MODULI SPACES OF FUNDAMENTAL GROUPS OF CURVES IN
POSITIVE CHARACTERISTIC I

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Abstract. In this series of papers, we investigate a new kind of anabelian phenomenon of curves over algebraically closed fields of positive characteristic which shows that the topological structures of moduli spaces of curves can be understood from open continuous homomorphisms of fundamental groups of curves. Let $p$ be a prime number, and let $\mathcal{M}_{g,n}$ be the coarse moduli space of the moduli stack over an algebraic closure of the finite field $\mathbb{F}_p$ classifying pointed stable curves of type $(g, n)$. We introduce a topological space $\Pi_{g,n}$ which we call the moduli space of admissible fundamental groups of pointed stable curves of type $(g, n)$ in characteristic $p$, whose underlying set consists of the set of isomorphism classes of the admissible fundamental groups of pointed stable curves of type $(g, n)$, and whose topology is determined by the sets of finite quotients of the admissible fundamental groups. By introducing a certain equivalence relation $\sim_{fe}$ on the underlying topological space $[\mathcal{M}_{g,n}]$ of $\mathcal{M}_{g,n}$ determined by Frobenius actions, we obtain a topological space $\mathcal{M}_{g,n}^{\text{adm}} \overset{\text{def}}{=} [\mathcal{M}_{g,n}] / \sim_{fe}$ whose topology is induced by the Zariski topology of $\mathcal{M}_{g,n}$. Moreover, there is a natural continuous map

$$\pi_{g,n}^{\text{adm}} : \mathcal{M}_{g,n} \to \Pi_{g,n}.$$

The topological space $\Pi_{g,n}$ gives us a new insight into the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p$ based on the following point of view: Every topological property concerning $\Pi_{g,n}$ is equivalent to an anabelian property concerning pointed stable curves over algebraically closed fields of characteristic $p$. Furthermore, the Homeomorphism Conjecture says that $\pi_{g,n}^{\text{adm}}$ is a homeomorphism, which is the main conjecture of the theory developed in the present series of papers. The Homeomorphism Conjecture generalizes all the conjectures in the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p$, and means that moduli spaces of curves can be reconstructed group-theoretically as topological spaces from the admissible fundamental groups of curves. One of main results of the present series of papers says that the Homeomorphism Conjecture holds when $\dim(\mathcal{M}_{g,n}) \leq 1$ (i.e., $(g, n) = (0, 3), (0, 4), (1, 1)$). In the present paper, we establish two fundamental tools to analyze the geometric behavior of curves from open continuous homomorphisms of admissible fundamental groups, which play central roles in the theory developed in the series of papers. Moreover, we prove that $\pi_{g,n}^{\text{adm}}([q])$ is a closed point of $\Pi_{g,n}$ when $[q]$ is a closed point of $\mathcal{M}_{0,n}$. In particular, we obtain that the Homeomorphism Conjecture holds when $(g, n) = (0, 3)$ or $(0, 4)$.

Keywords: pointed stable curve, admissible fundamental group, moduli space, anabelian geometry, positive characteristic.

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## Introduction

In the present paper, we study the anabelian geometry of curves over algebraically closed fields of positive characteristic.

### 0.1. Grothendieck’s anabelian philosophy.

In the 1980s, A. Grothendieck suggested a theory of arithmetic geometry called anabelian geometry (cf. [G1], [G2]), roughly speaking, which focuses on the following question: Can we reconstruct the geometric information of a variety group-theoretically from various versions of its algebraic fundamental groups?
group? Grothendieck calls varieties which can be completely determined by their fundamental groups *anabelian varieties*, and his “anabelian dream” consists in classifying the anabelian varieties in all dimensions over all fields. In the particular case of dimension 1, he conjectured that all smooth pointed stable curves (defined over certain fields) are anabelian varieties.

0.1.1. Let $p$ be a prime number, and let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of type $(g_X, n_X)$ over a field $k$ of characteristic char($k$), where $X$ denotes the underlying curve which is a semi-stable curve over $k$, $D_X$ denotes the set of marked points satisfying [K, Definition 1.1 (iv)], $g_X$ denotes the genus of $X$, and $n_X$ denotes the cardinality $\#D_X$ of $D_X$.

0.1.2. First, we explain some background of the anabelian geometry of curves. Suppose that $X^\bullet$ is smooth over $k$. When $k$ is an arithmetic field (e.g. a number field, a $p$-adic field, a finite field, etc.), Grothendieck’s anabelian conjectures for curves (or the Grothendieck conjectures for short), roughly speaking, are based on the following anabelian philosophy:

- **Weak Isom-version**: The isomorphism class of $X^\bullet$ can be determined group-theoretically from the isomorphism class of its algebraic fundamental group.
- **Isom-version**: The sets of isomorphisms of smooth pointed stable curves can be determined group-theoretically from the sets of isomorphisms of their algebraic fundamental groups.
- **Hom-version**: The sets of dominant morphisms of smooth pointed stable curves can be determined group-theoretically from the sets of open continuous homomorphisms of their algebraic fundamental groups.

The Grothendieck conjectures have been proven in many cases. For example, we have the following results: When $k$ is a number field, the conjecture was proved by H. Nakamura (weak Isom-version), A. Tamagawa (Isom-version), and S. Mochizuki (Hom-version) (cf. [Nakm1], [Nakm2], [T1], [M2]). When $k$ is a finitely generated field over the finite field $\mathbb{F}_p$, the Isom-version of the Grothendieck conjecture was proved by Tamagawa, Mochizuki, J. Stix, and M. Saida-Tamagawa (cf. [T1], [M4], [Sti1], [Sti2], [ST1], [ST3]). All the proofs of the Grothendieck conjectures for curves over arithmetic fields mentioned above require the use of the non-trivial outer Galois representations induced by the fundamental exact sequences of fundamental groups.

0.2. **Beyond the arithmetical action.** Next, let us return to the case where $X^\bullet$ is an arbitrary pointed stable curve, and suppose that $k$ is an algebraically closed field.

0.2.1. By choosing a suitable base point $x$ of $X^\bullet$, we have the admissible fundamental group

$$\pi_1^{\text{adm}}(X^\bullet, x)$$

of $X^\bullet$ (cf. Definition 1.2). Since we only focus on the isomorphism class of $\pi_1^{\text{adm}}(X^\bullet, x)$, for simplicity, we use the notation $\pi_1^{\text{adm}}(X^\bullet)$ to denote $\pi_1^{\text{adm}}(X^\bullet, x)$. In particular, if $X^\bullet$ is smooth over $k$, then $\pi_1^{\text{adm}}(X^\bullet)$ is naturally isomorphic to the tame fundamental group $\pi_1^{t}(X^\bullet)$. Write $\pi_1^{\text{adm}}(X^\bullet)^p$ for the maximal prime-to-$p$ quotient of $\pi_1^{\text{adm}}(X^\bullet)$ if char($k$) = $p$. 
The pro-finite group $\pi_1^{adm}(X^\bullet)$ (resp. $\pi_1^{adm}(X^\bullet,p')$) is isomorphic to the pro-finite completion (resp. pro-prime-to-$p$ completion) of the following group (cf. [V2, Théorème 2.2 (c)])

$$\langle a_1, \ldots, a_{g_X}, b_1, \ldots, b_{g_X}, c_1, \ldots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle$$

when $\text{char}(k) = 0$ (resp. $\text{char}(k) = p$). In the case of algebraically closed fields of characteristic 0, since the admissible fundamental groups of curves depend only on the types of curves, the anabelian geometry of curves does not exist in this situation. On the other hand, if $\text{char}(k) = p$, the situation is quite different from that in characteristic 0. The admissible fundamental group $\pi_1^{adm}(X^\bullet)$ is very mysterious and its structure is no longer known. In the remainder of the introduction, we assume that $k$ is an algebraically closed field of characteristic $p$.

0.2.2. After M. Raynaud and D. Harbater solved Abhyankar’s conjecture, Harbater asked whether or not the geometric information of a curve over $k$ can be carried out from its geometric fundamental groups (cf. [Ha1], [Ha2]). Since the late 1990s, some developments of F. Pop, Raynaud, Saidi, Tamagawa, and the author (cf. [PS], [R], [ST2], [T2], [T4], [T5], [Y2], [Y3], [Y4]) showed evidence for very strong anabelian phenomena for curves over algebraically closed fields of positive characteristic (cf. [T3] for more about this conjectural world based on Grothendieck’s anabelian philosophy mentioned in 0.1.2). In this situation, the Galois group of the base field is trivial, and the arithmetic fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field. This kind of anabelian phenomenon is the reason that we do not have an explicit description of the geometric fundamental group of any pointed stable curve in positive characteristic. Moreover, we may think that the anabelian geometry of curves over algebraically closed fields of characteristic $p$ is a theory based on the following rough consideration: The admissible fundamental group of a pointed stable curve over an algebraically closed field of characteristic $p$ must encode “moduli” of the curve.

0.3. A moduli version of the Weak Isom-version Conjecture. Let us explain the anabelian geometry of curves over algebraically closed fields of positive characteristic from the point of view of moduli spaces.

0.3.1. First, we fix some notation concerning moduli spaces of curves and admissible fundamental groups associated to points of moduli spaces.

Let $\overline{F}_p$ be an algebraically closed field of $F_p$, and let $\overline{M}_{g,n}$ be the moduli stack over $\overline{F}_p$ classifying pointed stable curves of type $(g, n)$ (i.e., the quotient stack of the moduli stack of $n$-pointed stable curves in the sense of [K] by the natural action of $n$-symmetric group $S_n$), $M_{g,n} \subseteq \overline{M}_{g,n}$ the open substack classifying smooth pointed stable curves, $M_{g,n}$ the coarse moduli space of $\overline{M}_{g,n}$, and $M_{g,n}$ the coarse moduli space of $M_{g,n}$. Let $q \in M_{g,n}$ be an arbitrary point, $k(q)$ the residue field of $M_{g,n}$, and $k_q$ an algebraically closed field which contains $k(q)$. Then the composition of natural morphisms

$$\text{Spec } k_q \rightarrow \text{Spec } k(q) \rightarrow \overline{M}_{g,n}$$

determines a pointed stable curve $X_{k_q}^\bullet$ of type $(g, n)$ over $k_q$. In particular, if $k_q$ is an algebraic closure of $k(q)$, we shall write $X_q^\bullet$ for $X_{k_q}^\bullet$. 

Write $\pi_1^{\text{adm}}(X_{k_q}^\bullet)$ for the admissible fundamental group $X_{k_q}^\bullet$ and $\Gamma_{X_{k_q}^\bullet}$ for the dual semi-graph of $X_{k_q}^\bullet$ (cf. [Y1, Definition 3.1]). Since the isomorphism classes of $\pi_1^{\text{adm}}(X_{k_q}^\bullet)$ and $\Gamma_{X_{k_q}^\bullet}$ do not depend on the choice of $k_q$, we shall denote by

$$\pi_1^{\text{adm}}(q), \Gamma_q$$

the admissible fundamental group $\pi_1^{\text{adm}}(X_{k_q}^\bullet)$ and the dual semi-graph $\Gamma_{X_{k_q}^\bullet}$, respectively. Moreover, we write $v(\Gamma_q)$, $e^{op}(\Gamma_q)$, and $e^{cl}(\Gamma_q)$ for the set of vertices of $\Gamma_q$, the set of open edges of $\Gamma_q$, and the set of closed edges of $\Gamma_q$, respectively, which correspond to the set of irreducible components of $X_{k_q}$, the set of marked points $D_{X_{k_q}}$, and the set of singular points $X_{k_q}^{\text{sing}}$ of $X_{k_q}$, respectively.

0.3.2. Let $\overline{\Pi}_{g,n}$ be the set of isomorphism classes (as profinite groups) of admissible fundamental groups of pointed stable curves of type $(g,n)$ over algebraically closed fields of characteristic $p$. Then the fundamental group functor $\pi_1^{\text{adm}}$ induces a natural surjective map from the underlying topological space $|\overline{\mathcal{M}}_{g,n}|$ of $\overline{\mathcal{M}}_{g,n}$ to $\overline{\Pi}_{g,n}$ as follows:

$$[\pi_1^{\text{adm}}]: |\overline{\mathcal{M}}_{g,n}| \to \overline{\Pi}_{g,n}, \ q \mapsto [\pi_1^{\text{adm}}(q)],$$

where $[\pi_1^{\text{adm}}(q)]$ denotes the isomorphism class of $\pi_1^{\text{adm}}(q)$.

Since the existence of Frobenius twists of pointed stable curves, the map $[\pi_1^{\text{adm}}]$ is not a bijection in general. For example, let $q, q' \in \overline{\mathcal{M}}_{g,n}$ be arbitrary points such that $X_q \setminus D_{X_q}$ is isomorphic to $X_{q'} \setminus D_{X_{q'}}$ as schemes (e.g. $X_q^\bullet$ is a Frobenius twist of $X_{q'}^\bullet$). Then we have that $[\pi_1^{\text{adm}}(q)] = [\pi_1^{\text{adm}}(q')]$.

0.3.3. In general, we introduce an equivalence relation $\sim_{fe}$ on $|\overline{\mathcal{M}}_{g,n}|$ which we call Frobenius equivalence (cf. [Y7, Definition 3.4] or Definition 3.1 of the present paper). Roughly speaking, $q_1 \sim_{fe} q_2$ for any points $q_1, q_2 \in \overline{\mathcal{M}}_{g,n}$ if there exists an isomorphism $\rho : \Gamma_{q_1} \to \Gamma_{q_2}$ of dual semi-graphs of $X_{q_1}^\bullet$ of $X_{q_2}^\bullet$ such that the pointed stable curves $\widetilde{X}_{q_1}, v_1$ and $\widetilde{X}_{q_2}, v_2$ associated to $v_1$ and $v_2 \overset{\text{def}}{=} \rho(v_1)$ (cf. 1.1.2), respectively, are isomorphic as schemes for every $v_1 \in v(\Gamma_{q_1})$. In particular, when $q_1 \in \overline{\mathcal{M}}_{g,n}$ (i.e., $X_{q_1}^\bullet$ is a non-singular curve), then $q_1 \sim_{fe} q_2$ if and only if $X_{q_1} \setminus D_{X_{q_1}}$ is isomorphic to $X_{q_2} \setminus D_{X_{q_2}}$ as schemes.

Moreover, [Y7, Proposition 3.7] shows that $[\pi_1^{\text{adm}}]$ factors through the following quotient set

$$\overline{\mathcal{M}}_{g,n} \overset{\text{def}}{=} |\overline{\mathcal{M}}_{g,n}|/\sim_{fe}.$$  

Then we obtain a natural surjective map

$$\pi_1^{\text{adm}}_{g,n} : \overline{\mathcal{M}}_{g,n} \to \overline{\Pi}_{g,n}$$

induced by $[\pi_1^{\text{adm}}]$.

0.3.4. One of the main conjectures in the theory of anabelian geometry of curves is the following weak Isom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic $p$ (or the Weak Isom-version Conjecture for short):

**Weak Isom-version Conjecture.** We maintain the notation introduced above. Then the surjective map

$$\pi_1^{\text{adm}}_{g,n} : \overline{\mathcal{M}}_{g,n} \to \overline{\Pi}_{g,n}, \ [q] \mapsto [\pi_1^{\text{adm}}(q)],$$

is a bijection, where $[q]$ denotes the image of $q$ of the natural quotient map $|\overline{\mathcal{M}}_{g,n}| \to \overline{\mathcal{M}}_{g,n}$.
The Weak Isom-version Conjecture was essentially formulated by Tamagawa in the case of smooth pointed stable curves, and by the author in the case of arbitrary pointed stable curves (cf. [T3], [Y7]), which means that the moduli spaces of curves in positive characteristic can be reconstructed group-theoretically as sets from isomorphism classes of admissible fundamental groups of pointed stable curves in positive characteristic.

The Weak Isom-version Conjecture is very difficult, which was only completely proved in the case where \((g,n) = (0,3)\) or \((0,4)\). More precisely, we have the following result obtained by Tamagawa and the author (cf. [T4, Theorem 0.2], [Y7, Theorem 3.8]):

**Theorem 0.1.** We maintain the notation introduced above. Write \(\mathcal{M}_{g,n}^{\text{cl}}\) for the images of the set of closed points of \(\bar{M}_{g,n}\). Then we have that \(\pi_{0,n}^{\text{adm}}(\mathcal{M}_{0,n}^{\text{cl}}) \cap \pi_{0,n}^{\text{adm}}(\mathcal{M}_{0,n}^{\text{cl}} \setminus \mathcal{M}_{0,n}^{\text{cl}}) = \emptyset\), and that

\[
\pi_{0,n}^{\text{adm}}|_{\mathcal{M}_{0,n}^{\text{cl}}} : \mathcal{M}_{0,n}^{\text{cl}} \to \Pi_{0,n}
\]

is an injection. In particular, the Weak Isom-version Conjecture holds when \((g,n) = (0,3)\) or \((0,4)\).

**Remark 0.1.1.** In other words, Theorem 0.1 is equivalent to the following anabelian result: Let \(q_1, q_2 \in \bar{M}_{0,n}\) be arbitrary points. Suppose that \(q_1\) is closed, and that \(\pi_1^{\text{adm}}(q_1)\) is isomorphic to \(\pi_1^{\text{adm}}(q_2)\) as profinite groups. Then we have \(q_1 \sim_{f.e.} q_2\).

**Remark 0.1.2.** Suppose that \(g\) is an arbitrary non-negative integer number. We also want to mention the following finiteness theorem (cf. [PS], [R], [T5], [Y2]): Let \([q] \in \mathcal{M}_{g,n}^{\text{cl}}\). Then we have

\[
\#((\pi_{2,n}^{\text{adm}})^{-1}([\pi_1^{\text{adm}}(q)])) < \infty.
\]

### 0.4. A new kind of anabelian phenomenon.

Until now the Weak Isom-version Conjecture is the ultimate goal of the anabelian geometry of curves over algebraically closed fields of characteristic \(p\), and except for [Y3], all of the researches focus on this conjecture (e.g. [PS], [R], [ST2], [T2], [T4], [T5], [Y2], [Y4]). Essentially, the Weak Isom-version Conjecture shares the same anabelian philosophy as Grothendieck originally suggested (i.e., the isomorphism classes of curves can be determined group-theoretically from the isomorphism classes of their fundamental groups), and this conjecture cannot give us any new insight into the anabelian phenomena of curves over algebraically closed fields of characteristic \(p\).

#### 0.4.1. Furthermore, when we try to formulate a “Hom-version” conjecture for curves over algebraically closed fields of characteristic \(p\) based on Grothendieck’s anabelian philosophy mentioned in 0.1.2 (i.e., an analogue of the conjecture posed in [G2, p289 (6)]), we see that the sets of dominate morphisms between pointed stable curves are empty, and that the sets of open continuous homomorphisms between their admissible fundamental groups are not empty in general (e.g. specialization homomorphisms of a non-isotrivial family of pointed stable curves). In fact, the existence of specialization homomorphisms is the reason that Tamagawa cannot formulate a “Hom-version” conjecture for tame fundamental groups of smooth pointed stable curves in general (cf. [T3, Remark 1.34]).
0.4.2. On the other hand, the results proved by the author in [Y3] show that it is possible that the sets of deformations of a smooth pointed stable curve can be reconstructed group-theoretically from the sets of open continuous homomorphisms of their admissible fundamental groups (i.e., the Weak Hom-version Conjecture posed in [Y3]). We maintain the notation introduced in 0.3.1. Let \( q_1, q_2 \in M_{g,n} \). Roughly speaking, the Weak Hom-version Conjecture says that a smooth pointed stable curve corresponding to a geometric point over \( q_2 \) can be deformed to a smooth pointed stable curve corresponding to a geometric point over \( q_1 \) if and only if the set of open continuous homomorphisms of admissible fundamental groups \( \text{Hom}_{\text{pro-gps}}(\pi_1^{\text{adm}}(q_1), \pi_1^{\text{adm}}(q_2)) \) is not empty.

Moreover, it implies the following new kind of anabelian phenomenon:

The topological structures of moduli spaces of curves in positive characteristic can be reconstructed group-theoretically from sets of open continuous homomorphisms of geometric fundamental groups of curves in positive characteristic.

The new kind of anabelian phenomenon cannot be explained by using Grothendieck’s original anabelian philosophy mentioned in 0.1.2, and motivates the theory developed in the present series of papers. Let us explain this in the remainder of the introduction.

0.5. Moduli spaces of admissible fundamental groups and the Homeomorphism Conjecture. We maintain the notation introduced in 0.3. Moreover, from now on, we shall regard \( \overline{M}_{g,n} \) as a topological space whose topology is induced naturally by the Zariski topology of \( |\overline{M}_{g,n}| \).

0.5.1. Let \( \mathcal{G} \) be the category of finite groups and \( G \in \mathcal{G} \) a finite group. We put

\[
U_{\Pi_{g,n},G} \overset{\text{def}}{=} \{ [\pi_1^{\text{adm}}(q)] \in \overline{\Pi}_{g,n} \mid \text{Hom}_{\text{surj}}(\pi_1^{\text{adm}}(q), G) \neq \emptyset \},
\]

where \( \text{Hom}_{\text{surj}}(\cdot, \cdot) \) denotes the set of surjective homomorphisms of profinite groups. We define a topological space \( (\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}}) \) group-theoretically from the set of isomorphism classes of admissible fundamental groups of pointed stable curves \( \Pi_{g,n} \), whose underlying set is \( \Pi_{g,n} \), and whose topology \( O_{\overline{\Pi}_{g,n}} \) is generated by \( \{ U_{\Pi_{g,n},G} \}_{G \in \mathcal{G}} \) as open subsets. For simplicity, we still use the notation \( \Pi_{g,n} \) to denote the topological space \( (\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}}) \), and shall say the topological space

\( \Pi_{g,n} \)

the moduli space of admissible fundamental groups of pointed stable curves of type \((g,n)\) over algebraically closed fields of characteristic \( p \) or the moduli space of admissible fundamental groups of type \((g,n)\) in characteristic \( p \) for short.

0.5.2. Theorem 3.6 of the present paper shows that the surjective map

\[
\pi_{g,n}^{\text{adm}} : \overline{\Pi}_{g,n} \to \overline{\Pi}_{g,n}
\]

is a continuous map. Moreover, we pose the following conjecture, which is the main conjecture of the theory developed in the present series of papers:

Homeomorphism Conjecture. We maintain the notation introduced above. Then we have that

\[
\pi_{g,n}^{\text{adm}} : \overline{\Pi}_{g,n} \to \overline{\Pi}_{g,n}
\]

is a homeomorphism.
0.5.3. The Homeomorphism Conjecture has a simple form if we only consider smooth pointed stable curves. Let $\mathbb{F}_p$ be the prime field of characteristic $p$, $M_{g,n,\mathbb{F}_p}$ the coarse moduli space of the moduli stack $\mathcal{M}_{g,n,\mathbb{F}_p}$ over $\mathbb{F}_p$ classifying smooth pointed stable curves of type $(g,n)$. Let $\Pi_{g,n} \subseteq \overline{\Pi}_{g,n}$ be the subset of isomorphism classes of admissible fundamental groups (= tame fundamental groups) of smooth pointed stable curves of type $(g,n)$. The subset $\Pi_{g,n}$ can be regarded as a topological space whose topology is induced by the topology of $\overline{\Pi}_{g,n}$ (in fact, $\Pi_{g,n}$ is an open subset of $\overline{\Pi}_{g,n}$ (cf. Proposition 3.7)). In this situation, the Homeomorphism Conjecture is equivalent to the following form: The natural map $M_{g,n,\mathbb{F}_p} \rightarrow \Pi_{g,n}$, $q \mapsto [\pi_1^{ad}(q)]$, is a homeomorphism.

0.5.4. Note that the Homeomorphism Conjecture is completely different from Grothendieck’s anabelian conjecture for moduli spaces of curves (i.e., a conjecture of Grothendieck based on a similar anabelian philosophy mentioned in 0.1.2 says that moduli spaces of curves are anabelian varieties in the sense of 0.1). Furthermore, the Homeomorphism Conjecture contains “moduli” information (i.e., classifications information) of curves, and Grothendieck’s anabelian conjecture for moduli spaces of curves does not contain any “moduli” information of curves.

0.5.5. The Homeomorphism Conjecture generalizes all the conjectures appeared in the theory of admissible (or tame) anabelian geometry of curves over algebraically closed fields of characteristic $p$, and means that

the moduli spaces of curves in positive characteristic can be reconstructed group-theoretically as topological spaces from sets of open continuous homomorphisms of admissible fundamental groups of pointed stable curves in positive characteristic.

This conjecture gives us a new insight into the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p$ based on the point of view:

The anabelian properties of pointed stable curves over algebraically closed fields of characteristic $p$ are equivalent to the topological properties of the topological space $\overline{\Pi}_{g,n}$.

This new point of view has raised a host of questions which cannot be seen if we only consider the Weak Isom-version Conjecture (e.g. Problem 3.9 of the present paper).

0.5.6. Next, let us explain the difference between the Weak Isom-version Conjecture and the Homeomorphism Conjecture from the aspect of group theory. The mean of anabelian geometry around the Weak Isom-version Conjecture (i.e., the theory developed in [PS], [R], [T2], [T4], [T5], [Y2], [Y4]) is the following:

Let $\mathcal{F}_i$, $i \in \{1,2\}$, be a geometric object in a certain category and $\Pi_{\mathcal{F}_i}$ the fundamental group associated to $\mathcal{F}_i$. Then the set of isomorphisms of geometric objects $\text{Isom}(\mathcal{F}_1, \mathcal{F}_2)$ can be understood from the set of isomorphisms of group-theoretical objects $\text{Isom}(\Pi_{\mathcal{F}_1}, \Pi_{\mathcal{F}_2})$. The term “anabelian” means that the geometric properties of a geometric object which can be determined by the isomorphism classes of its fundamental group. On the other hand, we do not know the relation of $\mathcal{F}_1$ and $\mathcal{F}_2$ if $\Pi_{\mathcal{F}_1}$ is not isomorphic to $\Pi_{\mathcal{F}_2}$.
In the theory developed in the present series of papers, we consider anabelian geometry in a completely different way. The mean of anabelian geometry around the Homeomorphism Conjecture is the following:

The relation of \( F_1 \) and \( F_2 \) in a certain moduli space can be understood from the set of open continuous homomorphisms \( \text{Hom}(F_1, F_2) \). Moreover, \( \text{Hom}(F_1, F_2) \) contains the geometric information of the moduli space. The term “anabelian” means the geometric properties of a certain moduli space of geometric objects (i.e., not only a single geometric object but also the moduli space of geometric objects) which can be determined by the set of open continuous homomorphisms of fundamental groups of geometric objects.

Thus, roughly speaking, the Weak Isom-version Conjecture is an “Isom-version” problem, and the Homeomorphism Conjecture is a “Hom-version” problem. In fact, the Weak Hom-version Conjecture for smooth pointed stable curves formulated in \([Y3]\) is equivalent to the Homeomorphism Conjecture when \( q \in M_{g,n} \). Similar to other theory in anabelian geometry, Hom-version problems are so much harder than the Isom-version problems.

0.6. Main result. Now, our main result of the present paper is as follows (see also Theorem 0.2):

**Theorem 0.2.** We maintain the notation introduced above. Let \( [q] \in \overline{\mathcal{M}_{0,n}} \) be an arbitrary closed point. Then \( \pi_{0,n}^{\text{adm}}([q]) \) is a closed point of \( \overline{\Pi}_{0,n} \). In particular, the Homeomorphism Conjecture holds when \((g,n) = (0,3) \) or \((0,4) \).

**Remark 0.2.1.** In this series of papers, we will prove the following results. In \([Y8]\), we will prove that the Homeomorphism Conjecture also holds when \((g,n) = (1,1) \). Then the Homeomorphism Conjecture holds when \( \dim(M_{g,n}) \leq 1 \). In \([Y9]\), we will prove a generalized version of Tamagawa’s essential dimension conjecture for closed points of \( \overline{M}_{1,n} \). In \([Y10]\), we will define clutching maps for moduli spaces of admissible fundamental groups, and prove the clutching maps are continuous maps.

0.6.1. Denote by \( \text{Hom}^{\text{open}}_{\text{pro-gps}}(-,-) \) and \( \text{Isom}^{\text{pro-gps}}(-,-) \) the set of open continuous homomorphisms of profinite groups and the set of isomorphisms of profinite groups, respectively. Then Theorem 0.2 follows from the following strong (Hom-version) anabelian result, which is a ultimate generalization of \([T2, \text{Theorem 0.2}]\) and \([T4, \text{Theorem 0.2}]\) when \( g = 0 \) and \( q_1 \) is closed (see also Theorem 6.6).

**Theorem 0.3.** Let \( q_1, q_2 \in \overline{M}_{0,n} \) be arbitrary points. Suppose that \( q_1 \) is closed. Then we have that

\[
\text{Hom}^{\text{open}}_{\text{pro-gps}}(\pi_1^{\text{adm}}(q_1), \pi_1^{\text{adm}}(q_2)) \neq \emptyset
\]

if and only if \( q_1 \sim_{f_c} q_2 \). In particular, if this is the case, we have that \( q_2 \) is a closed point, and that

\[
\text{Hom}^{\text{open}}_{\text{pro-gps}}(\pi_1^{\text{adm}}(q_1), \pi_1^{\text{adm}}(q_2)) = \text{Isom}^{\text{pro-gps}}(\pi_1^{\text{adm}}(q_1), \pi_1^{\text{adm}}(q_2)).
\]

**Remark 0.3.1.** In fact, in the present paper, we will prove a slightly stronger version of Theorem 0.3 by replacing \( \pi_1^{\text{adm}}(q_1) \) and \( \pi_1^{\text{adm}}(q_2) \) by the maximal pro-solvable quotients \( \pi_1^{\text{adm}}(q_1)_{\text{sol}} \) and \( \pi_1^{\text{adm}}(q_2)_{\text{sol}} \) of \( \pi_1^{\text{adm}}(q_1) \) and \( \pi_1^{\text{adm}}(q_2) \), respectively. Then we obtain a solvable version of Theorem 0.2 which is slightly stronger than Theorem 0.2. In particular, we
obtain that the Solvable Homeomorphism Conjecture (cf. 3.2.1) holds when \((g, n) = (0, 3)\) or \((0, 4)\).

**Remark 0.3.1.** Note that Theorem 0.3 is essentially different from Theorem 0.1. The reason is as follows: Let \(q_1, q_2 \in [\overline{M}_{g,n}]\) be arbitrary points such that \(q_1\) is not closed, and that \(q_2\) is a closed point contained in the topological closure of \(q_1\) in \([\overline{M}_{g,n}]\). Then every open continuous homomorphism \(\pi_1^{ad}(q_1) \to \pi_1^{ad}(q_2)\) is not an isomorphism (cf. [T5, Theorem 0.3]).

0.7. **Strategy of proof.** Next, we explain the method of proving Theorem 0.3 (or Theorem 0.2).

0.7.1. We establish two fundamental tools to analyze the geometric behavior of curves from open continuous homomorphisms of admissible fundamental groups, which play central roles in the theory of moduli spaces of admissible fundamental groups in positive characteristic.

0.7.2. The first tool is the following result, which says that the inertia subgroups and field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from arbitrary surjective open continuous homomorphisms of admissible fundamental groups (cf. Theorem 4.11 and Theorem 4.13 for more precise statements):

**Theorem 0.4.** Let \(X^*_i, i \in \{1, 2\}\), be a pointed stable curve of type \((g_{X_i}, n_{X_i})\) over an algebraically closed field \(k_i\) of characteristic \(p\), and \(\Gamma_{X^*_i}\) the dual semi-graph of \(X^*_i\). Let \(\Pi_{X^*_i}\) be either the admissible fundamental group \(\pi_1^{ad}(X^*_i)\) of \(X^*_i\) or the maximal pro-solvable quotient \(\pi_1^{ad}(X^*_i)^{sol}\) of \(\pi_1^{ad}(X^*_i)\), and \(I_i \subseteq \Pi_{X^*_i}\) an closed subgroup associated to an open edge of \(\Gamma_{X^*_i}\) (i.e., a closed subgroup which is (outer) isomorphic to the inertia subgroup of the marked point corresponding to an open edge of \(\Gamma_{X^*_i}\)). Suppose that \((g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})\). Let

\[ \phi : \Pi_{X^*_1} \to \Pi_{X^*_2} \]

be an arbitrary open continuous homomorphism of profinite groups. Then the following statements hold:

(i) \(\phi(I_1) \subseteq \Pi_{X^*_2}\) is a closed subgroup associated to an open edge of \(\Gamma_{X^*_2}\), and that there exists a closed subgroup \(I' \subseteq \Pi_{X^*_1}\) associated to an open edge of \(\Gamma_{X^*_1}\) such that \(\phi(I') = I_2\).

(ii) The field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from \(\Pi_{X^*_1}\), and that \(\phi\) induces a field isomorphism between the fields associated to \(I_1\) and \(\phi(I_1)\) group-theoretically.

One of main ingredients in the proof of Theorem 0.4 is a formula for the maximum generalized Hasse-Witt invariant \(\gamma^{max}(\Pi_{X^*_1})\) of prime-to-\(p\) cyclic admissible coverings of \(X^*_1\), which was proved by the author (cf. [Y6]).

0.7.3. The second tool is the following result, which we call combinatorial Grothendieck conjecture for open continuous homomorphisms, and which says that the geometry (i.e., topological and combinatorial structures) of pointed stable curves can be completely reconstructed group-theoretically from open continuous homomorphisms of admissible fundamental groups (cf. Theorem 5.30 for a more precise statement, and Theorem 5.26 for a more general form under certain assumptions):
Theorem 0.5. Let $X^*_i, i \in \{1, 2\}$, be a pointed stable curve of type $(0, n)$ over an algebraically closed field $k_i$ of characteristic $p$, and $\Gamma_{X_i^*}$ the dual semi-graph of $X_i^*$. Let $\Pi_{X_i^*}$ be the maximal pro-solvable quotient $\pi_1^{adm}(X_i^*)^{sol}$ of the admissible fundamental group $\pi_1^{adm}(X_i^*)$ of $X_i^*$ and $\Pi_i \subseteq \Pi_{X_i^*}$ a closed subgroup associated to a vertex (i.e., a closed subgroup which is (outer) isomorphic to the admissible fundamental group of the smooth pointed stable curve associated to a vertex of $\Gamma_{X_i^*}$), and $I_i \subseteq \Pi_{X_i^*}$ an closed subgroup associated to a closed edge (i.e., a closed subgroup which is (outer) isomorphic to the inertia subgroup of the node corresponding to a closed edge of $\Gamma_{X_i^*}$). Suppose that $\#e(I_{X_i^*}) = \#v(I_{X_i^*})$ and $\#e^{cl}(I_{X_i^*}) = \#e^{cl}(\Gamma_{X_i^*})$, where $\#(-)$ denotes the cardinality of ($-$). Let
\[
\phi : \Pi_{X_i^*} \to \Pi_{X_2^*}
\]
be an arbitrary open continuous homomorphism of profinite groups. Then the following statements hold:

(i) $\phi(I_1) \subseteq \Pi_{X_2^*}$ is a closed subgroup associated to a vertex of $\Gamma_{X_2^*}$, and that there exists a closed subgroup $\Pi' \subseteq \Pi_{X_2^*}$ associated to a vertex of $\Gamma_{X_2^*}$ such that $\phi(\Pi') = \Pi_2$.

(ii) $\phi(I_1) \subseteq \Pi_{X_2^*}$ is a closed subgroup associated to a closed edge of $\Gamma_{X_2^*}$, and that there exists a closed subgroup $I' \subseteq \Pi_{X_2^*}$ associated to a closed edge of $\Gamma_{X_i^*}$ such that $\phi(I') = I_2$.

(iii) $\phi$ induces an isomorphism
\[
\phi^{sg} : \Gamma_{X_1^*} \cong \Gamma_{X_2^*}
\]
of dual semi-graphs group-theoretically.

One of main ingredients in the proof of Theorem 0.5 is a formula for the limit of $p$-averages $\text{Avr}_p(\Pi_{X_i^*})$ of the admissible fundamental group of $X_i^*$, which was proved by Tamagawa and the author (cf. [T4], [Y5]).

0.7.4. The key observations of the proofs of Theorem 0.4 and Theorem 0.5 are the following: The inequalities of $\gamma^{max}(\Pi_{X_i^*})$ and $\text{Avr}_p(\Pi_{X_i^*})$ induced by $\phi$ play similar roles as the comparability of (outer) Galois representations in the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p$.

In fact, under certain assumptions, Theorem 0.5 also holds for arbitrary types (cf. Theorem 5.26 and Remark 5.26.1). Moreover, the author believes that Theorem 0.4, Theorem 0.5, and Theorem 5.26 will play important roles in the proof of the Homeomorphism Conjecture for arbitrary types.

0.7.5. By applying Theorem 0.4 and Theorem 0.5, we can prove Theorem 0.3 as follows:

Smooth case: $q_1 \in M_{0,n}$. In the proof of the Weak Isom-version Conjecture for smooth pointed stable curves of type $(0, n)$ over $\mathbb{F}_p$ (cf. [T2], [T4]), Tamagawa observed that the scheme structure of a smooth pointed stable curve of type $(0, n)$ over $\mathbb{F}_p$ is completely determined by its inertia subgroups of marked points and the field structures associated to the inertia subgroups. By constructing certain admissible coverings for $X_{q_1}^*$ and $X_{q_2}^*$, we apply Theorem 0.4 to prove that, when $X_{q_1}^*$ is non-singular, the scheme structure of $X_{q_2}^*$ can be determined by the scheme structure of $X_{q_1}^*$ via an open continuous homomorphism between their admissible fundamental groups. Then we obtain that $X_{q_1}^*$ is non-singular implies that $X_{q_2}^*$ is non-singular too. Then Theorem 0.3 follows from [Y3, Theorem 1.2] proved by the author.
Singular case: \( q_1 \in \overline{M_{0,n}} \setminus M_{0,n} \). By applying Theorem 0.4, the geometric operation (=removing a subset of marked points of a pointed stable curve and contracting the \((-1)\)-curves and the \((-2)\)-curves of a pointed semi-stable curve) can be translated to the group-theoretical operation (=quotient of a closed subgroup of the admissible fundamental group of a pointed stable curve, where the closed subgroup is generated by the inertia subgroups corresponding to a subset of marked points of the pointed stable curve). Then we can reduce Theorem 0.3 to the case where \( \#v(\Gamma_{q_1}) = \#v(\Gamma_{q_2}) \) and \( \#e^{cl}(\Gamma_{q_1}) = \#e^{cl}(\Gamma_{q_2}) \). Moreover, by applying Theorem 0.5, we can further reduce Theorem 0.3 to the case where \( q_1 \) and \( q_2 \) are contained in \( M_{0,n} \) (i.e., \( X_{q_1} \) and \( X_{q_2} \) are non-singular). Then Theorem 0.3 follows from [Y3, Theorem 1.2]. This completes the proof of our main theorem.

0.7.6. The present paper is organized as follows. In Section 1, we fix some notation concerning admissible coverings and admissible fundamental groups. In Section 2, we recall the definition of generalized Hasse-Witt invariants, a formula for maximum generalized Hasse-Witt invariants of prime-to-\( p \) admissible coverings, and a formula for limits of \( p \)-averages of admissible fundamental groups. In Section 3, we introduce the moduli spaces of admissible fundamental groups and formulate the Homeomorphism Conjecture. In Section 4, we prove Theorem 0.4. In Section 5, we prove Theorem 0.5. In Section 6, we prove our main theorem.

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PART I: FORMULATIONS OF MODULI SPACES OF ADMISSIBLE FUNDAMENTAL GROUPS

1. Admissible coverings and admissible fundamental groups

In this section, we recall some notation and definitions concerning admissible coverings and admissible fundamental groups.

1.1. Admissible coverings.

Definition 1.1. Let \( G \) be a semi-graph (cf. [Y6, Definition 2.1]).

(a) We shall denote by \( v(G) \), \( e^{op}(G) \), and \( e^{cl}(G) \) the set of vertices of \( G \), the set of open edges of \( G \), and the set of closed edges of \( G \), respectively.

(b) The semi-graph \( G \) can be regarded as a topological space with natural topology induced by \( \mathbb{R}^2 \). We define an one-point compactification \( G^{cpt} \) of \( G \) as follows: if \( e^{op}(G) = \emptyset \), we put \( G^{cpt} = G \); otherwise, the set of vertices of \( G^{cpt} \) is the disjoint union \( v(G^{cpt}) \overset{\text{def}}{=} v(G) \cup \{v_{\infty}\} \), the set of closed edges of \( G^{cpt} \) is \( e^{cl}(G^{cpt}) \overset{\text{def}}{=} e^{cl}(G) \cup e^{op}(G) \), the set of open edges of \( G \) is empty, and every edge \( e \in e^{op}(G) \subseteq e^{cl}(G^{cpt}) \) connects \( v_{\infty} \) with the vertex that is abutted by \( e \).

(c) Let \( v \in v(G) \). We shall say that \( G \) is 2-connected at \( v \) if \( G \setminus \{v\} \) is either empty or connected. Moreover, we shall say that \( G \) is 2-connected if \( G \) is 2-connected at each \( v \in \).
for the cardinality of \( Y \) the quotient morphism short if the following conditions are satisfied:

\[ k \text{ stable curve over } X \]

shall say that underlying curves induced by \( f \) to \( e \)

Moreover, write \( e \) denotes the singular locus of \( e \)

Moreover, write \( e \) denotes the singular locus of \( e \)

\[ E = (X, D_X) \]

be a pointed semi-stable curve of type \((g_X, n_X)\) over an algebraically closed field \( k \) of characteristic \( p \), where \( X \) denotes the underlying curve, \( D_X \) denotes the set of marked points, \( g_X \) denotes the genus of \( X \), and \( n_X \) denotes the cardinality \#\( D_X \) of \( D_X \). Write \( \Gamma_X \) for the dual semi-graph of \( X \) (cf. \[ Y1, Definition 3.1 \] for the definition of the dual semi-graph of a pointed semi-stable curve) and \( r_X \equiv \dim_\Q(H^1(\Gamma_X, \Q)) \) for the Betti number of the semi-graph \( \Gamma_X \). We shall say that \( X \) is a pointed stable curve over \( k \) if \( D_X \) satisfies \( \text{[K, Definition 1.1 (iv)]} \).

1.1.2. Let \( p \) be a prime number, and let \( X^\bullet = (X, D_X) \)

be a pointed semi-stable curve of type \((g_X, n_X)\) over an algebraically closed field \( k \) of characteristic \( p \), where \( X \) denotes the underlying curve, \( D_X \) denotes the set of marked points, \( g_X \) denotes the genus of \( X \), and \( n_X \) denotes the cardinality \#\( D_X \) of \( D_X \). Write \( \Gamma_X \) for the dual semi-graph of \( X^\bullet \) (cf. \[ Y1, Definition 3.1 \] for the definition of the dual semi-graph of a pointed semi-stable curve) and \( r_X \equiv \dim_\Q(H^1(\Gamma_X, \Q)) \) for the Betti number of the semi-graph \( \Gamma_X \). We shall say that \( X \) is a pointed stable curve over \( k \) if \( D_X \) satisfies \( \text{[K, Definition 1.1 (iv)]} \).

1.1.2. Let \( v \in v(\Gamma_X) \) and \( e \in e^{op}(\Gamma_X) \cup e^{cl}(\Gamma_X) \). We write \( X_v \) for the irreducible component of \( X \) corresponding to \( v \), write \( x_e \) for the node of \( X \) corresponding to \( e \) if \( e \in e^{cl}(\Gamma_X) \), and write \( x_e \) for the marked point of \( X \) corresponding to \( e \) if \( e \in e^{op}(\Gamma_X) \). Moreover, write \( \tilde{X}_v \) for the smooth compactification of \( U_{X_v} \equiv X_v \setminus X_v \)

\[ \sum_{e \in e^{op}(\Gamma_X) \cup e^{cl}(\Gamma_X)} b_e(v), \]

where \( b_e(v) \in \{0, 1, 2\} \) denotes the number of times that \( e \) meets \( v \). We put

\[ v(\mathbb{G})^{b \leq 1} \equiv \{ v \in v(\mathbb{G}) \mid b(v) \leq 1 \}, \]

and denote by \( e^{cl}(\mathbb{G})^{b \leq 1} \) the set of closed edges of \( \mathbb{G} \) which meet a vertex of \( v(\mathbb{G})^{b \leq 1} \).

1.1.2. Let \( v \in v(\Gamma_X) \) and \( e \in e^{op}(\Gamma_X) \cup e^{cl}(\Gamma_X) \). We write \( X_v \) for the irreducible component of \( X \) corresponding to \( v \), write \( x_e \) for the node of \( X \) corresponding to \( e \) if \( e \in e^{cl}(\Gamma_X) \), and write \( x_e \) for the marked point of \( X \) corresponding to \( e \) if \( e \in e^{op}(\Gamma_X) \). Moreover, write \( \tilde{X}_v \) for the smooth compactification of \( U_{X_v} \equiv X_v \setminus X_v^{sing} \), where \((-)^{sing}\) denotes the singular locus of \((-)\). We define a smooth pointed semi-stable curve of type \((g_v, n_v)\) over \( k \) to be

\[ \tilde{X}_v^\bullet = (\tilde{X}_v, D_{\tilde{X}_v}) \equiv (\tilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v). \]

We shall say that \( \tilde{X}_v^\bullet \) is the smooth pointed semi-stable curve of type \((g_v, n_v)\) associated to \( v \), or the smooth pointed semi-stable curve associated to \( v \) for short. In particular, we shall say that \( \tilde{X}_v^\bullet \) is the smooth pointed stable curve associated to \( v \) if \( \tilde{X}_v^\bullet \) is a pointed stable curve over \( k \).

Definition 1.2. Let \( Y^\bullet = (Y, D_Y) \) be a pointed semi-stable curve over \( k \), \( f^\bullet : Y^\bullet \to X^\bullet \) a finite morphism of pointed semi-stable curves over \( k \), and \( f : Y \to X \) the morphism of underlying curves induced by \( f^\bullet \).

We shall say \( f^\bullet \) a Galois admissible covering over \( k \) (or Galois admissible covering for short) if the following conditions are satisfied:

(i) There exists a finite group \( G \subseteq \text{Aut}_k(Y^\bullet) \) such that \( Y^\bullet / G = X^\bullet \), and \( f^\bullet \) is equal to the quotient morphism \( Y^\bullet \to Y^\bullet / G \).

(ii) For each \( y \in Y^{sm} \setminus D_Y \), \( f \) is étale at \( y \), where \((-)^{sm} \) denotes the smooth locus of \((-)\).

(iii) For any \( y \in Y^{sing} \), the image \( f(y) \) is contained in \( X^{sing} \).

(iv) For each \( y \in Y^{sing} \), we write \( D_y \subseteq G \) for the decomposition group of \( y \) and \#\( D_y \) for the cardinality of \( D_y \). Then we have that \((\#D_y, p) = 1\), and that the local morphism
between two nodes induced by $f$ may be described as follows:

\[
\begin{align*}
\hat{\mathcal{O}}_{X,f(y)} & \cong k[[u,v]]/(uv) \\
\hat{\mathcal{O}}_{Y,y} & \cong k[[s,t]]/(st)
\end{align*}
\]

where $\#(-)$ denotes the cardinality of $(-)$. Moreover, we have that $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$th root of unit.

(v) The local morphism between two marked points induced by $f$ may be described as follows:

\[
\begin{align*}
\hat{\mathcal{O}}_{X,f(y)} & \cong k[[a]] \\
\hat{\mathcal{O}}_{Y,y} & \cong k[[b]]
\end{align*}
\]

where $(m,p) = 1$ (i.e., a tamely ramified extension).

Moreover, we shall say $f^*$ an admissible covering if there exists a morphism of pointed semi-stable curves $h^* : W^* \to Y^*$ over $k$ such that the composite morphism $f^* \circ h^* : W^* \to X^*$ is a Galois admissible covering over $k$. We shall say an admissible covering $f^*$ étale if $f$ is an étale morphism.

Let $Z^*$ be a disjoint union of finitely many pointed semi-stable curves over $k$. We shall say that a morphism $f^*_Z : Z^* \to X^*$ over $k$ is a multi-admissible covering if the restriction of $f^*_Z$ to each connected component of $Z^*$ is admissible.

**Definition 1.3.** Let $f^* : Y^* \to X^*$ be an admissible covering over $k$ of degree $m$. Let $e \in e^{\text{op}}(\Gamma_X^*) \cup e^{\text{cl}}(\Gamma_X^*)$ and $x_e$ the closed point of $X$ corresponding to $e$. We put

\[
\begin{align*}
e^{\text{cl,ra}}_f & \overset{\text{def}}{=} \{ e \in e^{\text{cl}}(\Gamma_X^*) \mid \#f^{-1}(x_e) = 1 \}, \\
e^{\text{cl,ét}}_f & \overset{\text{def}}{=} \{ e \in e^{\text{cl}}(\Gamma_X^*) \mid \#f^{-1}(x_e) = m \}, \\
e^{\text{op,ra}}_f & \overset{\text{def}}{=} \{ e \in e^{\text{op}}(\Gamma_X^*) \mid \#f^{-1}(x_e) = 1 \}, \\
e^{\text{op,ét}}_f & \overset{\text{def}}{=} \{ e \in e^{\text{op}}(\Gamma_X^*) \mid \#f^{-1}(x_e) = m \}, \\
\nu^{\text{ra}}_f & \overset{\text{def}}{=} \{ v \in v(\Gamma_X^*) \mid \#\text{Irr}(f^{-1}(X_v)) = 1 \}, \\
\nu^{\text{sp}}_f & \overset{\text{def}}{=} \{ v \in v(\Gamma_X^*) \mid \#\text{Irr}(f^{-1}(X_v)) = m \},
\end{align*}
\]

where $\text{Irr}(-)$ denotes the set of irreducible components of $(-)$, “ra” means “ramification”, and “sp” means “split”. Note that if the Galois closure of $f^*$ is a Galois admissible covering whose Galois group is a $p$-group, then the definition of admissible coverings implies that $\#e^{\text{cl,ra}}_f = \#e^{\text{op,ra}}_f = 0$.

1.2. Admissible fundamental groups.
1.2.1. Let $\mathcal{C}$ be a category. We shall write $\text{Ob}(\mathcal{C})$ for the class of objects of $\mathcal{C}$, and write $\text{Hom}(\mathcal{C})$ for the class of morphisms of $\mathcal{C}$. We denote by
\[
\text{Cov}^{\text{adm}}(X^\bullet) \overset{\text{def}}{=} (\text{Ob}(\text{Cov}^{\text{adm}}(X^\bullet)),\text{Hom}(\text{Cov}^{\text{adm}}(X^\bullet)))
\]
the category which consists of the following data: (i) $\text{Ob}(\text{Cov}^{\text{adm}}(X^\bullet))$ consists of an empty object and all the pairs $(Z^\bullet, f^\bullet_Z : Z^\bullet \to X^\bullet)$, where $Z^\bullet$ is a disjoint union of finitely many pointed semi-stable curves over $k$, and $f^\bullet_Z$ is a multi-admissible covering over $k$; (ii) for any $(Z^\bullet, f^\bullet_Z), (Y^\bullet, f^\bullet_Y) \in \text{Ob}(\text{Cov}^{\text{adm}}(X^\bullet))$, we define
\[
\text{Hom}((Z^\bullet, f^\bullet_Z), (Y^\bullet, f^\bullet_Y)) \overset{\text{def}}{=} \{g^\bullet \in \text{Hom}_k(Z^\bullet, Y^\bullet) \mid f^\bullet_Y \circ g^\bullet = f^\bullet_Z\},
\]
where $\text{Hom}_k(Z^\bullet, Y^\bullet)$ denotes the set of $k$-morphisms of pointed semi-stable curves. It is well known that $\text{Cov}^{\text{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\text{sm}} \setminus D_X$, we obtain a fundamental group $\pi_1^{\text{adm}}(X^\bullet, x)$ which is called the \textit{admissible fundamental group} of $X^\bullet$. Since we only focus on the isomorphism class of $\pi_1^{\text{adm}}(X^\bullet, x)$, for simplicity, we omit the base point and denote the admissible fundamental group by
\[
\pi_1^{\text{adm}}(X^\bullet).
\]
Note that, by the definition of admissible coverings, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X^\bullet$ when $X^\bullet$ is smooth over $k$. Let $v \in v(\Gamma_{X^\bullet})$. Write $\pi_1^{\text{adm}}(\tilde{X}_v^\bullet)$ for the admissible fundamental group of the smooth pointed semi-stable curve $\tilde{X}_v^\bullet$ associated to $v$. Then we have a natural (outer) injection
\[
\pi_1^{\text{adm}}(\tilde{X}_v^\bullet) \hookrightarrow \pi_1^{\text{adm}}(X^\bullet).
\]
We shall denote by $\pi_1^{\text{adm}}(X), \pi_1^{\text{et}}(X), \pi_1^{\text{top}}(\Gamma_{X^\bullet})$ the admissible fundamental group of the pointed semi-stable curve $(X, \emptyset)$, the étale fundamental group of the underlying curve $X$ of $X^\bullet$, and the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, respectively. Then we have the following natural surjective open continuous homomorphisms (for suitable choices of base points):
\[
\pi_1^{\text{adm}}(X^\bullet) \twoheadrightarrow \pi_1^{\text{adm}}(X) \twoheadrightarrow \pi_1^{\text{et}}(X) \twoheadrightarrow \pi_1^{\text{top}}(\Gamma_{X^\bullet}).
\]
Note that the isomorphism classes of $\pi_1^{\text{adm}}(X^\bullet), \pi_1^{\text{adm}}(X), \pi_1^{\text{et}}(X), \pi_1^{\text{top}}(\Gamma_{X^\bullet})$ depend only on the pointed \textit{stable} curve associated to $X^\bullet$ (i.e., the pointed stable curve obtained by contracting the $(-1)$-curves and $(-2)$-curves of $X^\bullet$).

1.2.2. The admissible fundamental groups of pointed stable curves can be also described by using logarithmic geometry. Let $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ be the moduli stack over $\text{Spec} \mathbb{Z}$ parameterizing pointed stable curves of type $(g_X, n_X)$ and $\mathcal{M}_{g_X,n_X,\mathbb{Z}}$ the open substack of $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ parameterizing smooth pointed stable curves. Write $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ with the natural log structure associated to the divisor with normal crossings $\mathcal{M}_{g_X,n_X,\mathbb{Z}} \setminus \mathcal{M}_{g_X,n_X,\mathbb{Z}} \subset \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ relative to $\text{Spec} \mathbb{Z}$. The pointed stable curve $X^\bullet$ over $k$ induces a morphism $\text{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. Write $s_X^{\log}$ for the log scheme whose underlying scheme is $\text{Spec} k$, and whose log structure is the pulling-back log structure induced by the morphism $\text{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. We obtain a natural morphism $s_X^{\log} \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}^{\log}$ induced by the morphism $\text{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ and a stable log curve
\[
X^{\log} \overset{\text{def}}{=} s_X^{\log} \times_{\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}^{\log}} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}^{\log} + 1, \mathbb{Z}
\]
over \( s_X^{\log} \) whose underlying scheme is \( X \). Let \( Y^{\log} \rightarrow X^{\log} \) be an arbitrary finite Kummer log étale covering (cf. [I]). One can prove that there exists a Kummer log étale covering \( t_X^{\log} \rightarrow s_X^{\log} \) such that \( Y^{\log} \times_{s_X^{\log}} t_X^{\log} \rightarrow X^{\log} \times_{s_X^{\log}} t_X^{\log} \) is a log admissible covering (cf. [M1, §3.5 Definition]) over \( t_X^{\log} \). Then the admissible fundamental group of \( X^* \) does not depend on the log structure of \( X^{\log} \), and [M1, §3.11 Proposition] implies that the admissible fundamental group \( \pi_1^{\text{adm}}(X^*) \) of \( X^* \) is naturally isomorphic to the geometric log étale fundamental group of \( X^{\log} \) (i.e., \( \ker(\pi_1(X^{\log}) \rightarrow \pi_1(s_X^{\log})) \)). Then the maximal pro-prime-to-\( p \) quotient \( \pi_1^{\text{adm}}(X^*)' \) of \( \pi_1^{\text{adm}}(X^*) \) is isomorphic to the pro-prime-to-\( p \) completion of the following group (cf. [V, Théorème 2.2 (c)])

\[
\langle a_1, \ldots, a_{g,X}, b_1, \ldots, b_{g,X}, c_1, \ldots, c_{n,X} \mid \prod_{i=1}^{g,X} (a_i, b_i) \prod_{j=1}^{n,X} c_j = 1 \rangle.
\]

1.2.3. Let

\[
\pi_1^{\text{adm}}(X^*)^{\text{sol}}, \pi_1^{\text{adm}}(X)^{\text{sol}}, \pi_1^{\text{ét}}(X)^{\text{sol}}, \pi_1^{\text{top}}(\Gamma_X^*)^{\text{sol}}
\]

be the maximal pro-solvable quotients of \( \pi_1^{\text{adm}}(X^*) \), \( \pi_1^{\text{adm}}(X) \), \( \pi_1^{\text{ét}}(X) \), \( \pi_1^{\text{top}}(\Gamma_X^*) \), respectively. Then we obtain the following natural surjective open continuous homomorphisms

\[
\pi_1^{\text{adm}}(X^*)^{\text{sol}} \rightarrow \pi_1^{\text{adm}}(X)^{\text{sol}} \rightarrow \pi_1^{\text{ét}}(X)^{\text{sol}} \rightarrow \pi_1^{\text{top}}(\Gamma_X^*)^{\text{sol}}.
\]

We shall say

\[
\pi_1^{\text{adm}}(X^*)^{\text{sol}}
\]

the solvable admissible fundamental group of \( X^* \). Let \( v \in v(\Gamma_X^*) \). Write \( \pi_1^{\text{adm}}(\tilde{X}_v^*)^{\text{sol}} \) for the solvable admissible fundamental group of the smooth pointed semi-stable curve \( \tilde{X}_v^* \) associated to \( v \). Then the natural (outer) injection \( \pi_1^{\text{adm}}(\tilde{X}_v^*) \hookrightarrow \pi_1^{\text{adm}}(X^*) \) induces an (outer) homomorphism

\[
\pi_1^{\text{adm}}(\tilde{X}_v^*)^{\text{sol}} \rightarrow \pi_1^{\text{adm}}(X^*)^{\text{sol}}.
\]

We see that this homomorphism is an injection. Indeed, let \( \tilde{j}_v^* : \tilde{Y}_v^* \rightarrow \tilde{X}_v^* \) be a Galois admissible covering over \( k \) whose Galois group is an abelian group. Then we see immediately that there exists a Galois admissible covering \( g^* : Z^* \rightarrow X^* \) over \( k \) whose Galois group is a solvable group such that the following is satisfied: let \( Z_v \) be an irreducible component of \( Z^* \) such that \( g(Z_v) = X_v \); then the Galois admissible covering \( \tilde{Z}_v^* \rightarrow \tilde{X}_v^* \) over \( k \) induced by \( g^* \) factors through \( \tilde{j}_v^* \). This means that the homomorphism \( \pi_1^{\text{adm}}(\tilde{X}_v^*)^{\text{sol}} \rightarrow \pi_1^{\text{adm}}(X^*)^{\text{sol}} \) mentioned above is an injection.

1.2.4. In the remainder of the present paper, we shall denote by

\[
\Pi_{X^*}
\]

either \( \pi_1^{\text{adm}}(X^*) \) or \( \pi_1^{\text{adm}}(X^*)^{\text{sol}} \) unless indicated otherwise. If \( \Pi_{X^*} = \pi_1^{\text{adm}}(X^*) \), we denote by

\[
\Pi_{X^*}^{\text{cpt}} \overset{\text{def}}{=} \pi_1^{\text{adm}}(X), \quad \Pi_{X^*}^{\text{ét}} \overset{\text{def}}{=} \pi_1^{\text{ét}}(X), \quad \Pi_{X^*}^{\text{top}} \overset{\text{def}}{=} \pi_1^{\text{top}}(\Gamma_X^*).
\]

If \( \Pi_{X^*} = \pi_1^{\text{adm}}(X^*)^{\text{sol}} \), we denote by

\[
\Pi_{X^*}^{\text{cpt}} \overset{\text{def}}{=} \pi_1^{\text{adm}}(X)^{\text{sol}}, \quad \Pi_{X^*}^{\text{ét}} \overset{\text{def}}{=} \pi_1^{\text{ét}}(X)^{\text{sol}}, \quad \Pi_{X^*}^{\text{top}} \overset{\text{def}}{=} \pi_1^{\text{top}}(\Gamma_X^*)^{\text{sol}}.
\]
1.2.5. Let $H \subseteq \Pi_{X^*}$ be an arbitrary open subgroup. We write $X_H^*$ for the pointed semi-stable curve of type $(d_{X_H^*}, n_{X_H^*})$ over $k$ corresponding to $H$, $\Gamma_{X_H^*}$ for the dual semi-graph of $X_H^*$, and $r_{X_H^*}$ for the Betti number of $\Gamma_{X_H^*}$. Then we obtain an admissible covering

$$f_H^*: X_H^* \rightarrow X^*$$

over $k$ induced by the natural injection $H \hookrightarrow \Pi_{X^*}$, and obtain a natural map of dual semi-graphs

$$f_H^{\text{sg}}: \Gamma_{X_H^*} \rightarrow \Gamma_{X^*}$$

induced by $f_H^*$, where “$\text{sg}$” means “semi-graph”. Moreover, if $H$ is an open normal subgroup, then $\Gamma_{X_H^*}$ admits an action of $\Pi_{X^*}/H$ induced by the natural action of $\Pi_{X^*}/H$ on $X_H^*$. Note that the quotient of $\Gamma_{X_H^*}$ by $\Pi_{X^*}/H$ coincides with $\Gamma_{X^*}$, and that $H$ is isomorphic to the admissible fundamental group (resp. solvable admissible fundamental group) $\Pi_{X_H^*}$ of $X_H^*$ if $\Pi_{X^*} = \pi_1^{\text{adm}}(X^*)$ (resp. $\Pi_{X^*} = \pi_1^{\text{adm}}(X^*)^{\text{sol}}$). We also use the notation

$$H^{\text{opt}}, H^{\text{ét}}, H^{\text{top}}$$

to denote $\Pi_{X_H^*}^{\text{opt}}, \Pi_{X_H^*}^{\text{ét}},$ and $\Pi_{X_H^*}^{\text{top}}$, respectively.

1.2.6. We put

$$\widehat{\Pi} \triangleq \lim_{H \subseteq \Pi_{X^*} \text{ open}} X_H, \quad D_{\widehat{\Pi}} \triangleq \lim_{H \subseteq \Pi_{X^*} \text{ open}} D_{X_H}, \quad \Gamma_{\widehat{\Pi}} \triangleq \lim_{H \subseteq \Pi_{X^*} \text{ open}} \Gamma_{X_H^*}.$$ We shall say that

$$\widehat{X}^* = (\widehat{\Pi}, D_{\widehat{\Pi}})$$

is the universal admissible covering (resp. universal solvable admissible covering) of $X^*$ corresponding to $\Pi_{X^*}$ if $\Pi_{X^*} = \pi_1^{\text{adm}}(X^*)$ (resp. $\Pi_{X^*} = \pi_1^{\text{adm}}(X^*)^{\text{sol}}$), and that $\Gamma_{\widehat{\Pi}}$ is the dual semi-graph of $\widehat{X}^*$. Note that we have that $\text{Aut}(\widehat{\Pi}/X^*) = \Pi_{X^*}$, and that $\Gamma_{\widehat{\Pi}}$ admits a natural action of $\Pi_{X^*}$.

1.2.7. Let $v \in \nu(\Gamma_{X^*})$, $e \in e^{\text{op}}(\Gamma_{X^*}) \cup e^{\text{cl}}(\Gamma_{X^*})$, $\widehat{v} \in \nu(\Gamma_{\widehat{\Pi}})$ a vertex over $v$, and $\widehat{e} \in e^{\text{op}}(\Gamma_{\widehat{\Pi}}) \cup e^{\text{cl}}(\Gamma_{\widehat{\Pi}})$ an edge over $e$. We denote by

$$\Pi_{\widehat{e}} \subseteq \Pi_{X^*}, \quad I_{\widehat{e}} \subseteq \Pi_{X^*}$$

the stabilizer subgroups of $\widehat{v}$ and $\widehat{e}$, respectively. We see immediately that $\Pi_{\widehat{e}}$ is (outer) isomorphic to $\Pi_{\widehat{X}^*}$ of $\widehat{X}^*$, and that $I_{\widehat{e}}$ is (outer) isomorphic to an inertia subgroup associated to the closed point of $X$ corresponding to $e$. Then we have that $I_{\widehat{e}} \cong \widehat{\Pi}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal pro-prime-to-$p$ quotient of $(-)$. We put

$$\text{Ver}(\Pi_{X^*}) \triangleq \{ \Pi_{\widehat{e}} \}_{\widehat{e} \in \nu(\Gamma_{\widehat{X}^*})},$$

$$\text{Edg}^{\text{op}}(\Pi_{X^*}) \triangleq \{ I_{\widehat{e}} \}_{\widehat{e} \in e^{\text{op}}(\Gamma_{\widehat{X}^*})},$$

$$\text{Edg}^{\text{cl}}(\Pi_{X^*}) \triangleq \{ I_{\widehat{e}} \}_{\widehat{e} \in e^{\text{cl}}(\Gamma_{\widehat{X}^*})}.$$ Moreover, if $\widehat{e}$ abuts on $\widehat{v}$, then we have the following injections

$$I_{\widehat{e}} \hookleftarrow \Pi_{\widehat{e}} \hookleftarrow \Pi_{X^*}.$$ Note that $\text{Ver}(\Pi_{X^*}), \text{Edg}^{\text{op}}(\Pi_{X^*})$, and $\text{Edg}^{\text{cl}}(\Pi_{X^*})$ admit natural actions of $\Pi_{X^*}$ (i.e., the conjugacy actions), and that we have the following natural bijections

$$\text{Ver}(\Pi_{X^*})/\Pi_{X^*} \cong \nu(\Gamma_{X^*})$$
2. Maximum and averages of generalized Hasse-Witt invariants

In this section, we recall some results concerning Hasse-Witt invariants (or p-rank) and generalized Hasse-Witt invariants.


Definition 2.1. Let \( Z^* \) be a disjoint union of finitely many pointed semi-stable curves over \( k \). We define the p-rank (or Hasse-Witt invariant) of \( Z^* \) to be

\[
\sigma_Z \overset{\text{def}}{=} \dim_k(H^1_{\text{et}}(Z, \mathbb{F}_p)).
\]

In particular, if \( Z^* \) is a pointed semi-stable curve, then

\[
\sigma_Z = \dim_{\mathbb{F}_p}(\Pi_Z^{\text{ab}} \otimes \mathbb{F}_p),
\]

where \( \Pi_Z \) is either the admissible fundamental group or the solvable admissible fundamental group of \( Z^* \), and \((-)^{\text{ab}}\) denotes the abelianization of \((-)\).

2.1.1. Let \( X^* \) be a pointed stable curve of type \((g_X, n_X)\) over an algebraically closed field \( k \) of characteristic \( p > 0 \), \( \Gamma_X^* \), the dual semi-graph of \( X^* \), and \( \Pi_X^* \) either the admissible fundamental group or the solvable admissible fundamental group of \( X^* \). Let \( n \) be an arbitrary positive natural number prime to \( p \) and \( \mu_n \subseteq k^* \) the group of \( n \)th roots of unity. Fix a primitive \( n \)th root \( \zeta_n \), we may identify \( \frac{1}{n} \mathbb{Z} \) with \( \mathbb{Z} \) via the map \( i \). \( \zeta_n \). We have the following canonical decomposition

\[
H^1(X, \mathcal{O}_X^*) = H^1(X, \mathcal{O}_X)^{\text{st}} \oplus H^1(X, \mathcal{O}_X)^{\text{ni}},
\]

where \( F_{X_n} \) is a bijection on \( H^1(X, \mathcal{O}_X)^{\text{st}} \) and is nilpotent on \( H^1(X, \mathcal{O}_X)^{\text{ni}} \). Moreover, we have

\[
H^1(X, \mathcal{O}_X)^{\text{st}} = H^1(X, \mathcal{O}_X)/F_{X_n} \otimes_{\mathbb{F}_p} k,
\]

where \((-)^{F_{X_n}}\) denotes the subspace of \((-)\) on which \( F_{X_n} \) acts trivially. Then Artin-Schreier theory implies that we may identify

\[
H_\alpha \overset{\text{def}}{=} H^1_{\text{et}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k
\]

with the largest subspace of \( H^1(X, \mathcal{O}_X) \) on which \( F_{X_n} \) is a bijection.

The finite dimensional \( k \)-linear spaces \( H_\alpha \) is a finitely generated \( k[\mu_n] \)-module induced by the natural action of \( \mu_n \) on \( X_\alpha \). We have the following canonical decomposition

\[
H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha, i},
\]

where \( \zeta_n \in \mu_n \) acts on \( H_{\alpha, i} \) as the \( \zeta_n^i \)-multiplication. We define

\[
\gamma_{\alpha, i} \overset{\text{def}}{=} \dim_k(H_{\alpha, i}), \quad i \in \mathbb{Z}/n\mathbb{Z}.
\]
We shall say that \( \gamma_{\alpha,i}, i \in \mathbb{Z}/n\mathbb{Z} \), is a generalized Hasse-Witt invariant (cf. [Nakaj] for the case of smooth projective curves) of the cyclic multi-admissible covering \( X_{\alpha} \rightarrow X^\bullet \). Note that the decomposition above implies that

\[
\sigma_{X\alpha} = \dim_k(H_n) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.
\]

2.1.3. Let \( t \in \mathbb{N} \) be an arbitrary positive natural number, \( K_{p^t-1} \) the kernel of the natural surjection

\[
\Pi_X^\bullet \twoheadrightarrow \Pi_X^{ab} \otimes \mathbb{Z}/(p^t - 1)\mathbb{Z},
\]

and \( X_{K_{p^t-1}}^\bullet \) the pointed stable curve over \( k \) determined by \( K_{p^t-1} \). Next, we define two important invariants associated to \( X^\bullet \).

We put

\[
\max_{n \in \mathbb{N}} \text{s.t. } (n,p) = 1 \{ \gamma_{\alpha,i} \mid \alpha \in \text{Hom}(\Pi_X^{ab}, \mathbb{Z}/n\mathbb{Z}), \alpha \neq 0, i \in (\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}\},
\]

and shall say that \( \gamma_{\max}(X^\bullet) \) is the maximum generalized Hasse-Witt invariant of prime-to-\( p \) cyclic admissible coverings of \( X^\bullet \).

We put

\[
\text{Avr}_p(X^\bullet) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{\sigma_{X_{K_{p^t-1}}}}{\#((\Pi_X^{ab})^\bullet \otimes \mathbb{Z}/(p^t - 1)\mathbb{Z})},
\]

and shall say that \( \text{Avr}_p(X^\bullet) \) is the limit of \( p \)-averages of \( X^\bullet \).

2.2. Group-theoretical formulas for maximum and \( p \)-averages of generalized Hasse-Witt invariants.

2.2.1. Let \( \overline{F}_p \) be an arbitrary algebraic closure of the finite field \( F_p \), \( \chi \in \text{Hom}(\Pi_X^\bullet, \overline{F}_p^\times) \) such that \( \chi \neq 1 \), and \( \Pi_{\chi} \subseteq \Pi_X^\bullet \) the kernel of \( \chi \). The profinite group \( \Pi_{\chi} \) admits a natural action of \( \Pi_X^\bullet \) via conjugation. We put

\[
\text{Hom}(\Pi_X^\bullet, \mathbb{Z}/p\mathbb{Z})[\chi] = \{ a \in \text{Hom}(\Pi_X^\bullet, \mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{F}_p \mid \tau(a) = \chi(\tau)a \text{ for all } \tau \in \Pi_X^\bullet \},
\]

\[
\gamma_{\chi}(\text{Hom}(\Pi_X^\bullet, \mathbb{Z}/p\mathbb{Z})) \overset{\text{def}}{=} \text{dim}_{\mathbb{F}_p} \text{Hom}(\Pi_X^\bullet, \mathbb{Z}/p\mathbb{Z})[\chi]).
\]

We define the following two group-theoretical invariants associated to \( \Pi_X^\bullet \):

\[
\gamma_{\max}(\Pi_X^\bullet) \overset{\text{def}}{=} \max \{ \gamma_{\chi}(\text{Hom}(\Pi_X^\bullet, \mathbb{Z}/p\mathbb{Z})) \mid \chi \in \text{Hom}(\Pi_X^\bullet, \overline{F}_p^\times) \text{ such that } \chi \neq 1 \},
\]

\[
\text{Avr}_p(\Pi_X^\bullet) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{\text{dim}_{\mathbb{F}_p}(K_{p^t-1}^{ab} \otimes \mathbb{F}_p)}{\#((\Pi_X^\bullet)^{ab} \otimes \mathbb{Z}/(p^t - 1)\mathbb{Z})}.
\]

We see immediately that

\[
\gamma_{\max}(\Pi_X^\bullet) = \gamma_{\max}(X^\bullet),
\]

\[
\text{Avr}_p(\Pi_X^\bullet) = \text{Avr}_p(X^\bullet).
\]
Moreover, we have the following highly non-trivial formulas for $\gamma^{\max}(\Pi_X)$ and $\text{Avr}_p(\Pi_X)$ which were proved by Tamagawa and the author by using the theory of Raynaud-Tamagawa theta divisors (cf. [Y6, Theorem 4.6] for $\gamma^{\max}(\Pi_X)$, [T4, Theorem 0.5] for $\text{Avr}_p(\Pi_X)$ when $X$ is a smooth pointed stable curve over $k$, and [Y5, Theorem 5.2, Remark 5.2.1, and Remark 5.2.2] for $\text{Avr}_p(\Pi_X)$ when $X$ is an arbitrary pointed stable curve over $k$).

**Theorem 2.2.** We maintain the notation introduced above.

(a) We have

$$\gamma^{\max}(\Pi_X) = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

(b) Suppose that $\Gamma_X$ is 2-connected. Then we have

$$\text{Avr}_p(\Pi_X) = g_X - r_X - \#v(\Gamma_X)^{b \leq 1} + \#e^{cl}(\Gamma_X)^{b \leq 1}.$$  

**Remark 2.2.1.**

Suppose that $\Gamma_X$ is 2-connected. Note that $\#v(\Gamma_X)^{b \leq 1} \neq 0$ if one of the following conditions holds: (i) $X$ is smooth over $k$ and $\#e^{op}(\Gamma_X) \leq 1$; (ii) $\#v(\Gamma_X) = 2$, $\#e^{op}(\Gamma_X) = 0$, and $r_X = 0$. In the case (i), we have $\text{Avr}_p(\Pi_X) = g_X - 1$. In the case (ii), we have $\text{Avr}_p(\Pi_X) = g_X - 2 + 1 = g_X - 1$.

On the other hand, in the present paper, we will use the formula for $\text{Avr}_p(\Pi_X)$ when $\#v(\Gamma_X)^{b \leq 1} = \#e^{cl}(\Gamma_X)^{b \leq 1} = 0$.

**Lemma 2.3.** Let $X_i$, $i \in \{1, 2\}$, be a pointed stable curve of type $(g_X, n_X)$ over an algebraically closed field $k_i$ of characteristic $p$ and $\Pi_X$ either the admissible fundamental group of $X_i$ or the solvable admissible fundamental group of $X_i$. Let

$$\phi : \Pi_X \to \Pi_X$$

be an arbitrary surjective open continuous homomorphism of profinite groups, $H_2 \subseteq \Pi_X$, an arbitrary open normal subgroup, and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2)$. Then the following statements hold:

(a) We have

$$\gamma^{\max}(H_1) \geq \gamma^{\max}(H_2).$$

(b) Suppose that $(g_X, n_X) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Moreover, suppose either that $G \overset{\text{def}}{=} \Pi_X/H_2$ is a $p$-group, that $(\#G, p) = 1$, or that $G$ is a solvable group. Then we have

$$\text{Avr}_p(H_1) \geq \text{Avr}_p(H_2).$$

**Proof.** (a) Let $n \in \mathbb{Z}_{>0}$ be a positive natural number prime to $p$, and $\alpha_2 \in \text{Hom}(H_2^{ab}, \mathbb{Z}/n\mathbb{Z})$ such that $\alpha_2 \neq 0$. Let $j \in \mathbb{Z}/n\mathbb{Z}$ such that $\gamma_{a_2, j} = \gamma^{\max}(H_2)$. Write $Q_2$ for the kernel of the composition of the following homomorphisms

$$H_2 \to H_2^{ab} \overset{\alpha_2}{\to} \mathbb{Z}/n\mathbb{Z},$$

$$Q_1 \overset{\text{def}}{=} \phi^{-1}(Q_2),$$

and $\alpha_1 \in \text{Hom}(H_1^{ab}, \mathbb{Z}/n\mathbb{Z})$ for the homomorphism induced by $\phi|_{H_1}$ and $\alpha_2$. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Then $Q_1^{ab} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ admits a natural $\mathbb{F}_p[\mathbb{Z}/n\mathbb{Z}]$-module structure. Moreover, we see immediately that $\phi|_{H_1}$ induces a surjective homomorphism of $\mathbb{F}_p[\mathbb{Z}/n\mathbb{Z}]$-modules

$$Q_1^{ab} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \to Q_2^{ab} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p.$$
Then we obtain that \( \gamma_{\alpha_{1,j}} \geq \gamma_{\alpha_{2,j}} \). Thus, we have
\[
\gamma_{\alpha_{1}}^\text{max}(H_1) \geq \gamma_{\alpha_{2}}^\text{max}(H_2).
\]

(b) Let \( t \in \mathbb{N} \) be an arbitrary positive natural number, \( K_{H_i,p}^{-1} \) the kernel of the natural surjection
\[
H_i \twoheadrightarrow H_i^{ab} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}.
\]
Suppose that \( G \) is a \( p \)-group. We have that Galois admissible covering \( X_{H_i}^* \to X_i^* \) corresponding to \( H_i \) is étale. This implies that \( X_{H_1}^* \) and \( X_{H_2}^* \) are equal types. We obtain that
\[
\#(H_1^{ab} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}) = \#(H_2^{ab} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}).
\]
Suppose that \((\#G, p) = 1\). Since \( X_1^* \) and \( X_2^* \) are equal types, \( H_1^{p'} \) is isomorphic to \( H_2^{p'} \). We have that
\[
\#(H_1^{ab} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}) = \#(H_2^{ab} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}).
\]
Then \( \phi|_{H_i} \) implies that
\[
\text{Avr}_p(H_1) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{\dim_{\mathbb{F}_p}(K_{H_1,p}^{ab} \otimes \mathbb{F}_p)}{\#(H_1^{ab} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})} \geq \text{Avr}_p(H_2) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{\dim_{\mathbb{F}_p}(K_{H_2,p}^{ab} \otimes \mathbb{F}_p)}{\#(H_2^{ab} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})}.
\]
Suppose that \( G \) is solvable. Then the lemma follows immediately from the lemma if either \( G \) is a \( p \)-group, or \((\#G, p) = 1\). This completes the proof of the lemma. \( \square \)

3. Moduli spaces of admissible fundamental groups and the Homeomorphism Conjecture

In this section, we define the moduli spaces of fundamental groups and formulate the Homeomorphism Conjecture, which are main research objects of the series of papers.

3.1. The Weak Isom-version Conjecture. Let \( p \) be a prime number, \( \mathbb{F}_p \) the prime field of characteristic \( p \), and \( \overline{\mathbb{F}}_p \) an algebraically closed field of \( \mathbb{F}_p \). Let \( \overline{\mathcal{M}}_{g,n} \) be the moduli stack over \( \overline{\mathbb{F}}_p \) classifying pointed stable curves of type \((g, n)\) and \( \mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n} \) the open substack classifying smooth pointed stable curves. Let \( \overline{\mathcal{M}}_{g,n} \) and \( \mathcal{M}_{g,n} \) be the coarse moduli spaces of \( \overline{\mathcal{M}}_{g,n} \) and \( \mathcal{M}_{g,n} \), respectively.

3.1.1. Let \( q \in \overline{\mathcal{M}}_{g,n} \) be an arbitrary point, \( k(q) \) the residue field of \( \overline{\mathcal{M}}_{g,n} \), and \( k_q \) an algebraically closed field which contains \( k(q) \). Then the composition of natural morphisms \( \text{Spec} \ k_q \to \text{Spec} \ k(q) \to \overline{\mathcal{M}}_{g,n} \) defines a pointed stable curve \( X_{k_q}^* \) of type \((g, n)\) over \( k_q \). Write \( \pi_1^{\text{adm}}(X_{k_q}^*) \) for the admissible fundamental group \( X_{k_q}^* \) and \( \pi_1^{\text{adm}}(X_{k_q}^*)^{\text{sol}} \) for the solvable admissible fundamental group of \( X_{k_q}^* \). Since the isomorphism classes of \( \pi_1^{\text{adm}}(X_{k_q}^*) \) and \( \pi_1^{\text{adm}}(X_{k_q}^*)^{\text{sol}} \) do not depend on the choice of \( k_q \), we shall denote by
\[
\pi_1^{\text{adm}}(q), \ \pi_1^{\text{sol}}(q)
\]
the admissible fundamental groups \( \pi_1^{\text{adm}}(X_{K(q)}^*) \) and the solvable admissible fundamental group \( \pi_1^{\text{adm}}(X_{k_q}^*)^{\text{sol}} \), respectively. Moreover, we shall denote by
\[
X_{q}^*
\]
the pointed stable curve \( X_{k(q)}^* \) and \( \Gamma_q \) the dual semi-graph of \( X_{q}^* \), where \( \overline{k(q)} \) is an algebraic closure of \( k(q) \). Let \( v \in v(\Gamma_q) \). Then the smooth pointed stable curve \( \tilde{X}_{q,v}^* \) of type \((g_v, n_v)\)
associated to $v$ determines a morphism $\text{Spec} \bar{k}(q) \to M_{g_v,n_v}$. We shall write $q_v \in M_{g_v,n_v}$ for the image of the morphism and say $q_v$ the point of type $(g_v,n_v)$ associated to $v$.

3.1.2. We introduce an equivalent relation on the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$.

**Definition 3.1.** (a) Let $q_i \in M_{g_i,n_i}$, $i \in \{1,2\}$, be an arbitrary point. We shall say that $q_1$ is Frobenius equivalent to $q_2$ if $X_{q_1} \setminus D_{q_1}$ is isomorphic to $X_{q_2} \setminus D_{q_2}$ as schemes.

(b) Let $q_i \in \overline{M}_{g_i,n_i}$, $i \in \{1,2\}$, be an arbitrary point. We shall say that $q_1$ is Frobenius equivalent to $q_2$ if the following conditions are satisfied: (i) There exists an isomorphism $\rho : \Gamma_{q_1} \sim \Gamma_{q_2}$ of dual semi-graphs. (ii) Let $v_1 \in v(\Gamma_{q_1})$, $v_2 \overset{\text{def}}{=} \rho(v_1) \in v(\Gamma_{q_2})$, $q_{1,v_1}$ the point of type $(g_{v_1},n_{v_1})$ associated to $v_1$, and $q_{2,v_2}$ the point of type $(g_{v_2},n_{v_2})$ associated to $v_2$. We have that $q_{1,v_1}$ is Frobenius equivalent to $q_{2,v_2}$. (iii) Let $\rho_{v_1,v_2} : \Gamma_{q_{1,v_1}} \sim \Gamma_{q_{2,v_2}}$ be the isomorphism of dual semi-graphs induced by $\rho$. There exists a morphism $f_{v_1,v_2}^* : X_{q_{1,v_1}}^* \sim X_{q_{2,v_2}}^*$ such that the morphism $X_{q_{1,v_1}} \setminus D_{X_{q_{1,v_1}}} \to X_{q_{2,v_2}} \setminus D_{X_{q_{2,v_2}}}$ induced by $f_{v_1,v_2}^*$ is an isomorphism as schemes, and that the isomorphism of dual semi-graphs $f_{v_1,v_2}^* : \Gamma_{q_{1,v_1}} \sim \Gamma_{q_{2,v_2}}$ induced by $f_{v_1,v_2}^*$ coincides with $\rho_{v_1,v_2}$.

We shall denote by

$q_1 \sim_{f_e} q_2$

if $q_1$ is Frobenius equivalent to $q_2$. Note that $\sim_{f_e}$ is an equivalence relation on the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$.

(c) Let $q_i \in \overline{M}_{g_i,n_i}$, $i \in \{1,2\}$, be an arbitrary point, $k_i$ an algebraically closed field which contains $k(q_i)$, and $X_{q_i}^*$ the pointed stable curve of type $(g,n)$ over $k_i$. We shall say that $X_{q_i}^*$ is Frobenius equivalent to $X_{q_2}^*$ if $q_1$ is Frobenius equivalent to $q_2$.

The following result was proved by the author.

**Proposition 3.2.** Let $q_i \in \overline{M}_{g_i,n_i}$, $i \in \{1,2\}$, be an arbitrary point. Suppose that $q_1 \sim_{f_e} q_2$. Then we have that $\pi_1^\text{adm}(q_1)$ is isomorphic to $\pi_1^\text{adm}(q_2)$ as profinite groups. In particular, we have that $\pi_1^\text{sol}(q_1)$ is isomorphic to $\pi_1^\text{sol}(q_2)$ as profinite groups.

**Proof.** See [Y7, Proposition 3.7].

3.1.3. We put

$\mathcal{M}_{g,n} \overset{\text{def}}{=} M_{g,n}/\sim_{f_e} \subseteq \overline{M}_{g,n} \overset{\text{def}}{=} |\overline{M}_{g,n}|/\sim_{f_e}$,

$\Pi_{g,n} \overset{\text{def}}{=} \{ [\pi_1^\text{adm}(q)] \mid q \in M_{g,n} \} \subseteq \overline{\Pi}_{g,n} \overset{\text{def}}{=} \{ [\pi_1^\text{adm}(q)] \mid q \in \overline{M}_{g,n} \}$,

$\Pi_{g,n}^\text{sol} \overset{\text{def}}{=} \{ [\pi_1^\text{sol}(q)] \mid q \in M_{g,n} \} \subseteq \overline{\Pi}_{g,n}^\text{sol} \overset{\text{def}}{=} \{ [\pi_1^\text{sol}(q)] \mid q \in \overline{M}_{g,n} \}$,

where $[\pi_1^\text{adm}(q)]$ and $[\pi_1^\text{sol}(q)]$ denote the isomorphism classes (as profinite groups) of $\pi_1^\text{adm}(q)$ and $\pi_1^\text{sol}(q)$, respectively. Let $q \in \overline{M}_{g,n}$. We shall write $[q]$ for the image of $q$ in $\overline{\mathcal{M}}_{g,n}$. Then there are natural surjective maps of sets as follows:

$\text{sol} : \overline{\Pi}_{g,n} \to \overline{\Pi}_{g,n}^\text{sol}$, $[\pi_1^\text{adm}(q)] \mapsto [\pi_1^\text{sol}(q)]$, $\pi_{g,n}^\text{adm} : \overline{\mathcal{M}}_{g,n} \to \overline{\Pi}_{g,n}$, $[q] \mapsto [\pi_1^\text{adm}(q)]$, $\pi_{g,n}^\text{sol} \overset{\text{def}}{=} \text{sol} \circ \pi_{g,n}^\text{adm} : \overline{\mathcal{M}}_{g,n} \to \overline{\Pi}_{g,n}^\text{sol}$, $\pi_{g,n}^\text{adj} \overset{\text{def}}{=} \pi_{g,n}^\text{adm} \circ \overline{\mathcal{M}}_{g,n} \to \Pi_{g,n}$. 


where “t” means “tame”. Moreover, we have the following commutative diagrams:

\[
\begin{array}{c}
\mathcal{M}_{g,n} \xrightarrow{\pi_{g,n}^t} \Pi_{g,n} \\
\downarrow \hspace{1cm} \downarrow \\
\overline{\mathcal{M}}_{g,n} \xrightarrow{\pi_{g,n}^{adm}} \Pi_{g,n}
\end{array}
\]

and

\[
\begin{array}{c}
\mathcal{M}_{g,n} \xrightarrow{\pi_{g,n}^{sol}} \Pi_{g,n} \\
\downarrow \hspace{1cm} \downarrow \\
\overline{\mathcal{M}}_{g,n} \xrightarrow{\pi_{g,n}^{sol}} \Pi_{g,n}^\text{sol}
\end{array}
\]

where all vertical arrows are natural injections.

**Proposition 3.3.** We maintain the notation introduced above. Then we have

\[
\pi_{g,n}^{adm} : (\mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n}) \to \Pi_{g,n} \setminus \Pi_{g,n},
\]

\[
\pi_{g,n}^{sol} : (\mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n}) \to \Pi_{g,n}^\text{sol} \setminus \Pi_{g,n}^\text{sol}.
\]

**Proof.** The proposition follows immediately from [Y2, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1] (see also Theorem 4.2 of the present paper). \(\square\)

3.1.4. We may formulate a moduli version of the weak Isom-version of the Grothendieck conjecture for pointed stable curves over algebraically closed fields of characteristic \(p > 0\) (=the Weak Isom-version Conjecture) as follows:

**Weak Isom-version Conjecture.** We maintain the notation introduced above. Then we have that

\[
\pi_{g,n}^{adm} : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\Pi}_{g,n}
\]

is a bijection as sets.

The Weak Isom-version Conjecture was formulated by Tamagawa in the case of \(q \in M_{g,n}\), and by the author in the general case (cf. [T3], [Y7]), which is the ultimate goal of [PS], [R], [T2], [T4], [T5], and [Y2]. The Weak Isom-version Conjecture shows that moduli spaces of curves over algebraically closed fields of characteristic \(p\) can be reconstructed group-theoretically as “sets” from admissible fundamental groups of curves. Moreover, we have the following solvable version of the Weak Isom-version Conjecture which is slightly stronger than the original version.

**Solvable Weak Isom-version Conjecture.** We maintain the notation introduced above. Then we have that

\[
\pi_{g,n}^{sol} : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\Pi}_{g,n}^\text{sol}
\]

is a bijection as sets.
3.1.5. Write $\overline{\mathcal{M}}_{g,n}$ for the image of the set of closed points $\mathcal{M}_{g,n}$ of the natural map $|\mathcal{M}_{g,n}| \to \overline{\mathcal{M}}_{g,n}$. Then we have the following result.

**Theorem 3.4.** We maintain the notation introduced above. Then the following statements hold:

(a) We have that
\[ \pi_{g,n}^\text{sol} : \overline{\mathcal{M}}_{g,n} \to \Pi_{g,n}^{\text{sol}} \]

is quasi-finite (i.e., $\#(\pi_{g,n}^\text{sol}|_{\overline{\mathcal{M}}_{g,n}})^{-1}([\pi_1^\text{sol}(q)]) < \infty$ for every $[\pi_1^\text{sol}(q)] \in \Pi_{g,n}^{\text{sol}}$).

(b) Suppose that $g = 0$. Then we have that
\[ \pi_{g,n}^\text{sol} : \overline{\mathcal{M}}_{g,n} \to \Pi_{g,n}^{\text{sol}} \]

is an injection, and that
\[ \pi_{g,n}^\text{sol}(\overline{\mathcal{M}}_{g,n} \setminus \overline{\mathcal{M}}_{g,n}^\text{cl}) \subseteq \Pi_{g,n}^{\text{sol}} \setminus \pi_{g,n}^\text{sol}(\overline{\mathcal{M}}_{g,n}^\text{cl}). \]

In particular, the Solvable Weak Isomorphism Conjecture holds if $(g, n) = (0, 4)$.

**Proof.** Since [T4, Theorem 0.2] and [T5, Theorem 0.1] also hold for the maximal pro-solvable quotients of tame fundamental groups, the theorem follows immediately from [T4, Theorem 0.2], [T5, Theorem 0.1], [Y2, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1], and Proposition 3.3. \(\square\)

3.2. **Moduli spaces of admissible fundamental groups.** We maintain the notation introduced in 3.1. Moreover, we regard $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$ as topological spaces whose topologies are induced by the Zariski topologies of $|\mathcal{M}_{g,n}|$ and $|M_{g,n}|$, respectively.

3.2.1. Let $\mathcal{G}$ be the category of finite groups, $G \in \mathcal{G}$ an arbitrary finite group, $\Pi$ an arbitrary profinite group, and $\text{Hom}_{\text{surj}}(-, -)$ the set of surjective homomorphisms. We put
\[ U_{\Pi,n,G} \overset{\text{def}}{=} \{ [\pi_1^\text{adm}(q)] \in \Pi_{g,n} | \text{Hom}_{\text{surj}}(\pi_1^\text{adm}(q), G) \neq \emptyset \}, \]
\[ U_{\Pi,n,G}^{\text{sol}} \overset{\text{def}}{=} \{ [\pi_1^\text{adm}(q)] \in \Pi_{g,n} | \text{Hom}_{\text{surj}}(\pi_1^\text{adm}(q), G) \neq \emptyset \}, \]
\[ U_{\Pi,n,G}^{\text{sol}} \overset{\text{def}}{=} \{ [\pi_1^\text{sol}(q)] \in \Pi_{g,n}^{\text{sol}} | \text{Hom}_{\text{surj}}(\pi_1^\text{sol}(q), G) \neq \emptyset \}, \]
\[ U_{\Pi,n,G}^{\text{sol}} \overset{\text{def}}{=} \{ [\pi_1^\text{sol}(q)] \in \Pi_{g,n}^{\text{sol}} | \text{Hom}_{\text{surj}}(\pi_1^\text{sol}(q), G) \neq \emptyset \}. \]

Then we obtain the following topological spaces
\[ (\Pi_{g,n}, O_{\Pi_{g,n}}), (\Pi_{g,n}, O_{\Pi_{g,n}}), (\Pi_{g,n}^{\text{sol}}, O_{\Pi_{g,n}^{\text{sol}}}), (\Pi_{g,n}^{\text{sol}}, O_{\Pi_{g,n}^{\text{sol}}}) \]
whose topologies $O_{\Pi_{g,n}}$, $O_{\Pi_{g,n}}$, $O_{\Pi_{g,n}^{\text{sol}}}$, and $O_{\Pi_{g,n}^{\text{sol}}}$ are generated by $\{ U_{\Pi,n,G} \}_G \in \mathcal{G}$, $\{ U_{\Pi,n,G} \}_G \in \mathcal{G}$, $\{ U_{\Pi,n,G}^{\text{sol}} \}_G \in \mathcal{G}$, and $\{ U_{\Pi,n,G}^{\text{sol}} \}_G \in \mathcal{G}$ as open subsets, respectively. For simplicity, we still use the notation
\[ \Pi_{g,n}, \Pi_{g,n}^{\text{sol}}, \Pi_{g,n}^{\text{sol}}, \Pi_{g,n}^{\text{sol}}, \Pi_{g,n}^{\text{sol}} \]
to denote the topological spaces $(\Pi_{g,n}, O_{\Pi_{g,n}}), (\Pi_{g,n}, O_{\Pi_{g,n}}), (\Pi_{g,n}^{\text{sol}}, O_{\Pi_{g,n}^{\text{sol}}}),$ and $(\Pi_{g,n}^{\text{sol}}, O_{\Pi_{g,n}^{\text{sol}}})$, respectively.
Definition 3.5. We shall say that

$$\Pi_{g,n}, \text{ (resp. } \Pi_{g,n}^{\text{sol}})$$

is the moduli space of admissible fundamental groups of pointed stable curves (resp. solvable admissible fundamental groups) of type \((g, n)\) over algebraically closed fields of characteristic \(p\), or the moduli space of admissible fundamental groups (resp. solvable admissible fundamental groups) of type \((g, n)\) in characteristic \(p\) for short.

3.2.2. Let \(\mathcal{M}_{g,n,\log}^{\text{log}}\) be the log stack obtained by equipping \(\mathcal{M}_{g,n}\) with the natural log structure associated to the divisor with normal crossings \(\mathcal{M}_{g,n} \setminus \mathcal{M}_{g,n}\) relative to Spec \(\overline{\mathbb{F}}_p\). Let \(\mathcal{A}_d\) be the stack over Spec \(\overline{\mathbb{F}}_p\) defined as follows: For a scheme \(S\), the objects of \(\mathcal{A}_d(S)\) are HM-admissible coverings (cf. \([M1, \S 3.9 \text{ Definition}]\)) \(C^* \rightarrow D^*\) over \(S\) of degree \(d\) (note that if \(S\) is an algebraically closed field of characteristic \(p\), then HM-admissible coverings are equivalent to admissible coverings defined in Definition 1.2), where \(C^*\) is a pointed stable curve over \(S\), and \(D^*\) is a pointed stable curve of type \((g, n)\) over \(S\). By \([M1, \S 3.11 \text{ Proposition and } \S 3.22 \text{ Theorem}]\), the stack \(\mathcal{A}_d\) is a separated Deligne-Mumford stack of finite type over Spec \(\overline{\mathbb{F}}_p\). Moreover, \(\mathcal{A}_d\) is equipped with a canonical log structure \(\mathcal{M}_{\mathcal{A}_d} \rightarrow \mathcal{O}_{\mathcal{A}_d}\), together with a logarithmic morphism \(\mathcal{A}_d^{\text{log}} \overset{\text{def}}{=} (\mathcal{A}_d, \mathcal{M}_{\mathcal{A}_d}) \rightarrow \mathcal{M}_{g,n}^{\text{log}}\) (obtained by mapping \(C^* \rightarrow D^* \rightarrow D^*)\) which is log étale (not necessary proper).

Let \(G\) be an arbitrary finite group. For any HM-admissible covering \(C^* \rightarrow D^*\) over \(S\), \([M1, \S 3.10 \text{ and } \S 3.11]\) imply that \(C^* \rightarrow D^*\) can be extended to a log admissible covering \(C_{\log}^{\text{log}} \rightarrow D_{\log}^{\text{log}}\) over \(S_{\log}^{\text{log}}\) (cf. \([M1, \S 3.5 \text{ Definition}]\)). Since log admissible coverings are finite Kummer log étale coverings, we shall say \(C^* \rightarrow D^*\) over \(S\) a Galois HM-admissible covering with Galois group \(G\) if \(C_{\log}^{\text{log}} \rightarrow D_{\log}^{\text{log}}\) over \(S_{\log}^{\text{log}}\) is a Galois Kummer log étale covering with Galois group \(G\). Let \(\mathcal{A}_G\) be the substack of \(\mathcal{A}_{\#(G)}\) classifying Galois HM-admissible coverings with Galois group \(G\) which is a union of some connected components of \(\mathcal{A}_{\#(G)}\), and which is a separated Deligne-Mumford stack of finite type over Spec \(\overline{\mathbb{F}}_p\). Note that \(\mathcal{A}_G\) may be empty. Moreover, we shall denote by

\(\mathcal{A}_G^{\text{log}}\)

the log stack whose underlying stack is \(\mathcal{A}_G\), and whose log structure is the pulling-back log structure induced by \(\mathcal{A}_G \hookrightarrow \mathcal{A}_{\#(G)}\). Furthermore, we have a logarithmic morphism

\(\mathcal{A}_G^{\text{log}} \rightarrow \mathcal{M}_{g,n}^{\text{log}}\)

which is log étale (not necessary proper).

Theorem 3.6. We maintain the notation introduced above. Then we have that

\[ \pi_{g,n}^{\text{adm}} : \mathcal{M}_{g,n} \rightarrow \Pi_{g,n}, \]
\[ \pi_{g,n}^{\text{sol}} : \mathcal{M}_{g,n} \rightarrow \Pi_{g,n}^{\text{sol}} \]

are continuous maps.

Proof. We only need to treat the case \(\pi_{g,n}^{\text{adm}} : \mathcal{M}_{g,n} \rightarrow \Pi_{g,n}\). To verify the theorem, it is sufficient to prove that the composition of the natural maps

\[ \mathcal{M}_{g,n} \rightarrow \mathcal{M}_{g,n}^{\text{log}} \overset{\pi_{g,n}^{\text{adm}}}{\rightarrow} \Pi_{g,n}^{\text{log}} \]

is continuous.

Let $G$ be an arbitrary finite group. If $U_{\Pi_{g,n},G} = \emptyset$, the theorem is trivial. We may assume that $U_{\Pi_{g,n},G} \neq \emptyset$. Let $q \in \overline{M}_{g,n}$ such that $[\pi_{1,\text{adm}}(q)] \in U_{\Pi_{g,n},G}$, $\overline{k(q)}$ an algebraic closure of $k(q)$,

$$f_q^* : Y_q^* \to X_q^*$$

a Galois admissible covering over $\overline{k(q)}$ with Galois group $G$. Then we obtain a morphism

$$[f_q^*] : \text{Spec } \overline{k(q)} \to \mathcal{A}_G$$
determined by $f_q^*$. Let $U \to \mathcal{A}_G$ be an étale atlas. Then the morphism $\text{Spec } \overline{k(q)} \to \mathcal{A}_G$ factors through a morphism $\text{Spec } \overline{k(q)} \to U$. Write $q_U \in U$ for the image of the morphism $\text{Spec } \overline{k(q)} \to U$. Let $q'_U \in U$ be a closed point (i.e., an $\overline{\mathbb{F}}_p$-rational point) contained in the topological closure of $q_U$ in $U$ and $q' \in \overline{M}_{g,n}$ the image of $q'_U$ of $U \to \mathcal{A}_G \to \overline{M}_{g,n} \to \overline{M}_{g,n}$ which is a closed point of $\overline{M}_{g,n}$. Then we have $[\pi_{1,\text{adm}}(q')] \in U_{\Pi_{g,n},G}$. By replacing $q$ by $q'$, to verify the theorem, we only need to prove the theorem when $q$ is a closed point of $\overline{M}_{g,n}$.

Let $\mathcal{O}_{[f_q^*]}$ be the completion of strict henselization of $\mathcal{A}_G$ at $[f_q^*]$, $S \overset{\text{def}}{=} \mathcal{O}_{[f_q^*]}$, and $S^{\text{log}}$ the log scheme whose underlying scheme is $S$, and whose log structure is the pulling-back log structure of $\mathcal{A}^{\text{log}}_G$ induced by the natural morphism $S \to \mathcal{A}_G$ (cf. [M1, §3.23] for explicit descriptions of $S$ and $S^{\text{log}}$). Moreover, we have a Galois log admissible covering

$$f_S^{\text{log}} : Y_S^{\text{log}} \to X_S^{\text{log}}$$

over $S^{\text{log}}$ with Galois group $G$. On the other hand, by forgetting the log structure of $f_S^{\text{log}}$, we have a Galois HM-admissible covering $f_S^* : Y_S^* \to X_S^*$ over $S$ with Galois group $G$ whose closed fiber (i.e., the fiber over the closed point of $S$) is $f_q^*$. Since $\mathcal{A}_G$ is a Deligne-Mumford stack of finite type over $\text{Spec } \overline{\mathbb{F}}_p$, by applying [V1, Proposition 4.3 (1)], there exists a subring $A \subset \mathcal{O}_{[f_q^*]}$ which is of finite type over $\overline{\mathbb{F}}_p$ such that the Galois log admissible covering $f_S^{\text{log}}$ can be descended to a Galois Kummer log étale covering

$$f_E^{\text{log}} : Y_E^{\text{log}} \to X_E^{\text{log}}$$

over $E^{\text{log}}$ with Galois group $G$, where $E \overset{\text{def}}{=} \text{Spec } A$. By the construction, the pulling-back $f_E^{\text{log}} \times_{E^{\text{log}}} S^{\text{log}}$ via the natural morphism $S^{\text{log}} \to E^{\text{log}}$ is $f_S^{\text{log}}$. Moreover, by replacing $E$ by an open subset of $E$, we may assume that the underlying schemes $Y_E$ and $X_E$ are geometrically connected over each point $e \in E$. Then by forgetting the log structure of $f_E^{\text{log}}$, we obtain a Galois HM-admissible covering

$$f_E^* : Y_E^* \to X_E^*$$

over $E$ with Galois group $G$, and a morphism $E \to \mathcal{A}_G$ determined by $f_E^*$. Since $E$ is a scheme of finite type over $\text{Spec } \overline{\mathbb{F}}_p$, the image $W$ of $E \to \mathcal{A}_G \to \overline{M}_{g,n} \to \overline{M}_{g,n}$ is a constructible subset of $\overline{M}_{g,n}$ which contains $q$. Moreover, since the image of the composition of the natural morphisms $S \to \mathcal{A}_G \to \overline{M}_{g,n} \to \overline{M}_{g,n}$ is dense in $\overline{M}_{g,n}$, $W$ is a dense constructible subset of $\overline{M}_{g,n}$ which contains $q$. Then we have that

$$W = \bigsqcup_{i=1}^r W_i$$
is a finite disjoint union of local closed subsets \( \{W_i\}_{i=1,\ldots,r} \), of \( \overline{M}_{g,n} \). Without loss of generality, we may assume that \( q \in W_1 \). Since \( W_1 \) contains the image of \( S \), we obtain that \( W_1 \) is an open subset of \( \overline{M}_{g,n} \). This completes the proof of the theorem. □

**Remark 3.6.1.** In [Y10], we will give a constructive proof of Theorem 3.6, and use the constructive proof to study the continuity of clutching maps of moduli spaces of admissible fundamental groups defined in [Y4].

**Proposition 3.7.** We maintain the notation introduced above. Then the following statements hold.

(a) Let \( \pi_1^{\text{adm}}(q) \in \Pi_{g,n} \) and \( \pi_1^{\text{sol}}(q) \in \Pi_{g,n}^{\text{sol}} \) be arbitrary points. Then we have

\[
V([\pi_1^{\text{adm}}(q)]) = \{ [\pi_1^{\text{adm}}(q')] \in \Pi_{g,n} \mid \text{Hom}_{\text{surj}}(\pi_1^{\text{adm}}(q), \pi_1^{\text{adm}}(q')) \neq \emptyset \},
\]

\[
V([\pi_1^{\text{sol}}(q)]) = \{ [\pi_1^{\text{sol}}(q')] \in \Pi_{g,n}^{\text{sol}} \mid \text{Hom}_{\text{surj}}(\pi_1^{\text{sol}}(q), \pi_1^{\text{sol}}(q')) \neq \emptyset \},
\]

where \( V([\pi_1^{\text{adm}}(q)]) \) and \( V([\pi_1^{\text{sol}}(q)]) \) denote the topological closures of \([\pi_1^{\text{adm}}(q)]\) and \([\pi_1^{\text{sol}}(q)]\) in \( \Pi_{g,n} \) and \( \Pi_{g,n}^{\text{sol}} \), respectively.

(b) We have that

\[
\Pi_{g,n} \subseteq \Pi_{g,n}^{\text{adm}}, \quad \Pi_{g,n}^{\text{sol}} \subseteq \Pi_{g,n}^{\text{sol}}
\]

are open subsets.

(c) Let \( Z \) be an arbitrary irreducible closed subset of \( \overline{M}_{g,n} \). Then \( V(\pi_1^{\text{adm}}(Z)) \) and \( V(\pi_1^{\text{sol}}(Z)) \) are irreducible closed subsets of \( \Pi_{g,n} \) and \( \Pi_{g,n}^{\text{sol}} \), respectively, where \( V(\pi_1^{\text{adm}}(Z)) \) and \( V(\pi_1^{\text{sol}}(Z)) \) denote the topological closures of \( \pi_1^{\text{adm}}(Z) \) and \( \pi_1^{\text{sol}}(Z) \) in \( \Pi_{g,n} \) and \( \Pi_{g,n}^{\text{sol}} \), respectively. In particular, the topological spaces \( \Pi_{g,n} \) and \( \Pi_{g,n}^{\text{sol}} \) are irreducible.

(d) Let \( V \) be either an irreducible closed subset of \( \Pi_{g,n} \) or an irreducible closed subset of \( \Pi_{g,n}^{\text{sol}} \). Then \( V \) has a unique generic point.

**Proof.** (a) follows immediately from the definitions of \( O_{\Pi_{g,n}} \) and \( O_{\Pi_{g,n}^{\text{sol}}} \), respectively.

(b) Let \( [\pi_1^{\text{adm}}(q)] \in \Pi_{g,n} \) be an arbitrary point and \( \pi_1^{\text{adm}}(q) \) the set of finite quotients of \( \pi_1^{\text{adm}}(q) \). Moreover, since \( \pi_1^{\text{adm}}(q) \) is topologically finitely generated, we have a subset of open normal subgroups \( \{H_j\}_{j \in \mathbb{N}} \) of \( \pi_1^{\text{adm}}(q) \) such that \( H_{j_1} \subseteq H_{j_2} \) for any \( j_1 \geq j_2 \), and that

\[
\pi_1^{\text{adm}}(q) \cong \lim_{j \in \mathbb{N}} \pi_1^{\text{adm}}(q)/H_j.
\]

We put

\[
S(q) \overset{\text{def}}{=} \{ \pi_1^{\text{adm}}(q)/H_j, \; j \in \mathbb{N} \} \subseteq \pi_1^{\text{adm}}(q).
\]

We see that, in order to prove that \( \Pi_{g,n} \) is an open subset of \( \Pi_{g,n}^{\text{sol}} \), it is sufficient to prove that, for every point \( [q_2] \in M_{g,n} \), there exists a finite group \( G \in S(q_2) \) such that \( U_{\Pi_{g,n},G} \) is contained in \( \Pi_{g,n} \).

Suppose that \( U_{\Pi_{g,n},G} \cap (\Pi_{g,n} \setminus \Pi_{g,n}) \neq \emptyset \) for every \( G \in S(q_2) \). Since \( \pi_1^{\text{adm}} \) is continuous (cf. Theorem 3.6) and the set of generic points of \( M_{g,n} \setminus M_{g,n} \) is finite, there exists a generic point \( [q_1] \) of \( M_{g,n} \setminus M_{g,n} \) such that

\[
[q_1] \in \bigcap_{G \in S(q_2)} U_{\Pi_{g,n},G}.
\]
Then the set
\[ \text{Hom}_{\text{surj}}(\pi_1^{\text{adm}}(q_1), \pi_1^{\text{adm}}(q_2)) = \lim_{G \in S(q_2)} \text{Hom}_{\text{surj}}(\pi_1^{\text{adm}}(q_1), G) \]
is not empty. Thus, there is a surjective open continuous homomorphism \( \phi : \pi_1^{\text{adm}}(q_1) \to \pi_1^{\text{adm}}(q_2) \). Note that \( \phi \) induces an isomorphism of maximal prime-to-\( p \) quotients
\[ \phi^{p'} : \pi_1^{\text{adm}}(q_1)^{p'} \cong \pi_1^{\text{adm}}(q_2)^{p'} \].

By applying [Y2, Lemma 6.3], there exists an open characteristic subgroup \( H_1 \subseteq \pi_1^{\text{adm}}(q_1)^{p'} \) such that the pointed stable curve \( X_{H_1}^* \) of type \((g_{X_{H_1}},n_{X_{H_1}})\) over \( k_{q_1} \) corresponding to \( H_1 \) satisfying the following conditions: (1) \( \Gamma_{\text{cpt}}^{X_{H_1}} \) is 2-connected; (2) \( \#(v(\Gamma_{X_{H_1}})^{p'\leq 1}) = 0 \); (3) the Betti number \( r_{X_{H_1}} \) of the dual semi-graph of \( X_{H_1}^* \) is positive. Let \( H_2 \overset{\text{def}}{=} \phi^{p'}(H_1) \subseteq \pi_1^{\text{adm}}(q_2)^{p'} \). Then we obtain a smooth pointed stable curve \( X_{H_2}^* \) of type \((g_{X_{H_2}},n_{X_{H_2}})\) over \( k_{q_2} \) corresponding to \( H_2 \). Since \( H_1 \) is an open characteristic subgroup, we obtain that \((g_{X_{H_1}},n_{X_{H_1}}) = (g_{X_{H_2}},n_{X_{H_2}})\). Then Theorem 2.2 (b) and Lemma 2.3 (b) imply that
\[ r_{X_{H_1}} \leq 0. \]
This contradicts \( r_{X_{H_1}} > 0 \).

Similar arguments to the arguments given in the proof above imply that \( \Pi_{g,n}^{\text{sol}} \) is an open subset of \( \overline{\Pi}_{g,n}^{\text{sol}} \). This completes the proof of (b).

(c) is trivial.

(d) We only treat the case where \( V \) is an irreducible closed subset of \( \overline{\Pi}_{g,n} \). Let \( \text{Gen}(V) \) be the set of generic points of \( V \). Since every closed subset of \( \overline{\Pi}_{g,n} \) has a non-empty set of generic points, we have that \( \text{Gen}(V) \neq \emptyset \). Let \([\pi_1^{\text{adm}}(q_1)], [\pi_1^{\text{adm}}(q_2)] \in \text{Gen}(V)\) be arbitrary generic points. Let \( G \in \pi_1^{\text{adm}}(q_1) \) be an arbitrary finite group. Then \( U_{\Pi_{g,n},G} \cap V \neq \emptyset \). Thus, \([\pi_1^{\text{adm}}(q_2)] \in U_{\Pi_{g,n},G} \cap V \). This means that \( \pi_A^{\text{adm}}(q_1) \subseteq \pi_A^{\text{adm}}(q_2) \). Similar arguments to the arguments given in the proof above imply \( \pi_A^{\text{adm}}(q_1) \supseteq \pi_A^{\text{adm}}(q_2) \). Then we have
\[ \pi_A^{\text{adm}}(q_1) = \pi_A^{\text{adm}}(q_2). \]
Since admissible fundamental groups of pointed stable curves are topologically finitely generated, [FJ, Proposition 16.10.6] implies that \([\pi_1^{\text{adm}}(q_1)] = [\pi_1^{\text{adm}}(q_2)]\). This completes the proof of the proposition.

\[ \square \]

3.3. **The Homeomorphism Conjecture.** Next, we formulate the main conjectures of the theory developing in the present series of papers.

**Homeomorphism Conjecture.** We maintain the notation introduced above. Then we have that
\[ \pi_{g,n}^{\text{adm}} : \overline{\Pi}_{g,n} \to \Pi_{g,n} \]
is a homeomorphism.

Moreover, we have a solvable version of the Homeomorphism Conjecture as follows, which is slightly stronger than the original version.
Solvable Homeomorphism Conjecture. We maintain the notation introduced above. Then we have that

\[ \sigma_{g,n}^\text{sol} : \overline{\mathcal{M}}_{g,n} \to \overline{\Pi}_{g,n}^\text{sol} \]

is a homeomorphism.

The Homeomorphism Conjecture (or the Solvable Homeomorphism Conjecture) shows that moduli spaces of curves over algebraically closed fields of characteristic \( p > 0 \) can be reconstructed group-theoretically as “topological spaces” from admissible fundamental groups (or solvable admissible fundamental groups) of curves. Moreover, the conjectures give a new insight into the theory of anabelian geometry of curves over algebraically closed fields of characteristic \( p > 0 \) based on the following anabelian philosophy:

The topological space \( \overline{\mathcal{M}}_{g,n} \) (or \( \overline{\Pi}_{g,n}^\text{sol} \)) contains all anabelian informations of pointed stable curves of type \( (g,n) \) over algebraically closed fields of characteristic \( p > 0 \), and every topological property concerning the topological space \( \overline{\mathcal{M}}_{g,n} \) (or \( \overline{\Pi}_{g,n}^\text{sol} \)) can be regarded as an anabelian property of pointed stable curves of type \( (g,n) \) over algebraically closed fields of characteristic \( p > 0 \).

3.3.1. The main theorem of the present paper is the following, which will be proved in Section 6 (cf. Theorem 6.7).

**Theorem 3.8.** We maintain the notation introduced above. Let \([q] \in \overline{\mathcal{M}}_{0,n}^\text{cl}\) be an arbitrary closed point. Then \( \pi_{0,n}^{\text{adm}}([q]) \) and \( \pi_{0,n}^{\text{sol}}([q]) \) are closed points of \( \overline{\Pi}_{0,n} \) and \( \overline{\Pi}_{0,n}^\text{sol} \), respectively. In particular, the Homeomorphism Conjecture and the Solvable Homeomorphism Conjecture hold when \( (g,n) = (0,3) \) or \( (0,4) \).

3.3.2. On the other hand, the author is interested in the following problems concerning \( \overline{\Pi}_{g,n} \):

**Problem 3.9.** We maintain the notation introduced above.

1. Is \( \overline{\Pi}_{g,n} \) a noetherian topological space?
2. Let \([q] \in \overline{\mathcal{M}}_{g,n}^\text{cl}\). Is \( V(\pi_{g,n}^{\text{adm}}([q])) \) a finite set?
3. Let \( Z \) be an irreducible closed subset of \( \overline{\mathcal{M}}_{g,n} \). Is \( \pi_{g,n}^{\text{adm}}(Z) \) an irreducible closed subset of \( \overline{\Pi}_{g,n} \)?
4. Let \( i \in \{1,2\} \), and let \( V_{i,m} \subseteq \cdots \subseteq V_{i,1} \subseteq V_{i,0} \stackrel{df}{=} \overline{\Pi}_{g,n} \) be an arbitrary maximal chain of irreducible closed subsets of \( \overline{\Pi}_{g,n} \). Does \( m_1 = m_2 \) hold?
5. Let \( Z \) be an irreducible closed subset of \( \overline{\mathcal{M}}_{g,n} \). Does \( \dim(Z) = \dim(V(\pi_{g,n}^{\text{adm}}(Z))) \) hold? Here, \( \dim(V(\pi_{g,n}^{\text{adm}}(Z))) \) denotes the Krull dimension of \( V(\pi_{g,n}^{\text{adm}}(Z)) \). In particular, does \( \dim(\overline{\mathcal{M}}_{g,n}) = \dim(\overline{\Pi}_{g,n}) \) and \( \dim(V(\pi_{g,n}^{\text{adm}}([q]))) = 0 \) for every \([q] \in \overline{\mathcal{M}}_{g,n}^\text{cl}\) hold? Moreover, Is \( \pi_{g,n}^{\text{adm}}([q]) \) a closed point of \( \overline{\Pi}_{g,n} \) for every \([q] \in \overline{\mathcal{M}}_{g,n}^\text{cl}\)?
6. Does there exist a \( p \)-rank stratification on \( \overline{\Pi}_{g,n} \) which has purity?
7. Prove the Homeomorphism Conjecture. In particular, prove the Homeomorphism Conjecture for \( (0,n) \).

**Remark 3.9.1.** We may also ask the problems mentioned above for \( \overline{\Pi}_{g,n}^\text{sol} \).
Remark 3.9.2. Problem 3.9 (2) is equivalent to the following anabelian property of pointed stable curves:

Let \( [q] \in \overline{\mathcal{M}}_{g,n}^{cl} \). Then we have that the set

\[
\{ q' \in \overline{\mathcal{M}}_{g,n}^{cl} \mid \text{Hom}_{\text{surj}}(\pi_1^{\text{adm}}(q), \pi_1^{\text{adm}}(q')) \neq \emptyset \}
\]

is a finite set.

This is a generalized version of Theorem 3.4 (a).

Remark 3.9.3. We maintain the notation introduced above. Tamagawa’s Essential Dimension Conjecture (cf. [T3, Conjecture 5.3 (ii)]) says that:

Let \( i \in \{1, 2\} \), and let \( q_i \in \mathcal{M}_{g,n} \) and \( V(q_i) \) the topological closure of \( q_i \) in \( \overline{\mathcal{M}}_{g,n} \). Then we have that \( \dim(V(q_1)) = \dim(V(q_2)) \) if \( [\pi_1^{\text{adm}}(q_1)] = [\pi_1^{\text{adm}}(q_2)] \).

We see immediately that Problem 3.9 (5) is a generalized version of the Essential Dimension Conjecture.

Proposition 3.10. Let \( [q] \in \overline{\mathcal{M}}_{g,n}^{cl} \). Then we have that \( \dim(V(\pi_{g,n}^{\text{adm}}([q]))) = 0 \) if and only if \( \pi_{g,n}^{\text{adm}}([q]) \) is a closed point of \( \Pi_{g,n} \).

Proof. The “if” part of the proposition is trivial. We only need to prove the “only if” part of the proposition.

Let \( [\pi_1^{\text{adm}}(q')] \in V(\pi_{g,n}^{\text{adm}}([q])) \) be an arbitrary point and \( V(\pi_1^{\text{adm}}(q')) \) the topological closure of \( [\pi_1^{\text{adm}}(q')] \) in \( \overline{\mathcal{M}}_{g,n} \). Then we have that \( V([\pi_1^{\text{adm}}(q')]) \) is an irreducible closed subset which is contained in \( V(\pi_{g,n}^{\text{adm}}([q])) \). Since \( V(\pi_{g,n}^{\text{adm}}([q])) \) is an irreducible closed subset of dimension 0, we obtain that

\[
V(\pi_{g,n}^{\text{adm}}([q])) = V([\pi_1^{\text{adm}}(q')]).
\]

This means that there exist surjective open continuous homomorphisms

\[
\pi_1^{\text{adm}}(q) \twoheadrightarrow \pi_1^{\text{adm}}(q'),
\]

\[
\pi_1^{\text{adm}}(q') \twoheadrightarrow \pi_1^{\text{adm}}(q).
\]

Then we obtain \( \pi_A^{\text{adm}}(q) = \pi_A^{\text{adm}}(q') \). Since admissible fundamental groups of pointed stable curves are topologically finitely generated, [FJ, Proposition 16.10.6] implies that

\[
[\pi_1^{\text{adm}}(q)] = [\pi_1^{\text{adm}}(q')].
\]

Thus, we obtain \( V(\pi_{g,n}^{\text{adm}}([q])) = [\pi_1^{\text{adm}}(q)] \). This completes the proof of the proposition. \( \square \)

PART II: RECONSTRUCTIONS OF GEOMETRIC DATA FROM OPEN CONTINUOUS HOMOMORPHISMS

4. Reconstruction of inertia subgroups and field structures associated to marked points from open continuous homomorphisms

In this section, we will prove that the inertia subgroups and field structures associated to marked points can be reconstructed group-theoretically from surjective homomorphisms of admissible fundamental groups (or solvable admissible fundamental groups).

4.1. Anabelian reconstructions.
4.1.1. Let $\mathcal{P}$ be a category of profinite groups whose class of objects $\text{Ob}(\mathcal{P})$ consists of profinite groups, and whose class of morphisms $\text{Hom}_{\mathcal{P}}(\Pi, \Pi')$ is the class of open continuous homomorphisms of $\Pi$ and $\Pi'$. Let $\Pi \in \mathcal{P}$, and let $\mathfrak{S}_{\Pi}$ be a category whose class of objects $\text{Ob}(\mathfrak{S}_{\Pi})$ is a set of subgroups of $\Pi$, and whose class of morphisms $\text{Hom}_{\mathfrak{S}_{\Pi}}(H, H')$ for any $H, H' \in \mathfrak{S}_{\Pi}$ is defined as follows: the unique element of $\text{Hom}_{\mathfrak{S}_{\Pi}}(H, H')$ is the natural inclusion when $H \subseteq H'$; otherwise, $\text{Hom}_{\mathfrak{S}_{\Pi}}(H, H')$ is empty. We shall say that $\mathfrak{S}_{\Pi}$ is a category associated to $\Pi$.

4.1.2. Let $\mathcal{S}$ be a category whose class of objects $\text{Ob}(\mathcal{S})$ is the class of categories associated to profinite groups, and whose class of morphisms $\text{Hom}_{\mathcal{S}}(\mathfrak{S}_{\Pi}, \mathfrak{S}_{\Pi'})$ consists of the classes of functors defined as follows: $\theta_{\mathfrak{S}} \in \text{Hom}_{\mathcal{S}}(\mathfrak{S}_{\Pi}, \mathfrak{S}_{\Pi'})$ if there exists an open continuous homomorphism $\theta : \Pi \rightarrow \Pi'$ such that $\mathfrak{S}_{\Pi} = \{ H \overset{\text{def}}{=} \theta^{-1}(H') \}_{H' \in \mathfrak{S}_{\Pi'}}$, and that $\theta_{\mathfrak{S}} : \mathfrak{S}_{\Pi} \rightarrow \mathfrak{S}_{\Pi'}$, $H \mapsto H'$; otherwise, $\text{Hom}_{\mathcal{S}}(\mathfrak{S}_{\Pi}, \mathfrak{S}_{\Pi'})$ is empty.

There is a natural functor $\pi : \mathcal{S} \rightarrow \mathcal{P}$ defined as follows: Let $\mathfrak{S}_{\Pi}, \mathfrak{S}_{\Pi'} \in \mathcal{S}$ be categories associated to profinite groups $\Pi, \Pi'$, respectively; we have $\pi(\mathfrak{S}_{\Pi}) = \Pi$, $\pi(\mathfrak{S}_{\Pi'}) = \Pi'$, and $\pi(\theta_{\mathfrak{S}}) = \theta$. We see immediately that $\pi : \mathcal{S} \rightarrow \mathcal{P}$ is a fibred category over $\mathcal{P}$.

**Definition 4.1.** Let $i \in \{1, 2\}$, and let $\mathcal{F}_i$ be a geometric object (in a certain category), $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object $\mathcal{F}_i$, and $\mathfrak{S}_i \overset{\text{def}}{=} \mathfrak{S}_{\Pi_{\mathcal{F}_i}}$ a category associated to $\Pi_{\mathcal{F}_i}$. Let $\text{Inv}_{\mathcal{F}_i}$ be an invariant depending on the isomorphism class of $\mathcal{F}_i$ (in a certain category) and $\text{Add}_{\mathcal{F}_i}(\mathfrak{S}_i)$ an additional structure associated to $\mathfrak{S}_i$ (e.g., $\text{Add}_{\mathcal{F}_i}(\mathfrak{S}_i) = \mathfrak{S}_i$) on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on $\mathcal{F}_i$ and $\mathfrak{S}_i$.

(a) We shall say that $\text{Inv}_{\mathcal{F}_i}$ can be reconstructed group-theoretically from $\Pi_{\mathcal{F}_i}$ (or $\text{Add}_{\mathcal{F}_i}$) if $\Pi_{\mathcal{F}_i}$ induces $\text{Inv}_{\mathcal{F}_i}$ group-theoretically) if $\Pi_{\mathcal{F}_i} \cong \Pi_{\mathcal{F}_j}$ implies $\text{Inv}_{\mathcal{F}_i} = \text{Inv}_{\mathcal{F}_j}$.

(b) We shall say that $\text{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ can be reconstructed group-theoretically from $\Pi_{\mathcal{F}_2}$ (or $\text{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$) if $\Pi_{\mathcal{F}_2}$ induces $\text{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ group-theoretically) if every isomorphism $\theta : \Pi_{\mathcal{F}_1} \cong \Pi_{\mathcal{F}_2}$ induces a bijection $\theta_{\text{ad}} : \text{Add}_{\mathcal{F}_1}(\mathfrak{S}_1) \cong \text{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ which preserves the structures $\text{Add}_{\mathcal{F}_1}(\mathfrak{S}_1)$ and $\text{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$, where $\mathfrak{S}_1 \overset{\text{def}}{=} \Pi_{\mathcal{F}_1} \times_{\theta, \Pi_{\mathcal{F}_1}} \mathfrak{S}_2$ (i.e., the fiber product in the fibred category $\mathcal{S}$ over $\mathcal{P}$).

(c) Let $j_1, j_2 \in \{1, 2\}$ distinct from each other, and let $\theta : \Pi_{\mathcal{F}_{j_1}} \rightarrow \Pi_{\mathcal{F}_{j_2}}$ be an open continuous homomorphism of profinite groups and $\mathfrak{S}_i = \Pi_{\mathcal{F}_{j_i}} \times_{\theta, \Pi_{\mathcal{F}_{j_i}}} \mathfrak{S}_i$. We shall say that a map $\theta_{\text{ad}} : \text{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \rightarrow \text{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ can be reconstructed group-theoretically from $\theta : \Pi_{\mathcal{F}_{j_1}} \rightarrow \Pi_{\mathcal{F}_{j_2}}$ (or $\theta_{\text{ad}} : \text{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \rightarrow \text{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$) can be induced group-theoretically from $\theta : \Pi_{\mathcal{F}_{j_1}} \rightarrow \Pi_{\mathcal{F}_{j_2}}$, or $\theta : \Pi_{\mathcal{F}_{j_1}} \rightarrow \Pi_{\mathcal{F}_{j_2}}$ induces $\theta_{\text{ad}} : \text{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \rightarrow \text{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ group-theoretically) if the following holds: Let $\mathcal{F}'_i$ be a geometric object, $\Pi_{\mathcal{F}'_i}$ a profinite group associated to the geometric object $\mathcal{F}'_i$, $\theta_i : \Pi_{\mathcal{F}'_1} \cong \Pi_{\mathcal{F}'_2}$ an isomorphism of profinite groups, $\theta' : \Pi_{\mathcal{F}_1} \rightarrow \Pi_{\mathcal{F}_2}$, $\mathfrak{S}_i' \overset{\text{def}}{=} \Pi_{\mathcal{F}'_i} \times_{\theta_i, \Pi_{\mathcal{F}_i}} \mathfrak{S}_i$, $\text{Add}_{\mathcal{F}_i}'(\mathfrak{S}_i')$ an additional structure on the profinite group $\Pi_{\mathcal{F}_i}$. Moreover, suppose that we have the following commutative diagram of profinite groups:

\[
\begin{array}{ccc}
\Pi_{\mathcal{F}_1} & \xrightarrow{\theta'} & \Pi_{\mathcal{F}_2} \\
\theta_1 & & \theta_2 \\
\Pi_{\mathcal{F}_1} & \xrightarrow{\theta} & \Pi_{\mathcal{F}_2}.
\end{array}
\]
Then the commutative diagram of profinite groups above induces the following commutative diagram of additional structures

\[
\begin{array}{ccc}
\text{Add}_{F_1} (\mathcal{G}'_{j_1}) & \xrightarrow{\theta_{\text{ad}}} & \text{Add}_{F_2} (\mathcal{G}'_{j_2}) \\
\theta_{j_1, \text{ad}} & & \theta_{j_2, \text{ad}} \\
\text{Add}_{F_1} (\mathcal{G}_{j_1}) & \xrightarrow{\theta_{\text{ad}}} & \text{Add}_{F_2} (\mathcal{G}_{j_2})
\end{array}
\]

which preserves the structures of additional structures.

**Remark 4.1.1.** Let us explain the philosophy of **mono-anabelian geometry** introduced by Mochizuki. The classical point of view of anabelian geometry (i.e., the anabelian geometry considered in [G1], [G2]) focuses on a comparison between two geometric objects via their fundamental groups. Moreover, the term “group-theoretical”, in the classical point of view, means that “preserved by an arbitrary isomorphism between the fundamental groups under consideration”. We shall refer to the classical point of view as “bi-anabelian geometry”. Then Definition 4.1 is a definition from the point of view of bi-anabelian geometry.

On the other hand, mono-anabelian geometry focuses on the establishing a group-theoretic algorithm whose input datum is an abstract topological group which is isomorphic to the fundamental group of a given geometric object of interest (resp. a continuous homomorphism of abstract topological groups which are isomorphic to a continuous homomorphism of the fundamental groups of given geometric objects of interest), and whose output datum is a geometric object which is isomorphic to the given geometric object of interest (resp. a morphism of geometric objects which is isomorphic to a morphism of given geometric objects of interest). In the point of view of mono-anabelian geometry, the term “group-theoretic algorithm” is used to mean that “the algorithm in a discussion is phrased in language that only depends on the topological group structures of the fundamental groups under consideration”. Note that mono-anabelian results are stronger than bi-anabelian results.

We maintain the notation introduced in Definition 4.1. Then the mono-anabelian version of Definition 4.1 is as follows:

(a) We shall say that Inv$_F$ can be **mono-anabelian reconstructed** from $\Pi_F$ if there exists a group-theoretical algorithm whose input datum is $\Pi_F$, and whose output datum is Inv$_F (\mathcal{G})$.

(b) We shall say that Add$_F (\mathcal{G})$ can be **mono-anabelian reconstructed** from $\Pi_F$ if there exists a group-theoretical algorithm whose input datum is $\Pi_F$, and whose output datum is Add$_F$.

(c) Let $j_1, j_2 \in \{1, 2\}$ distinct from each other, and let $\theta : \Pi_{F_1} \rightarrow \Pi_{F_2}$ be an open continuous homomorphism of profinite groups and $\mathcal{G}_1 = \Pi_{F_1} \times_{\theta, \Pi_{F_2}} \mathcal{G}_2$. We shall say that a map (or a morphism) $\theta_{\text{ad}} : \text{Add}_{F_{j_1}} (\mathcal{G}_{j_1}) \rightarrow \text{Add}_{F_{j_2}} (\mathcal{G}_{j_2})$ can be **mono-anabelian reconstructed** from $\theta : \Pi_{F_1} \rightarrow \Pi_{F_2}$ if there exists a group-theoretical algorithm whose input datum is $\theta : \Pi_{F_1} \rightarrow \Pi_{F_2}$, and whose output datum is $\theta_{\text{ad}} : \text{Add}_{F_{j_1}} (\mathcal{G}_{j_1}) \rightarrow \text{Add}_{F_{j_2}} (\mathcal{G}_{j_2})$.

4.1.3. Let $i \in \{1, 2\}$, and let $X_i^\bullet = (X_i, D_{X_i})$ be a pointed stable curve of type $(g_{X_i}, n_{X_i})$ over an algebraically closed field $k_i$ of characteristic $p_i > 0$, $\Gamma_{X_i}$ the dual semi-graph of $X_i^\bullet$, and $\Pi_{X_i}$ either the admissible fundamental group or the solvable admissible fundamental
group of $X^\bullet_i$. The following result was proved by Tamagawa for smooth pointed stable curves and by the author for arbitrary pointed stable curves.

**Theorem 4.2.** We maintain the notation introduced above. Then the data $p_i, (g_{X_i}, n_{X_i}), \Pi_{X_i}^{et}, \Pi_{X_i}^{op}, \text{Ver}(\Pi_{X_i}^\bullet), \text{Edg}^{op}(\Pi_{X_i}^\bullet), \text{Edg}^{cl}(\Pi_{X_i}^\bullet)$, and $\Gamma_{X_i}^\bullet$ can be reconstructed group-theoretically from $\Pi_{X_i}^\bullet$.

**Proof.** See [Y2, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1], [T4, Theorem 0.1], and [Y6, Theorem 1.3]. □

**Remark 4.2.1.** [Y6, Theorem 1.3] gives a group-theoretical formula for $(g_{X_i}, n_{X_i})$. Then we obtain that the characteristic $p_i$ of $k_i$ and the type $(g_{X_i}, n_{X_i})$ can be mono-anabelian reconstructed from $\Pi_{X_i}^\bullet$. In fact, we have that $\Pi_{X_i}^{et}, \Pi_{X_i}^{op}, \text{Ver}(\Pi_{X_i}^\bullet), \text{Edg}^{op}(\Pi_{X_i}^\bullet), \text{Edg}^{cl}(\Pi_{X_i}^\bullet)$, and $\Gamma_{X_i}^\bullet$ can be mono-anabelian reconstructed from $\Pi_{X_i}^\bullet$ (see [Y4, Theorem 0.3]).

We do not use the term “mono-anabelian reconstruction” in the present paper. On the other hand, all of the results proved in Section 4 and Section 5 can be generalized to the case of mono-anabelian reconstructions. Moreover, mono-anabelian results will be used in [Y8], and play a fundamental role in [Y10].

4.2. Reconstructions of inertia subgroups associated to marked points from open continuous homomorphisms.

**Lemma 4.3.** We maintain the notation introduced above. Suppose that $p \overset{\text{def}}{=} p_1 = p_2$, that $(g_{X_i}, n_{X_i}) \overset{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let $\phi : \Pi_{X_i}^\bullet \to \Pi_{X_2}^\bullet$ be an arbitrary open continuous homomorphism. Then we have that $\phi$ is a surjection.

**Proof.** Let $\Pi_{X_2}^\bullet \overset{\text{def}}{=} \phi(\Pi_{X_i}^\bullet) \subseteq \Pi_{X_2}^\bullet$ be the image of $\phi$ which is an open subgroup of $\Pi_{X_2}^\bullet$. Let $X^\bullet_{\phi} = (X_{\phi}, D_{X_{\phi}})$ be the pointed stable curve of type $(g_{X_{\phi}}, n_{X_{\phi}})$ over $k_2$ induced by $\Pi_{\phi}$ and

$X^\bullet_{\phi} \to X^\bullet_{\phi}$

the admissible covering over $k_2$ induced by the natural inclusion $\Pi_{\phi} \hookrightarrow \Pi_{X_2}^\bullet$. The Riemann-Hurwitz formula implies that $g_{X_{\phi}} \geq g_X$ and $n_{X_{\phi}} \geq n_X$. Moreover, by applying Theorem 2.2 (a) and Lemma 2.3 (a), the natural surjection $\Pi_{X_i}^\bullet \to \Pi_{\phi}$ induced by $\phi$ imply that $g_X + n_X \geq g_{X_{\phi}} + n_{X_{\phi}}$. Then we have

$(g_{X_i}, n_{X_i}) = (g_{X_{\phi}}, n_{X_{\phi}})$.

This means that the admissible covering $X^\bullet_{\phi} \to X^\bullet_{\phi}$ is totally ramified over every marked point of $D_{X_2}$. Moreover, when $X^\bullet_{\phi} \to X^\bullet_{\phi}$ is totally ramified over every marked point of $D_{X_2}$, the Riemann-Hurwitz formula implies that $[\Pi_{X_1}^\bullet : \Pi_{\phi}] \neq 1$ if and only if $(g_{X_i}, n_{X_i}) = (0, 2)$. Since $X^\bullet_{\phi}$ is a pointed stable curve over $k_i$, we obtain $[\Pi_{X_i}^\bullet : \Pi_{\phi}] = 1$. Thus, $\phi$ is a surjection. □

4.2.1. In the remainder of this section, we suppose that $p \overset{\text{def}}{=} p_1 = p_2$, that $(g_{X_i}, n_{X_i}) \overset{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let

$\phi : \Pi_{X_i}^\bullet \to \Pi_{X_2}^\bullet$

be an arbitrary open continuous homomorphism. By Lemma 4.3, we see that $\phi$ is a surjective open continuous homomorphism. Let $\mathfrak{P}$ be the set of prime numbers, $\Sigma \subseteq
\( \mathfrak{P} \setminus \{p\} \) a subset, \( \Pi^{\Sigma}_{X^\bullet} \), the maximal pro-\( \Sigma \) quotient of \( \Pi_{X^\bullet} \), \( pr_{i}^{\Sigma} : \Pi_{X^\bullet} \to \Pi^{\Sigma}_{X^\bullet} \), the natural surjective homomorphism, and 

\[
\phi^{\Sigma} : \Pi^{\Sigma}_{X^\bullet} \xrightarrow{\sim} \Pi^{\Sigma}_{X^\bullet}
\]

the isomorphism induced by \( \phi \). In particular, if \( \Sigma = \mathfrak{P} \setminus \{p\} \), we use the notation \( \Pi^{\Sigma}_{X^\bullet} \)

and \( \phi^{\prime} : \Pi^{\prime}_{X^\bullet} \to \Pi^{\prime}_{X^\bullet} \) to denote \( \Pi^{\Sigma}_{X^\bullet} \) and \( \phi^{\Sigma} \), respectively.

**Lemma 4.4.** We maintain the notation introduced above. Then we have that \( \Pi^{\text{cpt}}_{X^\bullet} \) can be reconstructed group-theoretically from \( \Pi_{X^\bullet} \), and that the (surjective) open continuous homomorphism \( \phi : \Pi_{X^\bullet} \to \Pi^{\text{cpt}}_{X^\bullet} \) induces a surjective homomorphism

\[
\phi^{\text{cpt}} : \Pi^{\text{cpt}}_{X^\bullet} \to \Pi^{\text{cpt}}_{X^\bullet}
\]

group-theoretically. Moreover, the following commutative diagram of profinite groups

\[
\begin{array}{ccc}
\Pi_{X^\bullet} & \xrightarrow{\phi} & \Pi^{\text{cpt}}_{X^\bullet} \\
\downarrow & & \downarrow \\
\Pi^{\text{cpt}}_{X^\bullet} & \xrightarrow{\phi^{\text{cpt}}} & \Pi^{\text{cpt}}_{X^\bullet}
\end{array}
\]

can be reconstructed group-theoretically from \( \phi \).

**Proof.** By Theorem 4.2, we have that \( (g_{X}, n_{X}) \) can be reconstructed group-theoretically from \( \Pi_{X^\bullet} \). If \( n_{X} = 0 \), the lemma is trivial. Then we may assume that \( n_{X} > 0 \).

Let \( H_{i} \subseteq \Pi_{X^\bullet} \) be an open subgroup. Then the Riemann-Hurwitz formula implies that the admissible covering \( \chi_{H_{i}} : X_{H_{i}}^\bullet \to X_{1}^\bullet \) over \( k_{1} \) induced by \( H_{i} \subseteq \Pi_{X^\bullet} \) is étale over \( D_{X_{1}} \) if and only if \( g_{X_{H_{i}}} = [\Pi_{X^\bullet} : H_{i}](g_{X} - 1) + 1 \). We put

\[
\text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet}) \overset{\text{def}}{=} \{ H_{i} \subseteq \Pi_{X^\bullet} \text{ is an open normal subgroup} \} \quad \mid \quad g_{X_{H_{i}}} = [\Pi_{X^\bullet} : H_{i}](g_{X} - 1) + 1 \}
\]

\[
\subseteq \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet}) \overset{\text{def}}{=} \{ H_{i} \subseteq \Pi_{X^\bullet} \text{ is an open subgroup} \} \quad \mid \quad g_{X_{H_{i}}} = [\Pi_{X^\bullet} : H_{i}](g_{X} - 1) + 1 \}.
\]

By Theorem 4.2, we have that \( \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet}) \) and \( \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet}) \) can be reconstructed group-theoretically from \( \Pi_{X^\bullet} \). Since

\[
\Pi^{\text{cpt}}_{X^\bullet} \overset{\text{def}}{=} \Pi_{X^\bullet}/ \bigcap_{H_{i} \in \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet})} H_{i} = \Pi_{X^\bullet}/ \bigcap_{H_{i} \in \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet})} H_{i},
\]

we obtain that \( \Pi^{\text{cpt}}_{X^\bullet} \) can be reconstructed group-theoretically from \( \Pi_{X^\bullet} \).

Let \( H_{2} \in \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet}), H_{1} \overset{\text{def}}{=} \phi^{-1}(H_{2}), \) and \( G \overset{\text{def}}{=} \Pi_{X^\bullet}/H_{2} = \Pi_{X^\bullet}/H_{1}. \) We will prove that \( H_{1} \in \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet}) \). Let \( f_{H_{1}} : X_{H_{1}}^\bullet \to X_{1}^\bullet \) be the Galois admissible covering over \( k_{1} \) with Galois group \( G \) corresponding to \( H_{1} \), \( x_{1} \in D_{X_{1}} \) a marked point of \( X_{1}^\bullet \), and \( e_{f_{H_{1}}}(x_{1}) \) the ramification index of a point of \( f_{H_{1}}^{-1}(x_{1}) \). Since \( H_{2} \in \text{Et}_{D_{X_{1}}}^{\text{norm}}(\Pi_{X^\bullet}), \) we have
\(g_{X_2} = \#G(g_X - 1) + 1\) and \(n_{X_2} = \#Gn_X\). Then by applying the Riemann-Hurwitz formula, we obtain that

\[
g_{X_1} = \#G(g_X - 1) + 1 + \frac{1}{2} \sum_{x_1 \in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)}(e_{f_{H_1}}(x_1) - 1)
\]

and

\[
n_{X_1} = \sum_{x_1 \in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)}.
\]

By applying Theorem 2.2 (a) and Lemma 2.3 (a), the surjective homomorphism \(\phi|_1 : H_1 \rightarrow H_2\) induces that

\[
\gamma^{\text{max}}(H_1) + 2 = g_{X_1} + n_{X_1} \geq \gamma^{\text{max}}(H_2) + 2 = g_{X_2} + n_{X_2}.
\]

Then we obtain that

\[
g_{X_1} + n_{X_1} = \#G(g_X - 1) + 1 + \frac{1}{2} \sum_{x_1 \in D_{X_1}} \left( \#G - \frac{\#G}{e_{f_{H_1}}(x_1)} \right) + \sum_{x_1 \in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)}
\]

\[
\geq \#G(g_X - 1) + 1 + \#Gn_X.
\]

Thus, we have

\[
\sum_{x_1 \in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)} \geq \#Gn_X.
\]

Since \#\(D_{X_1} = n_X\), we see immediately that \(e_{f_{H_1}}(x_1) = 1\). This means that \(f_{H_1}^*\) is étale, and that

\[
H_1 \in \text{End}_{\text{norm}}(\Pi_{X_1}^*).
\]

Thus we may define the following surjective homomorphism

\[
\phi^{\text{cpt}} : \Pi_{X_1}^{\text{cpt}} \overset{\text{def}}{=} \Pi_{X_1}^*/\bigcap_{H_1 \in \text{End}_{\text{norm}}(\Pi_{X_1}^*)} H_1 \twoheadrightarrow \Pi_{X_2}^{\text{cpt}} \overset{\text{def}}{=} \Pi_{X_2}^*/\bigcap_{H_2 \in \text{End}_{\text{norm}}(\Pi_{X_2}^*)} H_2
\]

which is induced by \(\phi\) group-theoretically. Moreover, the commutative diagram

\[
\begin{array}{ccc}
\Pi_{X_1}^* & \xrightarrow{\phi} & \Pi_{X_2}^* \\
\downarrow & & \downarrow \\
\Pi_{X_1}^{\text{cpt}} & \xrightarrow{\phi^{\text{cpt}}} & \Pi_{X_2}^{\text{cpt}}
\end{array}
\]

follows immediately from the definition of \(\phi^{\text{cpt}}\). This completes the proof of the lemma. \(\square\)

**Lemma 4.5.** Let \(\ell\) be a prime number, and let \(H_2 \subseteq \Pi_{X_2}^*\) be an open normal subgroup and \(H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1}^*\). Suppose that \(G \overset{\text{def}}{=} \Pi_{X_1}^*/H_1 = \Pi_{X_2}^*/H_2\) is a cyclic group which is isomorphic to \(\mathbb{Z}/(\ell)\). Then we have that

\[(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).\]
Proof. Let \( f_H^* : X^*_H \to X^*_i \) be the Galois admissible covering over \( k_i \) with Galois group \( G \) corresponding to \( H \). Suppose that \( \ell = p \). Then the definition of admissible coverings implies that \( f_H^* \) is étale. Thus, we have that \((g_{X_H^1}, n_{X_H^1}) = (g_{X_H^2}, n_{X_H^2})\). Then we may suppose that \( \ell \neq p \).

By the Riemann-Hurwitz formula, we have
\[
g_{X_H^1} = \ell(g_X - 1) + 1 + \frac{1}{2} \#e_{f_H^1}(\ell - 1)
\]
and
\[
n_{X_H^1} = \#e_{f_H^1} + \ell(n_X - \#e_{f_H^1}).
\]
By applying Theorem 2.2 (a) and Lemma 2.3 (a), the surjective homomorphism \( \phi|_{H_i} : H_1 \to H_2 \) implies that
\[
\gamma_{\text{max}}(H_1) + 2 = g_{X_{H_i}} + n_{X_{H_i}} \geq \gamma_{\text{max}}(H_2) + 2 = g_{X_{H_2}} + n_{X_{H_2}}.
\]
Then we have
\[
\ell(g_X - 1) + 1 + \frac{1}{2} \#e_{f_H^1}(\ell - 1) + \#e_{f_H^1} + \ell(n_X - \#e_{f_H^1})
\]
\[
\geq \ell(g_X - 1) + 1 + \frac{1}{2} \#e_{f_H^2}(\ell - 1) + \#e_{f_H^2} + \ell(n_X - \#e_{f_H^2})
\]
\[
\geq \ell(g_X - 1) + 1 + \ell n_X + 1 + \frac{1}{2} (1 - \ell) \# e_{f_H^2}.
\]
Then we obtain that
\[
\# e_{f_H^1} \leq \# e_{f_H^2}.
\]
Let \( 0 \leq m \leq n_X \) be a positive natural number. We put
\[
\mathcal{N}_{i,m} \overset{\text{def}}{=} \{ N_i \subseteq \Pi_{X^*_i} \text{ is an open normal subgroup} \}
\]
\[
\mid \Pi_{X^*_i}/N_i \cong \mathbb{Z}/(\mathbb{Z} \text{ and } \# e_{f_{N_i}} = m) \}
\]
and
\[
\mathcal{N}_{i,\leq m} \overset{\text{def}}{=} \bigcup_{0 \leq j \leq m} \mathcal{N}_{i,j},
\]
where \( f_{N_i} \) denotes the Galois admissible covering over \( k_i \) corresponding to \( N_i \). The isomorphism \( \phi|_{N_i} \) induces a bijective map \( \phi^*_{\ell} : \mathcal{N}_{2,\leq n_X} \sim \mathcal{N}_{1,\leq n_X} \). To verify the lemma, it sufficient to prove that \( \phi^*_{\ell} \) induces a bijection
\[
\phi^*_{\ell} \mid_{N_{2,m}} : N_{2,m} \sim N_{1,m}.
\]
We note that since \((g_X, n_X) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})\), the isomorphism \( \phi|_{N_i} \) implies that \( \#N_{1,j} = \#N_{2,j} \) for each \( 0 \leq j \leq n_X \). Then by Lemma 4.4, we have a bijection \( \phi^*_{\ell} \mid_{N_{2,0}} : N_{2,0} \sim N_{1,0} \). We prove \( \phi^*_{\ell} \mid_{N_{2,m}} : N_{2,m} \sim N_{1,m} \) by induction on \( m \). Suppose that \( m \geq 1 \). The inequality \( \# e_{f_{N_2},\ell} \leq \# e_{f_{N_1},\ell} \) concerning the cardinality of branch locus implies that we have a bijection \( \phi^*_{\ell} \mid_{N_{2,\leq m}} : N_{2,\leq m} \sim N_{1,\leq m} \). By induction, \( \phi^*_{\ell} \mid_{N_{2,\leq m-1}} : N_{2,\leq m-1} \sim N_{1,\leq m-1} \) is a bijection. Then we obtain that
\[
\phi^*_{\ell} \mid_{N_{2,m}} : N_{2,m} \sim N_{1,m}.
\]
This completes the proof of the lemma. \( \square \)
Corollary 4.6. Let \( H_2 \subseteq \Pi X^\bullet_1 \) be an open normal subgroup and \( H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi X^\bullet_1 \). Suppose that \( G \overset{\text{def}}{=} \Pi X^\bullet_1 / H_1 = \Pi X^\bullet_2 / H_2 \) is a finite solvable group. Then we have that
\[
(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).
\]
Proof. The corollary follows immediately from Lemma 4.5.

Lemma 4.7. Let \( H_2 \subseteq \Pi X^\bullet_2 \) be an open normal subgroup and \( H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi X^\bullet_2 \). Suppose that \( H_2 \) contains the kernel of the natural homomorphism \( \Pi X^\bullet_2 \rightarrow \Pi X^\bullet_2 \) (i.e., the admissible covering corresponding to \( H_2 \) is étale over \( D_{X_2} \)). Then we have that
\[
(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).
\]
Proof. By Lemma 4.4, we have that \( H_1 \) contains the kernel of the natural homomorphism \( \Pi X^\bullet_1 \rightarrow \Pi X^\bullet_1 \) (i.e., the admissible covering corresponding to \( H_1 \) is étale over \( D_{X_1} \)). Then the lemma follows immediately from the Riemann-Hurwitz formula.

Definition 4.8. Let \( \Pi \) be an arbitrary profinite group and \( m, n \in \mathbb{N} \) positive natural numbers. We define the closed normal subgroup
\[
D_n(\Pi)
\]
of \( \Pi \) to be the topological closure of \( [\Pi, \Pi] \Pi^n \), where \( [\Pi, \Pi] \) denotes the commutator subgroup of \( \Pi \). Moreover, we define the closed normal subgroup
\[
D_n^{(m)}(\Pi)
\]
of \( \Pi \) inductively by \( D_n^{(1)}(\Pi) \overset{\text{def}}{=} D_n(\Pi) \) and \( D_n^{(j+1)}(\Pi) \overset{\text{def}}{=} D_n(D_n^{(j)}(\Pi)), \ j \in \{1, \ldots, m-1\} \). Note that \( \#(\Pi / D_n^{(m)}(\Pi)) \leq \infty \) when \( \Pi \) is topologically finitely generated.

Proposition 4.9. Let \( N_2 \subseteq \Pi X^\bullet_2 \) be an arbitrary open subgroup, \( N_1 \overset{\text{def}}{=} \phi^{-1}(N_2) \subseteq \Pi X^\bullet_1 \). Then there exist open normal subgroups \( H_2 \subseteq N_2 \subseteq \Pi X^\bullet_2 \) of \( \Pi X^\bullet_2 \) and \( H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq N_1 \subseteq \Pi X^\bullet_1 \) of \( \Pi X^\bullet_1 \) such that
\[
(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).
\]
Proof. Let \( M_i \) be an open normal subgroup of \( \Pi X^\bullet_i \), which is contained in \( N_i \). We put \( G \overset{\text{def}}{=} \Pi X^\bullet_1 / M_1 = \Pi X^\bullet_2 / M_2 \), and write \( m \) for \( [G : G_p] \), where \( G_p \) denotes a Sylow-\( p \) subgroup of \( G \). Then we have \( (m, p) = 1 \). Let \( f_{M_i} : X^\bullet_{M_i} \rightarrow X^\bullet_i \) be the Galois admissible covering over \( k_i \) with Galois group \( G \) corresponding to \( M_i \).

We put \( Q_2 \overset{\text{def}}{=} D_n^{(4)}(\Pi X^\bullet_2) \) and \( Q_1 \overset{\text{def}}{=} \phi^{-1}(Q_2) \). Note that since \( m \) is prime to \( p \) and \( \phi^p \) is an isomorphism, we have \( Q_1 = D_n^{(4)}(\Pi X^\bullet_1) \). Let \( H_i \overset{\text{def}}{=} M_i \cap Q_1 \subseteq \Pi X^\bullet_1 \). We denote by \( f_{Q_i} : X^\bullet_{Q_i} \rightarrow X^\bullet_i \), \( f_{H_i} : X^\bullet_{H_i} \rightarrow X^\bullet_i \) the Galois admissible covering over \( k_i \) with Galois group \( \Pi X^\bullet_1 / Q_i \) corresponding to \( Q_i \), and the Galois admissible covering over \( k_i \) with Galois group \( \Pi X^\bullet / H_i \) corresponding to \( H_i \), respectively. Note that \( f_{H_i} \) factors through \( f_{M_i} \) and \( f_{Q_i} \).

By applying [Y7, Lemma 3.2 and Lemma 3.3], we have that the ramification index of every point of \( D_{X_{Q_i}} \) is divided by \( m \). Then Abhyankar’s lemma implies that the Galois admissible covering
\[
g_i : X^\bullet_{H_i} \rightarrow X^\bullet_{Q_i}
\]
over $k_i$ induced by $H_i \subseteq Q_i$ is étale. Since $\Pi_{X_1} / Q_i$ is a finite solvable group, the proposition follows immediately from Corollary 4.6 and Lemma 4.7.

Lemma 4.10. (a) Let $J_i \subseteq \Pi_{X_i}$ be a closed subgroup which is isomorphic to $\hat{\mathbb{Z}}(1)^\ell$. Then the following conditions are equivalent:

(i) There exists a unique closed subgroup $I_i \in \text{Edg}^{op}(\Pi_{X_i})$ such that $J_i \subseteq I_i$.

(ii) There exists an open subgroup $N_i \subseteq \Pi_{X_i}$ such that there exists a cofinal system $\mathcal{C}_{N_i}$ of $N_i$ which consists of open normal subgroups $H_i \subseteq N_i \subseteq \Pi_{X_i}$ (i.e., $N_i \cong \lim_{\longleftarrow} H_i \in \mathcal{C}_{N_i} N_i / H_i$), the image of the composition of the natural homomorphisms

$$J_i \cap H_i \hookrightarrow H_i \rightarrow H_i^{\text{cpt,ab}}$$

is trivial for every $H_i \in \mathcal{C}_{N_i}$.

(b) Let $\ell$ be a prime number distinct from $p$, $I_i, J_i \in \text{Edg}^{op}(\Pi_{X_i})$ arbitrary closed subgroups, and $\Pi_{X_i}^{\ell}$ the maximal pro-$\ell$ quotient of $\Pi_{X_i}$. Write $\overline{T}_i$ and $\overline{J}_i$ for $\text{pr}^\ell(I_i)$ and $\text{pr}^\ell(J_i)$, respectively. Suppose that $\overline{T}_i = \overline{J}_i$. Then we have

$$I_i = J_i.$$

Proof. (a) (i) $\Rightarrow$ (ii) follows from [HM, Lemma 1.6]. For (ii) $\Rightarrow$ (i), since the proof of [HM, Lemma 1.6] implies that [HM, Lemma 1.6] also holds if there exists a cofinal system satisfying (ii), we complete the proof of (a).

(b) [M3, Proposition 1.2 (i)] implies that $I_i \cap J_i$ is trivial. Then we see that, by replacing $\Pi_{X_i}$ by a certain open subgroup of $\Pi_{X_i}$, there exists an open normal subgroup $N_i \subseteq \Pi_{X_i}$ such that $\#(\Pi_{X_i} / N_i) = \ell$, that $I_i \subseteq N_i$, and that $J_i \not\subseteq N_i$. This contradicts $\overline{T}_i = \overline{J}_i$. We complete the proof of (b).

4.2.2. Next, we prove the main result of this section.

Theorem 4.11. We maintain the notation introduced above. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X_1} \rightarrow \Pi_{X_2}$ induces a surjective map

$$\phi^{\text{edg,op}} : \text{Edg}^{op}(\Pi_{X_1}) \rightarrow \text{Edg}^{op}(\Pi_{X_2}),$$

group-theoretically. Moreover, $\phi$ induces a bijection

$$\phi^{\text{sg,op}} : \varepsilon^{op}(\Gamma_{X_1}) \xrightarrow{\sim} \varepsilon^{op}(\Gamma_{X_2})$$

of the sets of open edges of dual semi-graphs of $X_1$ and $X_2$ group-theoretically.

Proof. If $n_X = 0$, the theorem is trivial. Then we may assume that $n_X > 0$. Let $\mathcal{C}_{\Pi_{X_2}}$ be a cofinal system of $\Pi_{X_2}$ (i.e., $\mathcal{C}_{\Pi_{X_2}}$ consists of open normal subgroups of $\Pi_{X_2}$ such that $\Pi_{X_2} \xrightarrow{\sim} \lim_{\longleftarrow} H_2 \in \mathcal{C}_{\Pi_{X_2}} \Pi_{X_2} / H_2$). We put

$$\mathcal{C}_{\Pi_{X_1}} \overset{\text{def}}{=} \{ H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \mid H_2 \in \mathcal{C}_{\Pi_{X_2}} \}.$$  

Note that $\mathcal{C}_{\Pi_{X_1}}$ is not a cofinal system of $\Pi_{X_1}$ in general. Moreover, by applying Proposition 4.9, we may assume that

$$(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$$

holds for every $H_2 \in \mathcal{C}_{\Pi_{X_2}}$ and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \in \mathcal{C}_{\Pi_{X_1}}$. 

Let \( I_1 \in \text{Edg}^{\text{op}}(\Pi_{X_1^*}) \) and \( \phi(I_1) \subseteq \Pi_{X_2^*} \). We will prove that \( \phi(I_1) \in \text{Edg}^{\text{op}}(\Pi_{X_2^*}) \). Let \( H_2 \in C_{\Pi_{X_2^*}} \). By replacing \( \Pi_{X_1^*} \) and \( \phi \) by \( H_i \) and \( \phi|_{H_i} \), respectively, Lemma 4.4 implies that we have the following commutative diagram:

\[
\begin{array}{ccc}
I_1 \cap H_1 & \xrightarrow{\phi|_{I_1 \cap H_1}} & \phi(I_1) \cap H_2 \\
\downarrow & & \downarrow \\
H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\
\downarrow & & \downarrow \\
H_1^{\text{cpt,ab}} & \xrightarrow{\phi|_{H_1}^{\text{cpt,ab}}} & H_2^{\text{cpt,ab}}.
\end{array}
\]

Since \( I_1 \in \text{Edg}^{\text{op}}(\Pi_{X_1^*}) \), we have that \( I_1 \cap H_1 \leftrightarrow H_1 \rightarrow H_1^{\text{cpt,ab}} \) is trivial. Then the commutative diagram above implies that the natural morphism

\[ \phi(I_1) \cap H_2 \leftrightarrow H_2 \rightarrow H_2^{\text{cpt,ab}} \]

is trivial. Thus, by Lemma 4.10 (a), there exists \( I_2 \in \text{Edg}^{\text{op}}(\Pi_{X_2^*}) \) such that \( \phi(I_1) \subseteq I_2 \).

Let us prove \( \phi(I_1) = I_2 \). Suppose that \( \phi(I_1) \neq I_2 \). We put \( G \overset{\text{def}}{=} I_2/\phi(I_1) \). Note that \( G \) is a cyclic group, and that \((m, p) = 1\), where \( m \overset{\text{def}}{=} \#G \geq 2 \).

Suppose that \( g_X = 0 \). Then we have \( n_X \geq 3 \). Let \( N_2 \overset{\text{def}}{=} D_m(\Pi_{X_2}), N_1 \overset{\text{def}}{=} \phi^{-1}(N_2) = D_m(\Pi_{X_1}) \), and

\[
f_{N_i}^* : X_{N_i}^* \rightarrow X_i^*
\]

the Galois admissible covering over \( k_i \) corresponding to \( N_i \). Since the ramification index of each point of \( f_{N_i}^{-1}(D_{X_i}) \) is equal to \( m \), we have that

\[ I_1 \nsubseteq N_1, I_2 \nsubseteq N_2, \phi(I_1) \subseteq N_2. \]

On the other hand, the isomorphism of maximal pro-prime-to-\( p \) quotients \( \phi^p : \Pi_{X_1^*}^p \subseteq \Pi_{X_2^*}^p \) and \( I_1 \nsubseteq N_1 \) imply that \( \phi(I_1) \nsubseteq N_2 \). This contradicts \( \phi(I_1) \subseteq N_2 \). Then we obtain \( \phi(I_1) = I_2 \).

Suppose that \( g_X > 0 \). We put

\[ Q_2 \overset{\text{def}}{=} \ker(\Pi_{X_2^*} \rightarrow \Pi_{X_2^*}^{\text{cpt}} \rightarrow \Pi_{X_2^*}^{\text{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z}) \]

and \( Q_1 \overset{\text{def}}{=} \phi^{-1}(Q_2) \). Then Lemma 4.4 implies that \( Q_i = \ker(\Pi_{X_i^*} \rightarrow \Pi_{X_i^*}^{\text{cpt}} \rightarrow \Pi_{X_i^*}^{\text{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z}) \). Note that the assumption \( g_X > 0 \) implies that \( \Pi_{X_i^*}^{\text{cpt}} \rightarrow \Pi_{X_i^*}^{\text{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z} \) is not trivial. Then the nontrivial Galois admissible covering over \( k_i \) corresponding to \( Q_i \) is étale over \( D_{X_i} \). Moreover, we have \( I_i \subseteq Q_i \) and \( n_{X_{Q_i}} \geq 2 \). Let \( P_2 \overset{\text{def}}{=} D_m(Q_2), P_1 \overset{\text{def}}{=} \phi^{-1}(P_2) = D_m(Q_1) \), and

\[
g_i^* : X_i^* \rightarrow X_{Q_i}^*
\]

the Galois admissible covering over \( k_i \) corresponding to \( P_i \subseteq Q_i \). Since the ramification index of each point of \( g_i^{-1}(D_{X_{Q_i}}) \) is equal to \( m \), we have that

\[ I_1 \nsubseteq P_1, I_2 \nsubseteq P_2, \phi(I_1) \subseteq P_2. \]
On the other hand, the isomorphism of maximal pro-prime-to-$p$ quotients $\phi_1^{p'} : P_1^{p'} \rightarrow P_2^{p'}$ and $I_1 \subseteq P_1$ imply that $\phi(I_1) \subseteq P_2$. This contradicts $\phi(I_1) \subseteq P_2$. Then we obtain $\phi(I_1) = I_2$. Thus, we may define the following map

$$\phi_{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^*}) \rightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^*}), \ I_1 \mapsto I_2 \overset{\text{def}}{=} \phi(I_1).$$

Next, we will prove that $\phi_{\mathrm{edg,op}}$ is a surjection. Let $\ell$ be a prime number distinct from $p$ and $pr^{\ell}_I : \Pi^{\ell}_{X^*} \rightarrow \Pi^{\ell}_{N^*}$ the maximal pro-$\ell$ quotient. Let $J_2 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^*})$ be an arbitrary subgroup, $\mathcal{J}_2 \overset{\text{def}}{=} pr^{\ell}_I(J_2)$ the image of $J_2$, and $\mathcal{C}^{\ell}_{\Pi_{X^*}} \overset{\text{def}}{=} \{\mathcal{H}_I \overset{\text{def}}{=} pr^{\ell}_I(H_I) \}_{H_I \in \mathcal{C}_{X^*}}$, where $\mathcal{C}_{X^*}$ is the set of normal subgroups of $\Pi_{X^*}$ defined above. Note that $\mathcal{C}^{\ell}_{\Pi_{X^*}}$ is a cofinal system of $\Pi^{\ell}_{X^*}$, and that $\mathcal{H}_I = (\phi^{\ell})^{-1}(\mathcal{H}_I)$.

Let $\mathcal{H}_2 \in \mathcal{C}^{\ell}_{\Pi_{X^*}}$, $\mathcal{N}_2 \overset{\text{def}}{=} \mathcal{J}_2 \mathcal{H}_2 \supseteq \mathcal{H}_2$, $\mathcal{N}_1 \overset{\text{def}}{=} (\phi^{\ell})^{-1}(\mathcal{N}_2) \supseteq \mathcal{H}_1$, and $\mathcal{N}_i \overset{\text{def}}{=} (pr^{\ell}_I)^{-1}(\mathcal{N}_i)$. We have that $G \overset{\text{def}}{=} \mathcal{N}_1/\mathcal{H}_1 = \mathcal{N}_2/\mathcal{H}_2 = \mathcal{N}_2/\mathcal{H}_2$ is a cyclic $\ell$-group. Write $g_{H_i, N_i} : X^*_{H_i} \rightarrow X^*_{N_i}$ for the Galois admissible covering over $k_i$ with Galois group $G$. Since $J_2 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^*})$, we obtain that $g_{H_2, N_2}$ is totally ramified at a marked point of $X^*_{H_2}$. We put

$$\mathrm{Edg}^{\mathrm{op}, \ell, \mathrm{ab}}(N_i) \overset{\text{def}}{=} \{\text{the image of $I$ of the natural homomorphism $N_i \rightarrow N_i^{\ell, \mathrm{ab}} \mid I \in \mathrm{Edg}^{\mathrm{op}}(N_i)\} \}.$$ 

Note that $\#\mathrm{Edg}^{\mathrm{op}, \ell, \mathrm{ab}}(N_i) = n_{X_{N_i}}$. Then the composition of the following natural homomorphisms

$$\bigoplus_{I_{N_2} \in \mathrm{Edg}^{\mathrm{op}, \ell, \mathrm{ab}}(N_2)} I_{N_2} \rightarrow N_2^{\ell, \mathrm{ab}} \rightarrow G$$

is a surjection. By applying Lemma 4.4, we obtain that the isomorphism $\phi^{\ell}$ induces an isomorphism

$$\text{Im}( \bigoplus_{I_{N_1} \in \mathrm{Edg}^{\mathrm{op}, \ell, \mathrm{ab}}(N_1)} I_{N_1} \rightarrow N_1^{\ell, \mathrm{ab}} ) \overset{\sim}{\rightarrow} \text{Im}( \bigoplus_{I_{N_2} \in \mathrm{Edg}^{\mathrm{op}, \ell, \mathrm{ab}}(N_2)} I_{N_2} \rightarrow N_2^{\ell, \mathrm{ab}} ).$$

Then the composition of the following natural homomorphisms

$$\bigoplus_{I_{N_1} \in \mathrm{Edg}^{\mathrm{op}, \ell, \mathrm{ab}}(N_1)} I_{N_1} \rightarrow N_1^{\ell, \mathrm{ab}} \rightarrow G$$

is also a surjection. Since $G$ is a cyclic $\ell$-group, there exists $I'_{N_1} \in \mathrm{Edg}^{\mathrm{op}, \ell, \mathrm{ab}}(N_1)$ such that the composition $I'_{N_1} \hookrightarrow N_1^{\ell, \mathrm{ab}} \rightarrow G$ is a surjection. This means that $g_{H_1, N_1}$ is also totally ramified at a marked point of $X^*_{H_1}$.

We put

$$E_{\Pi_1} \overset{\text{def}}{=} \{x_1 \in D_{X_{H_1}} \mid g_{H_1, N_1} \text{ is totally ramified at $x_1$} \}.$$ 

Then we have that $E_{\Pi_1}$ is a non-empty finite set. Thus, we obtain

$$\lim_{\overline{\Pi_1} \in \overline{\mathcal{C}_{X^*}} \ni \Pi_1} E_{\Pi_1} \neq \emptyset.$$
This means that there exists \( J_1 \in \text{Edg}^{\text{op}}(\Pi_{X^*}) \) such that the image \( pr^f_2(\phi(J_1)) = \phi'(pr^f_1(J_1)) \) of \( J_1 \) of the composition of the natural homomorphisms

\[
\begin{align*}
\Pi_{X^*} & \overset{\phi}{\longrightarrow} \Pi_{X^*} \\
pr^f_1 \downarrow & \quad \quad \downarrow pr^f_2 \\
\Pi_{X^*} & \overset{\phi'}{\longrightarrow} \Pi_{X^*}
\end{align*}
\]

is equal to \( J_2 \). Since \( \phi(J_1) \in \text{Edg}^{\text{op}}(\Pi_{X^*}) \), by applying Lemma 4.10 (b), we have \( \phi(J_1) = J_2 \). Then \( \phi^{\text{edg}, \text{op}} \) is a surjection. Moreover, Theorem 4.2 implies that \( \text{Edg}^{\text{op}}(\Pi_{X^*}) \) can be reconstructed group-theoretically from \( \Pi_{X^*} \). This completes the proof of the first part of the theorem.

Let us prove the “moreover” part of the theorem. We see immediately that

\[
\phi^{\text{edg}, \text{op}} : \text{Edg}^{\text{op}}(\Pi_{X^*}) \to \text{Edg}^{\text{op}}(\Pi_{X^*})
\]

is compatible with the natural actions of \( \Pi_{X^*} \) and \( \Pi_{X^*} \), respectively. By using the surjectivity of \( \phi^{\text{edg}, \text{op}} \), we obtain immediately a surjection

\[
\phi^{\text{sg}, \text{op}} : e^{\text{op}}(\Gamma_{X^*}) \twoheadrightarrow \text{Edg}^{\text{op}}(\Pi_{X^*})/\Pi_{X^*} \to \text{Edg}^{\text{op}}(\Pi_{X^*})/\Pi_{X^*} \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X^*})
\]

of the sets of open edges of dual semi-graphs of \( X^*_1 \) and \( X^*_2 \), where \((-)^{\text{sg}}\) means “semi-graph”. Moreover, since \( n_X = \# e^{\text{op}}(\Gamma_{X^*}) = \# e^{\text{op}}(\Gamma_{X^*}) \), we have that \( \phi^{\text{sg}, \text{op}} \) is a bijection. On the other hand, Theorem 4.2 implies that \( e^{\text{op}}(\Gamma_{X^*}) \) can be reconstructed group-theoretically from \( \Pi_{X^*} \). This completes the proof of the theorem.

**Corollary 4.12.** We maintain the notation introduced above. Let \( H_2 \subseteq \Pi_{X^*} \) be an arbitrary open subgroup and \( H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X^*} \). Then we have that

\[
\gamma^{\max}(H_1) = \gamma^{\max}(H_2).
\]

**Proof.** By Theorem 4.11, we obtain that \( (g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}) \). Then Theorem 2.2 (a) implies that \( \gamma^{\max}(H_1) = \gamma^{\max}(H_2) \). □

### 4.3. Reconstructions of field structures from open continuous homomorphisms.

Let \( \tilde{X}_i = (\tilde{X}_i, D_{\tilde{X}_i}) \), \( i \in \{1, 2\} \), be the universal admissible (resp. the universal solvable admissible) covering associated to \( \Pi_{X^*} \) if \( \Pi_{X^*} \) is the admissible (resp. solvable admissible) fundamental group of \( X^* \). Let \( e_i \in e^{\text{op}}(\Gamma_{X^*}) \), \( \tilde{e}_i \in e^{\text{op}}(\Gamma_{\tilde{X}_i^*}) \) over \( e_i \), and \( I_{\tilde{e}_i} \in \text{Edg}^{\text{op}}(\Pi_{X^*}) \) such that \( \phi(I_{\tilde{e}_1}) = I_{\tilde{e}_2} \). Write \( \overline{\mathbb{F}}_{p,i} \) for the algebraic closure of \( \mathbb{F}_p \) in \( k_i \). We put

\[
\mathbb{F}_{\tilde{e}_i} \overset{\text{def}}{=} (I_{\tilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}_i) \cup \{*_{\tilde{e}_i}\},
\]

where \( \{*_{\tilde{e}_i}\} \) is an one-point set, and \( (\mathbb{Q}/\mathbb{Z})^{p'}_i \) denotes the prime-to-\( p \) part of \( \mathbb{Q}/\mathbb{Z} \) which can be canonically identified with

\[
\bigcup_{(p,m)=1} \mu_m(\overline{\mathbb{F}}_{p,i}).
\]

Moreover, \( \mathbb{F}_{\tilde{e}_i} \) can be identified with \( \overline{\mathbb{F}}_{p,i} \) as sets, hence, admits a structure of field, whose multiplicative group is \( I_{\tilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}_i \), and whose zero element is \( *_{\tilde{e}_i} \). An important consequence of Theorem 4.11 is as follows.
Theorem 4.13. We maintain the notation introduced above. Then the field structure of $\mathbb{F}_{e_1}$ can be reconstructed group-theoretically from $\Pi_{X^*_1}$. Moreover, $\phi$ induces a field isomorphism

$$\phi_{\mathbb{F}_{e_1},e_2}^{fd}: \mathbb{F}_{e_1} \cong \mathbb{F}_{e_2}$$

group-theoretically, where “fd” means “field”.

Proof. First, we claim that we may assume $n_X \geq 3$. If $g_X = 0$, then $n_X \geq 3$. Suppose that $g_X \geq 1$. Theorem 4.11 implies that $\phi: \Pi_{X^*_1} \rightarrow \Pi_{X^*_2}$ induces an open continuous surjection $\phi^{opt}: \Pi_{X^*_1}^{opt} \rightarrow \Pi_{X^*_2}^{opt}$. Let $H'_2 \subseteq \Pi_{X^*_2}^{opt}$ be an open normal subgroup such that $\#(\Pi_{X^*_2}^{opt}/H'_2) \geq 3$ and $H'_1 \overset{\text{def}}{=} (\phi^{opt})^{-1}(H'_2)$. Write $H_i \subseteq \Pi_{X^*_i}$, $i \in \{1, 2\}$, for the inverse image of $H_i'$ of the natural surjection $\Pi_{X^*_1} \rightarrow \Pi_{X^*_2}^{opt}$, and $X^*_i$, for the pointed stable curve of type $(g_{X^{(i)}, n_{X^{(i)}}})$ over $k_i$ corresponding to $H_i$. Note that $g_{X^{(i)}} = g_{X^{(i)**}} \geq 1$ and $n_{X^{(i)}} = n_{X^{(i)}} \geq 3$. By replacing $X^*_i$ by $X^*_i$ for $i \in \{1, 2\}$, we may assume that $n_X \geq 3$.

Second, we claim that we may assume $n_X = 3$. By applying Theorem 4.11, $\phi$ induces a bijection

$$\phi^{sg, op}: e^{op}(\Gamma_{X^*_1}) \cong e^{op}(\Gamma_{X^*_2}).$$

Let $E_{X^*_1} \overset{\text{def}}{=} \{e_{1,1}, e_{1,2}, e_{1,3}\} \subseteq e^{op}(\Gamma_{X^*_1})$ and $E_{X^*_2} \overset{\text{def}}{=} \phi^{sg, op}(E_{X^*_1}) \subseteq e^{op}(\Gamma_{X^*_2})$. Write $D_{X^*} \subseteq D_{X^*_i}$ for the set of marked points of $X^*_i$ corresponding to $E_{X^*_1}$. Then $(X_i, D_{X^*_i}^e)$, $i \in \{1, 2\}$, is a pointed stable curve of type $(g_X, 3)$ over $k_i$. Write $I_{X^*_i} \subseteq \{1, 2\}$, for the closed subgroup of $\Pi_{X^*_i}$ generated by the subgroups $I_{X^*_i} \subseteq \text{Edg}^{op}(\Pi_{X^*_i})$ where the image of $e$ in $e^{op}(\Gamma_{X^*_i})$ is contained in $e^{op}(\Gamma_{X^*_i}) \setminus E_{X^*_i}$. Then we have a natural isomorphism

$$\Pi_X = \Pi_{X^*_i}/I_{X^*_i}, i \in \{1, 2\}.$$}

Moreover, Theorem 4.11 implies that $\phi$ induces a surjective open continuous homomorphism

$$\phi'_i: \Pi_{X^*_i/1} \rightarrow \Pi_{X^*_2/1}.$$}

Thus, by replacing $X^*_i$, $\Pi_{X^*_i}$, and $\phi$ by $(X_i, D_{X^*_i}^e)$, $\Pi_{X^*_i/1}$, and $\phi'$, respectively, we may assume that $n_X = 3$.

Then the theorem follows from [Y6, Theorem 5.5].

Remark 4.13.1. Theorem 4.13 generalizes [T4, Proposition 5.3] and [Y3, Proposition 6.1] to the case of arbitrary pointed stable curves. [T4, Proposition 5.3] and [Y3, Proposition 6.1] play key roles in the proofs of weak Isom-version of the Grothendieck conjecture of smooth pointed stable curves over algebraically closed fields of characteristic $p > 0$ (cf. [T4, Theorem 0.2]) and weak Hom-version of the Grothendieck conjecture of smooth pointed stable curves over algebraically closed fields of characteristic $p > 0$ ([Y3, Theorem 1.2]), respectively.

5. Combinatorial Grothendieck conjecture for open continuous homomorphisms

In this section, we will prove a version of combinatorial Grothendieck conjecture for surjective open continuous homomorphisms under certain assumption, which is an analogue of Theorem 4.11 for topological data and combinatorial data associated to pointed stable curves. In the present section, we shall assume that all the fundamental groups of pointed stable curves are solvable admissible fundamental groups unless indicated otherwise.
5.1. Cohomology classes and sets of vertices. We maintain the notation introduced in Section 1. Let $X^\bullet$ be a pointed stable curve of type $(g_X, n_X)$ over an algebraically closed field $k$ of characteristic $p > 0$, $\Gamma_X^\bullet$ the dual semi-graph of $X^\bullet$, and $\Pi_X^\bullet$ the solvable admissible fundamental group of $X^\bullet$.

5.1.1. Let $\ell$ be a prime number. We put

$$v(\Gamma_X^\bullet)^{>0, \ell} \overset{\text{def}}{=} \{ v \in v(\Gamma_X^\bullet) \mid \dim_{\mathbb{F}_\ell}(\text{Hom}(\Pi_{X^\bullet}^{\ell}, \mathbb{F}_\ell)) > 0 \},$$

$$M_{X^\bullet}^{\text{et}} \overset{\text{def}}{=} \text{Hom}(\Pi_{X^\bullet}^{\ell}, \mathbb{F}_\ell),$$

$$M_{X^\bullet}^{\text{top}} \overset{\text{def}}{=} \text{Hom}(\Pi_{X^\bullet}^{\text{top}}, \mathbb{F}_\ell).$$

On the other hand, we have the natural isomorphisms $\text{Hom}(\Pi_{X^\bullet}^{\ell}, \mathbb{F}_\ell) \cong H^1_{\text{et}}(\tilde{X}_v, \mathbb{F}_\ell)$, $M_{X^\bullet}^{\text{et}} \cong H^1_{\text{et}}(X, \mathbb{F}_\ell)$, and $M_{X^\bullet}^{\text{top}} \cong H^1(\Gamma_X^\bullet, \mathbb{F}_\ell)$. In the theory of anabelian geometry, we want to emphasize the objects under consideration are arose from various fundamental groups. Then we do not use the standard notation $H^1_{\text{et}}(\tilde{X}_v, \mathbb{F}_\ell)$, $H^1_{\text{et}}(X, \mathbb{F}_\ell)$, and $H^1(\Gamma_X^\bullet, \mathbb{F}_\ell)$. Moreover, there is an injection $M_{X^\bullet}^{\text{top}} \hookrightarrow M_{X^\bullet}^{\text{et}}$ induced by the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{top}}$. We put

$$M_{X^\bullet}^{\text{nt}} \overset{\text{def}}{=} \text{coker}(M_{X^\bullet}^{\text{top}} \hookrightarrow M_{X^\bullet}^{\text{et}}),$$

where $(-)^{\text{nt}}$ means “non-top”. A non-zero element of $M_{X^\bullet}^{\text{nt}}$ corresponds to a Galois étale covering of the underlying curve $X$ of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ and an non-zero element of $M_{X^\bullet}^{\text{top}}$ corresponds to a Galois étale covering of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the map of dual semi-graphs is a topological covering.

5.1.2. Let $V_{X,\ell}^\bullet \subset M_{X^\bullet}^{\text{et}}$ be the subset of elements such that the image of $M_{X^\bullet}^{\text{et}} \to M_{X^\bullet}^{\text{nt}}$ is not 0. Then an element of $V_{X,\ell}^\bullet$ corresponds to a Galois étale covering of the underlying curve $X$ of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the map of dual semi-graphs is not a topological covering. Let $\alpha \in V_{X,\ell}^\bullet$ and

$$f_\alpha^\bullet : X_\alpha^\bullet \to X^\bullet$$

the Galois étale covering corresponding to $\alpha$. Denote by $\Gamma_{X_\alpha^\bullet}$ the dual semi-graph of $X_\alpha^\bullet$. We define a map

$$\iota : V_{X,\ell}^\bullet \to \mathbb{Z}_{>0}, \quad \alpha \mapsto \#v(\Gamma_{X_\alpha^\bullet}).$$

Furthermore, we put

$$V_{X,\ell}^\bullet \overset{\text{def}}{=} \{ \alpha \in V_{X,\ell}^\bullet \mid \iota \text{ attains its maximum} \}$$

$$= \{ \alpha \in V_{X,\ell}^\bullet \mid \iota(\alpha) = \ell \#v(\Gamma_{X^\bullet}) - \ell + 1 \}$$

$$= \{ \alpha \in V_{X,\ell}^\bullet \mid \#v_{f_\alpha} = 1 \}.$$
5.1.3. On the other hand, let $H \subseteq \Pi_X$ be an open subgroup. Write $f_H^{\ev}: \Gamma_{X_H} \to \Gamma_{X}$ for the map of dual semi-graphs induced by the admissible covering $f_H: X_H \to X$ over $k$ corresponding to $H$. We define a map

$$f_H^{\ev, \ell}: v(\Gamma_{X_H})^{>0, \ell} \to v(\Gamma_X)^{>0, \ell}$$

as follows: Let $v_H \in v(\Gamma_{X_H})^{>0, \ell}$ and $v \overset{\text{def}}{=} f_H^{\ev}(v_H) \in v(\Gamma_X)$. Then we have that $f_H^{\ev, \ell}(v_H) = v$ if $\dim_{\ell}(\Hom(\Pi_{X_H}, \mathbb{F}_\ell)) \neq 0$; otherwise, $f_H^{\ev, \ell}(v_H) = \emptyset$. Moreover, if $H \subseteq \Pi_X$ is an open normal subgroup, then $v(\Gamma_{X_H})^{>0, \ell}$ admits a natural action of $\Pi_X/H$.

**Proposition 5.1.** (a) We define a pre-equivalence relation $\sim$ on $V_{X, \ell}$ as follows:

Let $\alpha, \beta \in V_{X, \ell}^\times$. We have that $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda \alpha + \mu \beta \in V_{X, \ell}^\times$, $\lambda \alpha + \mu \beta \in V_{X, \ell}^\times$.

Then the pre-equivalence relation $\sim$ on $V_{X, \ell}$ is an equivalence relation.

(b) We denote by $V_{X, \ell}$ the quotient set of $V_{X, \ell}$ by $\sim$. Then we have a natural bijection

$$\kappa_{X, \ell}: V_{X, \ell} \overset{\sim}{\to} v(\Gamma_X)^{>0, \ell}, \ [\alpha] \mapsto v_\alpha,$$

where $[\alpha]$ denotes the equivalence class of $\alpha$.

(c) Let $\ell, \ell'$ be prime numbers distinct from each other. Suppose that $\ell \neq p$. Then we have a natural injection

$$V_{X, \ell} \hookrightarrow V_{X, \ell'},$$

which is a bijection if $\ell' \neq p$, and which fits into the following commutative diagram:

$$\begin{array}{ccc}
V_{X, \ell'} & \overset{\kappa_{X, \ell'}}{\longrightarrow} & v(\Gamma_X)^{>0, \ell'} \\
\downarrow & & \downarrow \\
V_{X, \ell} & \overset{\kappa_{X, \ell}}{\longrightarrow} & v(\Gamma_X)^{>0, \ell},
\end{array}$$

where the vertical map of the right-hand side is the natural injection induced by the definitions of $v(\Gamma_X)^{>0, \ell}$ and $v(\Gamma_X)^{>0, \ell'}$.

(d) Let $H \subseteq \Pi_X$ be an open subgroup. Suppose that $([\Pi_X] : H), \ell)$ = 1. Then the natural injection $H \hookrightarrow \Pi_X$ induces a map

$$\gamma_H^{\ev, \ell}: V_{X_H, \ell} \to V_{X, \ell}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc}
V_{X_H, \ell} & \overset{\kappa_{X_H, \ell}}{\longrightarrow} & v(\Gamma_{X_H})^{>0, \ell} \\
\gamma_H^{\ev, \ell} \downarrow & & \downarrow \gamma_H^{\ev, \ell} \\
V_{X, \ell} & \overset{\kappa_{X, \ell}}{\longrightarrow} & v(\Gamma_X)^{>0, \ell},
\end{array}$$

Moreover, suppose that $H \subseteq \Pi_X$ is an open normal subgroup. Then $V_{X_H, \ell}$ admits an action of $\Pi_X/H$ such that $\kappa_{X_H, \ell}$ is compatible with $\Pi_X/H$-actions (i.e., $\kappa_{X_H, \ell}$ is $\Pi_X/H$-equivariant).

**Proof.** See [Y4, Proposition 2.1 and Remark 2.1.1] for (a), (b), and (c). Let us explain (d). Let $[\alpha_X] \in V_{X, \ell}$. Then $\alpha_X$ induces an element

$$\beta_X^H = \sum_{\beta \in L_{\alpha_X}} c_\beta \beta \in \Hom(H, \mathbb{F}_\ell), \ c_\beta \in \mathbb{F}_\ell^\times.$$
via the natural homomorphism \( \text{Hom}(\Pi_{X^\bullet}, \mathbb{F}_\ell) \rightarrow \text{Hom}(H, \mathbb{F}_\ell) \), where \( L_{\alpha^X} \) is a subset of \( V_{X^H,\ell}^* \) such that, if \( \beta_1, \beta_2 \in L_{\alpha^X} \) distinct from each other, then \( [\beta_1] \neq [\beta_2] \).

Let \([\alpha_{X_H}] \in V_{X^H,\ell}\). Then we define \( \gamma_{H}^{\text{ver,}\ell}(\alpha_{X_H}) = [\alpha_X] \) if there exists \([\alpha_X] \in V_{X,\ell}\) such that there exists \( \beta \in L_{\alpha_X} \), and that \([\beta] = [\alpha_{X_H}] \) (i.e., \( \beta \sim \alpha_{X_H} \)). Otherwise, we put \( \gamma_{H}^{\text{ver,}\ell}(\alpha_{X_H}) = \emptyset \). It is easy to check that \( \gamma_{H}^{\text{ver,}\ell} \) is well-defined, and that the following diagram

\[
\begin{array}{ccc}
V_{X^H,\ell} & \xrightarrow{\kappa_{X^H,\ell}} & v(\Gamma_{X^H})^{>0,\ell} \\
\gamma_{H}^{\text{ver,}\ell} & \downarrow & \downarrow f_{H}^{\text{ver,}\ell} \\
V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_X)^{>0,\ell}
\end{array}
\]

is commutative.

Moreover, suppose that \( H \) is an open normal subgroup of \( \Pi_{X^\bullet} \). The natural exact sequence

\[ 1 \rightarrow H \rightarrow \Pi_{X^\bullet} \rightarrow \Pi_{X^\bullet}/H \rightarrow 1 \]

induces an outer representation

\[ \Pi_{X^\bullet}/H \rightarrow \text{Out}(H) \overset{\text{def}}{=} \frac{\text{Aut}(H)}{\text{Inn}(H)} \]

Then we obtain an action of \( \Pi_{X^\bullet}/H \) on \( V_{X^H,\ell} \subseteq \text{Hom}(H^{\ell*}, \mathbb{F}_\ell) \) induced by the outer representation. Let \( \sigma \in \Pi_{X^\bullet}/H \) and \( \alpha_{X_H} \alpha_{X_H} \in V_{X^H,\ell} \). Then we have that \( \alpha_{X_H} \sim \alpha_{X_H}^\prime \) if and only if \( \sigma(\alpha_{X_H}) \sim \sigma(\alpha_{X_H}^\prime) \). Thus, we obtain an action of \( \Pi_{X^\bullet}/H \) on \( V_{X^H,\ell} \) induced by the natural injection \( H \hookrightarrow \Pi_{X^\bullet} \). On the other hand, it is easy to check that the commutative diagram above is compatible with the \( \Pi_{X^\bullet}/H \)-actions. This completes the proof of the proposition. \( \square \)

**Remark 5.1.1.** By applying Theorem 4.2, we have that \( \Pi_{X^\bullet}^\text{et} \) and \( \Pi_{X^\bullet}^{\top} \) can be reconstructed group-theoretically from \( X^\bullet \). Then we obtain that \( V_{X,\ell} \) (or \( v(\Gamma_X)^{>0,\ell} \)) can be reconstructed group-theoretically from \( \Pi_{X^\bullet} \). Moreover, for every open subgroup \( H \subseteq \Pi_{X^\bullet} \), the map

\[ \gamma_{H}^{\text{ver,}\ell}: V_{X^H,\ell} \rightarrow V_{X,\ell} \]

constructed in Proposition 5.1 (d) can be reconstructed group-theoretically from the natural inclusion \( H \hookrightarrow \Pi_{X^\bullet} \).

### 5.2. Cohomology classes and sets of closed edges.

We maintain the notation introduced in Section 5.1. Moreover, in this subsection, we suppose that the genus of the normalization of each irreducible component of \( X \) is positive (i.e., \( v(\Gamma_X) = v(\Gamma_X)^{>0,\ell} \) if \( \ell \neq p \)), and that \( \Gamma_X^{\text{opt}} \) is 2-connected (cf. Definition 1.1).

#### 5.2.1. We shall say that

\[ \Xi_X^\bullet \overset{\text{def}}{=} (\ell, d, f_X^\bullet: Y^\bullet \rightarrow X^\bullet) \]

is an edge-triple associated to \( X^\bullet \) if the following conditions are satisfied:

(i) \( \ell \) and \( d \) are prime numbers distinct from each other and from \( p \).

(ii) \( \ell \equiv 1 \pmod{d} \); this means that all \( d \)th roots of unity are contained in \( \mathbb{F}_\ell \). Moreover, we write \( \mu_d \subseteq \mathbb{F}_\ell^* \) for the subgroup of \( d \)th roots of unity.
(iii) \( f_X^\bullet : Y^\bullet \to X^\bullet \) is a Galois admissible covering over \( k \) such that Galois group is isomorphic to \( \mu_d \), that \( f_X^\bullet \) is étale (i.e., \( f_X \) is étale), and that \( \# v^{op}_{f_X} = 0 \). Note that since \( v(\Gamma_X^\bullet) = v(\Gamma_X^\bullet)_{=0} \), we see that \( f_X^\bullet \) exists.

5.2.2. On the other hand, we shall say that

\[ \Sigma_{\Pi_X^\bullet} \overset{\text{def}}{=} (\ell, d, \alpha_{f_X}) \]

is an edge-triple associated to \( \Pi_X^\bullet \) if the following conditions are satisfied:

(i) \( \alpha_{f_X} \in \text{Hom}(\Pi_{X^\bullet}^{\text{et}}, F_d) \).

(ii) The composition of the following natural homomorphisms \( \Pi_{X^\bullet}^{\text{et}} \hookrightarrow \Pi_{X^\bullet}^{\text{et}} \overset{\alpha_{f_X}}{\to} F_d \) is a surjection for every \( v \in v(\Gamma_X^\bullet) \).

We see immediately that an edge-triple \( \Sigma_{X^\bullet} \) associated to \( X^\bullet \) is equivalent to an edge-triple \( \Sigma_{\Pi_X^\bullet} \) associated to \( \Pi_X^\bullet \). Moreover, \( f_X^\bullet \) is the Galois admissible covering corresponding to the kernel of the composition of the natural homomorphisms \( \Pi_{X^\bullet} \to \Pi_{X^\bullet}^{\text{et}} \overset{\alpha_{f_X}}{\to} F_d \).

5.2.3. In the remainder of the present subsection, we fix an edge-triple

\[ \Sigma_{\Pi_X^\bullet} \overset{\text{def}}{=} (\ell, d, \alpha_{f_X}) \]

associated to \( \Pi_X^\bullet \). Write \( \Sigma_{X^\bullet} \overset{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \to X^\bullet) \) for the edge-triple associated to \( X^\bullet \) corresponding to \( \Sigma_{\Pi_X^\bullet} \), \((g_Y, n_Y)\) for the type of \( Y^\bullet \), \( \Gamma_Y^\bullet \) for the dual semi-graph of \( Y^\bullet \), \( r_Y \) for the Betti number of \( \Gamma_Y^\bullet \), and \( \Pi_Y^\bullet \) for the kernel of the composition of the homomorphisms \( \Pi_{X^\bullet} \to \Pi_{X^\bullet}^{\text{et}} \overset{\alpha_{f_X}}{\to} F_d \).

5.2.4. We put

\[ M_{Y^\bullet} \overset{\text{def}}{=} \text{Hom}(\Pi_{Y^\bullet}, F_\ell). \]

There is a natural injection \( M_{Y^\bullet}^{\text{et}} \overset{\text{def}}{=} \text{Hom}(\Pi_{Y^\bullet}^{\text{et}}, F_\ell) \hookrightarrow M_{Y^\bullet} \) induced by the natural surjection \( \Pi_{Y^\bullet} \to \Pi_{Y^\bullet}^{\text{et}} \). Then we obtain an exact sequence

\[ 0 \to M_{Y^\bullet}^{\text{et}} \to M_{Y^\bullet} \to M_{Y^\bullet}^{\text{et}} \overset{\text{def}}{=} \text{coker}(M_{Y^\bullet}^{\text{et}} \to M_{Y^\bullet}) \to 0 \]

with a natural action of \( \mu_d \), where “ra” means “ramification”. For any element of \( M_{Y^\bullet} \), if the image of the element is not 0 in \( M_{Y^\bullet}^{\text{et}} \), then the Galois admissible covering of \( Y^\bullet \) with Galois group \( \mathbb{Z}/\ell \mathbb{Z} \) corresponding to the element is not étale.

5.2.5. Let \( M_{Y^\bullet}^{\text{et}, \mu_d} \subset M_{Y^\bullet}^{\text{et}} \) be the subset of elements on which \( \mu_d \) acts via the character \( \mu_d \to F_\ell^\times \). Let \( E^{\text{et}}_{2\Pi_X^\bullet} \subseteq M_{Y^\bullet} \) be the subset of elements whose images is nonzero elements of \( M_{Y^\bullet}^{\text{et}, \mu_d} \), and \( \alpha \in E^{\text{et}}_{2\Pi_X^\bullet} \). Write

\[ g_\alpha^\bullet : Y^\bullet \to Y^\bullet \]

for the Galois admissible covering over \( k \) corresponding to \( \alpha \). We define a map

\[ \epsilon : E^{\text{et}}_{2\Pi_X^\bullet} \to \mathbb{Z}_{\geq 0}, \alpha \mapsto \#(\epsilon^{\text{op}}(\Gamma_{Y^\bullet}) \cup \epsilon^{\text{cl}}(\Gamma_{Y^\bullet})), \]

where \( \Gamma_{Y^\bullet} \) denotes the dual semi-graph of \( Y^\bullet \). We put

\[ E^{\text{cl}, \epsilon}_{2\Pi_X^\bullet} \overset{\text{def}}{=} \{ \alpha \in E^{\text{et}}_{2\Pi_X^\bullet} \mid \# g_\alpha = 0, \# g_\alpha = d \}. \]

Note that \( E^{\text{cl}, \epsilon}_{2\Pi_X^\bullet} \) is not an empty set. For each \( \alpha \in E^{\text{cl}, \epsilon}_{2\Pi_X^\bullet} \), since the image of \( \alpha \) is contained in \( M_{Y^\bullet}^{\text{et}, \mu_d} \), we obtain that the action of \( \mu_d \) on the set \( \{ y_\ell \}_{\ell \in E^{\text{cl}, \epsilon}_{2\Pi_X^\bullet}} \subseteq \text{Nod}(Y^\bullet) \) is
transitive, where Nod(−) denotes the set of nodes of (−), and ye denotes the node of Y∗ corresponding to e. Then there exists a unique node xα of X∗ such that fX(ye) = xα for every ye ∈ {ye}e∈cl,α. We denote by eα ∈ ecl(ΓX∗) the closed edge corresponding to xα.

5.2.6. On the other hand, let H ⊆ ΠX∗ be an open subgroup. Write \( f_H^* : \Gamma X_H^* \to \Gamma X^* \) for the map of dual semi-graphs induced by the admissible covering \( f_H : X_H^* \to X^* \) over k corresponding to H. We shall denote by

\[ f^{\text{cl}}_H \overset{\text{def}}{=} f_H^* \big|_{\text{e}^{\text{cl}}(\Gamma X_H^*)} : \text{e}^{\text{cl}}(\Gamma X_H^*) \to \text{e}^{\text{cl}}(\Gamma X^*) \].

Moreover, if H ⊆ ΠX∗ is an open normal subgroup, then \( \text{e}^{\text{cl}}(\Gamma X_H^*) \) admits a natural action of \( \Pi X^*/H \).

**Proposition 5.2.** (a) We define a pre-equivalence relation \( \sim \) on \( E_{\text{cl}}^{\Pi X^*} \) as follows:

Let \( \alpha, \beta \in E_{\text{cl}}^{\Pi X^*} \). We have that \( \alpha \sim \beta \) if, for each \( \lambda, \mu \in \mathbb{F}^X \) for which

\[ \lambda \alpha + \mu \beta \in E_{\text{cl}}^{\Pi X^*} \], we have \( \lambda \alpha + \mu \beta \in E_{\text{cl}}^{\Pi X^*} \).

Then the pre-equivalence relation \( \sim \) on \( E_{\text{cl}}^{\Pi X^*} \) is an equivalence relation.

(b) We denote by \( E_{\text{cl}}^{\Pi X^*} \) the quotient set of \( E_{\text{cl}}^{\Pi X^*} \) by \( \sim \). Then we have a natural bijection

\[ \vartheta_{\text{cl}}^{\Pi X^*} : E_{\text{cl}}^{\Pi X^*} \overset{\sim}{\to} \text{e}^{\text{cl}}(\Gamma X^*), \ [\alpha] \mapsto e_\alpha, \]

where \([\alpha]\) denotes the equivalence class of \( \alpha \).

(c) Let \( \Sigma_{\text{cl}}^{\Pi X^*} \) be an arbitrary edge-triples associated to \( \Pi X^* \). Then we have a natural bijection

\[ E_{\text{cl}}^{\Sigma_{\text{cl}}^{\Pi X^*}} \overset{\sim}{\to} E_{\text{cl}}^{\Pi X^*} \]

which fits into the following commutative diagram:

\[ \begin{array}{ccc}
E_{\text{cl}}^{\Sigma_{\text{cl}}^{\Pi X^*}} & \overset{\vartheta_{\text{cl}}^{\Pi X^*}}{\longrightarrow} & \text{e}^{\text{cl}}(\Gamma X^*) \\
\downarrow & & \downarrow \\
E_{\text{cl}}^{\Pi X^*} & \overset{\vartheta_{\text{cl}}^{\Pi X^*}}{\longrightarrow} & \text{e}^{\text{cl}}(\Gamma X^*) \\
\end{array} \]

(d) Let H ⊆ ΠX∗ be an open subgroup. Suppose that \( ([\Pi X^* : H], \ell) = ([\Pi X^* : H], d) = 1 \). We have that \( \Sigma_{X^*} \) associated to \( \Pi X^* \) induces an edge-triple

\[ \Sigma_{X_H^*}^{\text{cl}} \overset{\text{def}}{=} (\ell, d, f_{X_H^*} : Y_{X_H^*}^{\text{cl}} \overset{\text{def}}{=} Y^* \times_{X^*} X_{H^*}^* \to X_{H^*}^*) \]

associated to \( X_H^* \), where \( Y^* \times_{X^*} X_{H}^* \) denotes the fiber product in the category of pointed stable curves. Write \( \Sigma_H^* \) for the edge-triple associated to H corresponding to \( \Sigma_{X_H^*}^{\text{cl}} \). Then the natural injection \( H \hookrightarrow \Pi X^* \) induces a surjective map

\[ \gamma_{\text{cl}}^{\Sigma_{\text{cl}}^{\Pi X^*}, H} : E_{\text{cl}}^{\Sigma_H^*} \to E_{\text{cl}}^{\Sigma_{\text{cl}}^{\Pi X^*}} \]
which fits into the following commutative diagram:

\[
\begin{array}{ccc}
E^\text{cl}_{\mathfrak{m}_H} & \xrightarrow{\vartheta_{\mathfrak{m}_H}} & e^\text{cl}(\Gamma_{X_H}^*) \\
\gamma^\text{cl}_{\Pi_{X^*}} & \downarrow f^\text{cl}_H & \\
E^\text{cl}_{\Pi_{X^*}} & \xrightarrow{\vartheta_{\Pi_{X^*}}} & e^\text{cl}(\Gamma_{X^*}).
\end{array}
\]

Moreover, suppose that \( H \subseteq \Pi_{X^*} \) is an open normal subgroup. Then \( E^\text{cl}_{\mathfrak{m}_H} \) admits an action of \( \Pi_{X^*}/H \) such that \( \vartheta_{\mathfrak{m}_H} \) is compatible with \( \Pi_{X^*}/H \)-actions (i.e., \( \vartheta_{\mathfrak{m}_H} \) is \( \Pi_{X^*}/H \)-equivariant).

**Proof.** See [Y4, Proposition 2.2 and Remark 2.2.1] for (a), (b), and (c). Let us explain (d). Let \( \alpha_X \in E^\text{cl}_{\Pi_{X^*}} \). Then \( \alpha_X \) induces an element

\[
\beta_{X_H} = \sum_{\beta \in J_{\alpha_X}} c_\beta \beta, \ c_\beta \in \mathbb{F}_\ell^x
\]

via the natural homomorphism \( \text{Hom}(\Pi_{Y^*}, \mathbb{F}_\ell) \to \text{Hom}(\Pi_{\mathcal{Y}_{X_H}^*}, \mathbb{F}_\ell) \), where \( \Pi_{\mathcal{Y}_{X_H}^*} \overset{\text{def}}{=} \Pi_{Y^*} \cap H \), and \( J_{\alpha_X} \) is a subset of \( E^\text{cl}_{\mathfrak{m}_H} \) such that, if \( \beta_1, \beta_2 \in J_{\alpha_X} \) distinct from each other, then \( [\beta_1] \neq [\beta_2] \).

Let \( [\alpha_{X_H}] \in E^\text{cl}_{\mathfrak{m}_H} \). We define

\[
\gamma^\text{cl}_{\Pi_{X^*}}([\alpha_{X_H}]) = [\alpha_X]
\]

if there exists \( \alpha_X \in E^\text{cl}_{\Pi_{X^*}} \) (note that \( \alpha_X \) always exists) such that \( [\beta] = [\alpha_{X_H}] \) for some \( \beta \in J_{\alpha_X} \). It is easy to check that \( \gamma^\text{cl}_{\Pi_{X^*}} \) is well-defined, and that the following diagram

\[
\begin{array}{ccc}
E^\text{cl}_{\mathfrak{m}_H} & \xrightarrow{\vartheta_{\mathfrak{m}_H}} & e^\text{cl}(\Gamma_{X_H}^*) \\
\gamma^\text{cl}_{\Pi_{X^*}} & \downarrow f^\text{cl}_H & \\
E^\text{cl}_{\Pi_{X^*}} & \xrightarrow{\vartheta_{\Pi_{X^*}}} & e^\text{cl}(\Gamma_{X^*})
\end{array}
\]

is commutative.

Moreover, suppose that \( H \) is an open normal subgroup of \( \Pi_{X^*} \). Since \( \Pi_{Y_{X_H}^*} \) is an open normal subgroup of \( \Pi_{X^*} \), we have

\[
\Pi_{X^*}/\Pi_{Y_{X_H}^*} \cong \Pi_{X^*}/H \times \mathbb{Z}/d\mathbb{Z}.
\]

Then the natural exact sequence

\[1 \to \Pi_{Y_{X_H}^*} \to \Pi_{X^*} \to \Pi_{X^*}/\Pi_{Y_{X_H}^*} \to 1\]

induces an outer representation \( \Pi_{X^*}/H \hookrightarrow \Pi_{X^*}/\Pi_{Y_{X_H}^*} \to \text{Out}(\Pi_{Y_{X_H}^*}) \). Thus, we obtain an action of \( \Pi_{X^*}/H \) on \( E^\text{cl}_{\mathfrak{m}_H} \subseteq \text{Hom}(\Pi_{Y_{X_H}^*}, \mathbb{F}_\ell) \) induced by the outer representation.

Let \( \sigma \in \Pi_{X^*}/H \) and \( \alpha_{X_H}, \alpha'_{X_H} \in E^\text{cl}_{\mathfrak{m}_H} \). We observe that \( \alpha_{X_H} \sim \alpha'_{X_H} \) if and only if \( \sigma(\alpha_{X_H}) \sim \sigma(\alpha'_{X_H}) \). Thus, we obtain an action of \( \Pi_{X^*}/H \) on \( E^\text{cl}_{\mathfrak{m}_H} \) induced by the natural injection \( H \hookrightarrow \Pi_{X^*} \). On the other hand, it is easy to check that the commutative diagram above is compatible with the \( \Pi_{X^*}/H \)-actions. This completes the proof of the proposition. \( \square \)
Remark 5.2.1. By applying Theorem 4.2, we have that \( \Pi^{\text{cl}}_{\Gamma} \) can be reconstructed group-theoretically from \( \Pi_X \). Then \( E^{\text{cl}}_{2\Pi_{X}} \) (or \( e^{\text{cl}}(\Gamma_X) \)) can be reconstructed group-theoretically from \( \Pi_X \). Moreover, for every open subgroup \( H \subseteq \Pi_X \), the map

\[
\gamma^{\text{cl}}_{2\Pi_{X},H} : E_{2\Pi_{X}}^{\text{cl}} \to E_{2\Pi_{X}}^{\text{cl}}
\]

constructed in Proposition 5.2 (d) can be reconstructed group-theoretically from the natural inclusion \( H \hookrightarrow \Pi_X \).

5.2.7. Next, we calculate the cardinality \( \#E^{\text{cl},*}_{2\Pi_{X}} \) of \( E^{\text{cl},*}_{2\Pi_{X}} \). We put

\[
E^{\text{cl},*}_{2\Pi_{X},e} \overset{\text{def}}{=} \{ \alpha \in E^{\text{cl},*}_{2\Pi_{X}} \mid e = e_\alpha \}, \quad e \in e^{\text{cl}}(\Gamma_X).
\]

Note that \( e = e_\alpha, \alpha \in E^{\text{cl},*}_{2\Pi_{X},e} \), means that the Galois admissible covering \( g^{\alpha}_e : Y^{*} \to Y^{*} \) over \( k \) induced by \( \alpha \) is (totally) ramified over \( f_X^{-1}(x_\alpha) \), where \( x_\alpha \) denotes the node of \( X \) corresponding to \( e \). Moreover, we have the following disjoint union

\[
E^{\text{cl},*}_{2\Pi_{X}} = \bigsqcup_{e \in e^{\text{cl}}(\Gamma_X)} E^{\text{cl},*}_{2\Pi_{X},e}.
\]

Let \( m \in \mathbb{Z}_{\geq 0} \) and \( e \in e^{\text{cl}}(\Gamma_X) \). We shall put

\[
E^{\text{cl},*\mid m}_{2\Pi_{X},e} \overset{\text{def}}{=} \{ \alpha \in E^{\text{cl},*}_{2\Pi_{X},e} \mid \#\eta_{\alpha} = m \}.
\]

Let \( e \in e^{\text{cl}}(\Gamma_X) \) be a closed edge. Write \( Y_e \) for the normalization of the underlying curve \( Y \) of \( Y^{*} \) at \( f_X^{-1}(x_\alpha) \) and

\[
nor_e : Y_e \to Y
\]

for the resulting normalization morphism. Since the genus of the normalization of each irreducible component of \( X^{*} \) is positive, we obtain that the genus of the normalization of each irreducible component of \( Y_e \) is also positive. Moreover, since \( \Gamma_X \) is 2-connected, \( Y_e \) is connected.

Lemma 5.3. We maintain the notation introduced above. Let \( e \in e^{\text{cl}}(\Gamma_X) \) be a closed edge. Then we have

\[
\#E^{\text{cl},*}_{2\Pi_{X},e} = \ell^{2g_Y - d + r_Y + 1} - \ell^{2g_Y - d + r_Y}.
\]

Moreover, we have

\[
\#E^{\text{cl},*}_{2\Pi_{X}} = \#e^{\text{cl}}(\Gamma_X)(\ell^{2g_Y - d + r_Y + 1} - \ell^{2g_Y - d + r_Y}).
\]

Proof. Write \( R_e \subseteq Y_e \) for the set of closed subset \( (f_X \circ \text{nor}_e)^{-1}(x_\alpha) \). Then \( E^{\text{cl},*}_{2\Pi_{X},e} \) can be naturally regarded as a subset of \( H^1_{\text{et}}(Y_e \setminus R_e, \mathbb{F}_\ell) \) via the natural open immersion \( Y_e \setminus R_e \hookrightarrow Y_e \). Write \( L_e \) for the \( \mathbb{F}_\ell \)-linear subspace spanned by \( E^{\text{cl},*}_{2\Pi_{X},e} \) in \( H^1_{\text{et}}(Y_e \setminus R_e, \mathbb{F}_\ell) \). Then we see that

\[
E^{\text{cl},*}_{2\Pi_{X},e} = L_e \setminus H^1_{\text{et}}(Y_e, \mathbb{F}_\ell).
\]

Write \( H^1_{\text{et}} \) for the cokernel of the natural inclusion \( H^1_{\text{et}}(Y_e, \mathbb{F}_\ell) \hookrightarrow L_e \). We obtain an exact sequence as follows:

\[
0 \to H^1_{\text{et}}(Y_e, \mathbb{F}_\ell) \to L_e \to H^1_{\text{et}} \to 0.
\]
On the other hand, since the action of $\mu_d$ on $f^{-1}(x_e)$ is transitive, the structure of the maximal pro-$\ell$ quotient $\Pi^1_e$, of $\Pi_{Y\bullet}$ implies that $\dim_{\mathbb{F}_\ell}(H_{\text{ra}}^1(Y_e, \mathbb{F}_\ell)) = 2(g_Y - d) - (r_Y - d) = 2g_Y - d - r_Y$, we obtain that

$$\#E_{\Pi^1_e,e}^{\text{cl}} = \ell^{2g_Y - d - r_Y + 1} - \ell^{2g_Y - d - r_Y}.$$ 

Thus, we have

$$\#E_{\Pi^1_e}^{\text{cl}} = \#e^{\text{cl}}(\Gamma_X\bullet)(\ell^{2g_Y - d - r_Y + 1} - \ell^{2g_Y - d - r_Y}).$$

This completes the proof of the lemma. \qed

5.2.8. We introduce some notation concerning open edges. We put

$$E_{\Pi^1_X\bullet}^{\text{op},*} \overset{\text{def}}{=} \{ \alpha \in E_{\Pi^1_X\bullet}^* \mid \#e^{\text{op},\text{ra}}_{g_\alpha} = d, \ #e^{\text{cl}}_{g_\alpha} = 0 \}.$$ 

Note that $E_{\Pi^1_X\bullet}^{\text{op},*}$ is not an empty set if $n_X \neq 0$. For each $\alpha \in E_{\Pi^1_X\bullet}^{\text{op},*}$, since the image of $\alpha$ is contained in $M_{\text{Y\bullet},\mu_d}$, we obtain that the action of $\mu_d$ on the set $\{y_e\}_{e \in e^{\text{op},\text{ra}}_{g_\alpha}} \subseteq D_Y$ is transitive, where $y_e$ denotes the marked point of $Y^\bullet$ corresponding to $e$. Then there exists a unique marked point $x_\alpha \in D_X$ of $X^\bullet$ such that $f_X(y_e) = x_\alpha$ for every $y_e \in \{y_e\}_{e \in e^{\text{op},\text{ra}}_{g_\alpha}}$. We denote by $e_\alpha \in e^\text{op}(\Gamma_X\bullet)$ the open edge corresponding to $x_\alpha$. Moreover, we put

$$E_{\Pi^1_X\bullet}^{\text{op},*}, e \overset{\text{def}}{=} \{ \alpha \in E_{\Pi^1_X\bullet}^{\text{op},*} \mid e = e_\alpha \}, \ e \in e^\text{op}(\Gamma_X\bullet).$$ 

Note that $e = e_\alpha$, $\alpha \in E_{\Pi^1_X\bullet}^{\text{op},*}, e$, means that the Galois admissible covering $g_\alpha^\bullet : Y_\alpha^\bullet \rightarrow Y^\bullet$ over $k$ induced by $\alpha$ is (totally) ramified over $f_X^{-1}(x_e)$, where $x_e$ denotes the marked point of $X^\bullet$ corresponding to $e$. Moreover, we have the following disjoint union

$$E_{\Pi^1_X\bullet}^{\text{op},*} = \bigsqcup_{e \in e^\text{op}(\Gamma_X\bullet)} E_{\Pi^1_X\bullet}^{\text{op},*}, e.$$ 

Let $m \in \mathbb{Z}_{\geq 0}$ and $e \in e^\text{op}(\Gamma_X\bullet)$. We shall put

$$E_{\Pi^1_X\bullet}^{\text{op},*, m} \overset{\text{def}}{=} \{ \alpha \in E_{\Pi^1_X\bullet}^{\text{op},*}, e \mid \#v_{g_\alpha}^{\text{op}} = m \}.$$ 

5.2.9. We introduce the following conditions concerning pointed stable curves. Let $W^\bullet$ be a pointed stable curve over $k$ of type $(g_W, n_W)$, $\Gamma_{W\bullet}$ the dual semi-graph of $W^\bullet$, and $\Pi_{W\bullet}$ the solvable admissible fundamental group of $W^\bullet$.

**Condition A.** We shall say that $W^\bullet$ satisfies Condition A if the following conditions are satisfied:

(i) The genus of the normalization of each irreducible component of $W$ is positive.

(ii) Every irreducible component of $W$ is smooth over $k$.

(iii) $\Gamma_{W\bullet}^{\text{cpt}}$ is 2-connected.

(iv) $\#(v(\Gamma_{W\bullet})^{k^{\leq 1}}) = 0$.

**Condition B.** We shall say that $W^\bullet$ satisfies Condition B if $\Gamma_{W^\bullet}^{\text{cpt}}$ is 2-connected for every open subgroup $H \subset \Pi_{W\bullet}$.

**Lemma 5.4.** Let $m$ be a positive natural number prime to $p$ and $H \overset{\text{def}}{=} D_m^{(3)}(\Pi_{W\bullet}) \subset \Pi_{W\bullet}$. Then $W_H^\bullet$ satisfies Condition A, and the Betti number of the dual semi-graph of $W_H^\bullet$ is positive.

**Proof.** The lemma follows from the structure of $\Pi_{W^\bullet}^p$. (cf. 1.2.2). \qed
5.3. **Reconstruction of sets of vertices, sets of closed edges, sets of genus, and sets of \( p \)-rank from open continuous homomorphisms.** In this subsection, we prove that sets of vertices, sets of closed edges, and sets of genus can be reconstructed group-theoretically from a surjective open continuous homomorphism of solvable admissible fundamental groups.

5.3.1. We fix some notation. Let \( i \in \{1, 2\} \), \( k_i \) an algebraically closed field of characteristic \( p > 0 \), and \( \ell \) a prime number distinct from \( p \). Let \( X_i \) be a pointed stable curve of type \((g_{X_i}, n_{X_i})\) over \( k_i \), \( X_i^\bullet \) the solvable admissible fundamental group of \( X_i \), \( \Gamma_i \) the dual semi-graph of \( X_i \), and \( r_{X_i} \) the Betti number of \( X_i \). Moreover, let \( v_i \in v(\Gamma_{X_i}^\bullet) \), \( \tilde{X}_{i,v_i} \) the smooth pointed stable curve of type \((g_{i,v_i}, n_{i,v_i})\) over \( k_i \) associated to \( v_i \), and \( \sigma_{i,v_i} \) the \( p \)-rank of \( \tilde{X}_{i,v_i}^\bullet \).

5.3.2. We introduce the following condition:

**Condition C.** We shall say that \( X_1 \) and \( X_2 \) satisfy Condition C if the following conditions are satisfied:

(i) \((g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})\);

(ii) \(#e(\Gamma_{X_1}^\bullet) = #e(\Gamma_{X_2}^\bullet)\);

(iii) \(#e(\Gamma_{X_1}^\bullet) = #e(\Gamma_{X_2}^\bullet)\).

5.3.3. In the remainder of the present subsection, we suppose that \( X_1 \) and \( X_2 \) satisfy Condition A, Condition B, and Condition C. Moreover, let \( \phi : \Pi_{X_1}^\bullet \to \Pi_{X_2}^\bullet \) be an arbitrary open continuous homomorphism of the solvable admissible fundamental groups of \( X_1^\bullet \) and \( X_2^\bullet \), and

\[
(g_X, n_X) \overset{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).
\]

Note that we have that \( r_{X_1} = r_{X_2} \), and that by Lemma 4.3, \( \phi \) is a surjective open continuous homomorphism. First, we have the following lemma.

**Lemma 5.5.** We maintain the notation introduced above. Then we have

\[
\text{Avr}_p(\Pi_{X_1}^\bullet) = g_X - r_X.
\]

**Proof.** The lemma follows immediately from Condition A and Theorem 2.2 (b). \( \square \)

5.3.4. Let \( G \) be a finite group such that \((\#G, p) = 1\) and

\[
f_i^\bullet : Y_i^\bullet \to X_i^\bullet
\]

a Galois admissible covering over \( k_i \) with Galois group \( G \). Let \( j \in \{1, 2\} \) such that \( i \neq j \). Then the isomorphism \( \phi^j : \Pi_{X_1}^\bullet \cong \Pi_{X_2}^\bullet \) induced by \( \phi \) implies that \( f_i^\bullet \) induces a Galois admissible covering

\[
f_j^\bullet : Y_j^\bullet \to X_j^\bullet
\]

over \( k_j \) with Galois group \( G \). We write \((g_Y, n_Y)\) for the type of \( Y_i^\bullet \), \( \Gamma_Y\) for the dual semi-graph of \( Y_i^\bullet \), and \( r_Y \) for the Betti number of \( \Gamma_Y^\bullet \).
Lemma 5.6. We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^*: Y_1^* \to X_1^*$ is étale, and that $\#v_{f_1} = m$. Then we have

$$\#e_{f_2}^{cl, ra} + \frac{1}{2}\#e_{f_2}^{op, ra} + \#v_{f_2}^{sp} \leq m.$$ 

Proof. Since $f_1^*$ is an étale covering, the Riemann-Hurwitz formula implies that

$$g_{Y_1} = \ell(g_X - 1) + 1,$$

$$g_{Y_2} = \ell(g_X - 1) + \frac{1}{2}(\ell - 1)\#e_{f_2}^{op, ra} + 1.$$ 

Then we obtain

$$g_{Y_1} - g_{Y_2} = \frac{1}{2}(\ell - 1)\#e_{f_2}^{op, ra}.$$ 

On the other hand, we have

$$r_{Y_1} = \ell\#e^{cl}(\Gamma_{X_1^*}) - \#v(\Gamma_{X_1^*}) + \#v^p_{f_1} - \ell\#v^p_{f_1} + 1$$

and

$$r_{Y_2} = \ell\#e^{cl, et} + \#e^{cl, ra} - \ell\#v^p_{f_2} - \#v^p_{f_2} + 1.$$ 

Since $\#e(\Gamma_{X_1^*}) = \#e(\Gamma_{X_2^*})$ and $\#v(\Gamma_{X_1^*}) = \#v(\Gamma_{X_2^*})$, we obtain that

$$r_{Y_1} - r_{Y_2} = (\ell - 1)\#e^{cl, ra} + (\ell - 1)(\#v^p_{f_1} - m).$$ 

Moreover, by applying Lemma 5.5 and Lemma 2.3 (b), we have

$$g_{Y_1} - g_{Y_2} \geq r_{Y_1} - r_{Y_2}.$$ 

Thus, we obtain

$$\#e_{f_2}^{cl, ra} + \frac{1}{2}\#e_{f_2}^{op, ra} + \#v_{f_2}^{sp} \leq m.$$ 

This completes the proof of the lemma. □

Corollary 5.7. We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^*: Y_1^* \to X_1^*$ is étale, and that $\#v_{f_1} = 0$. Then we have that $f_2^*: Y_2^* \to X_2^*$ is étale, and that $\#v_{f_2} = 0$.

Proof. The corollary follows immediately from Lemma 5.6. □

Corollary 5.8. We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^*: Y_1^* \to X_1^*$ is étale, and that $\#v_{f_1} = 1$. Then we have that $f_2^*: Y_2^* \to X_2^*$ is étale.

Proof. In order to verify the corollary, it is sufficient to prove that

$$\#e_{f_2}^{cl, ra} = \#e_{f_2}^{op, ra} = 0.$$ 

By applying Lemma 5.6, we have

$$\#e_{f_2}^{cl, ra} + \frac{1}{2}\#e_{f_2}^{op, ra} + \#v_{f_2}^{sp} \leq 1.$$ 

Suppose that $\#e_{f_2}^{cl, ra} \neq 0$. Since $X_2^*$ satisfies Condition A, the inequality above and the structures of the maximal prime-to-$p$ quotient of solvable admissible fundamental groups (cf. 1.2.2) imply that either (i) $\#e_{f_2}^{cl, ra} = 1$ and $\#e_{f_2}^{op, ra} \geq 2$, or (ii) $\#e_{f_2}^{cl, ra} \geq 2$ holds. Then we have $2\#e_{f_2}^{cl, ra} + \#e_{f_2}^{op, ra} + 2\#v_{f_2}^{sp} > 2$. Thus, we have $\#e_{f_2}^{cl, ra} = 0$. 

Suppose that \( \#e_{f_2}^{\text{op, ra}} \neq 0 \). Since \( \#e_{f_2}^{\text{cl, ra}} = 0 \), the inequality above implies that \( \#e_{f_2}^{\text{op, ra}} = 2 \). Let \( \ell' \neq p \) be a prime number distinct from \( \ell \), and let

\[
g_1^* : Z_1^* \to X_1^*
\]

be a Galois étale covering of over \( k_1 \) with Galois group \( \mathbb{Z}/\ell'\mathbb{Z} \) such that \( \#v_{g_1}^{\text{op}} = 0 \). Then Corollary 5.7 implies that the Galois admissible covering \( g_2^* : Z_2^* \to X_2^* \) over \( k_2 \) with Galois group \( \mathbb{Z}/\ell\mathbb{Z} \) induced by \( g_1^* \) is étale covering, and that \( \#v_{g_2}^{\text{op}} = 0 \). Write \( \Gamma_{Z_i^*} \) for the dual semi-graph of \( Z_i^* \). We obtain

\[
\#v(\Gamma_{X_i^*}) = \#v(\Gamma_{Z_1^*}) = \#v(\Gamma_{Z_2^*}) = \#v(\Gamma_{X_2^*}),
\]

\[
\ell' \#e_{\text{op}}(\Gamma_{X_i^*}) = \#e_{\text{op}}(\Gamma_{Z_1^*}) = \#e_{\text{op}}(\Gamma_{Z_2^*}) = \ell' \#e_{\text{op}}(\Gamma_{X_2^*}),
\]

\[
\ell' \#e_{\text{cl}}(\Gamma_{X_i^*}) = \#e_{\text{cl}}(\Gamma_{Z_1^*}) = \#e_{\text{cl}}(\Gamma_{Z_2^*}) = \ell' \#e_{\text{cl}}(\Gamma_{X_2^*}).
\]

We have that \( Z_1^* \) and \( Z_2^* \) satisfy Condition A, Condition B, and Condition C.

We denote by \( W_i^* \) \( \overset{\text{def}}{=} X_i^* \times_{X_i^*} Z_i^* \). Note that since \( \ell' \neq \ell \), we see that \( W_i^* \) is connected. Then \( f_i^* \) induces a Galois admissible covering

\[
h_i^* : W_i^* \to Z_i^*
\]

over \( k_i \) with Galois group \( \mathbb{Z}/\ell'\mathbb{Z} \). We have that \( h_1^* \) is étale, that \( \#v_{h_1}^{\text{op}} = 1 \), and that \( \#e_{h_2}^{\text{op, ra}} = 2\ell' \). Then Lemma 5.6 implies that

\[
1 < \#e_{h_2}^{\text{cl, ra}} + \frac{1}{2} \#e_{h_2}^{\text{op, ra}} + \#v_{h_2}^{\text{op}} = \#e_{h_2}^{\text{cl, ra}} + \ell' + \#v_{h_2}^{\text{op}} \leq 1.
\]

Thus, we obtain \( \#e_{f_2}^{\text{op, ra}} = 0 \). This completes the proof of the corollary. \( \square \)

5.3.5. We put

\[
M_{X_i^*}^{\text{def}} = \text{Hom}(\Pi_{X_i^*}, \mathbb{F}_\ell),
\]

\[
M_{X_i^*}^{\text{et}} = \text{Hom}(\Pi_{X_i^*}^{\text{et}}, \mathbb{F}_\ell),
\]

\[
M_{X_i^*}^{\text{top}} = \text{Hom}(\Pi_{X_i^*}^{\text{top}}, \mathbb{F}_\ell).
\]

Note that we have the following injections (or weight-monodromy filtration)

\[
M_{X_i^*}^{\text{top}} \hookrightarrow M_{X_i^*}^{\text{et}} \hookrightarrow M_{X_i^*} \quad (\text{or } M_{X_i^*}^{\text{top}} \subseteq M_{X_i^*}^{\text{et}} \subseteq M_{X_i^*})
\]

induced by the natural surjections \( \Pi_{X_i^*} \twoheadrightarrow \Pi_{X_i^*}^{\text{et}} \twoheadrightarrow \Pi_{X_i^*}^{\text{top}} \). Moreover, we have an isomorphism

\[
\psi_\ell : M_{X_2^*} \cong M_{X_1^*}
\]

induced by the isomorphism \( \phi_\ell : \Pi_{X_1^*} \cong \Pi_{X_2^*} \). Then we have the following propositions.

**Proposition 5.9.** We maintain the notation introduced above. Then the isomorphism

\[
\psi_\ell : M_{X_1^*} \cong M_{X_1^*}
\]

induces an isomorphism

\[
\psi_\ell^{\text{et}} : M_{X_2^*}^{\text{et}} \cong M_{X_1^*}^{\text{et}}
\]

group-theoretically. Moreover, we have the following commutative diagram:

\[
\begin{array}{ccc}
M_{X_2^*}^{\text{et}} & \xrightarrow{\psi_\ell^{\text{et}}} & M_{X_1^*}^{\text{et}} \\
\downarrow & & \downarrow \\
M_{X_2^*} & \xrightarrow{\psi_\ell} & M_{X_1^*}
\end{array}
\]
where the vertical arrows are injections.

**Proof.** To verify the proposition, it is sufficient to prove that \( \psi_{\ell}^{-1} : M_{X_1} \rightarrow M_{X_2} \) induces an isomorphism \( \psi_{\ell}^{-1,\text{ét}} : M_{X_1}^{\text{ét}} \rightarrow M_{X_2}^{\text{ét}} \) which fits into the following commutative diagram:

\[
\begin{array}{ccc}
M_{X_1}^{\text{ét}} & \xrightarrow{\psi_{\ell}^{-1,\text{ét}}} & M_{X_2}^{\text{ét}} \\
\downarrow & & \downarrow \\
M_{X_1} & \xrightarrow{\psi_{\ell}^{-1}} & M_{X_2},
\end{array}
\]

where the vertical arrows are injections.

Let \( \alpha_1 \in M_{X_1}^{\text{ét}} \) be a non-trivial element and \( f_{1,\alpha}^* : Y_1 \rightarrow X_1 \) the Galois étale covering over \( k_1 \) with Galois group \( \mathbb{Z}/\ell\mathbb{Z} \) corresponding to \( \alpha \). We put

\[
L_{X_1} \overset{\text{def}}{=} \{ \alpha_1 \in M_{X_1}^{\text{ét}} \mid \# v_{f_{1,\alpha_1}}^{\text{sp}} = 1 \}.
\]

We see that \( M_{X_1}^{\text{ét}} \) is spanned by \( L_{X_1} \) as an \( \mathbb{F}_\ell \)-linear space.

On the other hand, Corollary 5.8 implies that \( f_{1,\alpha}^* \) induces a Galois étale covering of \( X_2 \) over \( k_2 \) with Galois group \( \mathbb{Z}/\ell\mathbb{Z} \). This means that \( \psi_{\ell}^{-1} \) induces an injection of \( \mathbb{F}_\ell \)-linear spaces

\[
\psi_{\ell}^{-1,\text{ét}} : M_{X_1}^{\text{ét}} \hookrightarrow M_{X_2}^{\text{ét}}.
\]

Moreover, since \( \dim_{\mathbb{F}_\ell}(M_{X_1}^{\text{ét}}) = 2g_{X_1} - r_{X_1} = 2g_{X_2} - r_{X_2} = \dim_{\mathbb{F}_\ell}(M_{X_2}^{\text{ét}}) \), we obtain that

\[
\psi_{\ell}^{-1,\text{ét}} : M_{X_1}^{\text{ét}} \rightarrow M_{X_2}^{\text{ét}}
\]

is an isomorphism. This completes the proof of the proposition. \( \square \)

**Proposition 5.10.** We maintain the notation introduced above. Then the isomorphism \( \psi_{\ell} : M_{X_2} \rightarrow M_{X_1} \) induces an isomorphism

\[
\psi_{\ell}^{\text{top}} : M_{X_2}^{\text{top}} \rightarrow M_{X_1}^{\text{top}}
\]

group-theoretically. Moreover, we have the following commutative diagram:

\[
\begin{array}{ccc}
M_{X_2}^{\text{top}} & \xrightarrow{\psi_{\ell}^{\text{top}}} & M_{X_1}^{\text{top}} \\
\downarrow & & \downarrow \\
M_{X_2}^{\text{ét}} & \xrightarrow{\psi_{\ell}^{\text{ét}}} & M_{X_1}^{\text{ét}} \\
\downarrow & & \downarrow \\
M_{X_2} & \xrightarrow{\psi_{\ell}} & M_{X_1},
\end{array}
\]

where the vertical arrows are injections.

**Proof.** First, by Proposition 5.9, the isomorphism \( \psi_{\ell} : M_{X_2} \rightarrow M_{X_1} \) induces an isomorphism \( \psi_{\ell}^{\text{ét}} : M_{X_2}^{\text{ét}} \rightarrow M_{X_1}^{\text{ét}} \). Let \( \alpha_2 \in M_{X_2}^{\text{top}} \subseteq M_{X_2}^{\text{ét}} \) be a non-trivial element and

\[
f_{2,\alpha_2}^* : Y_2 \rightarrow X_2
\]

...
the Galois étale covering over \( k_2 \) with Galois group \( \mathbb{Z}/\ell \mathbb{Z} \) corresponding to \( \alpha_2 \). Then we obtain an element \( \alpha_1 \overset{\text{def}}{=} \psi^\text{et}_\ell(\alpha_2) \in M^\text{et}_{X_1^*} \). Write \( f^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \rightarrow X_1^* \) for the Galois étale covering over \( k_1 \) with Galois group \( \mathbb{Z}/\ell \mathbb{Z} \) corresponding to \( \alpha_1 \). Note that the types of \( Y^\bullet_{1,\alpha_1} \) and \( Y^\bullet_{2,\alpha_2} \) are equal, and that \( Y^\bullet_{1,\alpha_1} \) and \( Y^\bullet_{2,\alpha_2} \) satisfy Condition A.

Lemma 5.5 and Lemma 2.3 (b) imply that

\[
 r_{Y^\bullet_{1,\alpha_1}} \leq r_{Y^\bullet_{2,\alpha_2}},
\]

where \( r_{Y^\bullet_{1,\alpha_1}} \) and \( r_{Y^\bullet_{2,\alpha_2}} \) denote the Betti numbers of the dual semi-graphs of \( Y^\bullet_{1,\alpha_1} \) and \( Y^\bullet_{2,\alpha_2} \), respectively. Since \( \#v^\text{sp}_{f^\bullet_{2,\alpha_2}} = \#v(\Gamma_{X_1^*}) = \#v(\Gamma_{X_2^*}) \), the inequality implies \( \#v^\text{sp}_{f^\bullet_{1,\alpha_1}} = \#v(\Gamma_{X_1^*}) \). Thus, we have

\[
 \alpha_1 \in M^\text{top}_{X_1^*}.
\]

Then \( \alpha_1 \) induces an injection

\[
 \psi^\text{top}_\ell : M^\text{top}_{X_2^*} \rightarrow M^\text{top}_{X_1^*}.
\]

Moreover, since \( \dim \epsilon(M^\text{top}_{X_2^*}) = r_{X_2} = r_{X_1} = \dim \epsilon(M^\text{top}_{X_1^*}) \), we have that \( \psi^\text{top}_\ell \) is an isomorphism. This completes the proof of the proposition. \( \square \)

**Remark 5.10.1.** Proposition 5.9 and Proposition 5.10 mean that the weight-monodromy filtrations can be reconstructed group-theoretically from \( \phi \).

**Lemma 5.11.** We maintain the notation introduced above. Suppose that \( G \cong \mathbb{Z}/\ell \mathbb{Z} \), that \( f^\bullet_2 \) is étale, and that \( \#v^\text{ra}_{f^\bullet_2} = 1 \). Then we have that \( f^\bullet_1 \) is étale, and that \( \#v^\text{ra}_{f^\bullet_1} = 1 \).

**Proof.** By Proposition 5.9, we obtain that \( f^\bullet_1 \) is étale. This implies that \( g_{Y_1} = g_{Y_2} \), and that \( \#e^{\text{et}}(\Gamma_{Y_1^*}) = \ell \#e^{\text{el}}(\Gamma_{X_1^*}) = \ell \#e^{\text{el}}(\Gamma_{X_2^*}) = \#e^{\text{cl}}(\Gamma_{Y_2^*}) \). On the other hand, Lemma 5.5 and Lemma 2.3 (b) imply that

\[
 r_{Y_1} \leq r_{Y_2}.
\]

Thus, we obtain

\[
 \ell \#e^{\text{cl}}(\Gamma_{X_1^*}) - \ell(\#v(\Gamma_{X_1^*}) - \#v^\text{ra}_{f^\bullet_1}) - \#v^\text{ra}_{f^\bullet_1} + 1 \leq \ell \#e^{\text{cl}}(\Gamma_{X_2^*}) - \ell(\#v(\Gamma_{X_2^*}) - 1) - 1 + 1
\]

This implies that \( \#v^\text{ra}_{f^\bullet_1} \leq 1 \).

Suppose that \( \#v^\text{ra}_{f^\bullet_1} = 0 \). Let \( \alpha_{f_1} \in M^\text{top}_{X_1^*} \) be an element corresponding to \( f^\bullet_1 \). Then \( \alpha_{f_1} \in M^\text{top}_{X_1^*} \). Note that \( \alpha_{f_2} \overset{\text{def}}{=} (\psi^\text{et}_\ell)^{-1}(\alpha_{f_1}) \in M^\text{et}_{X_2^*} \) is the element corresponding to \( f^\bullet_2 \). Then Proposition 5.10 implies that \( \alpha_{f_2} \) is contained in \( M^\text{top}_{X_2^*} \). This means that \( \#v^\text{ra}_{f_2} = 0 \). This contradicts the assumption \( \#v^\text{ra}_{f^\bullet_2} = 1 \). Thus, we have \( \#v^\text{ra}_{f^\bullet_1} = 1 \). We complete the proof of the lemma. \( \square \)

5.3.6. We reconstruct the sets of vertices and the sets of genus of irreducible components group-theoretically from \( \phi \) as follows.

**Theorem 5.12.** We maintain the notation introduced above. Then the (surjective) open continuous homomorphism \( \phi : \Pi_{X_1^*} \rightarrow \Pi_{X_2^*} \) induces a bijection of the sets of vertices

\[
 \phi^{\text{sg,ver}} : v(\Gamma_{X_1^*}) \xrightarrow{\sim} v(\Gamma_{X_2^*})
\]

group-theoretically. Moreover, let \( v_1 \in v(\Gamma_{X_1^*}) \) and \( v_2 \overset{\text{def}}{=} \phi^{\text{sg,ver}}(v_1) \). Then we have

\[
 g_{1,v_1} = g_{2,v_2}.
\]
\textbf{Proof.} We maintain the notation introduced in Section 5.1. By applying Theorem 4.2, Proposition 5.9, and Proposition 5.10, we obtain that the following homomorphisms of the natural exact sequences can be induced group-theoretically from $\phi$:

\[
0 \longrightarrow M^{\text{top}}_{X_2^*} \longrightarrow M^{\text{et}}_{X_2^*} \longrightarrow M^{\text{et}}_{X_2^*} \longrightarrow 0
\]

Then we obtain $\psi^{\text{et}}(V^*_X, \ell) = V^*_{X_1, \ell}$, where $V^*_X, \ell$ denotes the subset of $M^{\text{et}}_{X_1^*}$ defined in 5.1.2.

Moreover, Lemma 5.11 implies that

\[
\psi^{\text{et}}(V^*_X, \ell) = V^*_{X_1, \ell}.
\]

Let $\alpha_2, \alpha_2' \in V^*_X, \ell$ distinct from each other such that $\alpha_2 \sim \alpha_2'$ (i.e., the equivalence relation defined in Proposition 5.1 (a)). By applying Lemma 5.11 again, for any $a, b \in F_\ell^+$, we see that $a\alpha_2 + b\alpha_2' \in V^*_X, \ell$ if and only $\psi^{\text{et}}(a\alpha_2 + b\alpha_2') = a\psi^{\text{et}}(\alpha_2) + b\psi^{\text{et}}(\alpha_2') \in V^*_X, \ell$. Thus, we obtain a bijection

\[
V^*_{X_2, \ell} \xrightarrow{\sim} V^*_{X_1, \ell}.
\]

Then the first part of the theorem follows from Proposition 5.1.

Next, let us prove the “moreover” part of the theorem. Let $v_i \in v(\Gamma_{X_1^*})$. We put

\[
L^v_{X_i^*} = \{ \alpha_i \in M^{\text{et}}_{X_i^*} \mid v_{f_{i, \alpha_i}} = \{v_i\}\},
\]

where $f_{i, \alpha_i}$ denotes the Galois admissible covering of $X_i^*$ over $k_i$ corresponding to $\alpha_i$.

Moreover, we denote by $[L^v_{X_i^*}]$ the image of $L^v_{X_i^*}$ in $M^{\text{et}}_{X_i^*}$. Then we have $\#[L^v_{X_i^*}] = \ell^{g_{l, v_i}} - 1$.

Suppose that $v_2 = \delta^{g_{l,v_2}}(v_1)$. Proposition 5.10 and Lemma 5.11 imply that $\psi^{\text{et}}$ induces an injection $[L^v_{X_2^*}] \hookrightarrow [L^v_{X_1^*}]$. Thus, we have

\[
\ell^{g_{l,v_2}} - 1 = \#[L^v_{X_2^*}] \leq \#[L^v_{X_1^*}] = \ell^{g_{l,v_1}} - 1.
\]

This means that $g_{l,v_2} \leq g_{l,v_1}$. On the other hand, since

\[
\sum_{v_1 \in v(\Gamma_{X_1^*})} g_{l,v_1} = g_X - r X_1 = g_X - r X_2 = \sum_{v_2 \in v(\Gamma_{X_2^*})} g_{l,v_2},
\]

we obtain

\[
g_{l,v_1} = g_{l,v_2}.
\]

This completes the proof of the theorem. \hfill \Box

5.3.7. Next, let us reconstruct the sets of closed edges from $\phi$. In the remainder of the present subsection, we fix an edge-triple

\[
\Sigma_{\Pi_{X_i^*}} \overset{\text{def}}{=} (\ell, d, \alpha_{f_{X_1}} : \Pi^{\text{et}}_{X_i^*} \rightarrow F_d)
\]

associated to $\Pi_{X_i^*}$. Then Corollary 5.7 implies that $\phi$ and the edge-triple $\Sigma_{\Pi_{X_i^*}}$ induce an edge-triple

\[
\Sigma_{\Pi_{X_i^*}} \overset{\text{def}}{=} (\ell, d, \alpha_{f_{X_2}} : \Pi^{\text{et}}_{X_2^*} \rightarrow F_d)
\]

associated to $\Pi_{X_i^*}$ group-theoretically. Write $\Pi_{Y_i^*}$ for the kernel of $\alpha_{f_{X_i}}$. The surjection $\phi : \Pi_{X_i^*} \rightarrow \Pi_{X_2^*}$ induces a surjection

\[
\phi_Y : \Pi_{Y_i^*} \rightarrow \Pi_{Y_2^*}.
\]
Moreover, the constructions of $Y_1^\bullet$ and $Y_2^\bullet$ imply that $Y_1^\bullet$ and $Y_2^\bullet$ satisfy Condition A, Condition B, and Condition C.

5.3.8. We put

$$M_{Y_1} \overset{\text{def}}{=} \text{Hom}(\Pi_{Y_1}, \mathbb{F}_\ell),$$

$$M_{Y_2}^{\text{et}} \overset{\text{def}}{=} \text{Hom}(\Pi_{Y_2}^{\text{et}}, \mathbb{F}_\ell),$$

$$M_{Y_2}^{\text{ra}} \overset{\text{def}}{=} M_{Y_2} / M_{Y_2}^{\text{et}}.$$  

Then, by Theorem 4.2 and Proposition 5.9, the following commutative diagram can be induced:

$$
\begin{array}{cccccc}
0 & \longrightarrow & M_{Y_2}^{\text{et}} & \longrightarrow & M_{Y_2} & \longrightarrow & M_{Y_2}^{\text{ra}} & \longrightarrow & 0 \\
\psi_{Y_2}^{\text{et}} & \downarrow & \psi_{Y_2} & \downarrow & \psi_{Y_2}^{\text{ra}} & \downarrow & \\
0 & \longrightarrow & M_{Y_2}^{\text{et}} & \longrightarrow & M_{Y_2}^* & \longrightarrow & M_{Y_2}^{\text{ra}} & \longrightarrow & 0,
\end{array}
$$

where all the vertical arrows are isomorphisms. Let $E_{\mathbb{Z}_\ell X_i}^* \overset{\text{def}}{=}$ the subset of $M_{Y_2}^*$ defined in 5.2.5. Since the actions of $\mu_d$ on the exact sequences are compatible with the isomorphisms appearing in the commutative diagram above, we have

$$\psi_{Y_2}(E_{\mathbb{Z}_\ell X_i}^*) = E_{\mathbb{Z}_\ell X_1}^*.$$

Let $m \in \mathbb{Z}_{\geq 0}$ and $e_i \in e^{cl}(\Gamma_{X_i^*})$. Recall that $E_{\mathbb{Z}_\ell X_i^*,e_i}^{cl,*}$ (cf. 5.2.7) is the subset of $E_{\mathbb{Z}_\ell X_i^*,e_i}^{cl,*}$ whose element $\alpha_i$ satisfies $\#n_{g_{i,\alpha_i}} = m$. Then we have the following lemma.

**Lemma 5.13.** We maintain the notation introduced above. Then we have

$$\psi_{Y_2}^{-1}(\bigsqcup_{e_i \in e^{cl}(\Gamma_{X_i^*})} E_{\mathbb{Z}_\ell X_i^*,e_i}^{cl,*}) \subseteq \bigsqcup_{e_2 \in e^{cl}(\Gamma_{X_2^*})} E_{\mathbb{Z}_\ell X_2^*,e_2}^{cl,*}.$$

Moreover, we have

$$\psi_{Y_2}^{-1}(E_{\mathbb{Z}_\ell X_1^*}^{cl,*}) = E_{\mathbb{Z}_\ell X_2^*}^{cl,*}.$$

**Proof.** Let $e_1 \in e^{cl}(\Gamma_{X_1^*})$ and $\alpha_1 \in E_{\mathbb{Z}_\ell X_1^*,e_1}^{cl,*}$. Then the Galois admissible covering $g_{1,\alpha_1} : Y_{1,\alpha} \rightarrow Y_1^\bullet$ over $k_1$ with Galois group $\mathbb{Z} / \ell \mathbb{Z}$ corresponding to $\alpha_1$ induces a Galois admissible covering $g_{2,\alpha_2} : Y_{2,\alpha_2} \rightarrow Y_2^\bullet$ over $k_2$ with Galois group $\mathbb{Z} / \ell \mathbb{Z}$. Write $\alpha_2 \in M_{Y_2}^*$ for the element corresponding to $g_{2,\alpha_2}$. We have

$$\alpha_2 \in E_{\mathbb{Z}_\ell X_2}^*.$$

Write $g_{Y_{1,\alpha_1}}$ for the genus of $Y_{1,\alpha_1}$; and $r_{Y_{1,\alpha_1}}$ for the Betti number of the dual semi-graph $\Gamma_{Y_{1,\alpha_1}}$. Then the Riemann-Hurwitz formula and Theorem 4.11 imply that

$$\begin{align*}
g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} &= \frac{1}{2}(\#e_{g_2,\alpha_2}^{\text{op},ra})(\ell - 1) = 0. \\
\end{align*}$$

On the other hand, we have

$$
\begin{align*}
r_{Y_{1,\alpha_1}} &= \ell(\#e_{g_2,\alpha_2}^{cl}(\Gamma_{Y_{1,\alpha_1}}) - d) + d - \#v(\Gamma_{Y_{1,\alpha_1}}) + 1, \\
r_{Y_{2,\alpha_2}} &= \ell \#e_{g_2,\alpha_2}^{cl,\text{et}} + \#e_{g_2,\alpha_2}^{cl,\text{ra}} - \ell \#e_{g_2,\alpha_2}^{cl,\text{sp}} - \#e_{g_2,\alpha_2}^{cl,\text{ra}} + 1.
\end{align*}
$$
Then Lemma 5.5 and Lemma 2.3 (b) imply that

\[ 0 = g_{Y_1,a_1} - g_{Y_2,a_2} \leq r_{Y_1,a_1} - r_{Y_2,a_2}. \]

Thus, we have

\[ \#e_{g_{2,a_2}}^{cl,ra} + \#v_{g_{2,a_2}}^{op,ra} + \frac{1}{2} \#e_{g_{2,a_2}}^{op,ra} = \#e_{g_{2,a_2}}^{cl,ra} + \#v_{g_{2,a_2}}^{op,ra} \leq d. \]

If \( \#e_{g_{2,a_2}}^{cl,ra} = 0 \), then \( g_{2,a_2} \) is étale. By replacing \( X_1^* \) and \( X_2^* \) by \( Y_1^* \) and \( Y_2^* \), respectively, Proposition 5.9 implies that \( g_{1,a_1} \) is also étale. This contradicts the definition of \( \alpha_1 \). Thus, we obtain \( \#e_{g_{2,a_2}}^{cl,ra} \neq 0 \).

If \( \#e_{g_{2,a_2}}^{cl,ra} \neq 0 \), then we have \( \#e_{g_{2,a_2}}^{cl,ra} = d \) and \( \#v_{g_{2,a_2}}^{op,ra} = \#e_{g_{2,a_2}}^{op,ra} = 0 \). This means that

\[ \alpha_2 \in \bigsqcup_{e_2 \in e^{cl}(\Gamma_{Y_2}^*)} E_{\Xi Y_2, e_2}^{cl, \ast, 0}. \]

Thus, we have

\[ \psi_{Y_i, \ell}^{-1} \left( \bigsqcup_{e_1 \in e^{cl}(\Gamma_{Y_1}^*)} E_{\Xi Y_1, e_1}^{\ast, 0} \right) \subseteq \bigsqcup_{e_2 \in e^{cl}(\Gamma_{Y_2}^*)} E_{\Xi Y_2, e_2}^{\ast, 0}. \]

Moreover, let \( \beta_i \in E_{\Xi Y_i}^{cl, \ast} \). Then \( \beta_i \) is a linear combination of some elements of

\[ \bigsqcup_{e_i \in e^{cl}(\Gamma_{Y_i}^*)} E_{\Xi Y_i, e_i}^{\ast, 0}. \]

Then we have \( \psi_{Y_i, \ell}^{-1}(E_{\Xi X_1}^{cl, \ast}) \subseteq E_{\Xi X_2}^{cl, \ast} \). On the other hand, since \( g_{Y_1} = g_{Y_2} \) and \( r_{Y_1} = r_{Y_2} \), Lemma 5.3 implies that \( \#\psi_{Y, \ell}^{-1}(E_{\Xi X_1}^{\ast, 0}) = \#E_{\Xi X_2}^{\ast, 0} \). Thus, we obtain

\[ \psi_{Y, \ell}^{-1}(E_{\Xi X_1}^{\ast, 0}) = E_{\Xi X_2}^{\ast, 0}. \]

This completes the proof of the lemma. \( \square \)

5.3.9. We reconstruct the sets of closed edges group-theoretically from \( \phi \) as follows.

**Theorem 5.14.** We maintain the notation introduced above. Then the (surjective) open continuous homomorphism \( \phi : \Pi X_1^* \to \Pi X_2^* \) induces a bijection of the sets of closed edges

\[ \phi^{sg, cl} : e^{cl}(\Gamma_{X_1}^*) \sim e^{cl}(\Gamma_{X_2}^*) \]

group-theoretically.

**Proof.** Let \( \alpha_2, \alpha_2' \in E_{\Xi X_2}^{cl, \ast} \) and \( \alpha_1 \overset{\text{def}}{=} \psi_{Y, \ell}(\alpha_2), \alpha_1' \overset{\text{def}}{=} \psi_{Y, \ell}(\alpha_2') \in E_{\Xi X_1}^{cl, \ast} \). Lemma 5.13 implies that \( \alpha_1 \sim \alpha_1' \) (i.e., the equivalence relation defined in Proposition 5.2 (a)) if and only if \( \alpha_2 \sim \alpha_2' \). Then the theorem follows from Proposition 5.2. \( \square \)
5.3.10. Next, let us reconstruct the sets of p-rank from \( \phi \). Note that the surjection \( \phi \) induces a surjection of the maximal pro-p quotients
\[
\phi^p : \Pi_{X_\bullet}^p \twoheadrightarrow \Pi_{X_{\bullet}^p}^p
\]
of solvable admissible fundamental groups. Then every Galois (étale) admissible covering \( h_2^\bullet : Z_2^\bullet \rightarrow X_2^\bullet \) over \( k_2 \) with Galois group \( \mathbb{Z}/p\mathbb{Z} \) induces a Galois (étale) admissible covering \( h_1^\bullet : Z_1^\bullet \rightarrow X_1^\bullet \) over \( k_1 \) with Galois group \( \mathbb{Z}/p\mathbb{Z} \). Moreover, \( \phi^p \) induces an injection
\[
\psi_p : N_{X_1^\bullet} \overset{\text{def}}{=} \text{Hom}(\Pi_{X_1^\bullet}^p, \mathbb{F}_p) \hookrightarrow N_{X_2^\bullet} \overset{\text{def}}{=} \text{Hom}(\Pi_{X_2^\bullet}^p, \mathbb{F}_p).
\]
We have the following lemmas.

**Lemma 5.15.** We maintain the notation introduced above. Suppose that \( \#v_{h_2}^{ra} = 0 \). Then we have that \( \#v_{h_1}^{ra} = 0 \). In particular, we obtain that
\[
\psi_p^{top} : N_{X_2^\bullet}^{top} \overset{\text{def}}{=} \text{Hom}(\Pi_{X_2^\bullet}^{top}, \mathbb{F}_p) \sim N_{X_1^\bullet}^{top} \overset{\text{def}}{=} \text{Hom}(\Pi_{X_1^\bullet}^{top}, \mathbb{F}_p)
\]
is an isomorphism.

**Proof.** Since \( h_2^\bullet \) is étale, the Riemann-Hurwitz formula implies that \( g_{Z_1} = g_{Z_2} \). Thus, similar arguments to the arguments given in the proofs of Proposition 5.10 imply that
\[
\#v_{h_1}^{ra} = 0.
\]
This completes the proof of the lemma. \( \square \)

**Lemma 5.16.** We maintain the notation introduced above. Suppose that \( \#v_{h_2}^{ra} = 1 \). Then we obtain that \( \#v_{h_1}^{ra} = 1 \).

**Proof.** Similar arguments to the arguments given in the proofs of Lemma 5.11 imply that \( \#v_{h_2}^{ra} \leq 1 \). If \( \#v_{h_2}^{ra} = 0 \), then the “in particular” part of Lemma 5.15 implies that \( \#v_{h_2}^{ra} = 0 \). This contradicts our assumption. Then we obtain that \( \#v_{h_1}^{ra} = 1 \). \( \square \)

5.3.11. We reconstruct the sets of p-rank of smooth pointed stable curves associated to vertices from \( \phi \) as follows.

**Theorem 5.17.** We maintain the notation introduced above. Then the (surjective) open continuous homomorphism \( \phi : \Pi_{X_\bullet} \twoheadrightarrow \Pi_{X_{\bullet}^p} \) induces an injection of the sets of vertices
\[
\psi_p^{sg,ver} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}
\]
group-theoretically. Moreover, let \( v_2 \in v(\Gamma_{X_2^\bullet})^{>0,p} \) and \( v_1 \overset{\text{def}}{=} \psi_p^{sg,ver}(v_2) \). Then we have
\[
\sigma_{2,v_2} \leq \sigma_{1,v_1}.
\]

**Proof.** Lemma 5.16 implies that
\[
\psi_p(V_{X_{2,p}^\bullet}) \subseteq V_{X_{1,p}^\bullet}.
\]
Let \( \alpha_2, \alpha_2' \in V_{X_{2,p}^\bullet} \) be elements distinct from each other such that \( \alpha_2 \sim \alpha_2' \). It is easy to see that \( a\alpha_2 + b\alpha_2' \in V_{X_{2,p}^\bullet} \) if and only if \( a\psi_p(\alpha_2) + b\psi_p(\alpha_2') \in V_{X_{1,p}^\bullet} \) for each \( a, b \in \mathbb{F}_p^\times \). Thus, by Proposition 5.1, we obtain an injection of the sets of vertices
\[
\psi_p^{sg,ver} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}.
\]
Let \( v_i \in v(\Gamma_{X_\bullet}^\bullet) \). We put
\[
L_{X_\bullet}^{v_i,p} \overset{\text{def}}{=} \{ \alpha_i \in N_{X_\bullet}^\bullet \mid v_{h_{1_i}^{ra}} = \{v_i\} \},
\]
where $h_{v_{1},\alpha_{i}}^{\star}$ denotes the Galois (étale) admissible covering corresponding to $\alpha_{i}$. Moreover, Lemma 5.16 implies that $\psi_{p}$ induces an injection $L_{X_{i}^{\star, p}}^{v_{1, p}} \hookrightarrow L_{X_{i}^{\star}}^{v_{1, p}}$.

We denote by $[L_{X_{i}^{\star}}^{v_{1, p}}]$ the image of $L_{X_{i}^{\star, p}}^{v_{1, p}}$ in $N_{X_{i}^{\star}}^{\star}/N_{X_{i}^{\star}}^{\text{top}}$. Then we have

$$\#[L_{X_{i}^{\star}}^{v_{1, p}}] = \sigma_{v_{1}, v_{2}} - 1.$$  

Suppose that $v_{1} \overset{\text{def}}{=} \psi_{p, \text{ver}}(v_{2})$. Lemma 5.15 implies that $\psi_{p}$ induces an injection

$$[L_{X_{i}^{\star}}^{v_{1, p}}] \hookrightarrow [L_{X_{i}^{\star}}^{v_{1, p}}].$$

Thus, we have

$$\sigma_{v_{1}, v_{2}} - 1 = \#[L_{X_{i}^{\star}}^{v_{1, p}}] \leq \#[L_{X_{i}^{\star}}^{v_{1, p}}] = \sigma_{v_{1}, v_{2}} - 1.$$  

This means that

$$\sigma_{v_{1}, v_{2}} \leq \sigma_{v_{1}, v_{2}}$$

for each $v_{2} \in v(\Gamma_{X_{i}^{\star}})^{>0,p}$. This completes the proof of the theorem. 

5.3.12. In the remainder of the present subsection, we prove a proposition which will be used in Section 5.5.

**Proposition 5.18.** We maintain the notation introduced above. Then the following statements hold:

(a) Let $S_{1}^{cl} \subseteq e^{cl}(\Gamma_{X_{i}^{\star}})$ be a subset of closed edges, $\alpha_{e_{1}} \in E_{\Sigma_{\Pi_{X_{i}^{\star}}}^{\text{cl}, e_{1}}}$ for every $e_{1} \in S_{1}^{cl}$,

$$\alpha_{1} \overset{\text{def}}{=} \sum_{e_{1} \in S_{1}^{cl}} \alpha_{e_{1}} \in E_{\Sigma_{\Pi_{X_{i}^{\star}}}}^{e_{1}},$$

and $g_{1, \alpha_{1}}^{\star} : Y_{1, \alpha_{1}}^{\star} \rightarrow Y_{1}^{\star}$ the Galois admissible covering over $k_{1}$ with Galois group $\mathbb{Z}/(\mathbb{Z}$ corresponding to $\alpha_{1}$. Let $\phi_{g, \text{cl}}^{\star} : e^{cl}(\Gamma_{X_{i}^{\star}}) \overset{\sim}{\rightarrow} e^{cl}(\Gamma_{X_{i}^{\star}})$ be the bijection of the sets of closed edges obtained in Theorem 5.14, $\alpha_{\phi_{g, \text{cl}}^{\star}(e_{1})} \in E_{\Sigma_{\Pi_{X_{i}^{\star}}}^{\text{cl}, \phi_{g, e_{1}}^{\star}(e_{1})}}^{e_{1}}$ the element induced by $\phi$ for every $e_{1} \in S_{1}^{cl}$,

$$\alpha_{2} \overset{\text{def}}{=} \sum_{e_{1} \in S_{1}^{cl}} \alpha_{\phi_{g, \text{cl}}^{\star}(e_{1})} \in E_{\Sigma_{\Pi_{X_{i}^{\star}}}^{e_{1}}},$$

and $g_{2, \alpha_{2}}^{\star} : Y_{2, \alpha_{2}}^{\star} \rightarrow Y_{2}^{\star}$ the Galois admissible covering over $k_{2}$ with Galois group $\mathbb{Z}/(\mathbb{Z}$ corresponding to $\alpha_{2}$. Suppose that $\#v_{1, \alpha_{1}}^{\text{op}} = 0$. Then we have that

$$\#v_{g_{1, \alpha_{1}}^{\star} \circ r_{1, \alpha_{2}}^{\star}}^{\text{op}, \alpha_{1}} = \#v_{g_{2, \alpha_{2}}^{\star}}^{\text{op}, \alpha_{2}} = 0.$$  

(b) Let $E_{\Sigma_{\Pi_{X_{i}^{\star}}}^{\text{op}, e_{1}, e_{1}}}^{\text{op}}$, $e_{1} \in e^{\text{op}}(\Gamma_{X_{i}^{\star}})$, be the set of cohomology classes defined in 5.2.8, and let $S_{1}^{\text{op}} \subseteq e^{\text{op}}(\Gamma_{X_{i}^{\star}})$ be a subset of open edges, $\alpha_{e_{1}} \in E_{\Sigma_{\Pi_{X_{i}^{\star}}}^{\text{op}, e_{1}}}$ for every $e_{1} \in S_{1}^{\text{op}}$,

$$\alpha_{1} \overset{\text{def}}{=} \sum_{e_{1} \in S_{1}^{\text{op}}} \alpha_{e_{1}} \in E_{\Sigma_{\Pi_{X_{i}^{\star}}}^{e_{1}}}^{e_{1}},$$

and $g_{1, \alpha_{1}}^{\star} : Y_{1, \alpha_{1}}^{\star} \rightarrow Y_{1}^{\star}$ the Galois admissible covering over $k_{1}$ with Galois group $\mathbb{Z}/(\mathbb{Z}$ corresponding to $\alpha_{1}$. Let $\phi_{g, \text{op}}^{\star} : e^{\text{op}}(\Gamma_{X_{i}^{\star}}) \overset{\sim}{\rightarrow} e^{\text{op}}(\Gamma_{X_{i}^{\star}})$ be the bijection of the sets of open
Thus, we have $g_{\phi^*e_{op}(e_1)} \in \mathcal{E}_{\mathcal{S}_1}^{\phi*0} \mathcal{X}_2^{\phi*0}$, the element induced by $\phi$ for every $e_1 \in \mathcal{S}_1^{op}$,

$$\alpha_2 \overset{\text{def}}{=} \sum_{e_1 \in \mathcal{S}_1^{op}} \alpha_{\phi^*e_{op}(e_1)} \in \mathcal{E}_{\mathcal{S}_1}^{\phi*0} \mathcal{X}_2^{\phi*0},$$

and $g^{\bullet}_{2\cdot o_2} : Y_{2\cdot o_2} \to Y_2^{\bullet}$ the Galois admissible covering over $k_2$ with Galois group $\mathbb{Z}/\ell \mathbb{Z}$ corresponding to $\alpha_2$. Suppose that $\#v_{g_{1\cdot o_1}} = 0$. Then we have that

$$\#e_{g_{2\cdot o_2}}^{cl, ra} = \#v_{g_{2\cdot o_2}}^{sp} = 0.$$

**Proof.** (a) Since $\#e_{g_{1\cdot o_1}}^{op, ra} = 0$, Theorem 4.11 imply that $\#e_{g_{2\cdot o_2}}^{op, ra} = 0$. On the other hand, we have

$$r_{Y_{1\cdot o_1}} = \ell(\#e_{\mathcal{S}_1}^{cl}(Y_1^{\bullet}) - d\#S_{1}^{cl}) + d\#S_{1}^{cl} - \#v(Y_1^{\bullet}) + 1,$$

$$r_{Y_{2\cdot o_2}} = \ell\#e_{g_{2\cdot o_2}}^{cl, et} + \#e_{g_{2\cdot o_2}}^{cl, ra} - \ell\#e_{g_{2\cdot o_2}}^{sp} - \#e_{g_{2\cdot o_2}}^{cl, ra} + 1.$$

Then Lemma 5.5 and Lemma 2.3 (b) imply that

$$0 = g_{Y_{1\cdot o_1}} - g_{Y_{2\cdot o_2}} \geq r_{Y_{1\cdot o_1}} - r_{Y_{2\cdot o_2}}.$$

Thus, we have

$$\#e_{g_{2\cdot o_2}}^{cl, ra} + \#v_{g_{2\cdot o_2}}^{sp} + \frac{1}{2}\#e_{g_{2\cdot o_2}}^{op, ra} = \#e_{g_{2\cdot o_2}}^{cl, ra} + \#v_{g_{2\cdot o_2}}^{sp} \leq d\#S_{1}^{cl}.$$

On the other hand, Lemma 5.13 implies that $\#e_{g_{2\cdot o_2}}^{cl, ra} = d\#S_{1}^{cl}$. Then we obtain $\#v_{g_{2\cdot o_2}}^{sp} = 0$. This completes the proof of (a).

(b) The Riemann-Hurwitz formula and Theorem 4.11 imply that

$$g_{Y_{1\cdot o_1}} - g_{Y_{2\cdot o_2}} = \frac{1}{2}(d\#S_{1}^{op} - \#e_{g_{2\cdot o_2}}^{op, ra})(\ell - 1) = 0.$$

On the other hand, we have

$$r_{Y_{1\cdot o_1}} = \ell\#e_{\mathcal{S}_1}^{cl}(Y_1^{\bullet}) - \#v(Y_1^{\bullet}) + 1,$$

$$r_{Y_{2\cdot o_2}} = \ell\#e_{g_{2\cdot o_2}}^{cl, et} + \#e_{g_{2\cdot o_2}}^{cl, ra} - \ell\#e_{g_{2\cdot o_2}}^{sp} - \#e_{g_{2\cdot o_2}}^{cl, ra} + 1.$$

Then Lemma 5.5 and Lemma 2.3 (b) imply that

$$g_{Y_{1\cdot o_1}} - g_{Y_{2\cdot o_2}} \geq r_{Y_{1\cdot o_1}} - r_{Y_{2\cdot o_2}}.$$

Thus, we have

$$\#e_{g_{2\cdot o_2}}^{cl, ra} + \#v_{g_{2\cdot o_2}}^{sp} + \frac{1}{2}\#e_{g_{2\cdot o_2}}^{op, ra} - \frac{d\#S_{1}^{op}}{2} \leq 0.$$

This means that

$$\#e_{g_{2\cdot o_2}}^{cl, ra} = \#v_{g_{2\cdot o_2}}^{sp} = 0.$$

We complete the proof of (b). \[\square\]

5.4. Reconstruction of commutative diagrams of sets of vertices, sets of open edges, and sets of closed edges from open continuous homomorphisms. We maintain the notation introduced in Section 5.3.
5.4.1. In the present subsection, we suppose that $X_1^*$ and $X_2^*$ satisfy Condition $A$, Condition $B$, and Condition $C$. Moreover, let

$$\phi : \Pi_{X_1^*} \rightarrow \Pi_{X_2^*}$$

be an arbitrary open continuous homomorphism of the solvable admissible fundamental groups of $X_1^*$ and $X_2^*$, and

$$g_X, n_X \overset{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

Note that we have $r_{X_1} = r_{X_2}$, and that by Lemma 4.3, $\phi$ is a surjection.

5.4.2. We fix some notation. Let $H_2$ be an open normal subgroup of $\Pi_{X_2^*}$, $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2)$ the open normal subgroup of $\Pi_{X_1^*}$, $G \overset{\text{def}}{=} \Pi_{X_1^*}/H_1 = \Pi_{X_2^*}/H_2$, and $\phi_{H_1}$ the surjection $\phi|_{H_1} : H_1 \rightarrow H_2$. Let $i \in \{1, 2\}$. We write

$$f^*_H : X^*_{H_i} \rightarrow X^*_{i}$$

for the Galois admissible covering over $k_i$ with Galois group $G$, $(g_{X_{H_i}}, n_{X_{H_i}})$ for the type of $X^*_{H_i}$, and $\Gamma_{X^*_{H_i}}$ for the dual semi-graph of $X^*_{H_i}$. Furthermore, we suppose that $X^*_{H_1}$ and $X^*_{H_2}$ satisfy Condition $A$, Condition $B$, and Condition $C$. Let $\ell$ and $d$ be prime numbers distinct from $p$ such that $\ell \neq d$ and $(\#G, \ell) = (\#G, d) = 1$, and let

$$\mathfrak{T}_{\Pi_{X_2^*}} \overset{\text{def}}{=} (\ell, d, \alpha_{fX_2} : \Pi_{X_2^*} \rightarrow \mathbb{F}_d)$$

be an edge-triple associated to $\Pi_{X_2^*}$ and $\mathfrak{T}_{X_2^*} \overset{\text{def}}{=} (\ell, d, f^*_X : Y^*_2 \rightarrow X^*_2)$ the edge-triple associated to $X^*_2$ corresponding to $\mathfrak{T}_{\Pi_{X_2^*}}$. By Corollary 5.7, we obtain an edge-triple

$$\mathfrak{T}_{\Pi_{X_1^*}} \overset{\text{def}}{=} (\ell, d, \alpha_{fX_1} : \Pi_{X_1^*} \rightarrow \mathbb{F}_d)$$

induced group-theoretically from $\phi$ and $\mathfrak{T}_{\Pi_{X_2^*}}$. We write $\mathfrak{T}_{X_1^*} \overset{\text{def}}{=} (\ell, d, f^*_X : Y^*_1 \rightarrow X^*_1)$ for the edge-triple associated to $X^*_1$ corresponding to $\mathfrak{T}_{\Pi_{X_1^*}}$. On the other hand, we put

$$Q_i \overset{\text{def}}{=} \ker(\Pi_{X_i^*} \rightarrow \Pi_{X_i^*} \overset{\alpha_{\phiX_i}}{\rightarrow} \mathbb{F}_d).$$

We have that $H_i \rightarrow H_i/(H_i \cap Q_i) \overset{\cong}{\rightarrow} \mathbb{F}_d$ factors through a homomorphism $\alpha_{fX_{H_i}} : H^*_i \rightarrow \mathbb{F}_d$. We see that

$$\mathfrak{T}_{H_i} \overset{\text{def}}{=} (\ell, d, \alpha_{fX_{H_i}})$$

is an edge-triple associated to $H_i$. Moreover, $\mathfrak{T}_{H_i}$ is induced group-theoretically from $H_i \subseteq \Pi_{X_i^*}$ and $\mathfrak{T}_{\Pi_{X_i^*}}$. Note that $\mathfrak{T}_{H_i}$ coincides with the edge-triple associated to $H_i$ induced group-theoretically from $\phi_{H_1}$ and $\mathfrak{T}_{H_2}$. Moreover, we denote by

$$\mathfrak{T}_{X_{H_i}^*} \overset{\text{def}}{=} (\ell, d, f^*_X : Y^*_{X_{H_i}} \overset{\text{def}}{=} Y^*_i \times_{X^*_i} X^*_H \rightarrow X^*_{H_i})$$

the edge-triple associated to $X^*_{H_i}$ corresponding to $\mathfrak{T}_{H_i}$. 
5.4.3. By applying Proposition 5.1, Remark 5.1.1, Proposition 5.2, and Remark 5.2.1, we have that the natural inclusion $H_i \hookrightarrow \Pi_{X_i^*}$ induces the following maps

$$\gamma_{H_i}^{\text{ver}, \ell} : V_{X_{H_i}, \ell} \to V_{X_i, \ell},$$
$$\gamma_{\Pi_{X_i^*}, H_i}^{\text{cl}} : E_{\Pi_{X_i^*}, H_i}^{\text{cl}} \to E_{\Pi_{X_i^*}}^{\text{cl}}$$

group-theoretically. We put

$$\gamma_{H_i}^{\text{ver}} : v(\Gamma_{X_{H_i}^*}) \xrightarrow{\kappa_{X_{H_i}^*}^{-1}} V_{X_{H_i}, \ell} \xrightarrow{\gamma_{H_i}^{\text{ver}, \ell}} V_{X_i, \ell} \xrightarrow{\kappa_{X_i, \ell}} v(\Gamma_{X_i^*}),$$
$$\gamma_{H_i}^{\text{cl}} : e^{\text{cl}}(\Gamma_{X_{H_i}^*}) \xrightarrow{\varphi_{\Pi_{X_i^*}}^{-1}} E_{\Pi_{X_i^*}, H_i}^{\text{cl}} \xrightarrow{\gamma_{\Pi_{X_i^*}, H_i}^{\text{cl}}} E_{\Pi_{X_i^*}}^{\text{cl}} \xrightarrow{\varphi_{\Pi_{X_i^*}}^{\text{cl}}} e^{\text{cl}}(\Gamma_{X_i^*}).$$

Then the maps $\gamma_{H_i}^{\text{ver}}$ and $\gamma_{H_i}^{\text{cl}}$ can be reconstructed group-theoretically from the inclusion $H_i \hookrightarrow \Pi_{X_i^*}$.

5.4.4. On the other hand, Theorem 4.2 implies that the sets $\text{Edg}^{\text{op}}(\Pi_{X_i^*})$ and $\text{Edg}^{\text{op}}(H_i)$ can be reconstructed group-theoretically from $\Pi_{X_i^*}$ and $H_i$, respectively. Note that we have a natural map

$$\text{Edg}^{\text{op}}(H_i) \to \text{Edg}^{\text{op}}(\Pi_{X_i^*})$$

induced by the natural inclusions of stabilizer subgroups. Moreover, this map compatible with the actions of $H_i$ and $\Pi_{X_i^*}$. Then we obtain a map

$$\gamma_{H_i}^{\text{op}} : e^{\text{op}}(\Gamma_{X_{H_i}^*}) \xrightarrow{\sim} \text{Edg}^{\text{op}}(H_i)/H_i \to \text{Edg}^{\text{op}}(\Pi_{X_i^*})/\Pi_{X_i^*} \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X_i^*})$$

which can be reconstructed by the inclusion $H_i \hookrightarrow \Pi_{X_i^*}$ group-theoretically.

5.4.5. By Theorem 4.11, Theorem 5.12, and Theorem 5.14, the following maps

$$\phi_{H_1}^{\text{sg}, \text{ver}} : v(\Gamma_{X_{H_1}^*}) \xrightarrow{\sim} v(\Gamma_{X_{H_2}^*}),$$
$$\phi_{H_1}^{\text{sg}, \text{op}} : e^{\text{op}}(\Gamma_{X_{H_1}^*}) \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X_{H_2}^*}),$$
$$\phi_{H_1}^{\text{sg}, \text{cl}} : e^{\text{cl}}(\Gamma_{X_{H_1}^*}) \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X_{H_2}^*}),$$
$$\phi_{\Pi_{X_i^*}}^{\text{ver}} : v(\Gamma_{X_i^*}) \xrightarrow{\sim} v(\Gamma_{X_i^*}),$$
$$\phi_{\Pi_{X_i^*}}^{\text{op}} : e^{\text{op}}(\Gamma_{X_i^*}) \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X_i^*}),$$
$$\phi_{\Pi_{X_i^*}}^{\text{cl}} : e^{\text{cl}}(\Gamma_{X_i^*}) \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X_i^*})$$

can be induced group-theoretically from $\phi_{H_1} : H_1 \to H_2$ and $\phi : \Pi_{X_1^*} \to \Pi_{X_2^*}$, respectively.

**Proposition 5.19.** We maintain the notation introduced above. Then the following diagrams

$$v(\Gamma_{X_{H_1}^*}) \xrightarrow{\phi_{H_1}^{\text{sg}, \text{ver}}} v(\Gamma_{X_{H_2}^*})$$
$$\gamma_{H_1}^{\text{ver}} \downarrow \quad \quad \quad \gamma_{H_2}^{\text{ver}} \downarrow$$
$$v(\Gamma_{X_i^*}) \xrightarrow{\phi_{\Pi_{X_i^*}}^{\text{sg}, \text{ver}}} v(\Gamma_{X_i^*})$$
where $t$ is this Galois admissible covering. Then we have

\[ e^{\text{op}}(\Gamma_{X_{H_1}^*}) \xrightarrow{\phi_{H_1}^{\text{op}}} e^{\text{op}}(\Gamma_{X_{H_2}^*}) \]
\[ \gamma_{H_1} \downarrow \quad \gamma_{H_2} \downarrow \]
\[ e^{\text{op}}(\Gamma_{X_1^*}) \xrightarrow{\phi_{H_1}^{\text{op}}} e^{\text{op}}(\Gamma_{X_2^*}), \]
\[ e^{\text{cl}}(\Gamma_{X_{H_1}^*}) \xrightarrow{\phi_{H_1}^{\text{cl}}} e^{\text{cl}}(\Gamma_{X_{H_2}^*}) \]
\[ \gamma_{H_1}^{\text{cl}} \downarrow \quad \gamma_{H_2}^{\text{cl}} \downarrow \]
\[ e^{\text{cl}}(\Gamma_{X_1^*}) \xrightarrow{\phi_{H_1}^{\text{cl}}} e^{\text{cl}}(\Gamma_{X_2^*}) \]

are commutative. Moreover, all the commutative diagrams above are compatible with the natural actions of $G$.

**Proof.** The commutativity of the second diagram follows immediately from Theorem 4.11. We treat the third diagram. To verify the commutativity of the third diagram, we only need to prove the commutativity of the following diagram

\[ e^{\text{cl}}(\Gamma_{X_{H_2}^*}) \xrightarrow{(\phi_{H_1}^{\text{cl}})^{-1}} e^{\text{cl}}(\Gamma_{X_{H_1}^*}) \]
\[ \gamma_{H_2}^{\text{cl}} \downarrow \quad \gamma_{H_1}^{\text{cl}} \downarrow \]
\[ e^{\text{cl}}(\Gamma_{X_1^*}) \xrightarrow{(\phi_{H_1}^{\text{cl}})^{-1}} e^{\text{cl}}(\Gamma_{X_2^*}). \]

Let $e_{H_2} \in e^{\text{cl}}(\Gamma_{X_{H_2}^*})$, $e_{H_1} \overset{\text{def}}{=} (\phi_{H_1}^{\text{cl}})^{-1}(e_{H_2}) \in e^{\text{cl}}(\Gamma_{X_{H_1}^*})$, $e_2 \overset{\text{def}}{=} \gamma_{H_2}^{\text{cl}}(e_{H_2}) \in e^{\text{cl}}(\Gamma_{X_2^*})$, $e_1 \overset{\text{def}}{=} (\gamma_{H_1}^{\text{cl}} \circ (\phi_{H_1}^{\text{cl}})^{-1})(e_{H_2}) \in e^{\text{cl}}(\Gamma_{X_1^*})$, and $e'_1 \overset{\text{def}}{=} (\phi_{H_1}^{\text{cl}})^{-1}(e_2) \in e^{\text{cl}}(\Gamma_{X_1^*})$. We will prove that $e_1 = e'_1$.

Write $S_{H_1}$ and $S_{H_2}$ for the sets $(\gamma_{H_1}^{\text{cl}})^{-1}(e'_1)$ and $(\gamma_{H_2}^{\text{cl}})^{-1}(e_2)$, respectively. Note that $e_{H_2} \in S_{H_2}$. To verify $e_1 = e'_1$, it is sufficient to prove that $e_{H_1} \in S_{H_1}$.

Let $\alpha_2 \in E_{\Sigma_{H_2}^{\text{cl}*},e_2}$. Then the proof of Lemma 5.13 implies that $\alpha_2$ induces an element

\[ \alpha_1 \in E_{\Sigma_{H_1}^{\text{cl}*},e'_1}. \]

Write $Y_\alpha^*$ for the pointed stable curve over $k_i$ corresponding to $\alpha_i$. We consider the Galois admissible covering

\[ Y_{\alpha_2}^* \times_{X_2^*} X_{H_2}^* \to Y_{X_{H_2}^*} \]

over $k_2$ with Galois group $\mathbb{Z}/\ell \mathbb{Z}$, and denote by $\beta_2$ the element of $E_{\Sigma_{H_2}}^{\text{cl}*}$ corresponding to this Galois admissible covering. Then we have

\[ \beta_2 = \sum_{c_2 \in S_{H_2}} t_{c_2} \beta_{c_2}, \]

where $t_{c_2} \in (\mathbb{Z}/\ell \mathbb{Z})^\times$ and $\beta_{c_2} \in E_{\Sigma_{H_2},c_2}^{\text{cl}*}$. Note that we have $t_{e_{H_2}} \neq 0$. On the other hand, the proof of Lemma 5.13 implies that $\beta_{c_2}$ induces an element $\beta(\phi_{H_1}^{\text{cl}})^{-1}(c_2) \in E_{\Sigma_{H_1}(\phi_{H_1}^{\text{cl}})^{-1}(e_2)}^{\text{cl}*}$. Then $\beta_2$ induces an element

\[ \beta_1 \overset{\text{def}}{=} \sum_{c_2 \in S_{X_{H_2}} \setminus \{e_{H_2}\}} t_{c_2} \beta(\phi_{H_1}^{\text{cl}})^{-1}(c_2) + t_{e_{H_2}} \beta_{e_{H_1}} \in E_{\Sigma_{H_1}}^{\text{cl}*}. \]
Note that since $\beta_1$ corresponds to the Galois admissible covering

$$Y_{\alpha_1} \times_{X_{\alpha_1}} X_{\alpha_1} \to Y_{\alpha_{H_1}}$$

over $k_1$ with Galois group $\mathbb{Z}/\ell \mathbb{Z}$, the composition of the Galois admissible coverings $Y_{\alpha_1} \times_{X_{\alpha_1}} X_{\alpha_1} \to Y_{\alpha_{H_1}}$ is ramified over $S_{H_1}$. This means that $e_{H_1}$ is contained in $S_{H_1}$.

Similar arguments to the arguments given in the proof above imply the first diagram is commutative. It is easy to check the “moreover” part of the proposition. This completes the proof of the proposition.

5.5. Combinatorial Grothendieck conjecture for open continuous homomorphisms. We maintain the notation introduced in Section 5.3.

5.5.1. In the present subsection, we suppose that $X_1$ and $X_2$ satisfy Condition A, Condition B, and Condition C unless indicated otherwise. Then we have $r_{X_1} = r_{X_2}$. We put $(g_X, n_X)_{\text{def}} = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let

$$\phi : \Pi_{X_1} \to \Pi_{X_2}$$

be an arbitrary open continuous homomorphism of the solvable admissible fundamental groups of $X_1$ and $X_2$. By Lemma 4.3, we have that $\phi$ is a surjection.

5.5.2. We fix some notation. Let $H_2$ be an open normal subgroup of $\Pi_{X_2}$, $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2)$ the open normal subgroup of $\Pi_{X_1}$, $G \overset{\text{def}}{=} \Pi_{X_1}/H_1 = \Pi_{X_2}/H_2$, and $\phi_{H_1} \overset{\text{def}}{=} \phi|_{H_1} : H_1 \to H_2$ the surjection induced by $\phi$. Let $i \in \{1, 2\}$. We write

$$f_{H_i} : X_{H_i} \to X_i$$

for the Galois admissible covering over $k_i$ with Galois group $G$, $(g_{X_{H_i}}, n_{X_{H_i}})$ for the type of $X_{H_i}$, $\Gamma_{X_{H_i}}$ for the dual semi-graph of $X_{H_i}$, and $r_{X_{H_i}}$ for the Betti number of $\Gamma_{X_{H_i}}$.

Lemma 5.20. We maintain the notation introduced above. Then $X_{H_i}$ satisfies Condition A, Condition B, and Condition C (i).

Proof. The first condition, the second condition, and the fourth condition of Condition A follow immediately from the definition of admissible coverings. Since $X_i$ satisfies Condition B and the third condition of Condition A, $X_{H_1}$ also satisfies Condition B and the third condition of Condition A. Moreover, Condition C (i) follows immediately from Theorem 4.11. This completes the proof of the lemma.

Lemma 5.21. We maintain the notation introduced above. Suppose that there exists an open normal subgroup $H_2' \subseteq H_2$ such that $X_{H_1}$ and $X_{H_2'}$ satisfy Condition A, Condition B, and Condition C, where $H_1' \overset{\text{def}}{=} \phi^{-1}(H_2') \subseteq H_1$. Then $X_{H_1}$ and $X_{H_2}$ satisfy Condition A, Condition B, and Condition C.

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X_{H_1}$ and $X_{H_2}$ satisfy Condition C (ii) and Condition C (iii).
Let \( G' \overset{\text{def}}{=} \Pi_{X_1^*}/H_1' = \Pi_{X_2^*}/H_2' \) and \( G'' \overset{\text{def}}{=} H_1/H_1' = H_2/H_2' \subseteq G' \). By applying Proposition 5.19, the following commutative diagrams

\[
\begin{array}{ccc}
    v(\Gamma_{X_1^*}^{H_1'}) & \xrightarrow{\phi_{H_1'}} & v(\Gamma_{X_2^*}^{H_2'}) \\
    \gamma_{H_1'} \downarrow & & \gamma_{H_2'} \downarrow \\
    v(\Gamma_{X_1^*}) & \xrightarrow{\phi_{H_1}} & v(\Gamma_{X_2^*}),
\end{array}
\]

\[
\begin{array}{ccc}
    e^{\text{op}}(\Gamma_{X_1^*}^{H_1'}) & \xrightarrow{\phi_{H_1'}^{\text{op}}} & e^{\text{op}}(\Gamma_{X_2^*}^{H_2'}) \\
    \gamma_{H_1'} \downarrow & & \gamma_{H_2'} \downarrow \\
    e^{\text{op}}(\Gamma_{X_1^*}) & \xrightarrow{\phi_{H_1}^{\text{op}}} & e^{\text{op}}(\Gamma_{X_2^*}),
\end{array}
\]

\[
\begin{array}{ccc}
    e^{\text{cl}}(\Gamma_{X_1^*}^{H_1'}) & \xrightarrow{\phi_{H_1'}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X_2^*}^{H_2'}) \\
    \gamma_{H_1'} \downarrow & & \gamma_{H_2'} \downarrow \\
    e^{\text{cl}}(\Gamma_{X_1^*}) & \xrightarrow{\phi_{H_1}^{\text{cl}}} & e^{\text{cl}}(\Gamma_{X_2^*})
\end{array}
\]

can be reconstructed group-theoretically from \( H_1' \leftrightarrow \Pi_{X_1^*}, \phi, \) and \( \phi_{H_1'} \overset{\text{def}}{=} \phi|_{H_1'} \). Moreover, the commutative diagrams are compatible with the actions of \( G'' \) and \( G' \). Then we obtain that

\[
\#v(\Gamma_{X_1^*}^{H_1'}) = \#(v(\Gamma_{X_2^*}^{H_2'})/G''') = \#(v(\Gamma_{X_2^*}^{H_2'})/G''),
\]

\[
\#e^{\text{op}}(\Gamma_{X_1^*}^{H_1'}) = \#(e^{\text{op}}(\Gamma_{X_2^*}^{H_2'})/G''') = \#(e^{\text{op}}(\Gamma_{X_2^*}^{H_2'})/G''),
\]

\[
\#e^{\text{cl}}(\Gamma_{X_1^*}^{H_1'}) = \#(e^{\text{cl}}(\Gamma_{X_2^*}^{H_2'})/G''') = \#(e^{\text{cl}}(\Gamma_{X_2^*}^{H_2'})/G'').
\]

This means that \( X_{H_1}^* \) and \( X_{H_2}^* \) satisfy Condition C. \( \square \)

**Lemma 5.22.** We maintain the notation introduced above. Suppose that \((\#G, p) = 1\), and that \( f_{H_2} \) is étale. Then \( X_{H_1}^* \) and \( X_{H_2}^* \) satisfy Condition A, Condition B, and Condition C.

**Proof.** By Lemma 5.20, to verify the lemma, we only need to prove that \( X_{H_1}^* \) and \( X_{H_2}^* \) satisfy Condition C (ii) and Condition C (iii). Moreover, since \( G \) is a finite solvable group, to verify the lemma, it is sufficient to prove the lemma when \( G \cong \mathbb{Z}/\ell\mathbb{Z} \), where \( \ell \) is a prime number distinct from \( p \). Thus, Proposition 5.9 implies that \( f_{H_1} \) is also étale.

We denote by \( H_2' \subseteq H_2 \) the inverse image \( D_1(\Pi_{X_2^*}^{H_2}) \) of the natural surjection \( \Pi_{X_2^*} \twoheadrightarrow \Pi_{X_2^*}^{H_2} \).

Then \( H_2' \) is an open normal subgroup of \( \Pi_{X_2^*} \). Let \( H_1 \overset{\text{def}}{=} \phi^{-1}(H_2') \subseteq H_1 \). We see that \( H_1 \) is equal to the inverse image \( D_1(\Pi_{X_1^*}^{H_2}) \) of the natural surjection \( \Pi_{X_1^*} \twoheadrightarrow \Pi_{X_1^*}^{H_2} \). Since \( X_1^* \) and \( X_2^* \) satisfy Condition C, Theorem 5.12 and the structures of the maximal prime-to-\( p \) quotients of solvable admissible fundamental groups (cf. 1.2.2) imply that \( X_{H_1}^* \) and \( X_{H_2}^* \) also satisfy Condition C. Then the lemma follows from Lemma 5.21. \( \square \)

**Lemma 5.23.** We maintain the notation introduced above. Suppose that \((\#G, p) = 1\). Then \( X_{H_1}^* \) and \( X_{H_2}^* \) satisfy Condition A, Condition B, and Condition C.
Proof. By Lemma 5.20, to verify the lemma, we only need to prove that \( X^*_{H_1} \) and \( X^*_{H_2} \) satisfy Condition C (ii) and Condition C (iii).

Since \( G \) is a finite solvable group, to verify the lemma, it is sufficient to prove the lemma when \( G \cong \mathbb{Z}/\ell\mathbb{Z} \), where \( \ell \) is a prime number distinct from \( p \).

Let \( \mathcal{T}_{\Pi X_*^1} = (\ell, d, \alpha_{f_{X_1}} : \Pi X_*^1 \to \mathbb{F}_d) \) be an edge-triple associated to \( \Pi X_*^1 \), \( \mathcal{T}_{\Pi X_*^2} = (\ell, d, \alpha_{f_{X_2}} : \Pi X_*^2 \to \mathbb{F}_d) \) be the edge-triple associated to \( \Pi X_*^2 \) induced by \( \phi \), and \( \mathcal{T}_{\Pi X_*^1} = (\ell, d, f_{X_*^1} : Y_*^1 \to X_*^1) \) the edge-triple associated to \( X_*^1 \) corresponding to \( \mathcal{T}_{\Pi X_*^1} \).

First, we suppose that \( f_{H_2} \) is étale over \( D_{X_2} \). Then Theorem 4.11 implies that \( f_{H_1} \) is also étale over \( D_{X_1} \). Let \( \alpha_1 \in E_{\mathcal{T}_{\Pi X_*^1}}^{cl,*,0} \), \( e_1 \in e^{cl}(\Gamma_{X_*^1}) \),

\[
\alpha_1 = \sum_{e_1 \in e^{cl}(\Gamma_{X_*^1})} \alpha_1 \in E_{\mathcal{T}_{\Pi X_*^1}}^{*,0},
\]

and \( g_{1,\alpha_1} : Y_*^{1,\alpha_1} \to Y_*^1 \) the Galois admissible covering over \( k_1 \) corresponding to \( \alpha_1 \). Note that we have \( \# e_{g_{1,\alpha_1}}^{op,ra} = \# e_{g_{1,\alpha_1}}^{sp} = 0 \). Let \( \phi^{op,cl} : e^{cl}(\Gamma_{X_*^1}) \to e^{cl}(\Gamma_{X_*^2}) \) be the bijection of the sets of closed edges obtained in Theorem 5.14, \( \alpha_{\phi^{op,cl}(e_1)} \in E_{\mathcal{T}_{\Pi X_*^2}}^{cl,*,0} \) the element induced by \( \phi \) for every \( e_1 \in e^{cl}(\Gamma_{X_*^1}) \),

\[
\alpha_2 = \sum_{e_1 \in e^{cl}(\Gamma_{X_*^1})} \alpha_{\phi^{op,cl}(e_1)} \in E_{\mathcal{T}_{\Pi X_*^2}}^{*,0},
\]

and \( g_{2,\alpha_2} : Y_*^{2,\alpha_2} \to Y_*^2 \) the Galois admissible covering over \( k_2 \) corresponding to \( \alpha_2 \). Then Proposition 5.18 (a) implies that

\[
\# e_{g_{2,\alpha_2}}^{op,ra} = \# e_{g_{2,\alpha_2}}^{sp} = 0.
\]

We obtain that \( g_{1,\alpha_1} \) is totally ramified over every node of \( Y_*^1 \), and that \( Y_*^{1,\alpha_1} \) and \( Y_*^{2,\alpha_2} \) satisfy Condition A, Condition B, and Condition C. Write \( N_i \subseteq \Pi X_*^i \) for the open normal subgroup corresponding to \( Y_*^{i,\alpha_i} \).

Let \( H_i \defeq H \cap N_i \) and \( X^*_{H_i} \) the pointed stable curve over \( k_i \) corresponding to \( H_i \). Note that \( X^*_{H_i} \) is isomorphic to a connected component of

\[
X^*_{H_i} \times X_*^1 \cdot Y_*^{g_{1,\alpha_i}}.
\]

We denote by \( h_*^i : X^*_{H_i} \to Y_*^{g_{1,\alpha_i}} \), the Galois admissible covering over \( k_i \) corresponding to the injection \( H_i \hookrightarrow N_i \). By applying Abhyankar’s lemma, \( f_{H_i} \) is étale over \( D_{X_i} \), implies that \( h_*^i \) is étale. Then the lemma follows from Lemma 5.21 and Lemma 5.22. This completes the proof of the lemma when \( f_{H_2} \) is étale over \( D_{X_2} \).

Next, let us prove the lemma in the general case. We take \( \beta_1 \in E_{\mathcal{T}_{\Pi X_*^1}}^{op,*,0} \) for every \( e_1 \in e^{op}(\Gamma_{X_*^1}) \) such that \( \# e_{g_{1,\beta_1}}^{op} = 0 \), where

\[
\beta_1 = \sum_{e_1 \in e^{op}(\Gamma_{X_*^1})} \beta_1 \in E_{\mathcal{T}_{\Pi X_*^1}}^{*,0},
\]

and \( g_{1,\beta_1} : Y_*^{1,\beta_1} \to Y_*^1 \) is the Galois admissible covering over \( k_1 \) corresponding to \( \beta_1 \). Note that we have that \( \# e_{g_{1,\beta_1}}^{cl,ra} = \# e_{g_{1,\beta_1}}^{sp} = 0 \). Let \( \phi^{op,cl} : e^{op}(\Gamma_{X_*^1}) \to e^{op}(\Gamma_{X_*^2}) \) be the bijection
of the sets of open edges obtained in Theorem 4.11, \( \beta_{\phi_{\psi,op}(e_1)} \in E^{op,*}_nX^*_{\psi,\phi_{\psi,op}(e_1)} \) the element induced by \( \phi \) for every \( e_1 \in e^{op}(\Gamma_{X^*}) \),

\[
\beta_2 \overset{\text{def}}{=} \sum_{e_1 \in e^{op}(\Gamma_{X^*})} \beta_{\phi_{\psi,op}(e_1)} \in E^{op}_nX^*_{\psi},
\]

and \( g_{2,\beta_2} : Y^*_{2,\beta_2} \to Y^*_2 \) the Galois admissible covering over \( k_2 \) corresponding to \( \beta_2 \). Then Proposition 5.18 (b) implies that

\[
\#v_{g_{2,\beta_2}}^{cl,ra} = \#v_{g_{2,\beta_2}}^{sp} = 0.
\]

We obtain that \( g_{i,\beta_i} \) is totally ramified over every marked point of \( Y_i \), and that \( Y^*_{1,\beta_1} \) and \( Y^*_{2,\beta_2} \) satisfy Condition A, Condition B, and Condition C. Write \( Q_i \subseteq \Pi_{X^*} \) for the open normal subgroup corresponding to \( Y^*_{i,\beta_i} \).

Let \( H''_i \overset{\text{def}}{=} H_i \cap Q_i \) and \( X''_{H''_i} \) the pointed stable curve over \( k_i \) corresponding to \( H''_i \). Note that \( X''_{H''_i} \) is isomorphic to a connected component of

\[
X^*_{H_i} \times X^*_{2,\beta_2} \cap Y^*_{g_{2,\beta_2}}.
\]

We denote by \( h^*_i : X''_{H''_i} \to X^*_{H_i} \) the Galois admissible covering over \( k_i \) corresponding to the injection \( H''_i \hookrightarrow Q_i \). By applying Abhyankar’s lemma, \( h^*_i \) is étale over \( D_{Y^*_{g_{2,\beta_2}}} \). By applying the lemma in the case where \( g_{i,\beta_i} \) is étale over \( D_1 \), we obtain that \( X''_{H''_i} \) and \( X^*_{H_i} \) satisfy Condition A, Condition B, and Condition C. Then the lemma follows from Lemma 5.21. We complete the proof of the lemma.

**Lemma 5.24.** We maintain the notation introduced above. Suppose that \( G \) is a \( p \)-group. Then \( X''_{H_i} \) and \( X^*_{H_i} \) satisfy Condition A, Condition B, and Condition C.

**Proof.** By Lemma 5.20, to verify the lemma, we only need to prove that \( X''_{H_i} \) and \( X^*_{H_i} \) satisfy Condition C (ii) and Condition C (iii).

To verify the lemma, without loss the generality, it is sufficient to treat the case where \( G \cong \mathbb{Z}/p\mathbb{Z} \). Since \( f^*_i \) is étale, \( X''_{H_i} \) and \( X^*_{H_i} \) satisfy Condition C (iii).

Let \( V_i \subseteq v(\Gamma_{X^*})^{>0,p} \) be the subset of vertices such that the natural (outer) homomorphism

\[
\Pi_{X^*_{v_i}} \to \Pi_{X^*_i} \to G \overset{\text{def}}{=} \Pi_{X^*_i}/H_i
\]

is non-trivial (since \( G \cong \mathbb{Z}/p\mathbb{Z} \), the homomorphism is a surjection). Then we obtain

\[
\#v(\Gamma_{X^*_{H_i}}) = p(\#v(\Gamma_{X^*_i}) - \#V_i) + \#V_i \quad \text{and} \quad \#v^{cl}(\Gamma_{X^*_{H_i}}) = p\#v^{cl}(\Gamma_{X^*_i}).
\]

Let

\[
\psi^p_{v,ver} : v(\Gamma_{X^*_i})^{>0,p} \hookrightarrow v(\Gamma_{X^*_i})^{>0,p}
\]

be the injection induced by \( \phi \), which is obtained in Theorem 5.17. We put

\[
V'_1 = \{ v^p_{v,ver}(v_2) \}_{v_2 \in V_2} \subseteq v(\Gamma_{X^*_i})^{>0,p}.
\]

By applying Lemma 5.16, we see that

\[
V_1 = V'_1.
\]

Thus, we have \( \#v(\Gamma_{X^*_{H_i}}) = \#v(\Gamma_{X^*_{H_i}}) \) and \( \#v^{cl}(\Gamma_{X^*_{H_i}}) = \#v^{cl}(\Gamma_{X^*_{H_i}}) \). This completes the proof of the lemma.

\[\square\]
Proposition 5.25. We maintain the notation introduced above. Then $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ satisfy Condition A, Condition B, and Condition C.

Proof. Since $G$ is a solvable group, the proposition follows from Lemma 5.23 and Lemma 5.24.

5.5.3. Next, we prove the main result of the present section which we call the combinatorial Grothendieck conjecture for open continuous homomorphisms.

Theorem 5.26. We maintain the notation introduced above. Then the surjective open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ induces the following surjective maps

$$\phi^{\text{ver}} : \text{Ver}(\Pi_{X_1^\bullet}) \to \text{Ver}(\Pi_{X_2^\bullet}),$$

$$\phi^{\text{edg}, \text{op}} : \text{Edg}^\text{op}(\Pi_{X_1^\bullet}) \to \text{Edg}^\text{op}(\Pi_{X_2^\bullet}),$$

$$\phi^{\text{edg}, \text{cl}} : \text{Edg}^\text{cl}(\Pi_{X_1^\bullet}) \to \text{Edg}^\text{cl}(\Pi_{X_2^\bullet})$$

group-theoretically. Moreover, $\phi$ induces an isomorphism

$$\phi^{\text{sg}} : \Gamma_{X_1^\bullet} \cong \Gamma_{X_2^\bullet}$$

of the dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$ group-theoretically.

Proof. By applying Theorem 4.11, the homomorphism $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ induces a surjective map $\phi^{\text{edg}, \text{op}} : \text{Edg}^\text{op}(\Pi_{X_1^\bullet}) \to \text{Edg}^\text{op}(\Pi_{X_2^\bullet})$ group-theoretically. We only need to treat the cases of $\phi^{\text{ver}}$ and $\phi^{\text{edg}, \text{cl}}$, respectively.

Let $C_{\Pi_{X_2^\bullet}}$ be a cofinal system of $\Pi_{X_2^\bullet}$ which consists of open normal subgroups of $\Pi_{X_2^\bullet}$. We put

$$C_{\Pi_{X_1^\bullet}} \overset{\text{def}}{=} \{ H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \mid H_2 \in C_{\Pi_{X_2^\bullet}} \}.$$ 

Note that $C_{\Pi_{X_1^\bullet}}$ is not a cofinal system of $\Pi_{X_1^\bullet}$ in general. Moreover, by applying Proposition 5.25, we have that $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ satisfy Condition A, Condition B, and Condition C for every $H_2 \in C_{\Pi_{X_2^\bullet}}$ and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \in C_{\Pi_{X_1^\bullet}}$.

We treat the case of $\phi^{\text{ver}}$. Let $\hat{X}_i^\bullet$ be the universal solvable admissible covering of $X_i^\bullet$ associated to $\Pi_{X_i^\bullet}$ and $\Gamma_{\hat{X}_i^\bullet}$ the dual semi-graph of $\hat{X}_i^\bullet$. Let $\hat{w}_1 \in v(\Gamma_{\hat{X}_1^\bullet})$ and $\Pi_{\hat{w}_1}$ the stabilizer subgroup of $\hat{w}_1$. Write $w_{H_1} \in v(\Gamma_{X_{H_1}^\bullet})$, $H_1 \in C_{\Pi_{X_1^\bullet}}$, for the image of $\hat{w}_1$. Proposition 5.19 implies that $\phi$ induces a cofinal system of vertices

$$C_{\hat{w}_2} \overset{\text{def}}{=} \{ w_{H_2} \overset{\text{def}}{=} \phi^{\text{ver}}(w_{H_1}) \mid H_2 \in C_{\Pi_{X_2^\bullet}} \},$$

which admits a natural action of $\Pi_{X_2^\bullet}$. Then we obtain an element $\hat{w}_2 \in v(\Gamma_{\hat{X}_2^\bullet})$. Moreover, the stabilizer of $C_{\hat{w}_2}$ is $\Pi_{\hat{w}_2}$. We see immediately that $\phi$ induces a surjective open continuous homomorphism

$$\phi|_{\Pi_{\hat{w}_1}} : \Pi_{\hat{w}_1} \to \Pi_{\hat{w}_2}$$

group-theoretically. Then we define

$$\phi^{\text{ver}} : \text{Ver}(\Pi_{X_1^\bullet}) \to \text{Ver}(\Pi_{X_2^\bullet}), \Pi_{\hat{w}_1} \mapsto \Pi_{\hat{w}_2}.$$
Next, we prove that $\phi^\text{ver}$ is a surjective map. Let $\widehat{v}_2 \in v(\Gamma_{X^*_2})$ and $\Pi_{\widehat{v}_2}$ the stabilizer subgroup of $\widehat{v}_2$. Write $v_{H_2} \in v(\Gamma_{X^*_2})$, $H_2 \in \mathcal{C}_{\Pi_{\widehat{v}_2}}$, for the image of $\widehat{v}_2$. Then we obtain a cofinal system of vertices

$$C_{\widehat{v}_2} \overset{\text{def}}{=} \{v_{H_2} \mid H_2 \in \mathcal{C}_{\Pi_{\widehat{v}_2}}\}$$

associated to $\widehat{v}_2$. The cofinal system $C_{\widehat{v}_2}$ admits a natural action of $\Pi_{\widehat{v}_2}$. We see immediately that the stabilizer of $C_{\widehat{v}_2}$ is equal to $\Pi_{\widehat{v}_2}$. Proposition 5.19 implies that $\phi$ and $C_{\widehat{v}_2}$ induce a set of

$$C' \overset{\text{def}}{=} \{v_{H_1} \overset{\text{def}}{=} (\phi^\text{sg,ver})^{-1}(v_{H_2}) \mid H_2 \in \mathcal{C}_{\Pi_{\widehat{v}_2}}\}$$

group-theoretically. By extending $C'$ to a cofinal system of vertices. Then we obtain an element $\widehat{v}_1 \in v(\Gamma_{X_1^*})$ such that the image of $\widehat{v}_1$ in $v(\Gamma_{X_{H_1}})$ is $v_{H_1}$. Thus, $\phi$ induces a surjective map

$$\phi|_{\Pi_{\widehat{v}_1}} : \Pi_{\widehat{v}_1} \rightarrow \Pi_{\widehat{v}_2}.$$ 

This means that $\phi^\text{ver}$ is a surjection.

By applying similar arguments to the arguments given in the proof above, we obtain that $\phi$ induces a surjective map $\phi^\text{edg,cl} : \text{Edg}^{\text{cl}}(\Pi_{X^*_1}) \rightarrow \text{Edg}^{\text{cl}}(\Pi_{X^*_2})$ group-theoretically. This completes the proof of the first part of the theorem.

The surjections $\phi^\text{ver}$, $\phi^\text{edg,op}$, and $\phi^\text{edg,cl}$ imply the following surjections

$$\phi^\text{sg,ver} : v(\Gamma_{X^*_1}) \rightarrow \text{Ver}(\Pi_{X^*_1})/\Pi_{X^*_1} \rightarrow \text{Ver}(\Pi_{X^*_2})/\Pi_{X^*_2} \rightarrow v(\Gamma_{X^*_2}),$$

$$\phi^\text{sg,op} : e^{\text{op}}(\Gamma_{X^*_1}) \rightarrow \text{Edg}^{\text{op}}(\Pi_{X^*_1})/\Pi_{X^*_1} \rightarrow \text{Edg}^{\text{op}}(\Pi_{X^*_2})/\Pi_{X^*_2} \rightarrow e^{\text{op}}(\Gamma_{X^*_2}),$$

$$\phi^\text{sg,cl} : e^{\text{cl}}(\Gamma_{X^*_1}) \rightarrow \text{Edg}^{\text{cl}}(\Pi_{X^*_1})/\Pi_{X^*_1} \rightarrow \text{Edg}^{\text{cl}}(\Pi_{X^*_2})/\Pi_{X^*_2} \rightarrow e^{\text{cl}}(\Gamma_{X^*_2}).$$

Since $X^*_1$ and $X^*_2$ satisfy Condition C, we have that $\phi^\text{sg,ver}$, $\phi^\text{sg,op}$, and $\phi^\text{sg,cl}$ are bijections. Furthermore, by applying [HM, Lemma 1.5, Lemma 1.7, and Lemma 1.9], $\phi$ induces an isomorphism of dual semi-graphs

$$\phi^\text{sg} : \Gamma_{X^*_1} \rightarrow \Gamma_{X^*_2}$$

group-theoretically. This completes the proof of the theorem. □

**Remark 5.26.1.** We maintain the notation introduced above. We see immediately that Theorem 5.26 does not hold without Condition C (e.g. $X^*_1$ is a generic curve of $\overline{M}_{g,n}$, and $X^*_2$ is a singular curve).

On the other hand, although the author cannot prove this at the present time, he believes that Theorem 5.26 also holds without Condition B (e.g. $n_{X^*_i} = 0$). The main difficult is that we do not have a precise formula for limits of $p$-averages of arbitrary pointed stable curves. Moreover, if the question of [Y5, Remark 4.10.2] is true, without too much difficulty, similar arguments to the arguments given in the proofs of this section imply that Theorem 5.26 holds without Condition B.

**Corollary 5.27.** We maintain the notation introduced above. Let $Q_2 \subseteq \Pi_{X^*_2}$ be an arbitrary open subgroup and $Q_1 \overset{\text{def}}{=} \phi^{-1}(Q_2) \subseteq \Pi_{X^*_1}$. Then we have

$$\text{Avr}_p(Q_1) = \text{Avr}_p(Q_2).$$

**Proof.** The corollary follows immediately from Theorem 5.26. □
5.5.4. In the remainder of this subsection, we will prove that if \( g_X = 0 \), Theorem 5.26 holds without Condition A and Condition B (cf. Theorem 5.30 below), which will play a key role in the proof of the main theorem of the next section. Furthermore, although the author cannot prove this at the present time, he also believes that Theorem 5.26 holds without Condition A and Condition B.

**Lemma 5.28.** Let \( E^* \) be a pointed stable curve of type \((0, n)\) over an algebraically closed field \( k \) of characteristic \( p > 0 \), \( \Pi_{E^*} \) the solvable admissible fundamental group of \( E^* \), and \( \ell \) a prime number such that \( \ell \neq p \), and that \( \ell > n \). We put

\[
\text{Edg}^{op, \ell, \text{ab}}(\Pi_{E^*}) \overset{\text{def}}{=} \{ \text{pr}^{\ell, \text{ab}}(I_x) \mid I_x \in \text{Edg}^{op}(\Pi_{E^*}) \} = \{ I_e \}_{e \in e^{op}(\Gamma_{E^*})},
\]

where \( \text{pr}^{\ell, \text{ab}} \) denotes the natural surjective homomorphism \( \Pi_{E^*} \rightarrow \Pi_{E^*}^{\ell, \text{ab}} \), and \( I_e \overset{\text{def}}{=} \text{pr}^{\ell, \text{ab}}(I_x) \). Let \( a_e \in I_e, e \in e^{op}(\Gamma_{E^*}) \), be a generator of \( I_e \) such that

\[
\prod_{e \in e^{op}(\Gamma_{E^*})} a_e = 1,
\]

and let \( \alpha : \Pi_{E^*}^{\ell, \text{ab}} \rightarrow \mathbb{F}_\ell \) be a surjection and \( r_e \overset{\text{def}}{=} \alpha(a_e) \). Write

\[
g^* : X^* \rightarrow E^*
\]

for the Galois admissible covering over \( k \) with Galois group \( \mathbb{Z}/\ell \mathbb{Z} \) corresponding to \( \alpha \). Suppose that \( r_e \neq 0 \) for every \( e \in e^{op}(\Gamma_{E^*}) \), and that

\[
\sum_{e \in e^{op}(\Gamma_{E^*})} r_e = \ell
\]

if we identify \( \mathbb{F}_\ell \) with \( \{0, 1, \ldots, \ell - 1\} \subseteq \mathbb{Z} \). Then \( g^* \) is totally ramified over every node and every marked point of \( E^* \). In particular, we have that the map of dual semi-graphs \( \Gamma_{X^*} \rightarrow \Gamma_{E^*} \) of \( X^* \) and \( E^* \) induced by \( g^* \) is an isomorphism (as semi-graphs), and that \( X^* \) satisfies Condition A.

**Proof.** We prove the lemma by induction on \( \#v(\Gamma_{E^*}) \). Suppose that \( \#v(\Gamma_{E^*}) = 1 \). Then the lemma is trivial.

Suppose that \( \#v(\Gamma_{E^*}) \geq 2 \). Let \( v_0 \in v(\Gamma_{E^*}) \) and \( \tilde{E}_{v_0} \) the smooth pointed stable curve associated to \( v_0 \) (cf. 1.1.2). Note that the underlying curve \( \tilde{E}_{v_0} \) coincides with the irreducible component of \( E \) corresponding to \( v_0 \). On the other hand, we define a pointed stable curve over \( k \) to be

\[
E_0^* = (E_0^{\text{def}} = E \setminus \tilde{E}_{v_0}, D_{E_0}^{\text{def}} = (D_E \cap E_0) \cup (E_0 \cap \tilde{E}_{v_0})),
\]

where \( E \setminus \tilde{E}_{v_0} \) denotes the topological closure of \( E \setminus \tilde{E}_{v_0} \) in \( E \). Then \( g^* \) induces Galois admissible coverings

\[
g^*_{v_0} : X_{v_0}^* \rightarrow \tilde{E}_{v_0}^*,
\]

\[
g_0^* : X_0^* \rightarrow E_0^*
\]

over \( k \) with Galois group \( \mathbb{Z}/\ell \mathbb{Z} \). To verify the lemma, we only need to prove that \( g^*_{v_0} \) and \( g_0^* \) are totally ramified over every node and every marked point of \( \tilde{E}_{v_0}^* \) and \( E_0^* \), respectively.
Let $\Pi_{E_0}$ and $\Pi_{E_0^*}$ be the solvable admissible fundamental groups of $\tilde{E}_0^*$ and $E_0^*$, respectively. Since $\Gamma_{E_0^*}^{\text{op}}$ is 2-connected, [Y5, Corollary 3.5] implies that the natural homomorphism

$$\theta_{e_0} : \Pi_{E_0^*}^{\ell,ab} \to \Pi_{E_0^*}^{\ell,ab}$$

is an injection. Let

$$\theta_0 : \Pi_{E_0^*}^{\ell,ab} \to \Pi_{E_0^*}^{\ell,ab}$$

be the homomorphism induced by the natural (outer) injective homomorphism $\Pi_{E_0^*} \to \Pi_{E_0^*}$ (in fact, $\theta_0$ is also an injection).

Let $\{x\} = E_0 \cap \tilde{E}_0$, $e_{v_0} \in e^{\text{op}}(\Gamma_{E_0^*})$ the open edge corresponding to $x$, $e_0 \in e^{\text{op}}(\Gamma_{E_0^*})$ the open edge corresponding to $x$, $\tilde{E}_{v_0}$ the universal solvable admissible covering of $\tilde{E}_{v_0}$, $\tilde{E}_0$, the universal solvable admissible covering of $E_0$, $\tilde{e}_0 \in e^{\text{op}}(\Gamma_{E_0^*})$ an element over $e_{v_0}$, and $e_0 \in e^{\text{op}}(\Gamma_{E_0^*})$ an element over $e_0$. We denote by $I_{e_0}$ the image of $I_{e_0}$ of $\Pi_{E_0^*} \to \Pi_{E_0^*}^{\ell,ab}$, and by $I_{e_0}$ the image of $I_{e_0}$ of $\Pi_{E_0^*} \to \Pi_{E_0^*}^{\ell,ab}$. We put

$$a_{e_{v_0}} = \prod_{e \in E_0^* \setminus \{e_{v_0}\}} a_e^{-1},$$

$$a_{e_0} = \prod_{e \in E_0^* \setminus \{e_0\}} a_e^{-1}.$$ Then $a_{e_{v_0}}$ and $a_{e_0}$ are generators of $I_{e_{v_0}}$ and $I_{e_0}$, respectively. Moreover, we put

$$\tilde{\alpha}_{e_{v_0}} : \Pi_{E_0^*}^{\ell,ab} \to \Pi_{E_0^*}^{\ell,ab} \to \mathbb{F}_\ell,$$

$$\alpha_0 : \Pi_{E_0^*}^{\ell,ab} \to \Pi_{E_0^*}^{\ell,ab} \to \mathbb{F}_\ell.$$ Then the structures of maximal pro-prime-to-$p$ quotients of solvable admissible fundamental groups (cf. 1.2.2) implies that

$$\tilde{\alpha}_{e_{v_0}}(a_{e_{v_0}}) = \ell - \sum_{e \in E_0^* \setminus \{e_{v_0}\}} r_e = \sum_{e \in E_0^* \setminus \{e_0\}} r_e,$$

$$\alpha_0(a_{e_0}) = \sum_{e \in E_0^* \setminus \{e_0\}} r_e.$$ Thus, by induction, we have that $g_{e_{v_0}}$ and $g_0^*$ are totally ramified over every node and every marked point of $\tilde{E}_{e_{v_0}}$ and $E_0^*$, respectively. We complete the proof of the lemma. \qed

**Lemma 5.29.** Let $E^*$ be a pointed stable curve of type $(0,n)$ over an algebraically closed field $k$ of characteristic $p > 0$. Then $E^*$ satisfies Condition B.

**Proof.** Let $f^* : W^* \to E^*$ be an arbitrary admissible covering over $k$, $\Gamma_{W^*}$ the dual semi-graph of $W^*$, and $f_{\text{reg}} : \Gamma_{W^*} \to \Gamma_{E^*}$ the map of dual semi-graphs of $W^*$ and $X^*$ induced by $f^*$. To verify the lemma, we only need to prove that $\Gamma_{W^*}^{\text{op}}$ is 2-connected.

Suppose that $f^*$ is trivial. Then the lemma follows from that $\Gamma_{E^*}^{\text{op}}$ is 2-connected.

Suppose that $f^*$ is non-trivial. Let $w \in v(\Gamma_{W^*})$ and $v \in v(\Gamma_{E^*})$. We denote by $\pi_0(w)$ and $\pi_0(v)$ the sets of connected components of $\Gamma_{W^*} \setminus \{w\}$ and $\Gamma_{E^*} \setminus \{v\}$, respectively.
Suppose that \( v = f^{ss}(w) \). Let \( C_w \in \pi_0(w) \). We see immediately that \( f^{ss}(C_w) \) is a connected component of \( \Gamma_{E^* \setminus \{ v \}} \). Write \( C_v \) for \( f^{ss}(C_w) \). Since \( C_v \cap e^{op}(\Gamma_{E^*}) \neq \emptyset \), we obtain that \( C_w \cap e^{op}(\Gamma_{E^*}) \neq \emptyset \). Thus, \( \Gamma_{E^*}^{edg} \) is 2-connected. This completes the proof of the lemma.

5.5.5. Moreover, Theorem 5.26 implies the following important result.

**Theorem 5.30.** Let \( i \in \{1, 2\} \), and let \( E_i^* \) be a pointed stable curve of type \((0, n)\) over \( k_i \) of characteristic \( p > 0 \), \( \Pi_{E_i^*} \) the solvable admissible fundamental group of \( E_i^* \), and

\[
\phi_E : \Pi_{E_1^*} \to \Pi_{E_2^*},
\]

an arbitrary open continuous homomorphism. Suppose that \( E_1^* \) and \( E_2^* \) satisfy Condition C. Then \( \phi_E : \Pi_{E_1^*} \to \Pi_{E_2^*} \) induces the following surjective maps

\[
\phi_E^{ver} : \text{Ver}(\Pi_{E_1^*}) \to \text{Ver}(\Pi_{E_2^*}),
\]

\[
\phi_E^{edg, op} : \text{Edg}^{op}(\Pi_{E_1^*}) \to \text{Edg}^{op}(\Pi_{E_2^*}),
\]

\[
\phi_E^{edg, cl} : \text{Edg}^{cl}(\Pi_{E_1^*}) \to \text{Edg}^{cl}(\Pi_{E_2^*})
\]

group-theoretically. Moreover, \( \phi_E \) induces an isomorphism

\[
\phi_E^{ss} : \Gamma_{E_1^*} \cong \Gamma_{E_2^*}
\]

of the dual semi-graphs of \( E_1^* \) and \( E_2^* \) group-theoretically.

**Proof.** Lemma 4.3 implies that \( \phi_E \) is a surjective map. By applying Theorem 4.11, the homomorphism \( \phi_E : \Pi_{E_1^*} \to \Pi_{E_2^*} \) induces a surjective map \( \phi^{edg, op} : \text{Edg}^{op}(\Pi_{E_1^*}) \to \text{Edg}^{op}(\Pi_{E_2^*}) \) group-theoretically. We only need to treat the cases of \( \phi_E^{ver} \) and \( \phi_E^{edg, cl} \), respectively.

Let \( \ell \) be a prime number such that \( \ell \neq p \), and that \( \ell >> n \). Let

\[
\alpha_2 : \Pi_{E_2^{ab}}^{edg} \to \mathbb{F}_\ell
\]

satisfying the assumptions of Lemma 5.28. Then Theorem 4.11 implies that \( \phi_E \) and \( \alpha_2 \) induces a surjection

\[
\alpha_1 : \Pi_{E_1^{ab}}^{edg} \to \mathbb{F}_\ell,
\]

which satisfies the assumptions of Lemma 5.28. Write \( g_i^* : X_i^* \to E_i^* \) for the Galois admissible covering over \( k_i \) with Galois group \( \mathbb{Z}/\ell\mathbb{Z} \). Then Lemma 5.28 and Lemma 5.29 imply that \( X_1^* \) and \( X_2^* \) satisfy Condition A, Condition B, and Condition C.

Write \( \Pi_{X_1^*} \subseteq \Pi_{E_1^*} \) for the open normal subgroup corresponding to \( g_1^* \). Let \( \Pi_{\tilde{E}_1} \in \text{Ver}(\Pi_{X_1^*}) \), \( \Pi_{\tilde{E}_1} \in \text{Edg}^{cl}(\Pi_{X_1^*}) \), \( \Pi_{\tilde{E}_1} \in \text{Ver}(\Pi_{E_1^*}) \), \( \Pi_{\tilde{E}_1} \in \text{Edg}^{cl}(\Pi_{E_1^*}) \) the unique element which contains \( \Pi_{\tilde{E}_1} \), and \( \Pi_{\tilde{E}_1} \in \text{Edg}^{cl}(\Pi_{E_1^*}) \) the unique element which contains \( I_{\tilde{E}_1} \). Since \( \Pi_{\tilde{E}_1} \) and \( I_{\tilde{E}_1} \) the normalizers of \( \Pi_{\tilde{E}_1} \) and \( I_{\tilde{E}_1} \) in \( \Pi_{E_1^*} \), respectively, the theorem follows immediately from Theorem 5.26. This completes the proof of the theorem.

\[\square\]
6. The Homeomorphism Conjecture for closed points when $g = 0$

We maintain the notation introduced in Section 3. In this section, we will prove that
$$\pi^{\mathrm{adm}}_{g,n}([q])$$
is a closed point of $\mathfrak{M}_{g,n}$ for every $[q] \in \mathfrak{M}^{\mathrm{cl}}_{g,n}$ if $g = 0$. In particular, the
Homeomorphism Conjecture holds when $(g,n) = (0,3), (0,4)$. In the present section, we shall assume that all the fundamental groups of pointed stable curves are solvable admissible fundamental groups unless indicated otherwise.

6.0.1. We fix some notation. Let $i \in \{1,2\}$, and let $X^\bullet_i$ be a pointed stable curve of

type $(0,n)$ over an algebraically closed field $k$. For simplicity, we shall use the notation
$X_{i,v_i}$ to denote the smooth pointed stable curve $\tilde{X}^\bullet_{i,v_i}$ of type $(0,n,v_i)$ over $k_i$ associated to $v_i \in v(\Gamma_{X^\bullet_i})$. On the other hand, let $\Pi^\bullet_i$ be the
solvable admissible fundamental group of $X^\bullet_i$ and
$$\phi : \Pi^\bullet_1 \to \Pi^\bullet_2$$
an arbitrary open continuous homomorphism. By Lemma 4.3, we see that $\phi$ is a surjective
open continuous homomorphism. Then $\phi$ induces an isomorphism
$$\phi^p : \Pi^p_{X^\bullet_1} \cong \Pi^p_{X^\bullet_2}$$
of the maximal prime-to-$p$ quotients of solvable admissible fundamental groups. Let $\hat{X}^\bullet_i$
be the universal solvable admissible covering of $X^\bullet_i$ corresponding to $\Pi^\bullet_i$, $\Gamma_{\hat{X}^\bullet_i}$ the
dual semi-graph of $\hat{X}^\bullet_i$, and $e_i \in e^{\mathrm{op}}(\Gamma_{X^\bullet_i})$. We put
$$\mathrm{Edg}_{e_i}^{\mathrm{op}}(\Pi_{X^\bullet_i}) \overset{\text{def}}{=} \{ \hat{e}_i \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet_i}) \mid \hat{e}_i \in e^{\mathrm{op}}(\Gamma_{\hat{X}^\bullet_i}) \text{ is an open edge over } e_i \}.$$Moreover, in the remainder of the present section, we shall suppose that $k_1$ is an algebraic
closure of $\mathbb{F}_p$.

6.0.2. Denote by
$$\mathrm{Hom}^{\mathrm{open}}_{\mathrm{pro-gps}}(-,-), \mathrm{Isom}^{\mathrm{pro-gps}}(-,-)$$
the set of open continuous homomorphisms of profinite groups and the set of continuous
isomorphisms of profinite groups, respectively. First, we have the following theorem which
was proved by the author (cf. [Y3, Theorem 1.2 and Remark 7.3.1]).

**Theorem 6.1.** We maintain the notation introduced above. Suppose that $X^\bullet_1$ and $X^\bullet_2$
are smooth over $k_1$ and $k_2$, respectively. Then we have that
$$\mathrm{Hom}^{\mathrm{open}}_{\mathrm{pro-gps}}(\Pi^\bullet_1, \Pi^\bullet_2) \neq \emptyset$$
if and only if $X^\bullet_1$ is Frobenius equivalent to $X^\bullet_2$. In particular, if this is the case, we have
that $X^\bullet_2$ can be defined over the algebraic closure of $\mathbb{F}_p$ in $k_2$, and that
$$\mathrm{Hom}^{\mathrm{open}}_{\mathrm{pro-gps}}(\Pi^\bullet_1, \Pi^\bullet_2) = \mathrm{Isom}^{\mathrm{pro-gps}}(\Pi^\bullet_1, \Pi^\bullet_2).$$

**Remark 6.1.1.** Let $[q] \in \mathfrak{M}^{\mathrm{cl}}_{0,n}$ be an arbitrary point. Theorem 6.1 and Proposition 3.7
(a) imply that
$$V(\pi^{\mathrm{sol}}_{0,n}([q])) \cap \Pi^{\mathrm{sol}}_{0,n} = \pi^{\mathrm{sol}}_{0,n}([q]).$$
Then we have that \( [\pi_1^{\text{sol}}(q)] \) is a closed point of \( \Pi_0^{\text{sol}} \). In particular,

\[
\pi_{0,4}^1 : \mathcal{M}_{0,4} \to \Pi_{0,4}, \quad \pi_{0,4}^{1,\text{sol}} : \mathcal{M}_{0,4} \to \Pi_{0,4}^{\text{sol}}
\]

are homeomorphisms. Note that Theorem 6.1 cannot tell us whether or not \( [\pi_1^{\text{sol}}(q)] \) is closed in \( \Pi_0^{\text{sol}} \). In fact, this is highly non-trivial, see Proposition 6.5 below.

**Lemma 6.2.** We maintain the notation introduced above. Suppose that \( X_1^* \) is a singular curve. Then \( X_2^* \) is also a singular curve.

**Proof.** Lemma 5.4 implies that there exists a Galois admissible covering

\[
f_1^* : Y_1^* \to X_1^*
\]

over \( k_1 \) with Galois group \( G \) such that \( (\#G, p) = 1 \), that the Betti number of the dual semi-graph of \( Y_1^* \) is positive, and that \( Y_1^* \) satisfies Condition A. Then \( \phi' \) induces a Galois admissible covering

\[
f_2^* : Y_2^* \to X_2^*
\]

over \( k_2 \) with Galois group \( G \). Write \( g_Y \), for the genus of \( Y_i^* \), and \( r_Y \) for the Betti number of the dual semi-graph of \( Y_i^* \).

By applying Theorem 4.11, we obtain that \( g_Y_1 = g_Y_2 \). Moreover, Theorem 2.2 and Lemma 2.3 (b) imply that

\[0 < r_Y_1 \leq r_Y_2\]

This means that \( X_2^* \) is a singular curve. We complete the proof of the lemma. \( \square \)

6.0.3. Let \( \overline{F}_p \) be an algebraic closure of the finite field \( F_p \), and let \( X^* \) be a smooth pointed stable curve of type \( (0, n) \) over \( \overline{F}_p \). We fix two marked points \( x_\infty, x_0 \in D_X \) distinct from each other. Moreover, we choose any field \( k' \cong \overline{F}_p \), and choose any isomorphism \( \varphi : X \isom \overline{F}_{k'} \) as schemes such that \( \varphi(x_\infty) = \infty \) and \( \varphi(x_0) = 0 \). Then the set of \( \overline{F}_p \)-rational points \( X(\overline{F}_p) \setminus \{x_\infty\} \isom \mathcal{A}_k^1(k') \) is equipped with a structure of \( \overline{F}_p \)-module via the bijection \( \varphi \). Note that since any \( k' \)-isomorphism of \( \mathcal{A}_k^1 \) fixing \( \infty \) and \( 0 \) is a scalar multiplication, the \( \overline{F}_p \)-module structure of \( X(\overline{F}_p) \setminus \{x_\infty\} \) does not depend on the choices of \( k' \) and \( \varphi \) but depends only on the choices of \( x_\infty \) and \( x_0 \). We shall say that \( X(\overline{F}_p) \setminus \{x_\infty\} \) is equipped with a structure of \( \overline{F}_p \)-module with respect to \( x_\infty \) and \( x_0 \). Then we have the following lemma.

**Lemma 6.3.** We maintain the notation introduced above. Suppose that \( X_1^* \) is smooth over \( k_1 \). Let \( e_{1,0}, e_{1,\infty} \in e^{\text{op}}(\Gamma_{X_1^*}) \) be open edges distinct from each other. Theorem 4.11 implies that \( \phi \) induces a bijection \( \phi^{\text{sg,op}} : e^{\text{op}}(\Gamma_{X_1^*}) \isom e^{\text{op}}(\Gamma_{X_2^*}) \) group-theoretically. We put \( e_{2,0} \defeq \phi^{\text{sg,op}}(e_{1,0}) \) and \( e_{2,\infty} \defeq \phi^{\text{sg,op}}(e_{1,\infty}) \). Let

\[
\sum_{e_{1} \in e^{\text{op}}(\Gamma_{X_1^*}) \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_{1}} x_{e_{1}} = x_{e_{1,0}}
\]

be a linear condition with respect to \( e_{1,\infty} \) and \( e_{1,0} \) on \( X_1^* \), where \( b_{e_{1}} \in \overline{F}_p \) for every \( e_1 \in e^{\text{op}}(\Gamma_{X_1^*}) \). Then the linear condition

\[
\sum_{e_{1} \in e^{\text{op}}(\Gamma_{X_2^*}) \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_{1}} x_{\phi^{\text{sg,op}}(e_{1})} = x_{\phi^{\text{sg,op}}(e_{1,0})} = x_{e_{2,0}}
\]

with respect to \( x_{e_{2,\infty}} \) and \( x_{e_{2,0}} \) on \( X_2^* \) also holds.
Proof. See [Y3, Lemma 7.1].

Lemma 6.4. Let $X^\bullet$ be a pointed stable curve of type $(0,n)$ over an algebraically closed field $k$ of characteristic $p > 0$ and $\ell \geq 3$ a prime number distinct from $p$. Then there exists a Galois admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ over $k$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the genus of $Y^\bullet$ is 0, and that there exists an irreducible component $Y_v$ of $Y$ satisfying $\#(Y_v \cap D_Y) \geq 3$.

Proof. Suppose that $X^\bullet$ is smooth over $k$. Then the lemma is trivial. We may suppose that $X^\bullet$ is singular. Since the type of $X^\bullet$ is $(0,n)$, there exists irreducible components $X_v_1$, $X_v_2$ of $X$ distinct from each other such that $\#(X_v_1 \cap D_X) \geq 2$ and $\#(X_v_2 \cap D_X) \geq 2$.

Let $x_1 \in X_v_1 \cap D_X$, $x_2 \in X_v_2 \cap D_X$, and

$$f^\bullet : Y^\bullet \to X^\bullet$$

a Galois admissible covering over $k$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that $f$ is totally ramified over $x_1$ and $x_2$, and that $f$ is étale over $D_X \setminus \{x_1, x_2\}$. We see immediately that the irreducible components $Y_v_1 \overset{\text{def}}{=} f^{-1}(X_v_1)$ and $Y_v_2 \overset{\text{def}}{=} f^{-1}(X_v_2)$ of $Y$ satisfy the conditions $\#(Y_v_1 \cap D_Y) \geq 3$ and $\#(Y_v_2 \cap D_Y) \geq 3$, respectively. Moreover, the Riemann-Hurwitz formula implies that the genus of $Y^\bullet$ is 0. This completes the proof of the lemma. □

6.0.4. Next, we generalize Theorem 6.1 to the case where we only assume that $X^\bullet_1$ is smooth over $k_1$.

Proposition 6.5. We maintain the notation introduced above. Suppose that $X^\bullet_1$ is smooth over $k_1$. Then $X^\bullet_1$ is Frobenius equivalent to $X^\bullet_2$. In particular, we have that $X^\bullet_2$ is smooth over $k_2$, and that $X^\bullet_2$ can be defined over the algebraic closure of $\mathbb{F}_p$ in $k_2$.

Proof. If $X^\bullet_2$ is smooth over $k_2$, the proposition follows immediately from Theorem 6.1. Then we may assume that $X^\bullet_2$ is singular (i.e., $\#v(\Gamma_{X^\bullet_2}) \geq 2$).

Step 1: We reduce the proposition to the case where $X^\bullet_1$ satisfies the conditions mentioned in Lemma 6.4.

Let $\ell \geq 3$ be a prime number distinct from $p$. Lemma 6.4 implies that there exists an open normal subgroup $H_2 \subseteq \Pi_{X^\bullet_2}$ such that $\Pi_{X^\bullet_2}/H_2 \cong \mathbb{Z}/\ell\mathbb{Z}$, that the Galois admissible covering $f^\bullet_{H_2} : X^\bullet_{H_2} \to X^\bullet_2$ corresponding to $H_2$ is totally ramified over two marked points of $X^\bullet_2$, and that there exists $w_{H_2} \in v(\Gamma_{X^\bullet_{H_2}})$ such that $\#(X_{H_2,w_{H_2}} \cap D_{X_{H_2}}) \geq 3$. Write $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X^\bullet_1}$ for the open subgroup and $f^\bullet_{H_1} : X^\bullet_{H_1} \to X^\bullet_1$ for the Galois admissible covering over $k_1$ corresponding to $H_1$. Theorem 4.11 implies that $f^\bullet_{H_1}$ is totally ramified over two marked points of $X^\bullet_1$, and that $n_{X_{H_1}} = n_{X_{H_2}}$. Since $f^\bullet_{H_1}$ is totally ramified over two marked points, we have that

$$g_{X_{H_1}} = g_{X_{H_2}} = 0.$$

If we can prove the proposition holds for $X^\bullet_{H_1}$, $X^\bullet_{H_2}$, and $\phi |_{H_1} : H_1 \to H_2$, then we obtain that $X^\bullet_2$ is also smooth over $k_2$. Then the proposition follows immediately from Theorem 6.1. Thus, by replacing $X^\bullet_1$, $X^\bullet_2$, and $\phi$ by $X^\bullet_{H_1}$, $X^\bullet_{H_2}$, and $\phi |_{H_1}$, respectively, we may assume that there exists $w_2 \in v(\Gamma_{X^\bullet_2})$ such that $\#(X_{2,w_2} \cap D_{X_2}) \geq 3$. 
Step 2: We construct a pointed stable curve $Z_1^*$ of type $(0, 5)$ over $k_i$ from $X_i^*$.

Let $e_{2,\infty}$, $e_{2,0}$, $e_{2,1} \in e^{op}(\Gamma_{X_i^*}) \cap e^{op}(\Gamma_{X_2^*})$ distinct from each other. Theorem 4.11 implies that $\phi$ induces a bijection

$$\phi^{sg, op} : e^{op}(\Gamma_{X_i^*}) \xrightarrow{\sim} e^{op}(\Gamma_{X_2^*})$$

group-theoretically. We put

$$e_{1,\infty} \overset{\text{def}}{=} (\phi^{sg, op})^{-1}(e_{2,\infty}), \ e_{1,0} \overset{\text{def}}{=} (\phi^{sg, op})^{-1}(e_{2,0}), \ e_{1,1} \overset{\text{def}}{=} (\phi^{sg, op})^{-1}(e_{2,1}).$$

Without loss of generality, we may assume that

$$x_{e_{1,\infty}} \overset{\text{def}}{=} \infty, \ x_{e_{1,0}} \overset{\text{def}}{=} 0, \ x_{e_{1,1}} \overset{\text{def}}{=} 1,$$

and that

$$X_1 = \mathbb{P}_{k_1}^1, \ X_{2,w_2} = \mathbb{P}_{k_2}^1.$$ 

Let $\pi_0(\Gamma_{X_i^*} \setminus \{w_2\})$ denote the set of connected components of $\Gamma_{X_i^*} \setminus \{w_2\}$ in $\Gamma_{X_2^*}$. Let $C_2 \in \pi_0(\Gamma_{X_2^*} \setminus \{w_2\})$. Since $X_i^*$ is a pointed stable curve of type $(0, n)$ over $k_2$, we have that $\#(C_2 \cap e^{op}(\Gamma_{X_i^*})) \geq 2$. Let $e_{2,C_2,1}$, $e_{2,C_2,2} \in C_2 \cap e^{op}(\Gamma_{X_i^*})$ be open edges distinct from each other. We put

$$e_{1,2} \overset{\text{def}}{=} (\phi^{sg, op})^{-1}(e_{2,C_2,1}) \in e^{op}(\Gamma_{X_i^*}),$$

$$e_{1,3} \overset{\text{def}}{=} (\phi^{sg, op})^{-1}(e_{2,C_2,2}) \in e^{op}(\Gamma_{X_i^*}).$$

We denote by $X_{2,C_2}$ the semi-stable subcurve of $X_2$ whose irreducible components are the irreducible components corresponding to the vertices of $\Gamma_{X_i^*}$ contained in $C_2$. Moreover, we write $e_{2,2}$ for the unique closed edge of $\Gamma_{X_2^*}$ connecting $w_2$ and $C_2$. Then the node $x_{e_{2,2}}$ corresponding to $e_{2,2}$ is the unique closed point of $X_2$ contained in $X_{2,w_2} \cap X_{2,C_2}$.

We put

$$Z_1^* = (Z_1 \overset{\text{def}}{=} X_1, D_{Z_1} \overset{\text{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,2}}, x_{e_{1,3}}\}),$$

$$Y_{1,1}^* = (Y_{1,1} \overset{\text{def}}{=} X_1, D_{Y_{1,1}} \overset{\text{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,3}}\}),$$

$$Y_{1,2}^* = (Y_{1,2} \overset{\text{def}}{=} X_1, D_{Y_{1,2}} \overset{\text{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,3}}\}),$$

$$Y_{2}^* = (Y_2 \overset{\text{def}}{=} X_{2,w_2}, D_{Y_2} \overset{\text{def}}{=} \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,2}}\}).$$

Moreover, we denote by $Z_2^*$ the pointed stable curve of type $(0, 5)$ over $k_2$ associated to the pointed semi-stable curve

$$(X_2, \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,c_2}}, x_{e_{2,c_2}}\})$$

over $k_2$ (i.e., the pointed stable curve obtained by contracting the $(-1)$-curves and the $(-2)$-curves of $(X_2, \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,c_2}}, x_{e_{2,c_2}}\})$). We see that $Z_2$ has two irreducible components $Z_{w_2}$ and $Z_{C_2}$ such that $Z_{w_2}$ is equal to $X_{2,w_2}$, that $x_{e_{2,2}} = Z_{w_2} \cap Z_{C_2}$, that $\{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}\} \subseteq Z_{w_2}$, and that $\{x_{e_{2,c_2}}, x_{e_{2,c_2}}\} \subseteq Z_{C_2}$.

Step 3: We prove that the solvable admissible fundamental groups and the natural homomorphisms between the solvable admissible fundamental groups of pointed stable curves constructing in Step 2 can be reconstructed group-theoretically from $\phi$.

Let

$$I_1 \subseteq \Pi_{X_i^*}, I_2 \subseteq \Pi_{X_2^*}.$$
be the closed subgroups generated by the inertia subgroups of
\[
\bigcup_{e_1 \in e^\text{op}(\Gamma_{X_1^*}) \setminus \{e_1, e_1, e_1, e_2, e_1, e_3\}} \text{Edg}_{e_1}^{\text{op}}(\Pi_{X_1^*}),
\]
\[
\bigcup_{e_2 \in e^\text{up}(\Gamma_{X_2^*}) \setminus \{e_2, e_2, e_2, e_2, e_2, e_2, e_2, e_2\}} \text{Edg}_{e_2}^{\text{up}}(\Pi_{X_2^*}),
\]
respectively,
\[
I_{1,1} \subseteq \Pi_{X_1^*}, I_{1,2} \subseteq \Pi_{X_1^*}
\]
the closed subgroups generated by the inertia subgroups of
\[
\bigcup_{e_1 \in e^\text{up}(\Gamma_{X_1^*}) \setminus \{e_1, e_1, e_1, e_1, e_1, e_2\}} \text{Edg}_{e_1}^{\text{up}}(\Pi_{X_1^*}),
\]
\[
\bigcup_{e_2 \in e^\text{up}(\Gamma_{X_2^*}) \setminus \{e_2, e_2, e_2, e_2, e_2, e_2, e_2, e_2\}} \text{Edg}_{e_2}^{\text{up}}(\Pi_{X_2^*}),
\]
respectively, and
\[
I_{2,1} \subseteq \Pi_{X_2^*}, I_{2,2} \subseteq \Pi_{X_2^*}
\]
the closed subgroups generated by the inertia subgroups of
\[
\bigcup_{e_2 \in e^\text{up}(\Gamma_{X_2^*}) \setminus \{e_2, e_2, e_2, e_2, e_2, e_2, e_2, e_2\}} \text{Edg}_{e_2}^{\text{up}}(\Pi_{X_2^*}),
\]
\[
\bigcup_{e_1 \in e^\text{up}(\Gamma_{X_1^*}) \setminus \{e_1, e_1, e_1, e_1, e_1, e_3\}} \text{Edg}_{e_1}^{\text{up}}(\Pi_{X_1^*}),
\]
respectively.

Then Theorem 4.11 implies that \(\phi(I_1) = I_2\), \(\phi(I_{1,1}) = I_{2,1}\), and \(\phi(I_{1,2}) = I_{2,2}\). Moreover, we see that \(\Pi_{X_1^*}/I_1\) and \(\Pi_{X_1^*}/I_2\) are (outer) isomorphic to the solvable admissible fundamental groups of \(Z_1^*\) and \(Z_2^*\), respectively, that \(\Pi_{X_1^*}/I_{1,1}\) and \(\Pi_{X_1^*}/I_{1,2}\) are (outer) isomorphic to the solvable admissible fundamental groups of \(Y_1^*\) and \(Y_2^*\), respectively, and that \(\Pi_{X_1^*}/I_{2,1}\) and \(\Pi_{X_1^*}/I_{2,2}\) are (outer) isomorphic to the solvable admissible fundamental group of \(Y_2^*\). Note that \(I_{1,1} \supseteq I_1 \subseteq I_{1,2}\) and \(I_{2,1} \supseteq I_2 \subseteq I_{2,2}\).

On the other hand, \(\phi\) induces the following surjective open continuous homomorphisms
\[
\tilde{\phi} : \Pi_{Z_1^*} \overset{\text{def}}{=} \Pi_{X_1^*}/I_1 \rightarrow \Pi_{Z_2^*} \overset{\text{def}}{=} \Pi_{X_2^*}/I_2,
\]
\[
\tilde{\phi}_{1,1} : \Pi_{Y_1^*} \overset{\text{def}}{=} \Pi_{X_1^*}/I_{1,1} \rightarrow \Pi_{Y_2^*} \overset{\text{def}}{=} \Pi_{X_2^*}/I_{2,1},
\]
\[
\tilde{\phi}_{1,2} : \Pi_{Y_1^*} \overset{\text{def}}{=} \Pi_{X_1^*}/I_{1,2} \rightarrow \Pi_{Y_2^*} \overset{\text{def}}{=} \Pi_{X_2^*}/I_{2,2}
\]
which fit into the following commutative diagram:

\[
\begin{array}{ccc}
\Pi Y_{1,1}^* & \xrightarrow{\bar{\phi}_{1,1}} & \Pi Y_{2}^* \\
\psi_{1,1} & & \psi_{2,1} \\
\Pi Z_{1}^* & \xrightarrow{\bar{\psi}} & \Pi Z_{2}^* \\
\psi_{1,2} & & \psi_{2,2} \\
\Pi Y_{1,2}^* & \xrightarrow{\bar{\phi}_{1,2}} & \Pi Y_{2}^*,
\end{array}
\]

where \(\psi_{1,1}, \psi_{1,2}, \psi_{2,1},\) and \(\psi_{2,2}\) denote the natural quotient homomorphisms.

Note that \(\psi_{2,1} \circ \bar{\phi} \neq \psi_{2,2} \circ \bar{\phi}\), and that the homomorphisms of maximal prime-to-\(p\) quotients of solvable admissible fundamental groups \(\bar{\phi}'_{1,1}, \bar{\phi}'\), and \(\bar{\phi}'_{1,2}\) induced by \(\bar{\phi}_{1,1}, \bar{\phi}\), and \(\bar{\phi}_{1,2}\), respectively, are isomorphisms. Moreover, we see that \(\psi_{2,1}(I_{e_2,c_2,1}) \in \text{Edg}_{e_2,c_2}^{op}(\Pi Y_{2}^*)\) and \(\psi_{2,2}(I_{e_2,c_2,2}) \in \text{Edg}_{e_2,c_2}^{op}(\Pi Y_{2}^*)\) for every \(I_{e_2,c_2,1} \in \text{Edg}_{e_2,c_2,1}^{op}(\Pi Z_{2}^*)\) and every \(I_{e_2,c_2,2} \in \text{Edg}_{e_2,c_2,2}^{op}(\Pi Z_{2}^*)\).

**Step 4:** We construct linear conditions associated to irreducible components of \(Z_{1}^*\).

Let \(\hat{e}_{i,0} \in \text{ed}(\Gamma_{X_{1}^*})\) be an open edge over \(e_{i,0}\). By applying Theorem 4.13,

\[
\mathbb{F}_{\hat{e}_{i,0}} \overset{\text{def}}{=} (I_{\hat{e}_{i,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_{x_0}^{\nu}) \sqcup \{e_{i,0}\}
\]

admits a structure of field which can be reconstructed group-theoretically from \(\Pi X_{1}^*\). Since we assume that \(k_{1}\) is an algebraic closure of \(\mathbb{F}_p\), we may suppose that \(k_{1} = \mathbb{F}_{\hat{e}_{i,0}}\). Moreover, we have that \(\phi\) induces a field isomorphism

\[
\phi_{\hat{e}_{i,0},e_{i,0}}^{\text{ed}} : \mathbb{F}_{\hat{e}_{i,0}} \xrightarrow{\sim} \mathbb{F}_{e_{i,0}}.
\]

group-theoretically. By [T2, Lemma 3.4], there exists a natural number \(m\) prime to \(p\) such that \(\mathbb{F}_p(\zeta_{1,m})\) contains \(m\)th roots of \(x_{e_{1,2}}, x_{e_{1,3}}\), where \(\zeta_{1,m}\) denotes a fixed primitive \(m\)th root of unity in \(\mathbb{F}_{e_{i,0}}\). Let \(s = [\mathbb{F}_p(\zeta_{1,m}) : \mathbb{F}_p]\). For each \(e_{1,u} \in \{e_{1,2}, e_{1,3}\}\), we fix an \(m\)th root \(x_{e_{1,u}}^{1/m}\) in \(\mathbb{F}_{e_{i,0}}\). Then we have

\[
x_{e_{1,u}}^{1/m} = \sum_{t=0}^{s-1} b_{1,u,t} \zeta_{1,m}^t, \quad u \in \{2, 3\},
\]

where \(b_{1,u,t} \in \mathbb{F}_p\) for each \(u \in \{2, 3\}\) and each \(t \in \{0, \ldots, s-1\}\). Note that since \(x_{e_{1,2}} \neq x_{e_{1,3}}\), there exists \(t' \in \{0, \ldots, s-1\}\) such that \(b_{1,2,t'} \neq b_{1,3,t'}\).

Let \(Z_{1} \setminus \{x_{e_{i,\infty}}\} = \text{Spec} \mathbb{F}_{e_{i,0}}[x_{1}]\),

\[
f_{Q_{1}} : Z_{1}^* \to Z_{1}^*
\]

the Galois admissible covering over \(\mathbb{F}_{e_{i,0}}\) with Galois group \(\mathbb{Z}/m\mathbb{Z}\) determined by the equation \(y_{1}^{m} = x_{1}\), and \(Q_{1} \subseteq \Pi Z_{2}^*\) the open normal subgroup induced by \(f_{Q_{1}}^{*}\). Then \(f_{Q_{1}}\) is totally ramified over \(\{x_{e_{1,0}} = 0, x_{e_{1,\infty}} = \infty\}\) and is étale over \(D_{Z_{1}} \setminus \{x_{e_{1,0}}, x_{e_{1,\infty}}\}\). Note that \(Z_{Q_{1}} = \mathbb{P}^{1}_{\mathbb{F}_{e_{i,0}}}\), and that the marked points of \(D_{Z_{Q_{1}}}\) over \(\{x_{e_{1,0}}, x_{e_{1,\infty}}\}\) are \(\{x_{e_{Q_{1},0}} \overset{\text{def}}{=} \)

\[
\cdots
\]
0, \( x_{eQ_1, \infty} \stackrel{\text{def}}{=} \infty \). We put

\[
x_{eQ_1, u} \stackrel{\text{def}}{=} \frac{1}{x_{e1, u}} \in D_{ZQ_1}, \ u \in \{2, 3\},
\]

and

\[
x_{e_{Q_1, 1}} \stackrel{\text{def}}{=} x_{e_{Q_1, 1}} \in D_{ZQ_1}, \ t \in \{0, \ldots, s - 1\}.
\]

Thus, we obtain a linear condition

\[
x_{eQ_1, u} = \sum_{t=0}^{s-1} b_{1, u, t} x_{e_{Q_1, 1}}, \ u \in \{2, 3\}
\]

with respect to \( x_{eQ_1, 0} \) and \( x_{eQ_1, \infty} \) on \( Z_{Q_1}^* \).

Since \((m, p) = 1\), there exists a unique open normal subgroup \( Q_2 \subseteq \Pi_{Z_2^*} \) such that \( \overline{\phi}^{-1}(Q_2) = Q_1 \). On the other hand, we put

\[
Q_{1, 1} \stackrel{\text{def}}{=} \psi_{1, 1}(Q_1) \subseteq \Pi_{Y_{1, 1}^*},
\]

\[
Q_{1, 2} \stackrel{\text{def}}{=} \psi_{1, 2}(Q_1) \subseteq \Pi_{Y_{1, 2}^*},
\]

\[
Q_{2, 1} \stackrel{\text{def}}{=} \psi_{2, 1}(Q_2) \subseteq \Pi_{Y_2^*},
\]

\[
Q_{2, 2} \stackrel{\text{def}}{=} \psi_{2, 2}(Q_2) \subseteq \Pi_{Y_2^*}.
\]

Note that the constructions of \( Q_1 \) and \( Q_2 \) imply that \( P_2 \stackrel{\text{def}}{=} Q_{2, 1} = Q_{2, 2} \). The commutative diagram of profinite groups constructed in Step 3 induces the following commutative diagram of profinite groups:

\[
\begin{array}{ccc}
Q_{1, 1} & \xrightarrow{\overline{\phi}_{Q_1, 1}} & P_2 \\
\psi_{Q_1, 1, 1} \downarrow & & \downarrow \psi_{Q_2, 2, 1} \\
Q_1 & \xrightarrow{\overline{\phi}_{Q_1}} & Q_2 \\
\psi_{Q_1, 1, 2} \downarrow & & \downarrow \psi_{Q_2, 2, 2} \\
Q_{1, 2} & \xrightarrow{\overline{\phi}_{Q_1, 2}} & P_2.
\end{array}
\]

Let \( j \in \{1, 2\} \). Write \( Y_{Q_1, j}^* \) for the pointed stable curve over \( k_1 \) corresponding to \( Q_{1,j} \). Then we see that \( e_{op}(\Gamma_{Y_{Q_1, j}^*}) \) can be regarded as a subset of \( e_{op}(\Gamma_{Z_1^*}) \) via \( \psi_{Q_1, 1, j} \). By applying Theorem 4.11 for \( \overline{\phi}_{Q_1}, \overline{\phi}_{Q_1, 1}, \) and \( \overline{\phi}_{Q_1, 2} \), respectively, the commutative diagram of profinite groups above implies that we may put

\[
e_{Q_2, \infty} \stackrel{\text{def}}{=} \overline{\phi}_{Q_1}(e_{Q_1, \infty}), \ e_{Q_2, 0} \stackrel{\text{def}}{=} \overline{\phi}_{Q_1}(e_{Q_1, 0}),
\]

\[
e'_{Q_2, 1} \stackrel{\text{def}}{=} \overline{\phi}_{Q_1}(e'_{Q_1, 1}), \ t \in \{0, \ldots, s - 1\},
\]

\[
e_{P_2, \infty} \stackrel{\text{def}}{=} \overline{\phi}_{Q_1, 1}(e_{Q_1, \infty}) = \overline{\phi}_{Q_1, 2}(e_{Q_1, \infty}), \ e_{P_2, 0} \stackrel{\text{def}}{=} \overline{\phi}_{Q_1, 1}(e_{Q_1, 0}) = \overline{\phi}_{Q_1, 2}(e_{Q_1, 0}),
\]

\[
e'_{P_2, 1} \stackrel{\text{def}}{=} \overline{\phi}_{Q_1, 1}(e'_{Q_1, 1}) = \overline{\phi}_{Q_1, 2}(e'_{Q_1, 1}), \ t \in \{0, \ldots, s - 1\},
\]

Moreover, we may identify \( e'_{Q_2, 1}, \ t \in \{0, \ldots, s - 1\} \), with \( e'_{P_2, 1} \) via \( \psi_{Q_2, 2, 1} \) (or \( \psi_{Q_2, 2, 2} \)).
We denote by \( \zeta_{2,m} \) the \( m \)-th Frobenius iterates of \( \zeta_{2,1} \). Then we have
\[
ex_{e,2,1} = x_{e,2,1}^t = \zeta_{2,m}, \quad t \in \{0, \ldots, s - 1\}.
\]
Let \( Y_{2}^* \) be the pointed stable curve over \( k_2 \) corresponding to \( P_2 \subseteq Y_{2}^* \). Moreover, by applying Lemma 6.3 for \( \phi_{Q_{1,1}} \), we obtain that
\[
ex_{e,2,0} = \sum_{t=0}^{s-1} b_{1,2,t} x_{e,2,1}^t
\]
with respect to \( x_{e,2,0} \) and \( x_{e,2,1} \) on \( Y_{2}^* \). On the other hand, by applying Lemma 6.3 for \( \phi_{Q_{1,2}} \), we obtain that
\[
ex_{e,2,0} = \sum_{t=0}^{s-1} b_{1,3,t} x_{e,2,1}^t
\]
with respect to \( x_{e,2,0} \) and \( x_{e,2,1} \) on \( Y_{2}^* \). This means that
\[
\sum_{t=0}^{s-1} b_{1,2,t} \zeta_{2,m} = \sum_{t=0}^{s-1} b_{1,3,t} \zeta_{2,m},
\]
which is impossible as \( b_{1,2,t} \neq b_{1,3,t} \) for some \( t' \in \{0, \ldots, s - 1\} \). Then we obtain that \( X_{2}^* \) is smooth over \( k_2 \). Thus, the proposition follows from Theorem 6.1. This completes the proof of the proposition. \( \square \)

6.0.5. Now, we prove the first form of our main theorem of the present paper.

**Theorem 6.6.** Let \( X_{i}^* \), \( i \in \{1, 2\} \), be an arbitrary pointed stable curve of type \((0, n)\) over an algebraically closed field \( k_1 \) of characteristic \( p > 0 \) and \( \Pi_{1}^* \) either the admissible fundamental group of \( X_{1}^* \) or the solvable admissible fundamental group of \( X_{2}^* \). Suppose that \( k_1 \) is an algebraic closure of \( \mathbb{F}_p \). Then we have that
\[
\text{Hom}_{\text{pro-gps}}^{\text{open}}(\Pi_{1}^*, \Pi_{2}^*) \neq \emptyset
\]
if and only if \( X_{1}^* \) is Frobenius equivalent to \( X_{2}^* \). In particular, if this is the case, we have that \( X_{2}^* \) can be defined over the algebraic closure of \( \mathbb{F}_p \) in \( k_2 \), and that
\[
\text{Hom}_{\text{pro-gps}}^{\text{open}}(\Pi_{1}^*, \Pi_{2}^*) = \text{Isom}_{\text{pro-gps}}(\Pi_{1}^*, \Pi_{2}^*).
\]

**Proof.** To verify the theorem, it is sufficient to prove the theorem when \( \Pi_{1}^* \) is the solvable admissible fundamental group of \( X_{1}^* \). The “if” part of the theorem follows from [Y7, Proposition 3.7]. Let us prove the “only if” part of the theorem. Suppose that \( \text{Hom}_{\text{pro-gps}}^{\text{open}}(\Pi_{1}^*, \Pi_{2}^*) \neq \emptyset \), and let \( \phi \in \text{Hom}_{\text{pro-gps}}^{\text{open}}(\Pi_{1}^*, \Pi_{2}^*) \) be an arbitrary element. Then Lemma 4.3 implies that \( \phi \) is a surjection.

Suppose that \( X_{1}^* \) is smooth over \( k_1 \). Then the theorem follows from Proposition 6.5. Thus, we may assume that \( X_{1}^* \) is a singular pointed stable curve.

Note that since \( X_{1}^* \) is singular, we have \( n = \#e^{\text{op}}(\Gamma_{1}^*) \geq 4 \). We prove the theorem by induction on \( \#e^{\text{op}}(\Gamma_{1}^*) \). Suppose that \( \#e^{\text{op}}(\Gamma_{1}^*) = 4 \). Since \( X_{1}^* \) is a singular pointed stable curve of type \((0, 4)\), we obtain that \( \#v(\Gamma_{1}^*) = 2 \) and \( \#e^{\text{cl}}(\Gamma_{1}^*) = 1 \). On the other hand, by applying Lemma 6.2, we obtain that \( X_{2}^* \) is also a singular pointed stable curve of type \((0, 4)\). Thus, we have that \( \#e^{\text{op}}(\Gamma_{2}^*) = 4 \), \( \#v(\Gamma_{2}^*) = 2 \), and \( \#e^{\text{cl}}(\Gamma_{2}^*) = 1 \). Then \( X_{1}^* \) and \( X_{2}^* \) satisfy Condition C defined in Section 5. Thus, by Theorem 5.30 and Theorem 6.1, we obtain that \( X_{1}^* \) is Frobenius equivalent to \( X_{2}^* \).
Suppose that \( \#e^{op}(\Gamma_{X^*}) \geq 5 \). Theorem 4.11 implies that \( \phi \) induces a bijection

\[
\phi^{sg, op} : e^{op}(\Gamma_{X^*}) \cong e^{op}(\Gamma_{X^*})
\]
group-theoretically. Let \( e_{1,n} \in e^{op}(\Gamma_{X^*}) \) and \( e_{2,n} \overset{\text{def}}{=} \phi^{sg, op}(e_{1,n}) \). We denote by

\[
Z_i^*
\]
the pointed stable curve of type \((0, n-1)\) over \( k \) associated to the pointed semi-stable curve \((X_i, D_{X_i} \setminus \{x_{e_{1,n}}\})\) (i.e., the pointed stable curve obtained by contracting the \((-1)\)-curves and the \((-2)\)-curves of \((X_i, D_{X_i} \setminus \{x_{e_{1,n}}\})\)).

Write \( I_{i,n} \subseteq \Pi_{X^*} \) for the closed subgroup generated by the subgroups contained in \( \text{Edg}^{op}_{e_{i,n}}(\Pi_{X^*}) \). Then we see that

\[
\Pi_{Z^*} \overset{\text{def}}{=} \Pi_{X^*}/I_{i,n}
\]
is (outer) isomorphic to the solvable admissible fundamental group of \( Z_i^* \). Moreover, Theorem 4.11 implies that \( \phi(I_{1,n}) = I_{2,n} \). Then \( \phi \) induces a surjective open continuous homomorphism

\[
\overline{\phi} : \Pi_{Z^*} \rightarrow \Pi_{Z^*}.
\]
By induction, we obtain that \( Z_i^* \) is Frobenius equivalent to \( Z_i^* \). Then \( \phi \) induces a bijection of dual semi-graphs

\[
\overline{\phi} : \Gamma_{Z^*} \cong \Gamma_{Z^*}.
\]
In particular, we put

\[
\overline{\phi}^{sg, ver} \overset{\text{def}}{=} \overline{\phi}^{sg} \mid_{v(\Gamma_{Z^*})} : v(\Gamma_{Z^*}) \cong v(\Gamma_{Z^*}),
\]

\[
\overline{\phi}^{sg, op} \overset{\text{def}}{=} \overline{\phi}^{sg} \mid_{e^{op}(\Gamma_{Z^*})} : e^{op}(\Gamma_{Z^*}) \cong e^{op}(\Gamma_{Z^*}).
\]
Note that \( v(\Gamma_{Z^*}) \), \( e^{op}(\Gamma_{Z^*}) \), the set of marked points \( D_{Z_i} \) of \( Z_i^* \) can be regarded naturally as subsets of \( v(\Gamma_{X^*}) \), \( e^{op}(\Gamma_{X^*}) \), the set of irreducible components of \( X_i \), the set of marked points \( D_{X_i} \) of \( X_i^* \) via the contracting morphism \((X_i, D_{X_i} \setminus \{x_{e_{1,n}}\}) \rightarrow Z_i^* \), respectively. Moreover, one of the following cases may occur: (i) \( \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) = \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) = \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) \); (ii) \( \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) = \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) = \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) \); (iv) \( \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) = \#v(\Gamma_{X^*}) = \#v(\Gamma_{Z^*}) \). Suppose that either (i) or (ii) holds. Then \( X^*_1 \) and \( X^*_2 \) satisfy Condition C defined in Section 5. Thus, by Theorem 5.30 and Theorem 6.1, we obtain that \( X^*_1 \) is Frobenius equivalent to \( X^*_2 \).

Suppose that (iii) holds. Let \( v_2 \in v(\Gamma_{X^*_1}) \) such that \( x_{e_{2,n}} \in X^*_{v_2} \overset{\text{def}}{=} X_{v_2} \) (i.e., the irreducible component of \( X_2 \) corresponding to \( v_2 \)). Since \( \#v(\Gamma_{X^*_2}) = \#v(\Gamma_{Z^*_2}) + 1 \), we have that \( \#(X_{v_2} \cap D_{X^*_2}) = 2 \). Note that \( \{v_2\} = v(\Gamma_{X^*_2}) \setminus v(\Gamma_{Z^*_2}) \).

Let \( x_{e_{2,n-1}} \in X_{v_2} \cap D_{X^*_2} \) be the marked point distinct from \( x_{e_{2,n}} \) and \( e_{2,n-1} \in e^{op}(\Gamma_{X^*_1}) \) the open edge corresponding to the marked point \( x_{e_{2,n-1}} \). On the other hand, let \( w_1 \in v(\Gamma_{X^*_1}) \) such that \( x_{e_{1,n}} \in X_{w_1} \overset{\text{def}}{=} X_{w_1} \). We put

\[
w_2 \overset{\text{def}}{=} \overline{\phi}^{sg, ver}(w_1) \in v(\Gamma_{Z^*_2}) \subseteq v(\Gamma_{X^*_2}),
\]

\[
e_{1,n-1} \overset{\text{def}}{=} (\overline{\phi}^{sg, op})^{-1}(e_{2,n-1}) \in e^{op}(\Gamma_{Z^*_1}) \subseteq e^{op}(\Gamma_{X^*_1}).
\]
Since $Z_1^*$ is a pointed stable curve of type $(0, n - 1)$, we have that

$$\#(X_{w_1} \cap D_{Z_1}) + \#(X_{w_1} \cap Z_1^{\text{sing}}) \geq 3.$$  

Then we see that there exist marked points $x_{e_1,n-2}, x_{e_1,n-3} \in D_{Z_1} \setminus \{x_{e_1,n-1}\}$ distinct from each other such that one of the following conditions is satisfied:

1. If $\#(X_{w_1} \cap D_{Z_1}) \geq 3$, then $x_{e_1,n-2}, x_{e_1,n-3} \in X_{w_1}$.
2. If $\#(X_{w_1} \cap D_{Z_1}) = 2$ and $x_{e_1,n-1} \notin X_{w_1}$, then $x_{e_1,n-2}, x_{e_1,n-3} \in X_{w_1}$.
3. If $\#(X_{w_1} \cap D_{Z_1}) = 1$ and $x_{e_1,n-1} \notin X_{w_1}$, then we have that $x_{e_1,n-3} \in X_{w_1}$, and that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 1$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is $\geq 2$) containing $x_{e_1,n-2}$ and $x_{e_1,n-3}$, respectively, are distinct from each other.
4. If $\#(X_{w_1} \cap D_{Z_1}) = 2$ and $x_{e_1,n-1} \in X_{w_1}$, then we have that $x_{e_1,n-3} \in X_{w_1}$, and that $x_{e_1,n-2}$ is contained in a connected component of $Z_1 \setminus X_{w_1}$.
5. If $\#(X_{w_1} \cap D_{Z_1}) = 1$ and $x_{e_1,n-1} \in X_{w_1}$, then we have that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 1$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is $\geq 2$) containing $x_{e_1,n-2}$ and $x_{e_1,n-3}$, respectively, are distinct from each other.
6. If $\#(X_{w_1} \cap D_{Z_1}) = 0$, then we have that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 0$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is $\geq 3$) containing $x_{e_1,n-1}, x_{e_1,n-2}$, and $x_{e_1,n-3}$, respectively, are distinct from each other.

Write $e_{1,n-2}$ and $e_{1,n-3} \in e_{\text{op}}(\Gamma_{Z_1^*})$ for the open edges corresponding to the marked points $x_{e_1,n-2}$ and $x_{e_1,n-3}$, respectively. We put

$$e_{2,n-2} \overset{\text{def}}{=} \varphi(\gamma_{e_{1,n-2}}), \ e_{2,n-3} \overset{\text{def}}{=} \varphi(\gamma_{e_{1,n-3}}).$$

Let $Y_i^*$ be the pointed stable curve of type $(0, 4)$ over $k_i$ associated to the pointed semi-stable curve

$$(X_1, \{x_{e_i,n}, x_{e_i,n-1}, x_{e_i,n-2}, x_{e_i,n-3}\}).$$

By the construction of the set of marked points $\{x_{e_i,n}, x_{e_i,n-1}, x_{e_i,n-2}, x_{e_i,n-3}\}$, we see that $Y_i^*$ is smooth over $k_i$ whose underlying curve is $X_{w_1}$, and that $Y_2^*$ is singular whose irreducible components are $X_{w_2} \overset{\text{def}}{=} X_{2,w_3}$ and $X_{2}^*$. Next, we will see that the solvable admissible fundamental groups and the natural homomorphisms between the solvable admissible fundamental groups of pointed stable curves constructing above can be reconstructed group-theoretically from $\phi$. Let $I_i \subseteq \Pi_{X_i^*}$ be the closed subgroup generated by the subgroups contained in

$$\bigcup_{e_i \in e_{\text{op}}(\Gamma_{X_i^*}) \setminus \{e_{i,n,e_{i,n-1},e_{i,n-2},e_{i,n-3}\}}} \text{Edg}_{e_{i,n}}^{\text{op}}(\Pi_{X_i^*}).$$

We see that

$$\Pi_{Y_i^*} \overset{\text{def}}{=} \Pi_{X_i^*}/I_i$$

is (outer) isomorphic to the solvable admissible fundamental group of $Y_i^*$. Moreover, Theorem 4.11 implies that $\phi(I_1) = I_2$. Then we obtain a surjective open continuous homomorphism

$$\tilde{\phi} : \Pi_{Y_1^*} \rightarrow \Pi_{Y_2^*}.$$  

This contradicts Proposition 6.5, since Proposition 6.5 implies that $Y_2^*$ is smooth over $k_2$. Then (iii) does not occur.
Suppose that (iv) holds. Similar arguments to the arguments given in the proof of (iii) imply that (iv) does not occur. More precisely, we have the following.

Let \( v_1 \in v(\Gamma X^*_i) \) such that \( x_{e_1,n} \in X_{v_1} \defeq X_{1,v_1} \). Since \( \# v(\Gamma X^*_i) = \# v(\Gamma Z^*_i) + 1 \), we have that \( \#(X_{n} \cap D_{X_i}) = 2 \). Note that \( \{ v_1 \} = v(\Gamma X^*_i) \setminus v(\Gamma Z^*_i) \).

Let \( x_{e_1,n-1} \in X_{v_1} \cap D_{X_i} \) be the marked point distinct from \( x_{e_1,n} \) and \( e_{1,n-1} \in e_{op}(\Gamma X^*_i) \) the open edge corresponding to the marked point \( x_{e_1,n-1} \). On the other hand, let \( w_2 \in v(\Gamma X^*_i) \) such that \( x_{e_2,n} \in X_{w_2} \defeq X_{2,w_2} \). We put

\[
\begin{align*}
  w_1 &= \defeq (\phi^{sg,ver})^{-1}(w_2) \in v(\Gamma Z^*_i) \subseteq v(\Gamma X^*_i), \\
  e_{2,n-1} &= \defeq (d_{e_{op}}^s)^{-1}(e_{1,n-1}) \in e_{op}(\Gamma Z^*_i) \subseteq e_{op}(\Gamma X^*_i).
\end{align*}
\]

Since \( Z^*_2 \) is a pointed stable curve of type \((0, n - 1)\), we have that

\[
\#(X_{w_2} \cap D_{Z_2}) + \#(X_{w_2} \cap Z_{2}^{sing}) \geq 3.
\]

Then we see that there exist marked points \( x_{e_2,n-2} \), \( x_{e_2,n-3} \in D_{Z_2} \setminus \{ x_{e_2,n-1} \} \) distinct from each other such that one of the following conditions is satisfied:

1. If \( \#(X_{w_2} \cap D_{Z_2}) \geq 3 \), then \( x_{e_2,n-2}, x_{e_2,n-3} \in X_{w_2} \).
2. If \( \#(X_{w_2} \cap D_{Z_2}) = 2 \) and \( x_{e_2,n-1} \notin X_{w_2} \), then \( x_{e_2,n-2}, x_{e_2,n-3} \in X_{w_2} \).
3. If \( \#(X_{w_2} \cap D_{Z_2}) = 1 \) and \( x_{e_2,n-1} \notin X_{w_2} \), then we have that \( x_{e_2,n-2}, x_{e_2,n-3} \in X_{w_2} \), and that the connected components of \( Z_2 \setminus X_{w_2} \) (note that since \( \#(X_{w_2} \cap D_{Z_2}) = 1 \), the cardinality of the set of connected components of \( Z_2 \setminus X_{w_2} \) is \( \geq 2 \)) containing \( x_{e_2,n-1} \) and \( x_{e_2,n-2} \), respectively, are distinct from each other.
4. If \( \#(X_{w_2} \cap D_{Z_2}) = 2 \) and \( x_{e_2,n-1} \in X_{w_2} \), then we have that \( x_{e_2,n-3} \in X_{w_2} \), and that \( x_{e_2,n-2} \) is contained in a connected component of \( Z_2 \setminus X_{w_2} \).
5. If \( \#(X_{w_2} \cap D_{Z_2}) = 1 \) and \( x_{e_2,n-1} \in X_{w_2} \), then we have that the connected components of \( Z_2 \setminus X_{w_2} \) (note that since \( \#(X_{w_2} \cap D_{Z_2}) = 1 \), the cardinality of the set of connected components of \( Z_2 \setminus X_{w_2} \) is \( \geq 2 \)) containing \( x_{e_2,n-2} \) and \( x_{e_2,n-3} \), respectively, are distinct from each other.
6. If \( \#(X_{w_2} \cap D_{Z_2}) = 0 \), then we have that the connected components of \( Z_2 \setminus X_{w_2} \) (note that since \( \#(X_{w_2} \cap D_{Z_2}) = 0 \), the cardinality of the set of connected components of \( Z_2 \setminus X_{w_2} \) is \( \geq 3 \)) containing \( x_{e_2,n-1}, x_{e_2,n-2}, x_{e_2,n-3} \), respectively, are distinct from each other.

Write \( e_{2,n-2} \) and \( e_{2,n-3} \in e_{op}(\Gamma Z^*_i) \) for the open edges corresponding to the marked points \( x_{e_2,n-2} \) and \( x_{e_2,n-3} \), respectively. We put

\[
\begin{align*}
  e_{1,n-2} &= \defeq (\phi^{sg,op})^{-1}(e_{2,n-2}), \\
  e_{1,n-3} &= \defeq (\phi^{sg,op})^{-1}(e_{2,n-3}).
\end{align*}
\]

Let \( Y^*_i \) be the pointed stable curve of type \((0, 4)\) over \( k_i \) associated to the pointed semi-stable curve

\[
(X_i; \{ x_{e_i,n}, x_{e_i,n-1}, x_{e_i,n-2}, x_{e_i,n-3} \}).
\]

By the construction of the set of marked points \( \{ x_{e_i,n}, x_{e_i,n-1}, x_{e_i,n-2}, x_{e_i,n-3} \} \), we see that \( Y^*_i \) is singular whose irreducible component are \( X_{w_1} \defeq X_{1,w_1} \) and \( X_{v_1} \), and that \( Y^*_2 \) is smooth over \( k_2 \) whose underlying curve is \( X_{w_2} \).

Let \( I_i \subseteq \Pi X^*_i \) be the closed subgroup generated by the subgroups contained in

\[
\bigcup_{e_i \in e_{op}(\Gamma X^*_i) \setminus \{ e_{i,n}, e_{i,n-1}, e_{i,n-2}, e_{i,n-3} \}} \text{Edg}^{op}_{e_i} (\Pi X^*_i).
\]
We see that
\[ \Pi Y_i \overset{\text{def}}{=} \Pi X_i / I_i \]
is (outer) isomorphic to the solvable admissible fundamental group of \( Y_i \). Moreover, Theorem 4.11 implies that \( \phi(I_1) = I_2 \). Then we obtain a surjective open continuous homomorphism
\[ \overline{\phi} : \Pi Y_i \rightarrow \Pi Y_i. \]
This contradicts Lemma 6.2, since Lemma 6.2 implies that \( Y_i \) is singular. Then (iv) does not occur. This completes the proof of the theorem.

6.0.6. Theorem 6.6 implies the following result concerning the Homeomorphism Conjecture which is the main theorem of the present paper.

**Theorem 6.7.** We maintain the notation introduced in Section 3. Let \([q] \in \mathcal{P}^{cl}_{0,n}\) be an arbitrary closed point. Then \( \pi_{0,n}^{\text{adm}}([q]) \) and \( \pi_{0,n}^{\text{sol}}([q]) \) are closed points of \( \Pi_{0,n} \) and \( \Pi_{0,n}^{\text{sol}} \), respectively. In particular, the Homeomorphism Conjecture and the Solvable Homeomorphism Conjecture hold when \((g;n) = (0,3)\) or \((0,4)\).

**Proof.** To verify the theorem, we only need to treat the case of solvable admissible fundamental groups.

Let \( V(\pi_{0,n}^{\text{sol}}([q])) \) be the topological closure of \( \pi_{0,n}^{\text{sol}}([q]) \) in \( \Pi_{0,n}^{\text{sol}} \) and \([\pi_{0,n}^{\text{sol}}(q')] \in V(\pi_{0,n}^{\text{sol}}([q]))\) an arbitrary point. Then by Proposition 3.7 (a), we obtain that there exists a surjective open continuous homomorphism
\[ \phi : \pi_{0,n}^{\text{sol}}(q) \rightarrow \pi_{0,n}^{\text{sol}}(q'). \]

Theorem 6.6 implies that \( q \sim_f q' \). Thus, we obtain that \([\pi_{0,n}^{\text{sol}}(q)] = [\pi_{0,n}^{\text{sol}}(q')]\). This means that \( V(\pi_{0,n}^{\text{sol}}([q])) = [\pi_{0,n}^{\text{sol}}(q)] \) is a closed point of \( \Pi_{0,n}^{\text{sol}} \). Moreover, the “in particular” part of the theorem follows from Theorem 3.4 (b). This completes the proof of the theorem. \( \square \)

**Remark 6.7.1.** In \([Y8]\), the author proved a similar result of Theorem 6.7 when \((g,n) = (1,1)\) and \( p > 2 \).


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