Moduli Spaces of Fundamental Groups of Curves in Positive Characteristic I

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Abstract

Let p be a prime number, and let $\overline{M}_{g,n}$ be the coarse moduli space of the moduli stack over an algebraic closure of the finite field $\mathbb{F}_p$ classifying pointed stable curves of type (g,n). In the present paper, we introduce a topological space $\overline{\Pi}_{g,n}$ which is called *the moduli space of admissible fundamental groups of pointed stable curves of type* (g,n) *over algebraically closed fields of characteristic* p, whose underlying set consists of the set of isomorphism classes of the admissible fundamental groups of pointed stable curves of type (g,n), and whose topology is determined by the sets of finite quotients of the admissible fundamental groups of pointed stable curves of type (g,n). By introducing a certain equivalence relation $\sim_{fe}$ on the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$, we obtain a topological space $\overline{\mathfrak{M}}_{g,n} \overset{\text{def}}{=} |\overline{M}_{g,n}|/\sim_{fe}$ whose topology is induced by the Zariski topology of $\overline{M}_{g,n}$. Moreover, there is a natural continuous map

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n}.$$

The topological space $\overline{\Pi}_{g,n}$ gives us a new insight into the theory of the anabelian geometry of curves over algebraically closed fields of characteristic p, and every topological property concerning $\overline{\Pi}_{g,n}$ can be regarded as an anabelian property concerning pointed stable curves over algebraically closed fields of characteristic p. Furthermore, we pose a conjecture (=*the Homeomorphism Conjecture*) which says that $\pi^{\mathrm{adm}}_{g,n}$ is a homeomorphism. The Homeomorphism Conjecture generalizes all the conjectures in the theory of anableian geometry of curves over algebraically closed fields of characteristic p, and means that moduli spaces of curves *can be reconstructed group-theoretically as topological spaces* from the admissible fundamental groups of curves. We prove that $\pi^{\mathrm{adm}}_{0,n}([q])$ is a closed point of $\overline{\Pi}_{0,n}$ when $[q]$ is a closed point of $\overline{\mathfrak{M}}_{0,n}$. In particular, we obtain that the Homeomorphism Conjecture holds when $(g,n) = (0,4)$.

Keywords: pointed stable curve, admissible fundamental group, moduli space, anabelian geometry; positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14F35, 14G32.

Contents

Introduction

In the present paper, we study the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of type (g_X, n_X) over a field k of characteristic $\mathrm{char}(k)$, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. First, we explain some background of the anabelian geometry of curves. Suppose that $X^\bullet$ is smooth over k. When k is an *arithmetic field* (e.g. number field, p-adic field, finite field, etc.), A. Grothendieck suggested a theory of arithmetic geometry called *anabelian geometry* (cf. [3], [4]), roughly speaking, in the case of curves, whose ultimate goal is the following question.

> Can we recover the isomorphism class of $X^\bullet$ group-theoretically from various versions of its fundamental group?

The various formulations of this question are called *Grothendieck's anabelian conjecture for curves* or the Grothendieck conjecture, for short. The Grothendieck conjecture has been proven in many cases. For example, the conjecture was proved by H. Nakamura,

A. Tamagawa, and S. Mochizuki in the case of number fields (cf. [14], [15], [26], [11]), and was proved by Tamagawa, J. Stix, and M. Saïdi-Tamagawa in the case of finitely generated fields over $\mathbb{F}_p$ (cf. [26], [24], [25], [22], [23]). All the proofs of the Grothendieck conjecture for curves over arithmetic fields mentioned above require the use of *the highly non-trivial outer Galois representations* induced by the fundamental exact sequences of fundamental groups.

Next, let us return to the case where $X^\bullet$ is an arbitrary pointed stable curve, and suppose that k is *an algebraically closed field.* By choosing a suitable base point of $X^\bullet$, we have the admissible fundamental group

$$\pi_1^{\mathrm{adm}}(X^\bullet)$$

of $X^\bullet$ (cf. Definition 1.2). In particular, if $X^\bullet$ is smooth over k, then $\pi_1^{\mathrm{adm}}(X^\bullet)$ is naturally (outer) isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(X^\bullet)$. Write $\pi_1^{\mathrm{adm}}(X^\bullet)^{p'}$ for the maximal prime-to-p quotient of $\pi_1^{\mathrm{adm}}(X^\bullet)$ if $\mathrm{char}(k) = p > 0$. The profinite group $\pi_1^{\mathrm{adm}}(X^\bullet)$ (resp. $\pi_1^{\mathrm{adm}}(X^\bullet)^{p'}$) is isomorphic to the profinite completion (resp. pro-prime-to-p completion) of the following group (cf. [33, Théorème 2.2 (c)])

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle$$

when $\mathrm{char}(k) = 0$ (resp. $\mathrm{char}(k) = p > 0$). In the case of algebraically closed fields of characteristic 0, since the admissible fundamental groups of curves depend only on the types of curves, the anabelian geometry of curves does not exist in this situation. On the other hand, if $\mathrm{char}(k) = p > 0$, the situation is quite different from that in characteristic 0. The admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet)$ is very mysterious and its structure is no longer known. In the remainder of the introduction, we assume that k is an algebraically closed field of characteristic $p > 0$.

Since the late 1990s, some developments of F. Pop, M. Raynaud, Saïdi, Tamagawa, J. Tong, and the author (cf. [16], [19], [27], [28], [29], [30], [31], [34], [35], [36]) showed evidence for very strong *anabelian phenomena* for curves over *algebraically closed fields of characteristic* $p > 0$. In this situation, the Galois group of the base field is trivial, and the *arithmetic* fundamental group coincides with the *geometric* fundamental group, thus there is a total absence of a Galois action of the base field. This kinds of anabelian phenomenon go beyond Grothendieck's anabelian geometry (because no Galois actions exist), and this is the reason that we do not have an explicit description of the geometric fundamental group of any pointed stable curve in positive characteristic. Moreover, we may think that the anabelian geometry of curves over algebraically closed fields of characteristic p is a theory based on the following rough consideration: The admissible fundamental group of a pointed stable curve over an algebraically closed field of characteristic p must encode *"moduli"* of the curve.

Let us explain the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ from the point of view of moduli spaces. Let $\overline{\mathbb{F}}_p$ be an algebraically closed field of $\mathbb{F}_p$, and let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over $\overline{\mathbb{F}}_p$ classifying pointed stable curves of type (g, n), $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ the open substack classifying smooth pointed stable curves, $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$, and $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$.

Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of $\overline{M}_{g,n}$, and k_q an algebraically closed field which contains $k(q)$. Then the composition of natural morphisms

$$\operatorname{Spec} k_q \to \operatorname{Spec} k(q) \to \overline{M}_{g,n}$$

determines a pointed stable curve $X^{\bullet}_{k_q}$ of type (g, n) over k_q. In particular, if k_q is an algebraic closure of $k(q)$, we shall write $X^{\bullet}_q$ for $X^{\bullet}_{k_q}$. Write $\pi_1^{\mathrm{adm}}(X^{\bullet}_{k_q})$ for the admissible fundamental group $X^{\bullet}_{k_q}$ and $\Gamma_{X^{\bullet}_{k_q}}$ for the dual semi-graph of $X^{\bullet}_{k_q}$. Since the isomorphism classes of $\pi_1^{\mathrm{adm}}(X^{\bullet}_{k_q})$ and $\Gamma_{X^{\bullet}_{k_q}}$ do not depend on the choices of k_q, we shall denote by

$$\pi_1^{\mathrm{adm}}(q),\ \Gamma_q$$

the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^{\bullet}_{k_q})$ and the dual semi-graph $\Gamma_{X^{\bullet}_{k_q}}$, respectively. Moreover, we write $v(\Gamma_q)$, $e^{\mathrm{op}}(\Gamma_q)$, and $e^{\mathrm{cl}}(\Gamma_q)$ for the set of vertices of Γ_q, the set of open edges of Γ_q, and the set of closed edges of Γ_q, respectively.

Let $\overline{\Pi}_{g,n}$ be the set of isomorphism classes (as profinite groups) of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of characteristic p. Then the fundamental group functor π_1^{adm} induces a natural sujective map from the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$ to $\overline{\Pi}_{g,n}$ as follows:

$$[\pi_1^{\mathrm{adm}}] : \overline{M}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n},\ q \mapsto [\pi_1^{\mathrm{adm}}(q)],$$

where $[\pi_1^{\mathrm{adm}}(q)]$ denotes the isomorphism class of $\pi_1^{\mathrm{adm}}(q)$. Note that the map $[\pi_1^{\mathrm{adm}}]$ is not a bijection in general. For example, let q, $q' \in \overline{M}_{g,n}$ be arbitrary points such that $X_q \setminus D_{X_q}$ is isomorphic to $X_{q'} \setminus D_{X_{q'}}$ as schemes (e.g. $X^{\bullet}_q$ is a Frobenius twist of $X^{\bullet}_{q'}$). Then we have that $[\pi_1^{\mathrm{adm}}(q)] = [\pi_1^{\mathrm{adm}}(q')]$. On the other hand, we introduces an equivalence relation $\sim_{fe}$ which is called *Frobenius equivalence* on $|\overline{M}_{g,n}|$ (cf. [39, Definition 3.4] or Definition 3.1 of the present paper). Roughly speaking, $q_1 \sim_{fe} q_2$ for any points $q_1, q_2 \in \overline{M}_{g,n}$ if there exists an isomorphism $\rho : \Gamma_{q_1} \xrightarrow{\sim} \Gamma_{q_2}$ of dual semi-graphs of $X^{\bullet}_{q_1}$ of $X^{\bullet}_{q_2}$ such that the pointed stable curves $\widetilde{X}^{\bullet}_{q_1,v_1}$ and $\widetilde{X}^{\bullet}_{q_2,v_2}$ associated to v_1 and $v_2 \overset{\mathrm{def}}{=} \rho(v_1)$ (cf. Section 1), respectively, are isomorphic as schemes for every $v_1 \in v(\Gamma_{q_1})$. In particular, when $q_1 \in M_{g,n}$ (i.e., $X^{\bullet}_{q_1}$ is a non-singular curve), then $q_1 \sim_{fe} q_2$ if and only if $X_{q_1} \setminus D_{X_{q_1}}$ is isomorphic to $X_{q_2} \setminus D_{X_{q_2}}$ as schemes. Moreover, [39, Proposition 3.7] shows that $[\pi_1^{\mathrm{adm}}]$ factors through the following quotient set

$$\overline{\mathfrak{M}}_{g,n} \overset{\mathrm{def}}{=} |\overline{M}_{g,n}| / \sim_{fe} .$$

Then we obtain a natural surjective map

$$\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

induced by $[\pi_1^{\mathrm{adm}}]$.

One of the main conjectures in the theory of anabelian geometry of curves is the following weak Isom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic p (=the Weak Isom-version Conjecture):

Weak Isom-version Conjecture . *We maintain the notation introduced above. Then the surjective map*

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}, \ \ [q] \mapsto [\pi_1^{\mathrm{adm}}(q)],$$

is a bijection, where $[q]$ *denotes the image of* q *of the natural quotient map* $|\overline{M}_{g,n}| \to \overline{\mathfrak{M}}_{g,n}$.

The Weak Isom-version Conjecture was formulated by Tamagawa in the case of smooth pointed stable curves, and by the author in the case of arbitrary pointed stable curves (cf. [28], [39]), which says that the moduli spaces of curves in positive characteristic *can be reconstructed group-theoretically as sets* from the isomorphism classes of admissible fundamental groups of pointed stable curves in positive characteristic. The Weak Isom-version Conjecture is very difficult, which was proved completely only in the case where $(g, n) = (0, 4)$. More precisely, we have the following result obtained by Tamagawa and the author (cf. [28, Theorem 0.2], [39, Theorem 3.8]):

Theorem 0.1. *We maintain the notation introduced above. Write* $\overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}$ *for the images of the set of closed points of* $|\overline{M}_{g,n}|$. *Then we have that* $\pi^{\mathrm{adm}}_{0,n}(\overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n}) \cap \pi^{\mathrm{adm}}_{0,n}(\overline{\mathfrak{M}}_{0,n} \setminus \overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n}) = \emptyset$, *and that*

$$\pi^{\mathrm{adm}}_{0,n}|_{\overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n}} : \overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n} \to \overline{\Pi}_{0,n}$$

is an injection. In particular, the Weak Isom-version Conjecture holds when $(g, n) = (0, 4)$.

Remark 0.1.1. In other words, Theorem 0.1 is equivalent to the following anabelian result:

> *Let* q_1 $q_2 \in \overline{M}_{0,n}$ *be an arbitrary points. Suppose that* q_1 *is closed, and that* $\pi_1^{\mathrm{adm}}(q_1)$ *is isomorphic to* $\pi_1^{\mathrm{adm}}(q_2)$ *as profinite groups. Then we have* $q_1 \sim_{fe} q_2$.

On the other hand, suppose that g is an arbitrary non-negative integer number. We also want to mention the following *finiteness theorem* (cf. [16], [19], [30], [31], [34]):

> *Let* $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}$. *Then we have* $\#((\pi^{\mathrm{adm}}_{g,n})^{-1}([\pi_1^{\mathrm{adm}}(q)]) \cap \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}) < \infty$.

At the time of writing, almost all of the researches concerning the anabelian geometry of curves over algebraically closed fields of characteristic p focus on the Weak Isom-version Conjecture, and the conjecture cannot give us any new insight into the anabelian phenomena of curves over algebraically closed fields of characteristic p. On the other hand, the results proved by the author in [35] show that

> it is possible that the *topological structures* of moduli spaces of curves in positive characteristic can be carried out group-theoretically from the geometric fundamental groups of curves in positive characteristic.

This is the main observation that motivated the theory developed in the present paper.

From now on, we shall regard $\overline{\mathfrak{M}}_{g,n}$ as a topological space whose topology is induced naturally by the Zariski topology of $|\overline{M}_{g,n}|$. Let $\mathscr{G}$ be the category of finite groups and $G \in \mathscr{G}$ a finite group. We put

$$U_{\overline{\Pi}_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{adm}}(q)] \in \overline{\Pi}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), G) \neq \emptyset\},$$

where $\mathrm{Hom}_{\mathrm{surj}}(-,-)$ denotes the set of surjective homomorphisms of profinite groups. We define a topological space

$$(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}})$$

group-theoretically from the set of isomorphism classes of admissible fundamental groups of pointed stable curves $\overline{\Pi}_{g,n}$, whose underlying set is $\overline{\Pi}_{g,n}$, and whose topology $O_{\overline{\Pi}_{g,n}}$ is generated by $\{U_{\overline{\Pi}_{g,n},G}\}_{G\in\mathscr{G}}$ as open subsets. For simplicity, we still use the notation $\overline{\Pi}_{g,n}$ to denote the topological space $(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}})$, and shall say

$$\overline{\Pi}_{g,n}$$

the moduli space of admissible fundamental groups of pointed stable curves of type (g, n) *over algebraically closed fields of characteristic* p or *the moduli space of admissible fundamental groups of type* (g, n) *in characteristic* p, for short. Theorem 3.5 (or Theorem 7.14) of the present paper implies that the surjective map

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is also a continuous map. Moreover, we pose the following conjecture, which is the main conjecture of the theory developed in the present paper:

Homeomorphism Conjecture . *We maintain the notation introduced above. Then we have that*

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is a homeomorphism.

The Homeomorphism Conjecture says that the moduli spaces of curves in positive characteristic *can be reconstructed group-theoretically as topological spaces* from the isomorphism classes of admissible fundamental groups of pointed stable curves in positive characteristic. This conjecture gives us a new insight into the theory of the anabelian geometry of curves over algebraically closed fields of characteristic p based on the following philosophy:

> The anabelian properties of pointed stable curves over algebraically closed fields of characteristic p are equivalent to the topological properties of the topological space $\overline{\Pi}_{g,n}$.

This new anabelian philosophy has raised a host of questions which cannot be seen if we only consider the Weak Isom-version Conjecture (e.g. Problem 3.9 of the present paper). Now, the main result of the present paper is as follows (see also Theorem 6.7):

Theorem 0.2. *We maintain the notation introduced above. Let* $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n}$ *be an arbitrary closed point. Then* $\pi^{\mathrm{adm}}_{0,n}([q])$ *is a closed point of* $\overline{\Pi}_{0,n}$*. In particular, the Homeomorphism Conjecture holds when* $(g, n) = (0, 4)$*.*

Remark 0.2.1. In [40], we will prove that the Homeomorphism Conjecture also holds when $(g, n) = (1, 1)$. Then the Homeomorphism Conjecture holds when the dimension of $\overline{M}_{g,n}$ is 1. In [41], by equipping the sets of inertia subgroups with certain orders, we define clutching morphisms and forgetting morphisms for moduli spaces of admissible fundamental groups, and prove the clutching morphisms and the forgetting morphisms are continuous maps.

We denote by $\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(-,-)$ and $\mathrm{Isom}_{\text{pro-gps}}(-,-)$ the set of open continuous homomorphisms of profinit groups and the set of isomorphisms of profinite groups, respectively. Then Theorem 0.2 follows immediately from the following strong anabelian result, which is a ultimate generalization of [28, Theorem 0.2] when $g = 0$ and q_1 is closed (see also Theorem 6.6).

Theorem 0.3. *Let $q_1,\ q_2 \in \overline{M}_{0,n}$ be arbitrary points. Suppose that q_1 is closed. Then we have that*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) \neq \emptyset$$

if and only if $q_1 \sim_{fe} q_2$. In particular, if this is the case, we have that q_2 is a closed point, and that

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) = \mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)).$$

Remark 0.3.1. In fact, in the present paper, we will prove a slightly stronger version of Theorem 0.3 by replacing $\pi_1^{\mathrm{adm}}(q_1)$ and $\pi_1^{\mathrm{adm}}(q_2)$ by the maximal pro-solvable quotients $\pi_1^{\mathrm{adm}}(q_1)^{\mathrm{sol}}$ and $\pi_1^{\mathrm{adm}}(q_2)^{\mathrm{sol}}$ of $\pi_1^{\mathrm{adm}}(q_1)$ and $\pi_1^{\mathrm{adm}}(q_2)$, respectively. Then we obtain a solvable version of Theorem 0.2 which is slightly stronger than Theorem 0.2. In particular, we obtain that *the Solvable Homeomorphism Conjecture* (cf. Section 3.2) holds when $(g, n) = (0, 4)$.

Remark 0.3.1. Note that Theorem 0.3 is essentially different from Theorem 0.1. The reason is as follows: Let $q_1,\ q_2 \in |\overline{M}_{g,n}|$ be arbitrary points such that q_1 is not closed, and that q_2 is a closed point contained in the topological closure of q_1 in $|\overline{M}_{g,n}|$. Then every open continuous homomorphism $\pi_1^{\mathrm{adm}}(q_1) \to \pi_1^{\mathrm{adm}}(q_2)$ is not an isomorphism (cf. [30, Theorem 0.3]).

Next, we explain the method of proving Theorem 0.3 (or Theorem 0.2). The first main step is to prove the following result, which says that the inertia subgroups and field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from arbitrary surjective open continuous homomorphisms of admissible fundamental groups (cf. Theorem 4.11 and Theorem 4.13 for more precise statements):

Theorem 0.4. *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic $p > 0$, and $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$. Let $\Pi_{X_i^\bullet}$ be either the admissible fundamental group $\pi_1^{\mathrm{adm}}(X_i^\bullet)$ of $X_i^\bullet$ or the maximal pro-solvable quotient $\pi_1^{\mathrm{adm}}(X_i^\bullet)^{\mathrm{sol}}$ of $\pi_1^{\mathrm{adm}}(X_i^\bullet)$, and $I_i \subseteq \Pi_{X_i^\bullet}$ an closed subgroup associated to an open edge of $\Gamma_{X_i^\bullet}$ (i.e., a closed subgroup which is (outer) isomorphic to the inertia subgroup of the marked point corresponding to an open edge of $\Gamma_{X_i^\bullet}$). Suppose that $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let*

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

be an arbitrary surjective open continuous homomorphism of profinite groups. Then the following statements hold:

(i) $\phi(I_1) \subseteq \Pi_{X_2^\bullet}$ is a closed subgroup associated to an open edge of $\Gamma_{X_2^\bullet}$, and that there exists a closed subgroup $I' \subseteq \Pi_{X_1^\bullet}$ associated to an open edge of $\Gamma_{X_1^\bullet}$ such that $\phi(I') = I_2$.

(ii) The field structures associated to inertia subgroups of marked points can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$, and that ϕ induces a field isomorphism between the fields associated to I_1 and $\phi(I_1)$ group-theoretically.

The main tool in the proof of Theorem 0.4 is a formula for the maximum generalized Hasse-Witt invariant $\gamma^{\max}(\Pi_{X_i^\bullet})$ of prime-to-p cyclic admissible coverings of $X_i^\bullet$, which was proved by the author in [38] by using the theory of Raynaud-Tamagawa theta divisors.

The second main step is the following result, which is called combinatorial Grothendieck conjecture for surjections (cf. Theorem 5.30 for a more precise statement):

Theorem 0.5. *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type $(0, n)$ over an algebraically closed field k_i of characteristic $p > 0$, and $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$. Let $\Pi_{X_i^\bullet}$ be the maximal pro-solvable quotient of the admissible fundamental group of $X_i^\bullet$ and $\Pi_i \subseteq \Pi_{X_i^\bullet}$ a closed subgroup associated to a vertex (i.e., a closed subgroup which is (outer) isomorphic to the admissible fundamental group of the smooth pointed stable curve associated to a vertex of $\Gamma_{X_i^\bullet}$), and $I_i \subseteq \Pi_{X_i^\bullet}$ an closed subgroup associated to a closed edge (i.e., a closed subgroup which is (outer) isomorphic to the inertia subgroup of the node corresponding to a closed edge of $\Gamma_{X_i^\bullet}$). Suppose that $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{X_2^\bullet})$ and $\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) = \#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$, where $\#(-)$ denotes the cardinality of $(-)$. Let*

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

be an arbitrary surjective open continuous homomorphism of profinite groups. Then the following statements hold:

(i) $\phi(\Pi_1) \subseteq \Pi_{X_2^\bullet}$ is a closed subgroup associated to a vertex of $\Gamma_{X_2^\bullet}$, and that there exists a closed subgroup $\Pi' \subseteq \Pi_{X_1^\bullet}$ associated to a vertex of $\Gamma_{X_1^\bullet}$ such that $\phi(\Pi') = \Pi_2$.

(ii) $\phi(I_1) \subseteq \Pi_{X_2^\bullet}$ is a closed subgroup associated to a closed edge of $\Gamma_{X_2^\bullet}$, and that there exists a closed subgroup $I' \subseteq \Pi_{X_1^\bullet}$ associated to a closed edge of $\Gamma_{X_1^\bullet}$ such that $\phi(I') = I_2$.

(iii) ϕ induces a bijection

$$\phi^{\mathrm{sg}} : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$$

of dual semi-graphs group-theoretically.

The main tool in the proof of Theorem 0.5 is a formula for the limit of p-averages $\mathrm{Avr}_p(\Pi_{X_i^\bullet})$ of the admissible fundamental group of $X_i^\bullet$, which was proved by Tamagawa and the author (cf. [28], [37]). The key observations of the proofs of Theorem 0.4 and Theorem 0.5 are as follows:

> The inequalities of $\gamma^{\max}(\Pi_{X_i^\bullet})$ and $\mathrm{Avr}_p(\Pi_{X_i^\bullet})$ induced by ϕ play roles of the comparability of (outer) Galois representations in the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

In fact, under certain assumptions, Theorem 0.5 also holds for arbitrary types (cf. Theorem 5.26 and Remark 5.26.1). Moreover, the author believes that Theorem 0.4, Theorem 0.5, and Theorem 5.26 will play important roles in the proof of the Homeomorphism Conjecture for arbitrary types.

By applying Theorem 0.4, the geometric operation (=removing a subset of marked points of a pointed stable curve and contracting the (-1)-curves and the (-2)-curves of a pointed semi-stable curve) can be translated to the group-theoretical operation (=quotient of a closed subgroup of the admissible fundamental group of a pointed stable curve, where the closed subgroup is generated by the inertia subgroups corresponding to a subset of marked points of the pointed stable curve). Then we can reduce Theorem 0.3 to the case where $\#v(\Gamma_{q_1}) = \#v(\Gamma_{q_2})$ and $\#e^{\rm cl}(\Gamma_{q_1}) = \#e^{\rm cl}(\Gamma_{q_2})$. Moreover, by applying Theorem 0.5, we can reduce Theorem 0.3 further to the case where q_1 and q_2 are contained in $M_{0,n}$ (i.e., $X^\bullet_{q_1}$ and $X^\bullet_{q_2}$ are non-singular). Then Theorem 0.3 follows immediately from [35, Theorem 1.2] proved by the author. This completes the proof of our main theorem.

The present paper is organized as follows. In Section 1, we fix some notation concerning admissible coverings and admissible fundamental groups. In Section 2, we recall the definition of generalized Hasse-Witt invariants, a formula for maximum generalized Hasse-Witt invariants of prime-to-p admissible coverings, and a formula for limits of p-averages of admissible fundamental groups. In Section 3, we introduce the moduli spaces of admissible fundamental groups and formulate the Homeomorphism Conjecture. In Section 4, we prove Theorem 0.4. In Section 5, we prove Theorem 0.5. In Section 6, we prove our main theorem. In Section 7, we prove the continuity of $\pi^{\rm adm}_{g,n}$.

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1 Admissible coverings and admissible fundamental groups

In this section, we recall some notation and definitions concerning admissible coverings and admissible fundamental groups.

Definition 1.1. Let $\mathbb{G}$ be a semi-graph (cf. [38, Definition 2.1]).

(a) We shall denote by $v(\mathbb{G})$, $e^{\rm op}(\mathbb{G})$, and $e^{\rm cl}(\mathbb{G})$ the set of vertices of $\mathbb{G}$, the set of open edges of $\mathbb{G}$, and the set of closed edges of $\mathbb{G}$, respectively.

(b) The semi-graph $\mathbb{G}$ can be regarded as a topological space with natural topology induced by $\mathbb{R}^2$. We define an *one-point compactification* $\mathbb{G}^{\rm cpt}$ of $\mathbb{G}$ as follows: if $e^{\rm op}(\mathbb{G}) = \emptyset$, we put $\mathbb{G}^{\rm cpt} = \mathbb{G}$; otherwise, the set of vertices of $\mathbb{G}^{\rm cpt}$ is the disjoint union $v(\mathbb{G}^{\rm cpt}) \overset{\rm def}{=} v(\mathbb{G}) \sqcup \{v_\infty\}$, the set of closed edges of $\mathbb{G}^{\rm cpt}$ is $e^{\rm cl}(\mathbb{G}^{\rm cpt}) \overset{\rm def}{=} e^{\rm cl}(\mathbb{G}) \cup e^{\rm op}(\mathbb{G})$, the set of open edges of $\mathbb{G}$ is empty, and every edge $e \in e^{\rm op}(\mathbb{G}) \subseteq e^{\rm cl}(\mathbb{G}^{\rm cpt})$ connects v_∞ with the vertex that is abutted by e.

(c) Let $v \in v(\mathbb{G})$. We shall say that $\mathbb{G}$ is 2-*connected* at v if $\mathbb{G} \setminus \{v\}$ is either empty or connected. Moreover, we shall say that $\mathbb{G}$ is 2-connected if $\mathbb{G}$ is 2-connected at each $v \in v(\mathbb{G})$. Note that, if $\mathbb{G}$ is connected, then $\mathbb{G}^{\rm cpt}$ is 2-connected at each $v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\rm cpt})$ if and only if $\mathbb{G}^{\rm cpt}$ is 2-connected. We put

$$b(v) \overset{\rm def}{=} \sum_{e \in e^{\rm op}(\mathbb{G}) \cup e^{\rm cl}(\mathbb{G})} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that e meets v. We put

$$v(\mathbb{G})^{b\leq 1} \stackrel{\text{def}}{=} \{v \in v(\mathbb{G}) \mid b(v) \leq 1\},$$

and denote by $e^{\text{cl}}(\mathbb{G})^{b\leq 1}$ the set of closed edges of $\mathbb{G}$ which meet a vertex of $v(\mathbb{G})^{b\leq 1}$.

Let p be a prime number, and let

$$X^\bullet = (X, D_X)$$

be a pointed *semi-stable* curve of type (g_X, n_X) over an algebraically closed field k of characteristic p, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ and $r_X \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$.

Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\text{op}}(\Gamma_{X^\bullet})$. Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \stackrel{\text{def}}{=} X_v \setminus X_v^{\text{sing}}$, where $(-)^{\text{sing}}$ denotes the singular locus of $(-)$. We define a smooth pointed semi-stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}^\bullet_v = (\widetilde{X}_v, D_{\widetilde{X}_v} \stackrel{\text{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v)).$$

We shall say that $\widetilde{X}^\bullet_v$ *is the smooth pointed semi-stable curve of type* (g_v, n_v) *associated to* v, or *the smooth pointed semi-stable curve associated to* v for short. In particular, we shall say that $\widetilde{X}^\bullet_v$ is the smooth pointed *stable* curve associated to v if $\widetilde{X}^\bullet_v$ is a pointed stable curve over k.

Definition 1.2. Let $Y^\bullet = (Y, D_Y)$ be a pointed semi-stable curve over k, $f^\bullet : Y^\bullet \to X^\bullet$ a *finite* morphism of pointed semi-stable curves over k, and $f : Y \to X$ the morphism of underlying curves induced by $f^\bullet$.

We shall say $f^\bullet$ a *Galois admissible covering* over k (or Galois admissible covering for short) if the following conditions are satisfied:

(i) There exists a finite group $G \subseteq \text{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$.

(ii) For each $y \in Y^{\text{sm}} \setminus D_Y$, f is étale at y, where $(-)^{\text{sm}}$ denotes the smooth locus of $(-)$.

(iii) For any $y \in Y^{\text{sing}}$, the image $f(y)$ is contained in X^{sing}.

(iv) For each $y \in Y^{\text{sing}}$, we write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y. Then we have that $(\#D_y, p) = 1$, and that the local morphism between two nodes induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^{\#D_y} \\ v & \mapsto & t^{\#D_y}, \end{array}$$

where $\#(-)$ denotes the cardinality of $(-)$. Moreover, we have that $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$th root of unit.

(v) The local morphism between two marked points induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[a]] & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m,p)=1$ (i.e., a tamely ramified extension).

Moreover, we shall say $f^\bullet$ an *admissible covering* if there exists a morphism of pointed semi-stable curves $h^\bullet : W^\bullet \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ h^\bullet : W^\bullet \to X^\bullet$ is a Galois admissible covering over k. We shall say an admissible covering $f^\bullet$ *étale* if f is an étale morphism.

Let $Z^\bullet$ be a disjoint union of finitely many pointed semi-stable curves over k. We shall say that a morphism $f_Z^\bullet : Z^\bullet \to X^\bullet$ over k is a *multi-admissible covering* if the restriction of $f_Z^\bullet$ to each connected component of $Z^\bullet$ is admissible.

Definition 1.3. Let $f^\bullet : Y^\bullet \to X^\bullet$ be an admissible covering over k of degree m. Let $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ and x_e the closed point of X corresponding to e. We put

$$e_f^{\mathrm{cl,ra}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid \# f^{-1}(x_e) = 1\},$$

$$e_f^{\mathrm{cl,\acute{e}t}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid \# f^{-1}(x_e) = m\},$$

$$e_f^{\mathrm{op,ra}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \mid \# f^{-1}(x_e) = 1\},$$

$$e_f^{\mathrm{op,\acute{e}t}} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \mid \# f^{-1}(x_e) = m\},$$

$$v_f^{\mathrm{ra}} \overset{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \#\mathrm{Irr}(f^{-1}(X_v)) = 1\},$$

$$v_f^{\mathrm{sp}} \overset{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \#\mathrm{Irr}(f^{-1}(X_v)) = m\},$$

where $\mathrm{Irr}(-)$ denotes the set of irreducible components of $(-)$, "ra" means "ramification", and "sp" means "split". Note that if the Galois closure of $f^\bullet$ is a Galois admissible covering whose Galois group is a p-group, then the definition of admissible coverings implies that $\# e_f^{\mathrm{cl,ra}} = \# e_f^{\mathrm{op,ra}} = 0$.

Let $\mathscr{C}$ be a category. We shall write $\mathrm{Ob}(\mathscr{C})$ for the class of objects of $\mathscr{C}$, and write $\mathrm{Hom}(\mathscr{C})$ for the class of morphisms of $\mathscr{C}$. We denote by

$$\mathrm{Cov}^{\mathrm{adm}}(X^\bullet) \overset{\mathrm{def}}{=} (\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)), \mathrm{Hom}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)))$$

the category which consists of the following data: (i) $\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$ consists of an empty object and all the pairs $(Z^\bullet, f_Z^\bullet : Z^\bullet \to X^\bullet)$, where $Z^\bullet$ is a disjoint union of finitely many pointed semi-stable curves over k, and $f_Z^\bullet$ is a multi-admissible covering over k; (ii) for any $(Z^\bullet, f_Z^\bullet)$, $(Y^\bullet, f_Y^\bullet) \in \mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$, we define

$$\mathrm{Hom}((Z^\bullet, f_Z^\bullet), (Y^\bullet, f_Y^\bullet)) \overset{\mathrm{def}}{=} \{g^\bullet \in \mathrm{Hom}_k(Z^\bullet, Y^\bullet) \mid f_Y^\bullet \circ g^\bullet = f_Z^\bullet\},$$

where $\mathrm{Hom}_k(Z^\bullet, Y^\bullet)$ denotes the set of k-morphisms of pointed semi-stable curves. It is well known that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point

$x \in X^{\mathrm{sm}} \setminus D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the *admissible fundamental group* of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by

$$\pi_1^{\mathrm{adm}}(X^\bullet).$$

Note that, by the definition of admissible coverings, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X^\bullet$ when $X^\bullet$ is smooth over k. Let $v \in v(\Gamma_{X^\bullet})$. Write $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)$ for the admissible fundamental group of the smooth pointed semi-stable curve $\widetilde{X}_v^\bullet$ associated to v. Then we have a natural (outer) injection

$$\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet) \hookrightarrow \pi_1^{\mathrm{adm}}(X^\bullet).$$

On the other hand, the structure of maximal pro-prime-to-p quotient $\pi_1^{\mathrm{adm}}(X^\bullet)^{p'}$ of $\pi_1^{\mathrm{adm}}(X^\bullet)$ is well-known, which is isomorphic to the pro-prime-to-p completion of the following group (cf. [V, Théorème 2.2 (c)])

$$\langle a_1, \ldots, a_{g_X}, b_1, \ldots, b_{g_X}, c_1, \ldots, c_{n_X} \mid \prod_{i=1}^{g_X}[a_i, b_i]\prod_{j=1}^{n_X} c_j = 1\rangle.$$

We shall denote by $\pi_1^{\mathrm{adm}}(X)$, $\pi_1^{\text{ét}}(X)$, $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})$ the admissible fundamental group of the pointed semi-stable curve $(X, \emptyset)$, the étale fundamental group of the underlying curve X of $X^\bullet$, and the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, respectively. Then we have the following natural surjective open continuous homomorphisms (for suitable choices of base points):

$$\pi_1^{\mathrm{adm}}(X^\bullet) \twoheadrightarrow \pi_1^{\mathrm{adm}}(X) \twoheadrightarrow \pi_1^{\text{ét}}(X) \twoheadrightarrow \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet}).$$

Note that the isomorphism classes of $\pi_1^{\mathrm{adm}}(X^\bullet)$, $\pi_1^{\mathrm{adm}}(X)$, $\pi_1^{\text{ét}}(X)$, and $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})$ depend only on the pointed *stable* curve associated to $X^\bullet$ (i.e., the pointed stable curve obtained by contracting the (-1)-curves and (-2)-curves of $X^\bullet$).

The admissible fundamental groups of pointed stable curves can be also described by using logarithmic geometry. Let $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ be the moduli stack over $\operatorname{Spec}\mathbb{Z}$ parameterizing pointed stable curves of type (g_X, n_X) and $\mathcal{M}_{g_X,n_X,\mathbb{Z}}$ the open substack of $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ parameterizing smooth pointed stable curves. Write $\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \setminus \mathcal{M}_{g_X,n_X,\mathbb{Z}} \subset \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ relative to $\operatorname{Spec}\mathbb{Z}$. The pointed stable curve $X^\bullet$ over k induces a morphism $\operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. Write $s_X^{\log}$ for the log scheme whose underlying scheme is $\operatorname{Spec} k$, and whose log structure is the pulling-back log structure induced by the morphism $\operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$. We obtain a natural morphism $s_X^{\log} \to \overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}$ induced by the morphism $\operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ and a stable log curve

$$X^{\log} \overset{\mathrm{def}}{=} s_X^{\log} \times_{\overline{\mathcal{M}}^{\log}_{g_X,n_X,\mathbb{Z}}} \overline{\mathcal{M}}^{\log}_{g_X,n_X+1,\mathbb{Z}}$$

over $s_X^{\log}$ whose underlying scheme is X. Let $Y^{\log} \to X^{\log}$ be an arbitrary Kummer log étale covering. One can prove that there exists a Kummer log étale covering $t_X^{\log} \to s_X^{\log}$ such that $Y^{\log} \times_{s_X^{\log}} t_X^{\log} \to X^{\log} \times_{s_X^{\log}} t_X^{\log}$ is a log admissible covering (cf. [9, §3.5 Definition]) over $t_X^{\log}$. Then the admissible fundamental group of $X^\bullet$ does not depend on the log structure of $X^{\log}$, and [9, §3.11 Proposition] implies that the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet)$ of $X^\bullet$ is naturally isomorphic to the geometric log étale fundamental group of $X^{\log}$ (i.e., $\ker(\pi_1(X^{\log}) \to \pi_1(s_X^{\log}))$).

Let

$$\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}},\ \pi_1^{\mathrm{adm}}(X)^{\mathrm{sol}},\ \pi_1^{\text{ét}}(X)^{\mathrm{sol}},\ \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}$$

be the maximal pro-solvable quotients of $\pi_1^{\mathrm{adm}}(X^\bullet)$, $\pi_1^{\mathrm{adm}}(X)$, $\pi_1^{\text{ét}}(X)$, $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})$, respectively. Then we obtain the following natural surjective open continuous homomorphisms

$$\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}} \twoheadrightarrow \pi_1^{\mathrm{adm}}(X)^{\mathrm{sol}} \twoheadrightarrow \pi_1^{\text{ét}}(X)^{\mathrm{sol}} \twoheadrightarrow \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}.$$

We shall say

$$\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$$

the *solvable admissible fundamental group* of $X^\bullet$. Let $v \in v(\Gamma_{X^\bullet})$. Write $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)^{\mathrm{sol}}$ for the solvable admissible fundamental group of the smooth pointed semi-stable curve $\widetilde{X}_v^\bullet$ associated to v. Then the natural (outer) injection $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet) \hookrightarrow \pi_1^{\mathrm{adm}}(X^\bullet)$ induces a homomorphism

$$\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)^{\mathrm{sol}} \to \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}.$$

We see that this homomorphism is an *injection*. Indeed, let $\widetilde{f}_v^\bullet : \widetilde{Y}_v^\bullet \to \widetilde{X}_v^\bullet$ be a Galois admissible covering over k whose Galois group is an abelian group. Then we see immediately that there exists a Galois admissible covering $g^\bullet : Z^\bullet \to X^\bullet$ over k whose Galois group is a solvable group such that the following is satisfied: let Z_v be an irreducible component of $Z^\bullet$ such that $g(Z_v) = X_v$; then the Galois admissible covering $\widetilde{Z}_v^\bullet \to \widetilde{X}_v^\bullet$ over k induced by $g^\bullet$ factors through $\widetilde{f}_v^\bullet$. This means that the homomorphism $\pi_1^{\mathrm{adm}}(\widetilde{X}_v^\bullet)^{\mathrm{sol}} \to \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$. mentioned above is an injection.

In the remainder of the present paper, *we shall denote by*

$$\Pi_{X^\bullet}$$

either $\pi_1^{\mathrm{adm}}(X^\bullet)$ *or* $\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$ *unless indicated otherwise.* If $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)$, we denote by

$$\Pi_{X^\bullet}^{\mathrm{cpt}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{adm}}(X),\ \Pi_{X^\bullet}^{\text{ét}} \overset{\mathrm{def}}{=} \pi_1^{\text{ét}}(X),\ \Pi_{X^\bullet}^{\mathrm{top}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet}).$$

If $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$, we denote by

$$\Pi_{X^\bullet}^{\mathrm{cpt}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{adm}}(X)^{\mathrm{sol}},\ \Pi_{X^\bullet}^{\text{ét}} \overset{\mathrm{def}}{=} \pi_1^{\text{ét}}(X)^{\mathrm{sol}},\ \Pi_{X^\bullet}^{\mathrm{top}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}.$$

Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup. We write $X_H^\bullet$ for the pointed semi-stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H, $\Gamma_{X_H^\bullet}$ for the dual semi-graph of $X_H^\bullet$, and r_{X_H} for the Betti number of $\Gamma_{X_H^\bullet}$. Then we obtain an admissible covering

$$f_H^\bullet : X_H^\bullet \to X^\bullet$$

over k induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, and obtain a natural morphism of dual semi-graphs

$$f_H^{\mathrm{sg}} : \Gamma_{X_H^\bullet} \to \Gamma_{X^\bullet}$$

induced by $f_H^\bullet$, where "sg" means "semi-graph". Moreover, if H is an open *normal* subgroup, then $\Gamma_{X_H^\bullet}$ admits an action of $\Pi_{X^\bullet}/H$ induced by the natural action of $\Pi_{X^\bullet}/H$ on $X_H^\bullet$. Note that the quotient of $\Gamma_{X_H^\bullet}$ by $\Pi_{X^\bullet}/H$ coincides with $\Gamma_{X^\bullet}$, and that H is isomorphic to the admissible fundamental group (resp. solvable admissible fundamental group) $\Pi_{X_H^\bullet}$ of $X_H^\bullet$ if $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)$ (resp. $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$). We also use the notation

$$H^{\mathrm{cpt}},\ H^{\text{ét}},\ H^{\mathrm{top}}$$

to denote $\Pi_{X_H^\bullet}^{\mathrm{cpt}}$, $\Pi_{X_H^\bullet}^{\text{ét}}$, and $\Pi_{X_H^\bullet}^{\mathrm{top}}$, respectively.

We put

$$\widehat{X} \stackrel{\mathrm{def}}{=} \varprojlim_{H\subseteq \Pi_{X^\bullet}\ \mathrm{open}} X_H,\ D_{\widehat{X}} \stackrel{\mathrm{def}}{=} \varprojlim_{H\subseteq \Pi_{X^\bullet}\ \mathrm{open}} D_{X_H},\ \Gamma_{\widehat{X}^\bullet} \stackrel{\mathrm{def}}{=} \varprojlim_{H\subseteq \Pi_{X^\bullet}\ \mathrm{open}} \Gamma_{X_H^\bullet}.$$

We shall say that

$$\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}})$$

is the universal admissible covering (resp. universal solvable admissible covering) of $X^\bullet$ corresponding to $\Pi_{X^\bullet}$ if $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)$ (resp. $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$), and that $\Gamma_{\widehat{X}^\bullet}$ is the dual semi-graph of $\widehat{X}^\bullet$. Note that we have that $\mathrm{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of $\Pi_{X^\bullet}$.

Let $v \in v(\Gamma_{X^\bullet})$, $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, $\widehat{v} \in v(\Gamma_{\widehat{X}^\bullet})$ a vertex over v, and $\widehat{e} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{\widehat{X}^\bullet})$ an edge over e. We denote by

$$\Pi_{\widehat{v}} \subseteq \Pi_{X^\bullet},\ I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$$

the stabilizer subgroups of $\widehat{v}$ and $\widehat{e}$, respectively. We see immediately that $\Pi_{\widehat{v}}$ is (outer) isomorphic to $\Pi_{\widetilde{X}_v^\bullet}$ of $\widetilde{X}_v^\bullet$, and that $I_{\widehat{e}}$ is (outer) isomorphic to an inertia subgroup associated to the closed point of X corresponding to e. Then we have that $I_{\widehat{e}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal pro-prime-to-p quotient of $(-)$. We put

$$\mathrm{Ver}(\Pi_{X^\bullet}) \stackrel{\mathrm{def}}{=} \{\Pi_{\widehat{v}}\}_{\widehat{v}\in v(\Gamma_{\widehat{X}^\bullet})},$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet}) \stackrel{\mathrm{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e}\in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet})},$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet}) \stackrel{\mathrm{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e}\in e^{\mathrm{cl}}(\Gamma_{\widehat{X}^\bullet})}.$$

Moreover, if $\widehat{e}$ abuts on $\widehat{v}$, then we have the following injections

$$I_{\widehat{e}} \hookrightarrow \Pi_{\widehat{v}} \hookrightarrow \Pi_{X^\bullet}.$$

Note that $\mathrm{Ver}(\Pi_{X^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})$, and $\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet})$ admit natural actions of $\Pi_{X^\bullet}$ (i.e., the conjugacy actions), and that we have the following natural bijections

$$\mathrm{Ver}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \stackrel{\sim}{\to} v(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \stackrel{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X^\bullet})/\Pi_{X^\bullet} \stackrel{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

2 Maximum and averages of generalized Hasse-Witt invariants

In this section, we recall some results concerning Hasse-Witt invariants (or p-rank) and generalized Hasse-Witt invariants.

Definition 2.1. Let $Z^\bullet$ be a disjoint union of finitely many pointed semi-stable curves over k. We define the *p-rank* (or *Hasse-Witt invariant*) of $Z^\bullet$ to be

$$\sigma_Z \stackrel{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(Z, \mathbb{F}_p)).$$

In particular, if $Z^\bullet$ is a pointed semi-stable curve, then

$$\sigma_Z = \dim_{\mathbb{F}_p}(\Pi^{\text{ab}}_{Z^\bullet} \otimes \mathbb{F}_p),$$

where $\Pi_{Z^\bullet}$ is either the admissible fundamental group or the solvable admissible fundamental group of $Z^\bullet$, and $(-)^{\text{ab}}$ denotes the abelianization of $(-)$.

Let $X^\bullet$ be a pointed stable curve over of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ either the admissible fundamental group or the solvable admissible fundamental group of $X^\bullet$. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ_n, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta_n^i \mapsto i$. Let $\alpha \in \text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$. We denote by $X^\bullet_\alpha = (X_\alpha, D_{X_\alpha})$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition (cf. [21, Section 9])

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha}) = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} \oplus H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}}$. Moreover, we have

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k,$$

where $(-)^{F_{X_\alpha}}$ denotes the subspace of $(-)$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify

$$H_\alpha \stackrel{\text{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$$

with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-vector spaces H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α. We have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta_n \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ_n^i-multiplication. We define

$$\gamma_{\alpha,i} \stackrel{\text{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z}.$$

We shall say that $\gamma_{\alpha,i}$, $i \in \mathbb{Z}/n\mathbb{Z}$, is a *generalized Hasse-Witt invariant* (cf. [13]) of the cyclic multi-admissible covering $X^\bullet_\alpha \to X^\bullet$. Note that the decomposition above implies that

$$\sigma_{X_\alpha} = \dim_k(H_\alpha) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.$$

Let $t \in \mathbb{N}$ be an arbitrary positive natural number, K_{p^t-1} the kernel of the natural surjection

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z},$$

and $X^\bullet_{K_{p^t-1}}$ the pointed stable curve over k determined by K_{p^t-1}. We define two important invariants associated to $X^\bullet$. We put

$$\gamma^{\max}(X^\bullet)$$

$$\overset{\mathrm{def}}{=} \max_{n \in \mathbb{N} \text{ s.t. } (n,p)=1} \{\gamma_{\alpha,i} \mid \alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}),\ \alpha \neq 0,\ i \in (\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}\},$$

and shall say that $\gamma^{\max}(X^\bullet)$ is *the maximum generalized Hasse-Witt invariant of prime-to-p cyclic admissible coverings of* $X^\bullet$. We put

$$\mathrm{Avr}_p(X^\bullet) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\sigma_{X_{K_{p^t-1}}}}{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})},$$

and shall say that $\mathrm{Avr}_p(X^\bullet)$ is *the limit of p-averages of* $X^\bullet$.

On the other hand, let $\overline{\mathbb{F}}_p$ be an arbitrary algebraic closure of the finite field $\mathbb{F}_p$, $\chi \in \mathrm{Hom}(\Pi_{X^\bullet}, \overline{\mathbb{F}}_p^\times)$ such that $\chi \neq 1$, and $\Pi_\chi \subseteq \Pi_{X^\bullet}$ the kernel of χ. The profinite group Π_χ admits a natural action of $\Pi_{X^\bullet}$ via conjugation. We put

$$\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})[\chi] \overset{\mathrm{def}}{=} \{a \in \mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \tau(a) = \chi(\tau)a$$

$$\text{for all } \tau \in \Pi_{X^\bullet}\},$$

$$\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})) \overset{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})[\chi]).$$

We define the following two *group-theoretical* invariants:

$$\gamma^{\max}(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \max\{\gamma_\chi(\mathrm{Hom}(\Pi_\chi, \mathbb{Z}/p\mathbb{Z})) \mid \chi \in \mathrm{Hom}(\Pi_{X^\bullet}, \overline{\mathbb{F}}_p^\times) \text{ such that } \chi \neq 1\},$$

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{p^t-1} \otimes \mathbb{F}_p)}{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})}.$$

We see immediately that

$$\gamma^{\max}(\Pi_{X^\bullet}) = \gamma^{\max}(X^\bullet),$$

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = \mathrm{Avr}_p(X^\bullet).$$

Moreover, we have the following important formulas for $\gamma^{\max}(\Pi_{X^\bullet})$ and $\mathrm{Avr}_p(\Pi_{X^\bullet})$ which were proved by Tamagawa and the author by using the theory of Raynaud-Tamagawa theta divisors (cf. [29, Theorem 0.5], [37, Theorem 5.2, Remark 5.2.1, and Remark 5.2.2], and [38, Theorem 1.4]).

Theorem 2.2. *We maintain the notation introduced above.*

(a) We have

$$\gamma^{\max}(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \text{if } n_X = 0, \\ g_X + n_X - 2, & \text{if } n_X \neq 0. \end{cases}$$

(b) Suppose that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected. Then we have

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#v(\Gamma_{X^\bullet})^{b\leq 1} + \#e^{\mathrm{cl}}(\Gamma_{X^\bullet})^{b\leq 1}.$$

Remark 2.2.1. Suppose that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected. Note that $\#v(\Gamma_{X^\bullet})^{b\leq 1} \neq 0$ if one of the following conditions holds: (i) $X^\bullet$ is smooth over k and $\#e^{\mathrm{op}}(\Gamma_{X^\bullet}) \leq 1$; (ii) $\#v(\Gamma_{X^\bullet}) = 2$, $\#e^{\mathrm{op}}(\Gamma_{X^\bullet}) = 0$, $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) = 1$, and $r_X = 0$. In the case (i), we have

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - 1.$$

In the case (ii), we have

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - 2 + 1 = g_X - 1.$$

Lemma 2.3. *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic $p > 0$ and $\Pi_{X_i^\bullet}$ either the admissible fundamental group of $X_i^\bullet$ or the solvable admissible fundamental group of $X_i^\bullet$. Let*

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

be an arbitrary surjective open continuous homomorphism of profinit groups, $H_2 \subseteq \Pi_{X_2^\bullet}$ an arbitrary open normal subgroup, and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2)$. Then the following statements hold:

(a) We have

$$\gamma^{\max}(H_1) \geq \gamma^{\max}(H_2).$$

(b) Suppose that $(g_X, n_X) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Moreover, suppose either that $G \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/H_2$ is a p-group, that $(\#G, p) = 1$, or that G is a solvable group. Then we have

$$\mathrm{Avr}_p(H_1) \geq \mathrm{Avr}_p(H_2).$$

Proof. (a) Let $n \in \mathbb{Z}_{>0}$ be a positive natural number prime to p, and $\alpha_2 \in \mathrm{Hom}(H_2^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ such that $\alpha_2 \neq 0$. Let $j \in \mathbb{Z}/n\mathbb{Z}$ such that $\gamma_{\alpha_{2,j}} = \gamma^{\max}(H_2)$. Write Q_2 for the kernel of the composition of the following homomorphisms

$$H_2 \twoheadrightarrow H_2^{\mathrm{ab}} \stackrel{\alpha_2}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z},$$

$Q_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(Q_2)$, and $\alpha_1 \in \mathrm{Hom}(H_1^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$ for the homomorphism induced by $\phi|_{H_1}$ and α_2. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Then $Q_i^{p,\mathrm{ab}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ admits a natural $\overline{\mathbb{F}}_p[\mathbb{Z}/n\mathbb{Z}]$-module structure. Moreover, we see immediately that $\phi|_{H_1}$ induces a surjective homomorphism of $\overline{\mathbb{F}}_p[\mathbb{Z}/n\mathbb{Z}]$-modules

$$Q_1^{p,\mathrm{ab}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \twoheadrightarrow Q_2^{p,\mathrm{ab}} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p.$$

Then we obtain that $\gamma_{\alpha_1,j} \geq \gamma_{\alpha_2,j}$. Thus, we have

$$\gamma^{\max}(H_1) \geq \gamma^{\max}(H_2).$$

(b) Let $t \in \mathbb{N}$ be an arbitrary positive natural number, K_{H_i,p^t-1} the kernel of the natural surjection

$$H_i \twoheadrightarrow H_i^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}.$$

Suppose that G is a p-group. We have that Galois admissible covering $X^\bullet_{H_i} \to X^\bullet_i$ corresponding to H_i is étale. This implies that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ are equal types. We obtain that

$$\#(H_1^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}) = \#(H_2^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}).$$

Suppose that $(\#G, p) = 1$. Since $X^\bullet_1$ and $X^\bullet_2$ are equal types, we have that

$$\#(H_1^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}) = \#(H_2^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z}).$$

Then $\phi|_{H_1}$ implies that

$$\mathrm{Avr}_p(H_1) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_1,p^t-1} \otimes \mathbb{F}_p)}{\#(H_1^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})}$$

$$\geq \mathrm{Avr}_p(H_2) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_2,p^t-1} \otimes \mathbb{F}_p)}{\#(H_2^{\mathrm{ab}} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z})}.$$

Suppose that G is solvable. Then the lemma follows immediately from the lemma if either G is a p-group, or $(\#G, p) = 1$. This completes the proof of the lemma. □

3 Moduli spaces of admissible fundamental groups and the Homeomorphism Conjecture

In this section, we define the moduli spaces of fundamental groups and formulate the Homeomorphism Conjecture, which are main research objects of the present paper.

3.1 The Weak Isom-version Conjecture

Let p be a prime number, $\mathbb{F}_p$ the finite field of characteristic p, and $\overline{\mathbb{F}}_p$ an algebraically closed field of $\mathbb{F}_p$. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over $\overline{\mathbb{F}}_p$ classifying pointed stable curves of type (g, n) and $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ the open substack classifying smooth pointed stable curves. Let $\overline{M}_{g,n}$ and $M_{g,n}$ be the coarse moduli spaces of $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}$, respectively.

Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of $\overline{M}_{g,n}$, and k_q an algebraically closed field which contains $k(q)$. Then the composition of natural morphisms

$$\mathrm{Spec}\, k_q \to \mathrm{Spec}\, k(q) \to \overline{M}_{g,n}$$

determines a pointed stable curve $X^\bullet_{k_q}$ of type (g, n) over k_q. Write $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})$ for the admissible fundamental group $X^\bullet_{k_q}$ and $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})^{\mathrm{sol}}$ for the solvable admissible fundamental group of $X^\bullet_{k_q}$. Since the isomorphism classes of $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})$ and $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})^{\mathrm{sol}}$ do not depend on the choices of k_q, we shall denote by

$$\pi_1^{\mathrm{adm}}(q),\ \pi_1^{\mathrm{sol}}(q)$$

the admissible fundamental groups $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})$ and the solvable admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet_{k_q})^{\mathrm{sol}}$, respectively. Moreover, we shall denote by

$$X^\bullet_q$$

the pointed stable curve $X^\bullet_{\overline{k(q)}}$ and Γ_q the dual semi-graph of $X^\bullet_q$, where $\overline{k(q)}$ is an algebraic closure of $k(q)$. Let $v \in v(\Gamma_q)$. Then the smooth pointed stable curve $\widetilde{X}^\bullet_{q,v}$ of type (g_v, n_v) associated to v determines a morphism

$$\mathrm{Spec}\,\overline{k(q)} \to M_{g_v,n_v}.$$

We shall write $q_v \in M_{g_v,n_v}$ for the image of the morphism and say q_v *the point of type* (g_v, n_v) *associated to* v.

Definition 3.1. (a) Let $q_i \in M_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point. We shall say that q_1 is *Frobenius equivalent* to q_2 if $X_{q_1} \setminus D_{X_{q_1}}$ is isomorphic to $X_{q_2} \setminus D_{X_{q_2}}$ as schemes.

(b) Let $q_i \in \overline{M}_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point. We shall say that q_1 is *Frobenius equivalent* to q_2 if the following conditions are satisfied:

(i) There exists an isomorphism $\rho : \Gamma_{q_1} \xrightarrow{\sim} \Gamma_{q_2}$ of dual semi-graphs.

(ii) Let $v_1 \in v(\Gamma_{q_1})$, $v_2 \overset{\mathrm{def}}{=} \rho(v_1) \in v(\Gamma_{q_2})$, q_{1,v_1} the point of type (g_{v_1}, n_{v_1}) associated to v_1, and q_{2,v_2} the point of type (g_{v_2}, n_{v_2}) associated to v_2. We have that q_{1,v_1} is Frobenius equivalent to q_{2,v_2}.

(iii) Let $\rho_{v_1,v_2} : \Gamma_{q_{1,v_1}} \xrightarrow{\sim} \Gamma_{q_{2,v_2}}$ be the isomorphism of dual semi-graphs induced by ρ. There exists a morphism $f^\bullet_{v_1,v_2} : X^\bullet_{q_{1,v_1}} \xrightarrow{\sim} X^\bullet_{q_{2,v_2}}$ such that the morphism $X_{q_{1,v_1}} \setminus D_{X_{q_{1,v_1}}} \to X_{q_{2,v_2}} \setminus D_{X_{q_{2,v_2}}}$ induced by $f^\bullet_{v_1,v_2}$ is an isomorphism as schemes, and that the isomorphism of dual semi-graphs $f^{\mathrm{sg}}_{v_1,v_2} : \Gamma_{q_{1,v_1}} \xrightarrow{\sim} \Gamma_{q_{2,v_2}}$ induced by $f^\bullet_{v_1,v_2}$ coincides with ρ_{v_1,v_2}.

We shall denote by

$$q_1 \sim_{fe} q_2$$

if q_1 is Frobenius equivalent to q_2. Note that $\sim_{fe}$ is an equivalence relation on the underlying topological space $|\overline{M}_{g,n}|$ of $\overline{M}_{g,n}$

(c) Let $q_i \in \overline{M}_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point, k_{q_i} an algebraically closed field which contains $k(q_i)$, and $X^\bullet_{k_{q_i}}$ the pointed stable curve of type (g, n) over k_{q_i}. We shall say that $X^\bullet_{k_{q_1}}$ *is Frobenius equivalent* to $X^\bullet_{k_{q_2}}$ if q_1 is Frobenius equivalent q_2.

The following result was proved by the author.

Proposition 3.2. *Let $q_i \in \overline{M}_{g,n}$, $i \in \{1, 2\}$, be an arbitrary point. Suppose that $q_1 \sim_{fe} q_2$. Then we have that $\pi_1^{\mathrm{adm}}(q_1)$ is isomorphic to $\pi_1^{\mathrm{adm}}(q_2)$ as profinite groups. In particular, we have that $\pi_1^{\mathrm{sol}}(q_1)$ is isomorphic to $\pi_1^{\mathrm{sol}}(q_2)$ as profinite groups.*

Proof. See [39, Proposition 3.7]. □

We put

$$\mathfrak{M}_{g,n} \overset{\text{def}}{=} |M_{g,n}|/\sim_{fe} \subseteq \overline{\mathfrak{M}}_{g,n} \overset{\text{def}}{=} |\overline{M}_{g,n}|/\sim_{fe},$$

$$\Pi_{g,n} \overset{\text{def}}{=} \{[\pi_1^{\text{adm}}(q)] \mid q \in M_{g,n}\} \subseteq \overline{\Pi}_{g,n} \overset{\text{def}}{=} \{[\pi_1^{\text{adm}}(q)] \mid q \in \overline{M}_{g,n}\},$$

$$\Pi_{g,n}^{\text{sol}} \overset{\text{def}}{=} \{[\pi_1^{\text{sol}}(q)] \mid q \in M_{g,n}\} \subseteq \overline{\Pi}_{g,n}^{\text{sol}} \overset{\text{def}}{=} \{[\pi_1^{\text{sol}}(q)] \mid q \in \overline{M}_{g,n}\},$$

where $[\pi_1^{\text{adm}}(q)]$ and $[\pi_1^{\text{sol}}(q)]$ denote the isomorphism classes (as profinite groups) of $\pi_1^{\text{adm}}(q)$ and $\pi_1^{\text{sol}}(q)$, respectively. Let $q \in \overline{M}_{g,n}$. We shall write $[q]$ for the image of q in $\overline{\mathfrak{M}}_{g,n}$. Then there are natural surjective maps of *sets* as follows:

$$sol : \overline{\Pi}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}^{\text{sol}},\ [\pi_1^{\text{adm}}(q)] \mapsto [\pi_1^{\text{sol}}(q)],$$

$$\pi_{g,n}^{\text{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n},\ [q] \mapsto [\pi_1^{\text{adm}}(q)],$$

$$\pi_{g,n}^{\text{sol}} \overset{\text{def}}{=} sol \circ \pi_{g,n}^{\text{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}^{\text{sol}},$$

$$\pi_{g,n}^{\text{t}} \overset{\text{def}}{=} \pi_{g,n}^{\text{adm}}|_{\mathfrak{M}_{g,n}} : \mathfrak{M}_{g,n} \twoheadrightarrow \Pi_{g,n},$$

$$\pi_{g,n}^{\text{t,sol}} \overset{\text{def}}{=} \pi_{g,n}^{\text{sol}}|_{\mathfrak{M}_{g,n}} : \mathfrak{M}_{g,n} \twoheadrightarrow \Pi_{g,n}^{\text{sol}},$$

where "t" means "tame". Moreover, we have the following commutative diagrams:

$$\begin{array}{ccc} \mathfrak{M}_{g,n} & \xrightarrow{\pi_{g,n}^{\text{t}}} & \Pi_{g,n} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n} & \xrightarrow{\pi_{g,n}^{\text{adm}}} & \overline{\Pi}_{g,n} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{M}_{g,n} & \xrightarrow{\pi_{g,n}^{\text{t,sol}}} & \Pi_{g,n}^{\text{sol}} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n} & \xrightarrow{\pi_{g,n}^{\text{sol}}} & \overline{\Pi}_{g,n}^{\text{sol}}, \end{array}$$

where all vertical arrows are natural injections.

Proposition 3.3. *We maintain the notation introduced above. Then we have*

$$\pi_{g,n}^{\text{adm}}(\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}) = \overline{\Pi}_{g,n} \setminus \Pi_{g,n},$$

$$\pi_{g,n}^{\text{sol}}(\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}) = \overline{\Pi}_{g,n}^{\text{sol}} \setminus \Pi_{g,n}^{\text{sol}}.$$

Proof. The proposition follows immediately from [34, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1] (see also Theorem 4.2 of the present paper). □

We may formulate the weak Isom-version of the Grothendieck conjecture for pointed stable curves over algebraically closed fields of characteristic $p > 0$ (=the Weak Isom-version Conjecture) as follows:

Weak Isom-version Conjecture . *We maintain the notation introduced above. Then we have that*

$$\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is a bijection as sets.

The Weak Isom-version Conjecture was formulated by Tamagawa in the case of $q \in M_{g,n}$, and by the author in the general case (cf. [28], [39]), which is the ultimate goal of [16], [19], [27], [28], [29], [30], [31], and [34]. The Weak Isom-version Conjecture shows that moduli spaces of curves over algebraically closed fields of characteristic $p > 0$ can be reconstructed group-theoretically as "*sets*" from admissible fundamental groups of curves.

Moreover, we have the following solvable version of the Weak Isom-version Conjecture which is slightly stronger than the original version.

Solvable Weak Isom-version Conjecture . *We maintain the notation introduced above. Then we have that*

$$\pi_{g,n}^{\mathrm{sol}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}^{\mathrm{sol}}$$

is a bijection as sets.

Write $\overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}$ for the image of the set of closed points $\overline{M}_{g,n}^{\mathrm{cl}}$ of the natural morphism $|\overline{M}_{g,n}| \twoheadrightarrow \overline{\mathfrak{M}}_{g,n}$. Then we have the following result.

Theorem 3.4. *We maintain the notation introduced above. Then the following statements hold:*

(a) We have that

$$\pi_{g,n}^{\mathrm{sol}}|_{\overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}} : \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}} \to \overline{\Pi}_{g,n}^{\mathrm{sol}}$$

is quasi-finite (i.e., $\#(\pi_{g,n}^{\mathrm{sol}}|_{\overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}})^{-1}([\pi_1^{\mathrm{sol}}(q)]) < \infty$ *for every* $[\pi_1^{\mathrm{sol}}(q)] \in \overline{\Pi}_{g,n}^{\mathrm{sol}}$*).*

(b) Suppose that $g = 0$*. Then we have that*

$$\pi_{g,n}^{\mathrm{sol}}|_{\overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}} : \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}} \to \overline{\Pi}_{g,n}^{\mathrm{sol}}$$

is an injection, and that

$$\pi_{g,n}^{\mathrm{sol}}(\overline{\mathfrak{M}}_{g,n} \setminus \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}) \subseteq \overline{\Pi}_{g,n}^{\mathrm{sol}} \setminus \pi_{g,n}^{\mathrm{sol}}(\overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}).$$

In particular, the Solvable Weak Isom-version Conjecture holds if $(g, n) = (0, 4)$*.*

Proof. Since [29, Theorem 0.2] and [30, Theorem 0.1] also hold for the maximal pro-solvable quotients of tame fundamental groups, the theorem follows immediately from [29, Theorem 0.2], [30, Theorem 0.1], [34, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1], and Proposition 3.3. □

3.2 Moduli spaces of admissible fundamental groups and the Homeomorphism Conjecture

We maintain the notation introduced in Section 3.1. Moreover, in the reminder of the present section, we regard $\overline{\mathfrak{M}}_{g,n}$ and $\mathfrak{M}_{g,n}$ as topological spaces whose topologies are induced by the Zariski topologies of $|\overline{M}_{g,n}|$ and $|M_{g,n}|$, respectively.

Let $\mathscr{G}$ be the category of finite groups, $G \in \mathscr{G}$ an arbitrary finite group, Π an arbitrary profinite group, and $\mathrm{Hom}_{\mathrm{surj}}(-,-)$ the set of surjective homomorphisms. We put

$$U_{\overline{\Pi}_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{adm}}(q)] \in \overline{\Pi}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), G) \neq \emptyset\},$$

$$U_{\Pi_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{adm}}(q)] \in \Pi_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), G) \neq \emptyset\},$$

$$U_{\overline{\Pi}^{\mathrm{sol}}_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{sol}}(q)] \in \overline{\Pi}^{\mathrm{sol}}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{sol}}(q), G) \neq \emptyset\},$$

$$U_{\Pi^{\mathrm{sol}}_{g,n},G} \overset{\mathrm{def}}{=} \{[\pi_1^{\mathrm{sol}}(q)] \in \Pi^{\mathrm{sol}}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{sol}}(q), G) \neq \emptyset\}.$$

Then we obtain the following topological spaces

$$(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}}),\ (\Pi_{g,n}, O_{\Pi_{g,n}}),$$

$$(\overline{\Pi}^{\mathrm{sol}}_{g,n}, O_{\overline{\Pi}^{\mathrm{sol}}_{g,n}}),\ (\Pi^{\mathrm{sol}}_{g,n}, O_{\Pi^{\mathrm{sol}}_{g,n}})$$

whose topologies $O_{\overline{\Pi}_{g,n}}$, $O_{\Pi_{g,n}}$, $O_{\overline{\Pi}^{\mathrm{sol}}_{g,n}}$, and $O_{\Pi^{\mathrm{sol}}_{g,n}}$ are generated by $\{U_{\overline{\Pi}_{g,n},G}\}_{G\in\mathscr{G}}$, $\{U_{\Pi_{g,n},G}\}_{G\in\mathscr{G}}$, $\{U_{\overline{\Pi}^{\mathrm{sol}}_{g,n},G}\}_{G\in\mathscr{G}}$, and $\{U_{\Pi^{\mathrm{sol}}_{g,n},G}\}_{G\in\mathscr{G}}$ as open subsets, respectively. For simplicity, we still use the notation

$$\overline{\Pi}_{g,n},\ \Pi_{g,n},\ \overline{\Pi}^{\mathrm{sol}}_{g,n},\ \Pi^{\mathrm{sol}}_{g,n}$$

to denote the topological spaces $(\overline{\Pi}_{g,n}, O_{\overline{\Pi}_{g,n}})$, $(\Pi_{g,n}, O_{\Pi_{g,n}})$, $(\overline{\Pi}^{\mathrm{sol}}_{g,n}, O_{\overline{\Pi}^{\mathrm{sol}}_{g,n}})$, and $(\Pi^{\mathrm{sol}}_{g,n}, O_{\Pi^{\mathrm{sol}}_{g,n}})$, respectively.

Theorem 3.5. *We maintain the notation introduced above. Then we have that*

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n},$$

$$\pi^{\mathrm{sol}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}^{\mathrm{sol}}_{g,n}$$

are continuous maps.

Proof. The theorem will be proved in Section 7, see Theorem 7.14. □

Proposition 3.6. *We maintain the notation introduced above. Then the following statements hold.*

(a) Let $[\pi_1^{\mathrm{adm}}(q)] \in \overline{\Pi}_{g,n}$ and $[\pi_1^{\mathrm{sol}}(q)] \in \overline{\Pi}^{\mathrm{sol}}_{g,n}$ be arbitrary points. Then we have

$$V([\pi_1^{\mathrm{adm}}(q)]) = \{[\pi_1^{\mathrm{adm}}(q')] \in \overline{\Pi}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), \pi_1^{\mathrm{adm}}(q')) \neq \emptyset\},$$

$$V([\pi_1^{\mathrm{sol}}(q)]) = \{[\pi_1^{\mathrm{sol}}(q')] \in \overline{\Pi}^{\mathrm{sol}}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{sol}}(q), \pi_1^{\mathrm{sol}}(q')) \neq \emptyset\},$$

where $V([\pi_1^{\mathrm{adm}}(q)])$ *and* $V([\pi_1^{\mathrm{sol}}(q)])$ *denote the topological closures of* $[\pi_1^{\mathrm{adm}}(q)]$ *and* $[\pi_1^{\mathrm{sol}}(q)]$ *in* $\overline{\Pi}_{g,n}$ *and* $\overline{\Pi}_{g,n}^{\mathrm{sol}}$*, respectively.*

(b) We have that

$$\Pi_{g,n} \subseteq \overline{\Pi}_{g,n},\ \Pi_{g,n}^{\mathrm{sol}} \subseteq \overline{\Pi}_{g,n}^{\mathrm{sol}}$$

are open subsets.

(c) Let Z *be an arbitrary irreducible closed subset of* $\overline{\mathfrak{M}}_{g,n}$*. Then* $V(\pi_{g,n}^{\mathrm{adm}}(Z))$ *and* $V(\pi_{g,n}^{\mathrm{sol}}(Z))$ *are irreducible closed subsets of* $\overline{\Pi}_{g,n}$ *and* $\overline{\Pi}_{g,n}^{\mathrm{sol}}$*, respectively, where* $V(\pi_{g,n}^{\mathrm{adm}}(Z))$ *and* $V(\pi_{g,n}^{\mathrm{sol}}(Z))$ *denote the topological closures of* $\pi_{g,n}^{\mathrm{adm}}(Z)$ *and* $\pi_{g,n}^{\mathrm{sol}}(Z)$ *in* $\overline{\Pi}_{g,n}$ *and* $\overline{\Pi}_{g,n}^{\mathrm{sol}}$*, respectively. In particular, the topological spaces* $\overline{\Pi}_{g,n}$ *and* $\overline{\Pi}_{g,n}^{\mathrm{sol}}$ *are irreducible.*

(d) Let V *be either an irreducible closed subset of* $\overline{\Pi}_{g,n}$ *or an irreducible closed subset of* $\overline{\Pi}_{g,n}^{\mathrm{sol}}$*. Then* V *has a unique generic point.*

Proof. (a) follows immediately from the definitions of $O_{\overline{\Pi}_{g,n}}$ and $O_{\overline{\Pi}_{g,n}^{\mathrm{sol}}}$, respectively.

(b) Let $[\pi_1^{\mathrm{adm}}(q)] \in \Pi_{g,n}$ be an arbitrary point and $\pi_A^{\mathrm{adm}}(q)$ the set of finite quotients of $\pi_1^{\mathrm{adm}}(q)$. Moreover, since $\pi_1^{\mathrm{adm}}(q)$ is topologically finitely generated, we have a subset of open normal subgroups $\{H_j\}_{j\in\mathbb{N}}$ of $\pi_1^{\mathrm{adm}}(q)$ such that $H_{j_1} \subseteq H_{j_2}$ for any $j_1 \geq j_2$, and that

$$\pi_1^{\mathrm{adm}}(q) \cong \varprojlim_{j\in\mathbb{N}} \pi_1^{\mathrm{adm}}(q)/H_j.$$

We put

$$S(q) \overset{\mathrm{def}}{=} \{\pi_1^{\mathrm{adm}}(q)/H_j,\ j \in \mathbb{N}\} \subseteq \pi_A^{\mathrm{adm}}(q).$$

We see that, in order to prove that $\Pi_{g,n}$ is an open subset of $\overline{\Pi}_{g,n}$, it is sufficient to prove that, for every point $[q_2] \in \mathfrak{M}_{g,n}$, there exists a finite group $G \in S(q_2)$ such that $U_{\overline{\Pi}_{g,n},G}$ is contained in $\Pi_{g,n}$.

Suppose that $U_{\overline{\Pi}_{g,n},G} \cap (\overline{\Pi}_{g,n} \setminus \Pi_{g,n}) \neq \emptyset$ for every $G \in S(q_2)$. Since $\pi_{g,n}^{\mathrm{adm}}$ is continuous (cf. Theorem 3.5) and the set of generic points of $\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}$ is finite, there exists a generic point $[q_1]$ of $\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}$ such that

$$[\pi_1^{\mathrm{adm}}(q_1)] \in \bigcap_{G\in S(q_2)} U_{\overline{\Pi}_{g,n},G}.$$

Then the set

$$\mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q_1), \pi_1^{\mathrm{adm}}(q_2)) = \varprojlim_{G\in S(q_2)} \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q_1), G)$$

is not empty. We obtain a surjective open continuous homomorphism $\phi : \pi_1^{\mathrm{adm}}(q_1) \twoheadrightarrow \pi_1^{\mathrm{adm}}(q_2)$. Note that ϕ induces an isomorphism of maximal prime-to-p quotients

$$\phi^{p'} : \pi_1^{\mathrm{adm}}(q_1)^{p'} \overset{\sim}{\to} \pi_1^{\mathrm{adm}}(q_2)^{p'}.$$

By applying [34, Lemma 6.3], there exists an open characteristic subgroup $H_1 \subseteq \pi_1^{\mathrm{adm}}(q_1)^{p'}$ such that the pointed stable curve $X_{H_1}^{\bullet}$ of type $(g_{X_{H_1}}, n_{X_{H_1}})$ over k_{q_1} corresponding to H_1 satisfying the following conditions: (1) $\Gamma_{X_{H_1}^{\bullet}}^{\mathrm{cpt}}$ is 2-connected; (2) $\#(v(\Gamma_{X_{H_1}^{\bullet}})^{b\leq 1}) =$

0; (3) the Betti number $r_{X_{H_1}}$ of the dual semi-graph of $X^\bullet_{H_1}$ is positive. Let $H_2 \stackrel{\text{def}}{=} (\phi^{p'})^{-1}(H_1) \subseteq \pi_1^{\text{adm}}(q_2)^{p'}$. Then we obtain a smooth pointed stable curve $X^\bullet_{H_2}$ of type $(g_{X_{H_2}}, n_{X_{H_2}})$ over k_{q_2} corresponding to H_2. Since H_i is an open characteristic subgroup, we obtain that $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then Theorem 2.2 (b) and Lemma 2.3 (b) imply that

$$r_{X_{H_1}} \leq 0.$$

This contradicts $r_{X_{H_1}} > 0$.

Similar arguments to the arguments given in the proof above imply that $\Pi^{\text{sol}}_{g,n}$ is an open subset of $\overline{\Pi}^{\text{sol}}_{g,n}$. This completes the proof of (b).

(c) is trivial.

(d) We only treat the case where V is an irreducible closed subset of $\overline{\Pi}_{g,n}$. Let $\text{Gen}(V)$ be the set of generic points of V. Since every closed subset of $\overline{\mathfrak{M}}_{g,n}$ has a set of generic points, we have that $\text{Gen}(V) \neq \emptyset$. Let $[\pi_1^{\text{adm}}(q_1)], [\pi_1^{\text{adm}}(q_2)] \in \text{Gen}(V)$ be arbitrary generic points. Let $G \in \pi_A^{\text{adm}}(q_1)$ be an arbitrary finite group. Then $U_{\overline{\Pi}_{g,n},G} \cap V \neq \emptyset$. Thus, $[\pi_1^{\text{adm}}(q_2)] \in U_{\overline{\Pi}_{g,n},G} \cap V$. This means that $\pi_A^{\text{adm}}(q_1) \subseteq \pi_A^{\text{adm}}(q_2)$. Similar arguments to the arguments given in the proof above imply $\pi_A^{\text{adm}}(q_1) \supseteq \pi_A^{\text{adm}}(q_2)$. Then we have

$$\pi_A^{\text{adm}}(q_1) = \pi_A^{\text{adm}}(q_2).$$

Since admissible fundamental groups of pointed stable curves are topologically finitely generated, [2, Proposition 16.10.6] implies that $[\pi_1^{\text{adm}}(q_1)] = [\pi_1^{\text{adm}}(q_2)]$. This completes the proof of the proposition. □

Definition 3.7. We shall say that

$$\overline{\Pi}_{g,n}$$

is *the moduli space of admissible fundamental groups of pointed stable curves of type* (g, n) *over algebraically closed fields of characteristic* p (or *the moduli space of admissible fundamental groups of type* (g, n) *in characteristic* p for short), and that

$$\overline{\Pi}^{\text{sol}}_{g,n}$$

is *the moduli space of solvable admissible fundamental groups of pointed stable curves of type* (g, n) *over algebraically closed fields of characteristic* $p > 0$ (or *the moduli space of solvable admissible fundamental groups of type* (g, n) *in characteristic* p for short). Moreover, we shall say that $O_{\overline{\Pi}_{g,n}}$ and $O_{\overline{\Pi}^{\text{sol}}_{g,n}}$ are the *anabelian topologies* of $\overline{\Pi}_{g,n}$ and $\overline{\Pi}^{\text{sol}}_{g,n}$, respectively.

Next, we formulate the main conjectures of the theory developing in the present paper.

Homeomorphism Conjecture . *We maintain the notation introduced above. Then we have that*

$$\pi^{\text{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$$

is a homeomorphism.

Moreover, we have a solvable version of the Homeomorphism Conjecture as follows, which is slightly stronger than the original version.

Solvable Homeomorphism Conjecture . *We maintain the notation introduced above. Then we have that*

$$\pi_{g,n}^{\mathrm{sol}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}^{\mathrm{sol}}$$

is a homeomorphism.

The Homeomorphism Conjecture (or the Solvable Homeomorphism Conjecture) shows that moduli spaces of curves over algebraically closed fields of characteristic $p > 0$ can be reconstructed group-theoretically as "*topological spaces* " from admissible fundamental groups (or solvable admissible fundamental groups) of curves. Moreover, the conjectures give a new insight into the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ based on the following anabelian philosophy:

> The topological space $\overline{\Pi}_{g,n}$ (or $\overline{\Pi}_{g,n}^{\mathrm{sol}}$) contains all anabelian informations of pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$, and every topological property concerning the topological space $\overline{\Pi}_{g,n}$ (or $\overline{\Pi}_{g,n}^{\mathrm{sol}}$) can be regarded as an anabelian property of pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$.

The main theorem of the present paper is the following, which will be proved in Section 6 (cf. Theorem 6.7).

Theorem 3.8. *We maintain the notation introduced above. Let $[q] \in \overline{\mathfrak{M}}_{0,n}^{\mathrm{cl}}$ be an arbitrary closed point. Then $\pi_{0,n}^{\mathrm{adm}}([q])$ and $\pi_{0,n}^{\mathrm{sol}}([q])$ are closed points of $\overline{\Pi}_{0,n}$ and $\overline{\Pi}_{0,n}^{\mathrm{sol}}$, respectively. In particular, the Homeomorphism Conjecture and the Solvable Homeomorphism Conjecture hold when $(g, n) = (0, 4)$.*

On the other hand, some major problems concerning $\overline{\Pi}_{g,n}$ are as follows:

Problem 3.9. *We maintain the notation introduced above.*

1. *Is $\overline{\Pi}_{g,n}$ a noetherian topological space?*

2. *Let $[q] \in \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}$. Is $V(\pi_{g,n}^{\mathrm{adm}}([q]))$ a finite set?*

3. *Let Z be an irreducible closed subset of $\overline{\mathfrak{M}}_{g,n}$. Is $\pi_{g,n}^{\mathrm{adm}}(Z)$ an irreducible closed subset of $\overline{\Pi}_{g,n}$?*

4. *Let $i \in \{1, 2\}$, and let $V_{i,m_i} \subseteq \cdots \subseteq V_{i,1} \subseteq V_{i,0} \stackrel{\mathrm{def}}{=} \overline{\Pi}_{g,n}$ be an arbitrary maximal chain of irreducible closed subsets of $\overline{\Pi}_{g,n}$. Does $m_1 = m_2$ hold?*

5. *Let Z be an irreducible closed subset of $\overline{\mathfrak{M}}_{g,n}$. Does $\dim(Z) = \dim(V(\pi_{g,n}^{\mathrm{adm}}(Z)))$ hold? Here, $\dim(V(\pi_{g,n}^{\mathrm{adm}}(Z)))$ denotes the Krull dimension of $V(\pi_{g,n}^{\mathrm{adm}}(Z))$. In particular, does $\dim(\overline{\mathfrak{M}}_{g,n}) = \dim(\overline{\Pi}_{g,n})$ and $\dim(V(\pi_{g,n}^{\mathrm{adm}}([q]))) = 0$ for every $[q] \in \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}$ hold? Moreover, Is $\pi_{g,n}^{\mathrm{adm}}([q])$ a closed point of $\overline{\Pi}_{g,n}$ for every $[q] \in \overline{\mathfrak{M}}_{g,n}^{\mathrm{cl}}$?*

6. *Prove the Homeomorphism Conjecture. In particular, prove the Homeomorphism Conjecture for $(0, n)$.*

Remark 3.9.1. We may also ask the problems mentioned above for $\overline{\Pi}^{\mathrm{sol}}_{g,n}$.

Remark 3.9.2. Problem 3.9 (2) is equivalent to the following anabelian property of pointed stable curves:

> Let $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}$. Then we have that the set
>
> $$\{q' \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n} \mid \mathrm{Hom}_{\mathrm{surj}}(\pi_1^{\mathrm{adm}}(q), \pi_1^{\mathrm{adm}}(q')) \neq \emptyset\}$$
>
> is a finite set.

This property is a generalized version of Theorem 3.4 (a).

Remark 3.9.3. We maintain the notation introduced above. Tamagawa posed a conjecture as follows (cf. [28, Conjecture 5.3 (ii)]), which is called *Essential Dimension Conjecture*:

> Let $i \in \{1, 2\}$, and let $q_i \in \mathfrak{M}_{g,n}$ and $V(q_i)$ the topological closure of q_i in $\overline{\mathfrak{M}}_{g,n}$. Then we have that $\dim(V(q_1)) = \dim(V(q_2))$ if $[\pi_1^{\mathrm{adm}}(q_1)] = [\pi_1^{\mathrm{adm}}(q_2)]$.

We see immediately that Problem 3.9 (5) is a generalized version of Tamagawa's Essential Dimension Conjecture.

Proposition 3.10. *Let* $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}$. *Then we have that* $\dim(V(\pi^{\mathrm{adm}}_{g,n}([q]))) = 0$ *if and only if* $\pi^{\mathrm{adm}}_{g,n}([q])$ *is a closed point of* $\overline{\Pi}_{g,n}$.

Proof. The "if" part of the proposition is trivial. We only need to prove the "only if" part of the proposition.

Let $[\pi_1^{\mathrm{adm}}(q')] \in V(\pi^{\mathrm{adm}}_{g,n}([q]))$ be an arbitrary point and $V([\pi_1^{\mathrm{adm}}(q')])$ the topological closure of $[\pi_1^{\mathrm{adm}}(q')]$ in $\overline{\Pi}_{g,n}$. Then we have that $V([\pi_1^{\mathrm{adm}}(q')])$ is an irreducible closed subset which is contained in $V(\pi^{\mathrm{adm}}_{g,n}([q]))$. Since $V(\pi^{\mathrm{adm}}_{g,n}([q]))$ is an irreducible closed subset, we obtain that

$$V(\pi^{\mathrm{adm}}_{g,n}([q])) = V([\pi_1^{\mathrm{adm}}(q')]).$$

This means that there exist surjective open continuous homomorphisms

$$\pi_1^{\mathrm{adm}}(q) \twoheadrightarrow \pi_1^{\mathrm{adm}}(q'),$$

$$\pi_1^{\mathrm{adm}}(q') \twoheadrightarrow \pi_1^{\mathrm{adm}}(q).$$

Then we obtain $\pi_A^{\mathrm{adm}}(q) = \pi_A^{\mathrm{adm}}(q')$. Since admissible fundamental groups of pointed stable curves are topologically finitely generated, [2, Proposition 16.10.6] implies that $[\pi_1^{\mathrm{adm}}(q)] = [\pi_1^{\mathrm{adm}}(q')]$. Thus, we obtain $V(\pi^{\mathrm{adm}}_{g,n}([q])) = [\pi_1^{\mathrm{adm}}(q)]$. This completes the proof of the proposition. □

4 Reconstruction of inertia subgroups and field structures from surjections

In this section, we will prove that the inertia subgroups assoicated to marked points can be reconstructed group-theoretically from surjective homomorphisms of admissible fundamental groups (or solvable admissible fundamental groups).

Let $\mathcal{P}$ be a category of profinite groups whose class of objects $\mathrm{Ob}(\mathcal{P})$ consists of profinite groups, and whose class of morphisms $\mathrm{Hom}_{\mathcal{P}}(\Pi, \Pi')$ is the class of open continuous homomorphisms of Π and Π'. Let $\Pi \in \mathcal{P}$, and let $\mathfrak{S}$ be a category whose class of objects $\mathrm{Ob}(\mathfrak{S})$ is a set of subgroups of Π, and whose class of morphisms $\mathrm{Hom}_{\mathfrak{S}}(H, H')$ for any $H, H' \in \mathfrak{S}$ is defined as follows: the unique element of $\mathrm{Hom}_{\mathfrak{S}}(H, H')$ is the natural inclusion when $H \subseteq H'$; otherwise, $\mathrm{Hom}_{\mathfrak{S}}(H, H')$ is empty. We shall say that $\mathfrak{S}$ is a category associated to Π.

Let $\mathfrak{S}$ and $\mathfrak{S}'$ be categories associated to profinite groups Π and Π', respectively. We define a class of functors $\mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}, \mathfrak{S}')$ as follows: $\mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}, \mathfrak{S}')$ is non-empty and $\theta_S \in \mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}, \mathfrak{S}')$ when there exists an open continuous homomorphism $\theta : \Pi \to \Pi'$ such that $\mathfrak{S} = \{H \overset{\mathrm{def}}{=} \theta^{-1}(H')\}_{H' \in \mathfrak{S}'}$, where $\theta_{\mathcal{S}} : \mathfrak{S} \to \mathfrak{S}'$, $H \mapsto H'$; otherwise, $\mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}, \mathfrak{S}')$ is empty. We denote by

$$\mathcal{S}$$

the category whose class of objects $\mathrm{Ob}(\mathcal{S})$ is the class of categories associated to profinite groups, and whose class of morphisms $\mathrm{Hom}_{\mathcal{S}}(\mathfrak{S}, \mathfrak{S}')$ consists of the classes of functors defined above. Then we have a natural functor $\pi : \mathcal{S} \to \mathcal{P}$ defined as follows: Let $\mathfrak{S}$, $\mathfrak{S}' \in \mathcal{S}$ be categories associated to profinite groups Π, Π', respectively; we have $\pi(\mathfrak{S}) = \Pi$, $\pi(\mathfrak{S}') = \Pi'$, and $\pi(\theta_{\mathcal{S}}) = \theta$. We see immediately that

$$\pi : \mathcal{S} \to \mathcal{P}$$

is a fibered category over $\mathcal{P}$.

Definition 4.1. Let $i \in \{1, 2\}$. Let $\mathcal{F}_i$ be a geometric object (in a certain category), $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object $\mathcal{F}_i$, and $\mathfrak{S}_i$ a category associated to $\Pi_{\mathcal{F}_i}$. Let $\mathrm{Inv}_{\mathcal{F}_i}$ be an invariant depending on the isomorphism class of $\mathcal{F}_i$ (in a certain category) and $\mathrm{Add}_{\mathcal{F}_i}(\mathfrak{S}_i)$ an additional structure associated to $\mathfrak{S}_i$ (e.g., $\mathrm{Add}_{\mathcal{F}_i}(\mathfrak{S}_i) = \mathfrak{S}_i$) on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on $\mathcal{F}_i$ and $\mathfrak{S}_i$.

(a) We shall say that $\mathrm{Inv}_{\mathcal{F}_i}$ can be *reconstructed group-theoretically* from $\Pi_{\mathcal{F}_i}$ (or $\Pi_{\mathcal{F}_i}$ induces $\mathrm{Inv}_{\mathcal{F}_i}$ group-theoretically) if $\Pi_{\mathcal{F}_1} \cong \Pi_{\mathcal{F}_2}$ implies $\mathrm{Inv}_{\mathcal{F}_1} = \mathrm{Inv}_{\mathcal{F}_2}$.

(b) We shall say that $\mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ can be *reconstructed group-theoretically* from $\Pi_{\mathcal{F}_2}$ (or $\Pi_{\mathcal{F}_2}$ induces $\mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ group-theoretically) if every isomorphism $\theta : \Pi_{\mathcal{F}_1} \xrightarrow{\sim} \Pi_{\mathcal{F}_2}$ induces a bijection $\theta_{\mathrm{ad}} : \mathrm{Add}_{\mathcal{F}_1}(\mathfrak{S}_1) \xrightarrow{\sim} \mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$ which preserves the structures $\mathrm{Add}_{\mathcal{F}_1}(\mathfrak{S}_1)$ and $\mathrm{Add}_{\mathcal{F}_2}(\mathfrak{S}_2)$, where $\mathfrak{S}_1 \overset{\mathrm{def}}{=} \Pi_{\mathcal{F}_1} \times_{\theta, \Pi_{\mathcal{F}_2}} \mathfrak{S}_2$ (i.e., the fiber product in the fibered category $\mathcal{S}$ over $\mathcal{P}$).

(c) Let j_1, $j_2 \in \{1, 2\}$ distinct from each other, and let $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ be an open continuous homomorphism of profinite groups and $\mathfrak{S}_1 = \Pi_{\mathcal{F}_1} \times_{\theta, \Pi_{\mathcal{F}_2}} \mathfrak{S}_2$. We shall say that a map $\theta_{\mathrm{ad}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ can be *reconstructed group-theoretically* from $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ (or $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ induces $\theta_{\mathrm{ad}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ group-theoretically) if the following condition is satisfied: Let $\mathcal{F}'_i$ be a geometric object, $\Pi_{\mathcal{F}'_i}$ a profinite group associated to the geometric object $\mathcal{F}'_i$, $\theta_i : \Pi_{\mathcal{F}'_i} \xrightarrow{\sim} \Pi_{\mathcal{F}_i}$ an isomorphism of profinite groups, $\theta' : \Pi_{\mathcal{F}'_1} \to \Pi_{\mathcal{F}'_2}$, $\mathfrak{S}'_i \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{F}'_i} \times_{\theta_i, \Pi_{\mathcal{F}_i}} \mathfrak{S}_i$, $\mathrm{Add}_{\mathcal{F}'_i}(\mathfrak{S}'_i)$ an additional structure on the profinite group $\Pi_{\mathcal{F}'_i}$. Suppose that the following commutative diagram of profinite groups holds

$$\begin{array}{ccc} \Pi_{\mathcal{F}'_{j_1}} & \xrightarrow{\theta'} & \Pi_{\mathcal{F}'_{j_2}} \\ \theta_{j_1}\big\downarrow & & \theta_{j_2}\big\downarrow \\ \Pi_{\mathcal{F}_{j_1}} & \xrightarrow{\theta} & \Pi_{\mathcal{F}_{j_2}}. \end{array}$$

Then the commutative diagram of profinite groups above induces the following commutative diagram of additional structures

$$\begin{array}{ccc} \mathrm{Add}_{\mathcal{F}'_{j_1}}(\mathfrak{S}'_{j_1}) & \xrightarrow{\theta'_{\mathrm{ad}}} & \mathrm{Add}_{\mathcal{F}'_{j_2}}(\mathfrak{S}'_{j_2}) \\ \theta_{j_1,\mathrm{ad}}\big\downarrow & & \theta_{j_2,\mathrm{ad}}\big\downarrow \\ \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) & \xrightarrow{\theta_{\mathrm{ad}}} & \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2}) \end{array}$$

which preserves the structures of additional structures.

Remark 4.1.1. Let us explain the philosophy of *mono-anabelian geometry* introduced by Mochizuki. The classical point of view of anabelian geometry (i.e., the anabelian geometry considered in [3], [4]) focuses on a comparison between two geometric objects via their fundamental groups. Moreover, the term "group-theoretical", in the classical point of view, means that "preserved by an arbitrary isomorphism between the fundamental groups under consideration". We shall refer to the classical point of view as "*bi-anabelian geometry*". Then Definition 4.1 is a definition from the point of view of bi-anabelian geometry.

On the other hand, mono-anabelian geometry focuses on the establishing a group-theoretic algorithm whose input datum is an abstract topological group which is isomorphic to the fundamental group of a given geometric object of interest (resp. a continuous homomorphism of abstract topological groups which are isomorphic to a continuous homomorphism of the fundamental groups of given geometric objects of interest), and whose output datum is a geometric object which is isomorphic to the given geometric object of interest (resp. a morphism of geometric objects which is isomorphic to a morphism of given geometric objects of interest). In the point of view of mono-anabelian geometry, the term "group-theoretic algorithm" is used to mean that "the algorithm in a discussion is phrased in language that only depends on the topological group structures of the fundamental groups under consideration". Note that mono-anabelian results are stronger than bi-anabelian results.

We maintain the notation introduced in Definition 4.1. Then the mono-anabelian version of Definition 4.1 is as follows:

(a) We shall say that $\mathrm{Inv}_{\mathcal{F}_i}$ can be *mono-anabelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Inv}_{\mathcal{F}_i}(\mathfrak{S}_i)$.

(b) We shall say that $\mathrm{Add}_{\mathcal{F}_i}(\mathfrak{S}_i)$ can be *mono-anabelian reconstructed* from $\Pi_{\mathcal{F}_i}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_i}$.

(c) Let j_1, $j_2 \in \{1, 2\}$ distinct from each other, and let $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ be an open continuous homomorphism of profinite groups and $\mathfrak{S}_1 = \Pi_{\mathcal{F}_1} \times_{\theta, \Pi_{\mathcal{F}_2}} \mathfrak{S}_2$. We shall say that a map (or a morphism) $\theta_{\mathrm{add}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$ can be *mono-anabelian reconstructed* from $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ if there exists a group-theoretical algorithm whose input datum is $\theta : \Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$, and whose output datum is $\theta_{\mathrm{add}} : \mathrm{Add}_{\mathcal{F}_{j_1}}(\mathfrak{S}_{j_1}) \to \mathrm{Add}_{\mathcal{F}_{j_2}}(\mathfrak{S}_{j_2})$.

Let $i \in \{1, 2\}$, and let $X_i^\bullet = (X_i, D_{X_i})$ be a pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic $p_i > 0$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and $\Pi_{X_i^\bullet}$ either the admissible fundamental group or the solvable admissible fundamental group of $X_i^\bullet$. The following result was proved by Tamagawa for smooth pointed stable curves and by the author for arbitrary pointed stable curves.

Theorem 4.2. *We maintain the notation introduced above. Then the characteristic p_i of k_i and the type (g_{X_i}, n_{X_i}) can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. Moreover, $\Pi^{\text{ét}}_{X_i^\bullet}$, $\Pi^{\text{top}}_{X_i^\bullet}$, $\mathrm{Ver}(\Pi_{X_i^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$, $\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_i^\bullet})$, and $\Gamma_{X_i^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$.*

Proof. See [34, Theorem 1.2, Remark 1.2.1, Remark 1.2.2, and Proposition 6.1], [29, Theorem 0.1], and [38, Theorem 1.3]. □

Remark 4.2.1. Note that [38, Theorem 1.3] gives a formula for (g_{X_i}, n_{X_i}). Then we obtain that the characteristic p_i of k_i and the type (g_{X_i}, n_{X_i}) can be *mono-anabelian reconstructed* from $\Pi_{X_i^\bullet}$. In fact, we have that $\Pi^{\text{ét}}_{X_i^\bullet}$, $\Pi^{\text{top}}_{X_i^\bullet}$, $\mathrm{Ver}(\Pi_{X_i^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$, $\mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_i^\bullet})$, and $\Gamma_{X_i^\bullet}$ can be *mono-anabelian reconstructed* from $\Pi_{X_i^\bullet}$ (see [36, Theorem 0.2]).

Remark 4.2.2. We do not use the term "mono-anabelian reconstruction" in the present paper. On the other hand, by applying Remark 4.2.1, all of the bi-anabelian results proved in the present paper can be generalized to the case of mono-anabelian reconstructions. Moreover, mono-anabelian results will be used in [40], and play a fundamental role in [41].

Lemma 4.3. *We maintain the notation introduced above. Suppose that $p \overset{\mathrm{def}}{=} p_1 = p_2$, that $(g_X, n_X) \overset{\mathrm{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ be an arbitrary open continuous homomorphism. Then we have that ϕ is a surjection.*

Proof. Let $\Pi_\phi \overset{\mathrm{def}}{=} \phi(\Pi_{X_1^\bullet}) \subseteq \Pi_{X_2^\bullet}$ be the image of ϕ which is an open subgroup of $\Pi_{X_2^\bullet}$. Let $X_\phi^\bullet = (X_\phi, D_{X_\phi})$ be the pointed stable curve of type (g_{X_ϕ}, n_{X_ϕ}) over k_2 induced by Π_ϕ and

$$X_\phi^\bullet \to X_2^\bullet$$

the admissible covering over k_2 induced by the natural inclusion $\Pi_\phi \hookrightarrow \Pi_{X_2^\bullet}$. The Riemann-Hurwitz formula implies that $g_{X_\phi} + n_{X_\phi} \geq g_X + n_X$. Moreover, by applying Theorem 2.2 (a) and Lemma 2.3 (a), the natural surjection $\Pi_{X_1^\bullet} \twoheadrightarrow \Pi_\phi$ induced by ϕ imply that $g_X \geq g_{X_\phi}$ and $n_X \geq n_{X_\phi}$. Then we have

$$(g_X, n_X) = (g_{X_\phi}, n_{X_\phi}).$$

This means that the admissible covering $X_\phi^\bullet \to X_2^\bullet$ is totally ramified over every marked point of D_{X_2}. Then the Riemann-Hurwitz formula implies that $[\Pi_{X_1^\bullet} : \Pi_\phi] \neq 1$ if and only if $(g_X, n_X) = (0, 2)$. Thus, we obtain that ϕ is a surjection. $\square$

In the remainder of this section, we suppose that $p \stackrel{\text{def}}{=} p_1 = p_2$, that $(g_X, n_X) \stackrel{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism. By Lemma 4.3, we see that ϕ is a *surjective* open continuous homomorphism. Let $\mathfrak{P}$ be the set of prime numbers, $\Sigma \subseteq \mathfrak{P} \setminus \{p\}$ a subset, $\Pi_{X_i^\bullet}^\Sigma$ the maximal pro-Σ quotient of $\Pi_{X_i^\bullet}$, $pr_i^\Sigma : \Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^\Sigma$ the natural surjective homomorphism, and

$$\phi^\Sigma : \Pi_{X_1^\bullet}^\Sigma \xrightarrow{\sim} \Pi_{X_2^\bullet}^\Sigma$$

the isomorphism induced by ϕ. In particular, if $\Sigma = \mathfrak{P} \setminus \{p\}$, we use the notation $\Pi_{X_i^\bullet}^{p'}$ and $\phi^{p'} : \Pi_{X_1^\bullet}^{p'} \xrightarrow{\sim} \Pi_{X_2^\bullet}^{p'}$ to denote $\Pi_{X_i^\bullet}^\Sigma$ and ϕ^Σ, respectively.

Lemma 4.4. *We maintain the notation introduced above. Then we have that $\Pi_{X_i^\bullet}^{\mathrm{cpt}}$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$, and that the (surjective) open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a surjective homomorphism*

$$\phi^{\mathrm{cpt}} : \Pi_{X_1^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_2^\bullet}^{\mathrm{cpt}}$$

group-theoretically. Moreover, the following commutative diagram of profinite groups

$$\begin{array}{ccc} \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\ \downarrow & & \downarrow \\ \Pi_{X_1^\bullet}^{\mathrm{cpt}} & \xrightarrow{\phi^{\mathrm{cpt}}} & \Pi_{X_2^\bullet}^{\mathrm{cpt}} \end{array}$$

can be reconstructed group-theoretically from ϕ.

Proof. By Theorem 4.2, we have that (g_X, n_X) can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. If $n_X = 0$, the lemma is trivial. Then we may assume that $n_X > 0$.

Let $H_i \subseteq \Pi_{X_i^\bullet}$ be an open subgroup. Then the Riemann-Hurwitz formula implies that the admissible covering

$$X_{H_i}^\bullet \to X_i^\bullet$$

over k_i induced by $H_i \subseteq \Pi_{X_i^\bullet}$ is étale over D_{X_i} if and only if $g_{X_{H_i}} = [\Pi_{X_i^\bullet} : H_i](g_X - 1) + 1$. We put

$$\begin{aligned} \mathrm{Et}_{D_{X_i}}^{\mathrm{norm}}(\Pi_{X_i^\bullet}) &\stackrel{\text{def}}{=} \{H_i \subseteq \Pi_{X_i^\bullet} \text{ is an open normal subgroup} \\ &\quad \mid g_{X_{H_i}} = [\Pi_{X_i^\bullet} : H_i](g_X - 1) + 1\} \\ \subseteq \mathrm{Et}_{D_{X_i}}(\Pi_{X_i^\bullet}) &\stackrel{\text{def}}{=} \{H_i \subseteq \Pi_{X_i^\bullet} \text{ is an open subgroup} \\ &\quad \mid g_{X_{H_i}} = [\Pi_{X_i^\bullet} : H_i](g_X - 1) + 1\}. \end{aligned}$$

By Theorem 4.2, we have that $\mathrm{Et}^{\mathrm{norm}}_{D_{X_i}}(\Pi_{X_i^\bullet})$ and $\mathrm{Et}_{D_{X_i}}(\Pi_{X_i^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. Since

$$\Pi^{\mathrm{cpt}}_{X_i^\bullet} \overset{\mathrm{def}}{=} \Pi_{X_i^\bullet}/\bigcap_{H_i\in \mathrm{Et}^{\mathrm{norm}}_{D_{X_i}}(\Pi_{X_i^\bullet})} H_i = \Pi_{X_i^\bullet}/\bigcap_{H_i\in \mathrm{Et}_{D_{X_i}}(\Pi_{X_i^\bullet})} H_i,$$

we obtain that $\Pi^{\mathrm{cpt}}_{X_i^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$.

Let $H_2 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_2}}(\Pi_{X_2^\bullet})$, $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2)$, and $G \overset{\mathrm{def}}{=} \Pi_{X_2^\bullet}/H_2 = \Pi_{X_1^\bullet}/H_1$. We will prove that $H_1 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_1}}(\Pi_{X_1^\bullet})$. Let

$$f^\bullet_{H_1} : X^\bullet_{H_1} \to X^\bullet_1$$

be the Galois admissible covering over k_1 with Galois group G corresponding to H_1, $x_1 \in D_{X_1}$ a marked point of $X_1^\bullet$, and $e_{f_{H_1}}(x_1)$ the ramification index of a point of $f^{-1}_{H_1}(x_1)$. Since $H_2 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_2}}(\Pi_{X_2^\bullet})$, we have $g_{X_{H_2}} = \#G(g_X - 1) + 1$ and $n_{X_{H_2}} = \#G n_X$. Then by applying the Riemann-Hurwitz formula, we obtain that

$$\begin{aligned} g_{X_{H_1}} &= \#G(g_X - 1) + 1 + \frac{1}{2}\cdot \sum_{x_1\in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)}(e_{f_{H_1}}(x_1) - 1) \\ &= \#G(g_X - 1) + 1 + \frac{1}{2}\cdot \sum_{x_1\in D_{X_1}} (\#G - \frac{\#G}{e_{f_{H_1}}(x_1)}) \end{aligned}$$

and

$$n_{X_{H_1}} = \sum_{x_1\in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)}.$$

By applying Theorem 2.2 (a) and Lemma 2.3 (a), the surjective homomorphism $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$ induces that

$$\gamma^{\max}(H_1) + 2 = g_{X_{H_1}} + n_{X_{H_1}} \geq \gamma^{\max}(H_2) + 2 = g_{X_{H_2}} + n_{X_{H_2}}.$$

Then we obtain that

$$\begin{aligned} g_{X_{H_1}} + n_{X_{H_1}} &= \#G(g_X - 1) + 1 + \frac{1}{2}\cdot \sum_{x_1\in D_{X_1}} (\#G - \frac{\#G}{e_{f_{H_1}}(x_1)}) + \sum_{x_1\in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)} \\ &= \#G(g_X - 1) + 1 + \frac{1}{2}\#G n_X + \frac{1}{2}\cdot \sum_{x_1\in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)} \\ &\geq \#G(g_X - 1) + 1 + \#G n_X. \end{aligned}$$

Thus, we have

$$\sum_{x_1\in D_{X_1}} \frac{\#G}{e_{f_{H_1}}(x_1)} \geq \#G n_X.$$

Since $\#D_{X_1} = n_X$, we see immediately that $e_{f_{H_1}}(x_1) = 1$. This means that $f^\bullet_{H_1}$ is étale, and that

$$H_1 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_1}}(\Pi_{X_1^\bullet}).$$

Thus we may define the following surjective homomorphism

$$\phi^{\mathrm{cpt}} : \Pi^{\mathrm{cpt}}_{X_1^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet} / \bigcap_{H_1 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_1}}(\Pi_{X_1^\bullet})} H_1 \twoheadrightarrow \Pi^{\mathrm{cpt}}_{X_2^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet} / \bigcap_{H_2 \in \mathrm{Et}^{\mathrm{norm}}_{D_{X_2}}(\Pi_{X_2^\bullet})} H_2$$

which is induced by ϕ group-theoretically. Moreover, the commutative diagram

$$\begin{array}{ccc} \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{cpt}}_{X_1^\bullet} & \xrightarrow{\phi^{\mathrm{cpt}}} & \Pi^{\mathrm{cpt}}_{X_2^\bullet} \end{array}$$

follows immediately from the definition of ϕ^{cpt}. This completes the proof of the lemma. □

Lemma 4.5. *Let ℓ be a prime number, and let $H_2 \subseteq \Pi_{X_2^\bullet}$ be an open normal subgroup and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$. Suppose that $G \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a cyclic group which is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. Then we have that*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. Let

$$f^\bullet_{H_i} : X^\bullet_{H_i} \to X^\bullet_i$$

be the Galois admissible covering over k_i with Galois group G corresponding to H_i. Suppose that $\ell = p$. Then the definition of admissible coverings implies that $f^\bullet_{H_i}$ is étale. Thus, we have that $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then we may suppose that $\ell \neq p$.

By the Riemann-Hurwitz formula, we have

$$g_{X_{H_i}} = \ell(g_X - 1) + 1 + \frac{1}{2}\#e^{\mathrm{op,ra}}_{f_{H_i}}(\ell - 1)$$

and

$$n_{X_{H_i}} = \#e^{\mathrm{op,ra}}_{f_{H_i}} + \ell(n_X - \#e^{\mathrm{op,ra}}_{f_{H_i}}).$$

By applying Theorem 2.2 (a) and Lemma 2.3 (a), the surjective homomorphism $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$ implies that

$$\gamma^{\max}(H_1) + 2 = g_{X_{H_1}} + n_{X_{H_1}} \geq \gamma^{\max}(H_2) + 2 = g_{X_{H_2}} + n_{X_{H_2}}.$$

Then we have

$$\ell(g_X - 1) + 1 + \frac{1}{2}\#e^{\mathrm{op,ra}}_{f_{H_1}}(\ell - 1) + \#e^{\mathrm{op,ra}}_{f_{H_1}} + \ell(n_X - \#e^{\mathrm{op,ra}}_{f_{H_1}})$$

$$= \ell(g_X - 1) + 1 + \ell n_X + \frac{1}{2}(1 - \ell)\#e^{\mathrm{op,ra}}_{f_{H_1}}$$

$$\geq \ell(g_X - 1) + 1 + \frac{1}{2}\#e^{\mathrm{op,ra}}_{f_{H_2}}(\ell - 1) + \#e^{\mathrm{op,ra}}_{f_{H_2}} + \ell(n_X - \#e^{\mathrm{op,ra}}_{f_{H_2}})$$

$$= \ell(g_X - 1) + 1 + \ell n_X + \frac{1}{2}(1-\ell)\#e^{\mathrm{op,ra}}_{f_{H_2}}.$$

Then we obtain that

$$\#e^{\mathrm{op,ra}}_{f_{H_1}} \leq \#e^{\mathrm{op,ra}}_{f_{H_2}}.$$

Let $0 \leq m \leq n_X$ be a positive natural number. We put

$$\mathcal{A}_{i,m} \stackrel{\mathrm{def}}{=} \{N_i \subseteq \Pi_{X_i^\bullet} \text{ is an open normal subgroup} \\ \mid \Pi_{X_i^\bullet}/N_i \cong \mathbb{Z}/\ell\mathbb{Z} \text{ and } \#e^{\mathrm{op,ra}}_{f_{N_i}} = m\}$$

and

$$\mathcal{A}_{i,\leq m} \stackrel{\mathrm{def}}{=} \bigcup_{0 \leq j \leq m} \mathcal{A}_{i,j},$$

where $f^\bullet_{N_i}$ denotes the Galois admissible covering over k_i corresponding to N_i. The isomorphism $\phi^{p'}$ induces a bijective map

$$\phi^*_\ell : \mathcal{A}_{2,\leq n_X} \xrightarrow{\sim} \mathcal{A}_{1,\leq n_X}.$$

To verify the lemma, it sufficient to prove that ϕ^*_ℓ induces a bijection

$$\phi^*_\ell|_{\mathcal{A}_{2,m}} : \mathcal{A}_{2,m} \xrightarrow{\sim} \mathcal{A}_{1,m}.$$

We note that since $(g_X, n_X) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$, the isomorphism $\phi^{p'}$ implies that $\#\mathcal{A}_{1,j} = \#\mathcal{A}_{2,j}$ for each $0 \leq j \leq n_X$. Then by Lemma 4.4, we have a bijection

$$\phi^*_\ell|_{\mathcal{A}_{2,0}} : \mathcal{A}_{2,0} \xrightarrow{\sim} \mathcal{A}_{1,0}.$$

We prove $\phi^*_\ell|_{\mathcal{A}_{2,m}} : \mathcal{A}_{2,m} \xrightarrow{\sim} \mathcal{A}_{1,m}$ holds by induction on m. Suppose that $m \geq 1$. The inequality $\#e^{\mathrm{op,ra}}_{f_{H_1}} \leq \#e^{\mathrm{op,ra}}_{f_{H_2}}$ concerning the cardinality of branch locus implies that we have a bijection $\phi^*_\ell|_{\mathcal{A}_{2,\leq m}} : \mathcal{A}_{2,\leq m} \xrightarrow{\sim} \mathcal{A}_{1,\leq m}$. By induction, $\phi^*_\ell|_{\mathcal{A}_{2,\leq m-1}} : \mathcal{A}_{2,\leq m-1} \xrightarrow{\sim} \mathcal{A}_{1,\leq m-1}$ is a bijection. Then we obtain that

$$\phi^*_\ell|_{\mathcal{A}_{2,m}} : \mathcal{A}_{2,m} \xrightarrow{\sim} \mathcal{A}_{1,m}.$$

This completes the proof of the lemma. □

Corollary 4.6. *Let $H_2 \subseteq \Pi_{X_2^\bullet}$ be an open normal subgroup and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$. Suppose that $G \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a finite solvable group. Then we have that*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. The corollary follows immediately from Lemma 4.5. □

Lemma 4.7. *Let $H_2 \subseteq \Pi_{X_2^\bullet}$ be an open normal subgroup and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$. Suppose that H_2 contains the kernel of the natural homomorphism $\Pi_{X_2^\bullet} \twoheadrightarrow \Pi^{\mathrm{cpt}}_{X_2^\bullet}$ (i.e., the admissible covering corresponding to H_2 is étale over D_{X_2}). Then we have that*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. By Lemma 4.4, we have that H_1 contains the kernel of the natural homomorphism $\Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_1^\bullet}^{\mathrm{cpt}}$ (i.e., the admissible covering corresponding to H_1 is étale over D_{X_1}). Then the lemma follows immediately from the Riemann-Hurwitz formula. $\square$

Definition 4.8. Let Π be an arbitrary profinite group and $m, n \in \mathbb{N}$ positive natural numbers. We define the closed normal subgroup

$$D_n(\Pi)$$

of Π to be the topological closure of $[\Pi, \Pi]\Pi^n$, where $[\Pi, \Pi]$ denotes the commutator subgroup of Π. Moreover, we define the closed normal subgroup

$$D_n^{(m)}(\Pi)$$

of Π inductively by $D_n^{(1)}(\Pi) \stackrel{\mathrm{def}}{=} D_n(\Pi)$ and $D_n^{(j+1)}(\Pi) \stackrel{\mathrm{def}}{=} D_n(D_n^{(j)}(\Pi))$, $j \in \{1, \ldots, m-1\}$. Note that $\#(\Pi/D_n^{(m)}(\Pi)) \leq \infty$ when Π is topologically finitely generated.

Proposition 4.9. *Let $N_2 \subseteq \Pi_{X_2^\bullet}$ be an arbitrary open subgroup, $N_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(N_2) \subseteq \Pi_{X_1^\bullet}$. Then there exist open normal subgroups $H_2 \subseteq N_2 \subseteq \Pi_{X_2^\bullet}$ of $\Pi_{X_2^\bullet}$ and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq N_1 \subseteq \Pi_{X_1^\bullet}$ of $\Pi_{X_1^\bullet}$ such that*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. Let M_i be an open normal subgroup of $\Pi_{X_i^\bullet}$ which is contained in N_i. We put $G \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/M_1 = \Pi_{X_2^\bullet}/M_2$, and write m for $[G : G_p]$, where G_p denotes a Sylow-p subgroup of G. Then we have $(m, p) = 1$. Let

$$f_{M_i}^\bullet : X_{M_i}^\bullet \to X_i^\bullet$$

be the Galois admissible covering over k_i with Galois group G corresponding to M_i.

We put $Q_2 \stackrel{\mathrm{def}}{=} D_m^{(4)}(\Pi_{X_2^\bullet})$ and $Q_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(Q_2)$. Note that since m is prime to p and $\phi^{p'}$ is an isomorphism, we have $Q_1 = D_m^{(4)}(\Pi_{X_1^\bullet})$. Let $H_i \stackrel{\mathrm{def}}{=} M_i \cap Q_i \subseteq \Pi_{X_i^\bullet}$. We denote by

$$f_{Q_i}^\bullet : X_{Q_i}^\bullet \to X_i^\bullet$$

and

$$f_{H_i}^\bullet : X_{H_i}^\bullet \to X_i^\bullet$$

the Galois admissible covering over k_i with Galois group $\Pi_{X_i^\bullet}/Q_i$ corresponding to Q_i, and the Galois admissible covering over k_i with Galois group $\Pi_{X_i^\bullet}/H_i$ corresponding to H_i, respectively. Note that $f_{H_i}^\bullet$ factors through $f_{M_i}^\bullet$ and $f_{Q_i}^\bullet$.

By applying [39, Lemma 3.2 and Lemma 3.3], we have that the ramification index of every point of $D_{X_{Q_i}}$ is divided by m. Then Abhyankar's lemma implies that the Galois admissible covering

$$g_i^\bullet : X_{H_i}^\bullet \to X_{Q_i}^\bullet$$

over k_i induced by $H_i \subseteq Q_i$ is étale. Since $\Pi_{X_i^\bullet}/Q_i$ is a finite solvable group, the proposition follows immediately from Corollary 4.6 and Lemma 4.7. $\square$

Lemma 4.10. *(a) Let $J_i \subseteq \Pi_{X_i}$ be a closed subgroup which is isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$. Then the following conditions are equivalent:*

(i) There exists a unique closed subgroup $I_i \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$ such that $J_i \subseteq I_i$.

(ii) There exists an open subgroup $N_i \subseteq \Pi_{X_i^\bullet}$ such that there exists a cofinal system $\mathcal{C}_{N_i}$ of N_i which consists of open normal subgroups $H_i \subseteq N_i \subseteq \Pi_{X_i^\bullet}$ of $\Pi_{X_i^\bullet}$ (i.e., $N_i \cong \varprojlim_{H_i \in \mathcal{C}_{N_i}} N_i/H_i$), the image of the composition of the natural homomorphisms

$$J_i \cap H_i \hookrightarrow H_i \to H_i^{\mathrm{cpt,ab}}$$

is trivial for every $H_i \in \mathcal{C}_{N_i}$.

(b) Let ℓ be a prime number such that $\ell \neq p$, $I_i, J_i \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$ arbitrary closed subgroups, and $\Pi^{\ell}_{X_i^\bullet}$ the maximal pro-ℓ quotient of $\Pi_{X_i^\bullet}$. Write $\overline{I}_i^{\ell}$ and $\overline{J}_i^{\ell}$ for $pr_i^{\ell}(I_i)$ and $pr_i^{\ell}(J_i)$, respectively. Suppose that $\overline{I}_i^{\ell} = \overline{J}_i^{\ell}$. Then we have

$$I_i = J_i.$$

Proof. By applying similar arguments to the arguments given in the proof of [7, Lemma 1.6], we obtain (a). On the other hand, (b) follows immediately from [12, Proposition 1.2 (i)]. □

Next, we prove the main result of this section.

Theorem 4.11. *We maintain the notation introduced above. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a surjective map*

$$\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet}),$$

group-theoretically. Moreover, ϕ induces a bijection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$$

of the sets of open edges of dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$ group-theoretically.

Proof. If $n_X = 0$, the theorem is trivial. Then we may assume that $n_X > 0$. Let $\mathcal{C}_{\Pi_{X_2^\bullet}}$ be a cofinal system of $\Pi_{X_2^\bullet}$ which consists of open normal subgroups of $\Pi_{X_2^\bullet}$. We put

$$\mathcal{C}_{\Pi_{X_1^\bullet}} \overset{\mathrm{def}}{=} \{H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \mid H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}\}.$$

Note that $\mathcal{C}_{\Pi_{X_1^\bullet}}$ is not a cofinal system of $\Pi_{X_1^\bullet}$ in general. Moreover, by applying Proposition 4.9, we may assume that

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$$

holds for every $H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}$ and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \in \mathcal{C}_{\Pi_{X_1^\bullet}}$.

Let $I_1 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})$ and $\phi(I_1) \subseteq \Pi_{X_2^\bullet}$. We will prove that $\phi(I_1) \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$. Let $H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}$. By replacing $\Pi_{X_i^\bullet}$ and ϕ by H_i and $\phi|_{H_1}$, respectively, Lemma 4.4 implies that we have the following commutative diagram:

$$\begin{array}{ccc} I_1 \cap H_1 & \xrightarrow{\phi|_{I_1 \cap H_1}} & \phi(I_1) \cap H_2 \\ \downarrow & & \downarrow \\ H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \downarrow & & \downarrow \\ H_1^{\mathrm{cpt,ab}} & \xrightarrow{\phi|_{H_1}^{\mathrm{cpt,ab}}} & H_2^{\mathrm{cpt,ab}}. \end{array}$$

Since $I_1 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})$, we have that $I_1 \cap H_1 \hookrightarrow H_1 \to H_1^{\mathrm{cpt,ab}}$ is trivial. Then the commutative diagram above implies that the natural morphism

$$\phi(I_1) \cap H_2 \hookrightarrow H_2 \to H_2^{\mathrm{cpt,ab}}$$

is trivial. Thus, by Lemma 4.10 (a), there exists $I_2 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$ such that $\phi(I_1) \subseteq I_2$.

Let us prove $\phi(I_1) = I_2$. Suppose that $\phi(I_1) \neq I_2$. We put $G \overset{\mathrm{def}}{=} I_2/\phi(I_1)$. Note that G is a cyclic group, and that $(m, p) = 1$, where $m \overset{\mathrm{def}}{=} \#G \geq 2$.

Suppose that $g_X = 0$. Then we have $n_X \geq 3$. Let $N_2 \overset{\mathrm{def}}{=} D_m(\Pi_{X_2})$, $N_1 \overset{\mathrm{def}}{=} \phi^{-1}(N_2) = D_m(\Pi_{X_1})$, and

$$f_{N_i}^\bullet : X_{N_i}^\bullet \to X_i^\bullet$$

the Galois admissible covering over k_i corresponding to N_i. Since the ramification index of each point of $f_{N_i}^{-1}(D_{X_i})$ is equal to m, we have that

$$I_1 \not\subseteq N_1,\ I_2 \not\subseteq N_2,\ \phi(I_1) \subseteq N_2.$$

On the other hand, the isomorphism of maximal pro-prime-to-p quotients $\phi^{p'} : \Pi_{X_1^\bullet}^{p'} \xrightarrow{\sim} \Pi_{X_2^\bullet}^{p'}$ and $I_1 \not\subseteq N_1$ imply that $\phi(I_1) \not\subseteq N_2$. This contradicts $\phi(I_1) \subseteq N_2$. Then we obtain $\phi(I_1) = I_2$.

Suppose that $g_X > 0$. We put

$$Q_2 \overset{\mathrm{def}}{=} \ker(\Pi_{X_2^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_2^\bullet}^{\mathrm{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z})$$

and $Q_1 \overset{\mathrm{def}}{=} \phi^{-1}(Q_2)$. Then Lemma 4.4 implies that $Q_1 = \ker(\Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_1^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_1^\bullet}^{\mathrm{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z})$. Note that the assumption $g_X > 0$ implies that $\Pi_{X_i^\bullet}^{\mathrm{cpt}} \twoheadrightarrow \Pi_{X_i^\bullet}^{\mathrm{cpt,ab}} \otimes \mathbb{Z}/m\mathbb{Z}$ is not trivial. Then the nontrivial Galois admissible covering over k_i corresponding to Q_i is étale over D_{X_i}. Moreover, we have $I_i \subseteq Q_i$ and

$$n_{X_{Q_i}} \geq 2.$$

Let $P_2 \overset{\mathrm{def}}{=} D_m(Q_2)$, $P_1 \overset{\mathrm{def}}{=} \phi^{-1}(P_2) = D_m(Q_1)$, and

$$g_i^\bullet : X_{P_i}^\bullet \to X_{Q_i}^\bullet$$

the Galois admissible covering over k_i corresponding to $P_i \subseteq Q_i$. Since the ramification index of each point of $g_i^{-1}(D_{X_{Q_i}})$ is equal to m, we have that

$$I_1 \nsubseteq P_1,\ I_2 \nsubseteq P_2,\ \phi(I_1) \subseteq P_2.$$

On the other hand, the isomorphism of maximal pro-prime-to-p quotients $\phi|_{P_1}^{p'} : P_1^{p'} \xrightarrow{\sim} P_2^{p'}$ and $I_1 \nsubseteq P_1$ imply that $\phi(I_1) \nsubseteq P_2$. This contradicts $\phi(I_1) \subseteq P_2$. Then we obtain $\phi(I_1) = I_2$. Thus, we may define the following map

$$\phi^{\text{edg,op}} : \text{Edg}^{\text{op}}(\Pi_{X_1^\bullet}) \to \text{Edg}^{\text{op}}(\Pi_{X_2^\bullet}),\ I_1 \mapsto I_2 \stackrel{\text{def}}{=} \phi(I_1).$$

Next, let us prove that $\phi^{\text{edg,op}}$ is a surjection. Let ℓ be a prime number distinct from p and $pr_i^\ell : \Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^\ell$. Let $J_2 \in \text{Edg}^{\text{op}}(\Pi_{X_2^\bullet})$ be an arbitrary subgroup, $\overline{J}_2^\ell \stackrel{\text{def}}{=} pr_2^\ell(J_2)$ the image of J_2, and $\mathcal{C}_{\Pi_{X_i^\bullet}}^\ell \stackrel{\text{def}}{=} \{\overline{H}_i \stackrel{\text{def}}{=} pr_2^\ell(H_i)\}_{H_i \in \mathcal{C}_{\Pi_{X_i^\bullet}}}$, where $\mathcal{C}_{\Pi_{X_i^\bullet}}$ is the set of normal subgroups of $\Pi_{X_i^\bullet}$ defined above. Note that $\mathcal{C}_{\Pi_{X_i^\bullet}}^\ell$ is a cofinal system of $\Pi_{X_i^\bullet}^\ell$, and that $\overline{H}_1 = (\phi^\ell)^{-1}(\overline{H}_2)$.

Let $\overline{H}_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}^\ell$, $\overline{N}_2 \stackrel{\text{def}}{=} \overline{J}_2^\ell \overline{H}_2 \supseteq \overline{H}_2$, $\overline{N}_1 \stackrel{\text{def}}{=} (\phi^\ell)^{-1}(\overline{N}_2) \supseteq \overline{H}_1$, and $N_i \stackrel{\text{def}}{=} (pr_i^\ell)^{-1}(\overline{N}_i)$. Note that $G \stackrel{\text{def}}{=} \overline{N}_1/\overline{H}_1 = N_1/H_1 = \overline{N}_2/\overline{H}_2 = N_2/H_2$ is a cyclic ℓ-group. Write

$$g_{H_i,N_i}^\bullet : X_{H_i}^\bullet \to X_{N_i}^\bullet$$

for the Galois admissible covering over k_i with Galois group G. Since $J_2 \in \text{Edg}^{\text{op}}(\Pi_{X_2^\bullet})$, we obtain that $g_{H_2,N_2}^\bullet$ is totally ramified at a marked point of $X_{H_2}^\bullet$. We put

$$\text{Edg}^{\text{op},\ell,\text{ab}}(N_i) \stackrel{\text{def}}{=} \{\text{the image of } I \text{ of}$$

$$\text{the natural homomorphism } N_i \twoheadrightarrow N_i^{\ell,\text{ab}} \mid I \in \text{Edg}^{\text{op}}(N_i)\}.$$

Note that $\#\text{Edg}^{\text{op},\ell,\text{ab}}(N_i) = n_{X_{N_i}}$. Then the composition of the following natural homomorphisms

$$\bigoplus_{I_{N_2} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_2)} I_{N_2} \to N_2^{\ell,\text{ab}} \twoheadrightarrow G$$

is a surjection. By applying Lemma 4.4, we obtain that the isomorphism ϕ^ℓ induces an isomorphism

$$\text{Im}(\bigoplus_{I_{N_1} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_1)} I_{N_1} \to N_1^{\ell,\text{ab}}) \xrightarrow{\sim} \text{Im}(\bigoplus_{I_{N_2} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_2)} I_{N_2} \to N_2^{\ell,\text{ab}}).$$

Then the composition of the following natural homomorphisms

$$\bigoplus_{I_{N_1} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_1)} I_{N_1} \to N_1^{\ell,\text{ab}} \twoheadrightarrow G$$

is also a surjection. Since G is a cyclic ℓ-group, there exists $I'_{N_1} \in \text{Edg}^{\text{op},\ell,\text{ab}}(N_1)$ such that the composition $I'_{N_1} \hookrightarrow N_1^{\ell,\text{ab}} \twoheadrightarrow G$ is a surjection. This means that $g_{H_1,N_1}^\bullet$ is also totally ramified at a marked point of $X_{H_1}^\bullet$.

We put

$$E_{\overline{H}_1} \overset{\text{def}}{=} \{x_1 \in D_{X_{H_1}} \mid g^{\bullet}_{H_1,N_1} \text{ is totally ramified at } x_1\}.$$

Then we have that $E_{\overline{H}_1}$ is a non-empty finite set. Thus, we obtain

$$\varprojlim_{\overline{H}_1 \in \mathcal{C}^{\ell}_{\Pi_{X_1^\bullet}}} E_{\overline{H}_1} \neq \emptyset.$$

This means that there exists $J_1 \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})$ such that the image $pr_2^{\ell}(\phi(J_1)) = \phi^{\ell}(pr_1^{\ell}(J_1))$ of J_1 of the composition of the natural homomorphisms

$$\begin{array}{ccc} \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet} \\ {\scriptstyle pr_1^{\ell}}\downarrow & & {\scriptstyle pr_2^{\ell}}\downarrow \\ \Pi^{\ell}_{X_1^\bullet} & \xrightarrow{\phi^{\ell}} & \Pi^{\ell}_{X_2^\bullet} \end{array}$$

is equal to $\overline{J}_2^{\ell}$. Since $\phi(J_1) \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$, by applying Lemma 4.10 (b), we have $\phi(J_1) = J_2$. Then $\phi^{\mathrm{edg,op}}$ is a surjection. Moreover, Theorem 4.2 implies that $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. This completes the proof of the first part of the theorem.

Let us prove the "moreover" part of the theorem. We see immediately that $\phi^{\mathrm{edg,op}}$: $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$ is compatible with the natural actions of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, respectively. By using the surjectivity of $\phi^{\mathrm{edg,op}}$, we obtain immediately a surjection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$$

of the sets of open edges of dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$, where $(-)^{\mathrm{sg}}$ means "semi-graph". Moreover, since $n_X = \#e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) = \#e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$, we have that $\phi^{\mathrm{sg,op}}$ is a bijection. On the other hand, Theorem 4.2 implies that $e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$. This completes the proof of the theorem. □

Corollary 4.12. *We maintain the notation introduced above. Let $H_2 \subseteq \Pi_{X_1^\bullet}$ be an arbitrary open subgroup and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_2^\bullet}$. Then we have that*

$$\gamma^{\max}(H_1) = \gamma^{\max}(H_2).$$

Proof. By Theorem 4.11, we obtain that $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then Theorem 2.2 (a) implies that $\gamma^{\max}(H_1) = \gamma^{\max}(H_2)$. □

Let $\widehat{X}_i^\bullet = (\widehat{X}_i, D_{\widehat{X}_i})$, $i \in \{1, 2\}$, be the universal admissible (resp. solvable admissible) covering associated to $\Pi_{X_i^\bullet}$ if $\Pi_{X_i^\bullet}$ is the admissible (resp. solvable admissible) fundamental group of $X_i^\bullet$. Let $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$, $\widehat{e}_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet})$ over e_i, and $I_{\widehat{e}_i} \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$ such that $\phi(I_{\widehat{e}_1}) = I_{\widehat{e}_2}$. Write $\overline{\mathbb{F}}_{p,i}$ for the algebraic closure of $\mathbb{F}_p$ in k_i. We put

$$\mathbb{F}_{\widehat{e}_i} \overset{\text{def}}{=} (I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widehat{e}_i}\},$$

where $\{*_{\widehat{e}_i}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_i^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with

$$\bigcup_{(p,m)=1} \mu_m(\overline{\mathbb{F}}_{p,i}).$$

Moreover, $\mathbb{F}_{\widehat{e}_i}$ can be identified with $\overline{\mathbb{F}}_{p,i}$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widehat{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}$, and whose zero element is $*_{\widehat{e}_i}$. An important consequence of Theorem 4.11 is as follows.

Theorem 4.13. *We maintain the notation introduced above. Then the field structure of* $\mathbb{F}_{\widehat{e}_i}$ *can be reconstructed group-theoretically from* $\Pi_{X_i^\bullet}$. *Moreover,* ϕ *induces a field isomorphism*

$$\phi^{\mathrm{fd}}_{\widehat{e}_1,\widehat{e}_2} : \mathbb{F}_{\widehat{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widehat{e}_2}$$

group-theoretically, where "fd" means "field".

Proof. By applying Theorem 4.11 and [38, Theorem 5.5], similar arguments to the arguments given in the proof of [35, Proposition 6.1] imply that the theorem holds. □

Remark 4.13.1. Theorem 4.13 generalizes [28, Proposition 5.3] and [35, Proposition 6.1] to the case of arbitrary pointed stable curves. [28, Proposition 5.3] and [35, Proposition 6.1] play key roles in the proofs of weak Isom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic $p > 0$ (cf. [28, Theorem 0.2]) and weak Hom-version of the Grothendieck conjecture of curves over algebraically closed fields of characteristic $p > 0$ ([35, Theorem 1.2]), respectively.

5 Combinatorial Grothendieck conjecture for surjections

In this section, we will prove a version of combinatorial Grothendicek conjecture for surjective open continuous homomorphisms under certain assumption, which is an analogue of Theorem 4.11 for topological data and combinatorial data associated to pointed stable curves. *In the present section, we shall assume that all the fundamental groups of pointed stable curves are solvable admissible fundamental groups unless indicated otherwise.*

5.1 Cohomology classes and sets of vertices

We maintain the notation introduced in Section 1. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ the solvable admissible fundamental group of $X^\bullet$.

Let ℓ be a prime number. We put

$$v(\Gamma_{X^\bullet})^{>0,\ell} \stackrel{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid \dim_{\mathbb{F}_\ell}(\mathrm{Hom}(\Pi^{\text{ét}}_{\widetilde{X}_v^\bullet}, \mathbb{F}_\ell)) > 0\},$$

$$M^{\text{ét}}_{X^\bullet} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\text{ét}}_{X^\bullet}, \mathbb{F}_\ell),$$

$$M_{X^\bullet}^{\mathrm{top}} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{top}}, \mathbb{F}_\ell).$$

On the other hand, we have the natural isomorphisms $\mathrm{Hom}(\Pi^{\text{ét}}_{\widetilde{X}_v^\bullet}, \mathbb{F}_\ell) \cong H^1_{\text{ét}}(\widetilde{X}_v, \mathbb{F}_\ell)$, $M^{\text{ét}}_{X^\bullet} \cong H^1_{\text{ét}}(X, \mathbb{F}_\ell)$, and $M^{\mathrm{top}}_{X^\bullet} \cong H^1(\Gamma_{X^\bullet}, \mathbb{F}_\ell)$. In the theory of anabelian geometry, we want to emphasize the objects under consideration are arose from various fundamental groups. Then we do not use the standard notation $H^1_{\text{ét}}(\widetilde{X}_v, \mathbb{F}_\ell)$, $H^1_{\text{ét}}(X, \mathbb{F}_\ell)$, and $H^1(\Gamma_{X^\bullet}, \mathbb{F}_\ell)$. Moreover, there is an injection $M^{\mathrm{top}}_{X^\bullet} \hookrightarrow M^{\text{ét}}_{X^\bullet}$ induced by the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top}}_{X^\bullet}$. We put

$$M^{\mathrm{nt}}_{X^\bullet} \overset{\mathrm{def}}{=} \mathrm{coker}(M^{\mathrm{top}}_{X^\bullet} \hookrightarrow M^{\text{ét}}_{X^\bullet}),$$

where $(-)^{\mathrm{nt}}$ means "non-top". A non-zero element of $M^{\text{ét}}_{X^\bullet}$ corresponds to a Galois étale covering of the underlying curve X of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ and an non-zero element of $M^{\mathrm{top}}_{X^\bullet}$ corresponds to a Galois étale covering of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the map of dual semi-graphs is a topological covering.

Let $V^*_{X,\ell} \subseteq M^{\text{ét}}_{X^\bullet}$ be the subset of elements such that the image of $M^{\text{ét}}_{X^\bullet} \twoheadrightarrow M^{\mathrm{nt}}_{X^\bullet}$ is not 0. Then an element of $V^*_{X,\ell}$ corresponds to a Galois étale covering of the underlying curve X of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the map of dual semi-graphs is not a topological covering. Let $\alpha \in V^*_{X,\ell}$ and

$$f^\bullet_\alpha : X^\bullet_\alpha \to X^\bullet$$

the Galois étale covering corresponding to α. Denote by $\Gamma_{X^\bullet_\alpha}$ the dual semi-graph of $X^\bullet_\alpha$. Then we obtain a map

$$\iota : V^*_{X,\ell} \to \mathbb{Z}_{>0},\ \alpha \mapsto \#v(\Gamma_{X^\bullet_\alpha}).$$

Furthermore, we put

$$V^\star_{X,\ell} \overset{\mathrm{def}}{=} \{\alpha \in V^*_{X,\ell} \mid \iota \text{ attains its maximum}\}$$
$$= \{\alpha \in V^*_{X,\ell} \mid \iota(\alpha) = \ell\#v(\Gamma_{X^\bullet}) - \ell + 1\}$$
$$= \{\alpha \in V^*_{X,\ell} \mid \#v^{\mathrm{ra}}_{f_\alpha} = 1\}.$$

For each $\alpha \in V^\star_{X,\ell}$, $\iota(\alpha) = \ell\#v(\Gamma_{X^\bullet}) - \ell + 1$ implies that there exists a unique irreducible component $Z \subseteq X_\alpha$ whose decomposition group under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is not trivial. Let $v_\alpha \in v(\Gamma_{X^\bullet})$ such that $X_{v_\alpha} = f_\alpha(Z)$. Then we have $v_\alpha \in v(\Gamma_{X^\bullet})^{>0,\ell}$. This means that $V^\star_{X,\ell} = \emptyset$ if and only if $v(\Gamma_{X^\bullet})^{>0,\ell} = \emptyset$.

On the other hand, let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Write $f^{\mathrm{sg}}_H : \Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by the admissible covering $f^\bullet_H : X^\bullet_H \to X^\bullet$ over k corresponding to H. We define a map

$$f^{\mathrm{ver},\ell}_H : v(\Gamma_{X^\bullet_H})^{>0,\ell} \to v(\Gamma_{X^\bullet})^{>0,\ell}$$

as follows: Let $v_H \in v(\Gamma_{X^\bullet_H})^{>0,\ell}$ and $v \overset{\mathrm{def}}{=} f^{\mathrm{sg}}_H(v_H) \in v(\Gamma_{X^\bullet_H})$. Then we have that $f^{\mathrm{ver},\ell}_H(v_H) = v$ if $\dim_{\mathbb{F}_\ell}(\mathrm{Hom}(\Pi^{\text{ét}}_{\widetilde{X}^\bullet_v}, \mathbb{F}_\ell)) \neq 0$; otherwise, $f^{\mathrm{ver},\ell}_H(v_H) = \emptyset$. Moreover, if $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup, then $v(\Gamma_{X^\bullet_H})^{>0,\ell}$ admits a natural action of $\Pi_{X^\bullet}/H$. Then we have the following proposition.

Proposition 5.1. *(a) We define a pre-equivalence relation* $\sim$ *on* $V^{\star}_{X,\ell}$ *as follows:*

> *Let* $\alpha, \beta \in V^{\star}_{X,\ell}$*. We have that* $\alpha \sim \beta$ *if, for each* $\lambda, \mu \in \mathbb{F}^{\times}_{\ell}$ *for which* $\lambda\alpha+\mu\beta \in V^{*}_{X,\ell}$*,* $\lambda\alpha + \mu\beta \in V^{\star}_{X,\ell}$*.*

Then the pre-equivalence relation $\sim$ *on* $V^{\star}_{X,\ell}$ *is an equivalence relation.*

(b) We denote by $V_{X,\ell}$ *the quotient set of* $V^{\star}_{X,\ell}$ *by* $\sim$*. Then we have a natural bijection*

$$\kappa_{X,\ell} : V_{X,\ell} \xrightarrow{\sim} v(\Gamma_{X^{\bullet}})^{>0,\ell},\ [\alpha] \mapsto v_{\alpha},$$

where $[\alpha]$ *denotes the equivalence class of* α*.*

(c) Let ℓ, ℓ' *be prime numbers distinct from each other. Suppose that* $\ell \neq p$*. Then we have a natural injection*

$$V_{X,\ell'} \hookrightarrow V_{X,\ell},$$

which is a bijection if $\ell' \neq p$*, and which fits into the following commutative diagram:*

$$\begin{array}{ccc} V_{X,\ell'} & \xrightarrow{\kappa_{X,\ell'}} & v(\Gamma_{X^{\bullet}})^{>0,\ell'} \\ \downarrow & & \downarrow \\ V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^{\bullet}})^{>0,\ell}, \end{array}$$

where the vertical map of the right-hand side is the natural injection induced by the definitions of $v(\Gamma_{X^{\bullet}})^{>0,\ell'}$ *and* $v(\Gamma_{X^{\bullet}})^{>0,\ell}$*.*

(d) Let $H \subseteq \Pi_{X^{\bullet}}$ *be an open subgroup. Suppose that* $([\Pi^{\bullet}_{X} : H], \ell) = 1$*. Then the natural injection* $H \hookrightarrow \Pi_{X^{\bullet}}$ *induces a map*

$$\gamma^{\mathrm{ver},\ell}_{H} : V_{X_H,\ell} \to V_{X,\ell}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} V_{X_H,\ell} & \xrightarrow{\kappa_{X_H,\ell}} & v(\Gamma_{X^{\bullet}_{H}})^{>0,\ell} \\ \gamma^{\mathrm{ver},\ell}_{H}\downarrow & & f^{\mathrm{ver},\ell}_{H}\downarrow \\ V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^{\bullet}})^{>0,\ell}. \end{array}$$

Moreover, suppose that $H \subseteq \Pi_{X^{\bullet}}$ *is an open normal subgroup. Then* $V_{X_H,\ell}$ *admits an action of* $\Pi_{X^{\bullet}}/H$ *such that* $\kappa_{X_H,\ell}$ *is compatible with* $\Pi_{X^{\bullet}}/H$*-actions (i.e.,* $\kappa_{X_H,\ell}$ *is* $\Pi_{X^{\bullet}}/H$*-equivariant).*

Proof. See [36, Proposition 2.1, Remark 2.1.1, and Remark 2.1.2] for (a), (b), and (d). The notation in the proof of (d) will be used in the remainder of the present paper, let us explain (d).

Let $[\alpha_X] \in V_{X,\ell}$. Then α_X induces an element

$$\beta_{X_H} = \sum_{\beta \in L_{\alpha_X}} c_{\beta}\beta \in \mathrm{Hom}(H, \mathbb{F}_{\ell}),\ c_{\beta} \in \mathbb{F}^{\times}_{\ell},$$

via the natural homomorphism $\mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{F}_\ell) \to \mathrm{Hom}(H, \mathbb{F}_\ell)$, where L_{α_X} is a subset of $V^\star_{X_H,\ell}$ such that, if $\beta_1, \beta_2 \in L_{\alpha_X}$ distinct from each other, then $[\beta_1] \neq [\beta_2]$.

Let $[\alpha_{X_H}] \in V_{X_H,\ell}$. Then we define

$$\gamma_H^{\mathrm{ver},\ell}([\alpha_{X_H}]) = [\alpha_X]$$

if there exists $[\alpha_X] \in V_{X,\ell}$ such that there exists $\beta \in L_{\alpha_X}$, and that $[\beta] = [\alpha_{X_H}]$ (i.e., $\beta \sim \alpha_{X_H}$). Otherwise, we put $\gamma_H^{\mathrm{ver},\ell}([\alpha_{X_H}]) = \emptyset$. It is easy to check that $\gamma_H^{\mathrm{ver},\ell}$ is well-defined, and that the following diagram

$$\begin{array}{ccc} V_{X_H,\ell} & \xrightarrow{\kappa_{X_H,\ell}} & v(\Gamma_{X_H^\bullet})^{>0,\ell} \\ {\scriptstyle \gamma_H^{\mathrm{ver},\ell}}\downarrow & & {\scriptstyle f_H^{\mathrm{ver},\ell}}\downarrow \\ V_{X,\ell} & \xrightarrow{\kappa_{X,\ell}} & v(\Gamma_{X^\bullet})^{>0,\ell} \end{array}$$

is commutative.

Moreover, suppose that H is an open normal subgroup of $\Pi_{X^\bullet}$. The natural exact sequence

$$1 \to H \to \Pi_{X^\bullet} \to \Pi_{X^\bullet}/H \to 1$$

induces an outer representation

$$\Pi_{X^\bullet}/H \to \mathrm{Out}(H) \overset{\mathrm{def}}{=} \frac{\mathrm{Aut}(H)}{\mathrm{Inn}(H)}.$$

Then we obtain an action of $\Pi_{X^\bullet}/H$ on

$$V^\star_{X_H,\ell} \subseteq \mathrm{Hom}(H^{\text{ét}}, \mathbb{F}_\ell)$$

induced by the outer representation. Let $\sigma \in \Pi_{X^\bullet}/H$ and $\alpha_{X_H}, \alpha'_{X_H} \in V^\star_{X_H,\ell}$. Then we have that $\alpha_{X_H} \sim \alpha'_{X_H}$ if and only if $\sigma(\alpha_{X_H}) \sim \sigma(\alpha'_{X_H})$. Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on $V_{X_H,\ell}$ induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. On the other hand, it is easy to check that the commutative diagram above is compatible with the $\Pi_{X^\bullet}/H$-actions. This completes the proof of the proposition. □

Remark 5.1.1. By applying Theorem 4.2, we have that $\Pi^{\text{ét}}_{X^\bullet}$ and $\Pi^{\mathrm{top}}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Then we obtain that $V_{X,\ell}$ (or $v(\Gamma_{X^\bullet})^{>0,\ell}$) can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, for every open subgroup $H \subseteq \Pi_{X^\bullet}$, the map

$$\gamma_H^{\mathrm{ver},\ell} : V_{X_H,\ell} \to V_{X,\ell}$$

constructed in Proposition 5.1 (d) can be reconstructed group-theoretically from the natural inclusion $H \hookrightarrow \Pi_{X^\bullet}$.

5.2 Cohomology classes and sets of closed edges

We maintain the notation introduced in Section 5.1. Moreover, in this subsection, we suppose that the genus of the normalization of each irreducible component of X is *positive* (i.e., $v(\Gamma_{X^\bullet}) = v(\Gamma_{X^\bullet})^{>0,\ell}$ if $\ell \neq p$), and that $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-*connected.*

We shall say that

$$\mathfrak{T}_{X^\bullet} \overset{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \to X^\bullet)$$

is an *edge-triple* associated to $X^\bullet$ if the following conditions are satisfied:

(i) ℓ and d are prime numbers distinct from each other and from p.

(ii) $\ell \equiv 1 \pmod d$; this means that all dth roots of unity are contained in $\mathbb{F}_\ell$. Moreover, we write $\mu_d \subseteq \mathbb{F}_\ell^\times$ for the subgroup of dth roots of unity.

(iii) $f_X^\bullet : Y^\bullet \to X^\bullet$ is a Galois admissible covering over k whose Galois group is isomorphic to μ_d such that $f_X^\bullet$ is étale, and that $\#v_{f_X}^{\text{sp}} = 0$. Note that since $v(\Gamma_{X^\bullet}) = v(\Gamma_{X^\bullet})^{>0,d}$, $f_X^\bullet$ exists.

On the other hand, we shall say that

$$\mathfrak{T}_{\Pi_{X^\bullet}} \overset{\text{def}}{=} (\ell, d, \alpha_{f_X})$$

is an *edge-triple* associated to $\Pi_{X^\bullet}$ if the following conditions are satisfied:

(i) $\alpha_{f_X} \in \mathrm{Hom}(\Pi_{X^\bullet}^{\text{ét}}, \mathbb{F}_d)$.

(ii) The composition of the following natural homomorphisms

$$\Pi_{\widetilde{X}_v^\bullet}^{\text{ét}} \hookrightarrow \Pi_{X^\bullet}^{\text{ét}} \overset{\alpha_{f_X}}{\to} \mathbb{F}_d$$

is a surjection for every $v \in v(\Gamma_{X^\bullet})$.

We see immediately that an edge-triple $\mathfrak{T}_{X^\bullet}$ associated to $X^\bullet$ is equivalent to an edge-triple $\mathfrak{T}_{\Pi_{X^\bullet}}$ associated to $\Pi_{X^\bullet}$, where the Galois admissible covering corresponding to the kernel of the composition of the natural homomorphisms $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \overset{\alpha_{f_X}}{\to} \mathbb{F}_d$ is $f_X^\bullet$.

In the remainder of the present subsection, we fix an edge-triple

$$\mathfrak{T}_{\Pi_{X^\bullet}} \overset{\text{def}}{=} (\ell, d, \alpha_{f_X})$$

associated to $\Pi_{X^\bullet}$. Write $\mathfrak{T}_{X^\bullet} \overset{\text{def}}{=} (\ell, d, f_X^\bullet : Y^\bullet \to X^\bullet)$ for the edge-triple associated to $X^\bullet$ corresponding to $\mathfrak{T}_{\Pi_{X^\bullet}}$, (g_Y, n_Y) for the type of $Y^\bullet$, $\Gamma_{Y^\bullet}$ for the dual semi-graph of $Y^\bullet$, r_Y for the Betti number of $\Gamma_{Y^\bullet}$, and $\Pi_{Y^\bullet}$ for the kernel of the composition of the homomorphisms $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \overset{\alpha_{f_X}}{\to} \mathbb{F}_d$.

We put

$$M_{Y^\bullet} \overset{\text{def}}{=} \mathrm{Hom}(\Pi_{Y^\bullet}, \mathbb{F}_\ell).$$

Note that there is a natural injection $M_{Y^\bullet}^{\text{ét}} \overset{\text{def}}{=} \mathrm{Hom}(\Pi_{Y^\bullet}^{\text{ét}}, \mathbb{F}_\ell) \hookrightarrow M_{Y^\bullet}$ induced by the natural surjection $\Pi_{Y^\bullet} \twoheadrightarrow \Pi_{Y^\bullet}^{\text{ét}}$. Then we obtain an exact sequence

$$0 \to M_{Y^\bullet}^{\text{ét}} \to M_{Y^\bullet} \to M_{Y^\bullet}^{\text{ra}} \overset{\text{def}}{=} \mathrm{coker}(M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}) \to 0$$

with a natural action of μ_d, where "ra" means "ramification". For any element of $M_{Y^\bullet}$, if the image of the element is not 0 in $M_{Y^\bullet}^{\text{ra}}$, then the Galois admissible covering of $Y^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to the element is not étale.

Let $M_{Y^\bullet,\mu_d}^{\text{ra}} \subseteq M_{Y^\bullet}^{\text{ra}}$ be the subset of elements on which μ_d acts via the character $\mu_d \hookrightarrow \mathbb{F}_\ell^\times$ and $E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^* \subseteq M_{Y^\bullet}$ the subset of elements that map to nonzero elements of $M_{Y^\bullet,\mu_n}^{\text{ra}}$. Let $\alpha \in E_{\mathfrak{T}_{\Pi_{X^\bullet}}}^*$. Write

$$g_\alpha^\bullet : Y_\alpha^\bullet \to Y^\bullet$$

for the Galois admissible covering over k corresponding to α. Then we obtain a map

$$\epsilon : E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \to \mathbb{Z}_{\geq 0},\ \alpha \mapsto \#(e^{\mathrm{op}}(\Gamma_{Y^\bullet_\alpha}) \cup e^{\mathrm{cl}}(\Gamma_{Y^\bullet_\alpha})),$$

where $\Gamma_{Y^\bullet_\alpha}$ denotes the dual semi-graph of $Y^\bullet_\alpha$. We put

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \stackrel{\mathrm{def}}{=} \{\alpha \in E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \mid \#e^{\mathrm{op,ra}}_{g_\alpha} = 0,\ \#e^{\mathrm{cl,ra}}_{g_\alpha} = d\}.$$

Note that $E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ is not an empty set. For each $\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, since the image of α is contained in $M^{\mathrm{ra}}_{Y^\bullet,\mu_d}$, we obtain that the action of μ_d on the set

$$\{y_e\}_{e \in e^{\mathrm{cl,ra}}_{g_\alpha}} \subseteq \mathrm{Nod}(Y^\bullet)$$

is transitive, where $\mathrm{Nod}(-)$ denotes the set of nodes of $(-)$, and y_e denotes the node of $Y^\bullet$ corresponding to e. Then there exists a unique node x_α of $X^\bullet$ such that $f_X(y_e) = x_\alpha$ for every $y_e \in \{y_e\}_{e \in e^{\mathrm{cl,ra}}_{g_\alpha}}$. We denote by $e_\alpha \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ the closed edge corresponding to x_α.

On the other hand, let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Write $f^{\mathrm{sg}}_H : \Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by the admissible covering $f^\bullet_H : X^\bullet_H \to X^\bullet$ over k corresponding to H. We shall denote by

$$f^{\mathrm{cl}}_H \stackrel{\mathrm{def}}{=} f^{\mathrm{sg}}_H|_{e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})} : e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

Moreover, if $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup, then $e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})$ admits a natural action of $\Pi_{X^\bullet}/H$. Then we have the following result.

Proposition 5.2. *(a) We define a pre-equivalence relation $\sim$ on $E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ as follows:*

> *Let $\alpha, \beta \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$. We have that $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in E^{*}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, we have $\lambda\alpha + \mu\beta \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$.*

Then the pre-equivalence relation $\sim$ on $E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ is an equivalence relation.

(b) We denote by $E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ the quotient set of $E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ by $\sim$. Then we have a natural bijection

$$\vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}} : E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X^\bullet}),\ [\alpha] \mapsto e_\alpha,$$

where $[\alpha]$ denotes the equivalent class of α.

(c) Let $\mathfrak{T}'_{\Pi_{X^\bullet}}$ be an arbitrary edge-triples associated to $\Pi_{X^\bullet}$. Then we have a natural bijection

$$E^{\mathrm{cl}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} \xrightarrow{\sim} E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}'_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta_{\mathfrak{T}'_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \\ \downarrow & & \| \\ E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}). \end{array}$$

(d) Let $H \subseteq \Pi_{X^\bullet}$ be an open subgroup. Suppose that $([\Pi_{X^\bullet} : H], \ell) = ([\Pi_{X^\bullet} : H], d) = 1$. We have that $\mathfrak{T}_{X^\bullet}$ associated to $\Pi_{X^\bullet}$ induces an edge-triple

$$\mathfrak{T}_{X_H^\bullet} \overset{\mathrm{def}}{=} (\ell, d, f_{X_H}^\bullet : Y_{X_H}^\bullet \overset{\mathrm{def}}{=} Y^\bullet \times_{X^\bullet} X_H^\bullet \to X_H^\bullet)$$

associated to $X_H^\bullet$, where $Y^\bullet \times_{X^\bullet} X_H^\bullet$ denotes the fiber product in the category of pointed stable curves. Write $\mathfrak{T}_H$ for the edge-triple associated to H corresponding to $\mathfrak{T}_{X_H^\bullet}$. Then the natural injection $H \hookrightarrow \Pi_{X^\bullet}$ induces a surjective map

$$\gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}, H} : E^{\mathrm{cl}}_{\mathfrak{T}_H} \twoheadrightarrow E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}_H} & \xrightarrow{\vartheta_{\mathfrak{T}_H}} & e^{\mathrm{cl}}(\Gamma_{X_H^\bullet}) \\ {\scriptstyle \gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}, H}} \downarrow & & {\scriptstyle f_H^{\mathrm{cl}}} \downarrow \\ E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}). \end{array}$$

Moreover, suppose that $H \subseteq \Pi_{X^\bullet}$ is an open normal subgroup. Then $E^{\mathrm{cl}}_{\mathfrak{T}_H}$ admits an action of $\Pi_{X^\bullet}/H$ such that $\vartheta_{\mathfrak{T}_H}$ is compatible with $\Pi_{X^\bullet}/H$-actions (i.e., $\vartheta_{\mathfrak{T}_H}$ is $\Pi_{X^\bullet}/H$-equivariant).

Proof. See [36, Proposition 2.2, Remark 2.2.1, and Remark 2.2.2] for (a), (b), and (c). The notation in the proof of (d) will be used in the remainder of the present paper, let us explain (d).

Let $\alpha_X \in E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$. Then α_X induces an element

$$\beta_{X_H} = \sum_{\beta \in J_{\alpha_X}} c_\beta \beta, \ c_\beta \in \mathbb{F}_\ell^\times$$

via the natural homomorphism $\mathrm{Hom}(\Pi_{Y_{X_H}^\bullet}, \mathbb{F}_\ell) \to \mathrm{Hom}(\Pi_{Y^\bullet}, \mathbb{F}_\ell)$, where $\Pi_{Y_{X_H}^\bullet} \overset{\mathrm{def}}{=} \Pi_{Y^\bullet} \cap H$, and J_{α_X} is a subset of $E^{\mathrm{cl},\star}_{\mathfrak{T}_H}$ such that, if $\beta_1, \beta_2 \in J_{\alpha_X}$ distinct from each other, then $[\beta_1] \neq [\beta_2]$. Let $[\alpha_{X_H}] \in E^{\mathrm{cl}}_{\mathfrak{T}_H}$. We define

$$\gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}, H}([\alpha_{X_H}]) = [\alpha_X]$$

if there exists $\alpha_X \in E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ such that there exists $\beta \in J_{\alpha_X}$, and that $[\beta] = [\alpha_{X_H}]$. It is easy to check that $\gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}, H}$ is well-defined, and that the following diagram

$$\begin{array}{ccc} E^{\mathrm{cl}}_{\mathfrak{T}_H} & \xrightarrow{\vartheta_{\mathfrak{T}_H}} & e^{\mathrm{cl}}(\Gamma_{X_H^\bullet}) \\ {\scriptstyle \gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}, H}} \downarrow & & {\scriptstyle f_H^{\mathrm{cl}}} \downarrow \\ E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\Pi_{X^\bullet}}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \end{array}$$

is commutative.

Moreover, suppose that H is an open normal subgroup of $\Pi_{X^\bullet}$. Since $\Pi_{Y^\bullet_{X_H}}$ is an open normal subgroup of $\Pi_{X^\bullet}$, we have

$$\Pi_{X^\bullet}/\Pi_{Y^\bullet_{X_H}} \cong \Pi_{X^\bullet}/H \times \mathbb{Z}/d\mathbb{Z}.$$

Then the natural exact sequence

$$1 \to \Pi_{Y^\bullet_{X_H}} \to \Pi_{X^\bullet} \to \Pi_{X^\bullet}/\Pi_{Y^\bullet_{X_H}} \to 1$$

induces an outer representation

$$\Pi_{X^\bullet}/H \hookrightarrow \Pi_{X^\bullet}/\Pi_{Y^\bullet_{X_H}} \to \mathrm{Out}(\Pi_{Y^\bullet_{X_H}}).$$

Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_H} \subseteq \mathrm{Hom}(\Pi_{Y^\bullet_{X_H}}, \mathbb{F}_\ell)$$

induced by the outer representation.

Let $\sigma \in \Pi_{X^\bullet}/H$ and α_{X_H}, $\alpha'_{X_H} \in E^{\mathrm{cl},\star}_{\mathfrak{T}_H}$. We obverse that $\alpha_{X_H} \sim \alpha'_{X_H}$ if and only if $\sigma(\alpha_{X_H}) \sim \sigma(\alpha'_{X_H})$. Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on $E^{\mathrm{cl}}_{\mathfrak{T}_H}$ induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. On the other hand, it is easy to check that the commutative diagram above is compatible with the $\Pi_{X^\bullet}/H$-actions. This completes the proof of the proposition. □

Remark 5.2.1. By applying Theorem 4.2, we have that $\Pi^{\text{ét}}_{X^\bullet}$ can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Then $E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ (or $e^{\mathrm{cl}}(\Gamma_{X^\bullet})$) can be reconstructed group-theoretically from $\Pi_{X^\bullet}$. Moreover, for every open subgroup $H \subseteq \Pi_{X^\bullet}$, the map

$$\gamma^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet},H}} : E^{\mathrm{cl}}_{\mathfrak{T}_H} \to E^{\mathrm{cl}}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$$

constructed in Proposition 5.2 (d) can be reconstructed group-theoretically from the natural inclusion $H \hookrightarrow \Pi_{X^\bullet}$.

Next, let us calculate the cardinality $\#E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ of $E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$. We put

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \mid e = e_\alpha\},\ e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

Note that $e = e_\alpha$, $\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e}$, means that the Galois admissible covering $g^\bullet_\alpha : Y^\bullet_\alpha \to Y^\bullet$ over k induced by α is (totally) ramified over $f_X^{-1}(x_e)$, where x_e denotes the closed point of X corresponding to e. Moreover, we have the following disjoint union

$$E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} = \bigsqcup_{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})} E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e}.$$

Let $m \in \mathbb{Z}_{\geq 0}$ and $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. We shall put

$$E^{\mathrm{cl},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \mid \#v^{\mathrm{sp}}_{g_\alpha} = m\}.$$

Let $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ be a closed edge. Write Y_e for the normalization of the underlying curve Y of $Y^\bullet$ at $f_X^{-1}(x_e)$ and

$$\mathrm{nor}_e : Y_e \to Y$$

for the resulting normalization morphism. Since the genus of the normalization of each irreducible component of $X^\bullet$ is positive, we obtain that the genus of the normalization of each irreducible component of Y_e is also positive. Moreover, since $\Gamma_{X^\bullet}$ is 2-connected, Y_e is connected. We have the following lemma.

Lemma 5.3. *We maintain the notation introduced above. Let $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ be a closed edge. Then we have*

$$\#E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e} = \ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}.$$

Moreover, we have

$$\#E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}} = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet})(\ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}).$$

Proof. Write $R_e \subseteq Y_e$ for the set of closed subset $(f_X \circ \mathrm{nor}_e)^{-1}(x_e)$. Then $E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}$ can be naturally regarded as a subset of $H^1_{\text{ét}}(Y_e \setminus R_e, \mathbb{F}_\ell)$ via the natural open immersion $Y_e \setminus R_e \hookrightarrow Y_e$. Write L_e for the $\mathbb{F}_\ell$-vector subspace spanned by $E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e}$ in $H^1_{\text{ét}}(Y_e \setminus R_e, \mathbb{F}_\ell)$. Then we see that

$$E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e} = L_e \setminus H^1_{\text{ét}}(Y_e, \mathbb{F}_\ell).$$

Write H_e^{ra} for the cokernel of the natural inclusion $H^1_{\text{ét}}(Y_e, \mathbb{F}_\ell) \hookrightarrow L_e$. We obtain an exact sequence as follows:

$$0 \to H^1_{\text{ét}}(Y_e, \mathbb{F}_\ell) \to L_e \to H_e^{\mathrm{ra}} \to 0.$$

On the other hand, since the action of μ_d on $f^{-1}(x_e)$ is translative, the structure of the maximal pro-ℓ quotient $\Pi^\ell_{Y^\bullet}$ of $\Pi_{Y^\bullet}$ implies that

$$\dim_{\mathbb{F}_\ell}(H_e^{\mathrm{ra}}) = 1.$$

Since

$$\dim_{\mathbb{F}_\ell}(H^1_{\text{ét}}(Y_e, \mathbb{F}_\ell)) = 2(g_Y - d) - (r_Y - d) = 2g_Y - d - r_Y,$$

we obtain that

$$\#E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}},e} = \ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}.$$

Thus, we have

$$\#E^{\mathrm{cl},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}} = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet})(\ell^{2g_Y-d-r_Y+1} - \ell^{2g_Y-d-r_Y}).$$

This completes the proof of the lemma. □

Next, we introduce some notation concerning open edges. We put

$$E^{\mathrm{op},\star}_{\mathfrak{I}_{\Pi_{X^\bullet}}} \overset{\mathrm{def}}{=} \{\alpha \in E^{*}_{\mathfrak{I}_{\Pi_{X^\bullet}}} \mid \#e^{\mathrm{op,ra}}_{g_\alpha} = d,\ \#e^{\mathrm{cl,ra}}_{g_\alpha} = 0\}.$$

Note that $E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$ is not an empty set if $n_X \neq 0$. For each $\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}}$, since the image of α is contained in $M^{\mathrm{ra}}_{Y^\bullet,\mu_d}$, we obtain that the action of μ_d on the set

$$\{y_e\}_{e\in e^{\mathrm{op,ra}}_{g_\alpha}} \subseteq D_Y$$

is transitive, where y_e denotes the marked point of $Y^\bullet$ corresponding to e. Then there exists a unique marked point $x_\alpha \in D_X$ of $X^\bullet$ such that $f_X(y_e) = x_\alpha$ for every $y_e \in \{y_e\}_{e\in e^{\mathrm{op,ra}}_{g_\alpha}}$. We denote by $e_\alpha \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ the open edge corresponding to x_α. Moreover, we put

$$E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} \mid e = e_\alpha\},\ e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}).$$

Note that $e = e_\alpha$, $\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e}$, means that the Galois admissible covering $g^\bullet_\alpha : Y^\bullet_\alpha \to Y^\bullet$ over k induced by α is (totally) ramified over $f_X^{-1}(x_e)$, where x_e denotes the closed point of X corresponding to e. Moreover, we have the following disjoint union

$$E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}}} = \bigsqcup_{e\in e^{\mathrm{op}}(\Gamma_{X^\bullet})} E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e}.$$

Let $m \in \mathbb{Z}_{\geq 0}$ and $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. We shall put

$$E^{\mathrm{op},\star,m}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \overset{\mathrm{def}}{=} \{\alpha \in E^{\mathrm{op},\star}_{\mathfrak{T}_{\Pi_{X^\bullet}},e} \mid \#v^{\mathrm{sp}}_{g_\alpha} = m\}.$$

Finally, we introduce the following conditions concerning pointed stable curves. Let $W^\bullet$ be a pointed stable curve over k of type (g_W, n_W), $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$, and $\Pi_{W^\bullet}$ the solvable admissible fundamental group of $W^\bullet$.

Condition A . We shall say that $W^\bullet$ satisfies Condition A if the following conditions are satisfied:

(i) the genus of the normalization of each irreducible component of W is positive;
(ii) every irreducible component of W is smooth over k;
(iii) $\Gamma^{\mathrm{cpt}}_{W^\bullet}$ is 2-connected;
(iv) $\#(v(\Gamma_{W^\bullet})^{b\leq 1}) = 0$.

Condition B . We shall say that $W^\bullet$ satisfies Condition B if $\Gamma^{\mathrm{cpt}}_{W^\bullet_H}$ is 2-connected for every open subgroup $H \subseteq \Pi_{W^\bullet}$.

Lemma 5.4. *Let m be a positive natural number prime to p and $H \overset{\mathrm{def}}{=} D^{(3)}_m(\Pi_{W^\bullet}) \subseteq \Pi_{W^\bullet}$. Then $W^\bullet_H$ satisfies Condition A, and the Betti number of the dual semi-graph of $W^\bullet_H$ is positive.*

Proof. The lemma follows from the structure of $\Pi^{p'}_{W^\bullet}$. □

5.3 Reconstruction of sets of vertices, sets of closed edges, sets of genus, and sets of p-rank from surjections

In this subsection, we prove that the sets of vertices, sets of closed edges, and sets of genus can be reconstructed group-theoretically from a surjective open continuous homomorphism of solvable admissible fundamental groups.

We fix some notation. Let $i \in \{1, 2\}$, k_i an algebraically closed field of characteristic $p > 0$, and ℓ a prime number distinct from p. Let $X_i^\bullet$ be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X_i^\bullet}$ the solvable admissible fundamental group of $X_i^\bullet$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X_i^\bullet}$. Moreover, let $v_i \in v(\Gamma_{X_i^\bullet})$, $\widetilde{X}_{i,v_i}^\bullet$ the smooth pointed stable curve of type (g_{i,v_i}, n_{i,v_i}) over k_i associated to v_i, and σ_{i,v_i} the p-rank of $\widetilde{X}_{i,v_i}^\bullet$. We introduce the following condition:

Condition C . We shall say that $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition C if the following conditions are satisfied:
(i) $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$;
(ii) $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{X_2^\bullet})$;
(iii) $\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) = \#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$.

In the remainder of the present subsection, we suppose that $X_1^\bullet$ *and* $X_2^\bullet$ *satisfy Condition A, Condition B, and Condition C*. Moreover, let

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism of the solvable admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$, and

$$(g_X, n_X) \overset{\mathrm{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

Note that we have that $r_{X_1} = r_{X_2}$, and that by Lemma 4.3, ϕ is a *surjective* open continuous homomorphism. First, we have the following lemma.

Lemma 5.5. *We maintain the notation introduced above. Then we have*

$$\mathrm{Avr}_p(\Pi_{X_i^\bullet}) = g_{X_i} - r_{X_i}.$$

Proof. The lemma follows immediately from Condition A and Theorem 2.2 (b). □

Let G be a finite group such that $(\#G, p) = 1$ and

$$f_i^\bullet : Y_i^\bullet \to X_i^\bullet$$

a Galois admissible covering over k_i with Galois group G. Let $j \in \{1, 2\}$ such that $i \neq j$. Then the isomorphism $\phi^{p'} : \Pi_{X_1^\bullet}^{p'} \overset{\sim}{\to} \Pi_{X_2^\bullet}^{p'}$ induced by ϕ implies that $f_i^\bullet$ induces a Galois admissible covering

$$f_j^\bullet : Y_j^\bullet \to X_j^\bullet$$

over k_j with Galois group G. We write (g_{Y_i}, n_{Y_i}) for the type of $Y_i^\bullet$, $\Gamma_{Y_i^\bullet}$ for the dual semi-graph of $Y_i^\bullet$, and r_{Y_i} for the Betti number of $\Gamma_{Y_i^\bullet}$.

Lemma 5.6. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ is étale, and that $\#v_{f_1}^{\mathrm{sp}} = m$. Then we have*

$$\#e_{f_2}^{\mathrm{cl,ra}} + \frac{1}{2}\#e_{f_2}^{\mathrm{op,ra}} + \#v_{f_2}^{\mathrm{sp}} \leq m.$$

Proof. Since $f_1^\bullet$ is an étale covering, the Riemann-Hurwitz formula implies that

$$g_{Y_1} = \ell(g_X - 1) + 1$$

and

$$g_{Y_2} = \ell(g_X - 1) + \frac{1}{2}(\ell - 1)\#e_{f_2}^{\mathrm{op,ra}} + 1.$$

Then we obtain

$$g_{Y_1} - g_{Y_2} = -\frac{1}{2}(\ell - 1)\#e_{f_2}^{\mathrm{op,ra}}.$$

On the other hand, we have

$$\begin{aligned} r_{Y_1} &= \ell\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) - \#v(\Gamma_{X_1^\bullet}) + \#v_{f_1}^{\mathrm{sp}} - \ell\#v_{f_1}^{\mathrm{sp}} + 1 \\ &= \ell\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) - \#v(\Gamma_{X_1^\bullet}) - (\ell - 1)m + 1 \end{aligned}$$

and

$$r_{Y_2} = \ell\#e_{f_2}^{\mathrm{cl,\acute{e}t}} + \#e_{f_2}^{\mathrm{cl,ra}} - \ell\#v_{f_2}^{\mathrm{sp}} - \#v_{f_2}^{\mathrm{ra}} + 1.$$

Since $\#e(\Gamma_{X_1^\bullet}) = \#e(\Gamma_{X_2^\bullet})$ and $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{X_2^\bullet})$, we obtain that

$$r_{Y_1} - r_{Y_2} = (\ell - 1)\#e_{f_2}^{\mathrm{cl,ra}} + (\ell - 1)(\#v_{f_2}^{\mathrm{sp}} - m)$$

Moreover, by applying Lemma 5.5 and Lemma 2.3 (b), we have

$$g_{Y_1} - g_{Y_2} \geq r_{Y_1} - r_{Y_2}.$$

Thus, we obtain

$$\#e_{f_2}^{\mathrm{cl,ra}} + \frac{1}{2}\#e_{f_2}^{\mathrm{op,ra}} + \#v_{f_2}^{\mathrm{sp}} \leq m.$$

This completes the proof of the lemma. □

Corollary 5.7. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ is étale, and that $\#v_{f_1}^{\mathrm{sp}} = 0$. Then we have that $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ is étale, and that $\#v_{f_2}^{\mathrm{sp}} = 0$.*

Proof. The corollary follows immediately from Lemma 5.6. □

Corollary 5.8. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ is étale, and that $\#v_{f_1}^{\mathrm{sp}} = 1$. Then we have that $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ is étale.*

Proof. In order to verify the corollary, it is sufficient to prove that

$$\#e_{f_2}^{\mathrm{cl,ra}} = \#e_{f_2}^{\mathrm{op,ra}} = 0.$$

By applying Lemma 5.6, we have

$$\#e_{f_2}^{\mathrm{cl,ra}} + \frac{1}{2}\#e_{f_2}^{\mathrm{op,ra}} + \#v_{f_2}^{\mathrm{sp}} \leq 1.$$

Suppose that $\#e_{f_2}^{\mathrm{cl,ra}} \neq 0$. Since $X_2^\bullet$ satisfies Condition A, the inequality above and the structures of the maxmial prime-to-p quotient of solvable admissible fundamental groups imply that either $\#e_{f_2}^{\rm cl,ra} = 1$ and $\#e_{f_2}^{\rm op,ra} \geq 2$ or $\#e_{f_2}^{\rm cl,ra} \geq 2$ holds. Then we have $\#e_{f_2}^{\rm cl,ra} + \frac{1}{2}\#e_{f_2}^{\rm op,ra} + \#v_{f_2}^{\rm sp} > 1$. Thus, we have $\#e_{f_2}^{\rm cl,ra} = 0$.

Suppose that $\#e_{f_2}^{\rm op,ra} \neq 0$. Since $\#e_{f_2}^{\rm cl,ra} = 0$, the inequality above implies that $\#e_{f_2}^{\rm op,ra} = 2$. Let $\ell' \neq p$ be a prime number distinct from ℓ, and let

$$g_1^\bullet : Z_1^\bullet \to X_1^\bullet$$

be a Galois étale covering of over k_1 with Galois group $\mathbb{Z}/\ell'\mathbb{Z}$ such that $\#v_{g_1}^{\rm sp} = 0$. Then Corollary 5.7 implies that the Galois admissible covering

$$g_2^\bullet : Z_2^\bullet \to X_2^\bullet$$

over k_2 with Galois group $\mathbb{Z}/\ell'\mathbb{Z}$ induced by $g_2^\bullet$ is étale covering, and that $\#v_{g_2}^{\rm sp} = 0$. Write $\Gamma_{Z_i^\bullet}$ for the dual semi-graphs of $Z_i^\bullet$. We obtain

$$\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{Z_1^\bullet}) = \#v(\Gamma_{Z_2^\bullet}) = \#v(\Gamma_{X_2^\bullet}),$$

$$\ell'\#e^{\rm op}(\Gamma_{X_1^\bullet}) = \#e^{\rm op}(\Gamma_{Z_1^\bullet}) = \#e^{\rm op}(\Gamma_{Z_2^\bullet}) = \ell'\#e^{\rm op}(\Gamma_{X_2^\bullet}),$$

$$\ell'\#e^{\rm cl}(\Gamma_{X_1^\bullet}) = \#e^{\rm cl}(\Gamma_{Z_1^\bullet}) = \#e^{\rm cl}(\Gamma_{Z_2^\bullet}) = \ell'\#e^{\rm cl}(\Gamma_{X_2^\bullet}).$$

We have that $Z_1^\bullet$ and $Z_2^\bullet$ satisfy Condition A, Condition B, and Condition C.

We denote by $W_i^\bullet \stackrel{\rm def}{=} Y_i^\bullet \times_{X_i^\bullet} Z_i^\bullet$. Note that since $\ell' \neq \ell$, we have that $W_i^\bullet$ is connected. Then $f_i^\bullet$ induces a Galois admissible covering

$$h_i^\bullet : W_i^\bullet \to Z_i^\bullet$$

over k_i with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. We have that $h_1^\bullet$ is étale, that $\#v_{h_1}^{\rm sp} = 1$, and that $\#e_{h_2}^{\rm op,ra} = 2\ell'$. Then Lemma 5.6 implies that

$$1 < \#e_{h_2}^{\rm cl,ra} + \frac{1}{2}\#e_{h_2}^{\rm op,ra} + \#v_{h_2}^{\rm sp} = \#e_{h_2}^{\rm cl,ra} + \ell' + \#v_{h_2}^{\rm sp} \leq 1.$$

Thus, we obtain $\#e_{f_2}^{\rm op,ra} = 0$. This completes the proof of the corollary. □

We put

$$M_{X_i^\bullet} \stackrel{\rm def}{=} \mathrm{Hom}(\Pi_{X_i^\bullet}, \mathbb{F}_\ell),$$

$$M_{X_i^\bullet}^{\rm ét} \stackrel{\rm def}{=} \mathrm{Hom}(\Pi_{X_i^\bullet}^{\rm ét}, \mathbb{F}_\ell),$$

$$M_{X_i^\bullet}^{\rm top} \stackrel{\rm def}{=} \mathrm{Hom}(\Pi_{X_i^\bullet}^{\rm top}, \mathbb{F}_\ell).$$

Note that we have the following injections (or weight-monodromy filtration)

$$M_{X_i^\bullet}^{\rm top} \hookrightarrow M_{X_i^\bullet}^{\rm ét} \hookrightarrow M_{X_i^\bullet} \ (\text{or } M_{X_i^\bullet}^{\rm top} \subseteq M_{X_i^\bullet}^{\rm ét} \subseteq M_{X_i^\bullet})$$

induced by the natural surjections $\Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^{\rm ét} \twoheadrightarrow \Pi_{X_i^\bullet}^{\rm top}$. Moreover, we have an isomorphism

$$\psi_\ell : M_{X_2^\bullet} \stackrel{\sim}{\to} M_{X_1^\bullet}$$

induced by the isomorphism $\phi^\ell : \Pi_{X_1^\bullet}^\ell \stackrel{\sim}{\to} \Pi_{X_2^\bullet}^\ell$. Then we have the following propositions.

Proposition 5.9. *We maintain the notation introduced above. Then the isomorphism* $\psi_\ell : M_{X_2^\bullet} \xrightarrow{\sim} M_{X_1^\bullet}$ *induces an isomorphism*

$$\psi_\ell^{\text{ét}} : M_{X_2^\bullet}^{\text{ét}} \xrightarrow{\sim} M_{X_1^\bullet}^{\text{ét}}$$

group-theoretically. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} M_{X_2^\bullet}^{\text{ét}} & \xrightarrow{\psi_\ell^{\text{ét}}} & M_{X_1^\bullet}^{\text{ét}} \\ \downarrow & & \downarrow \\ M_{X_2^\bullet} & \xrightarrow{\psi_\ell} & M_{X_1^\bullet}, \end{array}$$

where the vertical arrows are injections.

Proof. To verify the proposition, it is sufficient to prove that $\psi_\ell^{-1} : M_{X_1^\bullet} \xrightarrow{\sim} M_{X_2^\bullet}$ induces an isomorphism $\psi_\ell^{-1,\text{ét}} : M_{X_1^\bullet}^{\text{ét}} \xrightarrow{\sim} M_{X_2^\bullet}^{\text{ét}}$ which fits into the following commutative diagram:

$$\begin{array}{ccc} M_{X_1^\bullet}^{\text{ét}} & \xrightarrow{\psi_\ell^{-1,\text{ét}}} & M_{X_2^\bullet}^{\text{ét}} \\ \downarrow & & \downarrow \\ M_{X_1^\bullet} & \xrightarrow{\psi_\ell^{-1}} & M_{X_1^\bullet}, \end{array}$$

where the vertical arrows are injections.

Let $\alpha_1 \in M_{X_1^\bullet}^{\text{ét}}$ be a non-trivial element and $f_{1,\alpha}^\bullet : Y_{1,\alpha}^\bullet \to X_1^\bullet$ the Galois étale covering over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α. We put

$$L_{X_1^\bullet} \stackrel{\text{def}}{=} \{\alpha_1 \in M_{X_1^\bullet}^{\text{ét}} \mid \#v_{f_{1,\alpha_1}}^{\text{sp}} = 1\}.$$

We see that $M_{X_1^\bullet}^{\text{ét}}$ is spanned by $L_{X_1^\bullet}$ as an $\mathbb{F}_\ell$-vector space.

On the other hand, Corollary 5.8 implies that $f_{1,\alpha_1}^\bullet$ induces a Galois étale covering of $X_2^\bullet$ over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. This means that ψ_ℓ^{-1} induces an injection of $\mathbb{F}_\ell$-vector spaces

$$\psi_\ell^{-1,\text{ét}} : M_{X_1^\bullet}^{\text{ét}} \hookrightarrow M_{X_2^\bullet}^{\text{ét}}.$$

Moreover, since $\dim_{\mathbb{F}_\ell}(M_{X_1^\bullet}^{\text{ét}}) = 2g_{X_1} - r_{X_1} = 2g_{X_2} - r_{X_2} = \dim_{\mathbb{F}_\ell}(M_{X_2^\bullet}^{\text{ét}})$, we obtain that

$$\psi_\ell^{-1,\text{ét}} : M_{X_1^\bullet}^{\text{ét}} \xrightarrow{\sim} M_{X_2^\bullet}^{\text{ét}}$$

is an isomorphism. This completes the proof of the proposition. □

Proposition 5.10. *We maintain the notation introduced above. Then the isomorphism* $\psi_\ell : M_{X_2^\bullet} \xrightarrow{\sim} M_{X_1^\bullet}$ *induces an isomorphism*

$$\psi_\ell^{\text{top}} : M_{X_2^\bullet}^{\text{top}} \xrightarrow{\sim} M_{X_1^\bullet}^{\text{top}}$$

group-theoretically. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} M^{\mathrm{top}}_{X_2^\bullet} & \xrightarrow{\psi_\ell^{\mathrm{top}}} & M^{\mathrm{top}}_{X_1^\bullet} \\ \downarrow & & \downarrow \\ M^{\text{ét}}_{X_2^\bullet} & \xrightarrow{\psi_\ell^{\text{ét}}} & M^{\text{ét}}_{X_1^\bullet} \\ \downarrow & & \downarrow \\ M_{X_2^\bullet} & \xrightarrow{\psi_\ell} & M_{X_1^\bullet}, \end{array}$$

where the vertical arrows are injections.

Proof. First, by Proposition 5.9, the isomorphism $\psi_\ell : M_{X_2^\bullet} \xrightarrow{\sim} M_{X_1^\bullet}$ induces an isomorphism $\psi_\ell^{\text{ét}} : M^{\text{ét}}_{X_2^\bullet} \xrightarrow{\sim} M^{\text{ét}}_{X_1^\bullet}$. Let $\alpha_2 \in M^{\mathrm{top}}_{X_2^\bullet} \subseteq M^{\text{ét}}_{X_2^\bullet}$ be a non-trivial element and

$$f^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to X_2^\bullet$$

the Galois étale covering over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_2. Then we obtain an element $\alpha_1 \overset{\mathrm{def}}{=} \psi_\ell^{\text{ét}}(\alpha_2) \in M^{\text{ét}}_{X_1^\bullet}$. Write $f^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \to X_1^\bullet$ for the Galois étale covering over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_1. Note that the types of $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$ are equal, and that $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$ satisfy Condition A.

Lemma 5.5 and Lemma 2.3 (b) imply that

$$r_{Y_{1,\alpha_1}} \leq r_{Y_{2,\alpha_2}},$$

where $r_{Y_{1,\alpha_1}}$ and $r_{Y_{2,\alpha_2}}$ denote the Betti numbers of the dual semi-graphs of $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$, respectively. Since $\#v^{\mathrm{sp}}_{f_{2,\alpha_2}} = \#v(\Gamma_{X_2^\bullet}) = \#v(\Gamma_{X_1^\bullet})$, the inequality implies $\#v^{\mathrm{sp}}_{f_{1,\alpha_1}} = \#v(\Gamma_{X_1^\bullet})$. Thus, we have

$$\alpha_1 \in M^{\mathrm{top}}_{X_1^\bullet}.$$

Then α_1 induces an injection

$$\psi_\ell^{\mathrm{top}} : M^{\mathrm{top}}_{X_2^\bullet} \hookrightarrow M^{\mathrm{top}}_{X_1^\bullet}.$$

Moreover, since $\dim_{\mathbb{F}_\ell}(M^{\mathrm{top}}_{X_2^\bullet}) = r_{X_2} = r_{X_1} = \dim_{\mathbb{F}_\ell}(M^{\mathrm{top}}_{X_1^\bullet})$, we have that ψ_ℓ^{top} is an isomorphism. This completes the proof of the proposition. □

Remark 5.10.1. Proposition 5.9 and Proposition 5.10 mean that the weight-monodromy filtrations can be reconstructed group-theoretically from ϕ.

Lemma 5.11. *We maintain the notation introduced above. Suppose that $G \cong \mathbb{Z}/\ell\mathbb{Z}$, that $f_2^\bullet$ is étale, and that $\#v^{\mathrm{ra}}_{f_2} = 1$. Then we have that $f_1^\bullet$ is étale, and that $\#v^{\mathrm{ra}}_{f_1} = 1$.*

Proof. By Proposition 5.9, we obtain that $f_1^\bullet$ is étale. This implies that $g_{Y_1} = g_{Y_2}$, and that $\#e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet}) = \ell\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) = \ell\#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) = \#e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})$. On the other hand, Lemma 5.5 and Lemma 2.3 (b) imply that

$$r_{Y_1} \leq r_{Y_2}.$$

Thus, we obtain

$$\ell\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) - \ell(\#v(\Gamma_{X_1^\bullet}) - \#v^{\mathrm{ra}}_{f_1}) - \#v^{\mathrm{ra}}_{f_1} + 1 \leq \ell\#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) - \ell(\#v(\Gamma_{X_2^\bullet}) - 1) - 1 + 1$$

This implies that $\#v^{\mathrm{ra}}_{f_1} \leq 1$.

Suppose that $\#v^{\mathrm{ra}}_{f_1} = 0$. Let $\alpha_{f_1} \in M_{X_1^\bullet}$ be an element corresponding to $f_1^\bullet$. Then $\alpha_{f_1} \in M^{\mathrm{top}}_{X_1^\bullet}$. Note that $\alpha_{f_2} \overset{\mathrm{def}}{=} (\psi^{\text{ét}}_\ell)^{-1}(\alpha_{f_1}) \in M^{\text{ét}}_{X_2^\bullet}$ is the element corresponding to $f_2^\bullet$. Then Proposition 5.10 implies that α_{f_2} is contained in $M^{\mathrm{top}}_{X_2^\bullet}$. This means that $\#v^{\mathrm{ra}}_{f_2} = 0$. This contradicts the assumption $\#v^{\mathrm{ra}}_{f_2} = 1$. Thus, we have $\#v^{\mathrm{ra}}_{f_1} = 1$. We complete the proof of the lemma. □

We reconstruct the sets of vertices and the sets of genus of irreducible components group-theoretically from ϕ as follows.

Theorem 5.12. *We maintain the notation introduced above. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a bijection of the set of vertices*

$$\phi^{\mathrm{sg,ver}} : v(\Gamma_{X_1^\bullet}) \overset{\sim}{\to} v(\Gamma_{X_2^\bullet})$$

group-theoretically. Moreover, let $v_1 \in v(\Gamma_{X_1^\bullet})$ and $v_2 \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,vex}}(v_1)$. Then we have

$$g_{1,v_1} = g_{2,v_2}.$$

Proof. We maintain the notation introduced in Section 5.1. By applying Theorem 4.2, Proposition 5.9, and Proposition 5.10, we obtain that the following morphism of the natural exact sequences can be induced by ϕ group-theoretically:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{\mathrm{top}}_{X_2^\bullet} & \longrightarrow & M^{\text{ét}}_{X_2^\bullet} & \longrightarrow & M^{\mathrm{nt}}_{X_2^\bullet} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \psi^{\mathrm{top}}_\ell} & & \Big\downarrow{\scriptstyle \psi^{\text{ét}}_\ell} & & \Big\downarrow & & \\ 0 & \longrightarrow & M^{\mathrm{top}}_{X_1^\bullet} & \longrightarrow & M^{\text{ét}}_{X_1^\bullet} & \longrightarrow & M^{\mathrm{nt}}_{X_1^\bullet} & \longrightarrow & 0. \end{array}$$

Then we obtain

$$\psi^{\text{ét}}_\ell(V^*_{X_2,\ell}) = V^*_{X_1,\ell}.$$

Moreover, Lemma 5.11 implies that

$$\psi^{\text{ét}}_\ell(V^\star_{X_2,\ell}) = V^\star_{X_1,\ell}.$$

Let $\alpha_2,\ \alpha_2' \in V^\star_{X_2,\ell}$ distinct from each other such that $\alpha_2 \sim \alpha_2'$. By applying Lemma 5.11 again, for any $a,\ b \in \mathbb{F}_\ell^\times$, we see that $a\alpha_2 + b\alpha_2' \in V^\star_{X_2,\ell}$ if and only $\psi^{\text{ét}}_\ell(a\alpha_2 + b\alpha_2') = a\psi^{\text{ét}}_\ell(\alpha_2) + b\psi^{\text{ét}}_\ell(\alpha_2') \in V^\star_{X_1,\ell}$. Thus, we obtain a bijection

$$V_{X_2,\ell} \overset{\sim}{\to} V_{X_1,\ell}.$$

Then the first part of the theorem follows from Proposition 5.1.

Next, let us prove the "moreover" part of the theorem. Let $v_i \in v(\Gamma_{X_i^\bullet})$. We put

$$L^{v_i}_{X_i^\bullet} \overset{\mathrm{def}}{=} \{\alpha_i \in M^{\text{ét}}_{X_i^\bullet} \mid v^{\mathrm{ra}}_{f_{i,\alpha_i}} = \{v_i\}\},$$

where $f^\bullet_{i,\alpha_i}$ denotes the Galois admissible covering of $X^\bullet_i$ over k_i corresponding to α_i. Moreover, we denote by

$$[L^{v_i}_{X^\bullet_i}]$$

the image of $L^{v_i}_{X^\bullet_i}$ in $M^{\mathrm{nt}}_{X^\bullet_i}$. Then we have

$$\#[L^{v_i}_{X^\bullet_i}] = \ell^{g_{i,v_i}} - 1.$$

Suppose that $v_2 = \phi^{\mathrm{sg,ver}}(v_1)$. Proposition 5.10 and Lemma 5.11 imply that $\psi^{\text{ét}}_\ell$ induces an injection

$$[L^{v_2}_{X^\bullet_2}] \hookrightarrow [L^{v_1}_{X^\bullet_1}].$$

Thus, we have

$$\ell^{g_{2,v_2}} - 1 = \#[L^{v_2}_{X^\bullet_2}] \leq \#[L^{v_1}_{X^\bullet_1}] = \ell^{g_{1,v_1}} - 1.$$

This means that

$$g_{2,v_2} \leq g_{1,v_1}.$$

On the other hand, since

$$\sum_{v_1 \in v(\Gamma_{X^\bullet_1})} g_{1,v_1} = g_X - r_{X_1} = g_X - r_{X_2} = \sum_{v_2 \in v(\Gamma_{X^\bullet_2})} g_{2,v_2},$$

we obtain

$$g_{1,v_1} = g_{2,v_2}.$$

This completes the proof of the theorem. □

Next, let us reconstruct the sets of closed edges from ϕ. In the remainder of the present subsection, we fix an edge-triple

$$\mathfrak{T}_{\Pi_{X^\bullet_1}} \stackrel{\mathrm{def}}{=} (\ell, d, \alpha_{f_{X_1}} : \Pi^{\text{ét}}_{X^\bullet_1} \twoheadrightarrow \mathbb{F}_d)$$

associated to $\Pi_{X^\bullet_1}$. Then Corollary 5.7 implies that ϕ and the edge-triple $\mathfrak{T}_{\Pi_{X^\bullet_1}}$ induces an edge-triple

$$\mathfrak{T}_{\Pi_{X^\bullet_2}} \stackrel{\mathrm{def}}{=} (\ell, d, \alpha_{f_{X_2}} : \Pi^{\text{ét}}_{X^\bullet_2} \twoheadrightarrow \mathbb{F}_d)$$

associated to $\Pi_{X^\bullet_2}$ group-theoretically. Write $\Pi_{Y^\bullet_i}$ for the kernel of $\alpha_{f_{X_i}}$. The surjection $\phi : \Pi_{X^\bullet_1} \twoheadrightarrow \Pi_{X^\bullet_2}$ induces a sujection

$$\phi_Y : \Pi_{Y^\bullet_1} \twoheadrightarrow \Pi_{Y^\bullet_2}.$$

Moreover, the constructions of $Y^\bullet_1$ and $Y^\bullet_2$ imply that $Y^\bullet_1$ and $Y^\bullet_2$ satisfy Condition A, Condition B, and Condition C.

We put

$$M_{Y^\bullet_i} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{Y^\bullet_i}, \mathbb{F}_\ell),$$

$$M^{\text{ét}}_{Y^\bullet_i} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\text{ét}}_{Y^\bullet_i}, \mathbb{F}_\ell),$$

$$M^{\mathrm{ra}}_{Y^\bullet_i} \stackrel{\mathrm{def}}{=} M_{Y^\bullet_i}/M^{\text{ét}}_{Y^\bullet_i}.$$

Then, by Theorem 4.2 and Proposition 5.9, the following commutative diagram can be induced by ϕ_Y group-theoretically:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{\text{ét}}_{Y_2^\bullet} & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & M^{\text{ra}}_{Y_2^\bullet} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi^{\text{ét}}_{Y,\ell}} & & \downarrow{\scriptstyle \psi_{Y,\ell}} & & \downarrow & & \\ 0 & \longrightarrow & M^{\text{ét}}_{Y_1^\bullet} & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & M^{\text{ra}}_{Y_1^\bullet} & \longrightarrow & 0, \end{array}$$

where all the vertical arrows are isomorphisms. Let $E^*_{\mathfrak{T}_{\Pi_{X_i^\bullet}}}$ be the subset of $M_{Y_i^\bullet}$ defined in Section 5.2. Since the actions of μ_d on the exact sequences are compatible with the isomorphisms appearing in the commutative diagram above, we have

$$\psi_{Y,\ell}(E^*_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}) = E^*_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}.$$

Let $m \in \mathbb{Z}_{\geq 0}$ and $e_i \in e^{\text{cl}}(\Gamma_{X_i^\bullet})$. We recall that $E^{\text{cl},\star,m}_{\mathfrak{T}_{\Pi_{X_i^\bullet}},e_i}$ is the subset of $E^{\text{cl},\star}_{\mathfrak{T}_{\Pi_{X_i^\bullet}},e_i}$ whose element α_i satisfies $\#v^{\text{sp}}_{g_{i,\alpha_i}} = m$. Then we have the following lemma.

Lemma 5.13. *We maintain the notation introduced above. Then we have*

$$\psi^{-1}_{Y,\ell}\Big(\bigsqcup_{e_1 \in e^{\text{op}}(\Gamma_{X_1^\bullet})} E^{\text{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_1^\bullet}},e_1}\Big) \subseteq \bigsqcup_{e_2 \in e^{\text{op}}(\Gamma_{X_2^\bullet})} E^{\text{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_2^\bullet}},e_2}.$$

Moreover, we have

$$\psi^{-1}_{Y,\ell}(E^{\text{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}) = E^{\text{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}.$$

Proof. Let $e_1 \in e^{\text{cl}}(\Gamma_{X_1^\bullet})$ and $\alpha_1 \in E^{\text{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_1^\bullet}},e_1}$. Then the Galois admissible covering

$$g^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha} \to Y^\bullet_1$$

over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_1 induces a Galois admissible covering

$$g^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to Y^\bullet_2$$

over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Write $\alpha_2 \in M_{Y_2^\bullet}$ for the element corresponding to $g^\bullet_{2,\alpha_2}$. We have

$$\alpha_2 \in E^*_{\mathfrak{T}_{\Pi_{Y_2^\bullet}}}.$$

Write $g_{Y_{i,\alpha_i}}$ for the genus of $Y^\bullet_{i,\alpha_i}$, and $r_{Y_{i,\alpha_i}}$ for the Betti number of the dual semi-graph $\Gamma_{Y^\bullet_{i,\alpha_i}}$. Then the Riemann-Hurwitz formula and Theorem 4.11 imply that

$$g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} = -\frac{1}{2}(\#e^{\text{op,ra}}_{g_{2,\alpha_2}})(\ell - 1) = 0.$$

On the other hand, we have

$$r_{Y_{1,\alpha_1}} = \ell(\#e^{\text{cl}}(\Gamma_{Y_1^\bullet}) - d) + d - \#v(\Gamma_{Y_1^\bullet}) + 1$$

and

$$r_{Y_{2,\alpha_2}} = \ell\#e^{\mathrm{cl,ét}}_{g_{2,\alpha_2}} + \#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} - \ell\#v^{\mathrm{cl,sp}}_{g_{2,\alpha_2}} - \#v^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} + 1.$$

Then Lemma 5.5 and Lemma 2.3 (b) imply that

$$0 = g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} \geq r_{Y_{1,\alpha_1}} - r_{Y_{2,\alpha_1}}.$$

Thus, we have

$$\#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} + \#v^{\mathrm{sp}}_{g_{2,\alpha_2}} + \frac{1}{2}\#e^{\mathrm{op,ra}}_{g_{2,\alpha_2}} = \#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} + \#v^{\mathrm{sp}}_{g_{2,\alpha_2}} \leq d.$$

If $\#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} = 0$, then g_{2,α_2} is étale. By replacing $X_1^\bullet$ and $X_2^\bullet$ by $Y_1^\bullet$ and $Y_2^\bullet$, respectively, Proposition 5.9 implies that g_{1,α_1} is also étale. This contradicts the definition of α_1. Thus, we obtain $\#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} \neq 0$.

If $\#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} \neq 0$, then we have $\#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} = d$ and $\#v^{\mathrm{sp}}_{g_{2,\alpha_2}} = \#e^{\mathrm{op,ra}}_{g_{2,\alpha_2}} = 0$. This means that

$$\alpha_2 \in \bigsqcup_{e_2 \in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{Y_2^\bullet}}, e_2}.$$

Thus, we have

$$\psi^{-1}_{Y,\ell}\Big(\bigsqcup_{e_1 \in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{Y_1^\bullet}}, e_1}\Big) \subseteq \bigsqcup_{e_2 \in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{Y_2^\bullet}}, e_2}.$$

Moreover, let $\beta_i \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{Y_i^\bullet}}}$. Then β_i is a linear combination of the elements of

$$\bigsqcup_{e_i \in e^{\mathrm{cl}}(\Gamma_{Y_i^\bullet})} E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{Y_i^\bullet}}, e_i}.$$

Then we have

$$\psi^{-1}_{Y,\ell}(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}) \subseteq E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}.$$

On the other hand, since $g_{Y_1} = g_{Y_2}$ and $r_{Y_1} = r_{Y_2}$, Lemma 5.3 implies that $\#\psi^{-1}_{Y,\ell}(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}) = \#E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}$. Thus, we obtain

$$\psi^{-1}_{Y,\ell}(E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}) = E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}.$$

This completes the proof of the lemma. □

We reconstruct the sets of closed edges group-theoretically from ϕ as follows.

Theorem 5.14. *We maintain the notation introduced above. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a bijection of the set of closed edges*

$$\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$$

group-theoretically.

Proof. Let $\alpha_2, \alpha_2' \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_2^\bullet}}}$ and $\alpha_2 \overset{\mathrm{def}}{=} \psi_{Y,\ell}(\alpha_1')$, $\alpha_1' \overset{\mathrm{def}}{=} \psi_{Y,\ell}(\alpha_2') \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X_1^\bullet}}}$. We see immediately that $\alpha_1 \sim \alpha_1'$ if and only if $\alpha_2 \sim \alpha_2'$. Then the theorem follows from Lemma 5.13 and Proposition 5.2. □

Next, let us reconstruct the sets of p-rank from ϕ. Note that the surjection ϕ induces a surjection of the maximal pro-p quotients

$$\phi^p : \Pi^p_{X_1^\bullet} \twoheadrightarrow \Pi^p_{X_2^\bullet}$$

of solvable admissible fundamental groups. Then every Galois (étale) admissible covering $h_2^\bullet : Z_2^\bullet \to X_2^\bullet$ over k_2 with Galois group $\mathbb{Z}/p\mathbb{Z}$ induces a Galois (étale) admissible covering $h_1^\bullet : Z_1^\bullet \to X_1^\bullet$ over k_1 with Galois group $\mathbb{Z}/p\mathbb{Z}$. Moreover, ϕ^p induces an injection

$$\psi_p : N_{X_2^\bullet} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X_2^\bullet}, \mathbb{F}_p) \hookrightarrow N_{X_1^\bullet} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi_{X_1^\bullet}, \mathbb{F}_p).$$

We have the following lemmas.

Lemma 5.15. *We maintain the notation introduced above. Suppose that $\#v^{\mathrm{ra}}_{h_2} = 0$. Then we have that $h_1^\bullet$ is an étale covering, and that $\#v^{\mathrm{ra}}_{h_1} = 0$. In particular, we obtain that*

$$\psi_p^{\mathrm{top}} : N^{\mathrm{top}}_{X_2^\bullet} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\mathrm{top}}_{X_2^\bullet}, \mathbb{F}_p) \xrightarrow{\sim} N^{\mathrm{top}}_{X_1^\bullet} \overset{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\mathrm{top}}_{X_1^\bullet}, \mathbb{F}_p)$$

is an isomorphism.

Proof. Since $h_i^\bullet$ is étale, the Riemann-Hurwitz formula implies that

$$g_{Z_1} = g_{Z_2}.$$

Thus, similar arguments to the arguments given in the proofs of Proposition 5.10 imply that

$$\#v^{\mathrm{ra}}_{h_1} = 0.$$

This completes the proof of the lemma. □

Lemma 5.16. *We maintain the notation introduced above. Suppose that $\#v^{\mathrm{ra}}_{h_2} = 1$. Then we obtain that $h_1^\bullet$ is étale, and that $\#v^{\mathrm{ra}}_{h_1} = 1$.*

Proof. Similar arguments to the arguments given in the proofs of Lemma 5.11 imply that $\#v^{\mathrm{ra}}_{h_1} \leq 1$. If $\#v^{\mathrm{ra}}_{h_1} = 0$, then the "in particular" part of Lemma 5.15 implies that $\#v^{\mathrm{ra}}_{h_2} = 0$. This contradicts our assumption. Then we obtain that $\#v^{\mathrm{ra}}_{h_1} = 1$. □

We reconstruct the sets of p-rank of smooth pointed stable curves associated to vertices from ϕ as follows.

Theorem 5.17. *We maintain the notation introduced above. Then the (surjective) open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces an injection of the set of vertices*

$$\psi_p^{\mathrm{sg,ver}} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}$$

group-theoretically. Moreover, let $v_2 \in v(\Gamma_{X_2^\bullet})^{>0,p}$ and $v_1 \overset{\mathrm{def}}{=} \psi_p^{\mathrm{sg,vex}}(v_2)$. Then we have

$$\sigma_{2,v_2} \leq \sigma_{1,v_1}.$$

Proof. Lemma 5.16 implies that

$$\psi_p(V^{\star}_{X_2,p}) \subseteq V^{\star}_{X_1,p}.$$

Let $\alpha_2, \alpha_2' \in V^{\star}_{X_2,p}$ be elements distinct from each other such that $\alpha_2 \sim \alpha_2'$. It is easy to see that $a\alpha_2 + b\alpha_2' \in V^{\star}_{X_2,p}$ if and only if $a\psi_p(\alpha_2) + b\psi_p(\alpha_2') \in V^{\star}_{X_1,p}$ for each $a, b \in \mathbb{F}_p^{\times}$. Thus, by Proposition 5.1, we obtain an injection of the set of vertices

$$\psi_p^{\mathrm{sg,ver}} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}.$$

Let $v_i \in v(\Gamma_{X_i^\bullet})$. We put

$$L^{v_i,p}_{X_i^\bullet} \stackrel{\mathrm{def}}{=} \{\alpha_i \in N_{X_i^\bullet} \mid v^{\mathrm{ra}}_{h_{i,\alpha_i}} = \{v_i\}\},$$

where $h^\bullet_{i,\alpha_i}$ denotes the Galois (étale) admissible covering corresponding to α_i. Moreover, we denote by

$$[L^{v_i,p}_{X_i^\bullet}]$$

the image of $L^{v_i,p}_{X_i^\bullet}$ in $N_{X_i^\bullet}/N^{\mathrm{top}}_{X_i^\bullet}$, where $N^{\mathrm{top}}_{X_i^\bullet} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\Pi^{\mathrm{top}}_{X_i^\bullet}, \mathbb{F}_p)$. Then we have

$$\#[L^{v_i,p}_{X_i^\bullet}] = p^{\sigma_{i,v_i}} - 1.$$

Suppose that $v_1 \stackrel{\mathrm{def}}{=} \psi_p^{\mathrm{sg,ver}}(v_2)$. Lemma 5.15 implies that ψ_p induces an injection

$$[L^{v_2,p}_{X_2^\bullet}] \hookrightarrow [L^{v_1,p}_{X_1^\bullet}].$$

Thus, we have

$$p^{\sigma_{2,v_2}} - 1 = \#[L^{v_2,p}_{X_2^\bullet}] \leq \#[L^{v_1,p}_{X_1^\bullet}] = p^{\sigma_{1,v_1}} - 1.$$

This means that

$$\sigma_{2,v_2} \leq \sigma_{1,v_1}$$

for each $v_2 \in v(\Gamma_{X_2^\bullet})^{>0,p}$. This completes the proof of the theorem. □

In the remainder of the present subsection, we prove a proposition which will be used in Section 5.5.

Proposition 5.18. *We maintain the notation introduced above. Then the following statements hold:*

(a) Let $S_1^{\mathrm{cl}} \subseteq e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$ be a subset of closed edges, $\alpha_{e_1} \in E^{\mathrm{cl},\star,0}_{\mathfrak{I}_{\Pi_{X_1^\bullet}},e_1}$ for every $e_1 \in S_1^{\mathrm{cl}}$,

$$\alpha_1 \stackrel{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{cl}}} \alpha_{e_1} \in E^{*}_{\mathfrak{I}_{\Pi_{X_1^\bullet}}},$$

and $g^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \to Y^\bullet_1$ the Galois admissible covering over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_1. Let $\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$ be the bijection of the sets of closed *edges obtained in Theorem 5.14,* $\alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_2^\bullet}},\phi^{\mathrm{sg,cl}}(e_1)}$ *the element induced by* ϕ *for every* $e_1 \in S_1^{\mathrm{cl}}$*,*

$$\alpha_2 \stackrel{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{cl}}} \alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X_2^\bullet}}},$$

and $g^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to Y_2^\bullet$ *the Galois admissible covering over* k_2 *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α_2*. Suppose that* $\#v^{\mathrm{sp}}_{g_{1,\alpha_1}} = 0$*. Then we have that*

$$\#e^{\mathrm{op,ra}}_{g_{2,\alpha_2}} = \#v^{\mathrm{sp}}_{g_{2,\alpha_2}} = 0.$$

(b) Let $E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X_i^\bullet}},e_i}$*,* $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$*, be the set of cohomology classes defined in Section 5.2, and let* $S_1^{\mathrm{op}} \subseteq e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ *be a subset of closed edges,* $\alpha_{e_1} \in E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X_1^\bullet}},e_1}$ *for every* $e_1 \in S_1^{\mathrm{op}}$*,*

$$\alpha_1 \stackrel{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{op}}} \alpha_{e_1} \in E^*_{\mathfrak{T}_{\Pi_{X_1^\bullet}}},$$

and $g^\bullet_{1,\alpha_1} : Y^\bullet_{1,\alpha_1} \to Y_1^\bullet$ *the Galois admissible covering over* k_1 *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α_1*. Let* $\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ *be the bijection of the sets of open edges obtained in Theorem 4.11,* $\alpha_{\phi^{\mathrm{sg,op}}(e_1)} \in E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{X_2^\bullet}},\phi^{\mathrm{sg,op}}(e_1)}$ *the element induced by* ϕ *for every* $e_1 \in S_1^{\mathrm{op}}$*,*

$$\alpha_2 \stackrel{\mathrm{def}}{=} \sum_{e_1 \in S_1^{\mathrm{op}}} \alpha_{\phi^{\mathrm{sg,op}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X_2^\bullet}}},$$

and $g^\bullet_{2,\alpha_2} : Y^\bullet_{2,\alpha_2} \to Y_2^\bullet$ *the Galois admissible covering over* k_2 *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *corresponding to* α_2*. Suppose that* $\#v^{\mathrm{sp}}_{g_{1,\alpha_1}} = 0$*. Then we have that*

$$\#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} = \#v^{\mathrm{sp}}_{g_{2,\alpha_2}} = 0.$$

Proof. (a) The Riemann-Hurwitz formula and Theorem 4.11 imply that

$$g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} = -\frac{1}{2}(\#e^{\mathrm{op,ra}}_{g_{2,\alpha_2}})(\ell - 1) = 0.$$

Then we obtain $\#e^{\mathrm{op,ra}}_{g_{2,\alpha_2}} = 0$. On the other hand, we have

$$r_{Y_{1,\alpha_1}} = \ell(\#e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet}) - d\#S_1^{\mathrm{cl}}) + d\#S_1^{\mathrm{cl}} - \#v(\Gamma_{Y_1^\bullet}) + 1$$

and

$$r_{Y_{2,\alpha_2}} = \ell\#e^{\mathrm{cl,\acute{e}t}}_{g_{2,\alpha_2}} + \#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} - \ell\#v^{\mathrm{cl,sp}}_{g_{2,\alpha_2}} - \#v^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} + 1.$$

Then Lemma 5.5 and Lemma 2.3 (b) imply that

$$0 = g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} \geq r_{Y_{1,\alpha_1}} - r_{Y_{2,\alpha_1}}.$$

Thus, we have

$$\#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} + \#v^{\mathrm{sp}}_{g_{2,\alpha_2}} + \frac{1}{2}\#e^{\mathrm{op,ra}}_{g_{2,\alpha_2}} = \#e^{\mathrm{cl,ra}}_{g_{2,\alpha_2}} + \#v^{\mathrm{sp}}_{g_{2,\alpha_2}} \leq d\#S_1^{\mathrm{cl}}.$$

On the othe hand, Lemma 5.13 implies that $\#e^{\mathrm{cl,ra}}_{g_2,\alpha_2} = d\#S^{\mathrm{cl}}_1$. Then we obtain $\#v^{\mathrm{sp}}_{g_2,\alpha_2} = 0$. This completes the proof of (a).

(b) The Riemann-Hurwitz formula and Theorem 4.11 imply that

$$g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} = \frac{1}{2}(d\#S^{\mathrm{op}}_1 - \#e^{\mathrm{op,ra}}_{g_2,\alpha_2})(\ell - 1) = 0.$$

On the other hand, we have

$$r_{Y_{1,\alpha_1}} = \ell\#e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet}) - \#v(\Gamma_{Y_1^\bullet}) + 1$$

and

$$r_{Y_{2,\alpha_2}} = \ell\#e^{\mathrm{cl,ét}}_{g_2,\alpha_2} + \#e^{\mathrm{cl,ra}}_{g_2,\alpha_2} - \ell\#v^{\mathrm{sp}}_{g_2,\alpha_2} - \#v^{\mathrm{ra}}_{g_2,\alpha_2} + 1.$$

Then Lemma 5.5 and Lemma 2.3 (b) imply that

$$g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} \geq r_{Y_{1,\alpha_1}} - r_{Y_{2,\alpha_2}}.$$

Thus, we have

$$\#e^{\mathrm{cl,ra}}_{g_2,\alpha_2} + \#v^{\mathrm{sp}}_{g_2,\alpha_2} + \frac{1}{2}\#e^{\mathrm{op,ra}}_{g_2,\alpha_2} - \frac{d\#S^{\mathrm{op}}_1}{2} \leq 0.$$

This means that

$$\#e^{\mathrm{cl,ra}}_{g_2,\alpha_2} = \#v^{\mathrm{sp}}_{g_2,\alpha_2} = 0.$$

We complete the proof of (b). □

5.4 Reconstruction of commutative diagrams of sets of vertices, sets of open edges, and sets of closed edges from surjections

We maintain the notation introduced in Section 5.3. In the present subsection, we suppose that $X_1^\bullet$ and $X_2^\bullet$ *satisfy Condition A, Condition B, and Condition C.* Moreover, let

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism of the solvable admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$, and

$$(g_X, n_X) \stackrel{\mathrm{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

Note that we have $r_{X_1} = r_{X_2}$, and that by Lemma 4.3, ϕ is a surjection.

We fix some notation. Let H_2 be an open normal subgroup of $\Pi^\bullet_{X_2}$, $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2)$ the open normal subgroup of $\Pi_{X_1^\bullet}$, $G \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$, and ϕ_{H_1} the surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$. Let $i \in \{1, 2\}$. We write

$$f^\bullet_{H_i} : X^\bullet_{H_i} \to X_i^\bullet$$

for the Galois admissible covering over k_i with Galois group G, $(g_{X_{H_i}}, n_{X_{H_i}})$ for the type of $X^\bullet_{H_i}$, and $\Gamma_{X^\bullet_{H_i}}$ for the dual semi-graph of $X^\bullet_{H_i}$. Furthermore, *we suppose that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Let ℓ and d be prime numbers prime to p such that $\ell \neq d$ and $(\#G, \ell) = (\#G, d) = 1$, and let

$$\mathfrak{T}_{\Pi_{X_2^\bullet}} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_{X_2}} : \Pi_{X_2^\bullet}^{\text{ét}} \twoheadrightarrow \mathbb{F}_d)$$

be an edge-triple associated to $\Pi_{X_2^\bullet}$ and $\mathfrak{T}_{X_2^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_{X_2}^\bullet : Y_2^\bullet \to X_2^\bullet)$ the edge-triple associated to $X_2^\bullet$ corresponding to $\mathfrak{T}_{\Pi_{X_2^\bullet}}$. By Corollary 5.7, we obtain an edge-triple

$$\mathfrak{T}_{\Pi_{X_1^\bullet}} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_{X_1}} : \Pi_{X_1^\bullet}^{\text{ét}} \twoheadrightarrow \mathbb{F}_d)$$

induced group-theoretically from ϕ and $\mathfrak{T}_{\Pi_{X_2^\bullet}}$. We write $\mathfrak{T}_{X_1^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_{X_1}^\bullet : Y_1^\bullet \to X_1^\bullet)$ for the edge-triple associated to $X_1^\bullet$ corresponding to $\mathfrak{T}_{\Pi_{X_1^\bullet}}$. On the other hand, we put

$$Q_i \stackrel{\text{def}}{=} \ker(\Pi_{X_i^\bullet} \twoheadrightarrow \Pi_{X_i^\bullet}^{\text{ét}} \stackrel{\alpha_{f_{X_i}}}{\twoheadrightarrow} \mathbb{F}_d).$$

We have that $H_i \twoheadrightarrow H_i/(H_i \cap Q_i) \cong \mathbb{F}_d$ factors through $H_i^{\text{ét}}$. Write $\alpha_{f_{X_{H_i}}} : H_i^{\text{ét}} \twoheadrightarrow \mathbb{F}_d$ for this homomorphism. We see that

$$\mathfrak{T}_{H_i} \stackrel{\text{def}}{=} (\ell, d, \alpha_{f_{X_{H_i}}})$$

is an edge-triple associated to H_i which is induced group-theoretically from $H_i \subseteq \Pi_{X_i^\bullet}$ and $\mathfrak{T}_{\Pi_{X_i^\bullet}}$. Note that $\mathfrak{T}_{H_1}$ coincides with the edge-triple associated to H_1 induced group-theoretically from ϕ_{H_1} and $\mathfrak{T}_{H_2}$. Moreover, we denote by

$$\mathfrak{T}_{X_{H_i}^\bullet} \stackrel{\text{def}}{=} (\ell, d, f_{X_{H_i}}^\bullet : Y_{X_{H_i}}^\bullet \stackrel{\text{def}}{=} Y_i^\bullet \times_{X_i^\bullet} X_{H_i}^\bullet \to X_{H_i}^\bullet)$$

the edge-triple associated to $X_{H_i}^\bullet$ corresponding to $\mathfrak{T}_{H_i}$.

By applying Proposition 5.1, Remark 5.1.1, Proposition 5.2, and Remark 5.2.1, we have that the natural inclusion $H_i \hookrightarrow \Pi_{X_i^\bullet}$ induces the maps

$$\gamma_{H_i}^{\text{ver},\ell} : V_{X_{H_i},\ell} \to V_{X_i,\ell},$$

$$\gamma_{\mathfrak{T}_{\Pi_{X_i^\bullet}},H_i}^{\text{cl}} : E_{\mathfrak{T}_{H_i}}^{\text{cl}} \to E_{\mathfrak{T}_{\Pi_{X_i}}}^{\text{cl}}$$

group-theoretically. We put

$$\gamma_{H_i}^{\text{ver}} : v(\Gamma_{X_{H_i}^\bullet}) \stackrel{\kappa_{X_{H_i},\ell}^{-1}}{\stackrel{\sim}{\to}} V_{X_{H_i},\ell} \stackrel{\gamma_{H_i}^{\text{ver},\ell}}{\to} V_{X_i,\ell} \stackrel{\kappa_{X_i,\ell}}{\stackrel{\sim}{\to}} v(\Gamma_{X_i^\bullet}),$$

$$\gamma_{H_i}^{\text{cl}} : e^{\text{cl}}(\Gamma_{X_{H_i}^\bullet}) \stackrel{\vartheta_{\mathfrak{T}_{H_i}}^{-1}}{\stackrel{\sim}{\to}} E_{\mathfrak{T}_{H_i}}^{\text{cl}} \stackrel{\gamma_{\mathfrak{T}_{\Pi_{X_i^\bullet}},H_i}^{\text{cl}}}{\to} E_{\mathfrak{T}_{\Pi_{X_i^\bullet}}}^{\text{cl}} \stackrel{\vartheta_{\mathfrak{T}_{\Pi_{X_i^\bullet}}}}{\stackrel{\sim}{\to}} e^{\text{cl}}(\Gamma_{X_i^\bullet}).$$

Then the maps $\gamma_{H_i}^{\text{ver}}$ and $\gamma_{H_i}^{\text{cl}}$ can be reconstructed group-theoretically from the inclusion $H_i \hookrightarrow \Pi_{X_i^\bullet}$.

On the other hand, Theorem 4.2 implies that the sets $\mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$ and $\mathrm{Edg}^{\mathrm{op}}(H_i)$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$ and H_i, respectively. Note that we have a natural map

$$\mathrm{Edg}^{\mathrm{op}}(H_i) \to \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})$$

induced by the natural inclusions of the stabilizer subgroups. Moreover, this map compatible with the actions of H_i and $\Pi_{X_i^\bullet}$. Then we obtain a map

$$\gamma_{H_i}^{\mathrm{op}} : e^{\mathrm{op}}(\Gamma_{X_{H_i}^\bullet}) \xrightarrow{\sim} \mathrm{Edg}^{\mathrm{op}}(H_i)/H_i \to \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet})/\Pi_{X_i^\bullet} \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$$

which can be induced by the inclusion $H_i \hookrightarrow \Pi_{X_i^\bullet}$ group-theoretically.

By Theorem 4.11, Theorem 5.12, and Theorem 5.14, the following maps

$$\phi_{H_1}^{\mathrm{sg,ver}} : v(\Gamma_{X_{H_1}^\bullet}) \xrightarrow{\sim} v(\Gamma_{X_{H_2}^\bullet}),$$

$$\phi_{H_1}^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_{H_1}^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_{H_2}^\bullet}),$$

$$\phi_{H_1}^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_{H_1}^\bullet}) \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X_{H_2}^\bullet}),$$

$$\phi^{\mathrm{sg,ver}} : v(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} v(\Gamma_{X_2^\bullet}),$$

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet}),$$

$$\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$$

can be reconstructed group-theoretically from $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ and $\phi_{H_1} : H_1 \twoheadrightarrow H_2$, respectively.

Proposition 5.19. *We maintain the notation introduced above. Then the following diagrams*

$$\begin{array}{ccc} v(\Gamma_{X_{H_1}^\bullet}) & \xrightarrow{\phi_{H_1}^{\mathrm{sg,ver}}} & v(\Gamma_{X_{H_2}^\bullet}) \\ \gamma_{H_1}^{\mathrm{ver}} \downarrow & & \gamma_{H_2}^{\mathrm{ver}} \downarrow \\ v(\Gamma_{X_1^\bullet}) & \xrightarrow{\phi^{\mathrm{sg,ver}}} & v(\Gamma_{X_2^\bullet}), \end{array}$$

$$\begin{array}{ccc} e^{\mathrm{op}}(\Gamma_{X_{H_1}^\bullet}) & \xrightarrow{\phi_{H_1}^{\mathrm{sg,op}}} & e^{\mathrm{op}}(\Gamma_{X_{H_2}^\bullet}) \\ \gamma_{H_1}^{\mathrm{op}} \downarrow & & \gamma_{H_2}^{\mathrm{op}} \downarrow \\ e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) & \xrightarrow{\phi^{\mathrm{sg,op}}} & e^{\mathrm{op}}(\Gamma_{X_2^\bullet}), \end{array}$$

$$\begin{array}{ccc} e^{\mathrm{cl}}(\Gamma_{X_{H_1}^\bullet}) & \xrightarrow{\phi_{H_1}^{\mathrm{sg,cl}}} & e^{\mathrm{cl}}(\Gamma_{X_{H_2}^\bullet}) \\ \gamma_{H_1}^{\mathrm{cl}} \downarrow & & \gamma_{H_2}^{\mathrm{cl}} \downarrow \\ e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) & \xrightarrow{\phi^{\mathrm{sg,cl}}} & e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) \end{array}$$

are commutative. Moreover, all the commutative diagrams above are compatible with the natural actions of G.

Proof. The commutativity of the second diagram follows immediately from Theorem 4.11. We treat the third diagram. To verify the commutativity of the third diagram, we only need to prove the commutativity of the following diagram

$$\begin{array}{ccc} e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}}) & \xrightarrow{(\phi^{\mathrm{sg,cl}}_{H_1})^{-1}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_1}}) \\ \gamma^{\mathrm{cl}}_{H_2}\downarrow & & \gamma^{\mathrm{cl}}_{H_1}\downarrow \\ e^{\mathrm{cl}}(\Gamma_{X^\bullet_2}) & \xrightarrow{(\phi^{\mathrm{sg,cl}})^{-1}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet_1}). \end{array}$$

Let $e_{H_2} \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}})$, $e_{H_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,cl}}_{H_1})^{-1}(e_{H_2}) \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_1}})$, $e_2 \overset{\mathrm{def}}{=} \gamma^{\mathrm{cl}}_{H_2}(e_{H_2}) \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_2})$, $e_1 \overset{\mathrm{def}}{=} (\gamma^{\mathrm{cl}}_{H_1} \circ (\phi^{\mathrm{sg,cl}}_{H_1})^{-1})(e_{H_2}) \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_1})$, $e'_1 \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,cl}})^{-1}(e_2) \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_1})$. We will prove that $e_1 = e'_1$.

Write S_{H_1} and S_{H_2} for the sets $(\gamma^{\mathrm{cl}}_{H_1})^{-1}(e'_1)$ and $(\gamma^{\mathrm{cl}}_{H_2})^{-1}(e_2)$, respectively. Note that $e_{H_2} \in S_{H_2}$. To verify $e_1 = e'_1$, it is sufficient to prove that $e_{H_1} \in S_{H_1}$.

Let $\alpha_2 \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet_2}}, e_2}$. Then the proof of Lemma 5.13 implies that α_2 induces an element

$$\alpha_1 \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{\Pi_{X^\bullet_1}}, e'_1}.$$

Write $Y^\bullet_{\alpha_i}$ for the pointed stable curve over k_i corresponding to α_i. We consider the Galois admissible covering

$$Y^\bullet_{\alpha_2} \times_{X^\bullet_2} X^\bullet_{H_2} \to Y^\bullet_{X_{H_2}}$$

over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, and denote by β_2 the element of $E^*_{\mathfrak{T}_{H_2}}$ corresponding to this Galois admissible covering. Then we have

$$\beta_2 = \sum_{c_2 \in S_{H_2}} t_{c_2}\beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{H_2}, c_2}$. Note that we have $t_{e_{H_2}} \neq 0$. On the other hand, the proof of Lemma 5.13 implies that β_{c_2} induces an element $\beta_{(\phi^{\mathrm{cl}}_{H_1})^{-1}(c_2)} \in E^{\mathrm{cl},\star}_{\mathfrak{T}_{H_1}, (\phi^{\mathrm{cl}}_{H_1})^{-1}(c_2)}$. Then β_2 induces an element

$$\beta_1 \overset{\mathrm{def}}{=} \sum_{c_2 \in S_{X_{H_2}} \setminus \{e_{H_2}\}} t_{c_2}\beta_{(\phi^{\mathrm{cl}}_{H_1})^{-1}(c_2)} + t_{e_{H_2}}\beta_{e_{H_1}} \in E^*_{\mathfrak{T}_{H_1}}.$$

Note that since β_1 corresponds to the Galois admissible covering

$$Y^\bullet_{\alpha_1} \times_{X^\bullet_1} X^\bullet_{H_1} \to Y^\bullet_{X_{H_1}}$$

over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, the composition of the Galois admissible coverings $Y^\bullet_{\alpha_1} \times_{X^\bullet_1} X^\bullet_{H_1} \to Y^\bullet_{X_{H_1}} \overset{f^\bullet_{X_{H_1}}}{\to} X^\bullet_{H_1}$ is ramified over S_{H_1}. This means that e_{H_1} is contained in S_{H_1}.

Similar arguments to the arguments given in the proof above imply the first diagram is commutative. It is easy to check the "moreover" part of the lemma. This completes the proof of the proposition. □

5.5 Combinatorial Grothendieck conjecture for surjections

We maintain the notation introduced in Section 5.3. In the present subsection, we suppose that $X_1^\bullet$ *and* $X_2^\bullet$ *satisfy Condition A, Condition B, and Condition C* unless indicated otherwise. Then we have $r_{X_1} = r_{X_2}$. We put $(g_X, n_X) \overset{\text{def}}{=} (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Let

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

be an arbitrary open continuous homomorphism of the solvable admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$. By Lemma 4.3, we have that ϕ is a surjection.

We fix some notation. Let H_2 be an open normal subgroup of $\Pi^\bullet_{X_2}$, $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2)$ the open normal subgroup of $\Pi_{X_1^\bullet}$, $G \overset{\text{def}}{=} \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$, and $\phi_{H_1} \overset{\text{def}}{=} \phi|_{H_1} : H_1 \twoheadrightarrow H_2$ the surjection induced by ϕ. Let $i \in \{1, 2\}$. We write

$$f^\bullet_{H_i} : X^\bullet_{H_i} \to X_i^\bullet$$

for the Galois admissible covering over k_i with Galois group G, $(g_{X_{H_i}}, n_{X_{H_i}})$ for the type of $X^\bullet_{H_i}$, $\Gamma_{X^\bullet_{H_i}}$ for the dual semi-graph of $X^\bullet_{H_i}$, and $r_{X_{H_i}}$ for the Betti number of $\Gamma_{X^\bullet_{H_i}}$.

Lemma 5.20. *We maintain the notation introduced above. Then $X^\bullet_{H_i}$ satisfies Condition A, Condition B, Condition C (i).*

Proof. The first condition, the second condition, and the fourth condition of Condition A follow immediately from the definition of admissible coverings. Since $X_i^\bullet$ satisfies Condition B and the third condition of Condition A, $X^\bullet_{H_i}$ also satisfies Condition B and the third condition of Condition A. Moreover, Condition C (i) follows immediately from Theorem 4.11. This completes the proof of the lemma. □

Lemma 5.21. *We maintain the notation introduced above. Suppose that there exists an open normal subgroup $H_2' \subseteq H_2$ such that $X^\bullet_{H_1'}$ and $X^\bullet_{H_2'}$ satisfy Condition A, Condition B, and Condition C, where $H_1' \overset{\text{def}}{=} \phi^{-1}(H_1') \subseteq H_1$. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii).

Let $G' \overset{\text{def}}{=} \Pi_{X_1^\bullet}/H_1' = \Pi_{X_2^\bullet}/H_2'$ and $G'' \overset{\text{def}}{=} H_1/H_1' = H_2/H_2' \subseteq G'$. By applying Proposition 5.19, the following commutative diagrams

$$\begin{array}{ccc} v(\Gamma_{X^\bullet_{H_1'}}) & \xrightarrow{\phi^{\text{sg,ver}}_{H_1'}} & v(\Gamma_{X^\bullet_{H_2'}}) \\ \gamma^{\text{ver}}_{H_1'}\downarrow & & \gamma^{\text{ver}}_{H_2'}\downarrow \\ v(\Gamma_{X_1^\bullet}) & \xrightarrow{\phi^{\text{sg,ver}}} & v(\Gamma_{X_2^\bullet}), \end{array}$$

$$\begin{array}{ccc} e^{\text{op}}(\Gamma_{X^\bullet_{H_1'}}) & \xrightarrow{\phi^{\text{sg,op}}_{H_1'}} & e^{\text{op}}(\Gamma_{X^\bullet_{H_2'}}) \\ \gamma^{\text{op}}_{H_1'}\downarrow & & \gamma^{\text{op}}_{H_2'}\downarrow \\ e^{\text{op}}(\Gamma_{X_1^\bullet}) & \xrightarrow{\phi^{\text{sg,op}}} & e^{\text{op}}(\Gamma_{X_2^\bullet}), \end{array}$$

$$\begin{array}{ccc} e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_1}}) & \xrightarrow{\phi^{\mathrm{sg,cl}}_{H'_1}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_2}}) \\ \gamma^{\mathrm{cl}}_{H'_1}\Big\downarrow & & \gamma^{\mathrm{cl}}_{H'_2}\Big\downarrow \\ e^{\mathrm{cl}}(\Gamma_{X^\bullet_1}) & \xrightarrow{\phi^{\mathrm{sg,cl}}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet_2}) \end{array}$$

can be reconstructed group-theoretically from $H'_i \hookrightarrow \Pi_{X^\bullet_i}$, ϕ, and $\phi_{H'_1} \overset{\mathrm{def}}{=} \phi|_{H'_1}$. Moreover, the commutative diagrams are compatible with the actions of G'' and G'. Then we obtain that

$$\#v(\Gamma_{X^\bullet_{H_1}}) = \#(v(\Gamma_{X^\bullet_{H'_1}})/G'') = \#(v(\Gamma_{X^\bullet_{H'_2}})/G'') = \#v(\Gamma_{X^\bullet_{H_2}}),$$
$$\#e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_1}}) = \#(e^{\mathrm{op}}(\Gamma_{X^\bullet_{H'_1}})/G'') = \#(e^{\mathrm{op}}(\Gamma_{X^\bullet_{H'_2}})/G'') = \#e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_2}}),$$
$$\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_1}}) = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_1}})/G'') = \#(e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H'_2}})/G'') = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}}).$$

This means that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C. □

Lemma 5.22. *We maintain the notation introduced above. Suppose that $(\#G, p) = 1$, and that f_{H_2} is étale. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii). Moreover, since G is a finite solvable group, to verify the lemma, it is sufficient to prove the lemme when $G \cong \mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a prime number distinct from p. Thus, Proposition 5.9 implies that f_{H_1} is also étale.

We denote by $H'_2 \subseteq H_2$ the inverse image $D_\ell(\Pi^{\text{ét}}_{X^\bullet_2})$ of the natural surjection $\Pi_{X^\bullet_2} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet_2}$. Then H'_2 is an open normal subgroup of $\Pi_{X^\bullet_2}$. Let $H'_1 \overset{\mathrm{def}}{=} \phi^{-1}(H'_2) \subseteq H_1$. We see that H'_1 is equal to the inverse image $D_\ell(\Pi^{\text{ét}}_{X^\bullet_1})$ of the natural surjection $\Pi_{X^\bullet_1} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet_1}$. Since $X^\bullet_1$ and $X^\bullet_2$ satisfy Condition C, Theorem 5.12 and the structures of the maximal prime-to-p quotients of solvable admissible fundamental groups imply that $X^\bullet_{H'_1}$ and $X^\bullet_{H'_2}$ also satisfy Condition C. Then the lemma follows from Lemma 5.21. □

Lemma 5.23. *We maintain the notation introduced above. Suppose that $(\#G, p) = 1$. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii).

Since G is a finite solvable group, to verify the lemma, it is sufficient to prove the lemme when $G \cong \mathbb{Z}/\ell\mathbb{Z}$, where ℓ is a prime number distinct from p.

Let $\mathfrak{T}_{\Pi_{X^\bullet_2}} = (\ell, d, \alpha_{f_{X_2}} : \Pi^{\text{ét}}_{X^\bullet_2} \twoheadrightarrow \mathbb{F}_d)$ be an edge-triple associated to $\Pi_{X^\bullet_2}$, $\mathfrak{T}_{\Pi_{X^\bullet_1}} = (\ell, d, \alpha_{f_{X_1}} : \Pi^{\text{ét}}_{X^\bullet_1} \twoheadrightarrow \mathbb{F}_d)$ the edge-triple associated to $\Pi_{X^\bullet_1}$ induced by ϕ, and $\mathfrak{T}_{X^\bullet_i} = (\ell, d, f^\bullet_{X_i} : Y^\bullet_i \to X^\bullet_i)$ the edge-triple associated to $X^\bullet_i$ corresponding to $\mathfrak{T}_{\Pi_{X^\bullet_i}}$.

First, we suppose that f_{H_2} is étale over D_{X_2}. Then Theorem 4.11 implies that f_{H_1} is also étale over D_{X_1}. Let $\alpha_{e_1} \in E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{X^\bullet_1}}, e_1}$, $e_1 \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_1})$,

$$\alpha_1 \overset{\mathrm{def}}{=} \sum_{e_1 \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_1})} \alpha_{e_1} \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet_1}}},$$

and $g^\bullet_{1,\alpha_1}: Y^\bullet_{1,\alpha_1} \to Y^\bullet_1$ the Galois admissible covering over k_1 corresponding to α_1. Note that we have that $\#e^{\mathrm{op,ra}}_{g_{1,\alpha_1}} = \#v^{\mathrm{sp}}_{g_{1,\alpha_1}} = 0$. Let $\phi^{\mathrm{sg,cl}}: e^{\mathrm{cl}}(\Gamma_{X^\bullet_1}) \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X^\bullet_2})$ be the bijection of the sets of closed edges obtained in Theorem 5.14, $\alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^{\mathrm{cl},\star,0}_{\mathfrak{T}_{\Pi_{X^\bullet_2}},\phi^{\mathrm{sg,cl}}(e_1)}$ the element induced by ϕ for every $e_1 \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_1})$,

$$\alpha_2 \stackrel{\mathrm{def}}{=} \sum_{e_1 \in e^{\mathrm{cl}}(\Gamma_{X^\bullet_1})} \alpha_{\phi^{\mathrm{sg,cl}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet_2}}},$$

and $g^\bullet_{2,\alpha_2}: Y^\bullet_{2,\alpha_2} \to Y^\bullet_2$ the Galois admissible covering over k_2 corresponding to α_2. Then Proposition 5.18 (a) implies that

$$\#e^{\mathrm{op,ra}}_{g_{2,\alpha_2}} = \#v^{\mathrm{sp}}_{g_{2,\alpha_2}} = 0.$$

We obtain that g_{i,α_i} is totally ramified over every node of Y_i, and that $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$ satisfy Condition A, Condition B, and Condition C. Write $N_i \subseteq \Pi_{X^\bullet_i}$ for the open normal subgroup corresponding to $Y^\bullet_{i,\alpha_i}$.

Let $H'_i \stackrel{\mathrm{def}}{=} H_i \cap N_i$ and $X^\bullet_{H'_i}$ the pointed stable curve over k_i corresponding to H'_i. Note that $X^\bullet_{H'_i}$ is isomorphic naturally to a connected component of

$$X^\bullet_{H_i} \times_{X^\bullet_i} Y^\bullet_{g_{i,\alpha_i}}.$$

We denote by $h^\bullet_i: X^\bullet_{H'_i} \to Y^\bullet_{g_{i,\alpha_i}}$ the Galois admissible covering over k_i corresponding to the injection $H'_i \hookrightarrow N_i$. By applying Abhyankar's lemma, f_{H_i} is étale over D_{X_i} implies that h_i is étale. Then the lemma follows from Lemma 5.21 and Lemma 5.22. This completes the proof of the lemme when f_{H_2} is étale over D_{X_2}.

Next, let us prove the lemme in the general case. We take $\beta_{e_1} \in E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X^\bullet_1}},e_1}$ for every $e_1 \in e^{\mathrm{op}}(\Gamma_{X^\bullet_1})$ such that $\#v^{\mathrm{sp}}_{g_{1,\beta_1}} = 0$, where

$$\beta_1 \stackrel{\mathrm{def}}{=} \sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X^\bullet_1})} \beta_{e_1} \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet_1}}},$$

and $g^\bullet_{1,\beta_1}: Y^\bullet_{1,\beta_1} \to Y^\bullet_1$ is the Galois admissible covering over k_1 corresponding to β_1. Note that we have that $\#e^{\mathrm{cl,ra}}_{g_{1,\beta_1}} = \#v^{\mathrm{sp}}_{g_{1,\beta_1}} = 0$. Let $\phi^{\mathrm{sg,op}}: e^{\mathrm{op}}(\Gamma_{X^\bullet_1}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X^\bullet_2})$ be the bijection of the sets of open edges obtained in Theorem 4.11, $\beta_{\phi^{\mathrm{sg,op}}(e_1)} \in E^{\mathrm{op},\star,0}_{\mathfrak{T}_{\Pi_{X^\bullet_2}},\phi^{\mathrm{sg,op}}(e_1)}$ the element induced by ϕ for every $e_1 \in e^{\mathrm{op}}(\Gamma_{X^\bullet_1})$,

$$\beta_2 \stackrel{\mathrm{def}}{=} \sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X^\bullet_1})} \beta_{\phi^{\mathrm{sg,op}}(e_1)} \in E^*_{\mathfrak{T}_{\Pi_{X^\bullet_2}}},$$

and $g^\bullet_{2,\beta_2}: Y^\bullet_{2,\beta_2} \to Y^\bullet_2$ the Galois admissible covering over k_2 corresponding to β_2. Then Proposition 5.18 (b) implies that

$$\#e^{\mathrm{cl,ra}}_{g_{2,\beta_2}} = \#v^{\mathrm{sp}}_{g_{2,\beta_2}} = 0.$$

We obtain that g_{i,β_i} is totally ramified over every marked point of Y_i, and that $Y^\bullet_{1,\beta_1}$ and $Y^\bullet_{2,\beta_2}$ satisfy Condition A, Condition B, and Condition C. Write $Q_i \subseteq \Pi_{X_i^\bullet}$ for the open normal subgroup corresponding to $Y^\bullet_{i,\beta_i}$.

Let $H_i'' \stackrel{\text{def}}{=} H_i \cap Q_i$ and $X^\bullet_{H_i''}$ the pointed stable curve over k_i corresponding to H_i''. Note that $X^\bullet_{H_i''}$ is isomorphic naturally to a connected component of

$$X^\bullet_{H_i} \times_{X_i^\bullet} Y^\bullet_{g_{i,\beta_i}}.$$

We denote by $h_i^{*,\bullet} : X^\bullet_{H_i''} \to Y^\bullet_{g_{i,\beta_i}}$ the Galois admissible covering over k_i corresponding to the injection $H_i'' \hookrightarrow Q_i$. By applying Abhyankar's lemma, h_i^* is étale over $D_{Y_{g_{i,\beta_i}}}$. By applying the lemma in the case where g_{i,β_i} is étale over D_{Y_i}, we obtain that $X^\bullet_{H_1''}$ and $X^\bullet_{H_2''}$ satisfy Condition A, Condition B, and Condition C. Then the lemma follows from Lemma 5.21. We complete the proof of the lemma. □

Lemma 5.24. *We maintain the notation introduced above. Suppose that G is a p-group. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. By Lemma 5.20, to verify the lemma, we only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (ii) and Condition C (iii).

To verify the lemma, without loss the generality, it is sufficient to treat the case where $G \cong \mathbb{Z}/p\mathbb{Z}$. Since $f^\bullet_{H_i}$ is étale, $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition C (iii).

Let $V_i \subseteq v(\Gamma_{X_i^\bullet})^{>0,p}$ be the subset of vertices such that the natural homomorphism

$$\Pi_{\widetilde{X}^\bullet_{i,v_i}} \hookrightarrow \Pi_{X_i^\bullet} \twoheadrightarrow G \stackrel{\text{def}}{=} \Pi_{X_i^\bullet}/H_i$$

is non-trivial (since $G \cong \mathbb{Z}/p\mathbb{Z}$, the homomorphism is a surjection). Then we obtain $\#v(\Gamma_{X^\bullet_{H_i}}) = p(\#v(\Gamma_{X_i^\bullet}) - \#V_i) + \#V_i$ and $\#e^{\text{cl}}(\Gamma_{X^\bullet_{H_i}}) = p\#e^{\text{cl}}(\Gamma_{X_i^\bullet})$.

Let

$$\psi_p^{\text{sg,ver}} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}$$

be the injection induced by ϕ, which is obtained in Theorem 5.17. We put

$$V_1' \stackrel{\text{def}}{=} \{\psi_p^{\text{sg,ver}}(v_2)\}_{v_2 \in V_2} \subseteq v(\Gamma_{X_1^\bullet})^{>0,p}.$$

By applying Lemma 5.16, we see that

$$V_1 = V_1'.$$

Thus, we have $\#v(\Gamma_{X^\bullet_{H_1}}) = \#v(\Gamma_{X^\bullet_{H_2}})$ and $\#e^{\text{cl}}(\Gamma_{X^\bullet_{H_1}}) = \#e^{\text{cl}}(\Gamma_{X^\bullet_{H_2}})$. This completes the proof of the lemma. □

Proposition 5.25. *We maintain the notation introduced above. Then $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A, Condition B, and Condition C.*

Proof. Since G is a solvable group, the proposition follows from Lemma 5.23 and Lemma 5.24. □

Next, we prove the main result of the present section which is called combinatorial Grothendieck conjecture for surjections.

Theorem 5.26. *We maintain the notation introduced above. Then the surjective open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces surjective maps*

$$\phi^{\mathrm{ver}} : \mathrm{Ver}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Ver}(\Pi_{X_2^\bullet}),$$

$$\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet}),$$

$$\phi^{\mathrm{edg,cl}} : \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})$$

group-theoretically. Moreover, ϕ induces a bijection

$$\phi^{\mathrm{sg}} : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$$

of the dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$ group-theoretically.

Proof. By applying Theorem 4.11, the homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a surjective map $\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$ group-theoretically. We only need to treat the cases of ϕ^{ver} and $\phi^{\mathrm{edg,cl}}$, respectively.

Let $\mathcal{C}_{\Pi_{X_2^\bullet}}$ be a cofinal system of $\Pi_{X_2^\bullet}$ which consists of open normal subgroups of $\Pi_{X_2^\bullet}$. We put

$$\mathcal{C}_{\Pi_{X_1^\bullet}} \overset{\mathrm{def}}{=} \{H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \mid H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}\}.$$

Note that $\mathcal{C}_{\Pi_{X_1^\bullet}}$ is not a cofinal system of $\Pi_{X_1^\bullet}$ in general. Moreover, by applying Proposition 5.25, we have that $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ satisfy Condition A, Condition B, and Condition C for every $H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}$ and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \in \mathcal{C}_{\Pi_{X_1^\bullet}}$.

We treat the case of ϕ^{ver}. Let $\widehat{X}_i^\bullet$ be the universal solvable admissible covering of $X_i^\bullet$ associated to $\Pi_{X_i^\bullet}$ and $\Gamma_{\widehat{X}_i^\bullet}$ the dual semi-graph of $\widehat{X}_i^\bullet$. Let $\widehat{w}_1 \in v(\Gamma_{\widehat{X}_1^\bullet})$ and $\Pi_{\widehat{w}_1}$ the stabilizer subgroup of $\widehat{w}_1$. Write $w_{H_1} \in v(\Gamma_{X_{H_1}^\bullet})$, $H_1 \in \mathcal{C}_{\Pi_{X_1^\bullet}}$, for the image of $\widehat{w}_1$. Proposition 5.19 implies that ϕ induces a cofinal system of vertices

$$\mathcal{C}_{\widehat{w}_2} \overset{\mathrm{def}}{=} \{w_{H_2} \overset{\mathrm{def}}{=} \phi_{H_1}^{\mathrm{ver}}(w_{H_1})\}_{H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}},$$

which admits a natural action of $\Pi_{X_2^\bullet}$. Then we obtain an element $\widehat{w}_2 \in v(\Gamma_{\widehat{X}_2^\bullet})$. Moreover, the stabilizer of $\mathcal{C}_{\widehat{w}_2}$ is $\Pi_{\widehat{w}_2}$. We see immediately that ϕ induces a surjective open continuous homomorphism

$$\phi|_{\Pi_{\widehat{w}_1}} : \Pi_{\widehat{w}_1} \twoheadrightarrow \Pi_{\widehat{w}_2}$$

group-theoretically. Then we define

$$\phi^{\mathrm{ver}} : \mathrm{Ver}(\Pi_{X_1^\bullet}) \to \mathrm{Ver}(\Pi_{X_2^\bullet}),\ \Pi_{\widehat{w}_1} \mapsto \Pi_{\widehat{w}_2}.$$

Next, we prove that ϕ^{ver} is a surjective. Let $\widehat{v}_2 \in v(\Gamma_{\widehat{X}_2^\bullet})$ and $\Pi_{\widehat{v}_2}$ the stabilizer subgroup of $\widehat{v}_2$. Write $v_{H_2} \in v(\Gamma_{X_{H_2}^\bullet})$, $H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}$, for the image of $\widehat{v}_2$. Then we obtain a cofinal system of vertices

$$\mathcal{C}_{\widehat{v}_2} \overset{\mathrm{def}}{=} \{v_{H_2}\}_{H_2 \in \mathcal{C}_{\Pi_{X_2^\bullet}}}$$

associated to $\widehat{v}_2$. The cofinal system $\mathcal{C}_{\widehat{v}_2}$ admits a natural action of $\Pi_{X_2^\bullet}$. We see immediately that the stabilizer of $\mathcal{C}_{\widehat{v}_2}$ is equal to $\Pi_{\widehat{v}_2}$. Proposition 5.19 implies that ϕ and $\mathcal{C}_{\widehat{v}_2}$ induce a set of

$$\mathcal{C}' \stackrel{\text{def}}{=} \{v_{H_1} \stackrel{\text{def}}{=} (\phi^{\text{sg,vex}})^{-1}(v_{H_2})\}_{H_1 \in \mathcal{C}_{\Pi_{X_2^\bullet}}}$$

group-theoretically. By extending $\mathcal{C}'$ to a cofinal system of vertices. Then we obtain an element $\widehat{v}_1 \in v(\Gamma_{\widehat{X}_1^\bullet})$ such that the image of $\widehat{v}_1$ in $v(\Gamma_{X_{H_1}})$ is v_{H_1}. Thus, ϕ induces a surjective

$$\phi|_{\Pi_{\widehat{v}_1}} : \Pi_{\widehat{v}_1} \twoheadrightarrow \Pi_{\widehat{v}_2}.$$

This means that ϕ^{ver} is a surjection.

By applying similar arguments to the arguments given in the proof above, we obtain that ϕ induces a surjective map $\phi^{\text{edg,cl}} : \text{Edg}^{\text{cl}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \text{Edg}^{\text{cl}}(\Pi_{X_2^\bullet})$ group-theoretically. This completes the first part of the theorem.

The surjections ϕ^{ver}, $\phi^{\text{edg,op}}$, and $\phi^{\text{edg,cl}}$ imply the following surjections

$$\phi^{\text{sg,ver}} : v(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} \text{Ver}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \text{Ver}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \xrightarrow{\sim} v(\Gamma_{X_2^\bullet}),$$

$$\phi^{\text{sg,op}} : e^{\text{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} \text{Edg}^{\text{op}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \text{Edg}^{\text{op}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X_2^\bullet}),$$

$$\phi^{\text{sg,cl}} : e^{\text{cl}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} \text{Edg}^{\text{cl}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \twoheadrightarrow \text{Edg}^{\text{cl}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet} \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X_2^\bullet}).$$

Since $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition C, we have that $\phi^{\text{sg,ver}}$, $\phi^{\text{sg,op}}$, and $\phi^{\text{sg,cl}}$ are bijections. Furthermore, by applying [7, Lemma 1.5, Lemma 1.7, and Lemma 1.9], ϕ induces a bijection

$$\phi : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}.$$

This completes the proof of the theorem. □

Remark 5.26.1. We maintain the notation introduced above. We see immediately that Theorem 5.26 does not hold without Condition C (e.g. $X_1^\bullet$ is a generic curve of $\overline{\mathcal{M}}_{g,n}$, and $X_2^\bullet$ is a singular curve).

On the other hand, the author believes that Theorem 5.26 also holds without Condition B (e.g. $n_{X_i} = 0$). The main difficult is that we do not have a precise formula for limits of p-averages of arbitrary pointed stable curves. Moreover, if the question of [37, Remark 4.10.2] is true, without too much difficulty, similar arguments to the arguments given in the proofs of this section imply that Theorem 5.26 holds without Condition B.

Furthermore, the author also believes that Theorem 5.26 holds without Condition A and Condition B. For example, Theorem 5.30 below shows that Theorem 5.26 holds without Condition A and Condition B if $g_X = 0$. Moreover, Theorem 5.30 will play a key role in the proof of the main theorem of the present paper.

Corollary 5.27. *We maintain the notation introduced above. Let $Q_2 \subseteq \Pi_{X_2^\bullet}$ be an arbitrary open subgroup and $Q_1 \stackrel{\text{def}}{=} \phi^{-1}(Q_2) \subseteq \Pi_{X_1^\bullet}$. Then we have*

$$\text{Avr}_p(Q_1) = \text{Avr}_p(Q_2).$$

Proof. The corollary follows immediately from Theorem 5.26. □

Lemma 5.28. *Let $E^\bullet$ be a pointed stable curve of type $(0, n)$ over an algebraically closed field k of characteristic $p > 0$, $\Pi_{E^\bullet}$ the solvable admissible fundamental group of $E^\bullet$, and ℓ a prime number such that $\ell \neq p$, and that $\ell >> n$. We put*

$$\mathrm{Edg}^{\mathrm{op},\ell,\mathrm{ab}}(\Pi_{E^\bullet}) \stackrel{\mathrm{def}}{=} \{pr^{\ell,\mathrm{ab}}(I_{\widehat{e}}) \mid I_{\widehat{e}} \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{E^\bullet})\} = \{I_e\}_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})},$$

where $pr^{\ell,\mathrm{ab}}$ denotes the natural surjective homomorphism $\Pi_{E^\bullet} \twoheadrightarrow \Pi_{E^\bullet}^{\ell,\mathrm{ab}}$, and $I_e \stackrel{\mathrm{def}}{=} pr^{\ell,\mathrm{ab}}(I_{\widehat{e}})$. Let $a_e \in I_e$, $e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})$, be a generator of I_e such that

$$\prod_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})} a_e = 1,$$

and let $\alpha : \Pi_{E^\bullet}^{\ell,\mathrm{ab}} \twoheadrightarrow \mathbb{F}_\ell$ be a surjection and $r_e \stackrel{\mathrm{def}}{=} \alpha(a_e)$. Write

$$g^\bullet : X^\bullet \to E^\bullet$$

for the Galois admissible covering over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α. Suppose that $r_e \neq 0$ for every $e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})$, and that

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_{E^\bullet})} r_e = \ell.$$

Then $g^\bullet$ is totally ramified over every node and every marked point of $E^\bullet$. In particular, we have that the map of dual semi-graphs $\Gamma_{X^\bullet} \to \Gamma_{E^\bullet}$ of $X^\bullet$ and $E^\bullet$ induced by $g^\bullet$ is a bijection, and that $X^\bullet$ satisfies Condition A.

Proof. We prove the lemma by induction on $\#v(\Gamma_{E^\bullet})$. Suppose that $\#v(\Gamma_{E^\bullet}) = 1$. Then the lemma is trivial.

Suppose that $\#v(\Gamma_{E^\bullet}) \geq 2$. Let $v_0 \in v(\Gamma_{E^\bullet})$ and $\widetilde{E}^\bullet_{v_0}$ the smooth pointed stable curve associated to v_0. Note that the underlying curve $\widetilde{E}_{v_0}$ coincides with the irreducible component of E corresponding to v_0. On the other hand, we define a pointed stable curve over k to be

$$E_0^\bullet = (E_0 \stackrel{\mathrm{def}}{=} \overline{E \setminus \widetilde{E}_{v_0}}, D_{E_0} \stackrel{\mathrm{def}}{=} (D_E \cap E_0) \cup (E_0 \cap \widetilde{E}_{v_0})),$$

where $\overline{E \setminus \widetilde{E}_{v_0}}$ denotes the topological closure of $E \setminus \widetilde{E}_{v_0}$ in E. Then $g^\bullet$ induces Galois admissible coverings

$$g^\bullet_{v_0} : \widetilde{X}^\bullet_{v_0} \to \widetilde{E}^\bullet_{v_0},$$
$$g^\bullet_0 : X^\bullet_0 \to E^\bullet_0$$

over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. To verify the lemma, we only need to prove that $g^\bullet_{v_0}$ and $g^\bullet_0$ are totally ramified over every node and every marked point of $\widetilde{E}^\bullet_{v_0}$ and $E^\bullet_0$, respectively.

Let $\Pi_{\widetilde{E}^\bullet_{v_0}}$ and $\Pi_{E^\bullet_0}$ be the solvable admissible fundamental groups of $\widetilde{E}^\bullet_{v_0}$ and $E^\bullet_0$, respectively. Since $\Gamma^{\mathrm{cpt}}_{E^\bullet}$ is 2-connected, [37, Corollary 3.5] implies that the natural homomorphism

$$\theta_{v_0} : \Pi^{\ell,\mathrm{ab}}_{\widetilde{E}^\bullet_{v_0}} \to \Pi^{\ell,\mathrm{ab}}_{E^\bullet}$$

is an injection. Let

$$\theta_0 : \Pi_{E_0^\bullet}^{\ell,\mathrm{ab}} \to \Pi_{E^\bullet}^{\ell,\mathrm{ab}}$$

be the homomorphism induced by the natural (outer) injective homomorphism $\Pi_{E_0^\bullet} \hookrightarrow \Pi_{E^\bullet}$ (in fact, θ_0 is also an injection).

Let $\{x\} = E_0 \cap \widetilde{E}_{v_0}$, $e_{v_0} \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}_{v_0}^\bullet})$ the open edge corresponding to x, $e_0 \in e^{\mathrm{op}}(\Gamma_{E_0^\bullet})$ the open edge corresponding to x, $\widehat{\widetilde{E}_{v_0}^\bullet}$ the universal admissible covering of $\widetilde{E}_{v_0}^\bullet$, $\widehat{E}_0^\bullet$ the universal admissible covering of $E_0^\bullet$, $\widehat{e}_{v_0} \in v(\Gamma_{\widehat{\widetilde{E}_{v_0}^\bullet}})$ an element over e_{v_0}, and $\widehat{e}_0 \in v(\Gamma_{\widehat{E}_0^\bullet})$ an element over e_0. We denote by $I_{e_{v_0}}$ the image of $I_{\widehat{e}_{v_0}}$ of $\Pi_{\widetilde{E}_{v_0}^\bullet} \twoheadrightarrow \Pi_{\widetilde{E}_{v_0}^\bullet}^{\ell,\mathrm{ab}}$, and by I_{e_0} the image of $I_{\widehat{e}_0}$ of $\Pi_{E_0^\bullet} \twoheadrightarrow \Pi_{E_0^\bullet}^{\ell,\mathrm{ab}}$. We put

$$a_{e_{v_0}} = (\prod_{e \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}_{v_0}^\bullet}) \setminus \{e_{v_0}\}} a_e)^{-1},$$

$$a_{e_0} = (\prod_{e \in e^{\mathrm{op}}(\Gamma_{E_0^\bullet}) \setminus \{e_0\}} a_e)^{-1}.$$

Then $a_{e_{v_0}}$ and a_{e_0} are generators of $I_{e_{v_0}}$ and I_{e_0}, respectively. Moreover, we put

$$\widetilde{\alpha}_{v_0} : \Pi_{\widetilde{E}_{v_0}^\bullet}^{\ell,\mathrm{ab}} \stackrel{\theta_{v_0}}{\to} \Pi_{E^\bullet}^{\ell,\mathrm{ab}} \stackrel{\alpha}{\twoheadrightarrow} \mathbb{F}_\ell.$$

and

$$\alpha_0 : \Pi_{E_0^\bullet}^{\ell,\mathrm{ab}} \stackrel{\theta_0}{\to} \Pi_{E^\bullet}^{\ell,\mathrm{ab}} \stackrel{\alpha}{\twoheadrightarrow} \mathbb{F}_\ell.$$

Then the structures of maximal pro-prime-to-p quotients of solvable admissible fundamental groups implies that

$$\widetilde{\alpha}_{v_0}(a_{e_{v_0}}) = \ell - \sum_{e \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}_{v_0}^\bullet}) \setminus \{e_{v_0}\}} r_e = \sum_{e \in e^{\mathrm{op}}(\Gamma_{E_0^\bullet}) \setminus \{e_0\}} r_e$$

and

$$\alpha_0(a_{e_0}) = \sum_{e \in e^{\mathrm{op}}(\Gamma_{\widetilde{E}_{v_0}^\bullet}) \setminus \{e_{v_0}\}} r_e.$$

Thus, by induction, we have that $g_{v_0}^\bullet$ and $g_0^\bullet$ are totally ramified over every node and every marked point of $\widetilde{E}_{v_0}^\bullet$ and $E_0^\bullet$, respectively. We complete the proof of the lemma. □

Lemma 5.29. *Let $E^\bullet$ be a pointed stable curve of type $(0, n)$ over an algebraically closed field k of characteristic $p > 0$. Then $E^\bullet$ satisfies Condition B.*

Proof. Let $f^\bullet : W^\bullet \to E^\bullet$ be an arbitrary admissible covering over k, $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$, and $f^{\mathrm{sg}} : \Gamma_{W^\bullet} \to \Gamma_{E^\bullet}$ the map of dual semi-graphs of $W^\bullet$ and $X^\bullet$ induced by $f^\bullet$. To verify the lemma, we only need to prove that $\Gamma_{W^\bullet}^{\mathrm{cpt}}$ is 2-connected.

Suppose that $f^\bullet$ is trivial. Then the lemma follows from that $\Gamma_{E^\bullet}^{\mathrm{cpt}}$ is 2-connected.

Suppose that $f^\bullet$ is non-trivial. Let $w \in v(\Gamma_{W^\bullet})$ and $v \in v(\Gamma_{E^\bullet})$. We denote by $\pi_0(w)$ and $\pi_0(v)$ the sets of connected components of $\Gamma_{W^\bullet} \setminus \{w\}$ and $\Gamma_{E^\bullet} \setminus \{v\}$, respectively.

Suppose that $v = f^{\rm sg}(w)$. Let $C_w \in \pi_0(w)$. We see immediately that $f^{\rm sg}(C_w)$ is a connected component of $\Gamma_{E^\bullet} \setminus \{v\}$. Write C_v for $f^{\rm sg}(C_w)$. Since $C_v \cap e^{\rm op}(\Gamma_{E^\bullet}) \neq \emptyset$, we obtain that $C_w \cap e^{\rm op}(\Gamma_{W^\bullet}) \neq \emptyset$. Thus, $\Gamma^{\rm cpt}_{W^\bullet}$ is 2-connected. This completes the proof of the lemma. □

Moreover, Theorem 5.26 implies the following important result.

Theorem 5.30. *Let $i \in \{1, 2\}$, and let $E^\bullet_i$ be a pointed stable curve of type $(0, n)$ over k_i of characteristic $p > 0$, $\Pi_{E^\bullet_i}$ the solvable admissible fundamental group of $E^\bullet_i$, and*

$$\phi_E : \Pi_{E^\bullet_1} \to \Pi_{E^\bullet_2}$$

an arbitrary open continuous homomorphism. Suppose that $E^\bullet_1$ and $E^\bullet_2$ satisfy Condition C. Then $\phi_E : \Pi_{E^\bullet_1} \to \Pi_{E^\bullet_2}$ induces surjective maps

$$\phi^{\rm ver}_E : {\rm Ver}(\Pi_{E^\bullet_1}) \twoheadrightarrow {\rm Ver}(\Pi_{E^\bullet_2}),$$

$$\phi^{\rm edg,op}_E : {\rm Edg}^{\rm op}(\Pi_{E^\bullet_1}) \twoheadrightarrow {\rm Edg}^{\rm op}(\Pi_{E^\bullet_2}),$$

$$\phi^{\rm edg,cl}_E : {\rm Edg}^{\rm cl}(\Pi_{E^\bullet_1}) \twoheadrightarrow {\rm Edg}^{\rm cl}(\Pi_{E^\bullet_2})$$

group-theoretically. Moreover, ϕ_E induces a bijection

$$\phi^{\rm sg}_E : \Gamma_{E^\bullet_1} \overset{\sim}{\to} \Gamma_{E^\bullet_2}$$

of the dual semi-graphs of $E^\bullet_1$ and $E^\bullet_2$ group-theoretically.

Proof. Lemma 4.3 implies that ϕ_E is a surjective. By applying Theorem 4.11, the homomorphism $\phi_E : \Pi_{E^\bullet_1} \twoheadrightarrow \Pi_{E^\bullet_2}$ induces a surjective map $\phi^{\rm edg,op} : {\rm Edg}^{\rm op}(\Pi_{E^\bullet_1}) \twoheadrightarrow {\rm Edg}^{\rm op}(\Pi_{E^\bullet_2})$ group-theoretically. We only need to treat the cases of $\phi^{\rm ver}_E$ and $\phi^{\rm edg,cl}_E$, respectively.

Let ℓ be a prime number such that $\ell \neq p$, and that $\ell >> n$. Let

$$\alpha_2 : \Pi^{\ell,\rm ab}_{E^\bullet_2} \twoheadrightarrow \mathbb{F}_\ell$$

satisfying the assumptions of Lemma 5.28. Then Theorem 4.11 implies that ϕ_E and α_2 induces a surjection

$$\alpha_1 : \Pi^{\ell,\rm ab}_{E^\bullet_1} \twoheadrightarrow \mathbb{F}_\ell,$$

which satisfies the assumptions of Lemma 5.28. Write $g^\bullet_i : X^\bullet_i \to E^\bullet_i$ for the Galois admissible covering over k_i with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Then Lemma 5.28 and Lemma 5.29 imply that $X^\bullet_1$ and $X^\bullet_2$ satisfy Condition A, Condition B, and Condition C.

Write $\Pi_{X^\bullet_i} \subseteq \Pi_{E^\bullet_i}$ for the open normal subgroup corresponding to $g^\bullet_i$. Let $\Pi_{\widehat{v}_{X_i}} \in {\rm Ver}(\Pi_{X^\bullet_i})$, $I_{\widehat{e}_{X_i}} \in {\rm Edg}^{\rm cl}(\Pi_{X^\bullet_i})$, $\Pi_{\widehat{v}_i} \in {\rm Ver}(\Pi_{E^\bullet_i})$ the unique element which contains $\Pi_{\widehat{v}_{X_i}}$, and $I_{\widehat{e}_i} \in {\rm Edg}^{\rm cl}(\Pi_{E^\bullet_i})$ the unique element which contains $I_{\widehat{e}_{X_i}}$. Since $\Pi_{\widehat{v}_i}$ and $I_{\widehat{e}_i}$ the normalizers of $\Pi_{\widehat{v}_{X_i}}$ and $I_{\widehat{e}_{X_i}}$ in $\Pi_{E^\bullet_i}$, respectively, the theorem follows immediately from Theorem 5.26. This completes the proof of the theorem. □

6 The Homeomorphism Conjecture for closed points when $g = 0$

We maintain the notation introduced in Section 3. In this section, we will prove that $\pi^{\mathrm{adm}}_{g,n}([q])$ is a closed point of $\overline{\Pi}_{g,n}$ for every $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{g,n}$ if $g = 0$. In particular, the Homeomorphism Conjecture holds when $(g, n) = (0, 4)$. *In the present section, we shall assume that all the fundamental groups of pointed stable curves are solvable admissible fundamental groups unless indicated otherwise.*

We fix some notation. Let $i \in \{1, 2\}$, and let $X_i^\bullet$ be a pointed stable curve of type $(0, n)$ over an algebraically closed field k_i of characteristic $p > 0$, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X^\bullet}$. Note that $\Gamma_{X_i^\bullet}$ is a tree, and that X_{i,v_i} is isomorphic to $\mathbb{P}^1_{k_i}$ for every $v_i \in v(\Gamma_{X_i^\bullet})$. In particular, X_{i,v_i} is smooth over k_i. For simplicity, we shall use the notation $X^\bullet_{i,v_i}$ to denote the smooth pointed stable curve $\widetilde{X}^\bullet_{i,v_i}$ of type $(0, n_{i,v_i})$ over k_i associated to $v_i \in v(\Gamma_{X_i^\bullet})$. On the other hand, let $\Pi_{X_i^\bullet}$ be the solvable admissible fundamental group of $X_i^\bullet$ and

$$\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$$

an arbitrary open continuous homomorphism. By Lemma 4.3, we see that ϕ is a *surjective* open continuous homomorphism. Then ϕ induces an isomorphism

$$\phi^p : \Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet}$$

of the maximal prime-to-p quotients of solvable admissible fundamental groups. Let $\widehat{X}_i^\bullet$ be a universal solvable admissible covering of $X_i^\bullet$ corresponding to $\Pi_{X_i^\bullet}$, $\Gamma_{\widehat{X}_i^\bullet}$ the dual semi-graph of $\widehat{X}_i^\bullet$, and $e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$. We put

$$\mathrm{Edg}^{\mathrm{op}}_{e_i}(\Pi_{X_i^\bullet}) \overset{\mathrm{def}}{=} \{I_{\widehat{e}_i} \in \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet}) \mid \widehat{e}_i \in e^{\mathrm{op}}(\Gamma_{\widehat{X}_i^\bullet}) \text{ is an open edge over } e_i\}.$$

Moreover, in the remainder of the present section, we shall suppose that k_1 *is an algebraic closure of* $\mathbb{F}_p$.

Denote by

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(-,-),\ \mathrm{Isom}_{\text{pro-gps}}(-,-)$$

the set of open continuous homomorphisms of profinit groups and the set of continuous isomorphisms of profinite groups, respectively. First, we have the following theorem which was proved by the author (cf. [35, Theorem 1.2 and Remark 7.3.1]).

Theorem 6.1. *We maintain the notation introduced above. Suppose that $X_1^\bullet$ and $X_2^\bullet$ are smooth over k_1 and k_2, respectively. Then we have that*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset$$

if and only if $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$. In particular, if this is the case, we have that $X_2^\bullet$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_2, and that

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) = \mathrm{Isom}_{\text{pro-gps}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}).$$

Remark 6.1.1. Let $[q] \in \mathfrak{M}_{0,n}^{\mathrm{cl}}$ be an arbitrary point. Theorem 6.1 and Proposition 3.6 (a) imply that

$$V(\pi_{0,n}^{\mathrm{sol}}([q])) \cap \Pi_{0,n}^{\mathrm{sol}} = \pi_{0,n}^{\mathrm{sol}}([q]).$$

Then we have that $[\pi_1^{\mathrm{sol}}(q)]$ is a closed point of $\Pi_{0,n}^{\mathrm{sol}}$. In particular,

$$\pi_{0,4}^{\mathrm{t}} : \mathfrak{M}_{0,4} \twoheadrightarrow \Pi_{0,4},\ \pi_{0,4}^{\mathrm{t,sol}} : \mathfrak{M}_{0,4} \twoheadrightarrow \Pi_{0,4}^{\mathrm{sol}}$$

are homeomorphisms.

Lemma 6.2. *We maintain the notation introduced above. Suppose that $X_1^\bullet$ is a singular curve. Then $X_2^\bullet$ is also a singular curve.*

Proof. Lemma 5.4 implies that there exists a Galois admissible covering

$$f_1^\bullet : Y_1^\bullet \to X_1^\bullet$$

over k_1 with Galois group G such that $(\#G, p) = 1$, that the Betti number of the dual semi-graph of $Y_1^\bullet$ is positive, and that $Y_1^\bullet$ satisfies Condition A. Then $\phi^{p'}$ induces a Galois admissible covering

$$f_2^\bullet : Y_2^\bullet \to X_2^\bullet$$

over k_2 with Galois group G. Write g_{Y_i} for the genus of $Y_i^\bullet$, and r_{Y_i} for the Betti number of the dual semi-graph of $Y_i^\bullet$.

By applying Theorem 4.11, we obtain that $g_{Y_1} = g_{Y_2}$. Moreover, Theorem 2.2 and Lemma 2.3 (b) imply that

$$0 < r_{Y_1} \leq r_{Y_2}.$$

This means that $X_2^\bullet$ is a singular curve. We complete the proof of the lemma. □

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$, and let $X^\bullet$ be a *smooth* pointed stable curve of type $(0, n)$ over $\overline{\mathbb{F}}_p$. We fix two marked points x_∞, $x_0 \in D_X$ distinct from each other. Moreover, we choose any field $k' \cong \overline{\mathbb{F}}_p$, and choose any isomorphism $\varphi : X \xrightarrow{\sim} \mathbb{P}^1_{k'}$ as schemes such that $\varphi(x_\infty) = \infty$ and $\varphi(x_0) = 0$. Then the set of $\overline{\mathbb{F}}_p$-rational points $X(\overline{\mathbb{F}}_p) \setminus \{x_\infty\} \xrightarrow{\sim} \mathbb{A}^1_{k'}(k')$ is equipped with a structure of $\mathbb{F}_p$-module via the bijection φ. Note that since any k'-isomorphism of $\mathbb{P}^1_{k'}$ fixing ∞ and 0 is a scalar multiplication, the $\mathbb{F}_p$-module structure of $X(\overline{\mathbb{F}}_p) \setminus \{x_\infty\}$ does not depend on the choices of k' and φ but depends only on the choices of x_∞ and x_0. We shall say that $X(\overline{\mathbb{F}}_p) \setminus \{x_\infty\}$ *is equipped with a structure of* $\mathbb{F}_p$*-module with respect to* x_∞ *and* x_0. Then we have the following lemma.

Lemma 6.3. *We maintain the notation introduced above. Suppose that $X_i^\bullet$ is smooth over k_i. Let $e_{1,0}$, $e_{1,\infty} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ be open edges distinct from each other. Theorem 4.11 implies that ϕ induces a bijection $\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ group-theoretically. We put $e_{2,0} \stackrel{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{1,0})$ and $e_{2,\infty} \stackrel{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{1,\infty})$. Let*

$$\sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} x_{e_1} = x_{e_{1,0}}$$

be a linear condition with respect to $e_{1,\infty}$ *and* $e_{1,0}$ *on* $X_1^\bullet$*, where* $b_{e_1} \in \mathbb{F}_p$ *for every* $e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$*. Then the linear condition*

$$\sum_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} x_{\phi^{\mathrm{sg,op}}(e_1)} = x_{\phi^{\mathrm{sg,op}}(e_{1,0})} = x_{e_{2,0}}$$

with respect to $x_{e_{2,\infty}}$ *and* $x_{e_{2,0}}$ *on* $X_2^\bullet$ *also holds.*

Proof. See [35, Lemma 7.1]. □

Lemma 6.4. *Let* $X^\bullet$ *be a pointed stable curve of type* $(0, n)$ *over an algebraically closed field* k *of characteristic* $p > 0$ *and* $\ell \geq 3$ *a prime number distinct from* p*. Then there exists a Galois admissible covering* $f^\bullet : Y^\bullet \to X^\bullet$ *over* k *with Galois group* $\mathbb{Z}/\ell\mathbb{Z}$ *such that the genus of* $Y^\bullet$ *is* 0*, and that there exists an irreducible component* Y_v *of* Y *satisfying* $\#(Y_v \cap D_Y) \geq 3$.

Proof. Suppose that $X^\bullet$ is smooth over k. Then the lemma is trivial. We may suppose that $X^\bullet$ is singular. Since the type of $X^\bullet$ is $(0, n)$, there exists irreducible components X_{v_1}, X_{v_2} of X distinct from each other such that $\#(X_{v_1} \cap D_X) \geq 2$ and $\#(X_{v_2} \cap D_X) \geq 2$.

Let $x_1 \in X_{v_1} \cap D_X$, $x_2 \in X_{v_2} \cap D_X$, and

$$f^\bullet : Y^\bullet \to X^\bullet$$

a Galois admissible covering over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that f is totally ramified over x_1 and x_2, and that f is étale over $D_X \setminus \{x_1, x_2\}$. We see immediately that the irreducible components $Y_{v_1} \stackrel{\mathrm{def}}{=} f^{-1}(X_{v_1})$ and $Y_{v_2} \stackrel{\mathrm{def}}{=} f^{-1}(X_{v_2})$ of Y satisfy the conditions $\#(Y_{v_1} \cap D_Y) \geq 3$ and $\#(Y_{v_2} \cap D_Y) \geq 3$, respectively. Moreover, the Riemann-Hurwitz formula implies that the genus of $Y^\bullet$ is 0. This completes the proof of the lemma. □

Next, we generalize Theorem 6.1 to the case where we only assume that $X_1^\bullet$ is smooth over k_1.

Proposition 6.5. *We maintain the notation introduced above. Suppose that* $X_1^\bullet$ *is smooth over* k_1*. Then* $X_1^\bullet$ *is Frobenius equivalent to* $X_2^\bullet$*. In particular, we have that* $X_2^\bullet$ *is smooth over* k_2*, and that* $X_2^\bullet$ *can be defined over the algebraic closure of* $\mathbb{F}_p$ *in* k_2*.*

Proof. If $X_2^\bullet$ is smooth over k_2, the proposition follows immediately from Theorem 6.1. Then we may assume that $X_2^\bullet$ is singular (i.e., $\#v(\Gamma_{X_2^\bullet}) \geq 2$).

Let $\ell \geq 3$ be a prime number prime to p. Lemma 6.4 implies that there exists an open normal subgroup $H_2 \subseteq \Pi_{X_2^\bullet}$ such that $\Pi_{X_2^\bullet}/H_2 \cong \mathbb{Z}/\ell\mathbb{Z}$, that the Galois admissible covering $f_{H_2}^\bullet : X_{H_2}^\bullet \to X_2^\bullet$ corresponding to H_2 is totally ramified over two marked points of $X_2^\bullet$, and that there exists $w_{H_2} \in v(\Gamma_{X_{H_2}^\bullet})$ such that $\#(X_{H_2, w_{H_2}} \cap D_{X_{H_2}}) \geq 3$. Write $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$ for the open subgroup and $f_{H_1}^\bullet : X_{H_1}^\bullet \to X_1^\bullet$ the Galois admissible covering over k_1 corresponding to H_1. Theorem 4.11 implies that $f_{H_1}^\bullet$ is totally ramified over two marked points of $X_1^\bullet$, and that $n_{X_{H_1}} = n_{X_{H_2}}$. Since $f_{H_i}^\bullet$ is totally ramified over two marked points, we have that

$$g_{X_{H_1}} = g_{X_{H_2}} = 0.$$

If we can prove the proposition holds for $X^\bullet_{H_1}$, $X^\bullet_{H_2}$, and $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$, then we obtain that $X^\bullet_2$ is also smooth over k_2. Then the proposition follows immediately from Theorem 6.1. Thus, by replacing $X^\bullet_1$, $X^\bullet_2$, and ϕ by $X^\bullet_{H_1}$, $X^\bullet_{H_2}$, and $\phi|_{H_1}$, respectively, we may assume that there exists $w_2 \in v(\Gamma_{X^\bullet_2})$ such that $\#(X_{2,w_2} \cap D_{X_2}) \geq 3$.

Let $e_{2,\infty}$, $e_{2,0}$, $e_{2,1} \in e^{\mathrm{op}}(\Gamma_{X^\bullet_2}) \cap e^{\mathrm{op}}(\Gamma_{X^\bullet_{2,w_2}})$ distinct from each other. Theorem 4.11 implies that ϕ induces a bijection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X^\bullet_1}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X^\bullet_2})$$

group-theoretically. We put

$$e_{1,\infty} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{2,\infty}),\ e_{1,0} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{2,0}),\ e_{1,1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{2,1}).$$

Without loss of generality, we may assume that

$$x_{e_{i,\infty}} \overset{\mathrm{def}}{=} \infty,\ x_{e_{i,0}} \overset{\mathrm{def}}{=} 0,\ x_{e_{i,1}} \overset{\mathrm{def}}{=} 1,$$

and that

$$X_1 = \mathbb{P}^1_{k_1},\ X_{w_2} = \mathbb{P}^1_{k_2}.$$

Let $\pi_0(\Gamma_{X^\bullet_2} \setminus \{w_2\})$ denote the set of connected components of $\Gamma_{X^\bullet_2} \setminus \{w_2\}$ in $\Gamma_{X^\bullet_2}$. Let $C_2 \in \pi_0(\Gamma_{X^\bullet_2} \setminus \{w_2\})$. Since $X^\bullet_2$ is a pointed stable curve of type $(0, n)$ over k_2, we have that $\#(C_2 \cap e^{\mathrm{op}}(\Gamma_{X^\bullet_2})) \geq 2$. Let $e_{2,C_2,1}$, $e_{2,C_2,2} \in C_2 \cap e^{\mathrm{op}}(\Gamma_{X^\bullet_2})$ be open edges distinct from each other. We put

$$e_{1,2} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{2,C_2,1}) \in e^{\mathrm{op}}(\Gamma_{X^\bullet_1}),$$

$$e_{1,3} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{2,C_2,2}) \in e^{\mathrm{op}}(\Gamma_{X^\bullet_1}).$$

We denote by X_{2,C_2} the semi-stable subcurve of X_2 whose irreducible components are the irreducible components corresponding to the vertices of $\Gamma_{X^\bullet_2}$ contained in C_2. Moreover, we write $e_{2,2}$ for the unique closed edge of $\Gamma_{X^\bullet_2}$ connecting w_2 and C_2. Then the node $x_{e_{2,2}}$ corresponding to $e_{2,2}$ is the unique closed point of X_2 contained in $X_{w_2} \cap X_{2,C_2}$.

We put

$$Z^\bullet_1 = (Z_1 \overset{\mathrm{def}}{=} X_1, D_{Z_1} \overset{\mathrm{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,2}}, x_{e_{1,3}}\}),$$

$$Y^\bullet_{1,1} = (Y_{1,1} \overset{\mathrm{def}}{=} X_1, D_{Y_{1,1}} \overset{\mathrm{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,2}}\}),$$

$$Y^\bullet_{1,2} = (Y_{1,2} \overset{\mathrm{def}}{=} X_1, D_{Y_1} \overset{\mathrm{def}}{=} \{x_{e_{1,\infty}}, x_{e_{1,0}}, x_{e_{1,1}}, x_{e_{1,3}}\}),$$

$$Y^\bullet_2 = (Y_2 \overset{\mathrm{def}}{=} X_{w_2}, D_{Y_2} \overset{\mathrm{def}}{=} \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,2}}\}).$$

Moreover, we denote by $Z^\bullet_2$ the pointed stable curve of type $(0, 5)$ over k_2 associated to the pointed semi-stable curve

$$(X_2, \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,C_2,1}}, x_{e_{2,C_2,2}}\})$$

over k_2 (i.e., the pointed stable curve obtained by contracting the (-1)-curves and the (-2)-curves of $(X_2, \{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}, x_{e_{2,C_2,1}}, x_{e_{2,C_2,2}}\})$. We see that Z_2 has two irreducible components Z_{w_2} and Z_{C_2} such that Z_{w_2} is equal to X_{w_2}, that $x_{e_{2,2}} = Z_{w_2} \cap Z_{C_2}$, that $\{x_{e_{2,\infty}}, x_{e_{2,0}}, x_{e_{2,1}}\} \subseteq Z_{w_2}$, and that $\{x_{e_{2,C_2,1}}, x_{e_{2,C_2,2}}\} \subseteq Z_{C_2}$.

Next, we will see that the solvable admissible fundamental groups and the natural homomorphisms of the the solvable admissible fundamental groups of pointed stable curves constructing above can be reconstructed group-theoretically from ϕ. Let

$$I_1 \subseteq \Pi_{X_1^\bullet}, I_2 \subseteq \Pi_{X_2^\bullet}$$

be the closed subgroups generated by the inertia subgroups of

$$\bigcup_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}, e_{1,1}, e_{1,2}, e_{1,3}\}} \mathrm{Edg}^{\mathrm{op}}_{e_1}(\Pi_{X_1^\bullet}),$$

$$\bigcup_{e_2 \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \setminus \{e_{2,\infty}, e_{2,0}, e_{2,1}, e_{2,C_2,1}, e_{2,C_2,2}\}} \mathrm{Edg}^{\mathrm{op}}_{e_2}(\Pi_{X_2^\bullet}),$$

respectively,

$$I_{1,1} \subseteq \Pi_{X_1^\bullet}, I_{1,2} \subseteq \Pi_{X_1^\bullet}$$

the closed subgroups generated by the inertia subgroups of

$$\bigcup_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}, e_{1,1}, e_{1,2}\}} \mathrm{Edg}^{\mathrm{op}}_{e_1}(\Pi_{X_1^\bullet}),$$

$$\bigcup_{e_1 \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \setminus \{e_{1,\infty}, e_{1,0}, e_{1,1}, e_{1,3}\}} \mathrm{Edg}^{\mathrm{op}}_{e_1}(\Pi_{X_1^\bullet}),$$

respectively, and

$$I_{2,1} \subseteq \Pi_{X_2^\bullet},\ I_{2,2} \subseteq \Pi_{X_2^\bullet}$$

the closed subgroups generated by the inertia subgroups of

$$\bigcup_{e_2 \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \setminus \{e_{2,\infty}, e_{2,0}, e_{2,1}, e_{2,C_2,1}\}} \mathrm{Edg}^{\mathrm{op}}_{e_2}(\Pi_{X_2^\bullet}),$$

$$\bigcup_{e_2 \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \setminus \{e_{2,\infty}, e_{2,0}, e_{2,1}, e_{2,C_2,2}\}} \mathrm{Edg}^{\mathrm{op}}_{e_2}(\Pi_{X_2^\bullet}),$$

respectively.

Then Theorem 4.11 implies that $\phi(I_1) = I_2$, $\phi(I_{1,1}) = I_{2,1}$, and $\phi(I_{1,2}) = I_{2,2}$. Moreover, we see that $\Pi_{X_1^\bullet}/I_1$ and $\Pi_{X_2^\bullet}/I_2$ are (outer) isomorphic to the solvable admissible fundamental groups of $Z_1^\bullet$ and $Z_2^\bullet$, respectively, that $\Pi_{X_1^\bullet}/I_{1,1}$ and $\Pi_{X_1^\bullet}/I_{1,2}$ are (outer) isomorphic to the solvable admissible fundamental groups of $Y_{1,1}^\bullet$ and $Y_{1,2}^\bullet$, respectively, and that $\Pi_{X_2^\bullet}/I_{2,1}$ and $\Pi_{X_2^\bullet}/I_{2,2}$ are (outer) isomorphic to the solvable admissible fundamental group of $Y_2^\bullet$. Note that $I_{1,1} \supseteq I_1 \subseteq I_{1,2}$ and $I_{2,1} \supseteq I_2 \subseteq I_{2,2}$.

On the other hand, ϕ induces the following surjective open continuous homomorphisms

$$\overline{\phi} : \Pi_{Z_1^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_1^\bullet}/I_1 \twoheadrightarrow \Pi_{Z_2^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_2^\bullet}/I_2,$$

$$\overline{\phi}_{1,1} : \Pi_{Y^\bullet_{1,1}} \stackrel{\text{def}}{=} \Pi_{X^\bullet_1}/I_{1,1} \twoheadrightarrow \Pi_{Y^\bullet_2} \stackrel{\text{def}}{=} \Pi_{X^\bullet_2}/I_{2,1},$$

$$\overline{\phi}_{1,2} : \Pi_{Y^\bullet_{1,2}} \stackrel{\text{def}}{=} \Pi_{X^\bullet_1}/I_{1,2} \twoheadrightarrow \Pi_{Y^\bullet_2} \stackrel{\text{def}}{=} \Pi_{X^\bullet_2}/I_{2,2}$$

which fit into the following commutative diagram:

$$\begin{array}{ccc} \Pi_{Y^\bullet_{1,1}} & \xrightarrow{\overline{\phi}_{1,1}} & \Pi_{Y^\bullet_2} \\ {\scriptstyle \psi_{1,1}}\uparrow & & \uparrow{\scriptstyle \psi_{2,1}} \\ \Pi_{Z^\bullet_1} & \xrightarrow{\overline{\phi}} & \Pi_{Z^\bullet_2} \\ {\scriptstyle \psi_{1,2}}\downarrow & & \downarrow{\scriptstyle \psi_{2,2}} \\ \Pi_{Y^\bullet_{1,2}} & \xrightarrow{\overline{\phi}_{1,2}} & \Pi_{Y^\bullet_2}, \end{array}$$

where $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{2,1}$, and $\psi_{2,2}$ denote the natural quotient homomorphisms.

Note that $\psi_{2,1} \circ \overline{\phi} \neq \psi_{2,2} \circ \overline{\phi}$, and that the homomorphisms of maximal prime-to-p quotients of solvable admissible fundamental groups $\overline{\phi}^{p'}_{1,1}$, $\overline{\phi}^{p'}$, and $\overline{\phi}^{p'}_{1,2}$ induced by $\overline{\phi}_{1,1}$, $\overline{\phi}$, and $\overline{\phi}_{1,2}$, respectively, are isomorphisms. Moreover, we see that $\psi_{2,1}(I_{\widehat{e}_{2,C_{2},1}}) \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,2}}(\Pi_{Y^\bullet_2})$ and $\psi_{2,2}(I_{\widehat{e}_{2,C_{2},2}}) \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,2}}(\Pi_{Y^\bullet_2})$ for every $I_{\widehat{e}_{2,C_{2},1}} \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,C_{2},1}}(\Pi_{Z^\bullet_2})$ and every $I_{\widehat{e}_{2,C_{2},2}} \in \mathrm{Edg}^{\mathrm{op}}_{e_{2,C_{2},2}}(\Pi_{Z^\bullet_2})$.

Let $\widehat{e}_{i,0} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet_i})$ be an open edge over $e_{i,0}$. By applying Theorem 4.13,

$$\mathbb{F}_{\widehat{e}_{i,0}} \stackrel{\text{def}}{=} (I_{\widehat{e}_{i,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}_i) \sqcup \{*_{\widehat{e}_{i,0}}\}$$

admits a structure of field which can be reconstructed group-theoretically from $\Pi_{X^\bullet_i}$. Since we assume that k_1 is an algebraic closure of $\mathbb{F}_p$, we may suppose that $k_1 = \mathbb{F}_{\widehat{e}_{1,0}}$. Moreover, we have that ϕ induces a field isomorphism

$$\phi^{\mathrm{fd}}_{\widehat{e}_{1,0},\widehat{e}_{2,0}} : \mathbb{F}_{\widehat{e}_{1,0}} \xrightarrow{\sim} \mathbb{F}_{\widehat{e}_{2,0}}.$$

group-theoretically. By [T2, Lemma 3.4], there exists a natural number m prime to p such that $\mathbb{F}_p(\zeta_{1,m})$ contains mth roots of $x_{e_{1,2}}$, $x_{e_{1,3}}$, where $\zeta_{1,m}$ denotes a fixed primitive mth root of unity in $\mathbb{F}_{\widehat{e}_{1,0}}$. Let $s \stackrel{\text{def}}{=} [\mathbb{F}_p(\zeta_{1,m}) : \mathbb{F}_p]$. For each $e_{1,u} \in \{e_{1,2}, e_{1,3}\}$, we fix an mth root $x^{\frac{1}{m}}_{e_{1,u}}$ in $\mathbb{F}_{\widehat{e}_{1,0}}$. Then we have

$$x^{\frac{1}{m}}_{e_{1,u}} = \sum_{t=0}^{s-1} b_{1,u,t}\zeta^t_{1,m},\ u \in \{2,3\},$$

where $b_{1,u,t} \in \mathbb{F}_p$ for each $u \in \{2,3\}$ and each $t \in \{0, \ldots, s-1\}$. Note that since $x_{e_{1,2}} \neq x_{e_{1,3}}$, there exists $t' \in \{0, \ldots, s-1\}$ such that $b_{1,2,t'} \neq b_{1,3,t'}$.

Let $Z_1 \setminus \{x_{e_{1,\infty}}\} = \operatorname{Spec} \mathbb{F}_{\widehat{e}_{1,0}}[x_1]$,

$$f^\bullet_{Q_1} : Z^\bullet_{Q_1} \to Z^\bullet_1$$

the Galois admissible covering over $\mathbb{F}_{\widehat{e}_{1,0}}$ with Galois group $\mathbb{Z}/m\mathbb{Z}$ determined by the equation $y_1^m = x_1$, and $Q_1 \subseteq \Pi_{Z_1^\bullet}$ the open normal subgroup induced by $f^\bullet_{Q_1}$. Then f_{Q_1} is totally ramified over $\{x_{e_{1,0}} = 0, x_{e_{1,\infty}} = \infty\}$ and is étale over $D_{Z_1} \setminus \{x_{e_{1,0}}, x_{e_{1,\infty}}\}$. Note that $Z_{Q_1} = \mathbb{P}^1_{\mathbb{F}_{\widehat{e}_{1,0}}}$, and that the marked points of $D_{Z_{Q_1}}$ over $\{x_{e_{1,0}}, x_{e_{1,\infty}}\}$ are $\{x_{e_{Q_1,0}} \overset{\text{def}}{=} 0, x_{e_{Q_1,\infty}} \overset{\text{def}}{=} \infty\}$. We put

$$x_{e_{Q_1,u}} \overset{\text{def}}{=} x_{e_{1,u}}^{\frac{1}{m}} \in D_{Z_{Q_1}},\ u \in \{2,3\},$$

and

$$x_{e^t_{Q_1,1}} \overset{\text{def}}{=} \zeta^t_{1,m} \in D_{Z_{Q_1}},\ t \in \{0, \dots, s-1\}.$$

Thus, we obtain a linear condition

$$x_{e_{Q_1,u}} = \sum_{t=0}^{s-1} b_{1,u,t} x_{e^t_{Q_1,1}},\ u \in \{2,3\}$$

with respect to $x_{e_{Q_1,0}}$ and $x_{e_{Q_1,\infty}}$ on $Z^\bullet_{Q_1}$.

Since $(m,p) = 1$, there exists a unique open normal subgroup $Q_2 \subseteq \Pi_{Z_2^\bullet}$ such that $\overline{\phi}^{-1}(Q_2) = Q_1$. On the other hand, we put

$$Q_{1,1} \overset{\text{def}}{=} \psi_{1,1}(Q_1) \subseteq \Pi_{Y^\bullet_{1,1}},$$

$$Q_{1,2} \overset{\text{def}}{=} \psi_{1,2}(Q_1) \subseteq \Pi_{Y^\bullet_{1,2}},$$

$$Q_{2,1} \overset{\text{def}}{=} \psi_{2,1}(Q_2) \subseteq \Pi_{Y^\bullet_2},$$

$$Q_{2,2} \overset{\text{def}}{=} \psi_{2,2}(Q_2) \subseteq \Pi_{Y^\bullet_2}.$$

Note that the constructions of Q_1 and Q_2 imply that $P_2 \overset{\text{def}}{=} Q_{2,1} = Q_{2,2}$. The commutative diagram of profinite groups above induces the following commutative diagram of profinite groups:

$$\begin{array}{ccc} Q_{1,1} & \xrightarrow{\overline{\phi}_{Q_{1,1}}} & P_2 \\ \uparrow & & \uparrow \\ Q_1 & \xrightarrow{\overline{\phi}_{Q_1}} & Q_2 \\ \downarrow & & \downarrow \\ Q_{1,2} & \xrightarrow{\overline{\phi}_{Q_{1,2}}} & P_2. \end{array}$$

Let $j \in \{1,2\}$. Write $Y^\bullet_{Q_{1,j}}$ for the pointed stable curve over k_1 corresponding to $Q_{1,j}$. Then we see that $e^{\text{op}}(\Gamma_{Y^\bullet_{Q_{1,j}}})$ can be regarded as a subset of $e^{\text{op}}(\Gamma_{Z^\bullet_{Q_1}})$. By applying Theorem 4.11 for $\overline{\phi}_{Q_{1,1}}$ and $\overline{\phi}_{Q_{1,2}}$, respectively, the commutative diagram of profinite groups above implies that we may put

$$e_{P_2,\infty} \overset{\text{def}}{=} \overline{\phi}^{\text{sg,op}}_{Q_{1,1}}(e_{Q_1,\infty}) = \overline{\phi}^{\text{sg,op}}_{Q_{1,2}}(e_{Q_1,\infty}),\ e_{P_2,0} \overset{\text{def}}{=} \overline{\phi}^{\text{sg,op}}_{Q_{1,1}}(e_{Q_1,0}) = \overline{\phi}^{\text{sg,op}}_{Q_{1,2}}(e_{Q_1,0}),$$

$$e^t_{P_2,1} \overset{\text{def}}{=} \overline{\phi}^{\text{sg,op}}_{Q_{1,1}}(e^t_{Q_1,1}) = \overline{\phi}^{\text{sg,op}}_{Q_{1,2}}(e^t_{Q_1,1}),\ t \in \{0, \dots, s-1\},$$

$$e_{P_2,2} \overset{\text{def}}{=} \overline{\phi}^{\text{sg,op}}_{Q_{1,1}}(e_{Q_1,2}) = \overline{\phi}^{\text{sg,op}}_{Q_{1,2}}(e_{Q_1,3}).$$

We denote by $\zeta_{2,m} \overset{\text{def}}{=} \phi^{\text{fd}}_{\widehat{e}_{1,0},\widehat{e}_{2,0}}(\zeta_{1,m})$. Then we have

$$x_{e^t_{Q_2,1}} = \zeta^t_{2,m},\ t \in \{0, \dots, s-1\}.$$

Let $Y^\bullet_{P_2}$ be the pointed stable curve over k_2 corresponding to $P_2 \subseteq \Pi_{Y^\bullet_2}$. Moreover, by applying Lemma 6.3 for $\overline{\phi}_{Q_{1,1}}$, we obtain that

$$x_{e_{P_2,2}} = \sum_{t=0}^{s-1} b_{1,2,t} x_{e^t_{Q_2,1}}$$

with respect to $x_{e_{P_2,0}}$ and $x_{e_{P_2,\infty}}$ on $Y^\bullet_{P_2}$. On the other hand, by applying Lemma 6.3 for $\overline{\phi}_{Q_{1,2}}$, we obtain that

$$x_{e_{P_2,2}} = \sum_{t=0}^{s-1} b_{1,3,t} x_{e^t_{Q_2,1}}$$

with respect to $x_{e_{P_2,0}}$ and $x_{e_{P_2,\infty}}$ on $Y^\bullet_{P_2}$. This means that

$$\sum_{t=0}^{s-1} b_{1,2,t}\zeta^t_{2,m} = \sum_{t=0}^{s-1} b_{1,3,t}\zeta^t_{2,m},$$

which is impossible as $b_{1,2,t'} \neq b_{1,3,t'}$ for some $t' \in \{0, \dots, s-1\}$. Then we obtain that $X^\bullet_2$ is smooth over k_2. Thus, the proposition follows immediately from Theorem 6.1. This completes the proof of the proposition. □

Now, we can prove the first form of our main theorem of the present paper.

Theorem 6.6. *Let $X^\bullet_i$, $i \in \{1, 2\}$, be an arbitrary pointed stable curve of type $(0, n)$ over an algebraically closed field k_i of characteristic $p > 0$ and $\Pi_{X^\bullet_i}$ either the admissible fundamental group of $X^\bullet_i$ or the solvable admissible fundamental group of $X^\bullet_i$. Suppose that k_1 is an algebraic closure of $\mathbb{F}_p$. Then we have that*

$$\text{Hom}^{\text{open}}_{\text{pro-gps}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) \neq \emptyset$$

if and only if $X^\bullet_1$ is Frobenius equivalent to $X^\bullet_2$. In particular, if this is the case, we have that $X^\bullet_2$ can be defined over the algebraic closure of $\mathbb{F}_p$ in k_2, and that

$$\text{Hom}^{\text{open}}_{\text{pro-gps}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) = \text{Isom}_{\text{pro-gps}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}).$$

Proof. To verify the theorem, it is sufficient to prove the theorem when $\Pi_{X^\bullet_i}$ is the solvable admissible fundamental group of $X^\bullet_i$. The "if" part of the theorem follows from [39, Proposition 3.7]. Let us prove the "only if" part of the theorem. Suppose that $\text{Hom}^{\text{open}}_{\text{pro-gps}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) \neq \emptyset$, and let $\phi \in \text{Hom}^{\text{open}}_{\text{pro-gps}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2})$ be an arbitrary element. Then Lemma 4.3 implies that ϕ is a surjection.

Suppose that $X_1^\bullet$ is smooth over k_1. Then the theorem follows from Proposition 6.5. Thus, we may assume that $X_1^\bullet$ is a singular pointed stable curve.

Note that since $X_1^\bullet$ is singular, we have $n = \#e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \geq 4$. We prove the theorem by induction on $\#e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$. Suppose that $\#e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) = 4$. Since $X_1^\bullet$ is a singular pointed stable curve of type $(0, 4)$, we obtain that $\#v(\Gamma_{X_1^\bullet}) = 2$ and $\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) = 1$. On the other hand, by applying Lemma 6.2, we obtain that $X_2^\bullet$ is also a singular pointed stable curve of type $(0, 4)$. Thus, we have that $\#e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) = 4$, $\#v(\Gamma_{X_2^\bullet}) = 2$, and $\#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) = 1$. Then $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition C defined in Section 5. Thus, by Theorem 5.30 and Theorem 6.1, we obtain that $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$.

Suppose that $\#e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \geq 5$. Theorem 4.11 implies that ϕ induces a bijection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$$

group-theoretically. Let $e_{1,n} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ and $e_{2,n} \stackrel{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{1,n})$. We denote by

$$Z_i^\bullet$$

the pointed stable curve of type $(0, n-1)$ over k_i associated to the pointed semi-stable curve $(X_i, D_{X_i} \setminus \{x_{e_{i,n}}\})$ whose underlying curve Z_i can be regarded naturally as a subcurve of X_i. Write $I_{i,n} \subseteq \Pi_{X_i^\bullet}$ for the closed subgroup generated by the subgroups $\mathrm{Edg}^{\mathrm{op}}_{e_{i,n}}(\Pi_{X_i^\bullet})$. Then we see that

$$\Pi_{Z_i^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_i^\bullet}/I_{i,n}$$

is (outer) isomorphic to the solvable admissible fundamental group of $Z_i^\bullet$. Moreover, Theorem 4.11 implies that $\phi(I_{1,n}) = I_{2,n}$. Then ϕ induces a surjective open continuous homomorphism

$$\overline{\phi} : \Pi_{Z_1^\bullet} \twoheadrightarrow \Pi_{Z_2^\bullet}.$$

By induction, we obtain that $Z_1^\bullet$ is Frobenius equivalent to $Z_2^\bullet$. Then ϕ induces a bijection of dual semi-graphs

$$\overline{\phi}^{\mathrm{sg}} : \Gamma_{Z_1^\bullet} \xrightarrow{\sim} \Gamma_{Z_2^\bullet}.$$

In particular, we put

$$\overline{\phi}^{\mathrm{sg,ver}} = \overline{\phi}^{\mathrm{sg}}|_{v(\Gamma_{Z_1^\bullet})} : v(\Gamma_{Z_1^\bullet}) \xrightarrow{\sim} v(\Gamma_{Z_2^\bullet}),$$

$$\overline{\phi}^{\mathrm{sg,op}} = \overline{\phi}^{\mathrm{sg}}|_{e^{\mathrm{op}}(\Gamma_{Z_1^\bullet})} : e^{\mathrm{op}}(\Gamma_{Z_1^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{Z_2^\bullet}).$$

Note that $\Gamma_{Z_i^\bullet}$ can be regarded naturally as a sub-semi-graph of $\Gamma_{X_i^\bullet}$. Moreover, one of the following cases may occur: (i) $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{Z_1^\bullet}) = \#v(\Gamma_{X_2^\bullet}) = \#v(\Gamma_{Z_2^\bullet})$; (ii) $\#v(\Gamma_{X_1^\bullet}) - 1 = \#v(\Gamma_{Z_1^\bullet}) = \#v(\Gamma_{X_2^\bullet}) - 1 = \#v(\Gamma_{Z_2^\bullet})$; (iii) $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{Z_1^\bullet}) = \#v(\Gamma_{X_2^\bullet}) - 1 = \#v(\Gamma_{Z_2^\bullet})$; (iv) $\#v(\Gamma_{X_1^\bullet}) - 1 = \#v(\Gamma_{Z_1^\bullet}) = \#v(\Gamma_{X_2^\bullet}) = \#v(\Gamma_{Z_2^\bullet})$.

Suppose that either (i) or (ii) holds. Then $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition C defined in Section 5. Thus, by Theorem 5.30 and Theorem 6.1, we obtain that $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$.

Suppose that (iii) holds. Let $v_2 \in v(\Gamma_{X_2^\bullet})$ such that $x_{e_{2,n}} \in X_{v_2}$. Since $\#v(\Gamma_{X_2^\bullet}) = \#v(\Gamma_{Z_2^\bullet}) + 1$, we have that $\#(X_{v_2} \cap D_{X_2}) = 2$. Note that $\{v_2\} = v(\Gamma_{X_2^\bullet}) \setminus v(\Gamma_{Z_1^\bullet})$.

Let $x_{e_{2,n-1}} \in X_{v_2} \cap D_{X_2}$ be the marked point distinct from $x_{e_{2,n}}$ and $e_{2,n-1} \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ the open edge corresponding to the marked point $x_{e_{2,n-1}}$. On the other hand, let $w_1 \in v(\Gamma_{X_1^\bullet})$ such that $x_{e_{1,n}} \in X_{w_1}$. We put

$$w_2 \overset{\mathrm{def}}{=} \overline{\phi}^{\mathrm{sg,ver}}(w_1) \in v(\Gamma_{Z_2^\bullet}) \subseteq v(\Gamma_{X_2^\bullet}),$$

$$e_{1,n-1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{2,n-1}) \in e^{\mathrm{op}}(\Gamma_{Z_1^\bullet}) \subseteq e^{\mathrm{op}}(\Gamma_{X_1^\bullet}).$$

Since $Z_1^\bullet$ is a pointed stable curve of type $(0, n-1)$, we have that

$$\#(X_{w_1} \cap D_{Z_1}) + \#(X_{w_1} \cap Z_1^{\mathrm{sing}}) \geq 3.$$

Then we see that there exist marked points $x_{e_{1,n-2}}, x_{e_{1,n-3}} \in D_{Z_1} \setminus \{x_{e_{1,n-1}}\}$ distinct from each other such that the following conditions are satisfied:

(1) if $\#(X_{w_1} \cap D_{Z_1}) \geq 3$, then $x_{e_{1,n-2}}, x_{e_{1,n-3}} \in X_{w_1}$;

(2) if $\#(X_{w_1} \cap D_{Z_1}) = 2$ and $x_{e_{1,n-1}} \notin X_{w_1}$, then $x_{e_{1,n-2}}, x_{e_{1,n-3}} \in X_{w_1}$;

(3) if $\#(X_{w_1} \cap D_{Z_1}) = 1$ and $x_{e_{1,n-1}} \notin X_{w_1}$, then we have that $x_{e_{1,n-3}} \in X_{w_1}$, and that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 1$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is ≥ 2) containing $x_{e_{2,n-1}}$ and $x_{e_{n-2}}$ are distinct from each other;

(4) if $\#(X_{w_1} \cap D_{Z_1}) = 2$ and $x_{e_{1,n-1}} \in X_{w_1}$, then we have that $x_{e_{1,n-3}} \in X_{w_1}$, and that $x_{e_{1,n-2}}$ is contained in a connected component of $Z_1 \setminus X_{w_1}$;

(5) if $\#(X_{w_1} \cap D_{Z_1}) = 1$ and $x_{e_{1,n-1}} \in X_{w_1}$, then we have that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 1$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is ≥ 2) containing $x_{e_{1,n-2}}$ and $x_{e_{1,n-3}}$ are distinct from each other;

(6) if $\#(X_{w_1} \cap D_{Z_1}) = 0$, then we have that the connected components of $Z_1 \setminus X_{w_1}$ (note that since $\#(X_{w_1} \cap D_{Z_1}) = 0$, the cardinality of the set of connected components of $Z_1 \setminus X_{w_1}$ is ≥ 3) containing $x_{e_{1,n-1}}$, $x_{e_{1,n-2}}$, and $x_{e_{1,n-3}}$ are distinct from each other.

Write $e_{1,n-2}$ and $e_{1,n-3} \in e^{\mathrm{op}}(\Gamma_{Z_1^\bullet})$ for the open edges corresponding to the marked points $x_{e_{1,n-2}}$ and $x_{e_{1,n-3}}$, respectively. We put

$$e_{2,n-2} \overset{\mathrm{def}}{=} \overline{\phi}^{\mathrm{sg,op}}(e_{1,n-2}),\ e_{2,n-3} \overset{\mathrm{def}}{=} \overline{\phi}^{\mathrm{sg,op}}(e_{1,n-3}).$$

Let $Y_i^\bullet$ be the pointed stable curve of type $(0, 4)$ over k_i associated to the pointed semi-stable curve

$$(X_i, \{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}).$$

By the construction of the set of marked points $\{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}$, we see that $Y_1^\bullet$ is smooth over k_1 whose underlying curve is X_{w_1}, and that $Y_2^\bullet$ is singular whose irreducible components are X_{w_2} and X_{v_2}.

Next, we will see that the solvable admissible fundamental groups and the natural homomorphisms of the the solvable admissible fundamental groups of pointed stable curves constructing above can be reconstructed group-theoretically from ϕ. Let $I_i \subseteq \Pi_{X_i^\bullet}$ be the closed subgroups generated by the subgroups

$$\bigcup_{e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_{i,n}, e_{i,n-1}, e_{i,n-2}, e_{i,n-3}\}} \mathrm{Edg}^{\mathrm{op}}_{e_i}(\Pi_{X_i^\bullet}).$$

We see that

$$\Pi_{Y_i^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_i^\bullet}/I_i$$

is (outer) isomorphic to the solvable admissible fundamental group of $Y_i^\bullet$. Moreover, Theorem 4.11 implies that $\phi(I_1) = I_2$. Then we obtain a surjective open continuous homomorphism

$$\overline{\overline{\phi}} : \Pi_{Y_1^\bullet} \twoheadrightarrow \Pi_{Y_2^\bullet}.$$

This contradicts Proposition 6.5, since Proposition 6.5 implies that $Y_2^\bullet$ is smooth over k_2. Then (iii) does not occur.

Suppose that (iv) holds. Similar arguments to the arguments given in the proof of (iii) imply that (iv) does not occur. More precisely, we have the following.

Let $v_1 \in v(\Gamma_{X_1^\bullet})$ such that $x_{e_{1,n}} \in X_{v_1}$. Since $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{Z_1^\bullet}) + 1$, we have that $\#(X_{v_1} \cap D_{X_1}) = 2$. Note that $\{v_1\} = v(\Gamma_{X_1^\bullet}) \setminus v(\Gamma_{Z_1^\bullet})$.

Let $x_{e_{1,n-1}} \in X_{v_1} \cap D_{X_1}$ be the marked point distinct from $x_{e_{1,n}}$ and $e_{1,n-1} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ the open edge corresponding to the marked point $x_{e_{1,n-1}}$. On the other hand, let $w_2 \in v(\Gamma_{X_2^\bullet})$ such that $x_{e_{2,n}} \in X_{w_2}$. We put

$$w_1 \stackrel{\mathrm{def}}{=} (\overline{\phi}^{\mathrm{sg,ver}})^{-1}(w_2) \in v(\Gamma_{Z_1^\bullet}) \subseteq v(\Gamma_{X_1^\bullet}),$$

$$e_{2,n-1} \stackrel{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{1,n-1}) \in e^{\mathrm{op}}(\Gamma_{Z_2^\bullet}) \subseteq e^{\mathrm{op}}(\Gamma_{X_2^\bullet}).$$

Since $Z_2^\bullet$ is a pointed stable curve of type $(0, n-1)$, we have that

$$\#(X_{w_2} \cap D_{Z_2}) + \#(X_{w_2} \cap Z_2^{\mathrm{sing}}) \geq 3.$$

Then we see that there exist marked points $x_{e_{2,n-2}}, x_{e_{2,n-3}} \in D_{Z_2} \setminus \{x_{e_{2,n-1}}\}$ distinct from each other such that the following conditions are satisfied:

(1) if $\#(X_{w_2} \cap D_{Z_2}) \geq 3$, then $x_{e_{2,n-2}}, x_{e_{2,n-3}} \in X_{w_2}$;

(2) if $\#(X_{w_2} \cap D_{Z_2}) = 2$ and $x_{e_{2,n-1}} \notin X_{w_2}$, then $x_{e_{2,n-2}}, x_{e_{2,n-3}} \in X_{w_2}$;

(3) if $\#(X_{w_2} \cap D_{Z_2}) = 1$ and $x_{e_{2,n-1}} \notin X_{w_2}$, then we have that $x_{e_{2,n-3}} \in X_{w_2}$, and that the connected components of $Z_2 \setminus X_{w_2}$ (note that since $\#(X_{w_2} \cap D_{Z_2}) = 1$, the cardinality of the set of connected components of $Z_2 \setminus X_{w_2}$ is ≥ 2) containing $x_{e_{2,n-1}}$ and $x_{e_{n-2}}$ are distinct from each other;

(4) if $\#(X_{w_2} \cap D_{Z_2}) = 2$ and $x_{e_{2,n-1}} \in X_{w_2}$, then we have that $x_{e_{2,n-3}} \in X_{w_2}$, and that $x_{e_{2,n-2}}$ is contained in a connected component of $Z_2 \setminus X_{w_2}$;

(5) if $\#(X_{w_2} \cap D_{Z_2}) = 1$ and $x_{e_{2,n-1}} \in X_{w_2}$, then we have that the connected components of $Z_2 \setminus X_{w_2}$ (note that since $\#(X_{w_2} \cap D_{Z_2}) = 1$, the cardinality of the set of connected components of $Z_2 \setminus X_{w_2}$ is ≥ 2) containing $x_{e_{2,n-2}}$ and $x_{e_{n-3}}$ are distinct from each other;

(6) if $\#(X_{w_2} \cap D_{Z_2}) = 0$, then we have that the connected components of $Z_2 \setminus X_{w_2}$ (note that since $\#(X_{w_2} \cap D_{Z_2}) = 0$, the cardinality of the set of connected components of $Z_2 \setminus X_{w_2}$ is ≥ 3) containing $x_{e_{2,n-1}}$, $x_{e_{n-2}}$, and $x_{e_{2,n-3}}$ are distinct from each other.

Write $e_{2,n-2}$ and $e_{2,n-3} \in e^{\mathrm{op}}(\Gamma_{Z_2^\bullet})$ for the open edges corresponding to the marked points $x_{e_{2,n-2}}$ and $x_{e_{2,n-3}}$, respectively. We put

$$e_{1,n-2} \stackrel{\mathrm{def}}{=} (\overline{\phi}^{\mathrm{sg,op}})^{-1}(e_{2,n-2}),\ e_{1,n-3} \stackrel{\mathrm{def}}{=} (\overline{\phi}^{\mathrm{sg,op}})^{-1}(e_{2,n-3}).$$

Let $Y_i^\bullet$ be the pointed stable curve of type $(0,4)$ over k_i associated to the pointed semi-stable curve

$$(X_i, \{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}).$$

By the construction of the set of marked points $\{x_{e_{i,n}}, x_{e_{i,n-1}}, x_{e_{i,n-2}}, x_{e_{i,n-3}}\}$, we see that $Y_1^\bullet$ is singular whose irreducible component are X_{w_1} and X_{v_1}, and that $Y_2^\bullet$ is smooth over k_2 whose underlying curve is X_{w_2}.

Let $I_i \subseteq \Pi_{X_i^\bullet}$ be the closed subgroups generated by the subgroups

$$\bigcup_{e_i \in e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_{i,n}, e_{i,n-1}, e_{i,n-2}, e_{i,n-3}\}} \mathrm{Edg}^{\mathrm{op}}_{e_i}(\Pi_{X_i^\bullet}).$$

We see that

$$\Pi_{Y_i^\bullet} \stackrel{\mathrm{def}}{=} \Pi_{X_i^\bullet}/I_i$$

is (outer) isomorphic to the solvable admissible fundamental group of $Y_i^\bullet$. Moreover, Theorem 4.11 implies that $\phi(I_1) = I_2$. Then we obtain a surjective open continuous homomorphism

$$\overline{\overline{\phi}} : \Pi_{Y_1^\bullet} \twoheadrightarrow \Pi_{Y_2^\bullet}.$$

This contradicts Lemma 6.2, since Lemma 6.2 implies that $Y_2^\bullet$ is singular. Then (iv) does not occur. This completes the proof of the theorem. □

Theorem 6.6 implies the following result concerning the Homeomorphism Conjecture which is the main theorem of the present paper.

Theorem 6.7. *We maintain the notation introduced in Section 3. Let* $[q] \in \overline{\mathfrak{M}}^{\mathrm{cl}}_{0,n}$ *be an arbitrary closed point. Then* $\pi^{\mathrm{adm}}_{0,n}([q])$ *and* $\pi^{\mathrm{sol}}_{0,n}([q])$ *are closed points of* $\overline{\Pi}_{0,n}$ *and* $\overline{\Pi}^{\mathrm{sol}}_{0,n}$*, respectively. In particular, the Homeomorphism Conjecture and the Solvable Homeomorphism Conjecture hold when* $(g,n) = (0,4)$*.*

Proof. To verify the theorem, we only need to treat the case of solvable admissible fundamental groups.

Let $V(\pi^{\mathrm{sol}}_{0,n}([q]))$ be the topological closure of $\pi^{\mathrm{sol}}_{0,n}([q])$ in $\overline{\Pi}^{\mathrm{sol}}_{0,n}$ and $[\pi_1^{\mathrm{sol}}(q')] \in V(\pi^{\mathrm{sol}}_{0,n}([q]))$ an arbitrary point. Then by Proposition 3.6 (a), we obtain that there exists a surjective open continuous homomorphism

$$\phi : \pi_1^{\mathrm{sol}}(q) \twoheadrightarrow \pi_1^{\mathrm{sol}}(q').$$

Theorem 6.6 implies that $q \sim_{fe} q'$. Thus, we obtain that $[\pi_1^{\mathrm{sol}}(q)] = [\pi_1^{\mathrm{sol}}(q')]$. This means that $V(\pi^{\mathrm{sol}}_{0,n}([q])) = [\pi_1^{\mathrm{sol}}(q)]$ is a closed point of $\overline{\Pi}^{\mathrm{sol}}_{0,n}$. Moreover, the "in particular" part of the theorem follows from Theorem 3.4 (b). This completes the proof of the theorem. □

Remark 6.7.1. In [40], the author proves a similar result of Theorem 6.7 when $(g,n) = (1,1)$ and $p > 2$.

7 Continuity of $\pi^{\mathrm{adm}}_{g,n}$

In this section, we prove Theorem 3.5.

7.1 Moduli spaces of curves with level structures

We fix some notation. Let k be an algebraic closure of the finite field $\mathbb{F}_p$ and B, B' schemes over k. Let $X_B^\bullet = (X_B, D_{X_B})$ be a pointed stable curve of type (g, n) over B and $B' \to B$ a k-morphism. We shall write $X_{B'}^\bullet$ for $X_B^\bullet \times_B B'$ the pointed stable curve over B'. Let $f_B^\bullet : Y_B^\bullet \to X_B^\bullet$ be a finite flat morphism of pointed stable curves over B, $b \in B$, and $\bar{b} \to B$ a geometric point over b. We shall say $f_B^\bullet$ an admissible covering (resp. a Galois admissible covering) over B if $f_{\bar{b}}^\bullet \overset{\text{def}}{=} f_B^\bullet \times_B \bar{b} : Y_{\bar{b}}^\bullet \to X_{\bar{b}}^\bullet$ is an admissible covering (resp. a Galois admissible covering) over $\bar{b}$ for all $b \in B$.

Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack over $\operatorname{Spec}\mathbb{Z}$ parameterizing pointed stable curves of type (g, n) and $\mathcal{M}_{g,n,\mathbb{Z}}$ the open substack of $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ parameterizing smooth pointed stable curves. Let $\overline{\mathcal{M}}_{g,n} \overset{\text{def}}{=} \overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} k$, $\mathcal{M}_{g,n} \overset{\text{def}}{=} \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} k$, $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$, and $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$. We denote by $\overline{\omega}_{g,n} : \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ and $\omega_{g,n} : \mathcal{M}_{g,n} \to M_{g,n}$ the natural morphisms, respectively.

When $g = 0$, then $\mathcal{M}_{0,n}$ is a scheme over k. Thus, we have $\mathcal{M}_{0,n} = M_{0,n}$. Note that $M_{0,n}$ is a quasi-projective variety over k. In general, the coarse moduli space $M_{g,n}$ is not a fine moduli space. In order to build a family of curves over schemes, we use the level structures. Let $m \geq 3$ be an integer number prime to p.

Suppose that $g = 1$. We denote by $M_{1,1}^{(m)}$ the moduli stack over k classifying smooth pointed stable curves of type $(1, 1)$ with level m-structure (i.e., the moduli stack of elliptic curves in characteristic p with level m-structure). Then we have a natural morphism $\omega_{1,1}^{(m)} : M_{1,1}^{(m)} \to \mathcal{M}_{1,1} \to M_{1,1}$. Moreover, we have

$$M_{1,n}^{(m)} \overset{\text{def}}{=} M_{1,1}^{(m)} \times_{M_{1,1}} M_{1,n},$$

where $M_{1,n} \to M_{1,1}$ is the natural morphism induced by the forgetting morphism $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$ determined by forgetting the last $n-1$ marked points. Then we obtain a morphism

$$\omega_{1,n}^{(m)} : M_{1,n}^{(m)} \to M_{1,n}$$

determined by the second projection. Note that $M_{1,n}^{(m)}$ is a quasi-projective variety over k, and that $M_{1,n}^{(m)}(B)$ is the set of B-isomorphism classes of smooth pointed stable curves of type $(1, n)$ over B such that, by forgetting the last $n - 1$ marked points, the smooth pointed stable curves of type $(1, 1)$ are elliptic curves over B with level m-structure.

Suppose that $g \geq 2$. Let $M_{g,0}^{(m)}$ be the moduli stack over k classifying smooth pointed stable curves of type $(g, 0)$ with level m-structure. Then we obtain a natural morphism $\omega_{g,0}^{(m)} : M_{g,0}^{(m)} \to \mathcal{M}_{g,0} \to M_{g,0}$. Moreover, we have

$$M_{g,n}^{(m)} \overset{\text{def}}{=} M_{g,0}^{(m)} \times_{M_{g,0}} M_{g,n},$$

where $M_{g,n} \to M_{g,0}$ is the natural morphism induced by the forgetting morphism $\mathcal{M}_{g,n} \to \mathcal{M}_{g,0}$ determined by forgetting marked points. Then we obtain a morphism

$$\omega_{g,n}^{(m)} : M_{g,n}^{(m)} \to M_{g,n}$$

determined by the second projection. Note that $M_{g,n}^{(m)}$ is a quasi-projective variety over k, and that $M_{g,n}^{(m)}(B)$ is the set of B-isomorphism classes of smooth pointed stable curves of type (g,n) over B whose underlying curves are smooth projective curves of genus g over B with level m-structure.

For simplicity, we shall write $H_{g,n}$ for $M_{g,n}^{(m)}$ when $g \geq 1$, $H_{0,n}$ for $M_{0,n}$ when $g = 0$, $\omega_{g,n}^{(m)}$ for the morphism $\omega_{g,n}^{(m)} : H_{g,n} \overset{\text{def}}{=} M_{g,n}^{(m)} \to M_{g,n}$ when $g \geq 1$, and $\omega_{0,n}^{(m)}$ for $\mathrm{id}_{M_{0,n}} : H_{0,n} \overset{\text{def}}{=} M_{0,n} \to M_{0,n}$ when $g = 0$. Moreover, we denote by

$$X_{H_{g,n}}^{\bullet} = (X_{H_{g,n}}, D_{X_{H_{g,n}}})$$

the universal smooth pointed stable curve of type (g,n) over $H_{g,n}$ with a level m-structure $\tau_{H_{g,n}} \overset{\text{def}}{=} \tau_{H_{g,0}} \times_{H_{g,0}} H_{g,n}$ induced by the universal level m-structure

$$\tau_{H_{g,0}} : \mathrm{Pic}^0_{X_{H_{g,0}}/H_{g,0}}[m] \overset{\sim}{\to} (\mathbb{Z}/m\mathbb{Z})^{2g}_{H_{g,0}}$$

when $g \geq 2$, by the universal level m-structure

$$\tau_{H_{1,1}} : \mathrm{Pic}^0_{X_{H_{1,1}}/H_{1,1}}[m] \overset{\sim}{\to} (\mathbb{Z}/m\mathbb{Z})^{2}_{H_{1,1}}$$

when $g = 1$, and by the trivial level m-structure when $g = 0$.

7.2 The sets of finite quotients of admissible fundamental groups

We maintain the notation introduced in Section 3 and Section 7.1. Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $\pi_1^{\mathrm{adm}}(q)$ the admissible fundamental group of the pointed stable curve $X_q^{\bullet}$ over an algebraic closure of the residue field $k(q)$ of q, Γ_q the dual semi-graph of $X_q^{\bullet}$, and $\pi_A^{\mathrm{adm}}(q)$ the set of finite quotients of $\pi_1^{\mathrm{adm}}(q)$. Since $\pi_1^{\mathrm{adm}}(q)$ is topologically finitely generated, the isomorphism class of $\pi_1^{\mathrm{adm}}(q)$ is determined completely by $\pi_A^{\mathrm{adm}}(q)$ (cf. [FJ, Proposition 16.10.6]). First, we have the following lemmas.

Lemma 7.1. *Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary points such that $q_2 \in \overline{\{q_1\}}$, where $\overline{\{(-)\}}$ denotes the topological closure of $(-)$ in $\overline{M}_{g,n}$. Then we have that*

$$\pi_A^{\mathrm{adm}}(q_2) \subseteq \pi_A^{\mathrm{adm}}(q_1).$$

Proof. The lemma follows immediately from the specialization theorem of admissible fundamental groups of pointed stable curves (cf. [V2, Théorème 2.2]). □

Lemma 7.2. *Let S be a smooth variety over k, η_S the generic point of S, and $X_S^{\bullet}$ a smooth pointed stable curve over S. Let $Y_{\eta_S}^{\bullet}$ be a smooth pointed stable curve over η_S and*

$$f_{\eta_S}^{\bullet} : Y_{\eta_S}^{\bullet} \to X_{\eta_S}^{\bullet}$$

a Galois admissible covering over η_S. Then there exist an open subset $U \subseteq S$ and a Galois admissible covering

$$f_U^{\bullet} : Y_U^{\bullet} \to X_U^{\bullet}$$

of smooth pointed stable curves over U such that $f_U^{\bullet} \times_U \eta_S = f_{\eta_S}^{\bullet}$.

Proof. Write Y_S for the normalization of X_S in the function field of Y_{η_S} and D_{Y_S} for the set of the topological closures of points of $D_{Y_{\eta_S}}$ in Y_S. [5, Proposition 5] implies that, by replacing S by an open subset of S, we may assume that the fiber $Y_s \stackrel{\text{def}}{=} Y_S \times_S s$ is geometrically irreducible over every *closed* point $s \in S$.

The normalization $f_S : Y_S \to X_S$ induces a morphism

$$g_S \stackrel{\text{def}}{=} f_S|_{Y_S \setminus D_{Y_S}} : Y_S \setminus D_{Y_S} \to X_S \setminus D_{X_S}$$

over S. Since the restriction of g_S on the generic fiber η_S is a Galois étale covering over η_S, there exists an open subset $U \subseteq S$ such that g_U is a Galois étale covering over U. Thus, by replacing S by U, we may assume that g_S is a Galois étale covering. Since the fiber $Y_s \stackrel{\text{def}}{=} Y_S \times_S s$ is generically smooth over each $s \in S$, Y_s is geometrically irreducible over each point $s \in S$.

The normalization $f_S : Y_S \to X_S$ induces a morphism

$$g_S \stackrel{\text{def}}{=} f_S|_{Y_S \setminus D_{Y_S}} : Y_S \setminus D_{Y_S} \to X_S \setminus D_{X_S}$$

over S. Since the restriction of g_S on the generic fiber η_S is étale, there exists an open subset $U \subseteq S$ such that

$$g_u : Y_S \setminus D_{Y_S} \times_S u \to X_S \setminus D_{X_S} \times_S u$$

is étale for each $u \in U$. Thus, by replacing S by the open subset U, we may assume that g_S is étale. Since the fiber $Y_s \stackrel{\text{def}}{=} Y_S \times_S s$ is generically smooth over each $s \in S$, Y_s is geometrically irreducible over each point $s \in S$.

Let $X_S^{\log}$ be the log scheme over S whose underlying scheme is X_S, and whose log structure is determined by the marked points D_{X_S}. Since S is smooth over k, we see that $X_S^{\log}$ is log regular. Note that f_S is tamely ramified over the generic points of points D_{X_S}. Then the log purity (cf. [9, Theorem B]) implies that g_S extends uniquely to a Galois log étale morphism $f_S^{\log} : Y_S^{\log} \to X_S^{\log}$ over S. Let $Y_S^\bullet \stackrel{\text{def}}{=} (Y_S, D_S)$. Then $Y_S^\bullet$ is a smooth pointed stable curve over S. Thus, $f_S^{\log}$ induces a morphism $f_S^\bullet : Y_S^\bullet \to X_S^\bullet$ such that the restriction of $f_S^\bullet$ on η_S is equal to $f_{\eta_S}^\bullet$, and that $f_s : Y_s^\bullet \to X_s^\bullet$ induced by $f_S^\bullet$ is a Galois admissible covering over every $s \in S$. □

Lemma 7.3. *Let $q \in M_{g,n}$ be an arbitrary point, V_q^{sm} the topological closure of q in $M_{g,n}$, and $C \subseteq V_q^{\mathrm{sm,cl}}$ a subset of closed points of V_q^{sm}. Suppose that C is dense in V_q^{sm}. Then we have that*

$$\pi_A^{\mathrm{adm}}(q) = \bigcup_{c \in C} \pi_A^{\mathrm{adm}}(c).$$

Proof. If q is a closed point, then the lemma is trivial. We may assume that q is not a closed point. Lemma 7.1 implies that, to verify the lemma, it is sufficient to prove that, for any $G \in \pi_A^{\mathrm{adm}}(q)$, there exists a closed point $c \in C$ such that $G \in \pi_A^{\mathrm{adm}}(c)$.

Let $q^{(m)} \in (\omega_{g,n}^{(m)})^{-1}(q)$ be a point of $H_{g,n}$, $V_{q^{(m)}}$ the topological closure of $q^{(m)}$ in $H_{g,n}$, and $k(q^{(m)})$ the residue field of $q^{(m)}$ which is the function field of $V_{q^{(m)}}$. Write M' for the normalization of $V_{q^{(m)}}$ in $k(q^{(m)})$. Then there exists an open subset of $M \subseteq M'$

such that M is smooth over k. Moreover, the composition of the natural morphisms $M \hookrightarrow M' \to V_{q^{(m)}} \hookrightarrow H_{g,n}$ determines a smooth pointed stable curve $X^\bullet_M \stackrel{\text{def}}{=} X^\bullet_{H_{g,n}} \times_{H_{g,n}} M$ over M.

Let k_q be an algebraic closure of $k(q^{(m)})$. By the construction, k_q is also an algebraic closure of the residue field $k(q)$ of q. Let $Y^\bullet_{k_q} \to X^\bullet_{k_q}$ be a Galois admissible covering over k_q with Galois group G. By replacing $k(q^{(m)})$ by a finite extension l of $k(q^{(m)})$, the Galois admissible covering can be descended to a Galois admissible covering $Y^\bullet_l \to X^\bullet_l$ over l with Galois group G. Write N for the normalization of M in l, $X^\bullet_N$ for $X^\bullet_M \times_M N$, and $Y^\bullet_N$ for the normalization of $X^\bullet_N$ in the function field of $Y^\bullet_l$. Then we obtain a covering

$$Y^\bullet_N \to X^\bullet_N$$

over N such that the restriction induced by the natural morphism $\operatorname{Spec} l \to N$ is the Galois admissible covering $Y^\bullet_l \to X^\bullet_l$ over l with Galois group G. Since N is generically smooth over k, by replacing N by an open subset of N, we may assume that N is smooth over k. Thus, Lemma 7.2 implies that there exists an open subset $U \subseteq N$ such that the morphism

$$Y^\bullet_U \stackrel{\text{def}}{=} Y^\bullet_N \times_N U \to X^\bullet_U \stackrel{\text{def}}{=} X^\bullet_N \times_N U$$

is a Galois admissible covering over U with Galois group G.

We denote by

$$W \subseteq V^{\text{sm}}_q$$

the image of U of the composition of the natural morphisms $U \hookrightarrow N \to M \hookrightarrow M' \to V_{q^{(m)}} \hookrightarrow H_{g,n} \to M_{g,n}$, which is a dense constructible subset of V^{sm}_q. Then W contains an open subset W' of V^{sm}_q. Since C is dense in V^{sm}_q, we obtain that $W \cap C \neq \emptyset$. This means that, there exists a closed point $c \in C$ such that $G \in \pi^{\text{adm}}_A(c)$. We complete the proof of the lemma. $\square$

We maintain the notation introduced in the proof of Lemma 7.3. We shall denote by

$$U_q \subseteq U$$

the inverse image of W' of the composition of the natural morphisms $U \hookrightarrow N \to M \hookrightarrow M' \to V_{q^{(m)}} \hookrightarrow H_{g,n} \to M_{g,n}$, which is an open subset of U. Then the proof of Lemma 7.3 implies the following corollary.

Corollary 7.4. *We maintain the notation introduced in the proof of Lemma 7.3. Let $q \in M_{g,n}$ be an arbitrary point, k_q an algebraic closure of the residue field $k(q)$, V^{sm}_q the topological closure of q in $M_{g,n}$, and $f^\bullet_{k_q} : Y^\bullet_{k_q} \to X^\bullet_{k_q}$ a Galois admissible covering over k_q with Galois group G. Then there exist a smooth k-variety U_q and a finite morphism $U_q \to H_{g,n}$ (not necessary a surjection) such that the following conditions are satisfied:*

(i) the image of U_q of the composition of the natural morphisms $U_q \to H_{g,n} \stackrel{\omega^{(m)}_{g,n}}{\to} M_{g,n}$ is an open subset of V^{sm}_q;

(ii) the morphism $U_q \to H_{g,n}$ induces a smooth pointed stable curve

$$X^\bullet_{U_q} \stackrel{\text{def}}{=} X^\bullet_{H_{g,n}} \times_{H_{g,n}} U_q$$

over U_q with a level m-structure $\tau_{U_q} \stackrel{\mathrm{def}}{=} \tau_{H_{g,n}} \times_{H_{g,n}} U_q$;

(iii) there exists a Galois admissible covering $f^{\bullet}_{U_q} : Y^{\bullet}_{U_q} \to X^{\bullet}_{U_q}$ of smooth pointed stable curves over U_q with Galois group G such that $f^{\bullet}_{U_q} \times_{U_q} \operatorname{Spec} k_q = f^{\bullet}_{k_q}$.

In the remainder of this subsection, we will generalize Lemma 7.3 to the case where $q \in \overline{M}_{g,n}$.

Lemma 7.5. *Let S be a k-variety and $s_1, s_2 \in S$ two points of S such that $s_1 \neq s_2$ and $s_2 \in \overline{\{s_1\}}$. Then there exist a complete discrete valuation ring R and a morphism $\operatorname{Spec} R \to S$ such that the image of the morphism is $\{s_1, s_2\}$.*

Proof. It is easy to see that we may assume that s_1 is the generic point of S, and s_2 is a closed point of S. If $\dim(S) = 1$, then the lemma is trivial. We may assume that $\dim(S) \geq 2$.

Let $\overline{s}_1$ be a geometric point over s_1 and $\overline{S} \stackrel{\mathrm{def}}{=} S \times_S \overline{s}_1$. Then the natural morphisms $\overline{s}_1 \to s_1 \to S$ and $s_2 \to S$ induce a morphism $f_1 : \overline{s}_1 \to \overline{S}$ and $f_2 : s_2 \times_k \overline{s}_1 \to \overline{S}$, respectively. We denote by s'_1 the image of f_1, and by s'_2 the image of f_2. Note that s'_1, s'_2 are closed points of $\overline{S}$ and $s'_1 \neq s'_2$. Then there exists a curve $C \subseteq \overline{S}$ which contains s'_1, s'_2. Write η_C for the generic point of C. Thus, the image of the composition of the morphisms $\eta_C \hookrightarrow C \hookrightarrow \overline{S} \to S$ is s_1.

There exist a complete discrete valuation ring R and a morphism $\operatorname{Spec} R \to C$ such that the image of the morphism is $\{\eta_C, s'_2\}$. Then the desired morphism is the composition of the morphisms $\operatorname{Spec} R \to C \hookrightarrow \overline{S} \to S$. This completes the proof of the lemma. $\square$

Lemma 7.6. *Let R be a complete discrete valuation ring, K_R the quotient field of R, and k_R the residue field of R such that k_R is an algebraically closed field containing k. Let*

$$f^{\bullet}_{K_R} : Y^{\bullet}_{K_R} \to X^{\bullet}_{K_R}$$

be a Galois admissible covering over K_R with Galois group G. Write $\Gamma_{X^{\bullet}_{K_R}}$ for the dual semi-graph of $X^{\bullet}_{K_R}$ and $\Gamma_{Y^{\bullet}_{K_R}}$ for the dual semi-graph of $Y^{\bullet}_{K_R}$. Suppose that $\widetilde{X}^{\bullet}_{K_R,v_X}$, $v_X \in v(\Gamma_{X^{\bullet}_{K_R}})$, and $\widetilde{Y}^{\bullet}_{K_R,v_Y}$, $v_Y \in v(\Gamma_{Y^{\bullet}_{K_R}})$, have good reduction over R, where $\widetilde{X}^{\bullet}_{K_R,v_X}$ and $\widetilde{Y}^{\bullet}_{K_R,v_Y}$ denote the smooth pointed stable curves of types (g_{v_X}, n_{v_X}) and (g_{v_Y}, n_{v_Y}) associated to v_X and v_Y, respectively. Then there exists a Galois admissible covering

$$f^{\bullet}_R : Y^{\bullet}_R \to X^{\bullet}_R$$

over R with Galois group G such that $f^{\bullet}_{k_R} = f^{\bullet}_R \times_R k_R$.

Proof. The smooth pointed stable curve $\widetilde{Y}^{\bullet}_{K_R,v_Y}$ over K_R determines a morphism

$$c_{v_Y} : \operatorname{Spec} K_R \to \mathcal{M}_{g_{v_Y},n_{v_Y}}.$$

Write $c_{Y_{K_R}} : \operatorname{Spec} K_R \to \overline{\mathcal{M}}_{g_Y,n_Y}$ for the morphism determined by $Y^{\bullet}_{K_R}$ over K_R, where (g_Y, n_Y) denotes the type of $Y^{\bullet}_{K_R}$. Then the pointed stable curve $Y^{\bullet}_{K_R}$ determines a clutching morphism

$$\kappa_{Y_{K_R}} : \bigtimes_{v_Y \in v(\Gamma_{Y^{\bullet}_{K_R}})} \mathcal{M}_{g_{v_Y},n_{v_Y}} \to \overline{\mathcal{M}}_{g_Y,n_Y}$$

such that the composition of the morphisms

$$\kappa_{Y_{K_R}} \circ (\bigtimes_{v_Y \in v(\Gamma_{Y^\bullet_{K_R}})} c_{v_Y}) = c_{Y_{K_R}}.$$

We denote by $Y^\bullet_{R,v_Y}$ the pointed stable model of $Y^\bullet_{K_R,v_Y}$ over R which is a smooth pointed stable curve of type (g_{v_Y}, n_{v_Y}) over R. By applying the clutching morphism $\kappa_{Y_{K_R}}$, we may glue the pointed stable curves $\{Y^\bullet_{R,v_Y}\}_{v_Y \in v(\Gamma_{Y^\bullet_{K_R}})}$ in a way that is compatible with the gluing of $\{Y^\bullet_{K_R,v_Y}\}_{v_Y \in v(\Gamma_{Y^\bullet_{K_R}})}$ that gives rise to $Y^\bullet_{K_R}$. Then we obtain a pointed stable curve $Y^\bullet_R$ over R.

Since $Y^\bullet_{K_R}$ admits an action of G, we obtain an action of G on the pointed stable model $Y^\bullet_R$. Let $Z^\bullet_R \overset{\text{def}}{=} Y^\bullet_R/G$, $f^\bullet_R : Y^\bullet_R \to Z^\bullet_R$ the quotient morphism, $Z^\bullet_{K_R}$ the generic fiber over K_R, and $Z^\bullet_{k_R}$ the special fiber over k_R. [18, R, Appendice, Corollaire] (or [8, Proposition 10.3.48]) implies that $Z^\bullet_R$ is a pointed semi-stable curve over R. Moreover, since $f^\bullet_{K_R}$ is a Galois admissible covering over K_R with Galois group G, $Z^\bullet_{K_R}$ is isomorphic to $X^\bullet_{K_R}$ over K_R.

On the other hand, write $f^{\text{sg}}_{K_R} : \Gamma_{Y^\bullet_{K_R}} \to \Gamma_{X^\bullet_{K_R}}$ for the map of dual semi-graphs induced by $f^\bullet_{K_R}$. Note that, for every $v_X \in v(\Gamma_{X^\bullet_{K_R}})$ and every $v_Y \in (f^{\text{sg}}_{K_R})^{-1}(v_X)$, $f^\bullet_{K_R}$ can be extended to a Galois admissible covering $f^\bullet_{R,v_Y,v_X} : Y^\bullet_{R,v_Y} \to X^\bullet_{R,v_X}$ of smooth pointed stable curves over R with Galois group G. Then we obtain

$$Y^\bullet_{R,v_Y}/G \cong X^\bullet_{R,v_X}$$

over R. This implies that $Z^\bullet_{k_R}$ is a pointed stable curve over k_R. Then we have $X^\bullet_R \cong Z^\bullet_R$ over R. We complete the proof of the lemma. □

Proposition 7.7. *Let $q \in \overline{M}_{g,n}$ be an arbitrary point, V_q the topological closure of q in $\overline{M}_{g,n}$, and $G \in \pi^{\text{adm}}_A(q)$ a finite group. Then there exists a closed point $c \in V^{\text{cl}}_q$ such that Γ_q is isomorphic to Γ_c as dual semi-graphs, and that $G \in \pi^{\text{adm}}_A(c)$.*

Proof. If q is a closed point, then the proposition is trivial. To verify the proposition, we may assume that q is not a closed point. If $q \in M_{g,n}$, then the proposition follows from Lemma 7.3. Then we may assume that $q \in \overline{M}_{g,n} \setminus M_{g,n}$.

Let k_q be an algebraic closure of the residue field $k(q)$ of q. The natural morphism $\operatorname{Spec} k_q \to \operatorname{Spec} k(q) \to \overline{M}_{g,n}$ determines a pointed stable curve $X^\bullet_{k_q}$ over k_q. Let $\widetilde{X}^\bullet_{k_q,v_X}$, $v_X \in v(\Gamma_q)$, be the smooth pointed stable curve of type (g_{v_X}, n_{v_X}) associated to v_X.

Let $Y^\bullet_{k_q}$ be a pointed stable curve of type (g_Y, n_Y) over k_q, $f^\bullet_{k_q} : Y^\bullet_{k_q} \to X^\bullet_{k_q}$ a Galois admissible covering over k_q with Galois group G, $\Gamma_{Y^\bullet_{k_q}}$ the dual semi-graph of $Y^\bullet_{k_q}$, and $f^{\text{sg}}_{k_q} : \Gamma_{Y^\bullet_{k_q}} \to \Gamma_q$ the map of dual semi-graphs induced by $f^\bullet_{k_q}$. For each $v_X \in v(\Gamma_q)$, write $\widetilde{Y}^\bullet_{k_q,v_Y}$, $v_Y \in (f^{\text{sg}}_{k_q})^{-1}(v_X)$, for the smooth pointed stable curve of type (g_{v_Y}, n_{v_Y}) corresponding to v_Y. Then $f^\bullet_{k_q}$ induces a Galois multi-admissible covering

$$f^\bullet_{k_q,v_X} : \bigsqcup_{v_Y \in (f^{\text{sg}}_{k_q})^{-1}(v_X)} \widetilde{Y}^\bullet_{k_q,v_Y} \to \widetilde{X}^\bullet_{k_q,v_X}$$

over k_q with Galois group G. Note that $\bigsqcup_{v_Y \in (f^{\mathrm{sg}}_{k_q})^{-1}(v_X)} \widetilde{Y}^\bullet_{k_q,v_Y}$ admits an action of G induced by the action of G on $Y^\bullet_{k_q}$. This action induces an action of G on the set $(f^{\mathrm{sg}}_{k_q})^{-1}(v_X)$. For each $v_Y \in (f^{\mathrm{sg}}_{k_q})^{-1}(v_X)$, write G_{v_Y} for the inertia subgroup of v_Y. Then we obtain a Galois admissible covering

$$f^\bullet_{k_q,v_Y,v_X} : \widetilde{Y}^\bullet_{k_q,v_Y} \to \widetilde{X}^\bullet_{k_q,v_X},\ v_Y \in (f^{\mathrm{sg}}_{k_q})^{-1}(v_X),$$

over k_q with Galois group G_{v_Y}.

The pointed stable curves $X^\bullet_{k_q}$, $\{\widetilde{X}^\bullet_{k_q,v_X}\}_{v_X \in v(\Gamma_q)}$, $Y^\bullet_{k_q}$, and $\{\widetilde{Y}^\bullet_{k_q,v_Y}\}_{v_Y \in v(\Gamma_{Y^\bullet_{k_q}})}$ over k_q determine morphisms $c_{X_{k_q}} : \operatorname{Spec} k_q \to \overline{\mathcal{M}}_{g,n}$, $\{c_{v_X} : \operatorname{Spec} k_q \to \mathcal{M}_{g_{v_X},n_{v_X}}\}_{v_X \in v(\Gamma_q)}$, $c_{Y_{k_q}} : \operatorname{Spec} k_q \to \overline{\mathcal{M}}_{g_Y,n_Y}$, and $\{c_{v_Y} : \operatorname{Spec} k_q \to \mathcal{M}_{g_{v_Y},n_{v_Y}}\}_{v_Y \in v(\Gamma_{Y^\bullet_{k_q}})}$, respectively. Then the pointed stable curves $X^\bullet_{k_q}$ and $Y^\bullet_{k_q}$ over k_q induce clutching morphisms

$$\kappa_{X_{k_q}} : \bigtimes_{v_X \in v(\Gamma_q)} \mathcal{M}_{g_{v_X},n_{v_X}} \to \overline{\mathcal{M}}_{g,n},$$

$$\kappa_{Y_{k_q}} : \bigtimes_{v_Y \in v(\Gamma_{Y^\bullet_{k_q}})} \mathcal{M}_{g_{v_Y},n_{v_Y}} \to \overline{\mathcal{M}}_{g_Y,n_Y},$$

respectively, such that

$$\kappa_{X_{k_q}} \circ (\bigtimes_{v_X \in v(\Gamma_q)} c_{v_X}) = c_{X_{k_q}},$$

$$\kappa_{Y_{k_q}} \circ (\bigtimes_{v_Y \in v(\Gamma_{Y^\bullet_{k_q}})} c_{v_Y}) = c_{Y_{k_q}}.$$

On the other hand, the smooth pointed stable curve $\widetilde{X}^\bullet_{k_q,v_X}$, $v_X \in v(\Gamma_q)$, over k_q determines a morphism $\operatorname{Spec} k_q \to M_{g_{v_X},n_{v_X}}$, and we denote by $q_{v_X} \in M_{g_{v_X},n_{v_X}}$ the image of the morphism. Write $V^{\mathrm{sm}}_{q_{v_X}}$ for the topological closure of q_{v_X} in $M_{g_{v_X},n_{v_X}}$. Let $k_{q_{v_X}}$ be an algebraic closure of the residue field $k(q_{v_X})$ of q_{v_X}. Since the admissible coverings over algebraically closed fields do not depend on the choices of base fields, for each $v_Y \in (f^{\mathrm{sg}}_{k_q})^{-1}(v_X)$, $f^\bullet_{k_q,v_Y,v_X}$ induces a G_{v_Y}-Galois admissible covering

$$f^\bullet_{k_{q_{v_X}},v_Y,v_X} : \widetilde{Y}^\bullet_{k_{q_{v_X}},v_Y} \to \widetilde{X}^\bullet_{k_{q_{v_X}},v_X}$$

over $k_{q_{v_X}}$. Then Corollary 7.4 implies that there exist a smooth k-variety $U_{q_{v_X}}$ and a finite morphism $U_{q_{v_X}} \to H_{g_{v_X},n_{v_X}}$ (not necessary a surjection) such that the following conditions are satisfied:

(i) the image of $U_{q_{v_X}}$ of the composition of the morphisms $U_{q_{v_X}} \to H_{g_{v_X},n_{v_X}} \stackrel{\omega^{(m)}_{g_{v_X},n_{v_X}}}{\to} M_{g_{v_X},n_{v_X}}$ is an open subset of $V^{\mathrm{sm}}_{q_{v_X}}$;

(ii) the morphism $U_{q_{v_X}} \to H_{g_{v_X},n_{v_X}}$ induces a smooth pointed stable curve

$$X^\bullet_{U_{q_{v_X}},v_X} \stackrel{\mathrm{def}}{=} X^\bullet_{H_{g_{v_X},n_{v_X}}} \times_{H_{g_{v_X},n_{v_X}}} U_{q_{v_X}}$$

over $U_{q_{v_X}}$ with a level m-structure $\tau_{U_{q_{v_X}}} \stackrel{\mathrm{def}}{=} \tau_{H_{g_{v_X},n_{v_X}}} \times_{H_{g_{v_X},n_{v_X}}} U_{q_{v_X}}$;

(iii) for each $v_Y \in (f^{\mathrm{sg}}_{k_q})^{-1}(v_X)$, there exists a Galois admissible covering

$$f^\bullet_{U_{q_{v_X}}, v_Y, v_X} : Y^\bullet_{U_{q_{v_X}}, v_Y} \to X^\bullet_{U_{q_{v_X}}, v_X}$$

of smooth pointed stable curves over $U_{q_{v_X}}$ with Galois group G_{v_Y} such that $f^\bullet_{U_{q_{v_X}}, v_Y, v_X} \times_{U_{q_{v_X}}} \operatorname{Spec} k_{q_{v_X}}$ is equal to $f^\bullet_{k_{q_{v_X}}, v_X, v_Y}$.

Then the clutching morphism induces a morphism

$$\kappa : \bigtimes_{v_X \in v(\Gamma_q)} U_{q_{v_X}} \to \bigtimes_{v_X \in v(\Gamma_q)} H_{g_{v_X}, n_{v_X}} \to \bigtimes_{v_X \in v(\Gamma_q)} \mathcal{M}_{g_{v_X}, n_{v_X}} \overset{\kappa_{X_{k_q}}}{\to} \overline{\mathcal{M}}_{g,n} \overset{\varpi_{g,n}}{\to} \overline{M}_{g,n}$$

over k. Since the image of κ is a dense constructible subset of V_q, the image of κ contains an open subset

$$W \subseteq V_q.$$

Let c be a closed point of W. Note that Γ_c is isomorphic to Γ_q as dual semi-graphs. Then Lemma 7.5 implies that there exist a complete discrete valuation ring R with algebraically closed residue field and a morphism $\operatorname{Spec} R \to W$ such that the image of the morphism is $\{q, c\}$. By replacing R by a finite extension of R, there is a pointed stable curve $X^\bullet_R$ over R. Write K_R for the quotient field of R, $\overline{K}_R$ for an algebraic closure of K_R, and k_R for the residue field of R. Moreover, we may assume that $\overline{K}_R$ contains k_q. For each $v_X \in v(\Gamma_q)$, the smooth pointed stable curve

$$\widetilde{X}^\bullet_{\overline{K}_R, v_X} \overset{\mathrm{def}}{=} \widetilde{X}^\bullet_{k_q, v_X} \times_{k_q} \overline{K}_R$$

of type (g_{v_X}, n_{v_X}) over $\overline{K}_R$ determines a morphism

$$\operatorname{Spec} \overline{K}_R \to \mathcal{M}_{g_{v_X}, n_{v_X}} \to M_{g_{v_X}, n_{v_X}}.$$

Let $\operatorname{Spec} \overline{K}_R \to H_{g_{v_X}, n_{v_X}}$ be a morphism obtained by restricting the natural morphism

$$\bigsqcup \operatorname{Spec} \overline{K}_R = \operatorname{Spec} \overline{K}_R \times_{M_{g_{v_X}, n_{v_X}}} H_{g_{v_X}, n_{v_X}} \to H_{g_{v_X}, n_{v_X}}$$

on a connected component of $\operatorname{Spec} \overline{K}_R \times_{M_{g_{v_X}, n_{v_X}}} H_{g_{v_X}, n_{v_X}}$ such that the image of $\operatorname{Spec} \overline{K}_R \to H_{g_{v_X}, n_{v_X}}$ is contained in the image of $U_{q_{v_X}} \to H_{g_{v_X}, n_{v_X}}$. The morphism $\operatorname{Spec} \overline{K}_R \to H_{g_{v_X}, n_{v_X}}$ above induces a level m-structure $\tau_{\overline{K}_R} \overset{\mathrm{def}}{=} \tau_{H_{g_{v_X}, n_{v_X}}} \times_{H_{g_{v_X}, n_{v_X}}} \operatorname{Spec} \overline{K}_R$.

By replacing R by a finite extension of R, $\widetilde{X}^\bullet_{\overline{K}_R, v_X}$ descends to a smooth pointed stable curve

$$\widetilde{X}^\bullet_{K_R, v_X}$$

over K_R, and the level m-structure $\tau_{\overline{K}_R}$ descends to a level m-structure τ_{K_R} on the smooth pointed stable curve $\widetilde{X}^\bullet_{K_R, v_X}$ over K_R. Let

$$X^\bullet_{R, v_X}$$

be the pointed stable model of $\widetilde{X}^\bullet_{K_R, v_X}$ over R. Note that, by the construction, $X^\bullet_{R, v_X}$ is smooth over R. Then the level m-structure τ_{K_R} extends to a level m-structure τ_R.

Thus, for each $v_X \in v(\Gamma_q)$, the smooth pointed stable curve $X^\bullet_{R,v_X}$ over R with the level m-structure τ_R determines a morphism $\mathrm{Spec}\, R \to H_{g_{v_X},n_{v_X}}$ such that the image of the composition of the morphisms

$$\mathrm{Spec}\, R \to \bigtimes_{v_X \in v(\Gamma_q)} H_{g_{v_X},n_{v_X}} \to \bigtimes_{v_X \in v(\Gamma_q)} \mathcal{M}_{g_{v_X},n_{v_X}} \overset{\kappa_{X_{k_q}}}{\to} \overline{\mathcal{M}}_{g,n} \overset{\overline{\omega}_{g,n}}{\to} \overline{M}_{g,n}$$

is $\{q, c\}$. Moreover, since the morphism $U_{q_{v_X}} \to H_{g_{v_X},n_{v_X}}$, $v_X \in v(\Gamma_q)$, is finite, by replacing R by a finite extension of R, we may assume that the morphism $\mathrm{Spec}\, R \to H_{g_{v_X},n_{v_X}}$ obtained above factors through the morphism $U_{q_{v_X}} \to H_{g_{v_X},n_{v_X}}$. Thus, for each $v_X \in v(\Gamma_q)$ and each $v_Y \in (f^{\mathrm{sg}}_{k_q})^{-1}(v_X)$, we obtain a Galois covering

$$f^\bullet_{R,v_Y,v_X} \overset{\mathrm{def}}{=} f^\bullet_{U_{q_{v_X}},v_Y,v_X} \times_{U_{q_{v_X}}} \mathrm{Spec}\, R : Y^\bullet_{R,v_Y} \overset{\mathrm{def}}{=} Y^\bullet_{U_{q_{v_X}},v_Y} \times_{U_{q_{v_X}}} \mathrm{Spec}\, R$$

$$\to X^\bullet_{R,v_X} \overset{\mathrm{def}}{=} X^\bullet_{U_{q_{v_X}},v_X} \times_{U_{q_{v_X}}} \mathrm{Spec}\, R$$

of smooth pointed stable curves over R with Galois group G_{v_Y}. Moreover, the clutching morphism $\kappa_{Y_{k_q}}$ implies that we may glue $\{Y^\bullet_{R,v_Y}\}_{v_Y \in v(\Gamma_{Y^\bullet_{k_q}})}$ in a way that is compatible with the gluing of $\{Y^\bullet_{k_q,v_Y}\}_{v_Y \in v(\Gamma_{Y^\bullet_{k_q}})}$ that gives rise to $Y^\bullet_{k_q}$. Then we obtain a pointed stable curve $Y^\bullet_R$ over R such that the following conditions are satisfied: (i) $Y^\bullet_R \times_{K_R} \overline{K}_R \cong Y^\bullet_{k_q} \times_{k_q} \overline{K}_R$ over $\overline{K}_R$; (ii) there exists a Galois admissible covering $f^\bullet_{K_R} : Y^\bullet_{K_R} \to X^\bullet_{K_R}$ over K_R with Galois group G such that $f^\bullet_{K_R} \times_{K_R} \overline{K}_R = f^\bullet_{k_q} \times_{k_q} \overline{K}_R$.

Then by applying Lemma 7.6, there exists a Galois admissible covering $f^\bullet_R : Y^\bullet_R \to X^\bullet_R$ over R with Galois group G such that the restriction of $f^\bullet_R$ on the special fibers is a connected Galois admissible covering over k_R with Galois group G. This means that $G \in \pi^{\mathrm{adm}}_A(c)$. We complete the proof of the proposition. □

7.3 Continuity of $\pi^{\mathrm{adm}}_{g,n}$

In this subsection, we prove the continuity of $\pi^{\mathrm{adm}}_{g,n}$.

Lemma 7.8. *Let v be a closed point of $H_{g,n}$, $\widehat{\mathcal{O}}_{H_{g,n},v}$ the completion of the local ring $\mathcal{O}_{H_{g,n},v}$, $\widehat{V} \overset{\mathrm{def}}{=} \mathrm{Spec}\, \widehat{\mathcal{O}}_{H_{g,n},v}$ with the natural morphism $\widehat{V} \to H_{g,n}$, and $X^\bullet_{\widehat{V}} \overset{\mathrm{def}}{=} X^\bullet_{H_{g,n}} \times_{H_{g,n}} \widehat{V}$ the smooth pointed stable curve over $\widehat{V}$ with a level m-structure $\tau_{\widehat{V}} \overset{\mathrm{def}}{=} \tau_{H_{g,n}} \times_{H_{g,n}} \widehat{V}$. Let $Y^\bullet_{\widehat{V}}$ be a smooth pointed stable curve over $\widehat{V}$ and*

$$f^\bullet_{\widehat{V}} : Y^\bullet_{\widehat{V}} \to X^\bullet_{\widehat{V}}$$

a Galois admissible covering over $\widehat{V}$ with Galois group G. Then there exist a subring $A \subseteq \widehat{\mathcal{O}}_{H_{g,n},v}$, a morphism $\alpha_E : E \overset{\mathrm{def}}{=} \mathrm{Spec}\, A \to H_{g,n}$, and a morphism $f^\bullet_E : Y^\bullet_E \to X^\bullet_E \overset{\mathrm{def}}{=} X^\bullet_{H_{g,n}} \times_{H_{g,n}} E$ such that the following conditions are satisfied:

(i) $X^\bullet_E \times_E \widehat{V}$ is isomorphic to $X^\bullet_{\widehat{V}}$ over $\widehat{V}$, and the pulling-back of $f^\bullet_E \times_E \widehat{V}$ via the natural morphism $\widehat{V} \to E$ is isomorphic to $f^\bullet_{\widehat{V}}$ over $\widehat{V}$;

(ii) $f^\bullet_E$ is a Galois admissible covering with Galois group G over E.

Proof. By applying [32, Proposition 4.3 (1)], there exists a subring $A' \subseteq \widehat{\mathcal{O}}_{H_{g,n},v}$ which is of finite type over k such that $Y^\bullet_{E'}$ is smooth over $E' \stackrel{\text{def}}{=} \operatorname{Spec} A'$, that the Galois admissible covering $f^\bullet_{\widehat{V}}$ can be descended to a finite morphism (a Galois covering) $f^\bullet_{E'}$: $Y^\bullet_{E'} \to X^\bullet_{E'}$ over $E' \stackrel{\text{def}}{=} \operatorname{Spec} A'$ with a level m-structure $\tau_{E'}$ on $X^\bullet_{E'}$, and that $f^\bullet_{E'}|_{e'}$ is a Galois multi-admissible covering with Galois group G over every $e' \in E'$. Moreover, by the construction, the pulling-back $f^\bullet_{E'} \times_{E'} \widehat{V}$ via $\widehat{V} \to E'$ is isomorphic to $f^\bullet_{\widehat{V}}$ over $\widehat{V}$. Moreover, the smooth pointed stable curve $X^\bullet_{E'}$ over E' with the level m-structure $\tau_{E'}$ determines a morphism $\alpha_{E'} : E' \to H_{g,n}$.

We denote by $v_{E'} \in E'$ the image of $v \in \widehat{V}$ via the natural morphism $\widehat{V} \to E'$ which is a closed point of E'. [5, Proposition 5] implies that, there exists an affine open subset $v_{E'} \in E \subseteq E'$ such that the fiber $Y^\bullet_e \stackrel{\text{def}}{=} Y^\bullet_E \times_E e$ is geometrically irreducible over each *closed* point $e \in E$. We shall put $A \stackrel{\text{def}}{=} \mathcal{O}_E(E) \subseteq \widehat{\mathcal{O}}_{H_{g,n},v}$. Moreover, since the underlying curve of $Y^\bullet_e \stackrel{\text{def}}{=} Y^\bullet_E \times_E e$ is smooth over each e, we have that $Y^\bullet_e$ is geometrically irreducible over each point $e \in E$. Thus, for each point $e \in E$, the restriction of $f^\bullet_E \stackrel{\text{def}}{=} f^\bullet_{E'} \times_{E'} E$ on e is a Galois admissible covering over e with Galois group G. We put

$$\alpha_E \stackrel{\text{def}}{=} \alpha_{E'}|_E : E \to H_{g,n}.$$

Then we obtain the desired curve. This completes the proof of the lemma. □

Definition 7.9. Let $q \in \overline{M}_{g,n}$ be an arbitrary point. For each $G \in \pi_A^{\text{adm}}(q)$, we define

$$U_G \stackrel{\text{def}}{=} \{q' \in \overline{M}_{g,n} \mid G \in \pi_A^{\text{adm}}(q')\}.$$

Moreover, we put $U_G^{\text{sm}} \stackrel{\text{def}}{=} U_G \cap M_{g,n}$.

First, let us prove that U_G^{sm} is an open subset of $M_{g,n}$.

Proposition 7.10. *Let q be an arbitrary point of $M_{g,n}$ and $G \in \pi_A^{\text{adm}}(q)$. Then U_G^{sm} is an open subset of $M_{g,n}$.*

Proof. To verify the proposition, Lemma 7.3 (or Proposition 7.7) implies that it is sufficient to prove that, for every *closed point* $c \in U_G^{\text{sm}}$, there exists an open subset $c \in U_c \subseteq M_{g,n}$ which is contained in U_G^{sm}.

Let $v \in H_{g,n}$ be a closed point such that $\omega_{g,n}^{(m)}(v) = c$. We maintain the notation introduced in Lemma 7.8. Then we obtain an affine k-variety E and a morphism α_E : $E \to H_{g,n}$ over k such that $(\omega_{g,n}^{(m)} \circ \alpha_E)(v_{E'}) = c$. Moreover, since the image $\widehat{V}$ of the composition of the morphisms $\widehat{V} \to E \stackrel{\alpha_E}{\to} H_{g,n} \stackrel{\omega_{g,n}^{(m)}}{\to} M_{g,n}$ is dense in $M_{g,n}$, the image of the composition of the morphisms $E \stackrel{\alpha_E}{\to} H_{g,n} \stackrel{\omega_{g,n}^{(m)}}{\to} M_{g,n}$ is also a dense constructible subset of $M_{g,n}$.

Write W for the image of E in $M_{g,n}$. Since W is a constructible subset, we have that

$$W = \bigcup_{i=1}^{r} W_i$$

is a finite disjoint union of local closed subsets $\{W_i\}_{i=1,\dots,r}$, of $M_{g,n}$. Without loss of generality, we may assume that $c \in W_1$. Since W_1 contains the image of $\widehat{V}$, we obtain that W_1 is an open subset of $M_{g,n}$. This completes the proof of the proposition. $\square$

Remark 7.10.1. In [20, Section 4], Stevenson proved that U_G^{sm} contains an open subset of $M_{g,n}$ when $n = 0$.

In the remainder of this subsection, we generalize Proposition 7.10 to the case of arbitrary points of $\overline{M}_{g,n}$.

Lemma 7.11. *Let R be a complete discrete valuation ring, K_R the quotient field of R, and k_R the residue field of R such that k_R is an algebraically closed field containing k. Let $X_R^\bullet$ be a pointed stable curve of type (g, n) over R and*

$$f_{k_R}^\bullet : Y_{k_R}^\bullet \to X_{k_R}^\bullet$$

a Galois admissible covering over k_R with Galois group G. Then, by replacing R by a finite extension of R, there exist a pointed stable curve $Y_R^\bullet$ over R and a Galois admissible covering

$$f_R^\bullet : Y_R^\bullet \to X_R^\bullet$$

over R with Galois group such that $f_R^\bullet \times_R k_R = f_{k_R}^\bullet$.

Proof. Let $X_{\mathcal{M}'}^\bullet$ be the versal formal deformation of the special fiber $X_{k_R}^\bullet$ of $X_R^\bullet$ over $\mathcal{M}' \stackrel{\mathrm{def}}{=} \operatorname{Spec} \mathcal{O}_{k_R}[[t_1, \dots, t_{3g-3+n}]]$, where $\mathcal{O}_{k_R}$ is a regular local ring with maximal ideal $p\mathcal{O}_{k_R}$ and residue field k_R (cf. [1, p79]). The pointed stable curve $X_R^\bullet$ over R determines a morphism $\operatorname{Spec} R \to \mathcal{M}'$ such that $X_{\mathcal{M}'}^\bullet \times_{\mathcal{M}'} \operatorname{Spec} R$ is isomorphic to $X_R^\bullet$ over R. Moreover, since $R \cong k_R[[t]]$, the morphism $\operatorname{Spec} R \to \mathcal{M}'$ factors through a morphism

$$\operatorname{Spec} R \to \mathcal{M} \stackrel{\mathrm{def}}{=} \operatorname{Spec} k_R[[t_1, \dots, t_{3g-3+n}]].$$

The natural morphism $\mathcal{M} \to \mathcal{M}'$ induces a pointed stable curve $X_{\mathcal{M}}^\bullet \stackrel{\mathrm{def}}{=} X_{\mathcal{M}'}^\bullet \times_{\mathcal{M}'} \mathcal{M}$ over $\mathcal{M}$.

Let $\overline{\mathcal{M}}_{g,n}^{\log}$ be the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. Then we obtain a log scheme $\mathcal{M}^{\log}$ whose underlying scheme is $\mathcal{M}$, and whose log structure is the pulling-back log structure induced by the natural morphism $\mathcal{M} \to \mathcal{M}' \to \overline{\mathcal{M}}_{g,n}$. Moreover, we obtain a stable log curve

$$X_{\mathcal{M}}^{\log} \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,n+1}^{\log} \times_{\overline{\mathcal{M}}_{g,n}^{\log}} \mathcal{M}^{\log}$$

over $\mathcal{M}^{\log}$ whose underlying curve is $X_{\mathcal{M}}$. Note that $X_{\mathcal{M}}^{\log}$ is log regular.

By replacing $\mathcal{M}^{\log}$ by a finite log étale covering $\mathcal{N}^{\log}$, and replacing R by a finite extension of R, we obtain a morphism $\operatorname{Spec} R \to \mathcal{N}$ induced by the morphism $\operatorname{Spec} R \to \mathcal{M}$. We obtain a log scheme $s_{k_R}^{\log}$ whose underlying scheme is $s_{k_R} \stackrel{\mathrm{def}}{=} \operatorname{Spec} k_R$, and whose log structure is the pulling-back log structure induced by the composition of the morphisms $s_{k_R} \to \operatorname{Spec} R \to \mathcal{N}$. Moreover, the Galois admissible covering $f_{k_R}^\bullet$ determines a log étale covering $f_{k_R}^{\log}: Y_{k_R}^{\log} \to X_{k_R}^{\log}$ over $s_{k_R}^{\log}$ such that the underlying morphism of $f_{k_R}^{\log}$ is f_{k_R}. By applying [6, Corollary 1], there exists a Galois log étale covering

$$f_{\mathcal{N}}^{\log}: Y_{\mathcal{N}}^{\log} \to X_{\mathcal{N}}^{\log} \stackrel{\text{def}}{=} X_{\mathcal{M}}^{\log} \times_{\mathcal{M}^{\log}} \mathcal{N}^{\log}$$

with Galois group G over $\mathcal{N}^{\log}$ such that

$$f_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_{k_R}^{\log}: Y_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_{k_R}^{\log} \to X_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_{k_R}^{\log}$$

is isomorphic to $f_{k_R}^{\log}$ over $s_{k_R}^{\log}$. Furthermore, by replacing $\mathcal{N}^{\log}$ by a finite log étale covering of $\mathcal{N}^{\log}$, we may assume that the underlying morphism of $f_{\mathcal{N}}^{\log}$ is a morphism of pointed stable curves over $\mathcal{N}$.

Let $s_R^{\log}$ be the log scheme whose underlying scheme is $\operatorname{Spec} R$, and whose log structure is the pulling-back log structure induced by the morphism $\operatorname{Spec} R \to \mathcal{N}$. Then we obtain a log étale covering

$$f_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_R^{\log}: Y_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_R^{\log} \to X_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_R^{\log}$$

over $s_R^{\log}$. We denote by

$$f_R^{\bullet}: Y_R^{\bullet} \to X_R^{\bullet}$$

the morphism induced by the underlying morphism of $f_{\mathcal{N}^{\log}}^{\log} \times_{\mathcal{N}^{\log}} s_R^{\log}$ over R. Since the special fiber $Y_R^{\bullet}$ is geometrically connected, the Zariski main theorem implies that $Y_R^{\bullet} \times_R R'$ is connected for every finite extension R' of R. Thus, the generic fiber of $Y_R^{\bullet}$ is geometrically connected.

Let us prove that $f_R^{\bullet}$ is a Galois admissible covering over R with Galois group G. Note that we have a log scheme $s_{K_R}^{\log}$ whose underlying scheme is $s_{K_R} \stackrel{\text{def}}{=} \operatorname{Spec} K_R$, and whose log structure is the pulling-back log structure induced by the composition of the natural morphisms $s_{K_R} \to \operatorname{Spec} R \to \mathcal{N}$. Then we see that

$$f_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log}: Y_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log} \to X_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log}$$

is geometrically connected Galois log étale covering over $s_{K_R}^{\log}$. This means that the underlying morphism of $f_{\mathcal{N}}^{\log} \times_{\mathcal{N}^{\log}} s_{K_R}^{\log}$ induces a Galois admissible covering over K_R with Galois group G. This completes the proof of the lemma. □

Let $c \in \overline{M}_{g,n}$ be a *closed point* and $k_c \stackrel{\text{def}}{=} k$ the residue field of c. Then the closed point c determines a pointed stable curve

$$X_{k_c}^{\bullet} = (X_{k_c}, D_{X_{k_c}})$$

over k. For each $v_X \in v(\Gamma_c)$, let $\widetilde{X}_{k_c, v_X}^{\bullet}$ be the smooth pointed stable curve of type (g_{v_X}, n_{v_X}) over k_c associated to v_X. Then we obtain a morphism

$$c_{v_X}: \operatorname{Spec} k_c \to \mathcal{M}_{g_{v_X}, n_{v_X}}, \; v_X \in v(\Gamma_c),$$

determined by $\widetilde{X}^\bullet_{k_c,v_X}$ over k_c. Write $c_{X_{k_c}} : \operatorname{Spec} k_c \to \overline{\mathcal{M}}_{g,n}$ for the morphism induced by $X^\bullet_{k_c}$ over k_c. Moreover, $X^\bullet_{k_c}$ over k_c determines a clutching morphism

$$\kappa_{X_{k_c}} : \bigtimes_{v_X \in v(\Gamma_c)} \mathcal{M}_{g_{v_X},n_{v_X}} \to \overline{\mathcal{M}}_{g,n}$$

satisfying $\kappa_{X_{k_c}} \circ (\bigtimes_{v_X \in v(\Gamma_c)} c_{v_X}) = c_{X_{k_c}}$. We denote by

$$M_c \overset{\text{def}}{=} \operatorname{Im}(\bigtimes_{v \in v(\Gamma_c)} \mathcal{M}_{g_v,n_v} \overset{\kappa_{X_{k_c}}}{\to} \overline{\mathcal{M}}_{g,n} \overset{\overline{\omega}_{g,n}}{\to} \overline{M}_{g,n})$$

the image of the composition of the natural morphisms.

Lemma 7.12. *We maintain the notation introduced above. Let $G \in \pi_A^{\text{adm}}(c)$ be a finite group. Then*

$$U_G \cap M_c$$

contains an open subset of M_c in which c is contained.

Proof. Let $Y^\bullet_{k_c} = (Y_{k_c}, D_{Y_{k_c}})$ be a pointed stable curve of type (g_Y, n_Y) over k_c and $f^\bullet_{k_c} : Y^\bullet_{k_c} \to X^\bullet_{k_c}$ a Galois admissible covering over k_c with Galois group G. Write $\Gamma_{Y^\bullet_{k_c}}$ for the dual semi-graph of $Y^\bullet_{k_c}$, and $f^{\text{sg}}_{k_c} : \Gamma_{Y^\bullet_{k_c}} \to \Gamma_c$ for the map of dual semi-graphs induced by $f^\bullet_{k_c}$. For every $v_X \in v(\Gamma_c)$ and every $v_Y \in (f^{\text{sg}}_{k_c})^{-1}(v_X)$, let $\widetilde{Y}^\bullet_{k_c,v_Y}$ be the smooth pointed stable curve of type (g_{v_Y}, n_{v_Y}) over k_c associated to v_Y. Then $f^\bullet_{k_c}$ induces a Galois multi-admissible covering

$$f^\bullet_{k_c,v_X} : \bigsqcup_{v_Y \in (f^{\text{sg}}_{k_c})^{-1}(v_X)} \widetilde{Y}^\bullet_{k_c,v_Y} \to \widetilde{X}^\bullet_{k_c,v_X}$$

over k_c. Note that $\bigsqcup_{v_Y \in (f^{\text{sg}}_{k_c})^{-1}(v_X)} \widetilde{Y}^\bullet_{k_c,v_Y}$ admits an action of G induced by the action of G on $Y^\bullet_{k_c}$. This action induces an action of G on the set $(f^{\text{sg}}_{k_c})^{-1}(v_X)$. For each $v_Y \in (f^{\text{sg}}_{k_c})^{-1}(v_X)$, write $G_{v_Y} \subseteq G$ for the inertia subgroup of v_Y. Then we obtain a Galois admissible covering

$$f^\bullet_{k_c,v_Y,v_X} : \widetilde{Y}^\bullet_{k_c,v_Y} \to \widetilde{X}^\bullet_{k_c,v_X}$$

over k with Galois group G_{v_Y}. Write $c_{Y_{k_c}} : \operatorname{Spec} k_c \to \overline{\mathcal{M}}_{g_Y,n_Y}$ for the morphism determined by $Y^\bullet_{k_c}$ over k_c, and $c_{v_Y} : \operatorname{Spec} k_c \to \overline{\mathcal{M}}_{g_{v_Y},n_{v_Y}}$, $v_Y \in v(\Gamma_{Y^\bullet_{k_c}})$, for the morphism determined by $\widetilde{Y}^\bullet_{k_c,v_Y}$ over k_c. Then the pointed stable curve $Y^\bullet_{k_c}$ over k_c determines a clutching morphism

$$\kappa_{Y_{k_c}} : \bigtimes_{v_Y \in v(\Gamma_{Y^\bullet_{k_c}})} \mathcal{M}_{g_{v_Y},n_{v_Y}} \to \overline{\mathcal{M}}_{g_Y,n_Y}$$

satisfying $\kappa_{Y_{k_c}} \circ (\bigtimes_{v_Y \in v(\Gamma_{Y^\bullet_{k_c}})} c_{v_Y}) = c_{Y_{k_c}}$.

On the other hand, for each $v_X \in v(\Gamma_c)$, we denote by

$$q_{v_X} \in M_{g_{v_X},n_{v_X}}$$

the image of $\omega_{g_{v_X},n_{v_X}} \circ c_{v_X}$. Then the last paragraph of the proof of Proposition 7.10 implies that, for each $v_X \in v(\Gamma_c)$, there exist an affine k-variety $E_{q_{v_X}}$ and a morphism $\alpha_{E_{q_{v_X}}} : E_{q_{v_X}} \to H_{g_{v_X},n_{v_X}}$ satisfying the following conditions:

(i) the image of $\alpha_{E_{q_{v_X}}}$ contains an open subset $U_{q_{v_X}}$ of $H_{g_{v_X},n_{v_X}}$ such that the image $\omega^{(m)}_{g_{v_X},n_{v_X}}(U_{q_{v_X}}) \subseteq M_{g_{v_X},n_{v_X}}$ contains q_{v_X};

(ii) we have a smooth pointed stable curve $X^{\bullet}_{E_{q_{v_X}}} \stackrel{\text{def}}{=} X^{\bullet}_{H_{g_{v_X},n_{v_X}}} \times_{H_{g_{v_X},n_{v_X}}} E_{q_{v_X}}$ over $E_{q_{v_X}}$ with a level m-structure $\tau_{E_{q_{v_X}}} \stackrel{\text{def}}{=} \tau_{H_{g_v,n_v}} \times_{H_{g_{v_X},n_{v_X}}} E_{q_{v_X}}$;

(iii) for each $v_Y \in (f^{\text{sg}}_{k_c})^{-1}(v_X)$, there exists a Galois admissible covering

$$f^{\bullet}_{E_{q_{v_X}},v_Y,v_X} : Y^{\bullet}_{E_{q_{v_X}},v_Y} \to X^{\bullet}_{E_{q_{v_X}},v_X}$$

over $E_{q_{v_X}}$ with Galois group G_{v_Y} such that the pulling-back of $f^{\bullet}_{E_{q_{v_X}},v_Y,v_X}$ to every point of $(\omega^{(m)}_{g_{v_X},n_{v_X}} \circ \alpha_{E_{q_{v_X}}})^{-1}(q_{v_X})$ is isomorphic to the Galois admissible covering $f^{\bullet}_{k_c,v_Y,v_X}$ over k_c with Galois group G_{v_X}.

Then the image of the composition of the natural morphisms

$$\bigtimes_{v_X \in v(\Gamma_c)} U_{q_{v_X}} \hookrightarrow \bigtimes_{v_X \in v(\Gamma_c)} H_{g_{v_X},n_{v_X}} \to \bigtimes_{v_X \in v(\Gamma_c)} \mathcal{M}_{g_{v_X},n_{v_X}} \stackrel{\kappa_{X_{k_c}}}{\to} \overline{\mathcal{M}}_{g,n} \stackrel{\overline{\omega}_{g,n}}{\to} \overline{M}_{g,n}$$

contains an open subset $W_c \subseteq M_c$ in which c is contained. To verify the lemma, it is sufficient to prove that $G \in \pi^{\text{adm}}_A(c')$ for each *closed point* $c' \in W_c$.

Since W_c is a k-variety, there exists a k-curve $C' \subseteq W_c$ which contains c and c'. Write C for the normalization of C', c_1 for a closed point of C over c, and c_2 for a closed point of C over c'. Let R_i, $i \in \{1,2\}$, be a finite extension of $\widehat{\mathcal{O}}_{C,c_i}$ (then R_i is a complete discrete valuation ring), K_{R_i} the quotient field of R_i, $\overline{K}_{R_i}$ an algebraic closure of K_{R_i}, and $k_{R_i} = k$ the residue field of R_i.

By replacing R_1 by a finite extension of R_1, there is a smooth pointed stable curve $X^{\bullet}_{R_1}$ over R_1 whose special fiber $X^{\bullet}_{k_{R_1}}$ over the residue field $k_{R_1} = k$ of R_1 is isomorphic to $X^{\bullet}_{k_c}$ over k. Lemma 7.11 implies that the Galois admissible covering $f^{\bullet}_{k_c}$ over $k_c = k$ with Galois group G can be lifted to a Galois admissible covering

$$f^{\bullet}_{R_1} : Y^{\bullet}_{R_1} \to X^{\bullet}_{R_1}$$

over R_1 with Galois group G. Moreover, for each $v_X \in v(\Gamma_c)$ and each $v_Y \in (f^{\text{sg}}_{k_c})^{-1}(v_X)$, the Galois admissible covering over k with Galois group G_{v_Y} can be lifted to a Galois admissible covering

$$f^{\bullet}_{R_1,v_Y,v_X} : Y^{\bullet}_{R_1,v_Y} \to X^{\bullet}_{R_1,v_X}$$

over R_1 with Galois group G_{v_Y}. Write $q^{(m)}_{v_X} \in U_{q_{v_X}} \subseteq H_{g_{v_X},n_{v_X}}$ for a closed point over q_{v_X}. The level m-structure $\tau_{H_{g_{v_X},n_{v_X}}} \times_{H_{g_{v_X},n_{v_X}}} q^{(m)}_{v_X}$ on the special fiber of $X^{\bullet}_{R_1,v_X}$ can be extended to a level m-structure τ_{R_1,v_X} on $X^{\bullet}_{R_1,v_X}$. Then, for $v_X \in v(\Gamma_c)$, the pointed stable curve $X^{\bullet}_{R_1,v_X}$ with the level m-structure τ_{R_1,v_X} determines a morphism $l_{R_1,v_X} : \operatorname{Spec} R_1 \to H_{g_{v_X},n_{v_X}}$. Thus, $X^{\bullet}_{R_1,v_X}$ is isomorphic to $X^{\bullet}_{H_{g_{v_X},n_{v_X}}} \times_{H_{g_{v_X},n_{v_X}}} \operatorname{Spec} R_1$ over R_1. On the other hand, for each $v_X \in v(\Gamma_c)$ and each $v_Y \in (f^{\text{sg}}_{k_c})^{-1}(v_X)$, $f^{\bullet}_{R_1,v_Y,v_X}$ induces a Galois admissible covering $f_{\overline{K}_{R_1},v_Y,v_X} : Y^{\bullet}_{\overline{K}_{R_1},v_Y} \to X^{\bullet}_{\overline{K}_{R_1},v_X}$ over $\overline{K}_{R_1}$ with Galois group G_{v_Y}.

Let η_{v_X} be a generic point of $E_{q_{v_X}} \times_{H_{g_{v_X},n_{v_X}}} \operatorname{Spec} K_{R_1}$ and $s_{1,v_X} \in E_{q_{v_X}} \times_{H_{g_{v_X},n_{v_X}}} k_{R_1} \hookrightarrow E_{q_{v_X}}$ a closed point contained in $V_{\eta_{v_X}} \overset{\mathrm{def}}{=} \overline{\{\eta_{v_X}\}}$ such that $\alpha_{E_{q_{v_X}}}(s_{1,v_X})$ is equal to the image of

$$\operatorname{Spec} k_{R_1} \hookrightarrow \operatorname{Spec} R_1 \overset{l_{R_1,v_X}}{\to} H_{g_{v_X},n_{v_X}},$$

where $\overline{\{\eta_{v_X}\}}$ denotes the topological closure of η_{v_X} in $E_{q_{v_X}} \times_{H_{g_{v_X},n_{v_X}}} R_1$. Note that since $R_1 \cong k[[t]]$, the scheme-theoretic image of $l_{R_1,v}$ is a local ring of dimension 1. Moreover, since the residue field of η_{v_X} is a finite extension of K_{R_1}, $V_{\eta_{v_X}}$ is an one dimensional k-scheme. Write A_{1,v_X} for the normalization of $\widehat{\mathcal{O}}_{V_{\eta_{v_X}},s_{1,v_X}}$. Note that A_{1,v_X} is a complete discrete valuation ring, and the natural morphism $\operatorname{Spec} A_{1,v_X} \to \operatorname{Spec} R_1$ is finite. Then we may assume that $\overline{K}_{R_1}$ contains $A_{1,v}$. Thus, the geometric generic fiber of the Galois admissible covering

$$f^\bullet_{A_{1,v_X},v_Y,v_X} \overset{\mathrm{def}}{=} f^\bullet_{E_{q_{v_X}},v_Y,v_X} \times_{E_{q_{v_X}}} \operatorname{Spec} A_{1,v_X} : Y^\bullet_{A_{1,v_X},v_Y} \to X^\bullet_{A_{1,v_X},v_X}$$

over $\overline{K}_{R_1}$ with Galois group G_{v_Y} is isomorphic to $f^\bullet_{\overline{K}_{R_1},v_Y,v_X}$ over $\overline{K}_{R_1}$.

Let $\overline{K}$ be an algebraically closed field which contains $\overline{K}_{R_1}$ and $\overline{K}_{R_2}$. Since the admissible fundamental groups do not depend on the choices of base fields, the Galois admissible covering $f^\bullet_{R_1} \times_{R_1} \overline{K}_{R_1}$ over $\overline{K}_{R_1}$ with Galois group G induces a Galois admissible covering $f^\bullet_{\overline{K}_{R_2}} : Y^\bullet_{\overline{K}_{R_2}} \to X^\bullet_{\overline{K}_{R_2}}$ over $\overline{K}_{R_2}$ with Galois group G such that $f^\bullet_{\overline{K}_{R_1}} \times_{\overline{K}_{R_1}} \overline{K}$ is isomorphic to $f^\bullet_{\overline{K}_{R_2}} \times_{\overline{K}_{R_2}} \overline{K}$ over $\overline{K}$. By replacing R_2 by a finite extension of R_2, there is a pointed stable curve $X^\bullet_{R_2}$ over R_2 whose special fiber $X^\bullet_{k_{R_2}}$ over the residue field $k_{R_2} = k$ of R_2 is isomorphic to $X^\bullet_{k_{c'}}$ over k, and $f^\bullet_{\overline{K}_{R_2}}$ can be descended to a Galois admissible covering

$$f^\bullet_{K_{R_2}} : Y^\bullet_{K_{R_2}} \to X^\bullet_{K_{R_2}}$$

over K_{R_2} with Galois group G. Moreover, for each $v_X \in v(\Gamma_c)$ and each $v_Y \in (f^{\mathrm{sg}}_{k_c})^{-1}(v_X)$, $f^\bullet_{K_{R_2}}$ induces a Galois admissible covering $f^\bullet_{K_{R_2},v_Y,v_X} : Y^\bullet_{K_{R_2},v_Y} \to X^\bullet_{K_{R_2},v_X}$ over K_{R_2} with Galois group G_{v_Y}. By choosing a suitable level m-structure τ_{R_2,v_X} on $X^\bullet_{R_2,v_X}$, we obtain a morphism

$$l_{R_2,v_X} : \operatorname{Spec} R_2 \to H_{g_{v_X},n_{v_X}}$$

determined by $X^\bullet_{R_2,v_X}$ over R_2 and τ_{R_2,v_X} such that image is contained in $U_{q_{v_X}}$.

By replacing R_2 by a finite extension of R_2, we may assume that $Y^\bullet_{K_{R_2,v_Y}}$, $v_Y \in v(\Gamma_{Y^\bullet_{k_c}})$, has pointed stable reduction over R_2. Next, let us prove that $Y^\bullet_{K_{R_2,v_Y}}$ has good reduction over R_2.

If the image of l_{R_2,v_X} is a constant morphism, then $Y^\bullet_{K_{R_2,v_Y}}$ has good reduction over R_2. We may assume that l_{R_2,v_X} is not a constant morphism. Let η'_{v_X} be a generic point of $E_{q_{v_X}} \times_{H_{g_{v_X},n_{v_X}}} \operatorname{Spec} K_{R_2}$ and $s_{2,v_X} \in E_{q_{v_X}} \times_{H_{g_{v_X},n_{v_X}}} k_{R_2} \hookrightarrow E_{q_{v_X}}$ a closed point contained in $V_{\eta'_{v_X}} \overset{\mathrm{def}}{=} \overline{\{\eta'_{v_X}\}}$ such that $\alpha_{E_{q_{v_X}}}(s_{2,v_X})$ is equal to the image of

$$\operatorname{Spec} k_{R_2} \hookrightarrow \operatorname{Spec} R_2 \overset{l_{R_2,v_X}}{\to} H_{g_{v_X},n_{v_X}},$$

where $\overline{\{\eta'_{v_X}\}}$ denotes the topological closure of $\eta_{v'_X}$ in $E_{q_{v_X}} \times_{H_{g_{v_X}, n_{v_X}}} R_2$. Since $R_2 \cong k[[t]]$, the scheme-theoretic image of l_{R_2, v_X} is a local ring of dimension 1, $V_{\eta'_v}$ is an one dimension k-scheme. Write A_{2,v_X} for the normalization of $\widehat{\mathcal{O}}_{V_{\eta'_{v_X}}, s_{2,v_X}}$. Note that A_{2,v_X} is a complete discrete valuation ring, and the natural morphism $\operatorname{Spec} A_{2,v_X} \to \operatorname{Spec} R_2$ is finite. We may assume that $A_{2,v_X} \subseteq \overline{K}_{R_2}$. Thus, we obtain a Galois admissible covering

$$f^{\bullet}_{A_{2,v_X}, v_Y, v_X} \stackrel{\text{def}}{=} f^{\bullet}_{E_{q_{v_X}}, v_Y} \times_{E_{q_{v_X}}} \operatorname{Spec} A_{2,v_X} : Y^{\bullet}_{A_{2,v_X}, v_Y} \to X^{\bullet}_{A_{2,v_X}, v_X}$$

of smooth pointed stable curves over A_{2,v_X} with Galois group G_{v_Y} such that $f^{\bullet}_{A_{2,v_X}, v_Y, v_X} \times_{A_{2,v_X}} \overline{K}_{R_2}$ is isomorphic to $f^{\bullet}_{\overline{K}_{R_2}, v_Y, v_X}$ over $\overline{K}_{R_2}$. This implies that $Y^{\bullet}_{K_{R_2, v_Y}}$ has good reduction.

The clutching morphism $\kappa_{Y^{\bullet}_{k_c}}$ implies that we may glue $\{Y^{\bullet}_{R, v_Y}\}_{v_Y \in v(\Gamma_{Y^{\bullet}_{k_c}})}$ in a way that is compatible with the gluing of $\{Y^{\bullet}_{K_{R_2}, v_Y}\}_{v_Y \in v(\Gamma_{Y^{\bullet}_{K_{R_2}}})}$ that gives rise to $Y^{\bullet}_{K_{R_2}}$. Then we obtain a pointed stable curve $Y^{\bullet}_{R_2}$ over R_2 such that

(i) $Y^{\bullet}_{R_2} \times_{R_2} K_{R_2} \cong Y^{\bullet}_{K_{R_2}}$ over K_{R_2};

(ii) there exists a Galois admissible covering $f^{\bullet}_{K_{R_2}} : Y^{\bullet}_{K_{R_2}} \to X^{\bullet}_{K_{R_2}}$ of pointed stable curves over K_{R_2} which is a Galois admissible covering over K_{R_2} with Galois group G such that $f^{\bullet}_{K_{R_2}} \times_{K_{R_2}} \overline{K}$ is isomorphic to $f^{\bullet}_{K_{R_1}} \times_{K_{R_1}} \overline{K}$.

Then Lemma 7.6 implies that there exists a Galois admissible covering $f^{\bullet}_{R_2} : Y^{\bullet}_{R_2} \to X^{\bullet}_{R_2}$ over R_2 with Galois group G such that the restriction of $f^{\bullet}_{R_2}$ on the special fibers is a Galois admissible covering over k_{R_2} with Galois group G. Since $X^{\bullet}_{k_{R_2}}$ is isomorphic to $X^{\bullet}_{k_{c'}}$ over $k = k_{c'}$, we have that $G \in \pi_A^{\text{adm}}(c')$. This completes the proof of the lemma. □

Corollary 7.13. *We maintain the notation introduced in Lemma 7.12. Let $G \in \pi_A^{\text{adm}}(c)$ be a finite group. Then*

$$U_G \cap M_c$$

is an open subset of M_c.

Proof. The corollary follows immediately from Proposition 7.7 and Lemma 7.12. □

Next, we prove the main result of the present section.

Theorem 7.14. *Let q be an arbitrary point of $\overline{M}_{g,n}$ and $G \in \pi_A^{\text{adm}}(q)$. Then U_G is an open subset of $\overline{M}_{g,n}$. In particular, the maps*

$$\pi_{g,n}^{\text{adm}} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n},$$

$$\pi_{g,n}^{\text{sol}} : \overline{\mathfrak{M}}_{g,n} \to \overline{\Pi}_{g,n}^{\text{sol}}$$

defined in Section 3.2 are continuous.

Proof. For each $j \in \mathbb{Z}_{\geq 0}$, we put $M_j \stackrel{\text{def}}{=} \{q' \in \overline{M}_{g,n} \mid \#e^{\text{cl}}(\Gamma_{q'}) = j\} \subseteq \overline{M}_{g,n}$, and denote by $\text{Gen}(M_j)$ the set of generic points of M_j. Note that $M_0 = M_{g,n}$, and that $M_j = \emptyset$ if $j >> 0$. Since $M_{j'} \cap M_{j''} = \emptyset$ if $j' \neq j''$, we have that

$$\overline{M}_{g,n} = \bigsqcup_{j \in \mathbb{Z}_{\geq 0}} M_j.$$

Moreover, for each $\eta_j \in \mathrm{Gen}(M_j)$, we put $M_{\eta_j} \stackrel{\mathrm{def}}{=} V_{\eta_j} \cap M_j$, where V_{η_j} denotes the topological closure of η_j in $\overline{M}_{g,n}$. Since $M_{\eta'_j} \cap M_{\eta''_j} = \emptyset$ if $\eta'_j \neq \eta''_j$ for each $j \in \mathbb{Z}_{\geq 0}$, we obtain a disjoint union

$$M_j = \bigsqcup_{\eta_j \in \mathrm{Gen}(M_j)} M_{\eta_j}.$$

Then we obtain

$$\overline{M}_{g,n} = \bigsqcup_{j \in \mathbb{Z}_{\geq 0}} \bigsqcup_{\eta_j \in \mathrm{Gen}(M_j)} M_{\eta_j}.$$

Thus, we have

$$U_G = \bigsqcup_{j \in \mathbb{Z}_{\geq 0}} \bigsqcup_{\eta_j \in \mathrm{Gen}(M_j)} M_{\eta_j} \cap U_G.$$

Corollary 7.13 implies that $M_{\eta_j} \cap U_G$ is an open subset of M_{η_j}. This means that $M_{\eta_j} \cap U_G$ is a constructible subset of M_{η_j}, and $M_{\eta_j} \cap U_G$ is stable under generization in $M_{\eta_j} \cap U_G$. Since M_{η_j} is a constructible subset of $\overline{M}_{g,n}$, we obtain that U_G is a constructible subset of $\overline{M}_{g,n}$.

Let $j' \geq j''$. If $M_{\eta_{j'}}$ is contained in the topological closure of $M_{\eta_{j''}}$ in $\overline{M}_{g,n}$ and $M_{\eta_{j'}} \cap U_G \neq \emptyset$, then Lemma 7.1 implies that $M_{\eta_{j''}} \cap U_G \neq \emptyset$. Since $M_{\eta_j} \cap U_G$ is stable under generization in $M_{\eta_j} \cap U_G$, $j \in \mathbb{Z}_{\geq 0}$, we obtain that U_G is stable under generization in $\overline{M}_{g,n}$. Thus, U_G is an open subset of $\overline{M}_{g,n}$. This completes the proof of the theorem. $\square$

References

[1] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.* **36** (1969) 75-109.

[2] M. D. Fried, M. Jarden, Field arithmetic. Third edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete.* 3. Folge. A Series of Modern Surveys in Mathematics **11**. *Springer-Verlag*, Berlin, 2008.

[3] A. Grothendieck, La longue marche à travers la théorie de Galois, manuscript (1981).

[4] A. Grothendieck, Letter to G. Faltings (German), manuscript (1984).

[5] D. Harbater, Formal patching and adding branch points, *Amer. J. Math.* **115** (1993), 487-508.

[6] Y. Hoshi, The exactness of the log homotopy sequence. *Hiroshima Math. J.* **39** (2009), 61-121.

[7] Y. Hoshi, S. Mochizuki, On the combinatorial anabelian geometry of nodally non-degenerate outer representations, *Hiroshima Math. J.* **41** (2011), 275–342.

[8] Q, Liu, Algebraic geometry and arithmetic curves, Oxford Graduate Texts in Mathematics, **6**, Paperback new edition (2006), Oxford University Press, Oxford.

[9] S. Mochizuki, The geometry of the compactification of the Hurwitz scheme. *Publ. Res. Inst. Math. Sci.* **31** (1995), 355–441.

[10] S. Mochizuki, Extending families of curves over log regular schemes, *J. reine angew. Math.* **511** (1999), 43-71.

[11] S. Mochizuki, The local pro-p anabelian geometry of curves. *Invent. Math.* **138** (1999), 319–423.

[12] S. Mochizuki, A combinatorial version of the Grothendieck conjecture. *Tohoku Math. J.* (2) **59** (2007), 455–479.

[13] S. Nakajima, On generalized Hasse-Witt invariants of an algebraic curve, *Galois groups and their representations* (Nagoya 1981) (Y. Ihara, ed.), *Adv. Stud. Pure Math*, **2**, North-Holland Publishing Company, Amsterdam, 1983, 69-88.

[14] H. Nakamura, Rigidity of the arithmetic fundamental group of a punctured projective line. *J. Reine Angew. Math.* **405** (1990), 117-130.

[15] H. Nakamura, Galois rigidity of the étale fundamental groups of punctured projective lines. *J. Reine Angew. Math.* **411** (1990), 205-216.

[16] F. Pop, M. Saïdi, On the specialization homomorphism of fundamental groups of curves in positive characteristic. *Galois groups and fundamental groups,* 107–118, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.

[17] M. Raynaud, Sections des fibrés vectoriels sur une courbe. *Bull. Soc. math. France* **110** (1982), 103–125.

[18] M. Raynaud, p-groupes et réduction semi-stable des courbes, *The Grothendieck Festschrift, Vol. III,* 179-197, *Progr. Math.,* **88**, Birkhäuser Boston, Boston, MA, 1990.

[19] M. Raynaud, Sur le groupe fondamental d'une courbe complète en caractéristique $p > 0$. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 335-351, *Proc. Sympos. Pure Math.,* **70**, *Amer. Math. Soc., Providence, RI,* 2002.

[20] K. Stevenson, Galois groups of unramified covers of projective curves in characteristic p. *Journal of Algebra* **182** (1996), 770–804.

[21] J-P. Serre, Sur la topologie des variétés algébriques en caractéristique p. *Symp. Int. Top. Alg., Mexico* (1958), 24–53.

[22] M. Saïdi, A. Tamagawa, A prime-to-p version of Grothendieck's anabelian conjecture for hyperbolic curves over finite fields of characteristic $p > 0$. *Publ. Res. Inst. Math. Sci.* **45** (2009), 135–186.

[23] M. Saïdi, A. Tamagawa, A refined version of Grothendieck's anabelian conjecture for hyperbolic curves over finite fields. *J. Algebraic Geom.* **27** (2018), 383–448.

[24] J. Stix, Affine anabelian curves in positive characteristic. *Compositio Math.* **134** (2002), 75-85.

[25] J. Stix, Projective anabelian curves in positive characteristic and descent theory for log-étale covers. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2002. *Bonner Mathematische Schriften*, **354**. Universität Bonn, Mathematisches Institut, Bonn, 2002.

[26] A. Tamagawa, The Grothendieck conjecture for affine curves. *Compositio Math.* **109** (1997), 135–194.

[27] A. Tamagawa, On the fundamental groups of curves over algebraically closed fields of characteristic > 0. *Internat. Math. Res. Notices* (1999), 853–873.

[28] A. Tamagawa, Fundamental groups and geometry of curves in positive characteristic. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 297–333, *Proc. Sympos. Pure Math.*, **70**, *Amer. Math. Soc., Providence, RI,* 2002.

[29] A. Tamagawa, On the tame fundamental groups of curves over algebraically closed fields of characteristic > 0. *Galois groups and fundamental groups,* 47–105, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.

[30] A. Tamagawa, Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups. *J. Algebraic Geom.* **13** (2004), 675–724.

[31] J. Tong, Diviseur thêta et formes différentielles, *Math. Z.* **264** (2010), 521–569.

[32] I. Vidal, Morphismes log étales et descente par homéomorphismes universels, *C. R. Acad. Sci. Paris Sér. I Math.* **332** (2001), 239-244.

[33] I. Vidal, Contributions à la cohomologie étale des schémas et des log-schémas, Thèse, U. Paris-Sud (2001).

[34] Y. Yang, On the admissible fundamental groups of curves over algebraically closed fields of characteristic $p > 0$, *Publ. Res. Inst. Math. Sci.* **54** (2018), 649–678.

[35] Y. Yang, Tame anabelian geometry and moduli spaces of curves over algebraically closed fields of characteristic $p > 0$, preprint. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/

[36] Y. Yang, The combinatorial mono-anabelian geometry of curves over algebraically closed fields of positive characteristic I: combinatorial Grothendieck conjecture, preprint. See http://www.kurims.kyoto-u.ac.jp/~yuyang/

[37] Y. Yang, On the averages of generalized Hasse-Witt invariants of pointed stable curves in positive characteristic, *Math. Z.* **295** (2020), 1–45.

[38] Y. Yang, Maximum generalized Hasse-Witt invariants of pointed stable curves in positive characteristic, preprint. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/

[39] Y. Yang, On the existence of specialization isomorphisms of admissible fundamental groups in positive characteristic, to appear in *Math. Res. Lett.*

[40] Y. Yang, Moduli spaces of fundamental groups of curves in positive characteristic II, in preparation.

[41] Y. Yang, Moduli spaces of fundamental groups of curves in positive characteristic III, in preparation.

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