

$$\mathrm{Res}_{i,t} : \mathrm{Hom}_{\mathrm{gp}}(\Pi_{X_i^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}_{p^t}^\times) \to \mathrm{Hom}_{\mathrm{gp}}(\mathbb{F}_{p^t,\widehat{e}_i}^\times, \mathbb{F}_{p^t}^\times),$$

$$\Gamma_{i,t} : \mathrm{Hom}_{\mathrm{gp}}(\Pi_{X_i^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}_{p^t}^\times) \to \mathbb{Z}_{\geq 0}, \ \chi_i \mapsto \gamma_{\chi_i}(M_{\chi_i}),$$

where the map $\mathrm{Res}_{i,t}$ is the restriction with respect to the natural inclusion

$$\mathbb{F}_{p^t,\widehat{e}_i}^\times = I_{\widehat{e}_i} \otimes \mathbb{Z}/n\mathbb{Z} \hookrightarrow \Pi_{X_i^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}.$$

Let m_0 be the product of all prime numbers $\leq p - 2$ if $p \neq 2, 3$ and $m_0 = 1$ if $p = 2, 3$. Let t_0 be the order of p in the multiplicative group $(\mathbb{Z}/m_0\mathbb{Z})^\times$. We have the following claim (see [T2, Claim 5.4] for the case where $X_i^\bullet$ is smooth over k_i):

Claim A: There exists a constant $C(g_X)$ which depends only on g_X such that, for each natural number $t > \max\{\log_p(C(g_X) + 1), \log_p(\#(e^{\mathrm{cl}}(\Gamma_{X_i^\bullet})))\}$ divisible by t_0, we have

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{p^t,\widehat{e}_i}, \mathbb{F}_{p^t}) = \mathrm{Hom}_{\mathrm{gp}}^{\mathrm{surj}}(\mathbb{F}_{p^t,\widehat{e}_i}^\times, \mathbb{F}_{p^t}^\times) \setminus \mathrm{Res}_{i,t}(\Gamma_{i,t}^{-1}(\{g_X + 1\})), \ i \in \{1, 2\},$$

where $\mathrm{Hom}_{\mathrm{gp}}^{\mathrm{surj}}(-, -)$ denotes the set of surjections whose elements are contained in $\mathrm{Hom}_{\mathrm{gp}}(-, -)$.

By applying similar arguments to the arguments given in the proof of [T2, Claim 5.4], Claim A is equivalent to the following claim:

Claim B: Let $m \in (\mathbb{Z}/n\mathbb{Z})^\sim$ (2.2.5). Then the following statements are equivalent:
(i) We have $m \in \{p^r \mid r = 0, \dots, t - 1\}$.
(ii) We have that $(m, n) = 1$, and that, there does not exist $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ (2.2.5) and $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X_i^\bullet)$ (Definition 2.4 (i)) such that $\mathrm{ord}_{x_{e_i}}(D) = m$ and $\gamma_{(\alpha, D)} = g_X + 1$.

Let us prove Claim B. Firstly, we prove (i) $\Rightarrow$ (ii). If $s(D) = 1$ (2.2.5), then Lemma 2.7 implies $\gamma_{(\alpha, D)} \leq g_X$. Thus, we may assume that $s(D) = 2$. We put

$$D_{X_i} \stackrel{\mathrm{def}}{=} \{x_{i,1} \stackrel{\mathrm{def}}{=} x_{e_i}, x_{i,2}, x_{i,3}\}.$$

By [T2, Proposition 2.21 (iv-a)], either $\mathrm{ord}_{x_{i,1}}(D) = n$ or $\mathrm{ord}_{x_{i,3}}(D) = n$ holds. This is impossible as $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$. Next, we prove (ii) $\Rightarrow$ (i). Suppose that $m \notin$

$\{p^r \mid r = 0, \ldots, t-1\}$, and that $(m, n) = 1$. Since t is divisible by t_0, n is divisible by all prime numbers $\leq p - 2$. Then the assumption $(m, n) = 1$ implies that $m \notin \{ap^r \mid a = 0, \ldots, p-2, r = 0, \ldots, t-1\}$. Then [T2, Proposition 2.21 (iv-c)] and Theorem 5.7 imply that there exist $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X_i^\bullet)$ such that $\mathrm{ord}_{x_{e_i}}(D) = m$ and $\gamma_{(\alpha, D)} = g_X + 1$. This completes the proof of the claim. Then we complete the proof of the first part of the theorem.

Next, we prove the "moreover" part of the theorem. Let $\kappa_2 \in \mathrm{Hom}_{\mathrm{gp}}(\Pi_{X_2^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}_{p^t}^\times)$. Then we obtain a character

$$\kappa_1 \in \mathrm{Hom}_{\mathrm{gp}}(\Pi_{X_1^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{F}_{p^r}^\times)$$

induced by ϕ. Moreover, the surjection $\phi|_{H_{\kappa_1}}$ induces a surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$. Suppose that $\kappa_2 \in \Gamma_{2,r}^{-1}(\{g_X + 1\})$. The surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$ implies that $\gamma_{\kappa_1}(M_{\kappa_1}) = g_X + 1$. Namely, we have $\kappa_1 \in \Gamma_{1,r}^{-1}(\{g_X + 1\})$. On the other hand, the isomorphism $\phi|_{I_{\widehat{e}_1}} : I_{\widehat{e}_1} \xrightarrow{\sim} I_{\widehat{e}_2}$ induces an injection

$$\mathrm{Res}_{2,r}(\Gamma_{2,r}^{-1}(\{g_X + 1\})) \hookrightarrow \mathrm{Res}_{1,r}(\Gamma_{1,r}^{-1}(\{g_X + 1\})).$$

Since $\#(\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{p^r, \widehat{e}_1}, \mathbb{F}_{p^r})) = \#(\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{p^r, \widehat{e}_2}, \mathbb{F}_{p^r}))$, Claim A implies that $\phi|_{I_{\widehat{e}_1}}$ induces a bijection $\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{p^r, \widehat{e}_2}, \mathbb{F}_{p^r}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{p^r, \widehat{e}_1}, \mathbb{F}_{p^r})$. Thus, $\phi|_{I_{\widehat{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{\widehat{e}_2}, \overline{\mathbb{F}}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{\widehat{e}_1}, \overline{\mathbb{F}}_p).$$

If we choose $\overline{\mathbb{F}}_p = \mathbb{F}_{\widehat{e}_2}$, then the image of $\mathrm{id}_{\mathbb{F}_{\widehat{e}_2}}$ via the above bijection induces a field isomorphism

$$\phi_{\widehat{e}_1, \widehat{e}_2}^{\mathrm{fd}} : \mathbb{F}_{\widehat{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widehat{e}_2}.$$

This completes the proof of the theorem. $\qquad\square$

Remark 6.4.1. We maintain the notation introduced above. We would like to mention that the condition

"Suppose that $\phi(I_{\widehat{e}_1}) = I_{\widehat{e}_2}$, and that $n_X = 3$."

appeared in Theorem 6.4 can be *omitted* (i.e., $\phi(I_{\widehat{e}_1})$ is a stabilizer subgroup associated to some open edge of $e^{\mathrm{op}}(\Gamma_{\widehat{X}_2^\bullet})$ for an arbitrary open continuous homomorphism ϕ, see [Y7, Theorem 4.13] for a more precise statement). This is one of the main results of [Y7, Section 4] which is highly non-trivial, and which is proved by using Theorem 5.4.

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