MODULI SPACES OF FUNDAMENTAL GROUPS OF CURVES IN POSITIVE CHARACTERISTIC I

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Abstract. For pointed stable curves over algebraically closed fields of positive characteristic, we investigate a new kind of anabelian phenomenon that cannot be explained by Grothendieck's original anabelian philosophy.

We introduce a topological space that is determined by the isomorphism classes of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of positive characteristic. We show that there is a natural continuous map from the moduli space of pointed stable curves of type (g, n) to the above topological space. Moreover, we conjecture that the above continuous map is a homeomorphism (which we call the *homeomorphism conjecture*). The homeomorphism conjecture can be regarded as a *dictionary* between the geometry of curves and the anabelian properties of curves, and it supplies a point of view to see *what anabelian phenomena that we can reasonably expect* from curves over algebraically closed fields of positive characteristic. One of the main results of the present series of papers says that the homeomorphism conjecture holds for one-dimensional moduli spaces.

In the present paper, we establish precise connections between the geometric behaviors of curves and open continuous homomorphisms of their admissible fundamental groups, which play central roles in the theory developed in the series of papers. By using the precise connections, we prove the homeomorphism conjecture for closed points of moduli spaces when $g = 0$. In particular, we obtain the homeomorphism conjecture for one-dimensional moduli spaces when $g = 0$.

Keywords: pointed stable curve, admissible fundamental group, moduli space, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30, 14G17; Secondary 14H10, 14F35, 14G32.

Contents

Introduction

0.1. **Grothendieck's anabelian philosophy.** In the 1980s, A. Grothendieck suggested a theory of arithmetic geometry called anabelian geometry ([G]), roughly speaking, which focuses on the following question: Can we reconstruct the geometric information of a variety group-theoretically from various versions of its algebraic fundamental group? The varieties which can be completely determined by their fundamental groups are called "anabelian varieties" by Grothendieck. To classify the anabelian varieties in all dimensions over all fields is called "anabelian dream" of him. In the particular case of dimension 1, he conjectured that all smooth pointed stable curves (defined over certain fields) are anabelian varieties.

0.1.1. Let p be a prime number and $\#(-)$ the cardinality of $(-)$. Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of type (g_X, n_X) over a field k of characteristic $\mathrm{char}(k)$, where X denotes the underlying curve which is a semi-stable curve over k, D_X denotes the (finite) set of marked points satisfying [K, Definition 1.1 (iv)], g_X denotes the genus of X, and $n_X \overset{\mathrm{def}}{=} \#(D_X)$. In the introduction, "curves" means pointed stable curves unless indicated otherwise.

0.1.2. Suppose that $X^\bullet$ is smooth over k. When k is an "arithmetic" field (e.g. a number field, a p-adic field, a finite field, etc.), Grothendieck's anabelian conjectures for curves (or the Grothendieck conjectures for short), roughly speaking, are based on the following *anabelian philosophy* ([G]):

> **Weak Isom-version:** The isomorphism class of $X^\bullet$ can be determined group-theoretically from the isomorphism class of its algebraic fundamental group.
>
> **Isom-version:** The sets of isomorphisms of smooth pointed stable curves can be determined group-theoretically from the sets of isomorphisms of their algebraic fundamental groups.
>
> **Hom-version:** The sets of dominant morphisms of smooth pointed stable curves can be determined group-theoretically from the sets of open continuous homomorphisms of their algebraic fundamental groups.

Grothendieck's anabelian conjectures have been proven in many cases. For instance, we have the following results: When k is a number field, the conjecture was proved by H. Nakamura (weak Isom-version) ([Nakam1], [Nakam2]), A. Tamagawa (Isom-version) ([T1]), and S. Mochizuki (Hom-version) ([M2]). When k is a finitely generated field over the finite field $\mathbb{F}_p$, the Isom-version of the Grothendieck conjecture was proved by Tamagawa ([T1]), Mochizuki ([M4]), J. Stix ([Sti1], [Sti2]), and M. Saïdi-Tamagawa ([ST1], [ST3]). All the proofs of the Grothendieck conjectures for curves over arithmetic fields mentioned above require the use of *the non-trivial outer Galois representations* induced by the fundamental exact sequences of fundamental groups.

0.2. **Beyond the arithmetical actions.** Next, we consider the case where $X^\bullet$ is an arbitrary pointed stable curve, and suppose that k is *an algebraically closed field.*

0.2.1. By choosing a suitable base point x of $X^\bullet$, we have the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ of $X^\bullet$ (see 1.2.2). For simplicity, we shall write $\pi_1^{\mathrm{adm}}(X^\bullet)$ for $\pi_1^{\mathrm{adm}}(X^\bullet, x)$, since we only focus on the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet, x)$. In particular, if $X^\bullet$ is smooth over k, then $\pi_1^{\mathrm{adm}}(X^\bullet)$ is naturally isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(X^\bullet)$.

When $\mathrm{char}(k) = 0$, since the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet)$ depends only on the type (g_X, n_X), the anabelian geometry of curves does not exist in this situation. On the other hand, if $\mathrm{char}(k) = p$, the situation is quite different from that in characteristic 0. The admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet)$ is very mysterious and its structure is no longer known. In the remainder of the introduction, we assume that k is an algebraically closed field of characteristic p.

0.2.2. After M. Raynaud ([R1]) and D. Harbater ([Ha1]) proved Abhyankar's conjecture, Harbater asked whether or not the geometric information of a curve over k can be carried out from its geometric fundamental groups ([Ha2], [Ha3]). Since the late 1990s, some developments of Raynaud ([R3]), F. Pop-Saïdi ([PS]), Tamagawa ([T2], [T4], [T5]), and the author of the present paper ([Y2], [Y6]) showed evidence for very strong *anabelian phenomena for curves over algebraically closed fields of positive characteristic* (see [T3] for more about this conjectural world based on Grothendieck's anabelian philosophy mentioned in 0.1.2). In this situation, the arithmetic fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field. This kind of anabelian phenomenon is the reason why we do not have an explicit description of the geometric fundamental group of any pointed stable curve in positive characteristic. Moreover, we may think that the anabelian geometry of curves is a theory based on the following rough consideration: The admissible fundamental group of a pointed stable curve over an algebraically closed field of characteristic p must encode *"moduli"* of the curve.

0.3. **A moduli version of the weak Isom-version conjecture.** We reformulate the anabelian geometry of curves over algebraically closed fields of positive characteristic from the point of view of moduli spaces.

0.3.1. Firstly, we fix some notation concerning moduli spaces of curves and admissible fundamental groups associated to points of moduli spaces. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over $\overline{\mathbb{F}}_p$ classifying pointed stable curves of type (g, n) (i.e. the quotient stack of the moduli stack of n-pointed stable curves in the sense of [K] by the natural action of n-symmetric group), $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ the open substack classifying smooth pointed stable curves, $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$, and $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$.

Let $q \in \overline{M}_{g,n}$ be a point, $k(q)$ the residue field of $\overline{M}_{g,n}$, and k_q an algebraically closed field containing $k(q)$. Then the composition of natural morphisms $\mathrm{Spec}\, k_q \to$

$\mathrm{Spec}\, k(q) \to \overline{M}_{g,n}$ determines a pointed stable curve $X^{\bullet}_{k_q}$ of type (g, n) over k_q. In particular, if k_q is an algebraic closure of $k(q)$, we shall write $X^{\bullet}_{q}$ for $X^{\bullet}_{k_q}$. Let $\pi_1^{\mathrm{adm}}(X^{\bullet}_{k_q})$ be the admissible fundamental group of $X^{\bullet}_{k_q}$. Since the isomorphism class of $\pi_1^{\mathrm{adm}}(X^{\bullet}_{k_q})$ does not depend on the choice of k_q (1.2.4), we shall write $\pi_1^{\mathrm{adm}}(q)$ for the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^{\bullet}_{k_q})$.

Let $\overline{\Pi}_{g,n}$ be the set of isomorphism classes (as profinite groups) of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of characteristic p. Then the fundamental group functor π_1^{adm} induces a natural sujective map from the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$ to $\overline{\Pi}_{g,n}$ as follows: $[\pi_1^{\mathrm{adm}}] : |\overline{M}_{g,n}| \twoheadrightarrow \overline{\Pi}_{g,n},\ q \mapsto [\pi_1^{\mathrm{adm}}(q)]$, where $[\pi_1^{\mathrm{adm}}(q)]$ denotes the isomorphism class of $\pi_1^{\mathrm{adm}}(q)$.

Since the existence of Frobenius twists of pointed stable curves, the map $[\pi_1^{\mathrm{adm}}]$ is *not a bijection* in general. We introduce an equivalence relation $\sim_{fe}$ on $|\overline{M}_{g,n}|$ which we call *Frobenius equivalence* (see [Y4, Definition 3.4] or Definition 3.1 of the present paper). Moreover, [Y4, Proposition 3.7] shows that $[\pi_1^{\mathrm{adm}}]$ factors through the following quotient set $\overline{\mathfrak{M}}_{g,n} \overset{\mathrm{def}}{=} |\overline{M}_{g,n}|/ \sim_{fe}$. Then we obtain a natural surjective map

$$\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n},\ [q] \mapsto [\pi_1^{\mathrm{adm}}(q)],$$

induced by $[\pi_1^{\mathrm{adm}}]$, where $[q]$ denotes the image of q of the natural quotient map $|\overline{M}_{g,n}| \to \overline{\mathfrak{M}}_{g,n}$.

0.3.2. The "Weak Isom-version" mentioned in 0.1.2 can be successfully formulated for pointed stable curves over algebraically closed fields of characteristic p (see [T2], [T3] for the case of smooth pointed stable curves, and [Y4] for the case of arbitrary pointed stable curves). We shall refer to the formulation as the weak Isom-version conjecture:

Weak Isom-version Conjecture . *We maintain the notation introduced above. Then the surjective map*

$$\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is a bijection.

The weak Isom-version conjecture is one of the main conjectures in the theory of anabelian geometry of curves, which was only completely proved in the case where $(g, n) = (0, 3)$ or $(0, 4)$ (see [T4, Theorem 0.2], [Y4, Theorem 3.8], or Theorem 3.4 of the present paper).

Until now, the weak Isom-version conjecture is the ultimate goal of the anabelian geometry of curves over algebraically closed fields of characteristic p, all of the researches focus on this conjecture (e.g. [PS], [R3], [Sar], [ST2], [T2], [T4], [T5], [Y2], [Y6]). Essentially, the weak Isom-version conjecture *shares the same anabelian* *philosophy as Grothendieck originally suggested* (i.e. the “Weak Isom-version” mentioned in 0.1.2), and this conjecture cannot give us any new insight into the anabelian phenomena of curves over algebraically closed fields of characteristic p.

0.3.3. The “Isom-version” mentioned in 0.1.2 can be also successfully formulated for pointed stable curves over algebraically closed fields of characteristic p (e.g. see [T3, Conjecture 1.33] for the case of smooth pointed stable curves). At the time of writing, no results are known for this conjecture.

0.4. **A new kind of anabelian phenomenon.**

0.4.1. When Tamagawa tried to formulate a “Hom-version” conjecture for curves over algebraically closed fields of characteristic p based on Grothendieck's anabelian philosophy mentioned in 0.1.2 (i.e. an analogue of the conjecture posed in [G, p289 (6)]), he noted that the following phenomenon exists:

> Let $q_i \in M_{g,n}$, $i \in \{1, 2\}$, be a smooth pointed stable curve over an algebraically closed field k_i of characteristic $p > 0$ and $\pi_1^{\mathrm{adm}}(q_i)$ the admissible fundamental group (=the tame fundamental group) of $X^\bullet_{q_i}$. Then we have (e.g. specialization homomorphisms of a non-isotrivial family of pointed stable curves)
>
> $$\mathrm{Hom}^{\mathrm{dom}}(X^\bullet_{q_1}, X^\bullet_{q_2}) = \emptyset,\ \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) \neq \emptyset,$$
>
> where $\mathrm{Hom}^{\mathrm{dom}}(-,-)$ denotes the set of dominant morphisms of pointed stable curves, and $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(-,-)$ denotes the set of open continuous homomorphisms of profinite groups. This means that
>
> $$\mathrm{Hom}^{\mathrm{dom}}(X^\bullet_{q_1}, X^\bullet_{q_2}) \not\twoheadrightarrow \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)).$$

The above phenomenon means that if we only consider anabelian philosophy suggested originally by Grothendieck mentioned in 0.1.2, the relation of two pointed stable curves *cannot* be determined by the set of open continuous homomorphisms of their admissible fundamental groups, and the “Hom-version” conjecture (in the sense of 0.1.2) for curves over algebraically closed fields of characteristic p *does not* exist.

In fact, the existence of specialization homomorphisms is the reason that Tamagawa cannot formulate a “Hom-version” conjecture for tame fundamental groups of smooth pointed stable curves in general ([T3, Remark 1.34]).

0.4.2. On the other hand, the author of the present paper considered the following the fundamental question:

> Does there exist *a geometric explanation* (i.e. *an anabelian explanation*) for the group-theoretical object $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2))$?

We observed a new phenomenon that has never been seen before: It is possible that *the sets of deformations* of a smooth pointed stable curve can be reconstructed group-theoretically from open continuous homomorphisms of their admissible fundamental groups. Let $q_1, q_2 \in M_{g,n}$. This mean is that, roughly speaking, a smooth pointed stable curve corresponding to a geometric point over q_2 can be deformed to a smooth pointed stable curve corresponding to a geometric point over q_1 if and only if the set of open continuous homomorphisms of admissible fundamental groups $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2))$ is not empty.

Moreover, the above observation implies *a new kind of anabelian phenomenon that cannot be explained by using Grothendieck's original anabelian philosophy mentioned in 0.1.2*:

> The *topological structures* of moduli spaces of curves in positive characteristic are encoded in the *sets of open continuous homomorphisms* of geometric fundamental groups of curves in positive characteristic.

This new kind of anabelian phenomenon can be precisely captured by using the so-called *moduli spaces of admissible fundamental groups* and *the homeomorphism conjecture* introduced in the present paper. Let us briefly explain them in the next subsection of the introduction.

0.5. **The homeomorphism conjecture.** We maintain the notation introduced in 0.3. Moreover, from now on, we shall regard $\overline{\mathfrak{M}}_{g,n}$ as a topological space whose topology is induced naturally by the Zariski topology of $|\overline{M}_{g,n}|$.

0.5.1. Let $\mathscr{G}$ be the category of finite groups and $G \in \mathscr{G}$ a finite group. We put

$$U_{\overline{\Pi}_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{adm}}(q)] \in \overline{\Pi}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), G) \neq \emptyset\},$$

where $\mathrm{Hom}_{\mathrm{surj}}(-,-)$ denotes the set of surjective homomorphisms of profinite groups. We define a topological space $(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}})$ group-theoretically from $\overline{\Pi}_{g,n}$ as follows: The underlying set is $\overline{\Pi}_{g,n}$, and the topology $O_{\overline{\Pi}_{g,n}}$ is generated by $\{U_{\overline{\Pi}_{g,n},G}\}_{G\in\mathscr{G}}$ as open subsets. For simplicity of notation, we still use $\overline{\Pi}_{g,n}$ to denote the topological space $(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}})$, and call the topological space

$$\overline{\Pi}_{g,n}$$

the moduli space of admissible fundamental groups of type (g, n).

0.5.2. Theorem 3.6 of the present paper shows that the surjective map $\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$ is a *continuous* map. Moreover, we pose the following conjecture, which is the main conjecture of the theory developed in the present series of papers:

Homeomorphism Conjecture . *We maintain the notation introduced above. Then we have that the natural map*

$$\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is a homeomorphism.

0.5.3. *Remark.* The homeomorphism conjecture has a simpler form if we only consider smooth pointed stable curves. Let $\mathbb{F}_p$ be the prime field of characteristic p, $M_{g,n,\mathbb{F}_p}$ the coarse moduli space of the moduli stack $\mathcal{M}_{g,n,\mathbb{F}_p}$ over $\mathbb{F}_p$ classifying smooth pointed stable curves of type (g,n). Let $\Pi_{g,n} \subseteq \overline{\Pi}_{g,n}$ be the subset of isomorphism classes of admissible fundamental groups (=tame fundamental groups) of smooth pointed stable curves of type (g,n). The subset $\Pi_{g,n}$ can be regarded as a topological space whose topology is induced by the topology of $\overline{\Pi}_{g,n}$ (in fact, $\Pi_{g,n}$ is an open subset of $\overline{\Pi}_{g,n}$ (see Proposition 3.10 (b)). In this situation, the homeomorphism conjecture is equivalent to the following form: *The natural map* $M_{g,n,\mathbb{F}_p} \twoheadrightarrow \Pi_{g,n},\ q \mapsto [\pi_1^{\mathrm{adm}}(q)]$, *is a homeomorphism.*

0.6. Weak Isom-version Conjecture vs. Homeomorphism Conjecture.

0.6.1. Firstly, let us explain the difference between the the weak Isom-version conjecture and the homeomorphism conjecture from *the aspect of anabelian philosophy*.

The weak Isom-version conjecture means that the moduli spaces of curves in positive characteristic can be reconstructed group-theoretically *as sets* from *isomorphism classes* of admissible fundamental groups of pointed stable curves in positive characteristic.

On the other hand, the homeomorphism conjecture generalizes all the conjectures appeared in the theory of admissible (or tame) anabelian geometry of curves over algebraically closed fields of characteristic p, and means that the moduli spaces of curves in positive characteristic can be reconstructed group-theoretically *as topological spaces* from *sets of open continuous homomorphisms* of admissible fundamental groups of pointed stable curves in positive characteristic.

The moduli spaces of admissible fundamental groups and the homeomorphism conjecture shed some new light on the theory of the anabelian geometry of curves over algebraically closed fields of characteristic p based on the following *new anabelian philosophy*:

> The *anabelian properties* of pointed stable curves over algebraically closed fields of characteristic p are equivalent to the *topological properties* of the topological space $\overline{\Pi}_{g,n}$.

Since Tamagawa discovered that there also exists the anabelian geometry for certain smooth pointed stable curves over the algebraically closed fields of characteristic p, almost 30 years have passed. However, the weak Isom-version conjecture is still the only anabelian phenomenon that we know in this situation, and we cannot even imagine what phenomena arose from curves and their fundamental groups should be anabelian.

The above philosophy supplies a point of view to see *what anabelian phenomena that we can reasonably expect* for pointed stable curves over algebraically closed fields of characteristic p. This means that the homeomorphism conjecture is a *dictionary* between the geometry of pointed stable curves (or moduli spaces of curves) and the anabelian properties of pointed stable curves. For instance, it has raised a host of new questions (e.g. Section 3.4) concerning anabelian phenomena which cannot be seen if we only consider the weak Isom-version conjecture.

0.6.2. Next, let us explain the difference between the weak Isom-version conjecture and the homeomorphism conjecture from *the aspect of group theory*. The mean of anabelian geometry around the weak Isom-version conjecture (i.e. the theory developed in [PS], [R3], [Sar], [T2], [T4], [T5], [Y2], [Y6]) is the following: Let $\mathcal{F}_i$, $i \in \{1, 2\}$, be a geometric object in a certain category and $\Pi_{\mathcal{F}_i}$ the fundamental group associated to $\mathcal{F}_i$. Then the set of isomorphisms of geometric objects $\mathrm{Isom}(\mathcal{F}_1, \mathcal{F}_2)$ can be understood from the set of isomorphisms of group-theoretical objects $\mathrm{Isom}(\Pi_{\mathcal{F}_1}, \Pi_{\mathcal{F}_2})$. The term "*anabelian*" means that the geometric properties of a geometric object which can be determined by the isomorphism classes of its fundamental group. On the other hand, we do not know the relation of $\mathcal{F}_1$ and $\mathcal{F}_2$ if $\Pi_{\mathcal{F}_1}$ is not isomorphic to $\Pi_{\mathcal{F}_2}$.

In the theory developed in the present series of papers, we consider anabelian geometry in a completely different way. The mean of anabelian geometry around the homeomorphism conjecture is the following: The relation of $\mathcal{F}_1$ and $\mathcal{F}_2$ in a certain moduli space can be understood from a certain set of homomorphisms $\mathrm{Hom}(\Pi_{\mathcal{F}_1}, \Pi_{\mathcal{F}_2})$. Moreover, $\mathrm{Hom}(\Pi_{\mathcal{F}_1}, \Pi_{\mathcal{F}_2})$ contains the *deformation information* of $\mathcal{F}_2$ along $\mathcal{F}_1$. The term "*anabelian*" means the geometric properties of a certain *moduli space* of geometric objects (i.e. not only a single geometric object but also the moduli space of geometric objects) which can be determined by the set of open continuous homomorphisms of fundamental groups of geometric objects.

Thus, roughly speaking, the weak Isom-version conjecture is an "*Isom-version*" problem, and the homeomorphism conjecture is a "*Hom-version*" problem. Similar to other theory in anabelian geometry, Hom-version problems are so much harder than the Isom-version problems.

0.7. **Main result.**

0.7.1. Our main result of the present paper is as follows:

Theorem 0.1 (Theorem 6.7)**.** *We maintain the notation introduced above. Let* $[q] \in \overline{\mathfrak{M}}_{0,n}^{\mathrm{cl}}$ *be an arbitrary closed point. Then* $\pi_{0,n}^{\mathrm{adm}}([q])$ *is a closed point of* $\overline{\Pi}_{0,n}$*. In particular, the homeomorphism conjecture holds when* $(g, n) = (0, 3)$ *or* $(0, 4)$*.*

Denote by $\mathrm{Isom}_{\mathrm{pg}}(-,-)$ the set of isomorphisms of profinite groups. Then Theorem 0.1 follows from the following strong (Hom-version) anabelian result.

Theorem 0.2 (Theorem 6.6)**.** *Let q_1, $q_2 \in \overline{M}_{0,n}$ be arbitrary points. Suppose that q_1 is closed. Then we have that*

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) \neq \emptyset$$

if and only if $q_1 \sim_{fe} q_2$. In particular, if this is the case, we have that q_2 is a closed point, and that

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) = \mathrm{Isom}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)).$$

Remark 0.2.1. In fact, in the present paper, we will prove a slightly stronger version of Theorem 0.2 by replacing $\pi_1^{\mathrm{adm}}(q_1)$ and $\pi_1^{\mathrm{adm}}(q_2)$ by the maximal pro-solvable quotients $\pi_1^{\mathrm{adm}}(q_1)^{\mathrm{sol}}$ and $\pi_1^{\mathrm{adm}}(q_2)^{\mathrm{sol}}$ of $\pi_1^{\mathrm{adm}}(q_1)$ and $\pi_1^{\mathrm{adm}}(q_2)$, respectively. Then we obtain a solvable version of Theorem 0.1 which is slightly stronger than Theorem 0.1. In particular, we obtain that *the solvable homeomorphism conjecture* (see 3.3) holds when $(g, n) = (0, 3)$ or $(0, 4)$.

0.7.2. We will prove directly Theorem 0.1 (or Theorem 0.2) without the use of results concerning the weak Isom-version conjecture obtained in [T2], [T4], [Y2], and its proof is much harder than the proofs of the main results of [T2], [T4], [Y2] since we need to establish new connections between geometry of arbitrary (possibly singular) pointed stable curves and arbitrary open continuous homomorphisms of their fundamental groups which are not isomorphisms in general ([T5, Theorem 0.3], [Y2, Theorem 7.9]).

0.8. **Strategy of proof.** We briefly explain the method of proving Theorem 0.2 (or Theorem 0.1), whose tools are based on formulas concerning generalized Hasse-Witt invariants proved in [Y3], [Y5] and the theory of combinatorial anabelian geometry of curves in positive characteristic developed in [Y2], [Y6].

0.8.1. Firstly, we establish precise connections between the geometric behaviors of curves and open continuous homomorphisms of their admissible fundamental groups, which play central roles in the theory of moduli spaces of admissible fundamental groups in positive characteristic.

The first result is the following, which is the main theorems of Section 4 (see Theorem 4.11 and Theorem 4.13 for more precise statements):

Theorem 0.3. *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic p, and $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$. Let $\Pi_{X_i^\bullet}$ be either the admissible fundamental group $\pi_1^{\mathrm{adm}}(X_i^\bullet)$ of $X_i^\bullet$ or the maximal pro-solvable quotient $\pi_1^{\mathrm{adm}}(X_i^\bullet)^{\mathrm{sol}}$ of $\pi_1^{\mathrm{adm}}(X_i^\bullet)$, and $I_i \subseteq \Pi_{X_i^\bullet}$ a closed subgroup associated to an open edge of $\Gamma_{X_i^\bullet}$ (i.e. a closed subgroup which is (outer)* *isomorphic to the inertia subgroup of the marked point corresponding to an open edge of* $\Gamma_{X_i^\bullet}$*). Suppose that* $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$*. Let*

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism of profinite groups. Then the following statements hold:

(i) $\phi(I_1) \subseteq \Pi_{X_2^\bullet}$ *is a closed subgroup associated to an open edge of* $\Gamma_{X_2^\bullet}$*, and there exists a closed subgroup* $I' \subseteq \Pi_{X_1^\bullet}$ *associated to an open edge of* $\Gamma_{X_1^\bullet}$ *such that* $\phi(I') = I_2$*.*

(ii) The field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$*, and* ϕ *induces a field isomorphism between the fields associated to* I_1 *and* $\phi(I_1)$ *group-theoretically.*

Theorem 0.3 says that the inertia subgroups and field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from arbitrary surjective open continuous homomorphisms of admissible fundamental groups. One of the main ingredients in the proof of Theorem 0.3 is an explicit formula for the maximum generalized Hasse-Witt invariant $\gamma^{\max}(\Pi_{X_i^\bullet})$ of an arbitrary pointed stable curve $X_i^\bullet$, which was proved by the author by using the theory of Raynaud-Tamagawa theta divisors ([Y5, Theorem 5.4]).

The second result is a generalized version of combinatorial Grothendieck conjecture in positive characteristic. One of the main results of Section 5 is as follows, which says that *the combinatorial Grothendieck conjecture for open continuous homomorphisms* holds for pointed stable curves of type $(0, n)$ (see Theorem 5.30 for a more precise statement):

Theorem 0.4. *Let* $X_i^\bullet$*,* $i \in \{1, 2\}$*, be a pointed stable curve of type* $(0, n)$ *over an algebraically closed field* k_i *of characteristic* p*, and* $\Gamma_{X_i^\bullet}$ *the dual semi-graph of* $X_i^\bullet$*. Let* $\Pi_{X_i^\bullet}$ *be the maximal pro-solvable quotient* $\pi_1^{\mathrm{adm}}(X_i^\bullet)^{\mathrm{sol}}$ *of the admissible fundamental group* $\pi_1^{\mathrm{adm}}(X_i^\bullet)$ *of* $X_i^\bullet$ *and* $\Pi_i \subseteq \Pi_{X_i^\bullet}$ *a closed subgroup associated to a vertex (i.e. a closed subgroup which is (outer) isomorphic to the solvable admissible fundamental group of the smooth pointed stable curve associated to a vertex of* $\Gamma_{X_i^\bullet}$*), and* $I_i \subseteq \Pi_{X_i^\bullet}$ *a closed subgroup associated to a closed edge (i.e. a closed subgroup which is (outer) isomorphic to the inertia subgroup of the node corresponding to a closed edge of* $\Gamma_{X_i^\bullet}$*). Suppose that* $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_2^\bullet}))$ *and* $\#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) = \#(e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}))$*, where* $v(-)$ *denotes the set of vertices of* $(-)$ *and* $e^{\mathrm{cl}}(-)$ *denotes the set of closed edges of* $(-)$ *(see 1.1.1). Let*

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism of profinite groups. Then the following statements hold:

(i) $\phi(\Pi_1) \subseteq \Pi_{X_2^\bullet}$ *is a closed subgroup associated to a vertex of* $\Gamma_{X_2^\bullet}$, *and there exists a closed subgroup* $\Pi' \subseteq \Pi_{X_1^\bullet}$ *associated to a vertex of* $\Gamma_{X_1^\bullet}$ *such that* $\phi(\Pi') = \Pi_2$.

(ii) $\phi(I_1) \subseteq \Pi_{X_2^\bullet}$ *is a closed subgroup associated to a closed edge of* $\Gamma_{X_2^\bullet}$, *and there exists a closed subgroup* $I' \subseteq \Pi_{X_1^\bullet}$ *associated to a closed edge of* $\Gamma_{X_1^\bullet}$ *such that* $\phi(I') = I_2$.

(iii) ϕ *induces an isomorphism*

$$\phi^{\mathrm{sg}} : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$$

of dual semi-graphs group-theoretically.

Theorem 0.4 says that the geometry (i.e. topological and combinatorial data) of pointed stable curves can be completely reconstructed group-theoretically from open continuous homomorphisms of admissible fundamental groups. One of the main ingredients in the proof of Theorem 0.4 is an explicit formula for the limit of p-averages $\mathrm{Avr}_p(\Pi_{X_i^\bullet})$ of the admissible fundamental group of $X_i^\bullet$, which was proved by Tamagawa ([T4, Theorem 0.5]) and the author ([Y3, Theorem 1.3]) by using the theory of Raynaud-Tamagawa theta divisors .

In anabelian geometry, the geometric data of an geometric object can be represented by various subgroups of its fundamental group. Then, roughly speaking, Theorem 0.3 and Theorem 0.4 mean that the geometric data of $X_2^\bullet$ *can be controlled* by the geometric data of $X_1^\bullet$ if there exists an open continuous homomorphism between their admissible fundamental groups.

Remark. In fact, Theorem 0.4 is a consequence of a generalized result (see Theorem 5.26) which says that Theorem 0.4 also holds for arbitrary types under certain assumptions. Moreover, the author believes that the methods developed in Section 5 can be used to prove the combinatorial Grothendieck conjecture for open continuous homomorphisms without any assumptions (see Remark 5.26.1 and Remark 5.26.2), and that Theorem 0.3, Theorem 0.4, and Theorem 5.26 will play important roles in the proof of the homeomorphism conjecture for higher dimensional moduli spaces. For instance, in [Y8], we use Theorem 0.3 and Theorem 5.26 to prove the homeomorphism conjecture for $(g, n) = (1, 1)$.

0.8.2. By applying Theorem 0.3 and Theorem 0.4, we briefly sketch the proof of Theorem 0.2 as follows:

Case I: $q_1 \in M_{0,n}$. Over $\overline{\mathbb{F}}_p$, the scheme structure of a smooth pointed stable curve of type $(0, n)$ can be completely determined by its inertia subgroups of marked points and the field structures associated to the inertia subgroups via generalized Hasse-Witt invariants. By constructing certain admissible coverings for $X_{q_1}^\bullet$ and $X_{q_2}^\bullet$, we apply Theorem 0.3 to prove that, when $X_{q_1}^\bullet$ is nonsingular, the scheme structure of $X_{q_2}^\bullet$ can be determined by the scheme structure of $X_{q_1}^\bullet$ via an open continuous homomorphism between their admissible fundamental groups (see Proposition 6.2 and Proposition 6.5).

Case II: $q_1 \in \overline{M}_{0,n} \setminus M_{0,n}$. By applying Theorem 0.3, the geometric operation (=removing a subset of marked points of a pointed stable curve and contracting the (-1)-curves and the (-2)-curves of a pointed semi-stable curve) can be translated to the group-theoretical operation (=quotient of a closed subgroup of the admissible fundamental group of a pointed stable curve, where the closed subgroup is generated by the inertia subgroups corresponding to a subset of marked points of the pointed stable curve). Then we can reduce Theorem 0.2 to the case where $\#(v(\Gamma_{X^\bullet_{q_1}})) = \#(v(\Gamma_{X^\bullet_{q_2}}))$ and $\#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{q_1}})) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{q_2}}))$. Moreover, by applying Theorem 0.4, we can reduce Theorem 0.2 further to the case where q_1 and q_2 are contained in $M_{0,n}$ (i.e. $X^\bullet_{q_1}$ and $X^\bullet_{q_2}$ are nonsingular). Then Theorem 0.2 follows from the case where $q_1 \in M_{0,n}$.

0.9. **Structure of the present paper.** The present paper is organized as follows.

Part I (Formulations of moduli spaces of admissible fundamental groups) consists of Section 1∼3. In Section 1, we fix some notation concerning admissible coverings and admissible fundamental groups. In Section 2, we recall the definition of generalized Hasse-Witt invariants, a formula for maximum generalized Hasse-Witt invariants of prime-to-p admissible coverings, and a formula for limits of p-averages of admissible fundamental groups. In Section 3, we introduce the moduli spaces of admissible fundamental groups (resp. the moduli spaces of solvable admissible fundamental groups) and formulate the homeomorphism conjecture. We also pose some open problems that are of particular interest of the author. In particular, we formulate a generalized version of Tamagawa's essential dimension conjecture from the point of view of the theory of moduli spaces of fundamental groups (Section 3.4.1). Moreover, we prove some basic properties concerning the topology of $\overline{\Pi}_{g,n}$.

Part II (Reconstructions of geometric data from open continuous homomorphisms) consists of Section 4∼5. In Section 4, we prove Theorem 0.3. In Section 5, we prove the combinatorial Grothendieck conjecture for open continuous homomorphisms under certain conditions. As a consequence, by applying Theorem 0.3, we obtain Theorem 0.4.

Part III (Main result) consists of Section 6, and we prove our main theorem in this part.

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PART I: FORMULATIONS OF MODULI SPACES OF ADMISSIBLE FUNDAMENTAL GROUPS

1. Admissible coverings and admissible fundamental groups

In this section, we set up notation and terminology concerning admissible coverings and admissible fundamental groups.

1.1. **Admissible coverings.**

1.1.1. Let Γ be a semi-graph (see [Y5, 2.1.1] for a rough explanation).

(a) We shall denote by $v(\Gamma)$, $e^{\mathrm{op}}(\Gamma)$, and $e^{\mathrm{cl}}(\Gamma)$ the set of vertices of Γ, the set of open edges of Γ, and the set of closed edges of Γ, respectively.

(b) The semi-graph Γ can be regarded as a topological space with natural topology induced by $\mathbb{R}^2$. We define an *one-point compactification* Γ^{cpt} of Γ as follows: if $e^{\mathrm{op}}(\Gamma) = \emptyset$, we put $\Gamma^{\mathrm{cpt}} = \Gamma$; otherwise, the set of vertices of Γ^{cpt} is the disjoint union $v(\Gamma^{\mathrm{cpt}}) \stackrel{\mathrm{def}}{=} v(\Gamma) \sqcup \{v_\infty\}$, the set of closed edges of Γ^{cpt} is $e^{\mathrm{cl}}(\Gamma^{\mathrm{cpt}}) \stackrel{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma) \cup e^{\mathrm{cl}}(\Gamma)$, the set of open edges of Γ is empty, and every edge $e \in e^{\mathrm{op}}(\Gamma) \subseteq e^{\mathrm{cl}}(\Gamma^{\mathrm{cpt}})$ connects v_∞ with the vertex that is abutted by e.

(c) Let $v \in v(\Gamma)$. We shall say that Γ is 2-*connected* at v if $\Gamma \setminus \{v\}$ is either empty or connected. Moreover, we shall say that Γ is 2-connected if Γ is 2-connected at each $v \in v(\Gamma)$. Note that, if Γ is connected, then Γ^{cpt} is 2-connected at each $v \in v(\Gamma) \subseteq v(\Gamma^{\mathrm{cpt}})$ if and only if Γ^{cpt} is 2-connected. We put

$$b(v) \stackrel{\mathrm{def}}{=} \sum_{e \in e^{\mathrm{op}}(\Gamma) \cup e^{\mathrm{cl}}(\Gamma)} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that e meets v. We put

$$v(\Gamma)^{b \leq 1} \stackrel{\mathrm{def}}{=} \{v \in v(\Gamma) \mid b(v) \leq 1\},$$

and denote by $e^{\mathrm{cl}}(\Gamma)^{b \leq 1}$ the set of closed edges of Γ which meet some vertex of $v(\Gamma)^{b \leq 1}$.

1.1.2. Let p be a prime number, and let

$$X^\bullet = (X, D_X)$$

be a *pointed semi-stable curve* of type (g_X, n_X) over an algebraically closed field k of characteristic p, where X denotes the underlying curve, D_X denotes the (finite) set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#(D_X)$ of D_X. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ (see [Y1, Definition 3.1] for the definition of the dual semi-graph of a pointed semi-stable curve) and $r_X \overset{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$. We shall say that $X^\bullet$ is a *pointed stable curve* over k if D_X satisfies [K, Definition 1.1 (iv)].

1.1.3. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \overset{\text{def}}{=} X_v \setminus X_v^{\mathrm{sing}}$, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$. We define a smooth pointed semi-stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \overset{\text{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v)).$$

We shall call $\widetilde{X}_v^\bullet$ *the smooth pointed semi-stable curve of type* (g_v, n_v) *associated to* v, or the smooth pointed semi-stable curve associated to v for short. In particular, we shall say that $\widetilde{X}_v^\bullet$ is the smooth pointed *stable* curve associated to v if $\widetilde{X}_v^\bullet$ is a pointed stable curve over k.

1.1.4. We recall the definition of Mochizuki's admissible coverings of pointed stable curves (see also [M1, §3]). Let $Y^\bullet = (Y, D_Y)$ be a pointed semi-stable curve over k and $\Gamma_{Y^\bullet}$ the dual semi-graph of $Y^\bullet$. Let

$$f^\bullet : Y^\bullet \to X^\bullet$$

be a *surjective, generically étale, finite* morphism of pointed semi-stable curves over k such that $f(y)$ is a smooth (resp. singular) point of X if y is a smooth (resp. singular) point of Y. Write $f : Y \to X$ for the morphism of underlying curves induced by $f^\bullet$, and $f^{\mathrm{sg}} : \Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by $f^\bullet$, where "sg" means "semi-graph". Let $v \in v(\Gamma_{X^\bullet})$ and $w \in (f^{\mathrm{sg}})^{-1}(v) \subseteq v(\Gamma_{Y^\bullet})$. Then $f^\bullet$ induces a morphism of smooth pointed semi-stable curves

$$\widetilde{f}^\bullet_{w,v} : \widetilde{Y}^\bullet_w \to \widetilde{X}^\bullet_v$$

over k associated to w and v.

Definition 1.1. We shall say that $f^\bullet : Y^\bullet \to X^\bullet$ is a *Galois admissible covering* over k with Galois group G if the following conditions are satisfied:

(i) There exists a finite group $G \subseteq \mathrm{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$.

(ii) $\widetilde{f}^\bullet_{w,v}$ is a tame covering over k for each $v \in v(\Gamma_{X^\bullet})$ and each $w \in (f^{\mathrm{sg}})^{-1}(v)$.
(iii) For each $y \in Y^{\mathrm{sing}}$, we write $D_y \subseteq G$ for the decomposition group of y and τ for a generator of D_y. Then the local morphism between singular points induced by f is

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^{\#D_y} \\ v & \mapsto & t^{\#D_y}, \end{array}$$

and that $\tau(s) = \zeta_{\#(D_y)}s$ and $\tau(t) = \zeta^{-1}_{\#(D_y)})t$, where $\zeta_{\#(D_y)}$ is a primitive $\#(D_y)$th root of unity.

Moreover, we shall say that $f^\bullet$ is an *admissible covering* if there exists a morphism of pointed semi-stable curves $h^\bullet : W^\bullet \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ h^\bullet : W^\bullet \to X^\bullet$ is a Galois admissible covering over k.

Let $Z^\bullet$ be a disjoint union of finitely many pointed semi-stable curves over k. We shall say that a morphism $f^\bullet_Z : Z^\bullet \to X^\bullet$ over k is a *multi-admissible covering* if the restriction of $f^\bullet_Z$ to each connected component of $Z^\bullet$ is admissible, and that $f^\bullet_Z$ is *étale* if the underlying morphism of curves f_Z induced by $f^\bullet_Z$ is an étale morphism.

Remark 1.1.1. In [M1, §3.9 Definition], the admissible coverings defined in Definition 1.1 are called HM-admissible coverings (i.e. Harris-Mumford admissible coverings).

1.1.5. Let $f^\bullet : Y^\bullet \to X^\bullet$ be an admissible covering over k of degree m. Let $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ and x_e the closed point of X corresponding to e. We put

$$e_f^{\mathrm{cl,ra}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid \#(f^{-1}(x_e)) = 1\},$$

$$e_f^{\mathrm{cl,ét}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid \#(f^{-1}(x_e)) = m\},$$

$$e_f^{\mathrm{op,ra}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \mid \#(f^{-1}(x_e)) = 1\},$$

$$e_f^{\mathrm{op,ét}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \mid \#(f^{-1}(x_e)) = m\},$$

$$v_f^{\mathrm{ra}} \overset{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \#(\mathrm{Irr}(f^{-1}(X_v))) = 1\},$$

$$v_f^{\mathrm{sp}} \overset{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \#(\mathrm{Irr}(f^{-1}(X_v))) = m\},$$

where $\mathrm{Irr}(-)$ denotes the set of irreducible components of $(-)$, "ra" means "ramification", and "sp" means "split". Note that if the Galois closure of $f^\bullet$ is a Galois admissible covering whose Galois group is a p-group, then the definition of admissible coverings implies $\#(e_f^{\mathrm{cl,ra}}) = \#(e_f^{\mathrm{op,ra}}) = 0$.

1.2. **Admissible fundamental groups.** In this subsection, we recall some well-known properties concerning admissible fundamental groups of pointed semi-stable curves. There are many approaches to define admissible fundamental groups of pointed semi-stable curves (e.g. constructing Galois categories of admissible covering (by equipping certain isomorphisms of tangent base points of branches of nodes), Mochizuki's theory of semi-graphs of anabelioids, geometric log étale fundamental groups, etc.). In the present paper, we define admissible fundamental groups of pointed stable curves by using log geometry (see also [T6, §2]).

1.2.1. We maintain the notation introduced in 1.1.2. Let $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ be the moduli stack over $\operatorname{Spec}\mathbb{Z}$ parameterizing pointed stable curves of type (g_X, n_X) (i.e. the quotient stack of the moduli stack of n-pointed stable curves in the sense of [K] by the natural action of n-symmetric group) and $\mathcal{M}_{g_X,n_X,\mathbb{Z}}$ the open substack of $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ parameterizing smooth pointed stable curves. Write $\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \setminus \mathcal{M}_{g_X,n_X,\mathbb{Z}} \subset \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ relative to $\operatorname{Spec}\mathbb{Z}$.

Write $X^\bullet_{\mathrm{st}}$ for the pointed *stable* curve associated to $X^\bullet$ (i.e. the pointed stable curve obtained by contracting the (-1)-curves and (-2)-curves of $X^\bullet$). Then we obtain a morphism $s \overset{\mathrm{def}}{=} \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ determined by $X^\bullet_{\mathrm{st}} \to s$. Write $s^{\log}_{X_{\mathrm{st}}}$ for the log scheme whose underlying scheme is $\operatorname{Spec} k$, and whose log structure is the pulling-back log structure induced by the morphism $s \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. We obtain a natural morphism $s^{\log}_{X_{\mathrm{st}}} \to \overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ induced by the morphism $s \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ and a stable log curve

$$X^{\log}_{\mathrm{st}} \overset{\mathrm{def}}{=} s^{\log}_{X_{\mathrm{st}}} \times_{\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}} \overline{\mathcal{M}}^{\log}_{g_X,n_X+1,\mathbb{Z}}$$

over $s^{\log}_{X_{\mathrm{st}}}$ whose underlying scheme is X_{st}. Then there exists a log blow-up $X^{\log} \to X^{\log}_{\mathrm{st}}$ such that the underlying scheme of $X^{\log}$ is X.

1.2.2. Let $\widetilde{x}^{\log} \to X^{\log}$ be a log geometric point and $\widetilde{x}^{\log} \to X^{\log} \to X^{\log}_{\mathrm{st}}$ the composition morphism of the natural morphisms of log schemes. Moreover, suppose that the image of the morphism of underlying schemes of $\widetilde{x}^{\log} \to X^{\log}_{\mathrm{st}}$ is a smooth point of X_{st}. Write $x \to X$ and $x \to X_{\mathrm{st}}$ for the geometric points induced by the log geometric points. Then we have a surjective homomorphism of log étale fundamental groups $\pi_1(X^{\log}, \widetilde{x}^{\log}) \twoheadrightarrow \pi_1(s^{\log}_{X_{\mathrm{st}}}, \widetilde{x}^{\log})$ (see [I] for the general theory of log étale fundamental groups). We call

$$\pi_1^{\mathrm{adm}}(X^\bullet, x) \overset{\mathrm{def}}{=} \ker(\pi_1(X^{\log}, \widetilde{x}^{\log}) \twoheadrightarrow \pi_1(s^{\log}_{X_{\mathrm{st}}}, \widetilde{x}^{\log}))$$

the *admissible fundamental group* of $X^\bullet$ (i.e. the geometric log étale fundamental group of $X^{\log}$). It is well known that $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ independents the log structures of $X^{\log}$, and that there is a bijection between the set of open (resp. open normal) subgroups of $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ and the set of isomorphism classes of admissible (resp. Galois admissible) coverings of $X^\bullet$ over k.

On the other hand, by applying similar arguments to the arguments given above, we obtain the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet_{\mathrm{st}}, x)$ of $X^\bullet_{\mathrm{st}}$. Moreover, by [I, Theorem 6.10], we have $\pi_1^{\mathrm{adm}}(X^\bullet, x) \cong \pi_1^{\mathrm{adm}}(X^\bullet_{\mathrm{st}}, x)$.

Since we only focus on the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ in the present paper, for simplicity of notation, we omit the base point and denote by

$$\pi_1^{\mathrm{adm}}(X^\bullet)$$

the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$. Note that, by the definition of admissible coverings, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the *tame fundamental group* of $X^\bullet$ when $X^\bullet$ is *smooth* over k.

1.2.3. *Remark.* Unlike [T2], we *do not consider the étale fundamental group of* $X \setminus D_X$ in general for the following reasons: (i) The étale fundamental group is not a good invariant if $X^\bullet$ is singular (since it does not contain the ramification information of singular points of $X^\bullet$), and if we consider anabelian geometry from the point of view of moduli spaces (since there does not exist a good deformation theory for étale coverings of $X \setminus D_X$ in positive characteristic if $D_X \neq \emptyset$). (ii) The results of anabelian geometry of curves concerning étale fundamental groups are weaker than the results of anabelian geometry of curves concerning tame (or admissible) fundamental groups ([T2, Corollary 1.5]).

1.2.4. Let k' be an arbitrary algebraically closed field containing k. Then it is well known that $\pi_1^{\mathrm{adm}}(X^\bullet) \cong \pi_1^{\mathrm{adm}}(X^\bullet \times_k k')$. Moreover, by applying [V, Théorème 2.2 (c)], we obtain that $\pi_1^{\mathrm{adm}}(X^\bullet)$ is topologically finitely generated, and that the maximal pro-prime-to-p quotient $\pi_1^{\mathrm{adm}}(X^\bullet)^{p'}$ of $\pi_1^{\mathrm{adm}}(X^\bullet)$ is isomorphic to the pro-prime-to-p completion of the following group

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X}[a_i, b_i]\prod_{j=1}^{n_X} c_j = 1\rangle.$$

1.2.5. Let $v \in v(\Gamma_{X^\bullet})$. Write $\pi_1^{\mathrm{adm}}(\widetilde{X}^\bullet_v)$ for the admissible fundamental group (=the tame fundamental group) of the smooth pointed semi-stable curve $\widetilde{X}^\bullet_v$ associated to v. Then we have a natural (outer) injection

$$\pi_1^{\mathrm{adm}}(\widetilde{X}^\bullet_v) \hookrightarrow \pi_1^{\mathrm{adm}}(X^\bullet).$$

We shall denote by $\pi_1^{\mathrm{adm}}(X)$, $\pi_1^{\text{ét}}(X)$, $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})$ the admissible fundamental group of the pointed semi-stable curve $(X, \emptyset)$, the étale fundamental group of the underlying curve X of $X^\bullet$, and the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, respectively. Then we have the following natural surjective open continuous homomorphisms (for suitable choices of base points):

$$\pi_1^{\mathrm{adm}}(X^\bullet) \twoheadrightarrow \pi_1^{\mathrm{adm}}(X) \twoheadrightarrow \pi_1^{\text{ét}}(X) \twoheadrightarrow \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet}).$$

Note that the isomorphism classes of $\pi_1^{\mathrm{adm}}(X^\bullet)$, $\pi_1^{\mathrm{adm}}(X)$, $\pi_1^{\text{ét}}(X)$, and $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})$ depend only on the pointed *stable* curve associated to $X^\bullet$.

1.2.6. Let $\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$, $\pi_1^{\mathrm{adm}}(X)^{\mathrm{sol}}$, $\pi_1^{\text{ét}}(X)^{\mathrm{sol}}$, $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}$ be the maximal pro-solvable quotients of $\pi_1^{\mathrm{adm}}(X^\bullet)$, $\pi_1^{\mathrm{adm}}(X)$, $\pi_1^{\text{ét}}(X)$, $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})$, respectively. Then we obtain the following natural surjective open continuous homomorphisms

$$\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}} \twoheadrightarrow \pi_1^{\mathrm{adm}}(X)^{\mathrm{sol}} \twoheadrightarrow \pi_1^{\text{ét}}(X)^{\mathrm{sol}} \twoheadrightarrow \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}.$$

We shall call

$$\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$$

the *solvable admissible fundamental group* of $X^\bullet$.

Let $v \in v(\Gamma_{X^\bullet})$. Write $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)^{\mathrm{sol}}$ for the solvable admissible fundamental group of the smooth pointed semi-stable curve $\widetilde{X}_v^\bullet$ associated to v. Then the natural (outer) injection $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet) \hookrightarrow \pi_1^{\mathrm{adm}}(X^\bullet)$ induces an (outer) homomorphism

$$\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)^{\mathrm{sol}} \to \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}.$$

We see that this homomorphism is an *injection*. Indeed, it follows immediately from the following: Let $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ be a Galois admissible covering over k whose Galois group is an abelian group. Then we see that there exists a Galois admissible covering $g^\bullet : Z^\bullet \to X^\bullet$ over k whose Galois group is a solvable group such that the following is satisfied: let Z_v be an irreducible component of $Z^\bullet$ such that $g(Z_v) = X_v$; then the Galois admissible covering $\widetilde{Z}_v^\bullet \to \widetilde{X}_v^\bullet$ over k induced by $g^\bullet$ factors through $\widetilde{f}_v^\bullet$. This means that the homomorphism $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)^{\mathrm{sol}} \to \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$ mentioned above is an injection.

1.2.7. In the remainder of the present paper, *we shall denote by*

$$\Pi_{X^\bullet}$$

either $\pi_1^{\mathrm{adm}}(X^\bullet)$ *or* $\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$ *unless indicated otherwise.* If $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)$, we denote by

$$\Pi_{X^\bullet}^{\mathrm{cpt}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{adm}}(X),\ \Pi_{X^\bullet}^{\text{ét}} \overset{\mathrm{def}}{=} \pi_1^{\text{ét}}(X),\ \Pi_{X^\bullet}^{\mathrm{top}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet}).$$

If $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$, we denote by

$$\Pi_{X^\bullet}^{\mathrm{cpt}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{adm}}(X)^{\mathrm{sol}},\ \Pi_{X^\bullet}^{\text{ét}} \overset{\mathrm{def}}{=} \pi_1^{\text{ét}}(X)^{\mathrm{sol}},\ \Pi_{X^\bullet}^{\mathrm{top}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}.$$

1.2.8. Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup. We write $X_H^\bullet$ for the pointed semi-stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H, $\Gamma_{X_H^\bullet}$ for the dual semi-graph of $X_H^\bullet$, and r_{X_H} for the Betti number of $\Gamma_{X_H^\bullet}$. Then we obtain an admissible covering

$$f_H^\bullet : X_H^\bullet \to X^\bullet$$

over k induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, and obtain a natural map of dual semi-graphs

$$f_H^{\mathrm{sg}} : \Gamma_{X_H^\bullet} \to \Gamma_{X^\bullet}$$

induced by $f_H^\bullet$, where "sg" means "semi-graph". Moreover, if H is an open *normal* subgroup, then $\Gamma_{X_H^\bullet}$ admits an action of $\Pi_{X^\bullet}/H$ induced by the natural action of $\Pi_{X^\bullet}/H$ on $X_H^\bullet$. Note that the quotient of $\Gamma_{X_H^\bullet}$ by $\Pi_{X^\bullet}/H$ coincides with $\Gamma_{X^\bullet}$, and that H is isomorphic to the admissible fundamental group (resp. solvable admissible fundamental group) $\Pi_{X_H^\bullet}$ of $X_H^\bullet$ if $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)$ (resp. $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$). We also use the notation

$$H^{\mathrm{cpt}},\ H^{\text{ét}},\ H^{\mathrm{top}}$$

to denote $\Pi_{X_H^\bullet}^{\mathrm{cpt}}$, $\Pi_{X_H^\bullet}^{\text{ét}}$, and $\Pi_{X_H^\bullet}^{\mathrm{top}}$, respectively.

1.2.9. Let ℓ be a prime number. Let $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/\ell\mathbb{Z})$ be a non-trivial element. Then α induces a Galois admissible covering $f_\alpha^\bullet : X_\alpha^\bullet \to X^\bullet$ over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ (i.e. the Galois admissible covering of $X^\bullet$ corresponding to the open normal subgroup $\ker(\alpha) \subseteq \Pi_{X^\bullet}$). We call $f_\alpha^\bullet$ the Galois admissible covering of $X^\bullet$ corresponding to α.

On the other hand, let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois admissible covering with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Then there exists a non-trivial element $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/\ell\mathbb{Z})$ such that $f_\alpha^\bullet = f^\bullet$. We call α an element corresponding to (or induced by) $f^\bullet$.

1.2.10. We put

$$\widehat{X} \overset{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} X_H,\ D_{\widehat{X}} \overset{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} D_{X_H},\ \Gamma_{\widehat{X}^\bullet} \overset{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} \Gamma_{X_H^\bullet}.$$

We shall say that

$$\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}})$$

is the universal admissible covering (resp. universal solvable admissible covering) of $X^\bullet$ corresponding to $\Pi_{X^\bullet}$ if $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)$ (resp. $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$), and that $\Gamma_{\widehat{X}^\bullet}$ is the dual semi-graph of $\widehat{X}^\bullet$. Note that we have that $\mathrm{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of $\Pi_{X^\bullet}$.

1.2.11. Let $v \in v(\Gamma_{X^\bullet})$, $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, $\widehat{v} \in v(\Gamma_{\widehat{X}^\bullet})$ a vertex over v, and $\widehat{e} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{\widehat{X}^\bullet})$ an edge over e. We denote by

$$\Pi_{\widehat{v}} \subseteq \Pi_{X^\bullet},\ I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$$

the stabilizer subgroups of $\widehat{v}$ and $\widehat{e}$, respectively. We see immediately that $\Pi_{\widehat{v}}$ is (outer) isomorphic to $\Pi_{\widetilde{X}_v^\bullet}$ of $\widetilde{X}_v^\bullet$, and that $I_{\widehat{e}}$ is (outer) isomorphic to an inertia subgroup associated to the closed point of X corresponding to e. Then we have $I_{\widehat{e}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal pro-prime-to-p quotient of $(-)$. We put

$$\mathrm{Ver}(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \{\Pi_{\widehat{v}}\}_{\widehat{v} \in v(\Gamma_{\widehat{X}^\bullet})},$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet})},$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\mathrm{cl}}(\Gamma_{\widehat{X}^\bullet})}.$$

Moreover, if $\widehat{e}$ abuts on $\widehat{v}$, then we have the following injections

$$I_{\widehat{e}} \hookrightarrow \Pi_{\widehat{v}} \hookrightarrow \Pi_{X^\bullet}.$$

Note that $\mathrm{Ver}(\Pi_{X^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})$, and $\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet})$ admit natural actions of $\Pi_{X^\bullet}$ (i.e. the conjugacy actions), and that we have the following natural bijections

$$\mathrm{Ver}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \overset{\sim}{\to} v(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X^\bullet})$$

induced by $I_{\widehat{v}} \mapsto v$, $I_{\widehat{e}} \mapsto e$, $I_{\widehat{e}} \mapsto e$, respectively.

2. Maximum and averages of generalized Hasse-Witt invariants

In this section, we recall some results concerning Hasse-Witt invariants (or p-rank) and generalized Hasse-Witt invariants.

2.1. **Hasse-Witt invariants and generalized Hasse-Witt invariants.**

2.1.1. Let $Z^\bullet$ be a disjoint union of finitely many pointed semi-stable curves over k. We define the *p-rank* (or *Hasse-Witt invariant*) of $Z^\bullet$ to be

$$\sigma_Z \overset{\mathrm{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(Z, \mathbb{F}_p)).$$

In particular, if $Z^\bullet$ is a pointed semi-stable curve, then we have $\sigma_Z = \dim_{\mathbb{F}_p}(\Pi_{Z^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_p)$, where $\Pi_{Z^\bullet}$ is either the admissible fundamental group or the solvable admissible fundamental group of $Z^\bullet$, and $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$.

2.1.2. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p>0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ either the admissible fundamental group or the solvable admissible fundamental group of $X^\bullet$. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ_n, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta_n^i \mapsto i$.

2.1.3. Let $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$. We denote by $X_\alpha^\bullet = (X_\alpha, D_{X_\alpha})$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition ([Ser, Section 9])

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha}) = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}} \oplus H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{ni}}$. Moreover, we have $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}} = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k$, where $(-)^{F_{X_\alpha}}$ denotes the subspace of $(-)$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify $H_\alpha \overset{\mathrm{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$ with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-linear space H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α. We have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta_n \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ_n^i-multiplication. We call

$$\gamma_{\alpha,i} \overset{\mathrm{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z},$$

a *generalized Hasse-Witt invariant* (see [Nakaj], [T4] for the case of smooth pointed stable curves) of the cyclic multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. Note that the above decomposition implies

$$\sigma_{X_\alpha} = \dim_k(H_\alpha) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.$$

2.1.4. Let $t \in \mathbb{N}$ be an arbitrary positive natural number, K_{p^t-1} the kernel of the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}$, and $X^\bullet_{K_{p^t-1}}$ the pointed stable curve over k determined by K_{p^t-1}. Next, we define two important invariants associated to $X^\bullet$.

We shall call

$$\gamma^{\max}(X^\bullet) \overset{\mathrm{def}}{=} \max_{m \in \mathbb{N} \text{ s.t. } (m,p)=1} \{\gamma_{\alpha,i} \mid \alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/m\mathbb{Z}),\ \alpha \neq 0,\ i \in (\mathbb{Z}/m\mathbb{Z}) \setminus \{0\}\}$$

the maximum generalized Hasse-Witt invariant of prime-to-p cyclic admissible coverings of $X^\bullet$.

We shall call

$$\mathrm{Avr}_p(X^\bullet) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\sigma_{X_{K_{p^t-1}}}}{\#(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})}$$

the limit of p*-averages of* $X^\bullet$.

2.2. **Two group-theoretical formulas.** In this subsection, we recall two group-theoretical formulas for maximum and p-averages of generalized Hasse-Witt invariants proved by Tamagawa and the author. We maintain the notation and settings introduced in Section 2.1.

2.2.1. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$, $\chi \in \mathrm{Hom}(\Pi_{X^\bullet}, \overline{\mathbb{F}}_p^\times)$ such that $\chi \neq 1$, and $\Pi_\chi \subseteq \Pi_{X^\bullet}$ the kernel of χ. The profinite group Π_χ admits a natural action of $\Pi_{X^\bullet}$ via the conjugation action. We put

$$\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})[\chi] \overset{\mathrm{def}}{=} \{a \in \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \tau(a) = \chi(\tau)a \text{ for all } \tau \in \Pi_{X^\bullet}\},$$

$$\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})) \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})[\chi]).$$

We define the following group-theoretical invariants associated to $\Pi_{X^\bullet}$:

$$\gamma^{\max}(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \max\{\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})) \mid \chi \in \mathrm{Hom}(\Pi_{X^\bullet}, \overline{\mathbb{F}}_p^\times) \text{ such that } \chi \neq 1\},$$

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K_{p^t-1}^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})}.$$

We see immediately that

$$\gamma^{\max}(\Pi_{X^\bullet}) = \gamma^{\max}(X^\bullet),\ \mathrm{Avr}_p(\Pi_{X^\bullet}) = \mathrm{Avr}_p(X^\bullet).$$

2.2.2. We have the following highly non-trivial formulas for $\gamma^{\max}(\Pi_{X^\bullet})$ and $\mathrm{Avr}_p(\Pi_{X^\bullet})$, which were proved by applying the theory of Raynaud-Tamagawa theta divisors.

Theorem 2.1. *We maintain the notation introduced above.*

(a) We have

$$\gamma^{\max}(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

(b) Suppose that $\Gamma_{X^\bullet}^{\mathrm{cpt}}$ *is 2-connected (1.1.1 (b)). Then we have (see 1.1.1 (c) for* $v(\Gamma_{X^\bullet})^{b\leq 1}$, $e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1}$*)*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#(v(\Gamma_{X^\bullet})^{b\leq 1}) + \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1}).$$

Proof. (a) This is [Y5, Theorem 5.4]. (b) This follows immediately from the "in particular" part of [Y3, Theorem 1.3]. Note that our notation differs from that of [Y3, Theorem 1.3]. Moreover, if $\Gamma_{X^\bullet}^{\mathrm{cpt}}$ is 2-connected, then we have that $\#E_v^{>1} \leq 1$ for each $v \in v(\Gamma_{X^\bullet})$, and that $\#(V_{X^\bullet}^{\mathrm{tre}}) = \#(v(\Gamma_{X^\bullet})^{b\leq 1})$, $\#(V_{X^\bullet}^{\mathrm{tre}, g_v=0}) = 0$, and $\#(e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1}) = \#(E_{X^\bullet}^{\mathrm{tre}})$. $\square$

Remark 2.1.1. In the present paper, we will use the formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ when $\#(v(\Gamma_{X^\bullet})^{b\leq 1}) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1}) = 0$.

Lemma 2.2. *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic p and $\Pi_{X_i^\bullet}$ either the admissible fundamental group of $X_i^\bullet$ or the solvable admissible fundamental group of $X_i^\bullet$. Let*

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

be an arbitrary surjective open continuous homomorphism of profinite groups, $H_2 \subseteq \Pi_{X_2^\bullet}$ an arbitrary open normal subgroup, and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2)$. Then the following statements hold:

(a) We have

$$\gamma^{\max}(H_1) \geq \gamma^{\max}(H_2).$$

(b) Suppose that $(g_X, n_X) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Moreover, suppose that one of the following conditions are satisified:

- *$G \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/H_2$ is a p-group.*
- *$(\#(G), p) = 1$.*
- *G is a solvable group.*

Then we have

$$\mathrm{Avr}_p(H_1) \geq \mathrm{Avr}_p(H_2).$$

Proof. (a) Let $m \in \mathbb{Z}_{>0}$ be a positive natural number prime to p such that there exists $\alpha_2 \in \mathrm{Hom}(H_2^{\mathrm{ab}}, \mathbb{Z}/m\mathbb{Z})$ satisfying $\alpha_2 \neq 0$ and $\gamma_{\alpha_2,j} = \gamma^{\max}(H_2)$ for some $j \in (\mathbb{Z}/m\mathbb{Z}) \setminus \{0\}$. Write Q_2 for the kernel of the composition of the following homomorphisms $H_2 \twoheadrightarrow H_2^{\mathrm{ab}} \stackrel{\alpha_2}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}$, $Q_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(Q_2)$, and $\alpha_1 \in \mathrm{Hom}(H_1^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ for the homomorphism induced by $\phi|_{H_1}$ and α_2. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Then $Q_i^{p,\mathrm{ab}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ admits a natural $\overline{\mathbb{F}}_p[\mathbb{Z}/n\mathbb{Z}]$-module structure. Moreover, we see immediately that $\phi|_{H_1}$ induces a surjective homomorphism of $\overline{\mathbb{F}}_p[\mathbb{Z}/n\mathbb{Z}]$-modules

$$Q_1^{p,\mathrm{ab}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \twoheadrightarrow Q_2^{p,\mathrm{ab}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p.$$

Then we obtain that $\gamma_{\alpha_1,j} \geq \gamma_{\alpha_2,j}$. Thus, we have $\gamma^{\max}(H_1) \geq \gamma^{\max}(H_2)$.

(b) Let $t \in \mathbb{N}$ be an arbitrary positive natural number, K_{H_i,p^t-1} the kernel of the natural surjection $H_i \twoheadrightarrow H_i^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}$. Suppose that G is a p-group. We have that Galois admissible covering $X_{H_i}^\bullet \to X_i^\bullet$ corresponding to H_i is étale. This implies that $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ are equal types. We obtain

$$\#(H_1^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}) = \#(H_2^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}).$$

Suppose that $(\#(G), p) = 1$. Since $X_1^\bullet$ and $X_2^\bullet$ are equal types, $H_1^{p'}$ is isomorphic to $H_2^{p'}$. We have

$$\#(H_1^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}) = \#(H_2^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}).$$

Then $\phi|_{H_1}$ implies

$$\mathrm{Avr}_p(H_1) \stackrel{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_1,p^t-1}\otimes \mathbb{F}_p)}{\#(H_1^{\mathrm{ab}}\otimes \mathbb{Z}/(p^t-1)\mathbb{Z})} \geq \mathrm{Avr}_p(H_2) \stackrel{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_2,p^t-1}\otimes \mathbb{F}_p)}{\#(H_2^{\mathrm{ab}}\otimes \mathbb{Z}/(p^t-1)\mathbb{Z})}$$

if either G is a p-group, or $(\#(G), p) = 1$ holds.

Suppose that G is solvable. Then the lemma follows immediately from the lemma when either G is a p-group, or $(\#(G), p) = 1$. This completes the proof of the lemma. $\square$

3. Moduli spaces of fundamental groups and the homeomorphism conjecture

In this section, we define the moduli spaces of fundamental groups and formulate the homeomorphism conjecture, which are main objects of the series of the present papers.

3.1. **The weak Isom-version conjecture.** Let p be a prime number, $\mathbb{F}_p$ the prime field of characteristic p, and $\overline{\mathbb{F}}_p$ an algebraic closure of $\mathbb{F}_p$. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over $\overline{\mathbb{F}}_p$ classifying pointed stable curves of type (g, n) and $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ the open substack classifying smooth pointed stable curves. Let $\overline{M}_{g,n}$ and $M_{g,n}$ be the coarse moduli spaces of $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$, respectively.

3.1.1. Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of $\overline{M}_{g,n}$, and k_q an algebraically closed field containing $k(q)$. Then the composition of natural morphisms $\mathrm{Spec}\, k_q \to \mathrm{Spec}\, k(q) \to \overline{M}_{g,n}$ determines a pointed stable curve $X^\bullet_{k_q}$ of type (g, n) over k_q. Write $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})$ for the admissible fundamental group $X^\bullet_{k_q}$ and $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})^{\mathrm{sol}}$ for the solvable admissible fundamental group of $X^\bullet_{k_q}$. Since the isomorphism classes of $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})$ and $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})^{\mathrm{sol}}$ do not depend on the choice of k_q, we shall write

$$\pi_1^{\mathrm{adm}}(q),\ \pi_1^{\mathrm{sol}}(q)$$

for $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})$, $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})^{\mathrm{sol}}$, respectively. Moreover, we shall denote by

$$X^\bullet_q$$

the pointed stable curve $X^\bullet_{\overline{k(q)}}$ and Γ_q the dual semi-graph of $X^\bullet_q$, where $\overline{k(q)}$ is an algebraic closure of $k(q)$. Let $v \in v(\Gamma_q)$. Then the smooth pointed stable curve $\widetilde{X}^\bullet_{q,v}$ of type (g_v, n_v) associated to v determines a morphism $\mathrm{Spec}\, \overline{k(q)} \to M_{g_v,n_v}$. We shall write $q_v \in M_{g_v,n_v}$ for the image of the morphism and call q_v *the point of type* (g_v, n_v) *associated to* v.

3.1.2. We recall an equivalent relation on the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$ that was introduced in [Y4].

Definition 3.1. (a) Let $q_i \in M_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point. We shall say that q_1 is *Frobenius equivalent* to q_2 if $X_{q_1} \setminus D_{X_{q_1}}$ is isomorphic to $X_{q_2} \setminus D_{X_{q_2}}$ as schemes.

(b) Let $q_i \in \overline{M}_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point. We shall say that q_1 is *Frobenius equivalent* to q_2 if the following conditions are satisfied:

(i) There exists an isomorphism $\rho : \Gamma_{q_1} \xrightarrow{\sim} \Gamma_{q_2}$ of dual semi-graphs.

(ii) Let $v_1 \in v(\Gamma_{q_1})$, $v_2 \stackrel{\text{def}}{=} \rho(v_1) \in v(\Gamma_{q_2})$, q_{1,v_1} the point of type (g_{v_1}, n_{v_1}) associated to v_1, and q_{2,v_2} the point of type (g_{v_2}, n_{v_2}) associated to v_2. We have that q_{1,v_1} is Frobenius equivalent to q_{2,v_2}.

(iii) Let $\rho_{v_1,v_2} : \Gamma_{q_{1,v_1}} \xrightarrow{\sim} \Gamma_{q_{2,v_2}}$ be the isomorphism of dual semi-graphs induced by ρ. There exists an isomorphism $\phi_{v_1,v_2} : \pi_1^{\text{adm}}(q_{1,v_1}) \xrightarrow{\sim} \pi_1^{\text{adm}}(q_{2,v_2})$ such that the isomorphism of dual semi-graphs $\Gamma_{q_{1,v_1}} \xrightarrow{\sim} \Gamma_{q_{2,v_2}}$ induced by ϕ_{v_1,v_2} (cf. [T4, Theorem 5.2] or [Y2, Theorem 1.2 and Remark 1.2.1]) coincides with ρ_{v_1,v_2}.

We shall denote by

$$q_1 \sim_{fe} q_2$$

if q_1 is Frobenius equivalent to q_2. We see that $\sim_{fe}$ is an equivalence relation on the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$

(c) Let $q_i \in \overline{M}_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point, k_{q_i} an algebraically closed field containing $k(q_i)$, and $X^\bullet_{k_{q_i}}$ the pointed stable curve of type (g, n) over k_{q_i}. We shall say that $X^\bullet_{k_{q_1}}$ *is Frobenius equivalent* to $X^\bullet_{k_{q_2}}$ if q_1 is Frobenius equivalent to q_2.

The following result was proved by the author.

Proposition 3.2. *Let $q_i \in \overline{M}_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point. Suppose $q_1 \sim_{fe} q_2$. Then we have that $\pi_1^{\text{adm}}(q_1)$ is isomorphic to $\pi_1^{\text{adm}}(q_2)$ as profinite groups. In particular, we have that $\pi_1^{\text{sol}}(q_1)$ is isomorphic to $\pi_1^{\text{sol}}(q_2)$ as profinite groups.*

Proof. See [Y4, Proposition 3.7]. □

3.1.3. We put

$$\mathfrak{M}_{g,n} \stackrel{\text{def}}{=} |M_{g,n}|/\sim_{fe} \subseteq \overline{\mathfrak{M}}_{g,n} \stackrel{\text{def}}{=} |\overline{M}_{g,n}|/\sim_{fe},$$

$$\Pi_{g,n} \stackrel{\text{def}}{=} \{[\pi_1^{\text{adm}}(q)] \mid q \in M_{g,n}\} \subseteq \overline{\Pi}_{g,n} \stackrel{\text{def}}{=} \{[\pi_1^{\text{adm}}(q)] \mid q \in \overline{M}_{g,n}\},$$

$$\Pi^{\text{sol}}_{g,n} \stackrel{\text{def}}{=} \{[\pi_1^{\text{sol}}(q)] \mid q \in M_{g,n}\} \subseteq \overline{\Pi}^{\text{sol}}_{g,n} \stackrel{\text{def}}{=} \{[\pi_1^{\text{sol}}(q)] \mid q \in \overline{M}_{g,n}\},$$

where $[\pi_1^{\text{adm}}(q)]$ and $[\pi_1^{\text{sol}}(q)]$ denote the isomorphism classes (as profinite groups) of $\pi_1^{\text{adm}}(q)$ and $\pi_1^{\text{sol}}(q)$, respectively. Let $q \in \overline{M}_{g,n}$. We shall write $[q]$ for the image of q in $\overline{\mathfrak{M}}_{g,n}$. Then there are natural surjective maps of *sets* as follows:

$$sol : \overline{\Pi}_{g,n} \twoheadrightarrow \overline{\Pi}^{\text{sol}}_{g,n}, \ [\pi_1^{\text{adm}}(q)] \mapsto [\pi_1^{\text{sol}}(q)],$$

$$\pi^{\mathrm{adm}}_{g,n}: \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n},\ [q] \mapsto [\pi_1^{\mathrm{adm}}(q)],$$
$$\pi^{\mathrm{sol}}_{g,n} \overset{\mathrm{def}}{=} sol \circ \pi^{\mathrm{adm}}_{g,n}: \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}^{\mathrm{sol}}_{g,n},$$
$$\pi^{\mathrm{t}}_{g,n} \overset{\mathrm{def}}{=} \pi^{\mathrm{adm}}_{g,n}|_{\mathfrak{M}_{g,n}}: \mathfrak{M}_{g,n} \twoheadrightarrow \Pi_{g,n},$$
$$\pi^{\mathrm{t,sol}}_{g,n} \overset{\mathrm{def}}{=} \pi^{\mathrm{sol}}_{g,n}|_{\mathfrak{M}_{g,n}}: \mathfrak{M}_{g,n} \twoheadrightarrow \Pi^{\mathrm{sol}}_{g,n},$$

where "t" means "tame". Moreover, we have the following commutative diagrams:

$$\begin{array}{ccc} \mathfrak{M}_{g,n} & \xrightarrow{\pi^{\mathrm{t}}_{g,n}} & \Pi_{g,n} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n} & \xrightarrow{\pi^{\mathrm{adm}}_{g,n}} & \overline{\Pi}_{g,n}, \end{array}$$

$$\begin{array}{ccc} \mathfrak{M}_{g,n} & \xrightarrow{\pi^{\mathrm{t,sol}}_{g,n}} & \Pi^{\mathrm{sol}}_{g,n} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n} & \xrightarrow{\pi^{\mathrm{sol}}_{g,n}} & \overline{\Pi}^{\mathrm{sol}}_{g,n}, \end{array}$$

where all vertical arrows are natural injections.

Proposition 3.3. *We maintain the notation introduced above. Then we have*

$$\pi^{\mathrm{adm}}_{g,n}(\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}) = \overline{\Pi}_{g,n} \setminus \Pi_{g,n},\ \pi^{\mathrm{sol}}_{g,n}(\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}) = \overline{\Pi}^{\mathrm{sol}}_{g,n} \setminus \Pi^{\mathrm{sol}}_{g,n}.$$

Proof. The proposition follows immediately from [Y2, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1] (see also Theorem 4.2 of the present paper). □

3.1.4. We may formulate a moduli version of the weak Isom-version of the Grothendieck conjecture for pointed stable curves over algebraically closed fields of characteristic $p > 0$ (=the weak Isom-version conjecture) as follows:

Weak Isom-version Conjecture . *We maintain the notation introduced above. Then we have that*

$$\pi^{\mathrm{adm}}_{g,n}: \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is a bijection as sets.

Moreover, we have the following solvable version of the weak Isom-version conjecture which is slightly stronger than the original version.

Solvable Weak Isom-version Conjecture . *We maintain the notation introduced above. Then we have that*

$$\pi^{\mathrm{sol}}_{g,n}: \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}^{\mathrm{sol}}_{g,n}$$

is a bijection as sets.

3.1.5. Write $\overline{M}^{\mathrm{cl}}_{g,n}$ for the set of closed points of $\overline{M}_{g,n}$ and $\overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}$ for the image of $\overline{M}^{\mathrm{cl}}_{g,n}$ of the natural map $|\overline{M}_{g,n}| \twoheadrightarrow \overline{\mathfrak{M}}_{g,n}$. Then we have the following result.

Theorem 3.4. *We maintain the notation introduced above. Then the following statements hold:*

(a) We have that

$$\pi^{\mathrm{sol}}_{g,n}|_{\overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}} : \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n} \to \overline{\Pi}^{\mathrm{sol}}_{g,n}$$

is quasi-finite (i.e. $\#((\pi^{\mathrm{sol}}_{g,n}|_{\overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}})^{-1}([\pi_1^{\mathrm{sol}}(q)])) < \infty$ *for every* $[\pi_1^{\mathrm{sol}}(q)] \in \overline{\Pi}^{\mathrm{sol}}_{g,n}$*).*

(b) Suppose that $g = 0$*. Then we have that*

$$\pi^{\mathrm{sol}}_{g,n}|_{\overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}} : \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n} \to \overline{\Pi}^{\mathrm{sol}}_{g,n}$$

is an injection, and that

$$\pi^{\mathrm{sol}}_{g,n}(\overline{\mathfrak{M}}_{g,n} \setminus \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}) \subseteq \overline{\Pi}^{\mathrm{sol}}_{g,n} \setminus \pi^{\mathrm{sol}}_{g,n}(\overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}).$$

In particular, the weak Isom-version conjecture and the Solvable Weak Isom-version Conjecture hold if $(g, n) = (0, 4)$*.*

Proof. Since [T4, Theorem 0.2] and [T5, Theorem 0.1] also hold for the maximal pro-solvable quotients of tame fundamental groups, the theorem follows immediately from [T4, Theorem 0.2], [T5, Theorem 0.1], [Y2, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1], and Proposition 3.3. □

Remark 3.4.1. The result (a) is called "*finiteness theorem*". When $q \in M_{g,n}$, by using the theory of Raynaud's theta divisors, the finiteness theorem was proved by Raynaud ([R3]), Pop-Saïdi ([PS]) under certain assumptions, and by Tamagawa ([T5]) in general case. Furthermore, Tamagawa's result was generalized to the case where $q \in \overline{M}_{g,n}$ by the author ([Y2]) as an application of the combinatorial Grothendieck conjecture for curves in positive characteristic.

3.2. **Moduli spaces of admissible fundamental groups.** We maintain the notation introduced in 3.1. Moreover, we regard $\overline{\mathfrak{M}}_{g,n}$ and $\mathfrak{M}_{g,n}$ as topological spaces whose topologies are induced by the Zariski topologies of $|\overline{M}_{g,n}|$ and $|M_{g,n}|$, respectively.

3.2.1. Let $\mathscr{G}$ be the category of finite groups, $G \in \mathscr{G}$ an arbitrary finite group, and $\mathrm{Hom}_{\mathrm{surj}}(-,-)$ the set of surjective homomorphisms. We put

$$U_{\overline{\Pi}_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{adm}}(q)] \in \overline{\Pi}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), G) \neq \emptyset\},$$

$$U_{\Pi_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{adm}}(q)] \in \Pi_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), G) \neq \emptyset\},$$

$$U_{\overline{\Pi}^{\mathrm{sol}}_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{sol}}(q)] \in \overline{\Pi}^{\mathrm{sol}}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{sol}}(q), G) \neq \emptyset\},$$

$$U_{\Pi^{\mathrm{sol}}_{g,n},G} \stackrel{\mathrm{def}}{=} \{[\pi_1^{\mathrm{sol}}(q)] \in \Pi^{\mathrm{sol}}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{sol}}(q), G) \neq \emptyset\}.$$

Then we obtain the following topological spaces

$$(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}}),\ (\Pi_{g,n}, O_{\Pi_{g,n}}),\ (\overline{\Pi}^{\mathrm{sol}}_{g,n}, O_{\overline{\Pi}^{\mathrm{sol}}_{g,n}}),\ (\Pi^{\mathrm{sol}}_{g,n}, O_{\Pi^{\mathrm{sol}}_{g,n}})$$

whose topologies $O_{\overline{\Pi}_{g,n}}$, $O_{\Pi_{g,n}}$, $O_{\overline{\Pi}^{\mathrm{sol}}_{g,n}}$, and $O_{\Pi^{\mathrm{sol}}_{g,n}}$ are generated by $\{U_{\overline{\Pi}_{g,n},G}\}_{G\in\mathscr{G}}$, $\{U_{\Pi_{g,n},G}\}_{G\in\mathscr{G}}$, $\{U_{\overline{\Pi}^{\mathrm{sol}}_{g,n},G}\}_{G\in\mathscr{G}}$, and $\{U_{\Pi^{\mathrm{sol}}_{g,n},G}\}_{G\in\mathscr{G}}$ as open subsets, respectively. For simplicity of notation, we still use the notation

$$\overline{\Pi}_{g,n},\ \Pi_{g,n},\ \overline{\Pi}^{\mathrm{sol}}_{g,n},\ \Pi^{\mathrm{sol}}_{g,n}$$

to denote the topological spaces $(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}})$, $(\Pi_{g,n}, O_{\Pi_{g,n}})$, $(\overline{\Pi}^{\mathrm{sol}}_{g,n}, O_{\overline{\Pi}^{\mathrm{sol}}_{g,n}})$, and $(\Pi^{\mathrm{sol}}_{g,n}, O_{\Pi^{\mathrm{sol}}_{g,n}})$, respectively.

Definition 3.5. We call

$$\overline{\Pi}_{g,n},\ (\text{resp. } \overline{\Pi}^{\mathrm{sol}}_{g,n})$$

the moduli space of admissible fundamental groups of pointed stable curves (resp. solvable admissible fundamental groups) of type (g, n) *over algebraically closed fields of characteristic* p, or the moduli space of admissible fundamental groups (resp. solvable admissible fundamental groups) of type (g, n) in characteristic p for short.

3.2.2. *Continuous of the map* $\pi^{\mathrm{adm}}_{g,n}$. Let $\overline{\mathcal{M}}^{\log}_{g,n}$ be the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ relative to $\mathrm{Spec}\,\overline{\mathbb{F}}_p$. Let $\mathcal{A}_d$ be the stack over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$ defined as follows: For a scheme S, the objects of $\mathcal{A}_d(S)$ are HM-admissible coverings ([M1, §3.9 Definition]) $C^\bullet \to D^\bullet$ over S of degree d (note that if S is an algebraically closed field of characteristic p, then HM-admissible coverings are equivalent to the HM-admissible coverings defined in Definition 1.1), where $C^\bullet$ is a pointed stable curve over S, and $D^\bullet$ is a pointed stable curve of type (g, n) over S. By [M1, §3.11 Proposition and §3.22 Theorem], the stack $\mathcal{A}_d$ is a separated Deligne-Mumford stack of finite type over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$. Moreover, $\mathcal{A}_d$ is equipped with a canonical log structure $M_{\mathcal{A}_d} \to \mathcal{O}_{\mathcal{A}_d}$, together with a logarithmic morphism $\mathcal{A}^{\log}_d \stackrel{\mathrm{def}}{=} (\mathcal{A}_d, M_{\mathcal{A}_d}) \to \overline{\mathcal{M}}^{\log}_{g,n}$ (obtained by mapping $C^\bullet \to D^\bullet \mapsto D^\bullet$) which is log étale (not necessary proper).

Let G be an arbitrary finite group. For any HM-admissible covering $C^\bullet \to D^\bullet$ over S, [M1, §3.10 and §3.11] imply that $C^\bullet \to D^\bullet$ can be extended to a log admissible covering $C^{\log} \to D^{\log}$ over $S^{\log}$ ([M1, §3.5 Definition]). Since log admissible coverings are finite Kummer log étale coverings, we shall say $C^\bullet \to D^\bullet$ over S a Galois HM-admissible covering with Galois group G if $C^{\log} \to D^{\log}$ over $S^{\log}$ is a Galois Kummer log étale covering with Galois group G. Note that if S is an algebraically closed field k of characteristic p, then a Galois HM-admissible covering can be regarded as a

Galois admissible covering in the sense of Definition 1.1 by equipping certain sets of isomorphisms of k-isomorphisms of branches of singular points (1.1.4).

Let $\mathcal{A}_G$ be the substack of $\mathcal{A}_{\#(G)}$ classifying Galois HM-admissible coverings with Galois group G which is a union of some connected components of $\mathcal{A}_{\#(G)}$, and which is a separated Deligne-Mumford stack of finite type over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$. Note that $\mathcal{A}_G$ may be empty. Moreover, we shall denote by $\mathcal{A}_G^{\log}$ the log stack whose underlying stack is $\mathcal{A}_G$, and whose log structure is the pulling-back log structure induced by $\mathcal{A}_G \hookrightarrow \mathcal{A}_{\#(G)}$. Furthermore, we have a logarithmic morphism $\mathcal{A}_G^{\log} \to \overline{\mathcal{M}}_{g,n}^{\log}$ which is log étale (not necessary proper).

Theorem 3.6. *We maintain the notation introduced above. Then we have that*

$$\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n},\ \ \pi_{g,n}^{\mathrm{sol}} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n}^{\mathrm{sol}}$$

are continuous maps.

Proof. We only need to treat the case $\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n}$. To verify the theorem, it is sufficient to prove that the composition of the natural maps

$$\overline{M}_{g,n} \twoheadrightarrow \overline{\mathfrak{M}}_{g,n} \overset{\pi_{g,n}^{\mathrm{adm}}}{\twoheadrightarrow} \overline{\Pi}_{g,n}$$

is continuous.

Let G be an arbitrary finite group. If $U_{\overline{\Pi}_{g,n},G} = \emptyset$, then the theorem is trivial. We may assume $U_{\overline{\Pi}_{g,n},G} \neq \emptyset$. Let $q \in \overline{M}_{g,n}$ such that $[\pi_1^{\mathrm{adm}}(q)] \in U_{\overline{\Pi}_{g,n},G}$, $\overline{k(q)}$ an algebraic closure of $k(q)$, and

$$f_q^\bullet : Y_q^\bullet \to X_q^\bullet$$

a Galois admissible covering over $\overline{k(q)}$ with Galois group G. Then we obtain a morphism

$$[f_q^\bullet] : \mathrm{Spec}\,\overline{k(q)} \to \mathcal{A}_G$$

determined by $f_q^\bullet$. Let $U \to \mathcal{A}_G$ be an étale altas. Then the morphism $\mathrm{Spec}\,\overline{k(q)} \to \mathcal{A}_G$ factors through a morphism $\mathrm{Spec}\,\overline{k(q)} \to U$. Write $q_U \in U$ for the image of the morphism $\mathrm{Spec}\,\overline{k(q)} \to U$. Let $q'_U \in U$ be a closed point (i.e. an $\overline{\mathbb{F}}_p$-rational point) contained in the topological closure of q_U in U and $q' \in \overline{M}_{g,n}$ the image of q'_U of $U \to \mathcal{A}_G \to \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ which is a closed point of $\overline{M}_{g,n}$. Then we have $[\pi_1^{\mathrm{adm}}(q')] \in U_{\overline{\Pi}_{g,n},G}$. By replacing q by q', to verify the theorem, we only need to prove the theorem when q is a closed point of $\overline{M}_{g,n}$.

Let $\mathcal{O}_{[f_q^\bullet]}$ be the completion of strict henselization of $\mathcal{A}_G$ at $[f_q^\bullet]$, $S \overset{\mathrm{def}}{=} \mathrm{Spec}\,\mathcal{O}_{[f_q^\bullet]}$, and $S^{\log}$ the log scheme whose underlying scheme is S, and whose log structure is the pulling-back log structure of $\mathcal{A}_G^{\log}$ induced by the natural morphism $S \to \mathcal{A}_G$

(see [M1, §3.23] for explicit descriptions of S and $S^{\log}$). Moreover, we have a Galois log admissible covering

$$f_S^{\log} : Y_S^{\log} \to X_S^{\log}$$

over $S^{\log}$ with Galois group G. On the other hand, by forgetting the log structure of $f_S^{\log}$, we obtain a Galois HM-admissible covering $f_S^{\bullet} : Y_S^{\bullet} \to X_S^{\bullet}$ over S with Galois group G whose closed fiber (i.e. the fiber over the closed point of S) is $f_q^{\bullet}$.

Since $\mathcal{A}_G$ is a Deligne-Mumford stack of finite type over $\operatorname{Spec} \overline{\mathbb{F}}_p$, by applying [V1, Proposition 4.3 (1)], there exists a subring $A \subseteq \mathcal{O}_{[f_q^{\bullet}]}$ which is of finite type over $\overline{\mathbb{F}}_p$ such that the Galois log admissible covering $f_S^{\log}$ can be descended to a Galois Kummer log étale covering

$$f_E^{\log} : Y_E^{\log} \to X_E^{\log}$$

over $E^{\log}$ with Galois group G, where $E \stackrel{\text{def}}{=} \operatorname{Spec} A$. By the construction, the pulling-back $f_E^{\log} \times_{E^{\log}} S^{\log}$ via the natural morphism $S^{\log} \to E^{\log}$ is $f_S^{\log}$. Moreover, by replacing E by an open subset of E, we may assume that the underlying schemes Y_E and X_E are geometrically connected over each point $e \in E$. Then by forgetting the log structure of $f_E^{\log}$, we obtain a Galois HM-admissible covering

$$f_E^{\bullet} : Y_E^{\bullet} \to X_E^{\bullet}$$

over E with Galois group G, and a morphism $E \to \mathcal{A}_G$ determined by $f_E^{\bullet}$.

Since E is a scheme of finite type over $\operatorname{Spec} \overline{\mathbb{F}}_p$, the image W of $E \to \mathcal{A}_G \to \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ is a constructible subset of $\overline{M}_{g,n}$ containing q. Moreover, since the image of the composition of the natural morphisms $S \to \mathcal{A}_G \to \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ is dense in $\overline{M}_{g,n}$, W is a dense constructible subset of $\overline{M}_{g,n}$ containing q. Then we have that

$$W = \bigsqcup_{i=1}^{r} W_i$$

is a finite disjoint union of local closed subsets $\{W_i\}_{i=1,\dots,r}$, of $\overline{M}_{g,n}$. Without loss of generality, we may assume $q \in W_1$. Since W_1 contains the image of S, we obtain that W_1 is an open subset of $\overline{M}_{g,n}$. This completes the proof of the theorem. □

3.3. **The homeomorphism conjecture.**

Next, we formulate the main conjectures of the theory of moduli spaces of fundamental groups.

Homeomorphism Conjecture . *We maintain the notation introduced above. Then we have that the continous map*

$$\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is a homeomorphism.

Moreover, we have a solvable version of the homeomorphism conjecture as follows, which is slightly stronger than the original version.

Solvable Homeomorphism Conjecture . *We maintain the notation introduced above. Then we have that the continous map*

$$\pi^{\mathrm{sol}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}^{\mathrm{sol}}_{g,n}$$

is a homeomorphism.

Remark. Note that the (solvable) homeomorphism conjecture is completely different from Grothendieck's anabelian conjecture for moduli spaces of curves (i.e. a conjecture of Grothendieck based on a similar anabelian philosophy mentioned in 0.1.2 says that moduli spaces of curves are anabelian varieties in the sense of 0.1). Furthermore, the (solvable) homeomorphism conjecture contains "moduli" information (i.e. classifications information) of curves, and Grothendieck's anabelian conjecture for moduli spaces of curves does not contain any "moduli" information of curves.

3.3.1. The main theorem of the present paper is the following, which will be proved in Section 6.

Theorem 3.7 (Theorem 6.7)**.** *We maintain the notation introduced above. Let* $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n}$ *be an arbitrary closed point. Then* $\pi^{\mathrm{adm}}_{0,n}([q])$ *and* $\pi^{\mathrm{sol}}_{0,n}([q])$ *are closed points of* $\overline{\Pi}_{0,n}$ *and* $\overline{\Pi}^{\mathrm{sol}}_{0,n}$*, respectively. In particular, the homeomorphism conjecture and the solvable homeomorphism conjecture hold when* $(g, n) = (0, 3)$ *or* $(0, 4)$*.*

3.4. **Some open problems.** Based on the homeomorphism conjecture, many new open problems and new conjectures can be formulated. In the present subsection, we outlines a few open problems and conjectures concerning $\overline{\Pi}_{g,n}$ that are of particular interest to the author. Note that we may also formulate the problems and the conjectures mentioned below for $\overline{\Pi}^{\mathrm{sol}}_{g,n}$.

3.4.1. *Dimension and the generalized essential dimension conjecture.* Let V be an irreducible closed subset of $\overline{\Pi}_{g,n}$, $I \subseteq \mathbb{Z}_{>0}$ a (possibly infinite) subset, and $V_i \subseteq V, i \in I$, an irreducible closed subset of $\overline{\Pi}_{g,n}$. We shall call $\mathscr{C} \overset{\mathrm{def}}{=} \{V_i\}_{i\in I}$ a chain of irreducible closed subsets of V if $V_s \subseteq V_t$ and $V_s \neq V_t$ hold for all $s, t \in I$ such that $s > t$. We sall call $\mathscr{C}$ *a maximal chain of irreducible closed subsets of* V if the following holds:

- Let $\mathscr{C}' \overset{\mathrm{def}}{=} \{V'_i\}_{i\in I'}$ be a chain of irreducible closed subsets of V such that $\mathscr{C} \subseteq \mathscr{C}'$. Then we have $\mathscr{C} = \mathscr{C}'$.

Moreover, we put $\mathrm{length}(\mathscr{C}) \overset{\mathrm{def}}{=} \#(I)$ when $\mathscr{C}$ is a maximal chain of irreducible closed subsets of V.

Let $\mathscr{C}$ be a maximal chain of irreducible closed subsets of V. We define the dimension of V to be

$$\dim(V) \overset{\mathrm{def}}{=} \max\{\mathrm{length}(\mathscr{C}) \mid \mathscr{C} \text{ is a maximal chain}$$

of irreducible closed subsets of $V\}$.

We have the following problem:

Problem 3.8. *(i) Let V be an irreducible closed subset of $\overline{\Pi}_{g,n}$ and $\mathscr{C}_i$, $i \in \{1,2\}$, an arbitrary maximal chain of irreducible closed subsets of V. Does*

$$\mathrm{length}(\mathscr{C}_1) = \mathrm{length}(\mathscr{C}_2)$$

hold?

(ii) Let Z be an irreducible closed subset of $\overline{\mathfrak{M}}_{g,n}$ and $[q_Z]$ the generic point of Z. Does

$$\dim(Z) = \dim(V([\pi_1^{\mathrm{adm}}(q_Z)]))$$

hold? In particular, do $\dim(Z) < \infty$, $\dim(\overline{\mathfrak{M}}_{g,n}) = \dim(\overline{\Pi}_{g,n})$, and $\dim(V([\pi_1^{\mathrm{adm}}(q)])) = 0$ for every $[q] \in \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}$ hold? Moreover, Is $\pi_{g,n}^{\mathrm{adm}}([q])$ a closed point of $\overline{\Pi}_{g,n}$ for every $[q] \in \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}$?

We maintain the notation introduced above. Tamagawa's *essential dimension conjecture* (see [T3, Conjecture 5.3 (ii)] for the case where $[q_i] \in \mathfrak{M}_{g,n}$) says that:

> Let $i \in \{1,2\}$, and let $[q_i] \in \overline{\mathfrak{M}}_{g,n}$ and $V([q_i])$ the topological closure of $[q_i]$ in $\overline{\mathfrak{M}}_{g,n}$. Then we have $\dim(V([q_1])) = \dim(V([q_2]))$ if $[\pi_1^{\mathrm{adm}}(q_1)] = [\pi_1^{\mathrm{adm}}(q_2)]$.

We see immediately that Problem 3.8 (ii) is a generalized version of the essential dimension conjecture. To more conveniently compare with Tamagawa's essential dimension conjecture, we formulate a new conjecture as following:

Generalized essential dimension conjecture . *Let $i \in \{1,2\}$, and let $[q_i] \in \overline{\mathfrak{M}}_{g,n}$ and $V([q_i])$ the topological closure of $[q_i]$ in $\overline{\mathfrak{M}}_{g,n}$. Then we have*

$$\dim(V([q_1])) \geq \dim(V([q_2]))$$

if $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) \neq \emptyset$, where $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(-,-)$ denotes the set of open continuous homomorphisms of profinite groups.

At present, the essential dimension conjecture has been proved when $(g,n) \in \{(0,n),(1,1)\}$ and $[q_i]$, $i \in \{1,2\}$, is a closed point of $\overline{\mathfrak{M}}_{g,n}$ (see [Sar], [T4], [Y2]), and the generalized essential dimension conjecture has been proved when $(g,n) \in \{(0,n),(1,n),(2,0)\}$ and q_1 is a closed point of $\overline{\mathfrak{M}}_{g,n}$ (see Theorem 6.6 of the present paper and [HY, Theorem 1.3]).

3.4.2. *p-rank stratification and purity.* Let $0 \leq \sigma \leq g$ be an integral number. We put

$$\overline{\Pi}_{g,n}^{\sigma} \stackrel{\mathrm{def}}{=} \{[\Pi] \in \overline{\Pi}_{g,n} \mid \dim_{\mathbb{F}_p}(\Pi^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{F}_p) \leq \sigma\},$$

and call $\overline{\Pi}_{g,n}^{\sigma}$ *the p-rank stratum of $\overline{\Pi}_{g,n}$ with p-rank σ*. Note that $\overline{\Pi}_{g,n}^{\sigma}$ is a closed subset of $\overline{\Pi}_{g,n}$. Then we have the following problem:

Problem 3.9. *(i) Let S_i, $i \in \{1, 2\}$, be an arbitrary irreducible component of $\overline{\Pi}^{\sigma}_{g,n}$. Then does*

$$\dim(S_1) = \dim(S_2)$$

hold?

(ii) Let $1 \leq \sigma \leq g$, and let $S^{\sigma-1}$, S^{σ} be any irreducible components of $\overline{\Pi}^{\sigma-1}_{g,n}$, $\overline{\Pi}^{\sigma}_{g,n}$, respectively. Then does

$$\dim(S^{\sigma-1}) = \dim(S^{\sigma}) - 1$$

hold?

(iii) Let S be an arbitrary irreducible component of $\overline{\Pi}^{\sigma}_{g,n}$. Then does

$$\dim(S) = 2g + n - 3 + \sigma$$

hold?

The above problem is an analogue of the purity of the p-rank strata of the moduli stack $\overline{\mathcal{M}}_{g,n}$ (see [FG, Theorem 2.3]).

3.5. **Some results about the topology of $\overline{\Pi}_{g,n}$.** In this subsection, we prove some basic properties concerning the topology of $\overline{\Pi}_{g,n}$.

3.5.1. Firstly, we have the following proposition.

Proposition 3.10. *We maintain the notation introduced above. Then the following statements hold.*

(a) Let $[\pi_1^{\mathrm{adm}}(q)] \in \overline{\Pi}_{g,n}$ and $[\pi_1^{\mathrm{sol}}(q)] \in \overline{\Pi}^{\mathrm{sol}}_{g,n}$ be arbitrary points. Then we have

$$V([\pi_1^{\mathrm{adm}}(q)]) = \{[\pi_1^{\mathrm{adm}}(q')] \in \overline{\Pi}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), \pi_1^{\mathrm{adm}}(q')) \neq \emptyset\},$$

$$V([\pi_1^{\mathrm{sol}}(q)]) = \{[\pi_1^{\mathrm{sol}}(q')] \in \overline{\Pi}^{\mathrm{sol}}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{sol}}(q), \pi_1^{\mathrm{sol}}(q')) \neq \emptyset\},$$

where $V([\pi_1^{\mathrm{adm}}(q)])$ and $V([\pi_1^{\mathrm{sol}}(q)])$ denote the topological closures of $[\pi_1^{\mathrm{adm}}(q)]$ and $[\pi_1^{\mathrm{sol}}(q)]$ in $\overline{\Pi}_{g,n}$ and $\overline{\Pi}^{\mathrm{sol}}_{g,n}$, respectively.

(b) We have that

$$\Pi_{g,n} \subseteq \overline{\Pi}_{g,n}, \ \Pi^{\mathrm{sol}}_{g,n} \subseteq \overline{\Pi}^{\mathrm{sol}}_{g,n}$$

are open subsets.

(c) Let Z be an arbitrary irreducible closed subset of $\overline{\mathfrak{M}}_{g,n}$. Then $V(\pi^{\mathrm{adm}}_{g,n}(Z))$ and $V(\pi^{\mathrm{sol}}_{g,n}(Z))$ are irreducible closed subsets of $\overline{\Pi}_{g,n}$ and $\overline{\Pi}^{\mathrm{sol}}_{g,n}$, respectively, where $V(\pi^{\mathrm{adm}}_{g,n}(Z))$ and $V(\pi^{\mathrm{sol}}_{g,n}(Z))$ denote the topological closures of $\pi^{\mathrm{adm}}_{g,n}(Z)$ and $\pi^{\mathrm{sol}}_{g,n}(Z)$ in $\overline{\Pi}_{g,n}$ and $\overline{\Pi}^{\mathrm{sol}}_{g,n}$, respectively. In particular, the topological spaces $\overline{\Pi}_{g,n}$ and $\overline{\Pi}^{\mathrm{sol}}_{g,n}$ are irreducible.

(d) Let V be either an irreducible closed subset of $\overline{\Pi}_{g,n}$ or an irreducible closed subset of $\overline{\Pi}^{\mathrm{sol}}_{g,n}$. Then V has a unique generic point.

(e) Let $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}$*. Then we have that* $\dim(V(\pi^{\mathrm{adm}}_{g,n}([q]))) = 0$ *if and only if* $\pi^{\mathrm{adm}}_{g,n}([q])$ *is a closed point of* $\overline{\Pi}_{g,n}$*.*

Proof. (a) follows immediately from the definitions of $O_{\overline{\Pi}_{g,n}}$ and $O_{\overline{\Pi}^{\mathrm{sol}}_{g,n}}$, respectively.

(b) Let $[\pi_1^{\mathrm{adm}}(q)] \in \Pi_{g,n}$ be an arbitrary point and $\pi_A^{\mathrm{adm}}(q)$ the set of finite quotients of $\pi_1^{\mathrm{adm}}(q)$. Moreover, since $\pi_1^{\mathrm{adm}}(q)$ is topologically finitely generated, we have a subset of open normal subgroups $\{H_j\}_{j\in\mathbb{N}}$ of $\pi_1^{\mathrm{adm}}(q)$ such that $H_{j_1} \subseteq H_{j_2}$ for any $j_1 \geq j_2$, and that

$$\pi_1^{\mathrm{adm}}(q) \cong \varprojlim_{j\in\mathbb{N}} \pi_1^{\mathrm{adm}}(q)/H_j.$$

We put $S(q) \stackrel{\mathrm{def}}{=} \{\pi_1^{\mathrm{adm}}(q)/H_j,\ j \in \mathbb{N}\} \subseteq \pi_A^{\mathrm{adm}}(q)$. We see that, in order to prove that $\Pi_{g,n}$ is an open subset of $\overline{\Pi}_{g,n}$, it is sufficient to prove that, for every point $[q_2] \in \mathfrak{M}_{g,n}$, there exists a finite group $G \in S(q_2)$ such that $U_{\overline{\Pi}_{g,n},G}$ is contained in $\Pi_{g,n}$.

Suppose that $U_{\overline{\Pi}_{g,n},G} \cap (\overline{\Pi}_{g,n} \setminus \Pi_{g,n}) \neq \emptyset$ for all $G \in S(q_2)$. Since $\pi^{\mathrm{adm}}_{g,n}$ is continuous (i.e. Theorem 3.6) and the set of generic points of $\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}$ is finite, there exists a generic point $[q_1]$ of $\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}$ such that

$$[\pi_1^{\mathrm{adm}}(q_1)] \in \bigcap_{G\in S(q_2)} U_{\overline{\Pi}_{g,n},G}.$$

Then the set

$$\mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) = \varprojlim_{G\in S(q_2)} \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q_1), G)$$

is not empty. Thus, there is a surjective open continuous homomorphism $\phi : \pi_1^{\mathrm{adm}}(q_1) \twoheadrightarrow \pi_1^{\mathrm{adm}}(q_2)$. Note that ϕ induces an isomorphism of maximal prime-to-p quotients $\phi^{p'} : \pi_1^{\mathrm{adm}}(q_1)^{p'} \xrightarrow{\sim} \pi_1^{\mathrm{adm}}(q_2)^{p'}$.

By applying [Y2, Lemma 6.3], there exists an open characteristic subgroup $H_1 \subseteq \pi_1^{\mathrm{adm}}(q_1)^{p'}$ such that the pointed stable curve $X^\bullet_{H_1}$ of type $(g_{X_{H_1}}, n_{X_{H_1}})$ over k_{q_1} corresponding to H_1 satisfying the following conditions:

- $\Gamma^{\mathrm{cpt}}_{X^\bullet_{H_1}}$ is 2-connected;
- $\#(v(\Gamma_{X^\bullet_{H_1}})^{b\leq 1}) = 0$;
- the Betti number $r_{X_{H_1}}$ of the dual semi-graph of $X^\bullet_{H_1}$ is positive.

Let $H_2 \stackrel{\mathrm{def}}{=} \phi^{p'}(H_1) \subseteq \pi_1^{\mathrm{adm}}(q_2)^{p'}$. Then we obtain a smooth pointed stable curve $X^\bullet_{H_2}$ of type $(g_{X_{H_2}}, n_{X_{H_2}})$ over k_{q_2} corresponding to H_2. Since H_i is an open characteristic subgroup, we obtain $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then Theorem 2.1 (b) and Lemma 2.2 (b) imply $r_{X_{H_1}} \leq 0$. This contradicts $r_{X_{H_1}} > 0$.

Similar arguments to the arguments given in the above proof imply that $\Pi^{\rm sol}_{g,n}$ is an open subset of $\overline{\Pi}^{\rm sol}_{g,n}$. This completes the proof of (b).

(c) is trivial.

(d) We only treat the case where V is an irreducible closed subset of $\overline{\Pi}_{g,n}$. Let $\mathrm{Gen}(V)$ be the set of generic points of V. Since every closed subset of $\overline{\mathfrak{M}}_{g,n}$ has a non-empty set of generic points, we have $\mathrm{Gen}(V)\neq\emptyset$. Let $[\pi_1^{\rm adm}(q_1)]$, $[\pi_1^{\rm adm}(q_2)]\in \mathrm{Gen}(V)$ be arbitrary generic points. Let $\pi_A^{\rm adm}(-)$ be the set of finite quotinets of $\pi_1^{\rm adm}(-)$ and $G\in\pi_A^{\rm adm}(q_1)$ an arbitrary finite group. Then $U_{\overline{\Pi}_{g,n},G}\cap V\neq\emptyset$. Thus, $[\pi_1^{\rm adm}(q_2)]\in U_{\overline{\Pi}_{g,n},G}\cap V$. This means that $\pi_A^{\rm adm}(q_1)\subseteq\pi_A^{\rm adm}(q_2)$. Similar arguments to the arguments given in the above proof imply $\pi_A^{\rm adm}(q_1)\supseteq\pi_A^{\rm adm}(q_2)$. Then we have $\pi_A^{\rm adm}(q_1)=\pi_A^{\rm adm}(q_2)$. Since admissible fundamental groups of pointed stable curves are topologically finitely generated, [FJ, Proposition 16.10.6] implies $[\pi_1^{\rm adm}(q_1)]=[\pi_1^{\rm adm}(q_2)]$. This completes the proof of the proposition.

(e) The "if" part of the proposition is trivial. We only need to prove the "only if" part of the proposition.

Let $[\pi_1^{\rm adm}(q')]\in V(\pi_{g,n}^{\rm adm}([q]))$ be an arbitrary point and $V([\pi_1^{\rm adm}(q')])$ the topological closure of $[\pi_1^{\rm adm}(q')]$ in $\overline{\Pi}_{g,n}$. Then we have that $V([\pi_1^{\rm adm}(q')])$ is an irreducible closed subset contained in $V(\pi_{g,n}^{\rm adm}([q]))$. Since $V(\pi_{g,n}^{\rm adm}([q]))$ is an irreducible closed subset of dimension 0, we obtain

$$V(\pi_{g,n}^{\rm adm}([q]))=V([\pi_1^{\rm adm}(q')]).$$

This means that there exist surjective open continuous homomorphisms

$$\pi_1^{\rm adm}(q)\twoheadrightarrow\pi_1^{\rm adm}(q'),\ \pi_1^{\rm adm}(q')\twoheadrightarrow\pi_1^{\rm adm}(q).$$

Then we obtain $\pi_A^{\rm adm}(q)=\pi_A^{\rm adm}(q')$. Since admissible fundamental groups of pointed stable curves are topologically finitely generated, [FJ, Proposition 16.10.6] implies $[\pi_1^{\rm adm}(q)]=[\pi_1^{\rm adm}(q')]$. Thus, we obtain $V(\pi_{g,n}^{\rm adm}([q]))=[\pi_1^{\rm adm}(q)]$. This completes the proof of the proposition. □

3.5.2. Next, we prove that the dimension of $\overline{\Pi}_{g,n}$ has a low bound.

Proposition 3.11. *The topological space $\overline{\Pi}_{g,n}$ is noetherian and*

$$3g-3+n\leq\dim(\overline{\Pi}_{g,n}).$$

Proof. The noetherian property of $\overline{\Pi}_{g,n}$ follows immediately from the continuity of the map $\pi_{g,n}^{\rm adm}$ and the fact that $\overline{M}_{g,n}$ is noetherian.

Let Γ be an arbitray semi-graph and $\omega: v(\Gamma)\to\mathbb{Z}_{\geq 0}$ a map such that $(\Gamma,\omega)=(\Gamma_{X^\bullet},\omega_{X^\bullet})$ for some pointed stable curve $X^\bullet$ of type (g,n) over an algebraically closed k, where $\Gamma_{X^\bullet}$ denotes the dual semi-graph of $X^\bullet$, and $\omega_{X^\bullet}: v(\Gamma_{X^\bullet})\to$

$\{g_v\}_{v\in v(\Gamma_{X^\bullet})}$ is the map defined as $v \mapsto g_v$ (recall that g_v is the genus of the smooth pointed stable curve assoicated to v (1.1.3)). We put

$$C(\Gamma,\omega) \stackrel{\text{def}}{=} \{q \in \overline{M}_{g,n} \mid (\Gamma_{X_q^\bullet}, \omega_{X_q^\bullet}) = (\Gamma,\omega)\}$$

We have the following combinatorial stratification (e.g. see [C, Section 4.1])

$$\overline{M}_{g,n} = \bigsqcup_{(\Gamma,\omega)} C(\Gamma,\omega)$$

such that $C(\Gamma_1,\omega_1) \subseteq \overline{C(\Gamma_2,\omega_2)}$ if and only if $(\Gamma_1,\omega_1) \geq (\Gamma_2,\omega_2)$ (i.e. (Γ_2,ω_2) is a weighted contraction of (Γ_1,ω_1), see [C, (2.27)]). Then we see immediately that there exists a chain of irreducible components

$$\overline{S}_{3g-3+n} \subseteq \overline{S}_{3g-3+n-1} \subseteq \cdots \subseteq \overline{S}_1 \subseteq \overline{S}_0 = \overline{M}_{g,n},$$

where S_i, $i \in \{0,\dots,3g-3+n\}$, is an irreducible component of some $C(\Gamma,\omega)$ such that $S_i \neq S_j$ if $i \neq j$, and $\overline{S}_i$ denotes the topological closure of S_i in $\overline{M}_{g,n}$.

Let q_i, $i \in \{0,\dots,3g-3+n\}$, be the generic point of S_i. Then there exist surjections of the admissible fundamental groups

$$\pi_1^{\mathrm{adm}}(q_0) \twoheadrightarrow \pi_1^{\mathrm{adm}}(q_1) \twoheadrightarrow \dots \twoheadrightarrow \pi_1^{\mathrm{adm}}(q_{3g-3+n-1}) \twoheadrightarrow \pi_1^{\mathrm{adm}}(q_{3g-3+n}).$$

By [Y2, Theorem 1.2] or [Y6, Theorem 0.3], each surjection of admissible fundamental groups mentioned above is *not* an isomorphism since the dual semi-graphs of $\{X_{q_i}^\bullet\}_i$ are not equal. We have

$$V([\pi_1^{\mathrm{adm}}(q_{3g-3+n})]) \subseteq \cdots \subseteq V([\pi_1^{\mathrm{adm}}(q_1)]) \subseteq \overline{\Pi}_{g,n}$$

such that

$$V([\pi_1^{\mathrm{adm}}(q_i)]) \supseteq V([\pi_1^{\mathrm{adm}}(q_j)]),\ V([\pi_1^{\mathrm{adm}}(q_i)]) \neq V([\pi_1^{\mathrm{adm}}(q_j)])$$

if $i < j$. We complete the proof of (b). □

Remark 3.11.1. At the time of writing this paper, the author still does not know how to prove that $\dim(\overline{\Pi}_{g,n}) < \infty$.

PART II: RECONSTRUCTIONS OF GEOMETRIC DATA FROM OPEN CONTINUOUS HOMOMORPHISMS

4. Reconstructions of inertia subgroups and field structures

In this section, we prove that the inertia subgroups and field structures associated to marked points can be reconstructed group-theoretically from *open continuous homomorphisms* of admissible fundamental groups (or solvable admissible fundamental groups). The main results of the present section are Theorem 4.11 and Theorem 4.13.

4.1. **Anabelian reconstructions.**

4.1.1. Let $\mathcal{P}$ be a category of profinite groups whose class of objects $\mathrm{Ob}(\mathcal{P})$ consists of profinite groups, and whose class of morphisms $\mathrm{Hom}_{\mathcal{P}}(\Pi, \Pi')$ is the class of open continuous homomorphisms of Π and Π'. Let $\Pi \in \mathcal{P}$, and let $\mathfrak{S}_{\Pi}$ be a category whose class of objects $\mathrm{Ob}(\mathfrak{S}_{\Pi})$ is a set of subgroups of Π, and whose class of morphisms $\mathrm{Hom}_{\mathfrak{S}_{\Pi}}(H, H')$ for any $H, H' \in \mathfrak{S}_{\Pi}$ is defined as follows: the unique element of $\mathrm{Hom}_{\mathfrak{S}_{\Pi}}(H, H')$ is the natural inclusion when $H \subseteq H'$; otherwise, $\mathrm{Hom}_{\mathfrak{S}_{\Pi}}(H, H')$ is empty. We call $\mathfrak{S}_{\Pi}$ a category associated to Π.

4.1.2. Let $\mathcal{S}$ be a category whose class of objects $\mathrm{Ob}(\mathcal{S})$ is the class of categories associated to profinite groups, and whose class of morphisms $\mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}_{\Pi}, \mathfrak{S}_{\Pi'})$ consists of the classes of functors defined as follows: $\theta_{\mathcal{S}} \in \mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}_{\Pi}, \mathfrak{S}_{\Pi'})$ if there exists an open continuous homomorphism $\theta : \Pi \to \Pi'$ such that $\mathfrak{S}_{\Pi} = \{H \overset{\mathrm{def}}{=} \theta^{-1}(H')\}_{H' \in \mathfrak{S}_{\Pi'}}$, and that $\theta_{\mathcal{S}} : \mathfrak{S}_{\Pi} \to \mathfrak{S}_{\Pi'}$, $H \mapsto H'$; otherwise, $\mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}_{\Pi}, \mathfrak{S}_{\Pi'})$ is empty.

There is a natural functor $\pi : \mathcal{S} \to \mathcal{P}$ defined as follows: Let $\mathfrak{S}_{\Pi}$, $\mathfrak{S}_{\Pi'} \in \mathcal{S}$ be categories associated to profinite groups Π, Π', respectively; we have $\pi(\mathfrak{S}_{\Pi}) = \Pi$, $\pi(\mathfrak{S}_{\Pi'}) = \Pi'$, and $\pi(\theta_{\mathcal{S}}) = \theta$. We see immediately that $\pi : \mathcal{S} \to \mathcal{P}$ is a fibered category over $\mathcal{P}$.

Definition 4.1. Let $i \in \{1, 2\}$, and let $\mathcal{F}_i$ be a geometric object (in a certain category), $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object $\mathcal{F}_i$, and $\mathfrak{S}_i \overset{\mathrm{def}}{=} \mathfrak{S}_{\Pi_{\mathcal{F}_i}}$ a category associated to $\Pi_{\mathcal{F}_i}$. Let $\mathrm{Inv}_{\mathcal{F}_i}$ be an invariant depending on the isomorphism class of $\mathcal{F}_i$ (in a certain category) and $\mathrm{Add}_{\mathcal{F}_i}(\mathfrak{S}_i)$ an additional structure associated to $\mathfrak{S}_i$ (e.g. $\mathrm{Add}_{\mathcal{F}_i}(\mathfrak{S}_i) = \mathfrak{S}_i$) on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on $\mathcal{F}_i$ and $\mathfrak{S}_i$.

(a) We shall say that $\mathrm{Inv}_{\mathcal{F}_i}$ can be *reconstructed group-theoretically* from $\Pi_{\mathcal{F}_i}$ (or $\mathrm{Inv}_{\mathcal{F}_i}$ can be induced group-theoretically from $\Pi_{\mathcal{F}_i}$, or $\Pi_{\mathcal{F}_i}$ induces $\mathrm{Inv}_{\mathcal{F}_i}$ group-theoretically) if $\Pi_{\mathcal{F}_1} \cong \Pi_{\mathcal{F}_2}$ implies $\mathrm{Inv}_{\mathcal{F}_1} = \mathrm{Inv}_{\mathcal{F}_2}$.

(b) We shall say that $\mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ can be *reconstructed group-theoretically* from $\Pi_{\mathcal{F}_2}$ (or $\mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ can be induced group-theoretically from $\Pi_{\mathcal{F}_2}$, or $\Pi_{\mathcal{F}_2}$ induces $\mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ group-theoretically) if every isomorphism $\theta : \Pi_{\mathcal{F}_1} \overset{\sim}{\to} \Pi_{\mathcal{F}_2}$ induces a bijection $\theta_{\mathrm{ad}} : \mathrm{Add}_{\mathcal{F}_1}(\mathfrak{S}_1) \overset{\sim}{\to} \mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ which preserves the structures $\mathrm{Add}_{\mathcal{F}_1}(\mathfrak{S}_1)$ and $\mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$, where $\mathfrak{S}_1 \overset{\mathrm{def}}{=} \Pi_{\mathcal{F}_1} \times_{\theta, \Pi_{\mathcal{F}_2}} \mathfrak{S}_2$ (i.e. the fiber product in the fibered category $\mathcal{S}$ over $\mathcal{P}$).

(c) Let j_1, $j_2 \in \{1, 2\}$ distinct from each other, and let $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ be an open continuous homomorphism of profinite groups and $\mathfrak{S}_1 = \Pi_{\mathcal{F}_1} \times_{\theta, \Pi_{\mathcal{F}_2}} \mathfrak{S}_2$. We shall say that a map $\theta_{\mathrm{ad}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ can be *reconstructed group-theoretically* from $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ (or $\theta_{\mathrm{ad}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ can be induced group-theoretically from $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$, or $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ induces $\theta_{\mathrm{ad}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ group-theoretically) if the following holds: Let $\mathcal{F}'_i$,

$i \in \{1, 2\}$, be a geometric object, $\Pi_{\mathcal{F}'_i}$ a profinite group associated to the geometric object $\mathcal{F}'_i$, $\theta_i : \Pi_{\mathcal{F}'_i} \overset{\sim}{\to} \Pi_{\mathcal{F}_i}$ an isomorphism of profinite groups, $\theta' : \Pi_{\mathcal{F}'_1} \to \Pi_{\mathcal{F}'_2}$, $\mathfrak{S}'_i \overset{\mathrm{def}}{=} \Pi_{\mathcal{F}'_i} \times_{\theta_i, \Pi_{\mathcal{F}_i}} \mathfrak{S}_i$, $\mathrm{Add}_{\mathcal{F}'_i}(\mathfrak{S}'_i)$ the additional structure on the profinite group $\Pi_{\mathcal{F}'_i}$ induced by θ_i. Moreover, suppose that we have the following commutative diagram of profinite groups:

$$\begin{array}{ccc} \Pi_{\mathcal{F}'_1} & \xrightarrow{\theta'} & \Pi_{\mathcal{F}'_2} \\ {\scriptstyle \theta_1}\downarrow & & {\scriptstyle \theta_2}\downarrow \\ \Pi_{\mathcal{F}_1} & \xrightarrow{\theta} & \Pi_{\mathcal{F}_2}. \end{array}$$

Then the above commutative diagram of profinite groups induces the following commutative diagram of additional structures

$$\begin{array}{ccc} \mathrm{Add}_{\mathcal{F}'_{j_1}}(\mathfrak{S}'_{j_1}) & \xrightarrow{\theta'_{\mathrm{ad}}} & \mathrm{Add}_{\mathcal{F}'_{j_2}}(\mathfrak{S}'_{j_2}) \\ {\scriptstyle \theta_{j_1,\mathrm{ad}}}\downarrow & & {\scriptstyle \theta_{j_2,\mathrm{ad}}}\downarrow \\ \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) & \xrightarrow{\theta_{\mathrm{ad}}} & \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2}) \end{array}$$

which preserves the structures of additional structures.

Remark 4.1.1. Let us explain the theory of *mono-anabelian geometry* introduced by Mochizuki. The classical point of view of anabelian geometry (i.e. the anabelian geometry considered in [G]) focuses on a comparison between two geometric objects via their fundamental groups. Moreover, the term "group-theoretical", in the classical point of view, means that "preserved by an arbitrary isomorphism between the fundamental groups under consideration". We shall refer to the classical point of view as "*bi-anabelian geometry*". Then Definition 4.1 is a definition from the point of view of bi-anabelian geometry.

On the other hand, mono-anabelian geometry focuses on the establishing a group-theoretic algorithm whose input datum is an abstract topological group which is isomorphic to the fundamental group of a given geometric object of interest (resp. a continuous homomorphism of abstract topological groups which are isomorphic to a continuous homomorphism of the fundamental groups of given geometric objects of interest), and whose output datum is a geometric object which is isomorphic to the given geometric object of interest (resp. a morphism of geometric objects which is isomorphic to a morphism of given geometric objects of interest). In the point of view of mono-anabelian geometry, the term "group-theoretic algorithm" is used to mean that "the algorithm in a discussion is phrased in language that only depends on the topological group structures of the fundamental groups under consideration". Note that mono-anabelian results are stronger than bi-anabelian results.

We maintain the notation introduced in Definition 4.1. Then the mono-anabelian version of Definition 4.1 is as follows:

(a) We shall say that $\mathrm{Inv}_{\mathcal{F}_i}$ can be *mono-anabelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Inv}_{\mathcal{F}_i}$.

(b) We shall say that $\mathrm{Add}_{\mathcal{F}_i}(\mathfrak{S}_i)$ can be *mono-anabelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_i}$.

(c) Let $j_1, j_2 \in \{1,2\}$ distinct from each other, and let $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ be an open continuous homomorphism of profinite groups and $\mathfrak{S}_1 = \Pi_{\mathcal{F}_1} \times_{\theta,\Pi_{\mathcal{F}_2}} \mathfrak{S}_2$. We shall say that a map (or a morphism) $\theta_{\mathrm{add}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ can be *mono-anabelian reconstructed* from $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ if there exists a group-theoretical algorithm whose input datum is $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$, and whose output datum is $\theta_{\mathrm{add}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$.

4.1.3. Let $i \in \{1,2\}$, and let $X_i^\bullet = (X_i, D_{X_i})$ be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic $p_i > 0$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and $\Pi_{X_i^\bullet}$ either the admissible fundamental group or the solvable admissible fundamental group of $X_i^\bullet$. We have the following result:

Theorem 4.2. *We maintain the notation introduced in 1.2.7 and 1.2.11. Then the data*

$$p_i,\ (g_{X_i}, n_{X_i}),\ \Pi^{\text{ét}}_{X_i^\bullet},\ \Pi^{\mathrm{top}}_{X_i^\bullet},\ \mathrm{Ver}(\Pi_{X_i^\bullet}),\ \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet}),\ \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_i^\bullet}),\ \Gamma_{X_i^\bullet}$$

can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$.

Proof. See [Y2, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1] and [Y5, Theorem 6.3]. □

Remark 4.2.1. [Y5, Theorem 1.3] gives a group-theoretical formula for (g_{X_i}, n_{X_i}). Then we obtain that the characteristic p_i of k_i and the type (g_{X_i}, n_{X_i}) can be *mono-anabelian reconstructed* from $\Pi_{X_i^\bullet}$. In fact, we have that $\Pi^{\text{ét}}_{X_i^\bullet}$, $\Pi^{\mathrm{top}}_{X_i^\bullet}$, $\mathrm{Ver}(\Pi_{X_i^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$, $\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_i^\bullet})$, and $\Gamma_{X_i^\bullet}$ can be *mono-anabelian reconstructed* from $\Pi_{X_i^\bullet}$ (see [Y6, Theorem 0.3]).

We do not use the term "mono-anabelian reconstruction" in the present paper. On the other hand, all of the results proved in Section 4 and Section 5 can be generalized to the case of mono-anabelian reconstructions.

4.1.4. The following lemma will be used in the remainder of the present paper.

Lemma 4.3. *Suppose that* $p \overset{\mathrm{def}}{=} p_1 = p_2$ *and* $(g_X, n_X) \overset{\mathrm{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. *Let* $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ *be an arbitrary open continuous homomorphism. Then* ϕ *is a surjection.*

Proof. Let $\Pi_\phi \overset{\mathrm{def}}{=} \phi(\Pi_{X_1^\bullet}) \subseteq \Pi_{X_2^\bullet}$ be the image of ϕ which is an open subgroup of $\Pi_{X_2^\bullet}$. Let $X_\phi^\bullet = (X_\phi, D_{X_\phi})$ be the pointed stable curve of type (g_{X_ϕ}, n_{X_ϕ}) over k_2 induced by Π_ϕ and $X_\phi^\bullet \to X_2^\bullet$ the admissible covering over k_2 induced by the natural inclusion $\Pi_\phi \hookrightarrow \Pi_{X_2^\bullet}$. The Riemann-Hurwitz formula implies $g_{X_\phi} \geq g_X$ and $n_{X_\phi} \geq n_X$. Moreover, by considering the maximal prime-to-p quotients of $\Pi_{X_1^\bullet}$ and Π_ϕ, the natural surjection $\Pi_{X_1^\bullet} \twoheadrightarrow \Pi_\phi$ induced by ϕ implies $2g_X + n_X \geq 2g_{X_\phi} + n_{X_\phi}$. Then we have $(g_X, n_X) = (g_{X_\phi}, n_{X_\phi})$. This means that the admissible covering $X_\phi^\bullet \to X_2^\bullet$ is totally ramified over every marked point of D_{X_2}. Moreover, the Riemann-Hurwitz formula implies that $[\Pi_{X_2^\bullet} : \Pi_\phi] \neq 1$ and $(g_X, n_X) = (g_{X_\phi}, n_{X_\phi})$ if and only if $(g_X, n_X) = (0, 2)$. Since $X_i^\bullet$ is a pointed stable curve over k_i, we obtain $[\Pi_{X_2^\bullet} : \Pi_\phi] = 1$. Thus, ϕ is a surjection. □

4.2. **Reconstructions of inertia subgroups.**

4.2.1. **Settings.** We maintain the notation introduced in 4.1.3. In the remainder of this subsection, we suppose that $p \overset{\mathrm{def}}{=} p_1 = p_2$ and $(g_X, n_X) \overset{\mathrm{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism. By Lemma 4.3, we see that ϕ is a *surjective* open continuous homomorphism. Let $i \in \{1, 2\}$, and let $\mathfrak{P}$ be the set of prime numbers, $\Sigma \subseteq \mathfrak{P} \setminus \{p\}$ a subset, $\Pi_{X_i^\bullet}^\Sigma$ the maximal pro-Σ quotient of $\Pi_{X_i^\bullet}$, $pr_i^\Sigma : \Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^\Sigma$ the natural surjective homomorphism, and

$$\phi^\Sigma : \Pi_{X_1^\bullet}^\Sigma \xrightarrow{\sim} \Pi_{X_2^\bullet}^\Sigma$$

the isomorphism induced by ϕ. In particular, if $\Sigma = \mathfrak{P} \setminus \{p\}$, we use the notation $\Pi_{X_i^\bullet}^{p'}$ and $\phi^{p'} : \Pi_{X_1^\bullet}^{p'} \xrightarrow{\sim} \Pi_{X_2^\bullet}^{p'}$ to denote $\Pi_{X_i^\bullet}^\Sigma$ and ϕ^Σ, respectively.

4.2.2. Firstly, we have some lemmas concerning types of admissible coverings.

Lemma 4.4. *We maintain the notation introduced above. Then we have that* $\Pi_{X_i^\bullet}^{\mathrm{cpt}}$ *(1.2.7) can be reconstructed group-theoretically from* $\Pi_{X_i^\bullet}$*, and that the (surjective) open continuous homomorphism* $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ *induces a surjective open continuous homomorphism*

$$\phi^{\mathrm{cpt}} : \Pi_{X_1^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_2^\bullet}^{\mathrm{cpt}}$$

group-theoretically. Moreover, the following commutative diagram of profinite groups

$$\begin{array}{ccc} \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\ \downarrow & & \downarrow \\ \Pi_{X_1^\bullet}^{\mathrm{cpt}} & \xrightarrow{\phi^{\mathrm{cpt}}} & \Pi_{X_2^\bullet}^{\mathrm{cpt}} \end{array}$$

can be reconstructed group-theoretically from ϕ.

Proof. By Theorem 4.2, we have that (g_X, n_X) can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. If $n_X = 0$, the lemma is trivial. Then we may assume $n_X > 0$.

Let $H_i \subseteq \Pi_{X_i^\bullet}$ be an open subgroup. Then the Riemann-Hurwitz formula implies that the admissible covering $X_{H_i}^\bullet \to X_i^\bullet$ over k_i induced by $H_i \subseteq \Pi_{X_i^\bullet}$ is étale over D_{X_i} if and only if $g_{X_{H_i}} = [\Pi_{X_i^\bullet} : H_i](g_X - 1) + 1$. We put

$$\begin{aligned}\mathrm{Et}^{\mathrm{norm}}_{D_{X_i}}(\Pi_{X_i^\bullet}) &\overset{\mathrm{def}}{=} \{H_i \subseteq \Pi_{X_i^\bullet} \text{ is an open normal subgroup} \\ &\qquad \mid g_{X_{H_i}} = [\Pi_{X_i^\bullet} : H_i](g_X - 1) + 1\} \\ &\subseteq \mathrm{Et}_{D_{X_i}}(\Pi_{X_i^\bullet}) \overset{\mathrm{def}}{=} \{H_i \subseteq \Pi_{X_i^\bullet} \text{ is an open subgroup} \\ &\qquad \mid g_{X_{H_i}} = [\Pi_{X_i^\bullet} : H_i](g_X - 1) + 1\}.\end{aligned}$$

By Theorem 4.2, we have that $\mathrm{Et}^{\mathrm{norm}}_{D_{X_i}}(\Pi_{X_i^\bullet})$ and $\mathrm{Et}_{D_{X_i}}(\Pi_{X_i^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. Since

$$\Pi^{\mathrm{cpt}}_{X_i^\bullet} \overset{\mathrm{def}}{=} \Pi_{X_i^\bullet} / \bigcap_{H_i \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_i}}(\Pi_{X_i^\bullet})} H_i = \Pi_{X_i^\bullet} / \bigcap_{H_i \in \mathrm{Et}_{D_{X_i}}(\Pi_{X_i^\bullet})} H_i,$$

we obtain that $\Pi^{\mathrm{cpt}}_{X_i^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$.

Let $H_2 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_2}}(\Pi_{X_2^\bullet})$, $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2)$, and $G \overset{\mathrm{def}}{=} \Pi_{X_2^\bullet}/H_2 = \Pi_{X_1^\bullet}/H_1$. We will prove that $H_1 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_1}}(\Pi_{X_1^\bullet})$. Let $f^\bullet_{H_1} : X^\bullet_{H_1} \to X_1^\bullet$ be the Galois admissible covering over k_1 with Galois group G corresponding to H_1, $x_1 \in D_{X_1}$ a marked point of $X_1^\bullet$, and $e_{f_{H_1}}(x_1)$ the ramification index of a point of $f^{-1}_{H_1}(x_1)$. Since $H_2 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_2}}(\Pi_{X_2^\bullet})$, we have $g_{X_{H_2}} = \#(G)(g_X - 1) + 1$ and $n_{X_{H_2}} = \#(G) n_X$. By applying the Riemann-Hurwitz formula, we obtain

$$\begin{aligned}g_{X_{H_1}} &= \#(G)(g_X - 1) + 1 + \frac{1}{2} \cdot \sum_{x_1 \in D_{X_1}} \frac{\#(G)}{e_{f_{H_1}}(x_1)} (e_{f_{H_1}}(x_1) - 1) \\ &= \#(G)(g_X - 1) + 1 + \frac{1}{2} \cdot \sum_{x_1 \in D_{X_1}} (\#(G) - \frac{\#(G)}{e_{f_{H_1}}(x_1)}), \\ n_{X_{H_1}} &= \sum_{x_1 \in D_{X_1}} \frac{\#(G)}{e_{f_{H_1}}(x_1)}.\end{aligned}$$

By applying Theorem 2.1 (a) and Lemma 2.2 (a), the surjective homomorphism $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$ induces the following inequality (see 2.2.1 for $\gamma^{\max}(H_i)$):

$$\gamma^{\max}(H_1) + 2 = g_{X_{H_1}} + n_{X_{H_1}} \geq \gamma^{\max}(H_2) + 2 = g_{X_{H_2}} + n_{X_{H_2}}.$$

Then we obtain

$$\begin{aligned} g_{X_{H_1}} + n_{X_{H_1}} &= \#(G)(g_X - 1) + 1 + \frac{1}{2}\cdot \sum_{x_1 \in D_{X_1}} (\#(G) - \frac{\#(G)}{e_{f_{H_1}}(x_1)}) + \sum_{x_1\in D_{X_1}} \frac{\#(G)}{e_{f_{H_1}}(x_1)} \\ &= \#(G)(g_X - 1) + 1 + \frac{1}{2}\#(G)n_X + \frac{1}{2}\cdot \sum_{x_1 \in D_{X_1}} \frac{\#(G)}{e_{f_{H_1}}(x_1)} \\ &\geq \#(G)(g_X - 1) + 1 + \#(G)n_X. \end{aligned}$$

Thus, we have

$$\sum_{x_1\in D_{X_1}} \frac{\#(G)}{e_{f_{H_1}}(x_1)} \geq \#(G)n_X.$$

Since $\#(D_{X_1}) = n_X$, we see immediately that $e_{f_{H_1}}(x_1) = 1$. This means that $f^\bullet_{H_1}$ is étale, and that $H_1 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_1}}(\Pi_{X_1^\bullet})$. Thus we may define the following surjective homomorphism

$$\phi^{\mathrm{cpt}} : \Pi^{\mathrm{cpt}}_{X_1^\bullet} \overset{\mathrm{def}}{=} \Pi_{X_1^\bullet}/\bigcap_{H_1\in \mathrm{Et}^{\mathrm{norm}}_{D_{X_1}}(\Pi_{X_1^\bullet})} H_1 \twoheadrightarrow \Pi^{\mathrm{cpt}}_{X_2^\bullet} \overset{\mathrm{def}}{=} \Pi_{X_2^\bullet}/\bigcap_{H_2\in \mathrm{Et}^{\mathrm{norm}}_{D_{X_2}}(\Pi_{X_2^\bullet})} H_2$$

which is induced by ϕ group-theoretically. Moreover, the commutative diagram

$$\begin{array}{ccc} \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{cpt}}_{X_1^\bullet} & \xrightarrow{\phi^{\mathrm{cpt}}} & \Pi^{\mathrm{cpt}}_{X_2^\bullet} \end{array}$$

follows immediately from the definition of ϕ^{cpt}. This completes the proof of the lemma. □

Lemma 4.5. *Let ℓ be a prime number, $H_2 \subseteq \Pi_{X_2^\bullet}$ an open normal subgroup, and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$. Suppose that $G \overset{\mathrm{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a cyclic group which is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. Then we have*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. Let $i \in \{1, 2\}$, and let $f^\bullet_{H_i} : X^\bullet_{H_i} \to X^\bullet_i$ be the Galois admissible covering over k_i with Galois group G corresponding to H_i. Suppose that $\ell = p$. Then the definition of admissible coverings implies that $f^\bullet_{H_i}$ is étale. Thus, we have $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then we may suppose $\ell \neq p$.

By the Riemann-Hurwitz formula, we have (see 1.1.5 for $e^{\mathrm{op,ra}}_{f_{H_i}}$)

$$g_{X_{H_i}} = \ell(g_X - 1) + 1 + \frac{1}{2}\#(e^{\mathrm{op,ra}}_{f_{H_i}})(\ell - 1),$$

$$n_{X_{H_i}} = \#(e^{\text{op,ra}}_{f_{H_i}}) + \ell(n_X - \#(e^{\text{op,ra}}_{f_{H_i}})).$$

By applying Theorem 2.1 (a) and Lemma 2.2 (a), the surjective homomorphism $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$ implies

$$\gamma^{\max}(H_1) + 2 = g_{X_{H_1}} + n_{X_{H_1}} \geq \gamma^{\max}(H_2) + 2 = g_{X_{H_2}} + n_{X_{H_2}}.$$

Then we have

$$\ell(g_X - 1) + 1 + \frac{1}{2}\#(e^{\text{op,ra}}_{f_{H_1}})(\ell - 1) + \#(e^{\text{op,ra}}_{f_{H_1}}) + \ell(n_X - \#(e^{\text{op,ra}}_{f_{H_1}}))$$
$$= \ell(g_X - 1) + 1 + \ell n_X + \frac{1}{2}(1-\ell)\#(e^{\text{op,ra}}_{f_{H_1}})$$
$$\geq \ell(g_X - 1) + 1 + \frac{1}{2}\#(e^{\text{op,ra}}_{f_{H_2}})(\ell - 1) + \#(e^{\text{op,ra}}_{f_{H_2}}) + \ell(n_X - \#(e^{\text{op,ra}}_{f_{H_2}}))$$
$$= \ell(g_X - 1) + 1 + \ell n_X + \frac{1}{2}(1-\ell)\#(e^{\text{op,ra}}_{f_{H_2}}).$$

Then we obtain

$$\#(e^{\text{op,ra}}_{f_{H_1}}) \leq \#(e^{\text{op,ra}}_{f_{H_2}}).$$

Let $0 \leq m \leq n_X$. We put

$$\mathcal{N}_{i,m} \overset{\text{def}}{=} \{N_i \subseteq \Pi_{X_i^\bullet} \text{ is an open normal subgroup}$$
$$\mid \Pi_{X_i^\bullet}/N_i \cong \mathbb{Z}/\ell\mathbb{Z} \text{ and } \#(e^{\text{op,ra}}_{f_{N_i}}) = m\},$$
$$\mathcal{N}_{i,\leq m} \overset{\text{def}}{=} \bigcup_{0 \leq j \leq m} \mathcal{N}_{i,j}.$$

Here $f^\bullet_{N_i}$ denotes the Galois admissible covering over k_i corresponding to N_i. The isomorphism $\phi^{p'}$ induces a bijective map $\phi^*_\ell : \mathcal{N}_{2,\leq n_X} \overset{\sim}{\to} \mathcal{N}_{1,\leq n_X}$, $N_2 \mapsto \phi^{-1}(N_2)$. To verify the lemma, it sufficient to prove that ϕ^*_ℓ induces a bijection

$$\phi^*_\ell|_{\mathcal{N}_{2,m}} : \mathcal{N}_{2,m} \overset{\sim}{\to} \mathcal{N}_{1,m}.$$

We note that since $(g_X, n_X) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$, the isomorphism $\phi^{p'}$ implies $\#(\mathcal{N}_{1,j}) = \#(\mathcal{N}_{2,j})$ for each $0 \leq j \leq n_X$. Then by Lemma 4.4, we have a bijection $\phi^*_\ell|_{\mathcal{N}_{2,0}} : \mathcal{N}_{2,0} \overset{\sim}{\to} \mathcal{N}_{1,0}$. We prove $\phi^*_\ell|_{\mathcal{N}_{2,m}} : \mathcal{N}_{2,m} \overset{\sim}{\to} \mathcal{N}_{1,m}$ by induction on m. Suppose that $m \geq 1$. The inequality $\#(e^{\text{op,ra}}_{f_{H_1}}) \leq \#(e^{\text{op,ra}}_{f_{H_2}})$ concerning the cardinality of branch locus implies that we have a bijection $\phi^*_\ell|_{\mathcal{N}_{2,\leq m}} : \mathcal{N}_{2,\leq m} \overset{\sim}{\to} \mathcal{N}_{1,\leq m}$. By induction, $\phi^*_\ell|_{\mathcal{N}_{2,\leq m-1}} : \mathcal{N}_{2,\leq m-1} \overset{\sim}{\to} \mathcal{N}_{1,\leq m-1}$ is a bijection. Then we obtain

$$\phi^*_\ell|_{\mathcal{N}_{2,m}} : \mathcal{N}_{2,m} \overset{\sim}{\to} \mathcal{N}_{1,m}.$$

This completes the proof of the lemma. □

Corollary 4.6. *Let $H_2 \subseteq \Pi_{X_2^\bullet}$ be an open normal subgroup and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$. Suppose that $G \overset{\text{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a finite solvable group. Then we have*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. The corollary follows immediately from Lemma 4.5. □

Lemma 4.7. *Let $H_2 \subseteq \Pi_{X_2^\bullet}$ be an open normal subgroup and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$. Suppose that H_2 contains the kernel of the natural homomorphism $\Pi_{X_2^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}^{\text{cpt}}$ (i.e. the admissible covering corresponding to H_2 is étale over D_{X_2}). Then we have*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. By Lemma 4.4, we have that H_1 contains the kernel of the natural homomorphism $\Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_1^\bullet}^{\text{cpt}}$ (i.e. the admissible covering corresponding to H_1 is étale over D_{X_1}). Then the lemma follows immediately from the Riemann-Hurwitz formula. □

Definition 4.8. Let Π be an arbitrary profinite group and $m, n \in \mathbb{N}$ positive natural numbers. We define the closed normal subgroup $D_n(\Pi)$ of Π to be the topological closure of $[\Pi, \Pi]\Pi^n$, where $[\Pi, \Pi]$ denotes the commutator subgroup of Π. Moreover, we define the closed normal subgroup $D_n^{(m)}(\Pi)$ of Π inductively by $D_n^{(0)}(\Pi) \overset{\text{def}}{=} \Pi$, $D_n^{(1)}(\Pi) \overset{\text{def}}{=} D_n(\Pi)$, and $D_n^{(j+1)}(\Pi) \overset{\text{def}}{=} D_n(D_n^{(j)}(\Pi))$, $j \in \{1, \dots, m-1\}$. Note that $\#(\Pi/D_n^{(m)}(\Pi)) \leq \infty$ when Π is topologically finitely generated.

Proposition 4.9. *Let $N_2 \subseteq \Pi_{X_2^\bullet}$ be an arbitrary open subgroup and $N_1 \overset{\text{def}}{=} \phi^{-1}(N_2) \subseteq \Pi_{X_1^\bullet}$. Then there exist open normal subgroups $H_2 \subseteq N_2 \subseteq \Pi_{X_2^\bullet}$ of $\Pi_{X_2^\bullet}$ and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq N_1 \subseteq \Pi_{X_1^\bullet}$ of $\Pi_{X_1^\bullet}$ such that*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. If $n_X = 0$, then the proposition is trivial. We may assume that $n_X \geq 1$. Let $i \in \{1, 2\}$, and let M_i be an open normal subgroup of $\Pi_{X_i^\bullet}$ such that $M_i \subseteq N_i$ and $\phi^{-1}(M_2) = M_1$. By replacing N_i by M_i, we may assume that N_i is an open normal subgroup of $\Pi_{X_i^\bullet}$. We put $G \overset{\text{def}}{=} \Pi_{X_1^\bullet}/N_1 = \Pi_{X_2^\bullet}/N_2$. Write m for $[G : G_p]$, where G_p is a Sylow-p subgroup of G. Then we have $(m, p) = 1$.

Moreover, let m' be a natural number prime to p. Corollary 4.6 implies that by replacing $X_i^\bullet$ and N_i by $X^\bullet_{D_{m'}^{(2)}(\Pi_{X_i^\bullet})}$ and $N_i \cap D_{m'}^{(2)}(\Pi_{X_i^\bullet})$, respectively, we may assume that $g_X \geq 2$ and $n_X \geq 2$, and that there exists an irreducible component of $X_i^\bullet$ such that the genus of the normalization of the irreducible component is ≥ 2, where $X^\bullet_{D_{m'}^{(2)}(\Pi_{X_i^\bullet})}$ denotes the pointed stable curve over k_i corresponding to $D_{m'}^{(2)}(\Pi_{X_i^\bullet})$.

First, suppose that G is a *simple* finite group. By applying Corollary 4.6, we may assume that G is *non-abelian*. We have the following claim:

> **Claim:** To verify the proposition, we may assume that n_X is a positive *even* number.

Let us prove this claim. Suppose that $p \neq 2$. Let $R_2 \subseteq \Pi_{X_2^\bullet}$ be an open subgroup such that $\#(\Pi_{X_2^\bullet}/R_2) = 2$, and that $R_2 \supseteq \ker(\Pi_{X_2^\bullet} \twoheadrightarrow \Pi^{\mathrm{cpt}}_{X_2^\bullet})$ (i.e. the cyclic Galois admissible covering corresponding to R_2 is étale). Let $R_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(R_2) \subseteq \Pi_{X_1^\bullet}$. Then Corollary 4.6 implies that by replacing H_i and $\Pi_{X_i^\bullet}$ by $H_i \cap R_i$ and R_i, respectively, we may assume that n_X is a positive even number. Suppose that $p = 2$. Let $\ell >> 0$ be a prime number such that $(\ell, 2) = (\ell, \#(G)) = 1$. By [R2, Théorème 4.3.1], there exists an open normal subgroup $R_2^* \subseteq \Pi_{X_2^\bullet}$ such that $\#(\Pi_{X_2^\bullet}/R_2^*) = \ell$, $R_2^* \supseteq \ker(\Pi_{X_2^\bullet} \twoheadrightarrow \Pi^{\mathrm{cpt}}_{X_2^\bullet})$, and

$$\dim_{\mathbb{F}_p}(R_2^{*,\mathrm{ab}} \otimes \mathbb{F}_p) > 0.$$

Let $R_1^* \stackrel{\mathrm{def}}{=} \phi^{-1}(R_2^*) \subseteq \Pi_{X_1^\bullet}$. Then we have $\#(\Pi_{X_1^\bullet}/R_1^*) = \ell$ and $\dim_{\mathbb{F}_p}(R_1^{*,\mathrm{ab}} \otimes \mathbb{F}_p) > 0$. Thus, we may take an open normal subgroup $R_2' \subseteq R_2^*$ such that

$$\Pi_{X_2^\bullet}/R_2' \cong \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/\ell\mathbb{Z}.$$

We put $R_1' \stackrel{\mathrm{def}}{=} \phi^{-1}(R_2')$. Then the construction of R_1' implies that $\Pi_{X_1^\bullet}/R_1' \cong \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/\ell\mathbb{Z}$. Corollary 4.6 implies that by replacing H_i and $\Pi_{X_i^\bullet}$ by $H_i \cap R_i'$ and R_i', respectively, we may assume that n_X is a positive even number. This completes the proof of the claim.

Since n_X is a positive *even* number, there exists an open normal subgroup $Q_2 \subseteq \Pi_{X_2^\bullet}$ such that $\Pi_{X_2^\bullet}/Q_2 \cong \mathbb{Z}/m\mathbb{Z}$, and that the Galois admissible covering $f^\bullet_{Q_2} : X^\bullet_{Q_2} \to X_2^\bullet$ induced by Q_2 is totally ramified over every marked point of D_{X_2}. Write Q_1 for $\phi^{-1}(Q_2)$ and $f^\bullet_{Q_1} : X^\bullet_{Q_1} \to X_1^\bullet$ for the Galois admissible covering with Galois group $\Pi_{X_1^\bullet}/Q_1 \cong \mathbb{Z}/m\mathbb{Z}$ induced by Q_1. Then Corollary 4.6 implies that $f^\bullet_{Q_1}$ is totally ramified over every marked point of D_{X_1}. Let $H_i \stackrel{\mathrm{def}}{=} N_i \cap Q_i$ and $f^\bullet_{H_i} : X^\bullet_{H_i} \cong X^\bullet_{N_i} \times_{X_i^\bullet} X^\bullet_{Q_i} \to X_i^\bullet$ the Galois admissible covering over k_i with Galois group $G \times \mathbb{Z}/m\mathbb{Z}$. By Abhyankar's lemma, we obtain that the natural morphism $X^\bullet_{H_i} \to X^\bullet_{Q_i}$ induced by the inclusion $H_i \subseteq Q_i$ is étale over every marked point of $D_{X_{Q_i}}$. Then the proposition follows immediately from Corollary 4.6 and Lemma 4.7. This completes the proposition when G is a simple group.

Next, let us prove the proposition in the case where G is an arbitrary finite group. Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \stackrel{\mathrm{def}}{=} G$ be a sequence of subgroups of G such that G_j/G_{j-1} is a non-trivial simple group for all $j \in \{2, \ldots n\}$. In order to verify the proposition, it is sufficient to to prove the proposition when $n = 2$. Let P_2 be the kernel of the natural homomorphism $\Pi_{X_2^\bullet} \twoheadrightarrow G \twoheadrightarrow G_1$ and $P_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(P_2)$. Then by replacing G by G_1 and by applying the proposition for the simple group G_1, we obtain an open normal subgroup $T_2 \subseteq P_2 \subseteq \Pi_{X_2^\bullet}$ such that $(g_{X_{T_1}}, n_{X_{T_1}}) = (g_{X_{T_2}}, n_{X_{T_2}})$, where

$T_1 \overset{\text{def}}{=} \phi^{-1}(T_2)$, and $(g_{X_{T_i}}, n_{X_{T_i}})$ denotes the type of the pointed stable curve $X^\bullet_{T_i}$ corresponding to T_i.

If $T_i \subseteq N_i$, then we may put $H_i \overset{\text{def}}{=} T_i$. If N_i does not contain T_i, we put $O_i \overset{\text{def}}{=} T_i \cap N_i$. Then we have $T_i/O_i \cong G/G_1$. Note that G/G_1 is a simple group. Then the proposition follows from the proposition when we replace $X^\bullet_i$ and G by $X^\bullet_{T_i}$ and the simple group G/G_1, respectively. This completes the proof of the proposition. □

Lemma 4.10. *Let ℓ be a prime number distinct from p, $I_i, J_i \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet_i})$ arbitrary closed subgroups (see 1.2.11 for $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet_i})$), and $\Pi^\ell_{X^\bullet_i}$ the maximal pro-ℓ quotient of $\Pi_{X^\bullet_i}$. Write $\overline{I}^\ell_i$ and $\overline{J}^\ell_i$ for $pr^\ell_i(I_i)$ and $pr^\ell_i(J_i)$ (4.2.1), respectively. Suppose that $\overline{I}^\ell_i = \overline{J}^\ell_i$. Then we have*

$$I_i = J_i.$$

Proof. Suppose that $I_i \neq J_i$. [M3, Proposition 1.2 (i)] implies that $I_i \cap J_i$ is trivial. Then we see that, by replacing $\Pi_{X^\bullet_i}$ by a certain open subgroup of $\Pi_{X^\bullet_i}$, there exists an open normal subgroup $N_i \subseteq \Pi_{X^\bullet_i}$ such that $\#(\Pi_{X^\bullet_i}/N_i) = \ell$, that $I_i \subseteq N_i$, and that $J_i \not\subseteq N_i$. This contradicts $\overline{I}^\ell_i = \overline{J}^\ell_i$. We complete the proof of the lemma. □

4.2.3. Next, we prove the main result of this section.

Theorem 4.11. *We maintain the settings introduced in 4.2.1. Then the open continuous homomorphism $\phi : \Pi_{X^\bullet_1} \to \Pi_{X^\bullet_2}$ induces a surjective map (see 1.2.11 for $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet_i})$)*

$$\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet_1}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet_2}),$$

group-theoretically. Moreover, ϕ induces a bijection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X^\bullet_1}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X^\bullet_2})$$

of the sets of open edges of dual semi-graphs of $X^\bullet_1$ and $X^\bullet_2$ group-theoretically.

Proof. If $n_X = 0$, the theorem is trivial. Then we may assume $n_X > 0$. Let $\mathcal{C}_{\Pi_{X^\bullet_2}}$ be a cofinal system of $\Pi_{X^\bullet_2}$ (i.e. $\mathcal{C}_{\Pi_{X^\bullet_2}}$ consists of open normal subgroups of $\Pi_{X^\bullet_2}$ such that $\Pi_{X^\bullet_2} \xrightarrow{\sim} \varprojlim_{H_2 \in \mathcal{C}_{\Pi_{X^\bullet_2}}} \Pi_{X^\bullet_2}/H_2$). We put

$$\mathcal{C}_{\Pi_{X^\bullet_1}} \overset{\text{def}}{=} \{H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \mid H_2 \in \mathcal{C}_{\Pi_{X^\bullet_2}}\}.$$

Note that $\mathcal{C}_{\Pi_{X^\bullet_1}}$ is not a cofinal system of $\Pi_{X^\bullet_1}$ in general. Moreover, by applying Proposition 4.9, we may assume that $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$ holds for every $H_2 \in \mathcal{C}_{\Pi_{X^\bullet_2}}$ and every $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \in \mathcal{C}_{\Pi_{X^\bullet_1}}$.

Let $I_1 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})$ and $\phi(I_1) \subseteq \Pi_{X_2^\bullet}$. We will prove $\phi(I_1) \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$. Let $H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}$. By replacing $\Pi_{X_i^\bullet}$ and ϕ by H_i and $\phi|_{H_1}$, respectively, Lemma 4.4 implies that we have the following commutative diagram:

$$\begin{array}{ccc} I_1 \cap H_1 & \xrightarrow{\phi|_{I_1 \cap H_1}} & \phi(I_1) \cap H_2 \\ \downarrow & & \downarrow \\ H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \downarrow & & \downarrow \\ H_1^{\mathrm{cpt,ab}} & \xrightarrow{\phi|_{H_1}^{\mathrm{cpt,ab}}} & H_2^{\mathrm{cpt,ab}}. \end{array}$$

Since $I_1 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})$, we have that $I_1 \cap H_1 \hookrightarrow H_1 \to H_1^{\mathrm{cpt,ab}}$ is trivial. Then the above commutative diagram implies that the natural morphism

$$\phi(I_1) \cap H_2 \hookrightarrow H_2 \to H_2^{\mathrm{cpt,ab}}$$

is trivial. Thus, by [HM, Lemma 1.6], there exists $I_2 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$ such that $\phi(I_1) \subseteq I_2$.

Let us prove $\phi(I_1) = I_2$. Suppose that $\phi(I_1) \neq I_2$. We put $G \stackrel{\mathrm{def}}{=} I_2/\phi(I_1)$. Note that G is a cyclic group, and that $(m, p) = 1$, where $m \stackrel{\mathrm{def}}{=} \#(G) \geq 2$.

Suppose $g_X = 0$. Then we have $n_X \geq 3$. Let $N_2 \stackrel{\mathrm{def}}{=} D_m(\Pi_{X_2})$, $N_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(N_2) = D_m(\Pi_{X_1})$, and

$$f_{N_i}^\bullet : X_{N_i}^\bullet \to X_i^\bullet$$

the Galois admissible covering over k_i corresponding to N_i. Since the ramification index of each point of $f_{N_i}^{-1}(D_{X_i})$ is equal to m, we have

$$I_1 \not\subseteq N_1,\ I_2 \not\subseteq N_2,\ \phi(I_1) \subseteq N_2.$$

On the other hand, the isomorphism of maximal pro-prime-to-p quotients $\phi^{p'} : \Pi_{X_1^\bullet}^{p'} \xrightarrow{\sim} \Pi_{X_2^\bullet}^{p'}$ and $I_1 \not\subseteq N_1$ imply $\phi(I_1) \not\subseteq N_2$. This contradicts $\phi(I_1) \subseteq N_2$. Then we obtain $\phi(I_1) = I_2$.

Suppose that $g_X > 0$. We put

$$Q_2 \stackrel{\mathrm{def}}{=} \ker(\Pi_{X_2^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_2^\bullet}^{\mathrm{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z})$$

and $Q_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(Q_2)$. Then Lemma 4.4 implies $Q_1 = \ker(\Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_1^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_1^\bullet}^{\mathrm{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z})$. Note that the assumption $g_X > 0$ implies that $\Pi_{X_i^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_i^\bullet}^{\mathrm{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z}$ is not trivial. Then Q_i is an open normal subgroup of $\Pi_{X_i^\bullet}$. Moreover, the nontrivial

Galois admissible covering over k_i corresponding to Q_i is étale over D_{X_i}. Then we have $I_i \subseteq Q_i$ and $n_{X_{Q_i}} \geq 2$. Let $P_2 \stackrel{\text{def}}{=} D_m(Q_2)$, $P_1 \stackrel{\text{def}}{=} \phi^{-1}(P_2) = D_m(Q_1)$, and

$$g_i^\bullet : X_{P_i}^\bullet \to X_{Q_i}^\bullet$$

the Galois admissible covering over k_i corresponding to $P_i \subseteq Q_i$. Since the ramification index of each point of $g_i^{-1}(D_{X_{Q_i}})$ is equal to m, we have

$$I_1 \not\subseteq P_1, \ I_2 \not\subseteq P_2, \ \phi(I_1) \subseteq P_2.$$

On the other hand, the isomorphism of maximal pro-prime-to-p quotients $\phi|_{P_1}^{p'}$: $P_1^{p'} \xrightarrow{\sim} P_2^{p'}$ and $I_1 \not\subseteq P_1$ imply $\phi(I_1) \not\subseteq P_2$. This contradicts $\phi(I_1) \subseteq P_2$. Then we obtain $\phi(I_1) = I_2$. Thus, we may define the following map

$$\phi^{\text{edg,op}} : \text{Edg}^{\text{op}}(\Pi_{X_1^\bullet}) \to \text{Edg}^{\text{op}}(\Pi_{X_2^\bullet}), \ I_1 \mapsto I_2 \stackrel{\text{def}}{=} \phi(I_1).$$

Next, we will prove that $\phi^{\text{edg,op}}$ is a surjection. Let ℓ be a prime number distinct from p and $pr_i^\ell : \Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^\ell$ the maximal pro-ℓ quotient. Let $J_2 \in \text{Edg}^{\text{op}}(\Pi_{X_2^\bullet})$ be an arbitrary subgroup, $\overline{J}_2^\ell \stackrel{\text{def}}{=} pr_2^\ell(J_2)$ the image of J_2, and $\mathcal{C}_{\Pi_{X_i^\bullet}}^\ell \stackrel{\text{def}}{=} \{\overline{H}_i \stackrel{\text{def}}{=} pr_i^\ell(H_i)\}_{H_i \in \mathcal{C}_{\Pi_{X_i^\bullet}}}$, where $\mathcal{C}_{\Pi_{X_i^\bullet}}$ is the set of normal subgroups of $\Pi_{X_i^\bullet}$ defined above. Note that $\mathcal{C}_{\Pi_{X_i^\bullet}}^\ell$ is a cofinal system of $\Pi_{X_i^\bullet}^\ell$, and that $\overline{H}_1 = (\phi^\ell)^{-1}(\overline{H}_2)$.

Let $\overline{H}_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}^\ell$, $\overline{N}_2 \stackrel{\text{def}}{=} \overline{J}_2^\ell \overline{H}_2 \supseteq \overline{H}_2$, $\overline{N}_1 \stackrel{\text{def}}{=} (\phi^\ell)^{-1}(\overline{N}_2) \supseteq \overline{H}_1$, and $N_i \stackrel{\text{def}}{=} (pr_i^\ell)^{-1}(\overline{N}_i)$. We have that $G \stackrel{\text{def}}{=} \overline{N}_1/\overline{H}_1 = N_1/H_1 = \overline{N}_2/\overline{H}_2 = N_2/H_2$ is a cyclic ℓ-group. Write

$$g_{H_i,N_i}^\bullet : X_{H_i}^\bullet \to X_{N_i}^\bullet$$

for the Galois admissible covering over k_i with Galois group G. Since $J_2 \in \text{Edg}^{\text{op}}(\Pi_{X_2^\bullet})$, we obtain that $g_{H_2,N_2}^\bullet$ is totally ramified at a marked point of $X_{H_2}^\bullet$. We put

$$\text{Edg}^{\text{op},\ell,\text{ab}}(N_i) \stackrel{\text{def}}{=} \{\text{the image of } I \text{ of}$$

$$\text{the natural homomorphism } N_i \twoheadrightarrow N_i^{\ell,\text{ab}} \mid I \in \text{Edg}^{\text{op}}(N_i)\}.$$

We have $\#(\text{Edg}^{\text{op},\ell,\text{ab}}(N_i)) = n_{X_{N_i}}$. Then the composition of the following natural homomorphisms

$$\bigoplus_{I_{N_2} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_2)} I_{N_2} \to N_2^{\ell,\text{ab}} \twoheadrightarrow G$$

is a surjection. By applying Lemma 4.4, we obtain that the isomorphism ϕ^ℓ induces an isomorphism

$$\text{Im}(\bigoplus_{I_{N_1} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_1)} I_{N_1} \to N_1^{\ell,\text{ab}}) \xrightarrow{\sim} \text{Im}(\bigoplus_{I_{N_2} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_2)} I_{N_2} \to N_2^{\ell,\text{ab}}).$$

Then the composition of the following natural homomorphisms

$$\bigoplus_{I_{N_1} \in \mathrm{Edg}^{\mathrm{op},\ell,\mathrm{ab}}(N_1)} I_{N_1} \to N_1^{\ell,\mathrm{ab}} \twoheadrightarrow G$$

is also a surjection. Since G is a cyclic ℓ-group, there exists $I'_{N_1} \in \mathrm{Edg}^{\mathrm{op},\ell,\mathrm{ab}}(N_1)$ such that the composition $I'_{N_1} \hookrightarrow N_1^{\ell,\mathrm{ab}} \twoheadrightarrow G$ is a surjection. This means that $g^\bullet_{H_1,N_1}$ is also totally ramified at a marked point of $X^\bullet_{H_1}$.

We put

$$E_{\overline{H}_1} \stackrel{\mathrm{def}}{=} \{x_1 \in D_{X_{H_1}} \mid g^\bullet_{H_1,N_1} \text{ is totally ramified at } x_1\}.$$

Then we have that $E_{\overline{H}_1}$ is a non-empty finite set. Thus, we obtain

$$\varprojlim_{\overline{H}_1 \in \mathcal{C}^\ell_{\Pi_{X_1^\bullet}}} E_{\overline{H}_1} \neq \emptyset.$$

Note that we have a commutative diagram

$$\begin{array}{ccc} \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\ {\scriptstyle pr_1^\ell}\downarrow & & {\scriptstyle pr_2^\ell}\downarrow \\ \Pi^\ell_{X_1^\bullet} & \xrightarrow{\phi^\ell} & \Pi^\ell_{X_2^\bullet}. \end{array}$$

Then there exists $J_1 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})$ such that $pr_2^\ell(\phi(J_1)) = \phi^\ell(pr_1^\ell(J_1)) = \overline{J}_2^\ell$. Since $\phi(J_1) \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$, by applying Lemma 4.10, we have $\phi(J_1) = J_2$. Then $\phi^{\mathrm{edg,op}}$ is a surjection. Moreover, Theorem 4.2 implies that $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. This completes the proof of the first part of the theorem.

Let us prove the "moreover" part of the theorem. We see that

$$\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$$

is compatible with the natural actions of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, respectively. By using the surjectivity of $\phi^{\mathrm{edg,op}}$, we obtain a surjection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$$

of the sets of open edges of dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$, where $(-)^{\mathrm{sg}}$ means "semi-graph". Moreover, since $n_X = \#(e^{\mathrm{op}}(\Gamma_{X_1^\bullet})) = \#(e^{\mathrm{op}}(\Gamma_{X_2^\bullet}))$, we have that $\phi^{\mathrm{sg,op}}$ is a bijection. On the other hand, Theorem 4.2 implies that $e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. This completes the proof of the theorem. $\square$

Corollary 4.12. *We maintain the notation introduced above. Let $H_2 \subseteq \Pi_{X_1^\bullet}$ be an arbitrary open subgroup and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_2^\bullet}$. Then we have*

$$\gamma^{\max}(H_1) = \gamma^{\max}(H_2).$$

Proof. By Theorem 4.11, we obtain $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then Theorem 2.1 (a) implies $\gamma^{\max}(H_1) = \gamma^{\max}(H_2)$. □

4.3. **Reconstructions of field structures.**

4.3.1. **Settings.** We maintain the settings introduced in 4.2.1.

4.3.2. Let $\widehat{X}_i^\bullet = (\widehat{X}_i, D_{\widehat{X}_i})$, $i \in \{1, 2\}$, be the universal admissible (resp. the universal solvable admissible) covering associated to $\Pi_{X_i^\bullet}$ (1.2.10) if $\Pi_{X_i^\bullet}$ is the admissible (resp. solvable admissible) fundamental group of $X_i^\bullet$. Let $e_i \in e^{\text{op}}(\Gamma_{X_i^\bullet})$, $\widehat{e}_i \in e^{\text{op}}(\Gamma_{\widehat{X}_i^\bullet})$ over e_i, and $I_{\widehat{e}_i} \in \text{Edg}^{\text{op}}(\Pi_{X_i^\bullet})$ such that $\phi(I_{\widehat{e}_1}) = I_{\widehat{e}_2}$. Write $\overline{\mathbb{F}}_{p,i}$ for the algebraic closure of $\mathbb{F}_p$ in k_i. We put

$$\mathbb{F}_{\widehat{e}_i} \overset{\text{def}}{=} (I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widehat{e}_i}\},$$

where $\{*_{\widehat{e}_i}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_i^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with

$$\bigcup_{(p,m)=1} \mu_m(\overline{\mathbb{F}}_{p,i}).$$

Moreover, let $a_{\widehat{e}_i}$ be a generator of $I_{\widehat{e}_i}$. Then we have a natural bijection

$$I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \overset{\sim}{\to} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}, \ a_{\widehat{e}_i} \otimes 1 \mapsto 1 \otimes 1.$$

Thus, we obtain the following bijections

$$I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \overset{\sim}{\to} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \overset{\sim}{\to} \bigcup_{(p,m)=1} \mu_m(k_i) \overset{\sim}{\to} \overline{\mathbb{F}}_{p,i}^{\times}.$$

This means that $\mathbb{F}_{\widehat{e}_i}$ can be identified with $\overline{\mathbb{F}}_{p,i}$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}$, and whose zero element is $*_{\widehat{e}_i}$.

4.3.3. An important consequence of Theorem 4.11 is as follows.

Theorem 4.13. *We maintain the settings introduced in 4.2.1 and the notation introduced above. Then the field structure of $\mathbb{F}_{\widehat{e}_i}$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. Moreover, ϕ induces a field isomorphism*

$$\phi^{\text{fd}}_{\widehat{e}_1, \widehat{e}_2} : \mathbb{F}_{\widehat{e}_1} \overset{\sim}{\to} \mathbb{F}_{\widehat{e}_2}$$

group-theoretically, where "fd" means "field".

Proof. Firstly, we claim that we may assume $n_X \geq 3$. If $g_X = 0$, then $n_X \geq 3$. Suppose that $g_X \geq 1$. Theorem 4.11 implies that $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces an open continuous surjection $\phi^{\mathrm{cpt}} : \Pi^{\mathrm{cpt}}_{X_1^\bullet} \twoheadrightarrow \Pi^{\mathrm{cpt}}_{X_2^\bullet}$ (1.2.7). Let $H'_2 \subseteq \Pi^{\mathrm{cpt}}_{X_2^\bullet}$ be an open normal subgroup such that $\#(\Pi^{\mathrm{cpt}}_{X_2^\bullet}/H'_2) \geq 3$ and $H'_1 \overset{\mathrm{def}}{=} (\phi^{\mathrm{cpt}})^{-1}(H'_2)$. Write $H_i \subseteq \Pi_{X_i^\bullet}$, $i \in \{1, 2\}$, for the inverse image of H'_i of the natural surjection $\Pi_{X_2^\bullet} \twoheadrightarrow \Pi^{\mathrm{cpt}}_{X_2^\bullet}$, and $X^\bullet_{H_i}$ for the pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i corresponding to H_i. Note that $g_{X_{H_1}} = g_{X_{H_2}} \geq 1$ and $n_{X_{H_1}} = n_{X_{H_2}} \geq 3$. By replacing $X_i^\bullet$ by $X^\bullet_{H_i}$, we may assume $n_X \geq 3$.

Second, we claim that we may assume $n_X = 3$. By applying Theorem 4.11, ϕ induces a bijection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet}).$$

Let $E_{X_1} \overset{\mathrm{def}}{=} \{e_{1,1}, e_{1,2}, e_{1,3}\} \subseteq e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ and $E_{X_2} \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(E_{X_1}) \subseteq e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$. Write $D'_{X_i} \subseteq D_{X_i}$ for the set of marked points of $X_i^\bullet$ corresponding to E_{X_i}. Then (X_i, D'_{X_i}) is a pointed semi-stable curve of type $(g_X, 3)$ over k_i. Let $X^\bullet_{\mathrm{st},i}$ be the pointed stable curve of type $(g_X, 3)$ over k_i associated to (X_i, D'_{X_i}) (1.2.1). Write I_i for the closed subgroup of $\Pi_{X_i^\bullet}$ generated by the subgroups $I_{\widehat{e}} \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$, where the image of $\widehat{e}$ in $e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$ is contained in $e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus E_{X_i}$. Then we have a natural isomorphism

$$\Pi_{X^\bullet_{\mathrm{st},i}} \cong \Pi_{(X_i, D'_{X_i})} \cong \Pi_{X_i^\bullet}/I_i.$$

Moreover, Theorem 4.11 implies that ϕ induces a surjective open continuous homomorphism

$$\phi' : \Pi_{X^\bullet_{\mathrm{st},1}} \twoheadrightarrow \Pi_{X^\bullet_{\mathrm{st},2}}.$$

Thus, by replacing $X_i^\bullet$, $\Pi_{X_i^\bullet}$, and ϕ by $X^\bullet_{\mathrm{st},i}$, $\Pi_{X^\bullet_{\mathrm{st},i}}$, and ϕ', respectively, we may assume $n_X = 3$.

Then the theorem follows immediately from [Y5, Theorem 6.4 and Remark 6.4.1]. □

Remark 4.13.1. Theorem 4.11 and Theorem 4.13 were obtained by Tamagawa in a special case where $X_i^\bullet$, $i \in \{1, 2\}$, is *non-singular* and ϕ is an *isomorphism* ([T4, Theorem 5.2 and Proposition 5.3]). Those results is the most important step in Tamagawa's proof of the weak Isom-version conjecture for *smooth* pointed stable curves ([T4, Theorem 0.2]).

The formula for $\mathrm{Avr}_p(\Pi_{X_i^\bullet})$ of *smooth* pointed stable curves ([T4, Theorem 0.5]) plays a central role in Tamagawa's proofs of [T4, Theorem 5.2 and Proposition 5.3]. On the other hand, *even through ϕ is an isomorphism*, the methods of [T4] cannot be generalized to the case of *arbitrary* pointed stable curves, since $\mathrm{Avr}_p(\Pi_{X_i^\bullet})$ depends not only on the type (g_X, n_X) but also on the structure of the dual semi-graph $\Gamma_{X_i^\bullet}$ in general (see [Y3, Theorem 1.3 and Theorem 1.4]).

5. Combinatorial Grothendieck conjecture for open continuous homomorphisms

In this section, we will prove a version of combinatorial Grothendicek conjecture for open continuous homomorphisms under certain assumption. Moreover, in the present section, *all fundamental groups are solvable admissible fundamental groups unless indicated otherwise.* The main results of the present section are Theorem 5.26 and Theorem 5.30.

5.1. **Cohomology classes and sets of vertices.**

5.1.1. **Settings.** Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ the solvable admissible fundamental group of $X^\bullet$.

5.1.2. Let ℓ be a prime number. Recall that $\widetilde{X}_v^\bullet$ denotes the smooth pointed stable curve of type (g_v, n_v) associated to $v \in v(\Gamma_{X^\bullet})$ (1.1.3). We put (see 1.2.7 for $\Pi_{X^\bullet}^{\text{ét}}$, $\Pi_{X^\bullet}^{\text{top}}$)

$$v(\Gamma_{X^\bullet})^{>0,\ell} \overset{\text{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \dim_{\mathbb{F}_\ell}(\text{Hom}(\Pi^{\text{ét}}_{\widetilde{X}_v^\bullet}, \mathbb{Z}/\ell\mathbb{Z})) > 0\} = \{v \in v(\Gamma_{X^\bullet}) \mid g_v > 0\},$$

$$M_{X^\bullet}^{\text{ét}} \overset{\text{def}}{=} \text{Hom}(\Pi_{X^\bullet}^{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}),\ M_{X^\bullet}^{\text{top}} \overset{\text{def}}{=} \text{Hom}(\Pi_{X^\bullet}^{\text{top}}, \mathbb{Z}/\ell\mathbb{Z}).$$

On the other hand, we have the natural isomorphisms $\text{Hom}(\Pi^{\text{ét}}_{\widetilde{X}_v^\bullet}, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1_{\text{ét}}(\widetilde{X}_v, \mathbb{Z}/\ell\mathbb{Z})$, $M_{X^\bullet}^{\text{ét}} \cong H^1_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z})$, and $M_{X^\bullet}^{\text{top}} \cong H^1(\Gamma_{X^\bullet}, \mathbb{Z}/\ell\mathbb{Z})$. In the theory of anabelian geometry, since we want to emphasize the objects under consideration are arose from various fundamental groups, we do not use the standard notation $H^1_{\text{ét}}(\widetilde{X}_v, \mathbb{Z}/\ell\mathbb{Z})$, $H^1_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z})$, and $H^1(\Gamma_{X^\bullet}, \mathbb{Z}/\ell\mathbb{Z})$. Moreover, there is an injection $M_{X^\bullet}^{\text{top}} \hookrightarrow M_{X^\bullet}^{\text{ét}}$ induced by the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{top}}$. We put

$$M_{X^\bullet}^{\text{nt}} \overset{\text{def}}{=} \text{coker}(M_{X^\bullet}^{\text{top}} \hookrightarrow M_{X^\bullet}^{\text{ét}}),$$

where $(-)^{\text{nt}}$ means "non-top".

A non-zero element of $M_{X^\bullet}^{\text{ét}}$ corresponds to a Galois étale covering of the underlying curve X of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. A non-zero element of $M_{X^\bullet}^{\text{top}}$ corresponds to a Galois étale covering of the underlying curve X of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the map of dual semi-graphs is a topological covering.

5.1.3. Let $V_{X,\ell}^* \subseteq M_{X^\bullet}^{\text{ét}}$ be the subset of elements of $M_{X^\bullet}^{\text{ét}}$ whose images of $M_{X^\bullet}^{\text{ét}} \twoheadrightarrow M_{X^\bullet}^{\text{nt}}$ are not 0. Then an element of $V_{X,\ell}^*$ corresponds to a Galois étale covering of the underlying curve X of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the map of dual semi-graphs is not a topological covering.

Let $\alpha \in V_{X,\ell}^*$ and

$$f_\alpha^\bullet : X_\alpha^\bullet \to X^\bullet$$

the Galois étale covering corresponding to α. Denote by $\Gamma_{X^\bullet_\alpha}$ the dual semi-graph of $X^\bullet_\alpha$. We define a map

$$\iota : V^*_{X,\ell} \to \mathbb{Z}_{>0},\ \alpha \mapsto \#(v(\Gamma_{X^\bullet_\alpha})).$$

Furthermore, we put

$$V^\star_{X,\ell} \overset{\text{def}}{=} \{\alpha \in V^*_{X,\ell} \mid \iota \text{ attains its maximum}\} = \{\alpha \in V^*_{X,\ell} \mid \iota(\alpha) = \ell\#(v(\Gamma_{X^\bullet}))-\ell+1\}.$$

For each $\alpha \in V^\star_{X,\ell}$, $\iota(\alpha) = \ell\#(v(\Gamma_{X^\bullet})) - \ell + 1$ implies that there exists a unique irreducible component $Z \subseteq X_\alpha$ whose decomposition group under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is not trivial. Then we have (see 1.1.5 for $v^{\rm ra}_{f_\alpha}$)

$$V^\star_{X,\ell} = \{\alpha \in V^*_{X,\ell} \mid \#(v^{\rm ra}_{f_\alpha}) = 1\}.$$

Let v_α be the unique element of $v^{\rm ra}_{f_\alpha}$ (i.e. $X_{v_\alpha} = f_\alpha(Z)$). Then we have $v_\alpha \in v(\Gamma_{X^\bullet})^{>0,\ell}$. This means that $V^\star_{X,\ell} \neq \emptyset$ if and only if $v(\Gamma_{X^\bullet})^{>0,\ell} \neq \emptyset$.

5.1.4. Let S, S' be sets. We shall call $f : S \to S'$ a *quasi-map* if f is a map from some subset $S_1 \subseteq S$ to S'. Moreover, suppose that $S^{\max}$ is the maximal subset of S such that f is a map from $S^{\max}$ to S'. Let $S^* \overset{\text{def}}{=} S \setminus S^{\max}$. Then we shall write $f(s) = \emptyset$ for all $s \in S^*$.

Let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Write $f^{\rm sg}_H : \Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by the admissible covering $f^\bullet_H : X^\bullet_H \to X^\bullet$ over k corresponding to H. We define a quasi-map (i.e. we allow that an element maps to empty set)

$$f^{\text{ver},\ell}_H : v(\Gamma_{X^\bullet_H})^{>0,\ell} \to v(\Gamma_{X^\bullet})^{>0,\ell}$$

as follows: Let $v_H \in v(\Gamma_{X^\bullet_H})^{>0,\ell}$ and $v \overset{\text{def}}{=} f^{\rm sg}_H(v_H) \in v(\Gamma_{X^\bullet_H})$. Then we have $f^{\text{ver},\ell}_H(v_H) = v$ if $\dim_{\mathbb{F}_\ell}(\text{Hom}(\Pi^{\text{ét}}_{\widetilde{X}^\bullet_v}, \mathbb{Z}/\ell\mathbb{Z})) \neq 0$; otherwise, $f^{\text{ver},\ell}_H(v_H) = \emptyset$. Moreover, if $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup, then $v(\Gamma_{X^\bullet_H})^{>0,\ell}$ admits a natural action of $\Pi_{X^\bullet}/H$.

Proposition 5.1. *(a) We define a pre-equivalence relation $\sim$ on $V^\star_{X,\ell}$ as follows:*

> *Let $\alpha, \beta \in V^\star_{X,\ell}$. We have that $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}^\times_\ell$ for which $\lambda\alpha + \mu\beta \in V^*_{X,\ell}$, $\lambda\alpha + \mu\beta \in V^\star_{X,\ell}$.*

Then the pre-equivalence relation $\sim$ on $V^\star_{X,\ell}$ is an equivalence relation.

(b) We denote by $V_{X,\ell}$ the quotient set of $V^\star_{X,\ell}$ by $\sim$ defined in (a). Then we have a natural bijection

$$\kappa_{X,\ell} : V_{X,\ell} \xrightarrow{\sim} v(\Gamma_{X^\bullet})^{>0,\ell},\ [\alpha] \mapsto v_\alpha,$$

where $[\alpha]$ denotes the equivalence class of α.

(c) Let ℓ, ℓ' be prime numbers distinct from each other. Suppose that $\ell' \neq p$. Then we have a natural injection

$$V_{X,\ell} \hookrightarrow V_{X,\ell'},$$

which is a bijection if $\ell \neq p$, and which fits into the following commutative diagram:

$$\begin{array}{ccc} V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>0,\ell} \\ \downarrow & & \downarrow \\ V_{X,\ell'} & \xrightarrow{\kappa_{X,\ell'}} & v(\Gamma_{X^\bullet})^{>0,\ell'}, \end{array}$$

where the vertical map of the right-hand side is the natural injection induced by the definitions of $v(\Gamma_{X^\bullet})^{>0,\ell}$ and $v(\Gamma_{X^\bullet})^{>0,\ell'}$.

(d) Let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Suppose $([\Pi_{X^\bullet} : H], \ell) = 1$. Then the natural injection $H \hookrightarrow \Pi_{X^\bullet}$ induces a map

$$\gamma_H^{\mathrm{ver},\ell} : V_{X_H,\ell} \to V_{X,\ell}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} V_{X_H,\ell} & \xrightarrow{\kappa_{X_H,\ell}} & v(\Gamma_{X_H^\bullet})^{>0,\ell} \\ \gamma_H^{\mathrm{ver},\ell} \downarrow & & f_H^{\mathrm{ver},\ell} \downarrow \\ V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>0,\ell}. \end{array}$$

Moreover, suppose that $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup. Then $V_{X_H,\ell}$ admits an action of $\Pi_{X^\bullet}/H$ such that $\kappa_{X_H,\ell}$ is compatible with $\Pi_{X^\bullet}/H$-actions (i.e. $\kappa_{X_H,\ell}$ is $\Pi_{X^\bullet}/H$-equivariant).

Proof. See [Y6, Proposition 2.1, Remark 2.1.1, and Remark 2.1.2]. □

Remark 5.1.1. By applying Theorem 4.2, we have that $\Pi^{\text{ét}}_{X^\bullet}$, $\Pi^{\text{top}}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Then we obtain that $V_{X,\ell}$ (or $v(\Gamma_{X^\bullet})^{>0,\ell}$) can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, for every open subgroup $H \subseteq \Pi_{X^\bullet}$, the map

$$\gamma_H^{\mathrm{ver},\ell} : V_{X_H,\ell} \to V_{X,\ell}$$

constructed in Proposition 5.1 (d) can be reconstructed group-theoretically from the natural inclusion $H \hookrightarrow \Pi_{X^\bullet}$.

5.2. **Cohomology classes and sets of closed edges.**

5.2.1. **Settings.** We maintain the settings introduced in 5.1.1. Moreover, in this subsection, we suppose that the genus of the normalization of each irreducible component of X is *positive* (i.e. $v(\Gamma_{X^\bullet}) = v(\Gamma_{X^\bullet})^{>0,\ell}$ (5.1.2) if $\ell \neq p$), and that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-*connected* (see 1.1.1 (b) (c)).

5.2.2. We shall say that

$$\mathfrak{T}_{X^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \to X^\bullet)$$

is an *edge-triple* associated to $X^\bullet$ if the following conditions are satisfied:

(i) ℓ and d are prime numbers distinct from each other and from p.

(ii) $\ell \equiv 1 \pmod{d}$; this means that all dth roots of unity are contained in $\mathbb{F}_\ell$. Moreover, we write $\mu_d \subseteq \mathbb{F}_\ell^\times$ for the subgroup of dth roots of unity.

(iii) $f_X^\bullet : Y^\bullet \to X^\bullet$ is a Galois admissible covering over k such that the Galois group is isomorphic to μ_d, that $f_X^\bullet$ is étale (i.e. f_X is étale), and that $\#(v_{f_X}^{\text{sp}}) = 0$ (see 1.1.5 for $v_{f_X}^{\text{sp}}$). Note that since $v(\Gamma_{X^\bullet}) = v(\Gamma_{X^\bullet})^{>0,d}$, we see that $f_X^\bullet$ exists.

5.2.3. We maintain the prime numbers ℓ and d introduced in 5.2.2. On the other hand, we shall say that

$$\mathfrak{T}_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_X})$$

is an *edge-triple* associated to $\Pi_{X^\bullet}$ if the following conditions are satisfied (see 1.2.7 for $\Pi_{X^\bullet}^{\text{ét}}$):

(i) $\alpha_{f_X} \in \text{Hom}(\Pi_{X^\bullet}^{\text{ét}}, \mathbb{Z}/d\mathbb{Z})$.

(ii) The composition of the natural homomorphisms $\Pi_{\widetilde{X}_v^\bullet}^{\text{ét}} \hookrightarrow \Pi_{X^\bullet}^{\text{ét}} \stackrel{\alpha_{f_X}}{\to} \mathbb{Z}/d\mathbb{Z}$ is a surjection for every $v \in v(\Gamma_{X^\bullet})$.

We see immediately that an edge-triple $\mathfrak{T}_{X^\bullet}$ associated to $X^\bullet$ is equivalent to an edge-triple $\mathfrak{T}_{\Pi_{X^\bullet}}$ associated to $\Pi_{X^\bullet}$. Moreover, $f_X^\bullet$ is the Galois admissible covering corresponding to the kernel of the composition of the natural homomorphisms $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \stackrel{\alpha_{f_X}}{\to} \mathbb{Z}/d\mathbb{Z}$.

5.2.4. **Further settings.** In the remainder of the present subsection, we fix an edge-triple

$$\mathfrak{T}_{\Pi_{X^\bullet}} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_X})$$

associated to $\Pi_{X^\bullet}$. Write $\mathfrak{T}_{X^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \to X^\bullet)$ for the edge-triple associated to $X^\bullet$ corresponding to $\mathfrak{T}_{\Pi_{X^\bullet}}$, (g_Y, n_Y) for the type of $Y^\bullet$, $\Gamma_{Y^\bullet}$ for the dual semi-graph of $Y^\bullet$, r_Y for the Betti number of $\Gamma_{Y^\bullet}$ (1.1.2), and $\Pi_{Y^\bullet}$ for the kernel of the composition of the homomorphisms $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \stackrel{\alpha_{f_X}}{\to} \mathbb{Z}/d\mathbb{Z}$ (i.e. the admissible (or solvable admissible) fundamental group of $Y^\bullet$).

5.2.5. We put

$$M_{Y^\bullet} \stackrel{\text{def}}{=} \text{Hom}(\Pi_{Y^\bullet}, \mathbb{Z}/\ell\mathbb{Z}).$$

There is a natural injection $M_{Y^\bullet}^{\text{ét}} \stackrel{\text{def}}{=} \text{Hom}(\Pi_{Y^\bullet}^{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}) \hookrightarrow M_{Y^\bullet}$ induced by the natural surjection $\Pi_{Y^\bullet} \twoheadrightarrow \Pi_{Y^\bullet}^{\text{ét}}$. Then we obtain an exact sequence

$$0 \to M_{Y^\bullet}^{\text{ét}} \to M_{Y^\bullet} \to M_{Y^\bullet}^{\text{ra}} \stackrel{\text{def}}{=} \text{coker}(M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}) \to 0$$

with a natural action of μ_d, where "ra" means "ramification". For any element of $M_{Y^\bullet}$, if the image of the element is not 0 in $M^{\rm ra}_{Y^\bullet}$, then the Galois admissible covering of $Y^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to the element is not étale.

5.2.6. Let $M^{\rm ra}_{Y^\bullet,\mu_d} \subseteq M^{\rm ra}_{Y^\bullet}$ be the subset of elements on which μ_d acts via the character $\mu_d \hookrightarrow \mathbb{F}_\ell^\times$. Write $E^*_{\mathfrak{T}_{\Pi_{X^\bullet}}} \subseteq M_{Y^\bullet}$ for the subset of elements whose images are nonzero elements of $M^{\rm ra}_{Y^\bullet,\mu_n}$.

Let $\alpha \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet}}}$. Write

$$g^\bullet_\alpha : Y^\bullet_\alpha \to Y^\bullet$$

for the Galois admissible covering over k corresponding to α. We define a map

$$\epsilon : E^*_{\mathfrak{T}_{\Pi_{X^\bullet}}} \to \mathbb{Z}_{\geq 0},\ \alpha \mapsto \#(e^{\rm op}(\Gamma_{Y^\bullet_\alpha}) \cup e^{\rm cl}(\Gamma_{Y^\bullet_\alpha})),$$

where $\Gamma_{Y^\bullet_\alpha}$ denotes the dual semi-graph of $Y^\bullet_\alpha$. We put (see 1.1.5 for $e^{\rm op,ra}_{g_\alpha}$ and $e^{\rm cl,ra}_{g_\alpha}$)

$$E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \overset{\rm def}{=} \{\alpha \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet}}} \mid \#(e^{\rm op,ra}_{g_\alpha}) = 0,\ \#(e^{\rm cl,ra}_{g_\alpha}) = d\}.$$

Note that $E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ is not an empty set. For each $\alpha \in E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, since the image of α is contained in $M^{\rm ra}_{Y^\bullet,\mu_d}$, we obtain that the action of μ_d on the set $\{y_e\}_{e \in e^{\rm cl,ra}_{g_\alpha}} \subseteq {\rm Nod}(Y^\bullet)$ is transitive, where ${\rm Nod}(-)$ denotes the set of nodes of $(-)$, and y_e denotes the node of $Y^\bullet$ corresponding to e. Then there exists a unique node x_α of $X^\bullet$ such that $f_X(y_e) = x_\alpha$ for all $y_e \in \{y_e\}_{e \in e^{\rm cl,ra}_{g_\alpha}}$. We denote by $e_\alpha \in e^{\rm cl}(\Gamma_{X^\bullet})$ the closed edge corresponding to x_α.

5.2.7. On the other hand, let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Write $f^{\rm sg}_H : \Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by the admissible covering $f^\bullet_H : X^\bullet_H \to X^\bullet$ over k corresponding to H. We shall denote by

$$f^{\rm cl}_H \overset{\rm def}{=} f^{\rm sg}_H|_{e^{\rm cl}(\Gamma_{X^\bullet_H})} : e^{\rm cl}(\Gamma_{X^\bullet_H}) \to e^{\rm cl}(\Gamma_{X^\bullet}).$$

Moreover, if $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup, then $e^{\rm cl}(\Gamma_{X^\bullet_H})$ admits a natural action of $\Pi_{X^\bullet}/H$.

Proposition 5.2. *(a) We define a pre-equivalence relation $\sim$ on $E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ as follows:*

> *Let $\alpha, \beta \in E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$. We have that $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, we have $\lambda\alpha + \mu\beta \in E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$.*

Then the pre-equivalence relation $\sim$ on $E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ is an equivalence relation.

(b) We denote by $E^{\rm cl}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ the quotient set of $E^{\rm cl,\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ by $\sim$ defined in (a). Then we have a natural bijection

$$\vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}} : E^{\rm cl}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \overset{\sim}{\to} e^{\rm cl}(\Gamma_{X^\bullet}),\ [\alpha] \mapsto e_\alpha,$$

where $[\alpha]$ *denotes the equivalence class of* α.

(c) Let $\mathfrak{T}'_{\Pi_{X^\bullet}}$ *be an arbitrary edge-triples associated to* $\Pi_{X^\bullet}$. *Then we have a natural bijection*

$$E^{\mathrm{cl}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} \xrightarrow{\sim} E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta_{\mathfrak{T}'_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \\ \downarrow & & \| \\ E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}). \end{array}$$

(d) Let $H \subseteq \Pi_{X^\bullet}$ *be an open subgroup. Suppose that* $([\Pi_{X^\bullet} : H], \ell) = ([\Pi_{X^\bullet} : H], d) = 1$. *We have that* $\mathfrak{T}_{X^\bullet}$ *associated to* $\Pi_{X^\bullet}$ *induces an edge-triple*

$$\mathfrak{T}_{X^\bullet_H} \overset{\mathrm{def}}{=} (\ell, d, f^\bullet_{X_H} : Y^\bullet_{X_H} \overset{\mathrm{def}}{=} Y^\bullet \times_{X^\bullet} X^\bullet_H \to X^\bullet_H)$$

associated to $X^\bullet_H$, *where* $Y^\bullet \times_{X^\bullet} X^\bullet_H$ *denotes the fiber product in the category of pointed stable curves. Write* $\mathfrak{T}_H$ *for the edge-triple associated to* H *corresponding to* $\mathfrak{T}_{X^\bullet_H}$. *Then the natural injection* $H \hookrightarrow \Pi_{X^\bullet}$ *induces a surjective map*

$$\gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}, H} : E^{\mathrm{cl}}_{\mathfrak{T}_H} \twoheadrightarrow E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}_H} & \xrightarrow{\vartheta_{\mathfrak{T}_H}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) \\ {\scriptstyle \gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}, H}} \downarrow & & {\scriptstyle f^{\mathrm{cl}}_H} \downarrow \\ E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}). \end{array}$$

Moreover, suppose that $H \subseteq \Pi_{X^\bullet}$ *is an open normal subgroup. Then* $E^{\mathrm{cl}}_{\mathfrak{T}_H}$ *admits an action of* $\Pi_{X^\bullet}/H$ *such that* $\vartheta_{\mathfrak{T}_H}$ *is compatible with* $\Pi_{X^\bullet}/H$*-actions (i.e.* $\vartheta_{\mathfrak{T}_H}$ *is* $\Pi_{X^\bullet}/H$*-equivariant).*

Proof. See [Y6, Proposition 2.2, Remark 2.2.1, and Remark 2.2.2]. □

Remark 5.2.1. By applying Theorem 4.2, we have that $\Pi^{\text{ét}}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Then $E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ (or $e^{\mathrm{cl}}(\Gamma_{X^\bullet})$) can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, for every open subgroup $H \subseteq \Pi_{X^\bullet}$, the map

$$\gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}, H}} : E^{\mathrm{cl}}_{\mathfrak{T}_H} \to E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

constructed in Proposition 5.2 (d) can be reconstructed group-theoretically from the natural inclusion $H \hookrightarrow \Pi_{X^\bullet}$.

5.2.8. Next, we calculate the cardinality $\#(E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}})$ of $E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}}$. We put

$$E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}} \mid e = e_\alpha\},\ e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

Note that $e = e_\alpha$, $\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}$, means that the Galois admissible covering $g^\bullet_\alpha : Y^\bullet_\alpha \to Y^\bullet$ over k induced by α is (totally) ramified over $f_X^{-1}(x_e)$, where x_e denotes the node of X corresponding to e. Moreover, we have the following disjoint union

$$E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}} = \bigsqcup_{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})} E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}.$$

Let $m \in \mathbb{Z}_{\geq 0}$ and $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. We shall put

$$E^{\mathrm{cl},\star,m}_{\mathfrak{I}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e} \mid \#(v^{\mathrm{sp}}_{g_\alpha}) = m\}.$$

Let $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ be a closed edge. Write Y_e for the normalization of the underlying curve Y of $Y^\bullet$ at $f_X^{-1}(x_e)$ and $\mathrm{nor}_e : Y_e \to Y$ for the resulting normalization morphism. Since the genus of the normalization of each irreducible component of $X^\bullet$ is positive, we obtain that the genus of the normalization of each irreducible component of Y_e is also positive. Moreover, since $\Gamma_{X^\bullet}$ is 2-connected, Y_e is connected.

Lemma 5.3. *We maintain the notation introduced above. Let $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ be a closed edge. Then we have*

$$\#(E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}) = \ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}.$$

Moreover, we have

$$\#(E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}}) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}))(\ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}).$$

Proof. Write $R_e \subseteq Y_e$ for the set of closed subset $(f_X \circ \mathrm{nor}_e)^{-1}(x_e)$. Then $E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}$ can be naturally regarded as a subset of $H^1_{\text{ét}}(Y_e \setminus R_e, \mathbb{Z}/\ell\mathbb{Z})$ via the natural open immersion $Y_e \setminus R_e \hookrightarrow Y_e$. Write L_e for the $\mathbb{F}_\ell$-linear subspace spanned by $E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}$ in $H^1_{\text{ét}}(Y_e \setminus R_e, \mathbb{Z}/\ell\mathbb{Z})$. Then we see $E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e} = L_e \setminus H^1_{\text{ét}}(Y_e, \mathbb{Z}/\ell\mathbb{Z})$.

Write H^{ra}_e for the cokernel of the natural inclusion $H^1_{\text{ét}}(Y_e, \mathbb{Z}/\ell\mathbb{Z}) \hookrightarrow L_e$. We obtain an exact sequence as follows:

$$0 \to H^1_{\text{ét}}(Y_e, \mathbb{Z}/\ell\mathbb{Z}) \to L_e \to H^{\mathrm{ra}}_e \to 0.$$

On the other hand, since the action of μ_d on $f^{-1}(x_e)$ is translative, the structure of the maximal pro-ℓ quotient $\Pi^\ell_{Y^\bullet}$ of $\Pi_{Y^\bullet}$ (1.2.4) implies $\dim_{\mathbb{F}_\ell}(H^{\mathrm{ra}}_e) = 1$. Since $\dim_{\mathbb{F}_\ell}(H^1_{\text{ét}}(Y_e, \mathbb{Z}/\ell\mathbb{Z})) = 2(g_Y - d) - (r_Y - d) = 2g_Y - d - r_Y$, we obtain

$$\#(E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}) = \ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}.$$

Thus, we have

$$\#(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet}))(\ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}).$$

This completes the proof of the lemma. □

5.2.9. We also introduce some notation concerning open edges. We put

$$E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \overset{\mathrm{def}}{=} \{\alpha \in E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \mid \#(e^{\mathrm{op,ra}}_{g_\alpha}) = d,\ \#(e^{\mathrm{cl,ra}}_{g_\alpha}) = 0\}.$$

Note that $E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ is not an empty set if $n_X \neq 0$. For each $\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, since the image of α is contained in $M^{\mathrm{ra}}_{Y^\bullet,\mu_d}$, we obtain that the action of μ_d on the set $\{y_e\}_{e\in e^{\mathrm{op,ra}}_{g_\alpha}} \subseteq D_Y$ is transitive, where y_e denotes the marked point of $Y^\bullet$ corresponding to e. Then there exists a unique marked point $x_\alpha \in D_X$ of $X^\bullet$ such that $f_X(y_e) = x_\alpha$ for every $y_e \in \{y_e\}_{e\in e^{\mathrm{op,ra}}_{g_\alpha}}$. We denote by $e_\alpha \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ the open edge corresponding to x_α. Moreover, we put

$$E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \mid e = e_\alpha\},\ e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}).$$

Note that $e = e_\alpha$, $\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e}$, means that the Galois admissible covering $g^\bullet_\alpha : Y^\bullet_\alpha \to Y^\bullet$ over k induced by α is (totally) ramified over $f_X^{-1}(x_e)$, where x_e denotes the marked point of $X^\bullet$ corresponding to e. Moreover, we have the following disjoint union

$$E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} = \bigsqcup_{e\in e^{\mathrm{op}}(\Gamma_{X^\bullet})} E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e}.$$

Let $m \in \mathbb{Z}_{\geq 0}$ and $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. We shall put

$$E^{\mathrm{op},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \mid \#(v^{\mathrm{sp}}_{g_\alpha}) = m\}.$$

5.3. **Three conditions.** We introduce the following conditions concerning pointed stable curves. Moreover, one of the main results of the present section (Theorem 5.26) will be proved under those conditions.

5.3.1. Let $W_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve over k_i of type (g_{W_i}, n_{W_i}), $\Gamma_{W_i^\bullet}$ the dual semi-graph of $W_i^\bullet$, and $\Pi_{W_i^\bullet}$ the solvable admissible fundamental group of $W_i^\bullet$. Let $H_i \subseteq \Pi_{W_i^\bullet}$ be an open subgroup, $W^\bullet_{H_i}$ the admissible covering of $W_i^\bullet$ corresponding to H_i, and $\Gamma_{W^\bullet_{H_i}}$ the dual semi-graph of $W^\bullet_{H_i}$.

Condition A . We shall say that $W_i^\bullet$ satisfies Condition A if the following conditions are satisfied:

(i) The genus of the normalization of each irreducible component of W_i is positive.
(ii) Every irreducible component of W_i is smooth over k_i.
(iii) $\Gamma^{\mathrm{cpt}}_{W_i^\bullet}$ is 2-connected (1.1.1 (b) (c)).

(iv) $\#(v(\Gamma_{W_i^\bullet})^{b\leq 1}) = 0$ (1.1.1 (c)).

Condition B . We shall say that $W_i^\bullet$ satisfies Condition B if $\Gamma^{\rm cpt}_{W^\bullet_{H_i}}$ is 2-connected for every open subgroup $H \subseteq \Pi_{W^\bullet}$.

Condition C . We shall say that $W_1^\bullet$ and $W_2^\bullet$ satisfy Condition C if the following conditions are satisfied:

(i) $(g_{W_1}, n_{W_1}) = (g_{W_2}, n_{W_2})$.
(ii) $\#(v(\Gamma_{W_1^\bullet})) = \#(v(\Gamma_{W_2^\bullet}))$.
(iii) $\#(e^{\rm cl}(\Gamma_{W_1^\bullet})) = \#(e^{\rm cl}(\Gamma_{W_2^\bullet}))$.

5.3.2. We maintain the notation introduced above, then we have the following lemma.

Lemma 5.4. *Let $m >> 0$ be a positive natural number prime to p and $H_i \overset{\rm def}{=} D_m^{(3)}(\Pi_{W_i^\bullet}) \subseteq \Pi_{W_i^\bullet}$ (see Definition 4.8 for $D_m^{(3)}(\Pi_{W_i^\bullet})$). Then we have that $W^\bullet_{H_i}$ satisfies Condition A, and that the Betti number of the dual semi-graph of $W^\bullet_{H_i}$ is positive.*

Proof. If $W_i^\bullet$ is smooth over k_i, then the lemma is trivial. We may assume that $W_i^\bullet$ is singular. Let $Q_i \overset{\rm def}{=} D_m^{(2)}(\Pi_{W_i^\bullet}) \subseteq \Pi_{W_i^\bullet}$. By the structure of $\Pi^{p'}_{W^\bullet}$ (1.2.4), it is easy to see that $W^\bullet_{Q_i}$ satisfies Condition A (i) (ii) (iv), and that the Betti number of the dual semi-graph of $W^\bullet_{Q_i}$ is positive. Write $f^\bullet : W^\bullet_{H_i} \to W^\bullet_{Q_i}$ for the Galois admissible covering over k_i with Galois group G induced by the natural inclusion $H_i \hookrightarrow Q_i$ and $f^{\rm sg} : \Gamma_{W^\bullet_{H_i}} \to \Gamma_{W^\bullet_{Q_i}}$ for the map of dual semi-graphs of $W^\bullet_{H_i}$ and $W^\bullet_{Q_i}$ induced by $f^\bullet$.

Let $v \in v(\Gamma_{W^\bullet_{Q_i}})$ be an arbitrary vertex. Note that $\#((f^{\rm sg})^{-1}(v)) \geq 2$. Since $f^\bullet$ is Galois, to verify that $\Gamma^{\rm cpt}_{W^\bullet_{H_i}}$ is 2-connected, we only need to prove that $\Gamma^{\rm cpt}_{W^\bullet_{H_i}} \setminus \{w\}$ is connected for a vertex $w \in (f^{\rm sg})^{-1}(v)$. Moreover, since m is prime to p, to verify $\Gamma^{\rm cpt}_{W^\bullet_{H_i}} \setminus \{w\}$ is connected, we may assume $\#(v(\Gamma^\bullet_{W_{Q_i}})) = 2$ and $\#(e^{\rm cl}(\Gamma^\bullet_{W_{Q_i}})) \geq 2$.

Let $C, D \subseteq \Gamma^{\rm cpt}_{W^\bullet_{H_i}} \setminus \{w\}$ be connected components. Suppose that $C \neq D$. Note that since $f^\bullet$ is Galois and $\Pi^{\rm ét}_{\widetilde{W}^\bullet_{Q_i,v}}$ is not trivial (i.e. Condition A (i)), C is isomorphic to D as semi-graphs. Let $w' \in ((f^{\rm sg})^{-1}(v) \setminus \{w\}) \cap C$, and let $C_{w'}$ be a connected component of $C \setminus \{w'\}$ such that there exists a closed edge which meets $C_{w'}$ and w. Then we obtain that there exists a connected component C' of $\Gamma^{\rm cpt}_{W^\bullet_{H_i}} \setminus \{w'\}$ which contains w, D, and $C_{w'}$. On the other hand, since $f^\bullet$ is Galois, C' is isomorphic to D as semi-graphs, which is impossible as D and C' are finite semi-graphs. Then we have $C = D$. We complete the proof of the lemma. □

5.4. **Reconstructions of topological and combinatorial data.** In this subsection, we prove that sets of vertices, sets of closed edges, and sets of genus can be reconstructed group-theoretically from an open continuous homomorphism of solvable admissible fundamental groups. The main results of the present subsection are Theorem 5.12, Theorem 5.14, and Theorem 5.17.

5.4.1. **Settings.** Let $i \in \{1,2\}$, and let k_i be an algebraically closed field of characteristic $p>0$ and ℓ a prime number distinct from p. Let $X_i^\bullet$ be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X_i^\bullet}$ the solvable admissible fundamental group of $X_i^\bullet$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X_i^\bullet}$ (1.1.2). Moreover, let $v_i \in v(\Gamma_{X_i^\bullet})$, $\widetilde{X}^\bullet_{i,v_i}$ the smooth pointed stable curve of type (g_{i,v_i}, n_{i,v_i}) over k_i associated to v_i (1.1.3), and σ_{i,v_i} the p-rank of $\widetilde{X}^\bullet_{i,v_i}$ (2.1.1).

We suppose that $X_1^\bullet$ *and* $X_2^\bullet$ *satisfy Condition A, Condition B, and Condition C* introduced in 5.3.1. Moreover, let

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism of the solvable admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$, and

$$(g_X, n_X) \overset{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

Note that $r_{X_1} = r_{X_2}$, and that by Lemma 4.3, ϕ is a *surjective* open continuous homomorphism.

5.4.2. Firstly, we have the following lemma.

Lemma 5.5. *We maintain the notation introduced above. Then we have (see 2.2.1 for* $\mathrm{Avr}_p(\Pi_{X_i^\bullet})$*)*

$$\mathrm{Avr}_p(\Pi_{X_i^\bullet}) = g_{X_i} - r_{X_i}.$$

Proof. The lemma follows immediately from Condition A and Theorem 2.1 (b). □

5.4.3. Let $i, j \in \{1,2\}$ such that $i \neq j$, and let G be a finite group such that $(\#(G), p) = 1$ and

$$f_i^\bullet : Y_i^\bullet \to X_i^\bullet$$

a Galois admissible covering over k_i with Galois group G. Then the isomorphism $\phi^{p'} : \Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet}$ induced by ϕ (4.2.1) implies that $f_i^\bullet$ induces a Galois admissible covering

$$f_j^\bullet : Y_j^\bullet \to X_j^\bullet$$

over k_j with Galois group G. We write (g_{Y_i}, n_{Y_i}) for the type of $Y_i^\bullet$, $\Gamma_{Y_i^\bullet}$ for the dual semi-graph of $Y_i^\bullet$, and r_{Y_i} for the Betti number of $\Gamma_{Y_i^\bullet}$.

Lemma 5.6. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ is étale, and that $\#(v_{f_1}^{\rm sp}) = m$ (see 1.1.5 for $v_{f_1}^{\rm sp}$). Then we have (see 1.1.5 for $e_{f_2}^{\rm cl,ra}$, $e_{f_2}^{\rm op,ra}$)*

$$0 \leq \#(e_{f_2}^{\rm cl,ra}) + \frac{1}{2}\#(e_{f_2}^{\rm op,ra}) + \#(v_{f_2}^{\rm sp}) \leq m.$$

Proof. Since $f_1^\bullet$ is an étale covering, the Riemann-Hurwitz formula implies

$$g_{Y_1} = \ell(g_X - 1) + 1,$$
$$g_{Y_2} = \ell(g_X - 1) + \frac{1}{2}(\ell - 1)\#(e_{f_2}^{\rm op,ra}) + 1.$$

Then we obtain

$$g_{Y_1} - g_{Y_2} = -\frac{1}{2}(\ell - 1)\#(e_{f_2}^{\rm op,ra}).$$

On the other hand, we have

$$\begin{aligned} r_{Y_1} &= \ell\#(e^{\rm cl}(\Gamma_{X_1^\bullet})) - \#(v(\Gamma_{X_1^\bullet})) + \#(v_{f_1}^{\rm sp}) - \ell\#(v_{f_1}^{\rm sp}) + 1 \\ &= \ell\#(e^{\rm cl}(\Gamma_{X_1^\bullet})) - \#(v(\Gamma_{X_1^\bullet})) - (\ell - 1)m + 1, \\ r_{Y_2} &= \ell\#(e_{f_2}^{\rm cl,ét}) + \#(e_{f_2}^{\rm cl,ra}) - \ell\#(v_{f_2}^{\rm sp}) - \#(v_{f_2}^{\rm ra}) + 1. \end{aligned}$$

Since $\#(e(\Gamma_{X_1^\bullet})) = \#(e(\Gamma_{X_2^\bullet}))$ and $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_2^\bullet}))$, we obtain

$$r_{Y_1} - r_{Y_2} = (\ell - 1)\#(e_{f_2}^{\rm cl,ra}) + (\ell - 1)(\#v_{f_2}^{\rm sp} - m).$$

Moreover, by applying Lemma 5.5 and Lemma 2.2 (b), we have $g_{Y_1} - g_{Y_2} \geq r_{Y_1} - r_{Y_2}$. Thus, we obtain

$$0 \leq \#(e_{f_2}^{\rm cl,ra}) + \frac{1}{2}\#(e_{f_2}^{\rm op,ra}) + \#(v_{f_2}^{\rm sp}) \leq m.$$

This completes the proof of the lemma. □

Corollary 5.7. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ is étale, and that $\#(v_{f_1}^{\rm sp}) = 0$. Then we have that $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ is étale, and that $\#(v_{f_2}^{\rm sp}) = 0$.*

Proof. The corollary follows immediately from Lemma 5.6. □

Corollary 5.8. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ is étale, and that $\#(v_{f_1}^{\rm sp}) = 1$. Then we have that $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ is étale.*

Proof. In order to verify the corollary, it is sufficient to prove that $\#(e_{f_2}^{\rm cl,ra}) = \#(e_{f_2}^{\rm op,ra}) = 0$. By applying Lemma 5.6, we have

$$0 \leq \#(e_{f_2}^{\rm cl,ra}) + \frac{1}{2}\#(e_{f_2}^{\rm op,ra}) + \#(v_{f_2}^{\rm sp}) \leq 1.$$

Suppose that $\#(e_{f_2}^{\mathrm{cl,ra}}) \neq 0$. Since $X_2^\bullet$ satisfies Condition A, the above inequality and the structures of the maxmial prime-to-p quotient of solvable admissible fundamental groups (1.2.4) imply that either (i) $\#(e_{f_2}^{\mathrm{cl,ra}}) = 1$ and $\#(e_{f_2}^{\mathrm{op,ra}}) \geq 2$, or (ii) $\#(e_{f_2}^{\mathrm{cl,ra}}) \geq 2$ holds. Then we have $2\#(e_{f_2}^{\mathrm{cl,ra}}) + \#(e_{f_2}^{\mathrm{op,ra}}) + 2\#(v_{f_2}^{\mathrm{sp}}) > 2$. Thus, we have $\#(e_{f_2}^{\mathrm{cl,ra}}) = 0$.

Suppose $\#(e_{f_2}^{\mathrm{op,ra}}) \neq 0$. Since $\#(e_{f_2}^{\mathrm{cl,ra}}) = 0$, the above inequality implies $\#(e_{f_2}^{\mathrm{op,ra}}) = 2$. Let $\ell' \neq p$ be a prime number distinct from ℓ, and let

$$g_1^\bullet : Z_1^\bullet \to X_1^\bullet$$

be a Galois étale covering of over k_1 with Galois group $\mathbb{Z}/\ell'\mathbb{Z}$ such that $\#(v_{g_1}^{\mathrm{sp}}) = 0$. Then Corollary 5.7 implies that the Galois admissible covering $g_2^\bullet : Z_2^\bullet \to X_2^\bullet$ over k_2 with Galois group $\mathbb{Z}/\ell'\mathbb{Z}$ induced by $g_1^\bullet$ is étale covering, and that $\#(v_{g_2}^{\mathrm{sp}}) = 0$. Write $\Gamma_{Z_i^\bullet}$ for the dual semi-graph of $Z_i^\bullet$. We obtain

$$\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{Z_1^\bullet})) = \#(v(\Gamma_{Z_2^\bullet})) = \#(v(\Gamma_{X_2^\bullet})),$$
$$\ell'\#(e^{\mathrm{op}}(\Gamma_{X_1^\bullet})) = \#(e^{\mathrm{op}}(\Gamma_{Z_1^\bullet})) = \#(e^{\mathrm{op}}(\Gamma_{Z_2^\bullet})) = \ell'\#(e^{\mathrm{op}}(\Gamma_{X_2^\bullet})),$$
$$\ell'\#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) = \#(e^{\mathrm{cl}}(\Gamma_{Z_1^\bullet})) = \#(e^{\mathrm{cl}}(\Gamma_{Z_2^\bullet})) = \ell'\#(e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})).$$

We have that $Z_1^\bullet$ and $Z_2^\bullet$ satisfy Condition A, Condition B, and Condition C.

We denote by $W_i^\bullet \stackrel{\mathrm{def}}{=} Y_i^\bullet \times_{X_i^\bullet} Z_i^\bullet$. Note that since $\ell' \neq \ell$, we see that $W_i^\bullet$ is connected. Then $f_i^\bullet$ induces a Galois admissible covering

$$h_i^\bullet : W_i^\bullet \to Z_i^\bullet$$

over k_i with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. We have that $h_1^\bullet$ is étale, that $\#(v_{h_1}^{\mathrm{sp}}) = 1$, and that $\#(e_{h_2}^{\mathrm{op,ra}}) = 2\ell'$. Then Lemma 5.6 implies

$$1 < \#(e_{h_2}^{\mathrm{cl,ra}}) + \frac{1}{2}\#(e_{h_2}^{\mathrm{op,ra}}) + \#(v_{h_2}^{\mathrm{sp}}) = \#(e_{h_2}^{\mathrm{cl,ra}}) + \ell' + \#(v_{h_2}^{\mathrm{sp}}) \leq 1.$$

This is a contradiction. Thus, we obtain $\#(e_{f_2}^{\mathrm{op,ra}}) = 0$. This completes the proof of the corollary. □

5.4.4. We put (see 1.2.7 for $\Pi_{X_i^\bullet}^{\text{ét}}$, $\Pi_{X_i^\bullet}^{\mathrm{top}}$)

$$M_{X_i^\bullet} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X_i^\bullet}, \mathbb{Z}/\ell\mathbb{Z}),\ M_{X_i^\bullet}^{\text{ét}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X_i^\bullet}^{\text{ét}}, \mathbb{Z}/\ell\mathbb{Z}),\ M_{X_i^\bullet}^{\mathrm{top}} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X_i^\bullet}^{\mathrm{top}}, \mathbb{Z}/\ell\mathbb{Z}).$$

Note that we have the following injections (or weight-monodromy filtration)

$$M_{X_i^\bullet}^{\mathrm{top}} \hookrightarrow M_{X_i^\bullet}^{\text{ét}} \hookrightarrow M_{X_i^\bullet}\ (\text{or } M_{X_i^\bullet}^{\mathrm{top}} \subseteq M_{X_i^\bullet}^{\text{ét}} \subseteq M_{X_i^\bullet})$$

induced by the natural surjections $\Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^{\text{ét}} \twoheadrightarrow \Pi_{X_i^\bullet}^{\mathrm{top}}$. Moreover, we have an isomorphism

$$\psi_\ell : M_{X_2^\bullet} \xrightarrow{\sim} M_{X_1^\bullet}$$

induced by the isomorphism $\phi^\ell : \Pi_{X_1^\bullet}^\ell \xrightarrow{\sim} \Pi_{X_2^\bullet}^\ell$.

Proposition 5.9. *We maintain the notation introduced above. Then the isomorphism* $\psi_\ell : M_{X_2^\bullet} \xrightarrow{\sim} M_{X_1^\bullet}$ *induces an isomorphism*

$$\psi_\ell^{\text{ét}} : M_{X_2^\bullet}^{\text{ét}} \xrightarrow{\sim} M_{X_1^\bullet}^{\text{ét}}$$

group-theoretically. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} M_{X_2^\bullet}^{\text{ét}} & \xrightarrow{\psi_\ell^{\text{ét}}} & M_{X_1^\bullet}^{\text{ét}} \\ \downarrow & & \downarrow \\ M_{X_2^\bullet} & \xrightarrow{\psi_\ell} & M_{X_1^\bullet}, \end{array}$$

where all vertical arrows are injections.

Proof. To verify the proposition, it is sufficient to prove that $\psi_\ell^{-1} : M_{X_1^\bullet} \xrightarrow{\sim} M_{X_2^\bullet}$ induces an isomorphism $\psi_\ell^{-1,\text{ét}} : M_{X_1^\bullet}^{\text{ét}} \xrightarrow{\sim} M_{X_2^\bullet}^{\text{ét}}$ which fits into the following commutative diagram:

$$\begin{array}{ccc} M_{X_1^\bullet}^{\text{ét}} & \xrightarrow{\psi_\ell^{-1,\text{ét}}} & M_{X_2^\bullet}^{\text{ét}} \\ \downarrow & & \downarrow \\ M_{X_1^\bullet} & \xrightarrow{\psi_\ell^{-1}} & M_{X_2^\bullet}, \end{array}$$

where all vertical arrows are injections.

Let $\alpha_1 \in M_{X_1^\bullet}^{\text{ét}}$ be a non-trivial element and $f_{1,\alpha}^\bullet : Y_{1,\alpha}^\bullet \to X_1^\bullet$ the Galois étale covering over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α. We put

$$L_{X_1^\bullet} \stackrel{\text{def}}{=} \{\alpha_1 \in M_{X_1^\bullet}^{\text{ét}} \mid \#(v_{f_{1,\alpha_1}}^{\text{sp}}) = 1\}.$$

We see that $M_{X_1^\bullet}^{\text{ét}}$ is spanned by $L_{X_1^\bullet}$ as an $\mathbb{F}_\ell$-linear space.

On the other hand, Corollary 5.8 implies that $f_{1,\alpha_1}^\bullet$ induces a Galois étale covering of $X_2^\bullet$ over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. This means that ψ_ℓ^{-1} induces an injection of $\mathbb{F}_\ell$-linear spaces

$$\psi_\ell^{-1,\text{ét}} : M_{X_1^\bullet}^{\text{ét}} \hookrightarrow M_{X_2^\bullet}^{\text{ét}}.$$

Moreover, since $\dim_{\mathbb{F}_\ell}(M_{X_1^\bullet}^{\text{ét}}) = 2g_{X_1} - r_{X_1} = 2g_{X_2} - r_{X_2} = \dim_{\mathbb{F}_\ell}(M_{X_2^\bullet}^{\text{ét}})$, we obtain that $\psi_\ell^{-1,\text{ét}}$ is an isomorphism. This completes the proof of the proposition. □

Proposition 5.10. *We maintain the notation introduced above. Then the isomorphism* $\psi_\ell : M_{X_2^\bullet} \xrightarrow{\sim} M_{X_1^\bullet}$ *induces an isomorphism*

$$\psi_\ell^{\text{top}} : M_{X_2^\bullet}^{\text{top}} \xrightarrow{\sim} M_{X_1^\bullet}^{\text{top}}$$

group-theoretically. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} M^{\mathrm{top}}_{X_2^\bullet} & \xrightarrow{\psi_\ell^{\mathrm{top}}} & M^{\mathrm{top}}_{X_1^\bullet} \\ \downarrow & & \downarrow \\ M^{\text{ét}}_{X_2^\bullet} & \xrightarrow{\psi_\ell^{\text{ét}}} & M^{\text{ét}}_{X_1^\bullet} \\ \downarrow & & \downarrow \\ M_{X_2^\bullet} & \xrightarrow{\psi_\ell} & M_{X_1^\bullet}, \end{array}$$

where all vertical arrows are injections.

Proof. Firstly, by Proposition 5.9, the isomorphism $\psi_\ell : M_{X_2^\bullet} \overset{\sim}{\to} M_{X_1^\bullet}$ induces an isomorphism $\psi_\ell^{\text{ét}} : M^{\text{ét}}_{X_2^\bullet} \overset{\sim}{\to} M^{\text{ét}}_{X_1^\bullet}$. Let $\alpha_2 \in M^{\mathrm{top}}_{X_2^\bullet} \subseteq M^{\text{ét}}_{X_2^\bullet}$ be a non-trivial element and

$$f^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to X_2^\bullet$$

the Galois étale covering over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_2. Then we obtain an element $\alpha_1 \overset{\mathrm{def}}{=} \psi_\ell^{\text{ét}}(\alpha_2) \in M^{\text{ét}}_{X_1^\bullet}$. Write $f^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \to X_1^\bullet$ for the Galois étale covering over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_1. Note that the types of $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$ are equal, and that $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$ satisfy Condition A.

Lemma 5.5 and Lemma 2.2 (b) imply $r_{Y_{1,\alpha_1}} \leq r_{Y_{2,\alpha_2}}$, where $r_{Y_{1,\alpha_1}}$ and $r_{Y_{2,\alpha_2}}$ denote the Betti numbers of the dual semi-graphs of $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$, respectively. Since $\#(v^{\mathrm{sp}}_{f_{2,\alpha_2}}) = \#(v(\Gamma_{X_2^\bullet})) = \#(v(\Gamma_{X_1^\bullet}))$, the inequality implies $\#(v^{\mathrm{sp}}_{f_{1,\alpha_1}}) = \#(v(\Gamma_{X_1^\bullet}))$. Thus, we have $\alpha_1 \in M^{\mathrm{top}}_{X_1^\bullet}$. Then α_1 induces an injection

$$\psi_\ell^{\mathrm{top}} : M^{\mathrm{top}}_{X_2^\bullet} \hookrightarrow M^{\mathrm{top}}_{X_1^\bullet}.$$

Moreover, since $\dim_{\mathbb{F}_\ell}(M^{\mathrm{top}}_{X_2^\bullet}) = r_{X_2} = r_{X_1} = \dim_{\mathbb{F}_\ell}(M^{\mathrm{top}}_{X_1^\bullet})$, we have that ψ_ℓ^{top} is an isomorphism. This completes the proof of the proposition. □

Remark 5.10.1. Proposition 5.9 and Proposition 5.10 mean that the weight-monodromy filtrations can be reconstructed group-theoretically from ϕ.

Lemma 5.11. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_2^\bullet$ is étale, and that $\#(v^{\mathrm{ra}}_{f_2}) = 1$ (1.1.5). Then we have that $f_1^\bullet$ is étale, and that $\#(v^{\mathrm{ra}}_{f_1}) = 1$.*

Proof. By Proposition 5.9, we obtain that $f_1^\bullet$ is étale. This implies $g_{Y_1} = g_{Y_2}$, and $\#(e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})) = \ell\#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) = \ell\#(e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})) = \#(e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet}))$. On the other hand, Lemma 5.5 and Lemma 2.2 (b) imply $r_{Y_1} \leq r_{Y_2}$. Thus, we obtain

$$\ell\#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}))-\ell(\#(v(\Gamma_{X_1^\bullet}))-\#(v^{\mathrm{ra}}_{f_1}))-\#(v^{\mathrm{ra}}_{f_1})+1 \leq \ell\#(e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}))-\ell(\#(v(\Gamma_{X_2^\bullet}))-1)-1+1.$$

This implies $\#(v^{\mathrm{ra}}_{f_1}) \leq 1$.

Suppose that $\#(v^{\mathrm{ra}}_{f_1}) = 0$. Let $\alpha_{f_1} \in M_{X^\bullet_1}$ be an element corresponding to $f^\bullet_1$. Then $\alpha_{f_1} \in M^{\mathrm{top}}_{X^\bullet_1}$. Note that $\alpha_{f_2} \overset{\mathrm{def}}{=} (\psi^{\text{ét}}_\ell)^{-1}(\alpha_{f_1}) \in M^{\text{ét}}_{X^\bullet_2}$ is an element corresponding to $f^\bullet_2$. Then Proposition 5.10 implies that α_{f_2} is contained in $M^{\mathrm{top}}_{X^\bullet_2}$. This means that $\#(v^{\mathrm{ra}}_{f_2}) = 0$. This contradicts the assumption $\#(v^{\mathrm{ra}}_{f_2}) = 1$. Thus, we have $\#(v^{\mathrm{ra}}_{f_1}) = 1$. We complete the proof of the lemma. □

5.4.5. We reconstruct the sets of vertices and the sets of genus of irreducible components group-theoretically from ϕ as follows.

Theorem 5.12. *We maintain the settings introduced in 5.4.1. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X^\bullet_1} \twoheadrightarrow \Pi_{X^\bullet_2}$ induces a bijection of the sets of vertices*

$$\phi^{\mathrm{sg,ver}} : v(\Gamma_{X^\bullet_1}) \overset{\sim}{\to} v(\Gamma_{X^\bullet_2})$$

group-theoretically. Moreover, let $v_1 \in v(\Gamma_{X^\bullet_1})$ and $v_2 \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,vex}}(v_1)$. Then we have the following equality of genus:

$$g_{1,v_1} = g_{2,v_2}.$$

Proof. We maintain the notation introduced in Section 5.1. By applying Theorem 4.2, Proposition 5.9, and Proposition 5.10, we obtain that the following homomorphisms of the natural exact sequences can be induced group-theoretically from ϕ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{\mathrm{top}}_{X^\bullet_2} & \longrightarrow & M^{\text{ét}}_{X^\bullet_2} & \longrightarrow & M^{\mathrm{nt}}_{X^\bullet_2} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \psi^{\mathrm{top}}_\ell} & & \Big\downarrow{\scriptstyle \psi^{\text{ét}}_\ell} & & \Big\downarrow & & \\ 0 & \longrightarrow & M^{\mathrm{top}}_{X^\bullet_1} & \longrightarrow & M^{\text{ét}}_{X^\bullet_1} & \longrightarrow & M^{\mathrm{nt}}_{X^\bullet_1} & \longrightarrow & 0. \end{array}$$

Then we obtain $\psi^{\text{ét}}_\ell(V^*_{X_2,\ell}) = V^*_{X_1,\ell}$ (see 5.1.3 for $V^*_{X_i,\ell}$). Moreover, Lemma 5.11 implies (see 5.1.3 for $V^\star_{X_i,\ell}$)

$$\psi^{\text{ét}}_\ell(V^\star_{X_2,\ell}) = V^\star_{X_1,\ell}.$$

Let $\alpha_2, \alpha'_2 \in V^\star_{X_2,\ell}$ be elements distinct from each other such that $\alpha_2 \sim \alpha'_2$ (i.e. the equivalence relation defined in Proposition 5.1 (a)). By applying Lemma 5.11 again, for any $a, b \in \mathbb{F}^\times_\ell$, we see that $a\alpha_2 + b\alpha'_2 \in V^\star_{X_2,\ell}$ if and only if $\psi^{\text{ét}}_\ell(a\alpha_2 + b\alpha'_2) = a\psi^{\text{ét}}_\ell(\alpha_2) + b\psi^{\text{ét}}_\ell(\alpha'_2) \in V^\star_{X_1,\ell}$. Thus, we obtain a bijection (see Proposition 5.1 (b) for $V_{X_i,\ell}$)

$$V_{X_2,\ell} \overset{\sim}{\to} V_{X_1,\ell}.$$

Then the first part of the theorem follows from Proposition 5.1.

Next, let us prove the "moreover" part of the theorem. Let $v_i \in v(\Gamma_{X^\bullet_i})$. We put

$$L^{v_i}_{X^\bullet_i} \overset{\mathrm{def}}{=} \{\alpha_i \in M^{\text{ét}}_{X^\bullet_i} \mid v^{\mathrm{ra}}_{f_{i,\alpha_i}} = \{v_i\}\},$$

where $f^{\bullet}_{i,\alpha_i}$ denotes the Galois admissible covering of $X^{\bullet}_i$ over k_i corresponding to α_i. Moreover, we denote by $[L^{v_i}_{X^{\bullet}_i}]$ the image of $L^{v_i}_{X^{\bullet}_i}$ in $M^{\mathrm{nt}}_{X^{\bullet}_i}$. Then we have $\#([L^{v_i}_{X^{\bullet}_i}]) = \ell^{g_{i,v_i}} - 1$.

Suppose $v_2 = \phi^{\mathrm{sg,ver}}(v_1)$. Proposition 5.10 and Lemma 5.11 imply that $\psi^{\text{ét}}_{\ell}$ induces an injection $[L^{v_2}_{X^{\bullet}_2}] \hookrightarrow [L^{v_1}_{X^{\bullet}_1}]$. Thus, we have $\ell^{g_{2,v_2}} - 1 = \#([L^{v_2}_{X^{\bullet}_2}]) \leq \#([L^{v_1}_{X^{\bullet}_1}]) = \ell^{g_{1,v_1}} - 1$. This implies $g_{2,v_2} \leq g_{1,v_1}$. On the other hand, since

$$\sum_{v_1 \in v(\Gamma_{X^{\bullet}_1})} g_{1,v_1} = g_X - r_{X_1} = g_X - r_{X_2} = \sum_{v_2 \in v(\Gamma_{X^{\bullet}_2})} g_{2,v_2},$$

we obtain $g_{1,v_1} = g_{2,v_2}$. This completes the proof of the theorem. □

5.4.6. **Further settings.** Next, let us reconstruct the sets of closed edges from ϕ. In the remainder of the present subsection, we fix an edge-triple

$$\mathfrak{T}_{\Pi_{X^{\bullet}_1}} \overset{\mathrm{def}}{=} (\ell, d, \alpha_{f_{X_1}} : \Pi^{\text{ét}}_{X^{\bullet}_1} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z})$$

associated to $\Pi_{X^{\bullet}_1}$ (5.2.3). Then Corollary 5.7 implies that ϕ and the edge-triple $\mathfrak{T}_{\Pi_{X^{\bullet}_1}}$ induce an edge-triple

$$\mathfrak{T}_{\Pi_{X^{\bullet}_2}} \overset{\mathrm{def}}{=} (\ell, d, \alpha_{f_{X_2}} : \Pi^{\text{ét}}_{X^{\bullet}_2} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z})$$

associated to $\Pi_{X^{\bullet}_2}$ group-theoretically. Write $\Pi_{Y^{\bullet}_i}$ for the kernel of $\alpha_{f_{X_i}}$. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X^{\bullet}_1} \twoheadrightarrow \Pi_{X^{\bullet}_2}$ induces a (surjective) open continuous homomorphism

$$\phi_Y : \Pi_{Y^{\bullet}_1} \twoheadrightarrow \Pi_{Y^{\bullet}_2}.$$

Moreover, the constructions of $Y^{\bullet}_1$ and $Y^{\bullet}_2$ imply that $Y^{\bullet}_1$ and $Y^{\bullet}_2$ satisfy Condition A, Condition B, and Condition C (5.3.1).

5.4.7. We put

$$M_{Y^{\bullet}_i} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{Y^{\bullet}_i}, \mathbb{Z}/\ell\mathbb{Z}),\ M^{\text{ét}}_{Y^{\bullet}_i} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\text{ét}}_{Y^{\bullet}_i}, \mathbb{Z}/\ell\mathbb{Z}),\ M^{\mathrm{ra}}_{Y^{\bullet}_i} \overset{\mathrm{def}}{=} M_{Y^{\bullet}_i}/M^{\text{ét}}_{Y^{\bullet}_i}.$$

Then, by Theorem 4.2 and Proposition 5.9, the following commutative diagram can be induced group-theoretically from ϕ_Y:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{\text{ét}}_{Y^{\bullet}_2} & \longrightarrow & M_{Y^{\bullet}_2} & \longrightarrow & M^{\mathrm{ra}}_{Y^{\bullet}_2} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi^{\text{ét}}_{Y,\ell}} & & \downarrow{\scriptstyle \psi_{Y,\ell}} & & \downarrow & & \\ 0 & \longrightarrow & M^{\text{ét}}_{Y^{\bullet}_1} & \longrightarrow & M_{Y^{\bullet}_1} & \longrightarrow & M^{\mathrm{ra}}_{Y^{\bullet}_1} & \longrightarrow & 0, \end{array}$$

where all vertical arrows are isomorphisms. Let $E^{*}_{\mathfrak{T}_{\Pi_{X^{\bullet}_i}}}$ be the subset of $M_{Y^{\bullet}_i}$ defined in 5.2.6. Since the actions of μ_d on the exact sequences are compatible with the isomorphisms appearing in the above commutative diagram, we have

$$\psi_{Y,\ell}(E^{*}_{\mathfrak{I}_{\Pi_{X_2^\bullet}}}) = E^{*}_{\mathfrak{I}_{\Pi_{X_1^\bullet}}}.$$

Let $m \in \mathbb{Z}_{\geq 0}$ and $e_i \in e^{\mathrm{cl}}(\Gamma_{X_i^\bullet})$. Recall that $E^{\mathrm{cl},\star,m}_{\mathfrak{I}_{\Pi_{X_i^\bullet}},e_i}$ (5.2.8) is the subset of $E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X_i^\bullet}},e_i}$ whose element α_i satisfies $\#(v^{\mathrm{sp}}_{g_{i,\alpha_i}}) = m$. Then we have the following lemma.

Lemma 5.13. *We maintain the notation introduced above. Then we have*

$$\psi^{-1}_{Y,\ell}(\bigsqcup_{e_1\in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{I}_{\Pi_{X_1^\bullet}},e_1}) \subseteq \bigsqcup_{e_2\in e^{\mathrm{op}}(\Gamma_{X_2^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{I}_{\Pi_{X_2^\bullet}},e_2}.$$

Moreover, we have

$$\psi^{-1}_{Y,\ell}(E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X_1^\bullet}}}) = E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X_2^\bullet}}}.$$

Proof. Let $e_1 \in e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$ and $\alpha_1 \in E^{\mathrm{cl},\star,0}_{\mathfrak{I}_{\Pi_{X_1^\bullet}},e_1}$. Then the Galois admissible covering $g^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha} \to Y^\bullet_1$ over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_1 induces a Galois admissible covering $g^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to Y^\bullet_2$ over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Write $\alpha_2 \in M_{Y_2^\bullet}$ for an element corresponding to $g^\bullet_{2,\alpha_2}$. We have $\alpha_2 \in E^{*}_{\mathfrak{I}_{\Pi_{Y_2^\bullet}}}$. Write $g_{Y_{i,\alpha_i}}$ for the genus of $Y^\bullet_{i,\alpha_i}$ and $r_{Y_{i,\alpha_i}}$ for the Betti number of the dual semi-graph $\Gamma_{Y^\bullet_{i,\alpha_i}}$. Then the Riemann-Hurwitz formula and Theorem 4.11 imply

$$g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} = -\frac{1}{2}(\#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}))(\ell-1) = 0.$$

On the other hand, we have

$$r_{Y_{1,\alpha_1}} = \ell(\#(e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})) - d) + d - \#(v(\Gamma_{Y_1^\bullet})) + 1,$$

$$r_{Y_{2,\alpha_2}} = \ell\#(e^{\mathrm{cl,\acute{e}t}}_{g_{2,\alpha_2}}) + \#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) - \ell\#(v^{\mathrm{cl,sp}}_{g_{2,\alpha_2}}) - \#(v^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) + 1.$$

Then Lemma 5.5 and Lemma 2.2 (b) imply $0 = g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} \geq r_{Y_{1,\alpha_1}} - r_{Y_{2,\alpha_1}}$. Thus, we have

$$\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) + \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) + \frac{1}{2}\#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}) = \#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) + \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) \leq d.$$

If $\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) = 0$, then g_{2,α_2} is étale. By replacing $X_1^\bullet$ and $X_2^\bullet$ by $Y_1^\bullet$ and $Y_2^\bullet$, respectively, Proposition 5.9 implies that g_{1,α_1} is also étale. This contradicts the definition of α_1. Thus, we obtain $\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) \neq 0$.

If $\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) \neq 0$, then we have $\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) = d$ and $\#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) = \#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}) = 0$. This means

$$\alpha_2 \in \bigsqcup_{e_2\in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{I}_{\Pi_{Y_2^\bullet}},e_2}.$$

Thus, we have

$$\psi_{Y,\ell}^{-1}(\bigsqcup_{e_1\in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{Y_1^\bullet}},e_1}) \subseteq \bigsqcup_{e_2\in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{Y_2^\bullet}},e_2}.$$

Moreover, let $\beta_i \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{Y_i^\bullet}}}$. Then β_i is a linear combination of some elements of

$$\bigsqcup_{e_i\in e^{\mathrm{cl}}(\Gamma_{Y_i^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{Y_i^\bullet}},e_i}.$$

Then we have $\psi_{Y,\ell}^{-1}(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}) \subseteq E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}$. On the other hand, since $g_{Y_1} = g_{Y_2}$ and $r_{Y_1} = r_{Y_2}$, Lemma 5.3 implies $\#(\psi_{Y,\ell}^{-1}(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}})) = \#(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}})$. Thus, we obtain

$$\psi_{Y,\ell}^{-1}(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}) = E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}.$$

This completes the proof of the lemma. □

Now, we can reconstruct the sets of closed edges group-theoretically from ϕ as follows.

Theorem 5.14. *We maintain the settings introduced in 5.4.1 and 5.4.6. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a bijection of the sets of closed edges*

$$\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$$

group-theoretically.

Proof. Let $\alpha_2, \alpha_2' \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}$ and $\alpha_1 \overset{\mathrm{def}}{=} \psi_{Y,\ell}(\alpha_2)$, $\alpha_1' \overset{\mathrm{def}}{=} \psi_{Y,\ell}(\alpha_2') \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}$. Lemma 5.13 implies that $\alpha_1 \sim \alpha_1'$ (i.e. the equivalence relation defined in Proposition 5.2 (a)) if and only if $\alpha_2 \sim \alpha_2'$. Then the theorem follows from Proposition 5.2. □

5.4.8. Next, let us reconstruct the sets of p-rank from ϕ. Note that the surjection ϕ induces a surjection of the maximal pro-p quotients

$$\phi^p : \Pi^p_{X_1^\bullet} \twoheadrightarrow \Pi^p_{X_2^\bullet}$$

of solvable admissible fundamental groups. Then every Galois (étale) admissible covering $h_2^\bullet : Z_2^\bullet \to X_2^\bullet$ over k_2 with Galois group $\mathbb{Z}/p\mathbb{Z}$ induces a Galois (étale) admissible covering $h_1^\bullet : Z_1^\bullet \to X_1^\bullet$ over k_1 with Galois group $\mathbb{Z}/p\mathbb{Z}$. Moreover, ϕ^p induces an injection

$$\psi_p : N_{X_2^\bullet} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X_2^\bullet}, \mathbb{Z}/p\mathbb{Z}) \hookrightarrow N_{X_1^\bullet} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X_1^\bullet}, \mathbb{Z}/p\mathbb{Z}).$$

We have the following lemmas.

Lemma 5.15. *We maintain the notation introduced above. Suppose that* $\#(v^{\mathrm{ra}}_{h_2}) = 0$*. Then we have* $\#(v^{\mathrm{ra}}_{h_1}) = 0$*. In particular, we obtain that*

$$\psi^{\mathrm{top}}_p : N^{\mathrm{top}}_{X^\bullet_2} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\mathrm{top}}_{X^\bullet_2}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} N^{\mathrm{top}}_{X^\bullet_1} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\mathrm{top}}_{X^\bullet_1}, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism.

Proof. Since $h^\bullet_i$ is étale, the Riemann-Hurwitz formula implies $g_{Z_1} = g_{Z_2}$. Thus, similar arguments to the arguments given in the proofs of Proposition 5.10 imply $\#(v^{\mathrm{ra}}_{h_1}) = 0$. This completes the proof of the lemma. □

Lemma 5.16. *We maintain the notation introduced above. Suppose that* $\#(v^{\mathrm{ra}}_{h_2}) = 1$*. Then we obtain* $\#(v^{\mathrm{ra}}_{h_1}) = 1$*.*

Proof. Similar arguments to the arguments given in the proofs of Lemma 5.11 imply $\#(v^{\mathrm{ra}}_{h_1}) \leq 1$. If $\#(v^{\mathrm{ra}}_{h_1}) = 0$, then the "in particular" part of Lemma 5.15 implies $\#(v^{\mathrm{ra}}_{h_2}) = 0$. This contradicts our assumption. Then we obtain $\#(v^{\mathrm{ra}}_{h_1}) = 1$. □

Now, we can reconstruct the sets of p-rank of smooth pointed stable curves associated to vertices from ϕ as follows.

Theorem 5.17. *We maintain the settings introduced in 5.4.1. Then the (surjective) open continuous homomorphism* $\phi : \Pi_{X^\bullet_1} \twoheadrightarrow \Pi_{X^\bullet_2}$ *induces an injection of the sets of vertices (see 5.1.2 for* $v(\Gamma_{X^\bullet_i})^{>0,p}$*)*

$$\psi^{\mathrm{sg,ver}}_p : v(\Gamma_{X^\bullet_2})^{>0,p} \hookrightarrow v(\Gamma_{X^\bullet_1})^{>0,p}$$

group-theoretically. Moreover, let $v_2 \in v(\Gamma_{X^\bullet_2})^{>0,p}$ *and* $v_1 \overset{\mathrm{def}}{=} \psi^{\mathrm{sg,vex}}_p(v_2)$*. Then we have the following inequality of p-rank*

$$\sigma_{2,v_2} \leq \sigma_{1,v_1}.$$

Proof. Lemma 5.16 implies $\psi_p(V^\star_{X_2,p}) \subseteq V^\star_{X_1,p}$. Let $\alpha_2, \alpha'_2 \in V^\star_{X_2,p}$ be elements distinct from each other such that $\alpha_2 \sim \alpha'_2$. It is easy to see that $a\alpha_2 + b\alpha'_2 \in V^\star_{X_2,p}$ if and only if $a\psi_p(\alpha_2) + b\psi_p(\alpha'_2) \in V^\star_{X_1,p}$ for each $a, b \in \mathbb{F}^\times_p$. Thus, by Proposition 5.1, we obtain an injection of the sets of vertices

$$\psi^{\mathrm{sg,ver}}_p : v(\Gamma_{X^\bullet_2})^{>0,p} \hookrightarrow v(\Gamma_{X^\bullet_1})^{>0,p}.$$

Let $v_i \in v(\Gamma_{X^\bullet_i})$. We put

$$L^{v_i,p}_{X^\bullet_i} \overset{\mathrm{def}}{=} \{\alpha_i \in N_{X^\bullet_i} \mid v^{\mathrm{ra}}_{h_{i,\alpha_i}} = \{v_i\}\},$$

where $h^\bullet_{i,\alpha_i}$ denotes the Galois (étale) admissible covering corresponding to α_i. Moreover, Lemma 5.16 implies that ψ_p induces an injection $L^{v_2,p}_{X^\bullet_2} \hookrightarrow L^{v_1,p}_{X^\bullet_1}$.

We denote by $[L^{v_i,p}_{X^\bullet_i}]$ the image of $L^{v_i,p}_{X^\bullet_i}$ in $N_{X^\bullet_i}/N^{\mathrm{top}}_{X^\bullet_i}$. Then we have $\#([L^{v_i,p}_{X^\bullet_i}]) = p^{\sigma_{i,v_i}} - 1$. Suppose that $v_1 \overset{\mathrm{def}}{=} \psi^{\mathrm{sg,ver}}_p(v_2)$. Lemma 5.15 implies that ψ_p induces an injection $[L^{v_2,p}_{X_2^\bullet}] \hookrightarrow [L^{v_1,p}_{X_1^\bullet}]$. Thus, we have $p^{\sigma_{2,v_2}} - 1 = \#([L^{v_2,p}_{X_2^\bullet}]) \leq \#([L^{v_1,p}_{X_1^\bullet}]) = p^{\sigma_{1,v_1}} - 1$. This means that $\sigma_{2,v_2} \leq \sigma_{1,v_1}$ for each $v_2 \in v(\Gamma_{X_2^\bullet})^{>0,p}$. We complete the proof of the theorem. □

5.4.9. In the remainder of the present subsection, we prove a proposition which will be used in Section 5.6.

Proposition 5.18. *We maintain the notation introduced above. Then the following statements hold:*

(a) Let $S_1^{\mathrm{cl}} \subseteq e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$ *be a subset of closed edges,* $\alpha_{e_1} \in E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_1^\bullet}},e_1}$ *(5.2.8) for every* $e_1 \in S_1^{\mathrm{cl}}$,

$$\alpha_1 \overset{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{cl}}} \alpha_{e_1} \in E^*_{\mathfrak{T}_{\Pi_{X_1^\bullet}}} (5.2.6),$$

and $g^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \to Y_1^\bullet$ *the Galois admissible covering over* k_1 *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α_1. *Let* $\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$ *be the bijection of the sets of closed edges obtained in Theorem 5.14,* $\alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_2^\bullet}},\phi^{\mathrm{sg,cl}}(e_1)}$ *the element induced by* ϕ *for every* $e_1 \in S_1^{\mathrm{cl}}$,

$$\alpha_2 \overset{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{cl}}} \alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X_2^\bullet}}},$$

and $g^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to Y_2^\bullet$ *the Galois admissible covering over* k_2 *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α_2. *Suppose* $\#(v^{\mathrm{sp}}_{g_{1,\alpha_1}}) = 0$. *Then we have*

$$\#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}) = \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) = 0.$$

(b) Let $E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X_i^\bullet}},e_i}$, $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$, *be the set of cohomology classes defined in 5.2.9, and let* $S_1^{\mathrm{op}} \subseteq e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ *be a subset of open edges,* $\alpha_{e_1} \in E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X_1^\bullet}},e_1}$ *for every* $e_1 \in S_1^{\mathrm{op}}$,

$$\alpha_1 \overset{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{op}}} \alpha_{e_1} \in E^*_{\mathfrak{T}_{\Pi_{X_1^\bullet}}},$$

and $g^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \to Y_1^\bullet$ *the Galois admissible covering over* k_1 *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α_1. *Let* $\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ *be the bijection of the sets of open edges obtained in Theorem 4.11,* $\alpha_{\phi^{\mathrm{sg,op}}(e_1)} \in E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X_2^\bullet}},\phi^{\mathrm{sg,op}}(e_1)}$ *the element induced by* ϕ *for every* $e_1 \in S_1^{\mathrm{op}}$,

$$\alpha_2 \overset{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{op}}} \alpha_{\phi^{\mathrm{sg,op}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X_2^\bullet}}},$$

and $g^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to Y^\bullet_2$ *the Galois admissible covering over* k_2 *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α_2*. Suppose* $\#(v^{\mathrm{sp}}_{g_{1,\alpha_1}}) = 0$*. Then we have*

$$\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) = \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) = 0.$$

Proof. (a) Since $\#(e^{\mathrm{op,ra}}_{g_{1,\alpha_1}}) = 0$, Theorem 4.11 implies $\#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}) = 0$. On the other hand, we have

$$r_{Y_{1,\alpha_1}} = \ell(\#(e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})) - d\#(S_1^{\mathrm{cl}})) + d\#(S_1^{\mathrm{cl}}) - \#(v(\Gamma_{Y_1^\bullet})) + 1,$$

$$r_{Y_{2,\alpha_2}} = \ell\#(e^{\mathrm{cl,\acute{e}t}}_{g_{2,\alpha_2}}) + \#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) - \ell\#(v^{\mathrm{cl,sp}}_{g_{2,\alpha_2}}) - \#(v^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) + 1.$$

Then Lemma 5.5 and Lemma 2.2 (b) imply $0 = g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} \geq r_{Y_{1,\alpha_1}} - r_{Y_{2,\alpha_1}}$. Thus, we have

$$\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) + \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) + \frac{1}{2}\#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}) = \#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) + \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) \leq d\#(S_1^{\mathrm{cl}}).$$

On the othe hand, Lemma 5.13 implies $\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) = d\#(S_1^{\mathrm{cl}})$. Then we obtain $\#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) = 0$. This completes the proof of (a).

(b) The Riemann-Hurwitz formula and Theorem 4.11 imply

$$g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} = \frac{1}{2}(d\#(S_1^{\mathrm{op}}) - \#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}))(\ell - 1) = 0.$$

On the other hand, we have

$$r_{Y_{1,\alpha_1}} = \ell\#(e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})) - \#(v(\Gamma_{Y_1^\bullet})) + 1,$$

$$r_{Y_{2,\alpha_2}} = \ell\#(e^{\mathrm{cl,\acute{e}t}}_{g_{2,\alpha_2}}) + \#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) - \ell\#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) - \#(v^{\mathrm{ra}}_{g_{2,\alpha_2}}) + 1.$$

Then Lemma 5.5 and Lemma 2.2 (b) imply $g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} \geq r_{Y_{1,\alpha_1}} - r_{Y_{2,\alpha_2}}$. Thus, we have

$$\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) + \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) + \frac{1}{2}\#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}) - \frac{d\#(S_1^{\mathrm{op}})}{2} \leq 0.$$

This means that $\#(e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}}) = \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) = 0$. We complete the proof of (b). □

5.5. **Reconstructions of commutative diagrams of combinatorial data.** In this subsection, we prove that the commutative diagrams of sets of vertices, sets of open edges, and sets of closed edges induced by admissible coverings can be reconstructed from an open continuous homomorphism of solvable admissible fundamental groups. The main result of the present subsection is Proposition 5.19.

5.5.1. **Settings**. In the present subsection, we maintain the settings introduced in 5.4.1. Furthermore, we fix some notation as follows. Let H_2 be an open normal subgroup of $\Pi_{X_2^\bullet}$, $H_1 \stackrel{\text{def}}{=} \phi^{-1}(H_2)$ the open normal subgroup of $\Pi_{X_1^\bullet}$, $G \stackrel{\text{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$, and ϕ_{H_1} the surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$. Let $i \in \{1, 2\}$. We write

$$f^\bullet_{H_i} : X^\bullet_{H_i} \to X^\bullet_i$$

for the Galois admissible covering over k_i with Galois group G, $(g_{X_{H_i}}, n_{X_{H_i}})$ for the type of $X^\bullet_{H_i}$, and $\Gamma_{X^\bullet_{H_i}}$ for the dual semi-graph of $X^\bullet_{H_i}$. Furthermore, *we suppose that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C* (5.3.1).

5.5.2. Let ℓ and d be prime numbers distinct from p such that $\ell \neq d$ and $(\#(G), \ell) = (\#(G), d) = 1$, and let

$$\mathfrak{T}_{\Pi_{X_2^\bullet}} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_{X_2}} : \Pi^{\text{ét}}_{X_2^\bullet} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z})$$

be an edge-triple associated to $\Pi_{X_2^\bullet}$ (5.2.3) and $\mathfrak{T}_{X_2^\bullet} \stackrel{\text{def}}{=} (\ell, d, f^\bullet_{X_2} : Y_2^\bullet \to X_2^\bullet)$ the edge-triple associated to $X_2^\bullet$ corresponding to $\mathfrak{T}_{\Pi_{X_2^\bullet}}$ (5.2.2). By Corollary 5.7, we obtain an edge-triple

$$\mathfrak{T}_{\Pi_{X_1^\bullet}} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_{X_1}} : \Pi^{\text{ét}}_{X_1^\bullet} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z})$$

induced group-theoretically from ϕ and $\mathfrak{T}_{\Pi_{X_2^\bullet}}$. We write $\mathfrak{T}_{X_1^\bullet} \stackrel{\text{def}}{=} (\ell, d, f^\bullet_{X_1} : Y_1^\bullet \to X_1^\bullet)$ for the edge-triple associated to $X_1^\bullet$ corresponding to $\mathfrak{T}_{\Pi_{X_1^\bullet}}$. On the other hand, we put

$$Q_i \stackrel{\text{def}}{=} \ker(\Pi_{X_i^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X_i^\bullet} \stackrel{\alpha_{f_{X_i}}}{\twoheadrightarrow} \mathbb{Z}/d\mathbb{Z}).$$

We have that $H_i \twoheadrightarrow H_i/(H_i \cap Q_i) \cong \mathbb{Z}/d\mathbb{Z}$ factors through a homomorphism $\alpha_{f_{X_{H_i}}} : H_i^{\text{ét}} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$. We see that

$$\mathfrak{T}_{H_i} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_{X_{H_i}}})$$

is an edge-triple associated to H_i. Moreover, $\mathfrak{T}_{H_i}$ is induced group-theoretically from $H_i \subseteq \Pi_{X_i^\bullet}$ and $\mathfrak{T}_{\Pi_{X_i^\bullet}}$. Note that $\mathfrak{T}_{H_1}$ coincides with the edge-triple associated to H_1 induced group-theoretically from ϕ_{H_1} and $\mathfrak{T}_{H_2}$. Moreover, we denote by

$$\mathfrak{T}_{X^\bullet_{H_i}} \stackrel{\text{def}}{=} (\ell, d, f^\bullet_{X_{H_i}} : Y^\bullet_{X_{H_i}} \stackrel{\text{def}}{=} Y_i^\bullet \times_{X_i^\bullet} X^\bullet_{H_i} \to X^\bullet_{H_i})$$

the edge-triple associated to $X^\bullet_{H_i}$ corresponding to $\mathfrak{T}_{H_i}$.

5.5.3. By applying Proposition 5.1, Remark 5.1.1, Proposition 5.2, and Remark 5.2.1, we have that the natural inclusion $H_i \hookrightarrow \Pi_{X_i^\bullet}$ induces the following maps

$$\gamma_{H_i}^{\mathrm{ver},\ell} : V_{X_{H_i},\ell} \to V_{X_i,\ell},\ \gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X_i^\bullet}},H_i} : E^{\mathrm{cl}}_{\mathfrak{T}_{H_i}} \to E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X_i}}}$$

group-theoretically. We put

$$\gamma_{H_i}^{\mathrm{ver}} : v(\Gamma_{X^\bullet_{H_i}}) \overset{\kappa^{-1}_{X_{H_i},\ell}}{\overset{\sim}{\to}} V_{X_{H_i},\ell} \overset{\gamma_{H_i}^{\mathrm{ver},\ell}}{\to} V_{X_i,\ell} \overset{\kappa_{X_i,\ell}}{\overset{\sim}{\to}} v(\Gamma_{X_i^\bullet}),$$

$$\gamma_{H_i}^{\mathrm{cl}} : e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_i}}) \overset{\vartheta^{-1}_{\mathfrak{T}_{H_i}}}{\overset{\sim}{\to}} E^{\mathrm{cl}}_{\mathfrak{T}_{H_i}} \overset{\gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X_i^\bullet}},H_i}}{\to} E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X_i^\bullet}}} \overset{\vartheta_{\mathfrak{T}_{\Pi_{X_i^\bullet}}}}{\overset{\sim}{\to}} e^{\mathrm{cl}}(\Gamma_{X_i^\bullet}).$$

Then the maps $\gamma_{H_i}^{\mathrm{ver}}$ and $\gamma_{H_i}^{\mathrm{cl}}$ can be reconstructed group-theoretically from the inclusion $H_i \hookrightarrow \Pi_{X_i^\bullet}$.

On the other hand, Theorem 4.2 implies that the sets $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$ and $\mathrm{Edg}^{\mathrm{op}}(H_i)$ (1.2.11) can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$ and H_i, respectively. Note that we have a natural map

$$\mathrm{Edg}^{\mathrm{op}}(H_i) \to \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$$

induced by the natural inclusions of stabilizer subgroups. Moreover, this map compatible with the actions of H_i and $\Pi_{X_i^\bullet}$. Then we obtain a map

$$\gamma_{H_i}^{\mathrm{op}} : e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_i}}) \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(H_i)/H_i \to \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})/\Pi_{X_i^\bullet} \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$$

which can be reconstructed by the inclusion $H_i \hookrightarrow \Pi_{X_i^\bullet}$ group-theoretically.

Moreover, by Theorem 4.11, Theorem 5.12, and Theorem 5.14, the following maps

$$\phi_{H_1}^{\mathrm{sg,ver}} : v(\Gamma_{X^\bullet_{H_1}}) \overset{\sim}{\to} v(\Gamma_{X^\bullet_{H_2}}),\ \phi_{H_1}^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_1}}) \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_2}}),\ \phi_{H_1}^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_1}}) \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}}),$$

$$\phi^{\mathrm{sg,ver}} : v(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} v(\Gamma_{X_2^\bullet}),\ \phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X_2^\bullet}),\ \phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$$

can be induced group-theoretically from $\phi_{H_1} : H_1 \twoheadrightarrow H_2$ and $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$, respectively.

We have the following result:

Proposition 5.19. *We maintain the notation introduced above. Then the following diagrams*

$$\begin{array}{ccc} v(\Gamma_{X^\bullet_{H_1}}) & \xrightarrow{\phi_{H_1}^{\mathrm{sg,ver}}} & v(\Gamma_{X^\bullet_{H_2}}) \\ {\scriptstyle \gamma_{H_1}^{\mathrm{ver}}}\downarrow & & {\scriptstyle \gamma_{H_2}^{\mathrm{ver}}}\downarrow \\ v(\Gamma_{X_1^\bullet}) & \xrightarrow{\phi^{\mathrm{sg,ver}}} & v(\Gamma_{X_2^\bullet}), \end{array}$$

$$\begin{array}{ccc} e^{\mathrm{op}}(\Gamma_{X^{\bullet}_{H_1}}) & \xrightarrow{\phi^{\mathrm{sg,op}}_{H_1}} & e^{\mathrm{op}}(\Gamma_{X^{\bullet}_{H_2}}) \\ \gamma^{\mathrm{op}}_{H_1}\big\downarrow & & \gamma^{\mathrm{op}}_{H_2}\big\downarrow \\ e^{\mathrm{op}}(\Gamma_{X^{\bullet}_1}) & \xrightarrow{\phi^{\mathrm{sg,op}}} & e^{\mathrm{op}}(\Gamma_{X^{\bullet}_2}), \end{array}$$

$$\begin{array}{ccc} e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{H_1}}) & \xrightarrow{\phi^{\mathrm{sg,cl}}_{H_1}} & e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{H_2}}) \\ \gamma^{\mathrm{cl}}_{H_1}\big\downarrow & & \gamma^{\mathrm{cl}}_{H_2}\big\downarrow \\ e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_1}) & \xrightarrow{\phi^{\mathrm{sg,cl}}} & e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_2}) \end{array}$$

are commutative. Moreover, the above commutative diagrams are compatible with the natural actions of G.

Proof. The commutativity of the second diagram follows immediately from Theorem 4.11 (in fact, the second commutative diagram holds without Condition A, Condition B, and Condition C). We treat the third diagram. To verify the commutativity of the third diagram, we only need to prove the commutativity of the following diagram

$$\begin{array}{ccc} e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{H_2}}) & \xrightarrow{(\phi^{\mathrm{sg,cl}}_{H_1})^{-1}} & e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{H_1}}) \\ \gamma^{\mathrm{cl}}_{H_2}\big\downarrow & & \gamma^{\mathrm{cl}}_{H_1}\big\downarrow \\ e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_2}) & \xrightarrow{(\phi^{\mathrm{sg,cl}})^{-1}} & e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_1}). \end{array}$$

Let $e_{H_2} \in e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{H_2}})$, $e_{H_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,cl}}_{H_1})^{-1}(e_{H_2}) \in e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{H_1}})$, $e_2 \overset{\mathrm{def}}{=} \gamma^{\mathrm{cl}}_{H_2}(e_{H_2}) \in e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_2})$, $e_1 \overset{\mathrm{def}}{=} (\gamma^{\mathrm{cl}}_{H_1} \circ (\phi^{\mathrm{sg,cl}}_{H_1})^{-1})(e_{H_2}) \in e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_1})$, and $e'_1 \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,cl}})^{-1}(e_2) \in e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_1})$. We will prove that $e_1 = e'_1$.

Write S_{H_1} and S_{H_2} for the sets $(\gamma^{\mathrm{cl}}_{H_1})^{-1}(e'_1)$ and $(\gamma^{\mathrm{cl}}_{H_2})^{-1}(e_2)$, respectively. Note that $e_{H_2} \in S_{H_2}$. To verify $e_1 = e'_1$, it is sufficient to prove that $e_{H_1} \in S_{H_1}$.

Let $\alpha_2 \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^{\bullet}_2}}, e_2}$ (5.2.8). Then the proof of Lemma 5.13 implies that α_2 induces an element $\alpha_1 \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^{\bullet}_1}}, e'_1}$. Write $Y^{\bullet}_{\alpha_i} \to Y^{\bullet}_i$ for the Galois admissible covering over k_i corresponding to α_i. We consider the Galois admissible covering

$$Y^{\bullet}_{\alpha_2} \times_{X^{\bullet}_2} X^{\bullet}_{H_2} \to Y^{\bullet}_{X_{H_2}}$$

over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, and denote by β_2 an element of $E^{*}_{\mathfrak{T}_{H_2}}$ (5.2.6) corresponding to this Galois admissible covering. Then we have

$$\beta_2 = \sum_{c_2 \in S_{H_2}} t_{c_2} \beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in E^{\mathrm{cl},\star}_{\mathfrak{I}_{H_2},c_2}$. Note that we have $t_{e_{H_2}} \neq 0$. On the other hand, the proof of Lemma 5.13 implies that β_{c_2} induces an element $\beta_{(\phi^{\mathrm{cl}}_{H_1})^{-1}(c_2)} \in E^{\mathrm{cl},\star}_{\mathfrak{I}_{H_1},(\phi^{\mathrm{cl}}_{H_1})^{-1}(c_2)}$. Then β_2 induces an element

$$\beta_1 \overset{\mathrm{def}}{=} \sum_{c_2 \in S_{H_2} \setminus \{e_{H_2}\}} t_{c_2}\beta_{(\phi^{\mathrm{cl}}_{H_1})^{-1}(c_2)} + t_{e_{H_2}}\beta_{e_{H_1}} \in E^{*}_{\mathfrak{I}_{H_1}}.$$

Note that since β_1 is an element corresponding to the Galois admissible covering

$$Y^\bullet_{\alpha_1} \times_{X_1^\bullet} X^\bullet_{H_1} \to Y^\bullet_{X_{H_1}}$$

over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, the composition of the Galois admissible coverings $Y^\bullet_{\alpha_1} \times_{X_1^\bullet} X^\bullet_{H_1} \to Y^\bullet_{X_{H_1}} \overset{f^\bullet_{X_{H_1}}}{\to} X^\bullet_{H_1}$ is ramified over S_{H_1}. This means that e_{H_1} is contained in S_{H_1}.

Similar arguments to the arguments given in the above proof imply the first diagram is commutative. It is easy to check the "moreover" part of the proposition. This completes the proof of the proposition. □

5.6. **Combinatorial Grothendieck conjecture.** In this subsection, we prove a version of combinatorial Grothendieck conjecture for *open continuous homomorphisms* under certain assumptions. The main results of the present subsection are Theorem 5.26 and Theorem 5.30.

5.6.1. **Settings**. In the present subsection, we maintain the settings introduced in 5.4.1. Moreover, we fix some notation as follows. Let H_2 be an open normal subgroup of $\Pi^\bullet_{X_2}$, $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2)$ the open normal subgroup of $\Pi_{X_1^\bullet}$, $G \overset{\mathrm{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$, and $\phi_{H_1} \overset{\mathrm{def}}{=} \phi|_{H_1} : H_1 \twoheadrightarrow H_2$ the surjection induced by ϕ. Let $i \in \{1, 2\}$. We write

$$f^\bullet_{H_i} : X^\bullet_{H_i} \to X^\bullet_i$$

for the Galois admissible covering over k_i with Galois group G, $(g_{X_{H_i}}, n_{X_{H_i}})$ for the type of $X^\bullet_{H_i}$, $\Gamma_{X^\bullet_{H_i}}$ for the dual semi-graph of $X^\bullet_{H_i}$, and $r_{X_{H_i}}$ for the Betti number of $\Gamma_{X^\bullet_{H_i}}$.

5.6.2. Firstly, we prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C introduced in 5.3.1 (see Proposition 5.25 below).

Lemma 5.20. *We maintain the notation introduced above. Then $X^\bullet_{H_i}$ satisfies Condition A, Condition B, and Condition C (i).*

Proof. The first condition, the second condition, and the fourth condition of Condition A follow immediately from the definition of admissible coverings. Since $X^\bullet_i$

satisfies Condition B and the third condition of Condition A, $X^\bullet_{H_i}$ also satisfies Condition B and the third condition of Condition A. Moreover, Condition C (i) follows immediately from Theorem 4.11. This completes the proof of the lemma. □

Lemma 5.21. *We maintain the notation introduced above. Suppose that there exists an open normal subgroup $H'_2 \subseteq H_2$ such that $X^\bullet_{H'_1}$ and $X^\bullet_{H'_2}$ satisfy Condition A, Condition B, and Condition C, where $H'_1 \stackrel{\text{def}}{=} \phi^{-1}(H'_2) \subseteq H_1$. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii).

Let $G' \stackrel{\text{def}}{=} \Pi_{X^\bullet_1}/H'_1 = \Pi_{X^\bullet_2}/H'_2$ and $G'' \stackrel{\text{def}}{=} H_1/H'_1 = H_2/H'_2 \subseteq G'$. By applying Proposition 5.19, the following commutative diagrams

$$\begin{array}{ccc} v(\Gamma_{X^\bullet_{H'_1}}) & \xrightarrow{\phi^{\mathrm{sg,ver}}_{H'_1}} & v(\Gamma_{X^\bullet_{H'_2}}) \\ \gamma^{\mathrm{ver}}_{H'_1}\downarrow & & \gamma^{\mathrm{ver}}_{H'_2}\downarrow \\ v(\Gamma_{X^\bullet_1}) & \xrightarrow{\phi^{\mathrm{sg,ver}}} & v(\Gamma_{X^\bullet_2}), \end{array}$$

$$\begin{array}{ccc} e^{\mathrm{op}}(\Gamma_{X^\bullet_{H'_1}}) & \xrightarrow{\phi^{\mathrm{sg,op}}_{H'_1}} & e^{\mathrm{op}}(\Gamma_{X^\bullet_{H'_2}}) \\ \gamma^{\mathrm{op}}_{H'_1}\downarrow & & \gamma^{\mathrm{op}}_{H'_2}\downarrow \\ e^{\mathrm{op}}(\Gamma_{X^\bullet_1}) & \xrightarrow{\phi^{\mathrm{sg,op}}} & e^{\mathrm{op}}(\Gamma_{X^\bullet_2}), \end{array}$$

$$\begin{array}{ccc} e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_1}}) & \xrightarrow{\phi^{\mathrm{sg,cl}}_{H'_1}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_2}}) \\ \gamma^{\mathrm{cl}}_{H'_1}\downarrow & & \gamma^{\mathrm{cl}}_{H'_2}\downarrow \\ e^{\mathrm{cl}}(\Gamma_{X^\bullet_1}) & \xrightarrow{\phi^{\mathrm{sg,cl}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet_2}) \end{array}$$

can be reconstructed group-theoretically from $H'_i \hookrightarrow \Pi_{X^\bullet_i}$, ϕ, and $\phi_{H'_1} \stackrel{\text{def}}{=} \phi|_{H'_1}$. Moreover, the commutative diagrams are compatible with the actions of G'' and G'. Then we obtain

$$\#(v(\Gamma_{X^\bullet_{H_1}})) = \#(v(\Gamma_{X^\bullet_{H'_1}})/G'') = \#(v(\Gamma_{X^\bullet_{H'_2}})/G'') = \#(v(\Gamma_{X^\bullet_{H_2}})),$$

$$\#(e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_1}})) = \#(e^{\mathrm{op}}(\Gamma_{X^\bullet_{H'_1}})/G'') = \#(e^{\mathrm{op}}(\Gamma_{X^\bullet_{H'_2}})/G'') = \#(e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_2}})),$$

$$\#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_1}})) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_1}})/G'') = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_2}})/G'') = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}})).$$

This means that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C. □

Lemma 5.22. *We maintain the notation introduced above. Suppose that* $(\#(G), p) = 1$*, and that* f_{H_2} *is étale. Then* $X^\bullet_{H_1}$ *and* $X^\bullet_{H_2}$ *satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii). Moreover, since G is a finite solvable group, to verify the lemma, it is sufficient to prove the lemme when $G \cong \mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a prime number distinct from p. Thus, Proposition 5.9 implies that f_{H_1} is also étale.

We denote by $H'_2 \subseteq H_2$ the inverse image of $D_\ell(\Pi^{\text{ét}}_{X^\bullet_2})$ (Definition 4.8) of the natural surjection $\Pi_{X^\bullet_2} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet_2}$. Then H'_2 is an open normal subgroup of $\Pi_{X^\bullet_2}$. Let $H'_1 \overset{\text{def}}{=} \phi^{-1}(H'_2) \subseteq H_1$. We see that H'_1 is equal to the inverse image of $D_\ell(\Pi^{\text{ét}}_{X^\bullet_1})$ of the natural surjection $\Pi_{X^\bullet_1} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet_1}$. Since $X^\bullet_1$ and $X^\bullet_2$ satisfy Condition C, Theorem 5.12 and the structures of the maximal prime-to-p quotients of solvable admissible fundamental groups (1.2.4) imply that $X^\bullet_{H'_1}$ and $X^\bullet_{H'_2}$ also satisfy Condition C. Then the lemma follows from Lemma 5.21. □

Lemma 5.23. *We maintain the notation introduced above. Suppose that* $(\#(G), p) = 1$*. Then* $X^\bullet_{H_1}$ *and* $X^\bullet_{H_2}$ *satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii).

Since G is a finite solvable group, to verify the lemma, it is sufficient to prove the lemme when $G \cong \mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a prime number distinct from p.

Let $\mathfrak{T}_{\Pi_{X^\bullet_2}} = (\ell, d, \alpha_{f_{X_2}} : \Pi^{\text{ét}}_{X^\bullet_2} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z})$ be an edge-triple associated to $\Pi_{X^\bullet_2}$ (5.2.3), $\mathfrak{T}_{\Pi_{X^\bullet_1}} = (\ell, d, \alpha_{f_{X_1}} : \Pi^{\text{ét}}_{X^\bullet_1} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z})$ the edge-triple associated to $\Pi_{X^\bullet_1}$ induced by ϕ, and $\mathfrak{T}_{X^\bullet_i} = (\ell, d, f^\bullet_{X_i} : Y^\bullet_i \to X^\bullet_i)$ the edge-triple associated to $X^\bullet_i$ corresponding to $\mathfrak{T}_{\Pi_{X^\bullet_i}}$ (5.2.2).

Firstly, we suppose that f_{H_2} is étale over D_{X_2}. Then Theorem 4.11 implies that f_{H_1} is also étale over D_{X_1}. Let $\alpha_{e_1} \in E^{\text{cl},\star,0}_{\mathfrak{T}_{\Pi_{X^\bullet_1}}, e_1}$ (5.2.8), $e_1 \in e^{\text{cl}}(\Gamma_{X^\bullet_1})$,

$$\alpha_1 \overset{\text{def}}{=} \sum_{e_1 \in e^{\text{cl}}(\Gamma_{X^\bullet_1})} \alpha_{e_1} \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet_1}}} (5.2.6),$$

and $g^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \to Y^\bullet_1$ the Galois admissible covering over k_1 corresponding to α_1. Note that we have $\#(e^{\text{op,ra}}_{g_{1,\alpha_1}}) = \#(v^{\text{sp}}_{g_{1,\alpha_1}}) = 0$ (Definition 1.1.5). Let $\phi^{\text{sg,cl}} : e^{\text{cl}}(\Gamma_{X^\bullet_1}) \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X^\bullet_2})$ be the bijection of the sets of closed edges obtained in Theorem

5.14, $\alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_2^\bullet}},\phi^{\mathrm{sg,cl}}(e_1)}$ the element induced by ϕ for every $e_1 \in e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$,

$$\alpha_2 \overset{\mathrm{def}}{=} \sum_{e_1 \in e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})} \alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X_2^\bullet}}},$$

and $g^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to Y^\bullet_2$ the Galois admissible covering over k_2 corresponding to α_2. Then Proposition 5.18 (a) implies $\#(e^{\mathrm{op,ra}}_{g_{2,\alpha_2}}) = \#(v^{\mathrm{sp}}_{g_{2,\alpha_2}}) = 0$. We obtain that g_{i,α_i} is totally ramified over every node of Y_i, and that $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$ satisfy Condition A, Condition B, and Condition C. Write $N_i \subseteq \Pi_{X_i^\bullet}$ for the open normal subgroup corresponding to $Y^\bullet_{i,\alpha_i}$.

Let $H'_i \overset{\mathrm{def}}{=} H_i \cap N_i$ and $X^\bullet_{H'_i}$ the pointed stable curve over k_i corresponding to H'_i. Note that $X^\bullet_{H'_i}$ is isomorphic to a connected component of

$$X^\bullet_{H_i} \times_{X_i^\bullet} Y^\bullet_{i,\alpha_i}.$$

We denote by $h^\bullet_i : X^\bullet_{H'_i} \to Y^\bullet_{i,\alpha_i}$ the Galois admissible covering over k_i corresponding to the injection $H'_i \hookrightarrow N_i$. By applying Abhyankar's lemma, f_{H_i} is étale over D_{X_i} implies that h_i is étale. Then the lemma follows from Lemma 5.21 and Lemma 5.22. This completes the proof of the lemme when f_{H_2} is étale over D_{X_2}.

Next, let us prove the lemme in the general case. We take $\beta_{e_1} \in E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X_1^\bullet}},e_1}$ for every $e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ such that $\#(v^{\mathrm{sp}}_{g_{1,\beta_1}}) = 0$, where

$$\beta_1 \overset{\mathrm{def}}{=} \sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})} \beta_{e_1} \in E^*_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}.$$

Write $g^\bullet_{1,\beta_1} : Y^\bullet_{1,\beta_1} \to Y^\bullet_1$ for the Galois admissible covering over k_1 corresponding to β_1. Note that we have $\#(e^{\mathrm{cl,ra}}_{g_{1,\beta_1}}) = \#(v^{\mathrm{sp}}_{g_{1,\beta_1}}) = 0$. Let $\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ be the bijection of the sets of open edges obtained in Theorem 4.11, $\beta_{\phi^{\mathrm{sg,op}}(e_1)} \in E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X_2^\bullet}},\phi^{\mathrm{sg,op}}(e_1)}$ the element induced by ϕ for every $e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$,

$$\beta_2 \overset{\mathrm{def}}{=} \sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})} \beta_{\phi^{\mathrm{sg,op}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X_2^\bullet}}},$$

and $g^\bullet_{2,\beta_2} : Y^\bullet_{2,\beta_2} \to Y^\bullet_2$ the Galois admissible covering over k_2 corresponding to β_2. Then Proposition 5.18 (b) implies $\#(e^{\mathrm{cl,ra}}_{g_{2,\beta_2}}) = \#(v^{\mathrm{sp}}_{g_{2,\beta_2}}) = 0$. We obtain that g_{i,β_i} is totally ramified over every marked point of Y_i, and that $Y^\bullet_{1,\beta_1}$ and $Y^\bullet_{2,\beta_2}$ satisfy Condition A, Condition B, and Condition C. Write $Q_i \subseteq \Pi_{X_i^\bullet}$ for the open normal subgroup corresponding to $Y^\bullet_{i,\beta_i}$.

Let $H_i'' \overset{\text{def}}{=} H_i \cap Q_i$ and $X^\bullet_{H_i''}$ the pointed stable curve over k_i corresponding to H_i''. Note that $X^\bullet_{H_i''}$ is isomorphic to a connected component of

$$X^\bullet_{H_i} \times_{X^\bullet_i} Y^\bullet_{i,\beta_i}.$$

We denote by $h_i^{*,\bullet} : X^\bullet_{H_i''} \to Y^\bullet_{i,\beta_i}$ the Galois admissible covering over k_i corresponding to the injection $H_i'' \hookrightarrow Q_i$. By applying Abhyankar's lemma, h_i^* is étale over $D_{Y_{i,\beta_i}}$. By applying the lemma in the case where h_i^* is étale over $D_{Y_{i,\beta_i}}$, we obtain that $X^\bullet_{H_1''}$ and $X^\bullet_{H_2''}$ satisfy Condition A, Condition B, and Condition C. Then the lemma follows from Lemma 5.21. We complete the proof of the lemma. □

Lemma 5.24. *We maintain the notation introduced above. Suppose that G is a p-group. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii).

To verify the lemma, without loss the generality, it is sufficient to treat the case where $G \cong \mathbb{Z}/p\mathbb{Z}$. Since $f^\bullet_{H_i}$ is étale, $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (iii).

Let $V_i \subseteq v(\Gamma_{X^\bullet_i})^{>0,p}$ (5.1.2) be the subset of vertices such that the natural (outer) homomorphism

$$\Pi_{\widetilde{X}^\bullet_{i,v_i}} \hookrightarrow \Pi_{X^\bullet_i} \twoheadrightarrow G \overset{\text{def}}{=} \Pi_{X^\bullet_i}/H_i$$

is non-trivial (since $G \cong \mathbb{Z}/p\mathbb{Z}$, the homomorphism is a surjection) for all $v_i \in V_i$, where $\Pi_{\widetilde{X}^\bullet_{i,v_i}}$ is the admissible fundamental group of the smooth pointed stable curve $\widetilde{X}^\bullet_{i,v_i}$ associated to v_i (1.1.3). Then we obtain $\#(v(\Gamma_{X^\bullet_{H_i}})) = p(\#(v(\Gamma_{X^\bullet_i})) - \#(V_i)) + \#(V_i)$ and $\#(e^{\text{cl}}(\Gamma_{X^\bullet_{H_i}})) = p\#(e^{\text{cl}}(\Gamma_{X^\bullet_i}))$.

Theorem 5.17 implies that we have an injection

$$\psi_p^{\text{sg,ver}} : v(\Gamma_{X^\bullet_2})^{>0,p} \hookrightarrow v(\Gamma_{X^\bullet_1})^{>0,p}$$

induced by ϕ. We put

$$V_1' \overset{\text{def}}{=} \{\psi_p^{\text{sg,ver}}(v_2)\}_{v_2 \in V_2} \subseteq v(\Gamma_{X^\bullet_1})^{>0,p}.$$

By applying Lemma 5.15 and Lemma 5.16, we see that $V_1 = V_1'$. Thus, we have $\#(v(\Gamma_{X^\bullet_{H_1}})) = \#(v(\Gamma_{X^\bullet_{H_2}}))$ and $\#(e^{\text{cl}}(\Gamma_{X^\bullet_{H_1}})) = \#(e^{\text{cl}}(\Gamma_{X^\bullet_{H_2}}))$. This completes the proof of the lemma. □

Proposition 5.25. *We maintain the notation introduced above. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. Since G is a solvable group, the proposition follows from Lemma 5.23 and Lemma 5.24. □

5.6.3. Next, we prove the main result of the present section which we call the combinatorial Grothendieck conjecture for *open continuous homomorphisms*.

Theorem 5.26. *We maintain the settings introduced in 5.4.1. Then the open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ induces the following surjective maps (see 1.2.11 for $\mathrm{Ver}(\Pi_{X_i^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$, and $\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_i^\bullet})$)*

$$\phi^{\mathrm{ver}} : \mathrm{Ver}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Ver}(\Pi_{X_2^\bullet}),\ \phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet}),$$

$$\phi^{\mathrm{edg,cl}} : \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})$$

group-theoretically. Moreover, ϕ induces an isomorphism

$$\phi^{\mathrm{sg}} : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$$

of the dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$ group-theoretically.

Proof. By applying Theorem 4.11, the homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a surjective map $\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$ group-theoretically. We only need to treat the cases of ϕ^{ver} and $\phi^{\mathrm{edg,cl}}$, respectively.

Let $\mathcal{C}_{\Pi_{X_2^\bullet}}$ be a cofinal system of $\Pi_{X_2^\bullet}$ which consists of open normal subgroups of $\Pi_{X_2^\bullet}$. We put

$$\mathcal{C}_{\Pi_{X_1^\bullet}} \overset{\mathrm{def}}{=} \{H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \mid H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}\}.$$

Note that $\mathcal{C}_{\Pi_{X_1^\bullet}}$ is not a cofinal system of $\Pi_{X_1^\bullet}$ in general. Moreover, by applying Proposition 5.25, we have that $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ satisfy Condition A, Condition B, and Condition C for every $H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}$ and every $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \in \mathcal{C}_{\Pi_{X_1^\bullet}}$.

We treat the case of ϕ^{ver}. Let $\widehat{X}_i^\bullet$ be the universal solvable admissible covering of $X_i^\bullet$ associated to $\Pi_{X_i^\bullet}$ and $\Gamma_{\widehat{X}_i^\bullet}$ the dual semi-graph of $\widehat{X}_i^\bullet$. Let $\widehat{w}_1 \in v(\Gamma_{\widehat{X}_1^\bullet})$ and $\Pi_{\widehat{w}_1}$ the stabilizer subgroup of $\widehat{w}_1$. Write $w_{H_1} \in v(\Gamma_{X_{H_1}^\bullet})$, $H_1 \in \mathcal{C}_{\Pi_{X_1^\bullet}}$, for the image of $\widehat{w}_1$. Proposition 5.19 implies that ϕ induces a cofinal system of vertices

$$\mathcal{C}_{\widehat{w}_2} \overset{\mathrm{def}}{=} \{w_{H_2} \overset{\mathrm{def}}{=} \phi_{H_1}^{\mathrm{ver}}(w_{H_1})\}_{H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}},$$

which admits a natural action of $\Pi_{X_2^\bullet}$. Then we obtain an element $\widehat{w}_2 \in v(\Gamma_{\widehat{X}_2^\bullet})$. Moreover, the stabilizer of $\mathcal{C}_{\widehat{w}_2}$ is $\Pi_{\widehat{w}_2}$. We see immediately that ϕ induces a surjective open continuous homomorphism

$$\phi|_{\Pi_{\widehat{w}_1}} : \Pi_{\widehat{w}_1} \twoheadrightarrow \Pi_{\widehat{w}_2}$$

group-theoretically. Then we define

$$\phi^{\mathrm{ver}} : \mathrm{Ver}(\Pi_{X_1^\bullet}) \to \mathrm{Ver}(\Pi_{X_2^\bullet}),\ \Pi_{\widehat{w}_1} \mapsto \Pi_{\widehat{w}_2}.$$

Next, we prove that ϕ^{ver} is a surjective map. Let $\widehat{v}_2 \in v(\Gamma_{\widehat{X}_2^\bullet})$ and $\Pi_{\widehat{v}_2}$ the stabilizer subgroup of $\widehat{v}_2$. Write $v_{H_2} \in v(\Gamma_{X^\bullet_{H_2}})$, $H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}$, for the image of $\widehat{v}_2$. Then we obtain a cofinal system of vertices

$$\mathcal{C}_{\widehat{v}_2} \overset{\mathrm{def}}{=} \{v_{H_2}\}_{H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}}$$

associated to $\widehat{v}_2$. The cofinal system $\mathcal{C}_{\widehat{v}_2}$ admits a natural action of $\Pi_{X_2^\bullet}$. We see immediately that the stabilizer of $\mathcal{C}_{\widehat{v}_2}$ is equal to $\Pi_{\widehat{v}_2}$. Proposition 5.19 implies that ϕ and $\mathcal{C}_{\widehat{v}_2}$ induce a set of vertices

$$\mathcal{C}' \overset{\mathrm{def}}{=} \{v_{H_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,vex}})^{-1}(v_{H_2})\}_{H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}}$$

group-theoretically. By extending $\mathcal{C}'$ to a cofinal system of vertices, we obtain an element $\widehat{v}_1 \in v(\Gamma_{\widehat{X}_1^\bullet})$ such that the image of $\widehat{v}_1$ in $v(\Gamma_{X_{H_1}})$ is v_{H_1}. Thus, ϕ induces a surjective map

$$\phi|_{\Pi_{\widehat{v}_1}} : \Pi_{\widehat{v}_1} \twoheadrightarrow \Pi_{\widehat{v}_2}.$$

This means that ϕ^{ver} is a surjection.

By applying similar arguments to the arguments given in the above proof, we obtain that ϕ induces a surjective map $\phi^{\mathrm{edg,cl}} : \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})$ group-theoretically. This completes the proof of the first part of the theorem.

We prove the "moreover" part of the theorem. The surjections ϕ^{ver}, $\phi^{\mathrm{edg,op}}$, and $\phi^{\mathrm{edg,cl}}$ imply the following surjections

$$\phi^{\mathrm{sg,ver}} : v(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} \mathrm{Ver}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \mathrm{Ver}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \overset{\sim}{\to} v(\Gamma_{X_2^\bullet}),$$

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X_2^\bullet}),$$

$$\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}).$$

Since $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition C, we have that $\phi^{\mathrm{sg,ver}}$, $\phi^{\mathrm{sg,op}}$, and $\phi^{\mathrm{sg,cl}}$ are bijections. Let $\widehat{e}_1 \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_1^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{\widehat{X}_1^\bullet})$ and $\widehat{v}_1 \in v(\Gamma_{\widehat{X}_1^\bullet})$ such that $\widehat{e}_1$ abuts on $\widehat{v}_1$. Then we have $I_{\widehat{e}_1} \subseteq \Pi_{\widehat{v}_1}$, $\phi^{\mathrm{edg,op}}(I_{\widehat{e}_1}) \subseteq \phi^{\mathrm{ver}}(\Pi_{\widehat{v}_1})$ if $\widehat{e}_1 \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_1^\bullet})$, and $\phi^{\mathrm{edg,cl}}(I_{\widehat{e}_1}) \subseteq \phi^{\mathrm{ver}}(\Pi_{\widehat{v}_1})$ if $\widehat{e}_1 \in e^{\mathrm{cl}}(\Gamma_{\widehat{X}_1^\bullet})$. By applying [HM, Lemma 1.5, Lemma 1.7, and Lemma 1.9], ϕ induces an isomorphism of dual semi-graphs

$$\phi^{\mathrm{sg}} : \Gamma_{X_1^\bullet} \overset{\sim}{\to} \Gamma_{X_2^\bullet}$$

group-theoretically. This completes the proof of the theorem. □

Remark 5.26.1. We maintain the notation introduced above. We see immediately that Theorem 5.26 does not hold without Condition C (e.g. $X_1^\bullet$ is a generic curve of $\overline{\mathcal{M}}_{g,n}$, and $X_2^\bullet$ is a singular curve).

On the other hand, although the author cannot prove this at the present time, he believes that Theorem 5.26 also holds without Condition B (e.g. $n_{X_i} = 0$). The main difficult is that we do not have a precise formula for limits of p-averages of arbitrary pointed stable curves. Moreover, if the question of [Y3, Remark 4.10.2] is true, without too much difficulty, similar arguments to the arguments given in the proofs of this section imply that Theorem 5.26 holds without Condition B.

Remark 5.26.2. We maintain the notation introduced above. Suppose $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$ (i.e. we *do not* need to assume that Condition A, Condition B, and Condition C (ii) (iii) hold).

In [Y7], the author of the present paper formulated a new conjecture called *the group-theoretical specialization conjecture* (see [Y7, Section 3.1.3]). The conjecture establishes a precise description of the relationship between the various data (i.e. combinatorial data, topological data, and geometric data) associated to pointed stable curves (see [Y7, Definition 2.5]) and the open continuous homomorphisms of their admissible fundamental groups, and it will be played a central role to study *the homeomorphism conjecture for higher dimensional moduli spaces* (see [Y7, Introduction]). Moreover, the group-theoretical specialization conjecture is the ultimate generalization of the combinatorial Grothendieck conjecture in positive characteristic, and [T4, Theorem 0.1 and Theorem 5.2], [Y2, Theorem 1.2], Theorem 4.11, Theorem 5.26, and Theorem 5.30 of the present paper are special cases of this conjecture.

Corollary 5.27. *We maintain the notation introduced above. Let $Q_2 \subseteq \Pi_{X_2^\bullet}$ be an arbitrary open subgroup and $Q_1 \overset{\text{def}}{=} \phi^{-1}(Q_2) \subseteq \Pi_{X_1^\bullet}$. Then we have (see 2.2.1 for $\mathrm{Avr}_p(Q_i)$)*

$$\mathrm{Avr}_p(Q_1) = \mathrm{Avr}_p(Q_2).$$

Proof. The corollary follows immediately from Theorem 5.26. □

5.6.4. In the remainder of this subsection, we will prove that if $g_X = 0$, Theorem 5.26 holds without Condition A and Condition B (see Theorem 5.30), which will play a key role in the proof of the main theorem of the present paper. Furthermore, although the author cannot prove this at the present time, he also believes that Theorem 5.26 holds without Condition A and Condition B.

Lemma 5.28. *Let $E^\bullet = (E, D_E)$ be a pointed stable curve of type $(0, n)$ over an algebraically closed field k of characteristic $p > 0$, $\Pi_{E^\bullet}$ the solvable admissible fundamental group of $E^\bullet$, and $\ell >> n$ a prime number distinct from p. We put*

$$\mathrm{Edg}^{\mathrm{op},\ell,\mathrm{ab}}(\Pi_{E^\bullet}) \overset{\text{def}}{=} \{pr^{\ell,\mathrm{ab}}(I_{\widehat{e}}) \mid I_{\widehat{e}} \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{E^\bullet})\} = \{I_e\}_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})},$$

where $pr^{\ell,\mathrm{ab}}$ denotes the natural surjective homomorphism $\Pi_{E^\bullet} \twoheadrightarrow \Pi_{E^\bullet}^{\ell,\mathrm{ab}}$, and $I_e \overset{\text{def}}{=} pr^{\ell,\mathrm{ab}}(I_{\widehat{e}})$. Let $a_e \in I_e$, $e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})$, be a generator of I_e such that

$$\prod_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})} a_e = 1,$$

and let $\alpha : \Pi^{\ell,\mathrm{ab}}_{E^\bullet} \twoheadrightarrow \mathbb{Z}/\ell\mathbb{Z}$ *be a surjection and* $r_e \overset{\mathrm{def}}{=} \alpha(a_e)$*. Write* $g^\bullet : X^\bullet \to E^\bullet$ *for the Galois admissible covering over* k *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α*. Suppose that* $r_e \neq 0$ *for every* $e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})$*, and that*

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})} r_e = \ell$$

if we identify $\mathbb{Z}/\ell\mathbb{Z}$ *with* $\{0, 1, \ldots, \ell-1\} \subseteq \mathbb{Z}$*. Then* $g^\bullet$ *is totally ramified over every node and every marked point of* $E^\bullet$*. In particular, we have that the map of dual semi-graphs* $\Gamma_{X^\bullet} \to \Gamma_{E^\bullet}$ *of* $X^\bullet$ *and* $E^\bullet$ *induced by* $g^\bullet$ *is an isomorphism (as semi-graphs), and that* $X^\bullet$ *satisfies Condition A.*

Proof. We prove the lemma by induction on $\#v(\Gamma_{E^\bullet})$. Suppose that $\#v(\Gamma_{E^\bullet}) = 1$. Then the lemma is trivial.

Suppose that $\#v(\Gamma_{E^\bullet}) \geq 2$. Let $v_0 \in v(\Gamma_{E^\bullet})$ and $\widetilde{E}^\bullet_{v_0}$ the smooth pointed stable curve associated to v_0 (1.1.3). Note that the underlying curve $\widetilde{E}_{v_0}$ coincides with the irreducible component of E corresponding to v_0. On the other hand, we define a pointed stable curve over k to be

$$E^\bullet_0 = (E_0 \overset{\mathrm{def}}{=} \overline{E \setminus \widetilde{E}_{v_0}}, D_{E_0} \overset{\mathrm{def}}{=} (D_E \cap E_0) \cup (E_0 \cap \widetilde{E}_{v_0})),$$

where $\overline{E \setminus \widetilde{E}_{v_0}}$ denotes the topological closure of $E \setminus \widetilde{E}_{v_0}$ in E. Then $g^\bullet$ induces the following Galois admissible coverings

$$g^\bullet_{v_0} : \widetilde{X}^\bullet_{v_0} \to \widetilde{E}^\bullet_{v_0}, \ g^\bullet_0 : X^\bullet_0 \to E^\bullet_0$$

over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. To verify the lemma, we only need to prove that $g^\bullet_{v_0}$ and $g^\bullet_0$ are totally ramified over every node and every marked point of $\widetilde{E}^\bullet_{v_0}$ and $E^\bullet_0$, respectively.

Let $\Pi_{\widetilde{E}^\bullet_{v_0}}$ and $\Pi_{E^\bullet_0}$ be the solvable admissible fundamental groups of $\widetilde{E}^\bullet_{v_0}$ and $E^\bullet_0$, respectively. Since $\Gamma^{\mathrm{cpt}}_{E^\bullet}$ is 2-connected, [Y3, Corollary 3.5] implies that the natural homomorphism $\theta_{v_0} : \Pi^{\ell,\mathrm{ab}}_{\widetilde{E}^\bullet_{v_0}} \to \Pi^{\ell,\mathrm{ab}}_{E^\bullet}$ is an injection. Let $\theta_0 : \Pi^{\ell,\mathrm{ab}}_{E^\bullet_0} \to \Pi^{\ell,\mathrm{ab}}_{E^\bullet}$ be the homomorphism induced by the natural (outer) injective homomorphism $\Pi_{E^\bullet_0} \hookrightarrow \Pi_{E^\bullet}$ (in fact, θ_0 is also an injection).

Let $\{x\} = E_0 \cap \widetilde{E}_{v_0}$, $e_{v_0} \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}^\bullet_{v_0}})$ the open edge corresponding to x, $e_0 \in e^{\mathrm{op}}(\Gamma_{E^\bullet_0})$ the open edge corresponding to x, $\widehat{\widetilde{E}^\bullet_{v_0}}$ the universal solvable admissible covering of $\widetilde{E}^\bullet_{v_0}$, $\widehat{E}^\bullet_0$ the universal solvable admissible covering of $E^\bullet_0$, $\widehat{e}_{v_0} \in e^{\mathrm{op}}(\Gamma_{\widehat{\widetilde{E}^\bullet_{v_0}}})$ an element over e_{v_0}, and $\widehat{e}_0 \in e^{\mathrm{op}}(\Gamma_{\widehat{E}^\bullet_0})$ an element over e_0. We denote by $I_{e_{v_0}}$ the image of $I_{\widehat{e}_{v_0}}$ of $\Pi_{\widetilde{E}^\bullet_{v_0}} \twoheadrightarrow \Pi^{\ell,\mathrm{ab}}_{\widetilde{E}^\bullet_{v_0}}$, and by I_{e_0} the image of $I_{\widehat{e}_0}$ of $\Pi_{E^\bullet_0} \twoheadrightarrow \Pi^{\ell,\mathrm{ab}}_{E^\bullet_0}$. We put

$$a_{e_{v_0}} = (\prod_{e \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}^\bullet_{v_0}}) \setminus \{e_{v_0}\}} a_e)^{-1},\ a_{e_0} = (\prod_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet_0}) \setminus \{e_0\}} a_e)^{-1}.$$

Then $a_{e_{v_0}}$ and a_{e_0} are generators of $I_{e_{v_0}}$ and I_{e_0}, respectively. Moreover, we put

$$\widetilde{\alpha}_{v_0} : \Pi^{\ell,\mathrm{ab}}_{\widetilde{E}^\bullet_{v_0}} \overset{\theta_{v_0}}{\to} \Pi^{\ell,\mathrm{ab}}_{E^\bullet} \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}/\ell\mathbb{Z},\ \alpha_0 : \Pi^{\ell,\mathrm{ab}}_{E^\bullet_0} \overset{\theta_0}{\to} \Pi^{\ell,\mathrm{ab}}_{E^\bullet} \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}/\ell\mathbb{Z}.$$

Then the structures of maximal pro-prime-to-p quotients of solvable admissible fundamental groups (1.2.4) imply

$$\widetilde{\alpha}_{v_0}(a_{e_{v_0}}) = \ell - \sum_{e \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}^\bullet_{v_0}}) \setminus \{e_{v_0}\}} r_e = \sum_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet_0}) \setminus \{e_0\}} r_e,\ \alpha_0(a_{e_0}) = \sum_{e \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}^\bullet_{v_0}}) \setminus \{e_{v_0}\}} r_e.$$

Thus, by induction, we have that $g^\bullet_{v_0}$ and $g^\bullet_0$ are totally ramified over every node and every marked point of $\widetilde{E}^\bullet_{v_0}$ and $E^\bullet_0$, respectively. We complete the proof of the lemma. □

Lemma 5.29. *Let $E^\bullet$ be a pointed stable curve of type $(0, n)$ over an algebraically closed field k of characteristic $p > 0$. Then $E^\bullet$ satisfies Condition B.*

Proof. Let $f^\bullet : W^\bullet \to E^\bullet$ be an arbitrary admissible covering over k, $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$, and $f^{\mathrm{sg}} : \Gamma_{W^\bullet} \to \Gamma_{E^\bullet}$ the map of dual semi-graphs of $W^\bullet$ and $X^\bullet$ induced by $f^\bullet$. To verify the lemma, we only need to prove that $\Gamma^{\mathrm{cpt}}_{W^\bullet}$ is 2-connected.

Suppose that $f^\bullet$ is trivial. Then the lemma follows from that $\Gamma^{\mathrm{cpt}}_{E^\bullet}$ is 2-connected.

Suppose that $f^\bullet$ is non-trivial. Let $w \in v(\Gamma_{W^\bullet})$ and $v \in v(\Gamma_{E^\bullet})$. We denote by $\pi_0(w)$ the set of connected components of $\Gamma_{W^\bullet} \setminus \{w\}$. Suppose $v = f^{\mathrm{sg}}(w)$. Let $C_w \in \pi_0(w)$ be an arbitrary connected component. We see immediately that $f^{\mathrm{sg}}(C_w) \cap e^{\mathrm{op}}(\Gamma_{E^\bullet}) \neq \emptyset$. Then we obtain $C_w \cap e^{\mathrm{op}}(\Gamma_{W^\bullet}) \neq \emptyset$. Thus, we have $\#(\pi_0(w)) = 1$. This means that $\Gamma^{\mathrm{cpt}}_{W^\bullet}$ is 2-connected. We complete the proof of the lemma. □

5.6.5. Theorem 5.26 implies the following important result.

Theorem 5.30. *Let $i \in \{1, 2\}$, and let $E^\bullet_i$ be a pointed stable curve of type $(0, n)$ over k_i of characteristic $p > 0$, $\Pi_{E^\bullet_i}$ the solvable admissible fundamental group of $E^\bullet_i$, and*

$$\phi_E : \Pi_{E^\bullet_1} \to \Pi_{E^\bullet_2}$$

an arbitrary open continuous homomorphism. Suppose that $E^\bullet_1$ and $E^\bullet_2$ satisfy Condition C. Then $\phi_E : \Pi_{E^\bullet_1} \to \Pi_{E^\bullet_2}$ induces the following surjective maps

$$\phi^{\mathrm{ver}}_E : \mathrm{Ver}(\Pi_{E^\bullet_1}) \twoheadrightarrow \mathrm{Ver}(\Pi_{E^\bullet_2}),\ \phi^{\mathrm{edg,op}}_E : \mathrm{Edg}^{\mathrm{op}}(\Pi_{E^\bullet_1}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{E^\bullet_2}),$$
$$\phi^{\mathrm{edg,cl}}_E : \mathrm{Edg}^{\mathrm{cl}}(\Pi_{E^\bullet_1}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{cl}}(\Pi_{E^\bullet_2})$$

group-theoretically. Moreover, ϕ_E induces an isomorphism

$$\phi_E^{\mathrm{sg}} : \Gamma_{E_1^\bullet} \xrightarrow{\sim} \Gamma_{E_2^\bullet}$$

of the dual semi-graphs of $E_1^\bullet$ and $E_2^\bullet$ group-theoretically.

Proof. Lemma 4.3 implies that ϕ_E is a surjective map. By applying Theorem 4.11, the homomorphism $\phi_E : \Pi_{E_1^\bullet} \twoheadrightarrow \Pi_{E_2^\bullet}$ induces a surjective map $\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{E_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{E_2^\bullet})$ group-theoretically. We only need to treat the cases of ϕ_E^{ver} and $\phi_E^{\mathrm{edg,cl}}$, respectively.

Let ℓ be a prime number such that $\ell \neq p$, and that $\ell >> n$. Let $\alpha_2 : \Pi_{E_2^\bullet}^{\ell,\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/\ell\mathbb{Z}$ satisfying the assumptions of Lemma 5.28. Then Theorem 4.11 implies that ϕ_E and α_2 induce a surjection $\alpha_1 : \Pi_{E_1^\bullet}^{\ell,\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/\ell\mathbb{Z}$ satisfying the assumptions of Lemma 5.28 too. Write $g_i^\bullet : X_i^\bullet \to E_i^\bullet$ for the Galois admissible covering over k_i with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Then Lemma 5.28 and Lemma 5.29 imply that $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition A, Condition B, and Condition C.

Write $\Pi_{X_i^\bullet} \subseteq \Pi_{E_i^\bullet}$ for the open normal subgroup corresponding to $g_i^\bullet$. Let $\Pi_{\widehat{v}_{X_i}} \in \mathrm{Ver}(\Pi_{X_i^\bullet})$, $I_{\widehat{e}_{X_i}} \in \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_i^\bullet})$, $\Pi_{\widehat{v}_i} \in \mathrm{Ver}(\Pi_{E_i^\bullet})$ the unique element containing $\Pi_{\widehat{v}_{X_i}}$, and $I_{\widehat{e}_i} \in \mathrm{Edg}^{\mathrm{cl}}(\Pi_{E_i^\bullet})$ the unique element containing $I_{\widehat{e}_{X_i}}$. Since $\Pi_{\widehat{v}_i}$ and $I_{\widehat{e}_i}$ are the normalizers of $\Pi_{\widehat{v}_{X_i}}$ and $I_{\widehat{e}_{X_i}}$ in $\Pi_{E_i^\bullet}$, respectively, the theorem follows immediately from Theorem 5.26. This completes the proof of the theorem. $\square$

PART III: MAIN RESULT

6. The homeomorphism conjecture for closed points when $g = 0$

We maintain the notation introduced in 3.1.3. In this section, we will prove that $\pi_{g,n}^{\mathrm{adm}}([q])$ (resp. $\pi_{g,n}^{\mathrm{sol}}([q])$) is a closed point of $\overline{\Pi}_{g,n}$ (resp. $\overline{\Pi}_{g,n}^{\mathrm{sol}}$) for every $[q] \in \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}$ if $g = 0$. In particular, the homeomorphism conjecture (resp. the solvable homeomorphism conjecture) holds when $(g, n) = (0, 3), (0, 4)$. In the present section, *all fundamental groups are solvable admissible fundamental groups unless indicated otherwise.* The main results of the present section are Theorem 6.6 and Theorem 6.7.

6.0.1. **Settings**. We fix some notation. Let $i \in \{1, 2\}$, and let k_i be an algebraically closed field of characteristic $p > 0$ and $\overline{\mathbb{F}}_{p,i}$ the algebraic closure of $\mathbb{F}_p$ in k_i. Let $X_i^\bullet$ be a pointed stable curve of type $(0, n)$ over k_i, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X^\bullet}$. Note that $\Gamma_{X_i^\bullet}$ is a tree, and that the irreducible component X_{i,v_i} corresponding to $v_i \in v(\Gamma_{X_i^\bullet})$ is isomorphic to $\mathbb{P}^1_{k_i}$. In particular, X_{i,v_i} is smooth over k_i. For simplicity of notation, we shall use the notation $X_{i,v_i}^\bullet$ to denote the smooth pointed stable curve $\widetilde{X}_{i,v_i}^\bullet$ of type $(0, n_{i,v_i})$ over k_i associated to

$v_i \in v(\Gamma_{X_i^\bullet})$ (1.1.3). Let $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X_i^\bullet})$. We shall denote by x_{e_i} the closed point of X_i corresponding to e_i.

On the other hand, let $\Pi_{X_i^\bullet}$ be the solvable admissible fundamental group of $X_i^\bullet$ and

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

an arbitrary open continuous homomorphism. By Lemma 4.3, we see that ϕ is a *surjective* open continuous homomorphism. Then ϕ induces an isomorphism

$$\phi^p : \Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet}$$

of the maximal prime-to-p quotients of solvable admissible fundamental groups. Let $\widehat{X}_i^\bullet$ be the universal solvable admissible covering of $X_i^\bullet$ corresponding to $\Pi_{X_i^\bullet}$, $\Gamma_{\widehat{X}_i^\bullet}$ the dual semi-graph of $\widehat{X}_i^\bullet$, and $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$. We put

$$\mathrm{Edg}^{\mathrm{op}}_{e_i}(\Pi_{X_i^\bullet}) \overset{\mathrm{def}}{=} \{I_{\widehat{e}_i} \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet}) \mid \widehat{e}_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet}) \text{ is an open edge over } e_i\}.$$

Moreover, in the present section, we shall suppose that k_1 *is an algebraic closure of* $\mathbb{F}_p$ (i.e. $k_1 = \overline{\mathbb{F}}_{p,1}$).

We denote by $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(-,-)$ and $\mathrm{Isom}_{\mathrm{pg}}(-,-)$ the set of open continuous homomorphisms of profinite groups and the set of continuous isomorphisms of profinite groups, respectively.

6.1. **Smooth case.** In this subsection, we maintain the settings introduced in 6.0.1 and assume that $X_i^\bullet$ is smooth over k_i. We recall some results obtained in [HYZ] which will be used in the remainder of the present paper.

6.1.1. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$, and let $X^\bullet$ be a *smooth* pointed stable curve of type $(0, n)$ over $\overline{\mathbb{F}}_p$. We fix two marked points x_∞, $x_0 \in D_X$ distinct from each other. Moreover, we choose any field $k' \cong \overline{\mathbb{F}}_p$, and choose any isomorphism $\varphi : X \xrightarrow{\sim} \mathbb{P}^1_{k'}$ as schemes such that $\varphi(x_\infty) = \infty$ and $\varphi(x_0) = 0$. Then the set of $\overline{\mathbb{F}}_p$-rational points $X(\overline{\mathbb{F}}_p) \setminus \{x_\infty\} \xrightarrow{\sim} \mathbb{A}^1_{k'}(k')$ is equipped with a structure of $\mathbb{F}_p$-module via the bijection φ. Note that since any k'-isomorphism of $\mathbb{P}^1_{k'}$ fixing ∞ and 0 is a scalar multiplication, the $\mathbb{F}_p$-module structure of $X(\overline{\mathbb{F}}_p) \setminus \{x_\infty\}$ does not depend on the choices of k' and φ but depends only on the choices of x_∞ and x_0. We call that $X(\overline{\mathbb{F}}_p) \setminus \{x_\infty\}$ *is equipped with a structure of* $\mathbb{F}_p$*-module with respect to* x_∞ *and* x_0. Then we have the following lemma.

Lemma 6.1. *We maintain the notation introduced above. Suppose that* $X_i^\bullet$ *is smooth over* k_i*. Let* $e_{1,0}$*,* $e_{1,\infty} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ *be open edges distinct from each other.*

Theorem [4.11](#) implies that ϕ induces a bijection $\phi^{\mathrm{sg,op}}: e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ group-theoretically. We put $e_{2,0} \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{1,0})$ and $e_{2,\infty} \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{1,\infty})$. Let

$$\sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} x_{e_1} = x_{e_{1,0}}$$

be a linear condition with respect to $x_{e_{1,\infty}}$ and $x_{e_{1,0}}$ on $X_1^\bullet$, where $b_{e_1} \in \mathbb{F}_p$ for every $e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$. Then we have the following linear condition

$$\sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} x_{\phi^{\mathrm{sg,op}}(e_1)} = x_{\phi^{\mathrm{sg,op}}(e_{1,0})} = x_{e_{2,0}}$$

with respect to $x_{e_{2,\infty}}$ and $x_{e_{2,0}}$ on $X_2^\bullet$.

Proof. This is Lemma 4.2 of [HYZ]. □

Remark 6.1.1. Note that, if $X_1 = \mathbb{P}^1_{k_1}$, then the linear condition mentioned in Lemma 6.1 is

$$\sum_{x_1 \in D_{X_1} \setminus \{\infty, 0\}} b_{e_1} x_1 = 0$$

with respect to ∞ and 0.

6.1.2. One of the main result of [HYZ] is the following result:

Proposition 6.2. *We maintain the notation introduced above. Suppose that $X_1^\bullet$ and $X_2^\bullet$ are smooth over k_1 and k_2, respectively. Then we have that*

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset$$

if and only if $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$ (Definition 3.1 (c)). In particular, if this is the case, we have that $X_2^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_2, and that

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) = \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}).$$

Proof. This is Theorem 4.3 (ii) of [HYZ]. □

Remark 6.2.1. Let $[q] \in \mathfrak{M}^{\mathrm{cl}}_{0,n}$ be an arbitrary point. Proposition 6.2 and Proposition 3.10 (a) imply $V(\pi^{\mathrm{sol}}_{0,n}([q])) \cap \Pi^{\mathrm{sol}}_{0,n} = \pi^{\mathrm{sol}}_{0,n}([q])$. Then we have that $[\pi_1^{\mathrm{sol}}(q)]$ is a closed point of $\Pi^{\mathrm{sol}}_{0,n}$. In particular,

$$\pi^{\mathrm{t}}_{0,4}: \mathfrak{M}_{0,4} \twoheadrightarrow \Pi_{0,4},\ \pi^{\mathrm{t,sol}}_{0,4}: \mathfrak{M}_{0,4} \twoheadrightarrow \Pi^{\mathrm{sol}}_{0,4}$$

are homeomorphisms. Note that Proposition 6.2 *cannot* tell us whether or not $[\pi_1^{\mathrm{sol}}(q)]$ is closed in $\overline{\Pi}^{\mathrm{sol}}_{0,n}$. In fact, this is highly non-trivial, see Proposition 6.5 below.

6.2. **General case.** We maintain the settings introduced in 6.0.1. In this subsection, we generalize Proposition 6.2 to the case where $X_i^\bullet$ is an arbitrary pointed stable curve of type $(0, n)$.

6.2.1. Firstly, we have the following lemmas.

Lemma 6.3. *We maintain the notation introduced above. Suppose that $X_1^\bullet$ is a singular curve. Then $X_2^\bullet$ is also a singular curve.*

Proof. Lemma 5.4 implies that there exists a Galois admissible covering $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ over k_1 with Galois group G such that $(\#(G), p) = 1$, that the Betti number of the dual semi-graph of $Y_1^\bullet$ is positive, and that $Y_1^\bullet$ satisfies Condition A. Then $\phi^{p'}$ induces a Galois admissible covering $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ over k_2 with Galois group G. Write g_{Y_i} for the genus of $Y_i^\bullet$ and r_{Y_i} for the Betti number of the dual semi-graph of $Y_i^\bullet$.

By applying Theorem 4.11, we obtain $g_{Y_1} = g_{Y_2}$. Moreover, Theorem 2.1 and Lemma 2.2 (b) imply $0 < r_{Y_1} \leq r_{Y_2}$. This means that $X_2^\bullet$ is a singular curve. We complete the proof of the lemma. □

Lemma 6.4. *Let $X^\bullet$ be a pointed stable curve of type $(0, n)$ over an algebraically closed field k of characteristic $p > 0$ and $\ell \geq 3$ a prime number distinct from p. Then there exists a Galois admissible covering $f^\bullet : Y^\bullet \to X^\bullet$ over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the genus of $Y^\bullet$ is 0 and $\#(Y_v \cap D_Y) \geq 3$ for some irreducible component Y_v of Y.*

Proof. Suppose that $X^\bullet$ is smooth over k. Then the lemma is trivial. We may suppose that $X^\bullet$ is singular. Since $X^\bullet$ is of type $(0, n)$, there exist irreducible components X_{v_1}, X_{v_2} of X distinct from each other such that $\#(X_{v_1} \cap D_X) \geq 2$ and $\#(X_{v_2} \cap D_X) \geq 2$.

Let $x_1 \in X_{v_1} \cap D_X$, $x_2 \in X_{v_2} \cap D_X$, and let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois admissible covering over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that f is totally ramified over x_1 and x_2, and that f is étale over $D_X \setminus \{x_1, x_2\}$. We see immediately that the irreducible components $Y_{v_1} \stackrel{\text{def}}{=} f^{-1}(X_{v_1})$ and $Y_{v_2} \stackrel{\text{def}}{=} f^{-1}(X_{v_2})$ of Y satisfy the conditions $\#(Y_{v_1} \cap D_Y) \geq 3$ and $\#(Y_{v_2} \cap D_Y) \geq 3$, respectively. Moreover, the Riemann-Hurwitz formula implies that the genus of $Y^\bullet$ is 0. This completes the proof of the lemma. □

6.2.2. Next, we generalize Proposition 6.2 to the case where we only assume that $X_1^\bullet$ is smooth over k_1.

Proposition 6.5. *We maintain the notation introduced above. Suppose that $X_1^\bullet$ is smooth over k_1. Then $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$ (Definition 3.1 (c)). In particular, we have that $X_2^\bullet$ is smooth over k_2, and that $X_2^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_2.*

Proof. If $X_2^\bullet$ is smooth over k_2, the proposition follows immediately from Proposition 6.2. Then we may assume that $X_2^\bullet$ is singular (i.e. $\#(v(\Gamma_{X_2^\bullet})) \geq 2$).

Step 1: We reduce the proposition to the case where $X_i^\bullet$ satisfies the conditions mentioned in Lemma 6.4.

Let $\ell \geq 3$ be a prime number distinct from p. Lemma 6.4 implies that there exists an open normal subgroup $H_2 \subseteq \Pi_{X_2^\bullet}$ such that $\Pi_{X_2^\bullet}/H_2 \cong \mathbb{Z}/\ell\mathbb{Z}$, that the Galois admissible covering $f_{H_2}^\bullet : X_{H_2}^\bullet \to X_2^\bullet$ corresponding to H_2 is totally ramified over two marked points of $X_2^\bullet$, and that there exists $w_{H_2} \in v(\Gamma_{X_{H_2}^\bullet})$ satisfying $\#(X_{H_2,w_{H_2}} \cap D_{X_{H_2}}) \geq 3$. Write $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$ for the open subgroup and $f_{H_1}^\bullet : X_{H_1}^\bullet \to X_1^\bullet$ for the Galois admissible covering over k_1 corresponding to H_1. Theorem 4.11 implies that $f_{H_1}^\bullet$ is totally ramified over two marked points of $X_1^\bullet$, and that $n_{X_{H_1}} = n_{X_{H_2}}$. Since $f_{H_i}^\bullet$ is totally ramified over two marked points, we have $g_{X_{H_1}} = g_{X_{H_2}} = 0$.

If we can prove the proposition holds for $X_{H_1}^\bullet$, $X_{H_2}^\bullet$, and $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$, then we obtain that $X_2^\bullet$ is also smooth over k_2. Then the proposition follows immediately from Proposition 6.2. Thus, by replacing $X_1^\bullet$, $X_2^\bullet$, and ϕ by $X_{H_1}^\bullet$, $X_{H_2}^\bullet$, and $\phi|_{H_1}$, respectively, we may assume that $\#(X_{2,w_2} \cap D_{X_2}) \geq 3$ for some $w_2 \in v(\Gamma_{X_2^\bullet})$.

Step 2: We construct a pointed stable curve $Z_i^\bullet$ of type $(0, 5)$ over k_i from $X_i^\bullet$.

Let $e_{2,\infty}$, $e_{2,0}$, $e_{2,1} \in e^{\text{op}}(\Gamma_{X_2^\bullet}) \cap e^{\text{op}}(\Gamma_{X_{2,w_2}^\bullet})$ distinct from each other. By Theorem 4.11, ϕ induces a bijection

$$\phi^{\text{sg,op}} : e^{\text{op}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} e^{\text{op}}(\Gamma_{X_2^\bullet})$$

group-theoretically. We put

$$e_{1,\infty} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{2,\infty}),\ e_{1,0} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{2,0}),\ e_{1,1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{2,1}).$$

Without loss of generality, we may assume

$$x_{e_{i,\infty}} \overset{\text{def}}{=} \infty,\ x_{e_{i,0}} \overset{\text{def}}{=} 0,\ x_{e_{i,1}} \overset{\text{def}}{=} 1,\ X_1 = \mathbb{P}^1_{k_1},\ X_{2,w_2} = \mathbb{P}^1_{k_2}.$$

Let $\pi_0(\Gamma_{X_2^\bullet} \setminus \{w_2\})$ denote the set of connected components of $\Gamma_{X_2^\bullet} \setminus \{w_2\}$ in $\Gamma_{X_2^\bullet}$. Let $C_2 \in \pi_0(\Gamma_{X_2^\bullet} \setminus \{w_2\})$. Since $X_2^\bullet$ is a pointed stable curve of type $(0, n)$ over k_2, we have $\#(C_2 \cap e^{\text{op}}(\Gamma_{X_2^\bullet})) \geq 2$. Let $e_{2,C_2,1}$, $e_{2,C_2,2} \in C_2 \cap e^{\text{op}}(\Gamma_{X_2^\bullet})$ be open edges distinct from each other. We put

$$e_{1,2} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{2,C_2,1}) \in e^{\text{op}}(\Gamma_{X_1^\bullet}),\ e_{1,3} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{2,C_2,2}) \in e^{\text{op}}(\Gamma_{X_1^\bullet}).$$

We denote by X_{2,C_2} the semi-stable subcurve of X_2 whose irreducible components are the irreducible components corresponding to the vertices of $\Gamma_{X_2^\bullet}$ contained in

C_2. Moreover, we write $e_{2,2}$ for the unique closed edge of $\Gamma_{X_2^\bullet}$ connecting w_2 and C_2. Then the node $x_{e_{2,2}}$ corresponding to $e_{2,2}$ is the unique closed point of X_2 contained in $X_{2,w_2} \cap X_{2,C_2}$.

We put

$$Z_1^\bullet = (Z_1 \stackrel{\text{def}}{=} X_1, D_{Z_1} \stackrel{\text{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,2}}, x_{e_{1,3}}\}),$$

$$Y_{1,1}^\bullet = (Y_{1,1} \stackrel{\text{def}}{=} X_1, D_{Y_{1,1}} \stackrel{\text{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,2}}\}),$$

$$Y_{1,2}^\bullet = (Y_{1,2} \stackrel{\text{def}}{=} X_1, D_{Y_{1,2}} \stackrel{\text{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,3}}\}),$$

$$Y_2^\bullet = (Y_2 \stackrel{\text{def}}{=} X_{2,w_2}, D_{Y_2} \stackrel{\text{def}}{=} \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,2}}\}).$$

Moreover, we denote by $Z_2^\bullet$ the pointed stable curve of type $(0,5)$ over k_2 associated to the pointed semi-stable curve

$$(X_2, \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,C_2,1}}, x_{e_{2,C_2,2}}\})$$

over k_2 (i.e. the pointed stable curve obtained by contracting the (-1)-curves and the (-2)-curves of $(X_2, \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,C_2,1}}, x_{e_{2,C_2,2}}\})$. We see that Z_2 has two irreducible components Z_{w_2} and Z_{C_2} such that Z_{w_2} is equal to X_{2,w_2}, that $\{x_{e_{2,2}}\} = Z_{w_2} \cap Z_{C_2}$, that $\{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}\} \subseteq Z_{w_2}$, and that $\{x_{e_{2,C_2,1}}, x_{e_{2,C_2,2}}\} \subseteq Z_{C_2}$.

Step 3: We prove that the solvable admissible fundamental groups and the natural homomorphisms between the solvable admissible fundamental groups of pointed stable curves constructing in Step 2 can be reconstructed group-theoretically from ϕ.

Let $I_1 \subseteq \Pi_{X_1^\bullet}, I_2 \subseteq \Pi_{X_2^\bullet}$ be the closed subgroups generated by the inertia subgroups of

$$\bigcup_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}, e_{1,1}, e_{1,2}, e_{1,3}\}} \mathrm{Edg}^{\mathrm{op}}_{e_1}(\Pi_{X_1^\bullet}),$$

$$\bigcup_{e_2 \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \setminus \{e_{2,\infty}, e_{2,0}, e_{2,1}, e_{2,C_2,1}, e_{2,C_2,2}\}} \mathrm{Edg}^{\mathrm{op}}_{e_2}(\Pi_{X_2^\bullet}),$$

respectively, $I_{1,1} \subseteq \Pi_{X_1^\bullet}, I_{1,2} \subseteq \Pi_{X_1^\bullet}$ the closed subgroups generated by the inertia subgroups of

$$\bigcup_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}, e_{1,1}, e_{1,2}\}} \mathrm{Edg}^{\mathrm{op}}_{e_1}(\Pi_{X_1^\bullet}),$$

$$\bigcup_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}, e_{1,1}, e_{1,3}\}} \mathrm{Edg}^{\mathrm{op}}_{e_1}(\Pi_{X_1^\bullet}),$$

respectively, and $I_{2,1} \subseteq \Pi_{X_2^\bullet}$, $I_{2,2} \subseteq \Pi_{X_2^\bullet}$ the closed subgroups generated by the inertia subgroups of

$$\bigcup_{e_2 \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \setminus \{e_{2,\infty}, e_{2,0}, e_{2,1}, e_{2,C_2,1}\}} \mathrm{Edg}^{\mathrm{op}}_{e_2}(\Pi_{X_2^\bullet}),$$

$$\bigcup_{e_2 \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \setminus \{e_{2,\infty}, e_{2,0}, e_{2,1}, e_{2,C_2,2}\}} \mathrm{Edg}^{\mathrm{op}}_{e_2}(\Pi_{X_2^\bullet}),$$

respectively.

Then Theorem 4.11 implies $\phi(I_1) = I_2$, $\phi(I_{1,1}) = I_{2,1}$, and $\phi(I_{1,2}) = I_{2,2}$. Moreover, we see that $\Pi_{X_1^\bullet}/I_1$ and $\Pi_{X_2^\bullet}/I_2$ are (outer) isomorphic to the solvable admissible fundamental groups of $Z_1^\bullet$ and $Z_2^\bullet$, respectively, that $\Pi_{X_1^\bullet}/I_{1,1}$ and $\Pi_{X_1^\bullet}/I_{1,2}$ are (outer) isomorphic to the solvable admissible fundamental groups of $Y_{1,1}^\bullet$ and $Y_{1,2}^\bullet$, respectively, and that $\Pi_{X_2^\bullet}/I_{2,1}$ and $\Pi_{X_2^\bullet}/I_{2,2}$ are (outer) isomorphic to the solvable admissible fundamental group of $Y_2^\bullet$. Note that $I_{1,1} \supseteq I_1 \subseteq I_{1,2}$ and $I_{2,1} \supseteq I_2 \subseteq I_{2,2}$.

On the other hand, ϕ induces the following surjective open continuous homomorphisms

$$\overline{\phi} : \Pi_{Z_1^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/I_1 \twoheadrightarrow \Pi_{Z_2^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/I_2,$$

$$\overline{\phi}_{1,1} : \Pi_{Y_{1,1}^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/I_{1,1} \twoheadrightarrow \Pi_{Y_2^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/I_{2,1},$$

$$\overline{\phi}_{1,2} : \Pi_{Y_{1,2}^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/I_{1,2} \twoheadrightarrow \Pi_{Y_2^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/I_{2,2}$$

which fit into the following commutative diagram:

$$\begin{array}{ccc}
\Pi_{Y_{1,1}^\bullet} & \xrightarrow{\overline{\phi}_{1,1}} & \Pi_{Y_2^\bullet} \\
\psi_{1,1}\uparrow & & \psi_{2,1}\uparrow \\
\Pi_{Z_1^\bullet} & \xrightarrow{\overline{\phi}} & \Pi_{Z_2^\bullet} \\
\psi_{1,2}\downarrow & & \psi_{2,2}\downarrow \\
\Pi_{Y_{1,2}^\bullet} & \xrightarrow{\overline{\phi}_{1,2}} & \Pi_{Y_2^\bullet},
\end{array}$$

where $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{2,1}$, and $\psi_{2,2}$ denote the natural quotient homomorphisms.

Note that $\psi_{2,1} \circ \overline{\phi} \neq \psi_{2,2} \circ \overline{\phi}$, and that the homomorphisms of maximal prime-to-p quotients of solvable admissible fundamental groups $\overline{\phi}^{p'}_{1,1}$, $\overline{\phi}^{p'}$, and $\overline{\phi}^{p'}_{1,2}$ induced by $\overline{\phi}_{1,1}$, $\overline{\phi}$, and $\overline{\phi}_{1,2}$, respectively, are isomorphisms. Moreover, we see that $\psi_{2,1}(I_{\widehat{e}_{2,C_2,1}}) \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,2}}(\Pi_{Y_2^\bullet})$ and $\psi_{2,2}(I_{\widehat{e}_{2,C_2,2}}) \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,2}}(\Pi_{Y_2^\bullet})$ for every $I_{\widehat{e}_{2,C_2,1}} \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,C_2,1}}(\Pi_{Z_2^\bullet})$ and every $I_{\widehat{e}_{2,C_2,2}} \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,C_2,2}}(\Pi_{Z_2^\bullet})$.

Step 4: We construct linear conditions associated to irreducible components of $Z_i^\bullet$.

Let $\widehat{e}_{i,0} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet})$ be an open edge over $e_{i,0}$. By applying Theorem 4.13,

$$\mathbb{F}_{\widehat{e}_{i,0}} \overset{\mathrm{def}}{=} (I_{\widehat{e}_{i,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widehat{e}_{i,0}}\}$$

admits a structure of field which can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. Since we assume that k_1 is an algebraic closure of $\mathbb{F}_p$, we may suppose that $k_1 = \mathbb{F}_{\widehat{e}_{1,0}}$. Moreover, Theorem 4.13 implies that ϕ induces a field isomorphism

$$\phi^{\mathrm{fd}}_{\widehat{e}_{1,0},\widehat{e}_{2,0}} : \mathbb{F}_{\widehat{e}_{1,0}} \overset{\sim}{\to} \mathbb{F}_{\widehat{e}_{2,0}}$$

group-theoretically. We see that there exists a natural number m prime to p such that $\mathbb{F}_p(\zeta_{m,1})$ contains mth roots of $x_{e_{1,2}}$, $x_{e_{1,3}}$, where $\zeta_{m,1}$ denotes a fixed primitive mth root of unity in $\mathbb{F}_{\widehat{e}_{1,0}}$. Let $s \overset{\mathrm{def}}{=} [\mathbb{F}_p(\zeta_{m,1}) : \mathbb{F}_p]$. For each $e_{1,u} \in \{e_{1,2}, e_{1,3}\}$, we fix an mth root $x_{e_{1,u}}^{\frac{1}{m}}$ in $\mathbb{F}_{\widehat{e}_{1,0}}$. Then we have

$$x_{e_{1,u}}^{\frac{1}{m}} = \sum_{t=0}^{s-1} b_{1,u,t}\zeta_{m,1}^t,\ u \in \{2,3\},$$

where $b_{1,u,t} \in \mathbb{F}_p$ for each $u \in \{2,3\}$ and each $t \in \{0,\ldots,s-1\}$. Note that since $x_{e_{1,2}} \neq x_{e_{1,3}}$, there exists $t' \in \{0,\ldots,s-1\}$ such that $b_{1,2,t'} \neq b_{1,3,t'}$.

Let $Z_1 \setminus \{x_{e_{1,\infty}}\} = \mathrm{Spec}\,\mathbb{F}_{\widehat{e}_{1,0}}[\mathsf{x}_1]$, and let $f^\bullet_{Q_1} : Z^\bullet_{Q_1} \to Z_1^\bullet$ be the Galois admissible covering over $\mathbb{F}_{\widehat{e}_{1,0}}$ with Galois group $\mathbb{Z}/m\mathbb{Z}$ determined by the equation $\mathsf{y}_1^m = \mathsf{x}_1$ and $Q_1 \subseteq \Pi_{Z_1^\bullet}$ the open normal subgroup induced by $f^\bullet_{Q_1}$. Then f_{Q_1} is totally ramified over $\{x_{e_{1,0}} = 0, x_{e_{1,\infty}} = \infty\}$ and is étale over $D_{Z_1} \setminus \{x_{e_{1,0}}, x_{e_{1,\infty}}\}$. Note that $Z_{Q_1} = \mathbb{P}^1_{\mathbb{F}_{\widehat{e}_{1,0}}}$, and that the marked points of $D_{Z_{Q_1}}$ over $\{x_{e_{1,0}}, x_{e_{1,\infty}}\}$ are $\{x_{e_{Q_1,0}} \overset{\mathrm{def}}{=} 0, x_{e_{Q_1,\infty}} \overset{\mathrm{def}}{=} \infty\}$. We put

$$x_{e_{Q_1,u}} \overset{\mathrm{def}}{=} x_{e_{1,u}}^{\frac{1}{m}} \in D_{Z_{Q_1}},\ u \in \{2,3\},$$

$$x_{e^t_{Q_1,1}} \overset{\mathrm{def}}{=} \zeta^t_{m,1} \in D_{Z_{Q_1}},\ t \in \{0,\ldots,s-1\}.$$

Thus, we obtain a linear condition

$$x_{e_{Q_1,u}} = \sum_{t=0}^{s-1} b_{1,u,t} x_{e^t_{Q_1,1}},\ u \in \{2,3\},$$

with respect to $x_{e_{Q_1,0}}$ and $x_{e_{Q_1,\infty}}$ on $Z^\bullet_{Q_1}$.

Since $(m,p)=1$, there exists a unique open normal subgroup $Q_2 \subseteq \Pi_{Z_2^\bullet}$ such that $\overline{\phi}^{-1}(Q_2)=Q_1$. On the other hand, we put

$$Q_{1,1} \overset{\text{def}}{=} \psi_{1,1}(Q_1) \subseteq \Pi_{Y_{1,1}^\bullet},\ Q_{1,2} \overset{\text{def}}{=} \psi_{1,2}(Q_1) \subseteq \Pi_{Y_{1,2}^\bullet},$$

$$Q_{2,1} \overset{\text{def}}{=} \psi_{2,1}(Q_2) \subseteq \Pi_{Y_2^\bullet},\ Q_{2,2} \overset{\text{def}}{=} \psi_{2,2}(Q_2) \subseteq \Pi_{Y_2^\bullet}.$$

Note that the constructions of Q_1 and Q_2 imply $P_2 \overset{\text{def}}{=} Q_{2,1}=Q_{2,2}$. The commutative diagram of profinite groups constructed in Step 3 induces the following commutative diagram of profinite groups:

$$\begin{array}{ccc} Q_{1,1} & \xrightarrow{\overline{\phi}_{Q_{1,1}}} & P_2 \\ {\scriptstyle\psi_{Q_1,1,1}}\uparrow & & {\scriptstyle\psi_{Q_2,2,1}}\uparrow \\ Q_1 & \xrightarrow{\overline{\phi}_{Q_1}} & Q_2 \\ {\scriptstyle\psi_{Q_1,1,2}}\downarrow & & {\scriptstyle\psi_{Q_2,2,2}}\downarrow \\ Q_{1,2} & \xrightarrow{\overline{\phi}_{Q_{1,2}}} & P_2. \end{array}$$

Let $j \in \{1,2\}$. Write $Y_{Q_{1,j}}^\bullet$ for the pointed stable curve over k_1 corresponding to $Q_{1,j}$. Then we see that $e^{\mathrm{op}}(\Gamma_{Y_{Q_{1,j}}^\bullet})$ can be regarded as a subset of $e^{\mathrm{op}}(\Gamma_{Z_{Q_1}^\bullet})$ via $\psi_{Q_1,1,j}$. By applying Theorem 4.11 for $\overline{\phi}_{Q_1}$, $\overline{\phi}_{Q_{1,1}}$, and $\overline{\phi}_{Q_{1,2}}$, respectively, the above commutative diagram of profinite groups implies that we may put

$$e_{Q_2,\infty} \overset{\text{def}}{=} \overline{\phi}_{Q_1}^{\mathrm{sg,op}}(e_{Q_1,\infty}),\ e_{Q_2,0} \overset{\text{def}}{=} \overline{\phi}_{Q_1}^{\mathrm{sg,op}}(e_{Q_1,0}),$$

$$e_{Q_2,1}^t \overset{\text{def}}{=} \overline{\phi}_{Q_1}^{\mathrm{sg,op}}(e_{Q_1,1}^t),\ t \in \{0,\dots,s-1\},$$

$$e_{P_2,\infty} \overset{\text{def}}{=} \overline{\phi}_{Q_{1,1}}^{\mathrm{sg,op}}(e_{Q_1,\infty}) = \overline{\phi}_{Q_{1,2}}^{\mathrm{sg,op}}(e_{Q_1,\infty}),\ e_{P_2,0} \overset{\text{def}}{=} \overline{\phi}_{Q_{1,1}}^{\mathrm{sg,op}}(e_{Q_1,0}) = \overline{\phi}_{Q_{1,2}}^{\mathrm{sg,op}}(e_{Q_1,0}),$$

$$e_{P_2,1}^t \overset{\text{def}}{=} \overline{\phi}_{Q_{1,1}}^{\mathrm{sg,op}}(e_{Q_1,1}^t) = \overline{\phi}_{Q_{1,2}}^{\mathrm{sg,op}}(e_{Q_1,1}^t),\ t \in \{0,\dots,s-1\},$$

$$e_{P_2,2} \overset{\text{def}}{=} \overline{\phi}_{Q_{1,1}}^{\mathrm{sg,op}}(e_{Q_1,2}) = \overline{\phi}_{Q_{1,2}}^{\mathrm{sg,op}}(e_{Q_1,3}).$$

Moreover, we may identify $e_{Q_2,1}^t$, $t \in \{0,\dots,s-1\}$, with $e_{P_2,1}^t$ via $\psi_{Q_2,2,1}$ (or $\psi_{Q_2,2,2}$).

We denote by $\zeta_{m,2} \overset{\text{def}}{=} \phi^{\mathrm{fd}}_{\widehat{e}_{1,0},\widehat{e}_{2,0}}(\zeta_{m,1})$. Without loss of generality, we may assume $x_{e_{Q_2,1}^1} = \zeta_{m,2}$. Then we have

$$x_{e_{P_2,1}^t} = x_{e_{Q_2,1}^t} = \zeta_{m,2}^t,\ t \in \{0,\dots,s-1\}.$$

Let $Y^\bullet_{P_2}$ be the pointed stable curve over k_2 corresponding to $P_2 \subseteq \Pi_{Y^\bullet_2}$. Moreover, by applying Lemma 6.1 for $\overline{\phi}_{Q_{1,1}}$, we obtain

$$x_{e_{P_2,2}} = \sum_{t=0}^{s-1} b_{1,2,t} x_{e^t_{P_2,1}}$$

with respect to $x_{e_{P_2,0}}$ and $x_{e_{P_2,\infty}}$ on $Y^\bullet_{P_2}$. On the other hand, by applying Lemma 6.1 for $\overline{\phi}_{Q_{1,2}}$, we obtain

$$x_{e_{P_2,2}} = \sum_{t=0}^{s-1} b_{1,3,t} x_{e^t_{P_2,1}}$$

with respect to $x_{e_{P_2,0}}$ and $x_{e_{P_2,\infty}}$ on $Y^\bullet_{P_2}$. This means that

$$\sum_{t=0}^{s-1} b_{1,2,t}\zeta^t_{m,2} = \sum_{t=0}^{s-1} b_{1,3,t}\zeta^t_{m,2},$$

which is impossible as $b_{1,2,t'} \neq b_{1,3,t'}$ for some $t' \in \{0, \dots, s-1\}$. Then we obtain that $X^\bullet_2$ is smooth over k_2. Thus, the proposition follows from Proposition 6.2. This completes the proof of the proposition. □

6.2.3. Now, we prove the first form of the main theorem of the present paper.

Theorem 6.6. *Let $X^\bullet_i$, $i \in \{1, 2\}$, be an arbitrary pointed stable curve of type $(0, n)$ over an algebraically closed field k_i of characteristic $p > 0$ and $\Pi_{X^\bullet_i}$ either the admissible fundamental group of $X^\bullet_i$ or the solvable admissible fundamental group of $X^\bullet_i$. Suppose that k_1 is an algebraic closure of $\mathbb{F}_p$. Then we have that*

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) \neq \emptyset$$

if and only if $X^\bullet_1$ is Frobenius equivalent to $X^\bullet_2$ (Definition 3.1 (c)). In particular, if this is the case, we have that $X^\bullet_2$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_2, and that

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) = \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}).$$

Proof. To verify the theorem, it is sufficient to prove the theorem when $\Pi_{X^\bullet_i}$ is the solvable admissible fundamental group of $X^\bullet_i$. The "if" part of the theorem follows from [Y4, Proposition 3.7]. Let us prove the "only if" part of the theorem. Suppose that $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) \neq \emptyset$, and let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2})$ be an arbitrary open continuous homomorphism. Then Lemma 4.3 implies that ϕ is a surjection.

Suppose that $X^\bullet_1$ is smooth over k_1. Then the theorem follows from Proposition 6.5. Thus, we may assume that $X^\bullet_1$ is a singular pointed stable curve.

Note that since $X^\bullet_1$ is singular, we have $n = \#(e^{\mathrm{op}}(\Gamma_{X^\bullet_1})) \geq 4$. We prove the theorem by induction on $\#(e^{\mathrm{op}}(\Gamma_{X^\bullet_1}))$. Suppose that $\#(e^{\mathrm{op}}(\Gamma_{X^\bullet_1})) = 4$. Since $X^\bullet_1$ is a singular pointed stable curve of type $(0, 4)$, we obtain $\#(v(\Gamma_{X^\bullet_1})) = 2$ and

$\#(e^{\rm cl}(\Gamma_{X_1^\bullet}))=1$. On the other hand, by applying Lemma 6.3, we obtain that $X_2^\bullet$ is also a singular pointed stable curve of type $(0,4)$. Thus, we have $\#(e^{\rm op}(\Gamma_{X_2^\bullet}))=4$, $\#(v(\Gamma_{X_2^\bullet}))=2$, and $\#(e^{\rm cl}(\Gamma_{X_2^\bullet}))=1$. Then $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition C defined in 5.3.1. Thus, by Theorem 5.30 and Proposition 6.2, we obtain that $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$.

Suppose that $\#(e^{\rm op}(\Gamma_{X_1^\bullet}))\geq 5$. Theorem 4.11 implies that ϕ induces a bijection

$$\phi^{\rm sg,op}:e^{\rm op}(\Gamma_{X_1^\bullet})\xrightarrow{\sim}e^{\rm op}(\Gamma_{X_2^\bullet})$$

group-theoretically. Let $e_{1,n}\in e^{\rm op}(\Gamma_{X_1^\bullet})$ and $e_{2,n}\overset{\rm def}{=}\phi^{\rm sg,op}(e_{1,n})$. We denote by $Z_i^\bullet$ the pointed stable curve of type $(0,n-1)$ over k_i associated to the pointed semi-stable curve $(X_i, D_{X_i}\setminus\{x_{e_{i,n}}\})$ (i.e. the pointed stable curve obtained by contracting the (-1)-curves and the (-2)-curves of $(X_i, D_{X_i}\setminus\{x_{e_{i,n}}\})$).

Write $I_{i,n}\subseteq\Pi_{X_i^\bullet}$ for the closed subgroup generated by the subgroups contained in $\mathrm{Edg}^{\rm op}_{e_{i,n}}(\Pi_{X_i^\bullet})$. Then we see that $\Pi_{Z_i^\bullet}\overset{\rm def}{=}\Pi_{X_i^\bullet}/I_{i,n}$ is (outer) isomorphic to the solvable admissible fundamental group of $Z_i^\bullet$. Moreover, Theorem 4.11 implies $\phi(I_{1,n})=I_{2,n}$. Then ϕ induces a surjective open continuous homomorphism

$$\overline{\phi}:\Pi_{Z_1^\bullet}\twoheadrightarrow\Pi_{Z_2^\bullet}.$$

By induction, we obtain that $Z_1^\bullet$ is Frobenius equivalent to $Z_2^\bullet$. Then ϕ induces a bijection of dual semi-graphs

$$\overline{\phi}^{\rm sg}:\Gamma_{Z_1^\bullet}\xrightarrow{\sim}\Gamma_{Z_2^\bullet}.$$

In particular, we put

$$\overline{\phi}^{\rm sg,ver}\overset{\rm def}{=}\overline{\phi}^{\rm sg}|_{v(\Gamma_{Z_1^\bullet})}:v(\Gamma_{Z_1^\bullet})\xrightarrow{\sim}v(\Gamma_{Z_2^\bullet}),$$

$$\overline{\phi}^{\rm sg,op}\overset{\rm def}{=}\overline{\phi}^{\rm sg}|_{e^{\rm op}(\Gamma_{Z_1^\bullet})}:e^{\rm op}(\Gamma_{Z_1^\bullet})\xrightarrow{\sim}e^{\rm op}(\Gamma_{Z_2^\bullet}).$$

Note that $v(\Gamma_{Z_i^\bullet})$, $e^{\rm op}(\Gamma_{Z_i^\bullet})$, the set of irreducible components of Z_i, the set of marked points D_{Z_i} of $Z_i^\bullet$ can be regarded naturally as subsets of $v(\Gamma_{X_i^\bullet})$, $e^{\rm op}(\Gamma_{X_i^\bullet})$, the set of irreducible components of X_i, the set of marked points D_{X_i} of $X_i^\bullet$ via the contracting morphism $(X_i, D_{X_i}\setminus\{x_{e_{i,n}}\})\to Z_i^\bullet$, respectively. Moreover, we see that one of the following cases may occur:

(i) $\#(v(\Gamma_{X_1^\bullet}))=\#(v(\Gamma_{Z_1^\bullet}))=\#(v(\Gamma_{X_2^\bullet}))=\#(v(\Gamma_{Z_2^\bullet}))$;
(ii) $\#(v(\Gamma_{X_1^\bullet}))-1=\#(v(\Gamma_{Z_1^\bullet}))=\#(v(\Gamma_{X_2^\bullet}))-1=\#(v(\Gamma_{Z_2^\bullet}))$;
(iii) $\#(v(\Gamma_{X_1^\bullet}))=\#(v(\Gamma_{Z_1^\bullet}))=\#(v(\Gamma_{X_2^\bullet}))-1=\#(v(\Gamma_{Z_2^\bullet}))$;
(iv) $\#(v(\Gamma_{X_1^\bullet}))-1=\#(v(\Gamma_{Z_1^\bullet}))=\#(v(\Gamma_{X_2^\bullet}))=\#(v(\Gamma_{Z_2^\bullet}))$.

Suppose that either (i) or (ii) holds. Then $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition C defined in 5.3.1. Thus, by Theorem 5.30 and Proposition 6.2, we obtain that $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$.

Suppose that (iii) holds. Let $v_2 \in v(\Gamma_{X_2^\bullet})$ such that $x_{e_{2,n}} \in X_{v_2} \stackrel{\text{def}}{=} X_{2,v_2}$ (i.e. the irreducible component of X_2 corresponding to v_2). Since $\#(v(\Gamma_{X_2^\bullet})) = \#(v(\Gamma_{Z_2^\bullet}))+1$, we have $\#(X_{v_2} \cap D_{X_2}) = 2$. Note that $\{v_2\} = v(\Gamma_{X_2^\bullet}) \setminus v(\Gamma_{Z_1^\bullet})$.

Let $x_{e_{2,n-1}} \in X_{v_2} \cap D_{X_2}$ be the marked point distinct from $x_{e_{2,n}}$ and $e_{2,n-1} \in e^{\text{op}}(\Gamma_{X_2^\bullet})$ the open edge corresponding to the marked point $x_{e_{2,n-1}}$. On the other hand, let $w_1 \in v(\Gamma_{X_1^\bullet})$ such that $x_{e_{1,n}} \in X_{w_1} \stackrel{\text{def}}{=} X_{1,w_1}$. We put

$$w_2 \stackrel{\text{def}}{=} \overline{\phi}^{\text{sg,ver}}(w_1) \in v(\Gamma_{Z_2^\bullet}) \subseteq v(\Gamma_{X_2^\bullet}),$$

$$e_{1,n-1} \stackrel{\text{def}}{=} (\overline{\phi}^{\text{sg,op}})^{-1}(e_{2,n-1}) \in e^{\text{op}}(\Gamma_{Z_1^\bullet}) \subseteq e^{\text{op}}(\Gamma_{X_1^\bullet}).$$

Since $Z_1^\bullet$ is a pointed stable curve of type $(0, n-1)$, we have

$$\#(X_{w_1} \cap D_{Z_1}) + \#(X_{w_1} \cap Z_1^{\text{sing}}) \geq 3.$$

Then we see that there exist marked points $x_{e_{1,n-2}}, x_{e_{1,n-3}} \in D_{Z_1} \setminus \{x_{e_{1,n-1}}\}$ distinct from each other such that one of the following conditions is satisfied:

(1) If $\#(X_{w_1} \cap D_{Z_1}) \geq 3$, then $x_{e_{1,n-2}}, x_{e_{1,n-3}} \in X_{w_1}$.

(2) If $\#(X_{w_1} \cap D_{Z_1}) = 2$ and $x_{e_{1,n-1}} \notin X_{w_1}$, then $x_{e_{1,n-2}}, x_{e_{1,n-3}} \in X_{w_1}$.

(3) If $\#(X_{w_1} \cap D_{Z_1}) = 1$ and $x_{e_{1,n-1}} \notin X_{w_1}$, then we have that $x_{e_{1,n-3}} \in X_{w_1}$, and that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 1$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is ≥ 2) containing $x_{e_{1,n-1}}$ and $x_{e_{1,n-2}}$, respectively, are distinct from each other.

(4) If $\#(X_{w_1} \cap D_{Z_1}) = 2$ and $x_{e_{1,n-1}} \in X_{w_1}$, then we have that $x_{e_{1,n-3}} \in X_{w_1}$, and that $x_{e_{1,n-2}}$ is contained in a connected component of $Z_1 \setminus X_{w_1}$.

(5) If $\#(X_{w_1} \cap D_{Z_1}) = 1$ and $x_{e_{1,n-1}} \in X_{w_1}$, then we have that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 1$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is ≥ 2) containing $x_{e_{1,n-2}}$ and $x_{e_{1,n-3}}$, respectively, are distinct from each other.

(6) If $\#(X_{w_1} \cap D_{Z_1}) = 0$, then we have that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 0$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is ≥ 3) containing $x_{e_{1,n-1}}$, $x_{e_{1,n-2}}$, and $x_{e_{1,n-3}}$, respectively, are distinct from each other.

Write $e_{1,n-2}$ and $e_{1,n-3} \in e^{\text{op}}(\Gamma_{Z_1^\bullet})$ for the open edges corresponding to the marked points $x_{e_{1,n-2}}$ and $x_{e_{1,n-3}}$, respectively. We put

$$e_{2,n-2} \stackrel{\text{def}}{=} \overline{\phi}^{\text{sg,op}}(e_{1,n-2}),\ e_{2,n-3} \stackrel{\text{def}}{=} \overline{\phi}^{\text{sg,op}}(e_{1,n-3}).$$

Let $Y_i^\bullet$ be the pointed stable curve of type $(0, 4)$ over k_i associated to the pointed semi-stable curve

$$(X_i, \{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}).$$

By the construction of the set of marked points $\{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}$, we see that $Y_1^\bullet$ is smooth over k_1 whose underlying curve is X_{w_1}, and that $Y_2^\bullet$ is singular whose irreducible components are $X_{w_2} \overset{\text{def}}{=} X_{2,w_2}$ and X_{v_2}.

Next, we will see that the solvable admissible fundamental groups and the natural homomorphisms between the solvable admissible fundamental groups of pointed stable curves constructing above can be reconstructed group-theoretically from ϕ. Let $I_i \subseteq \Pi_{X_i^\bullet}$ be the closed subgroup generated by the subgroups contained in

$$\bigcup_{e_i \in e^{\text{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_{i,n}, e_{i,n-1}, e_{i,n-2}, e_{i,n-3}\}} \text{Edg}^{\text{op}}_{e_i}(\Pi_{X_i^\bullet}).$$

We see that $\Pi_{Y_i^\bullet} \overset{\text{def}}{=} \Pi_{X_i^\bullet}/I_i$ is (outer) isomorphic to the solvable admissible fundamental group of $Y_i^\bullet$. Moreover, Theorem 4.11 implies $\phi(I_1) = I_2$. Then we obtain a surjective open continuous homomorphism $\overline{\overline{\phi}} : \Pi_{Y_1^\bullet} \twoheadrightarrow \Pi_{Y_2^\bullet}$. This contradicts Proposition 6.5, since Proposition 6.5 implies that $Y_2^\bullet$ is smooth over k_2. Then (iii) does not occur.

Suppose that (iv) holds. Similar arguments to the arguments given in the proof of (iii) imply that (iv) does not occur. More precisely, we have the following.

Let $v_1 \in v(\Gamma_{X_1^\bullet})$ such that $x_{e_{1,n}} \in X_{v_1} \overset{\text{def}}{=} X_{1,v_1}$. Since $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{Z_1^\bullet}))+1$, we have $\#(X_{v_1} \cap D_{X_1}) = 2$. Note that $\{v_1\} = v(\Gamma_{X_1^\bullet}) \setminus v(\Gamma_{Z_1^\bullet})$.

Let $x_{e_{1,n-1}} \in X_{v_1} \cap D_{X_1}$ be the marked point distinct from $x_{e_{1,n}}$ and $e_{1,n-1} \in e^{\text{op}}(\Gamma_{X_1^\bullet})$ the open edge corresponding to the marked point $x_{e_{1,n-1}}$. On the other hand, let $w_2 \in v(\Gamma_{X_2^\bullet})$ such that $x_{e_{2,n}} \in X_{w_2} \overset{\text{def}}{=} X_{2,w_2}$. We put

$$w_1 \overset{\text{def}}{=} (\overline{\phi}^{\text{sg,ver}})^{-1}(w_2) \in v(\Gamma_{Z_1^\bullet}) \subseteq v(\Gamma_{X_1^\bullet}),$$

$$e_{2,n-1} \overset{\text{def}}{=} \overline{\phi}^{\text{sg,op}}(e_{1,n-1}) \in e^{\text{op}}(\Gamma_{Z_2^\bullet}) \subseteq e^{\text{op}}(\Gamma_{X_2^\bullet}).$$

Since $Z_2^\bullet$ is a pointed stable curve of type $(0, n-1)$, we have

$$\#(X_{w_2} \cap D_{Z_2}) + \#(X_{w_2} \cap Z_2^{\text{sing}}) \geq 3.$$

Then we see that there exist marked points $x_{e_{2,n-2}}$, $x_{e_{2,n-3}} \in D_{Z_2} \setminus \{x_{e_{2,n-1}}\}$ distinct from each other such that one of the following conditions is satisfied:

(1) If $\#(X_{w_2} \cap D_{Z_2}) \geq 3$, then $x_{e_{2,n-2}}$, $x_{e_{2,n-3}} \in X_{w_2}$.

(2) If $\#(X_{w_2} \cap D_{Z_2}) = 2$ and $x_{e_{2,n-1}} \notin X_{w_2}$, then $x_{e_{2,n-2}}$, $x_{e_{2,n-3}} \in X_{w_2}$.

(3) If $\#(X_{w_2} \cap D_{Z_2}) = 1$ and $x_{e_{2,n-1}} \notin X_{w_2}$, then we have that $x_{e_{2,n-3}} \in X_{w_2}$, and that the connected components of $Z_2 \setminus X_{w_2}$ (note that since $\#(X_{w_2} \cap D_{Z_2}) = 1$, the cardinality of the set of connected components of $Z_2 \setminus X_{w_2}$ is ≥ 2) containing $x_{e_{2,n-1}}$ and $x_{e_{2,n-2}}$, respectively, are distinct from each other.

(4) If $\#(X_{w_2} \cap D_{Z_2}) = 2$ and $x_{e_{2,n-1}} \in X_{w_2}$, then we have that $x_{e_{2,n-3}} \in X_{w_2}$, and that $x_{e_{2,n-2}}$ is contained in a connected component of $Z_2 \setminus X_{w_2}$.

(5) If $\#(X_{w_2} \cap D_{Z_2}) = 1$ and $x_{e_{2,n-1}} \in X_{w_2}$, then we have that the connected components of $Z_2 \setminus X_{w_2}$ (note that since $\#(X_{w_2} \cap D_{Z_2}) = 1$, the cardinality of the set of connected components of $Z_2 \setminus X_{w_2}$ is ≥ 2) containing $x_{e_{2,n-2}}$ and $x_{e_{2,n-3}}$, respectively, are distinct from each other.

(6) If $\#(X_{w_2} \cap D_{Z_2}) = 0$, then we have that the connected components of $Z_2 \setminus X_{w_2}$ (note that since $\#(X_{w_2} \cap D_{Z_2}) = 0$, the cardinality of the set of connected components of $Z_2 \setminus X_{w_2}$ is ≥ 3) containing $x_{e_{2,n-1}}$, $x_{e_{2,n-2}}$, and $x_{e_{2,n-3}}$, respectively, are distinct from each other.

Write $e_{2,n-2}$ and $e_{2,n-3} \in e^{\mathrm{op}}(\Gamma_{Z_2^\bullet})$ for the open edges corresponding to the marked points $x_{e_{2,n-2}}$ and $x_{e_{2,n-3}}$, respectively. We put

$$e_{1,n-2} \overset{\mathrm{def}}{=} (\overline{\phi}^{\mathrm{sg,op}})^{-1}(e_{2,n-2}),\ e_{1,n-3} \overset{\mathrm{def}}{=} (\overline{\phi}^{\mathrm{sg,op}})^{-1}(e_{2,n-3}).$$

Let $Y_i^\bullet$ be the pointed stable curve of type $(0,4)$ over k_i associated to the pointed semi-stable curve

$$(X_i, \{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}).$$

By the construction of the set of marked points $\{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}$, we see that $Y_1^\bullet$ is singular whose irreducible component are $X_{w_1} \overset{\mathrm{def}}{=} X_{1,w_1}$ and X_{v_1}, and that $Y_2^\bullet$ is smooth over k_2 whose underlying curve is X_{w_2}.

Let $I_i \subseteq \Pi_{X_i^\bullet}$ be the closed subgroup generated by the subgroups contained in

$$\bigcup_{e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_{i,n}, e_{i,n-1}, e_{i,n-2}, e_{i,n-3}\}} \mathrm{Edg}^{\mathrm{op}}_{e_i}(\Pi_{X_i^\bullet}).$$

We see that $\Pi_{Y_i^\bullet} \overset{\mathrm{def}}{=} \Pi_{X_i^\bullet}/I_i$ is (outer) isomorphic to the solvable admissible fundamental group of $Y_i^\bullet$. Moreover, Theorem 4.11 implies $\phi(I_1) = I_2$. Then we obtain a surjective open continuous homomorphism $\overline{\overline{\phi}} : \Pi_{Y_1^\bullet} \twoheadrightarrow \Pi_{Y_2^\bullet}$. This contradicts Lemma 6.3, since Lemma 6.3 implies that $Y_2^\bullet$ is singular. Then (iv) does not occur. This completes the proof of the theorem. □

6.2.4. Theorem 6.6 implies the following result concerning the homeomorphism conjecture formulated in 3.3.

Theorem 6.7. *We maintain the notation introduced in 3.1.3 and 3.2.1. Let* $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n}$ *be an arbitrary closed point. Then* $\pi^{\mathrm{adm}}_{0,n}([q])$ *and* $\pi^{\mathrm{sol}}_{0,n}([q])$ *are closed points of* $\overline{\Pi}_{0,n}$ *and* $\overline{\Pi}^{\mathrm{sol}}_{0,n}$*, respectively. In particular, the homeomorphism conjecture and the solvable homeomorphism conjecture hold when* $(g,n) = (0,3)$ *or* $(0,4)$*.*

Proof. To verify the theorem, we only need to treat the case of solvable admissible fundamental groups.

Let $V(\pi_{0,n}^{\mathrm{sol}}([q]))$ be the topological closure of $\pi_{0,n}^{\mathrm{sol}}([q])$ in $\overline{\Pi}_{0,n}^{\mathrm{sol}}$ and $[\pi_1^{\mathrm{sol}}(q')] \in V(\pi_{0,n}^{\mathrm{sol}}([q]))$ an arbitrary point. Then by Proposition 3.10 (a), we obtain that there exists a surjective open continuous homomorphism

$$\phi : \pi_1^{\mathrm{sol}}(q) \twoheadrightarrow \pi_1^{\mathrm{sol}}(q').$$

Theorem 6.6 implies $q \sim_{fe} q'$. Thus, we obtain $[\pi_1^{\mathrm{sol}}(q)] = [\pi_1^{\mathrm{sol}}(q')]$. This means that $V(\pi_{0,n}^{\mathrm{sol}}([q])) = [\pi_1^{\mathrm{sol}}(q)]$ is a closed point of $\overline{\Pi}_{0,n}^{\mathrm{sol}}$. □

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