On the Averages of Generalized Hasse-Witt Invariants of Pointed Stable Curves in Positive Characteristic

Yu Yang

À la mémoire du Professeur Michel Raynaud

Abstract

In the present paper, we study fundamental groups of curves in positive characteristic. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. In the present paper, we study a kind of group-theoretical invariant $\mathrm{Avr}_p(\Pi_{X^\bullet})$ associated to the isomorphism class of $\Pi_{X^\bullet}$ called the limit of p-averages of $\Pi_{X^\bullet}$, which plays a central role in the theory of anabelian geometry of curves over algebraically closed fields of positive characteristic. Without any assumptions concerning $\Gamma_{X^\bullet}$, we give a lower bound and a upper bound of $\mathrm{Avr}_p(\Pi_{X^\bullet})$. In particular, we prove an explicit formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ under a certain assumption concerning $\Gamma_{X^\bullet}$ which generalizes a formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ obtained by A. Tamagawa. Moreover, if $X^\bullet$ is a component-generic pointed stable curve, we prove an explicit formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ without any assumptions concerning $\Gamma_{X^\bullet}$, which can be regarded as an averaged analogue of the results of S. Nakajima, B. Zhang, and E. Ozman-R. Pries concerning p-rank of abelian étale coverings of projective generic curves for admissible coverings of component-generic pointed stable curves.

Keywords: pointed stable curve, admissible fundamental group, generalized Hasse-Witt invariant, Raynaud-Tamagawa theta divisor, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14G32.

Contents

1 Introduction

In the present paper, we study admissible fundamental groups of pointed stable curves over algebraically closed fields of positive characteristic. Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k. Here X denotes the underlying curve of $X^\bullet$, and D_X denotes the set of marked points of $X^\bullet$. Write U_X for $X \setminus D_X$, $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$, $v(\Gamma_{X^\bullet})$ for the set of vertices of $\Gamma_{X^\bullet}$, and r_X for the Betti number of $\Gamma_{X^\bullet}$. Moreover, by choosing a suitable base point of $X^\bullet$, we obtain the admissible fundamental group

$$\Pi_{X^\bullet}$$

of $X^\bullet$ (cf. Definition 2.2). In particular, $\Pi_{X^\bullet}$ is naturally (outer) isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(U_X)$ if $X^\bullet$ is smooth over k.

Write $\Pi^{p'}_{X^\bullet}$ for the maximal prime-to-p quotient of $\Pi^{p'}_{X^\bullet}$ if the characteristic $\mathrm{char}(k)$ of k is $p > 0$. We denote by

$$\Pi \overset{\mathrm{def}}{=} \begin{cases} \Pi_{X^\bullet}, & \text{if } \mathrm{char}(k) = 0, \\ \Pi^{p'}_{X^\bullet}, & \text{if } \mathrm{char}(k) = p > 0. \end{cases}$$

Then the structures of Π are well-known, which are isomorphic to the profinite completion and the maximal prime-to-p quotient of the profinite completion of the following free group (cf. [G, XIII.2.12], [V, Théorème 2.2 (c)])

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X}[a_i, b_i]\prod_{j=1}^{n_X} c_j = 1\rangle$$

if $\mathrm{char}(k) = 0$ and $\mathrm{char}(k) = p$, respectively. In particular, $\Pi_{X^\bullet}$ and $\Pi^{p'}_{X^\bullet}$ are free profinite groups with $2g_X + n_X - 1$ generators if $n_X > 0$ and with $2g_X$ generators if $n_X = 0$. Note that we cannot determine whether U_X is affine (i.e., $n_X \neq 0$) or not group-theoretically from the isomorphism class of Π. Moreover, (g_X, n_X) cannot be determined group-theoretically from the isomorphism class of Π.

If $\mathrm{char}(k) = p > 0$, $\Pi_{X^\bullet}$ is very mysterious, and the structure of $\Pi_{X^\bullet}$ is no longer known. In the remainder of the introduction, we assume that $\mathrm{char}(k) = p > 0$. First, since all the admissible coverings in positive characteristic can be lifted to characteristic

0 (cf. [V, Théorème 2.2 (a)]), we obtain that $\Pi_{X^\bullet}$ is topologically finitely generated. Then the isomorphism class of $\Pi_{X^\bullet}$ is determined by the set of finite quotients of $\Pi_{X^\bullet}$ (cf. [FJ, Proposition 16.10.6]). Moreover, the theory developed in [T1] and [Y1] implies that the isomorphism class of $X^\bullet$ as a scheme can possibly be determined by not only the isomorphism class of $\Pi_{X^\bullet}$ as a profinite group but also the isomorphism class of the maximal pro-solvable quotient of $\Pi_{X^\bullet}$ as a profinite group. Then we may ask the following question.

Which finite solvable group can appear as a quotient of $\Pi_{X^\bullet}$?

Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup and $X_H^\bullet = (X_H, D_{X_H})$ the pointed stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H. We have an important invariant associated to $X_H^\bullet$ (or H) called *p-rank* (or *Hasse-Witt invariant*) which is defined to be

$$\sigma(X_H^\bullet) \overset{\text{def}}{=} \dim_{\mathbb{F}_p}(H^{\text{ab}} \otimes \mathbb{F}_p),$$

where $(-)^{\text{ab}}$ denotes the abelianization of $(-)$. Note that we have $\sigma(X_H^\bullet) \leq g_{X_H}$. Roughly speaking, $\sigma(X_H^\bullet)$ controls the finite quotients of $\Pi_{X^\bullet}$ which are extensions of the group $\Pi_{X^\bullet}/H$ by p-groups. Since the structures of maximal prime-to-p quotients of admissible fundamental groups have been known, in order to solve the question mentioned above, we need compute the p-rank $\sigma(X_H^\bullet)$ when $\Pi_{X^\bullet}/H$ is abelian. If $\Pi_{X^\bullet}/H$ is a p-group, then $\sigma(X_H^\bullet)$ can be computed by applying the Deuring-Shafarevich formula (cf. [C]). If $\Pi_{X^\bullet}/H$ is not a p-group, the situation of $\sigma(X_H^\bullet)$ is very complicated. The Deuring-Shafarevich formula implies that, to compute $\sigma(X_H^\bullet)$, we only need to assume that $\Pi_{X^\bullet}/H$ is a prime-to-p group.

Suppose that $n_X = 0$ (i.e., $X^\bullet = X$), and that $X^\bullet$ is smooth over k. If $X^\bullet$ is a curve corresponding to a geometric generic point of moduli space (i.e., a geometric generic curve), S. Nakajima (cf. [N]) proved that, if $\Pi_{X^\bullet}/H$ is a cyclic group with a prime-to-p order, then $\sigma(X_H^\bullet)$ attains the maximum g_{X_H} (i.e., $X_H^\bullet$ is *ordinary*). Moreover, B. Zhang (cf. [Z]) extended Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is an arbitrary abelian group. Recently, E. Ozman and R. Pries (cf. [OP]) generalized Nakajima's result to the case where $\Pi_{X^\bullet}/H$ is a cyclic group with a prime order distinct from p, and $X^\bullet$ is a curve corresponding to an arbitrary geometric point of p-rank stratas of moduli space. Let $n \in \mathbb{N}$ such that $(n, p) = 1$. In other words, the results of Nakajima, Zhang, and Ozman-Pries show that, for *each* connected Galois étale covering of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$, the *generalized Hasse-Witt invariants* (cf. [N]) associated to non-trivial characters of $\mathbb{Z}/n\mathbb{Z}$ attain the maximum $g_X - 1$ except for the eigenspaces associated with eigenvalue 1. In particular, every cyclic connected Galois étale covering with a prime order distinct from p is *new-ordinary*. However, if $X^\bullet$ is not geometric generic, $\sigma(X_H^\bullet)$ cannot be computed explicitly in general. On the other hand, M. Raynaud (cf. [R]) developed his theory of theta divisors and proved that, if $n >> 0$, then the generalized Hasse-Witt invariants attain the maximum $g_X - 1$ for *almost* all of the connected Galois étale coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. This means that

$$\lim_{n\to\infty} \frac{\#\text{Rev}^{\text{ét}}_{n,g_X-1}(X^\bullet)}{\#\text{Rev}^{\text{ét}}_{n}(X^\bullet)} = 1 \text{ and } \lim_{n\to\infty} \frac{\#(\text{Rev}^{\text{ét}}_{n}(X^\bullet) \setminus \text{Rev}^{\text{ét}}_{n,g_X-1}(X^\bullet))}{\#\text{Rev}^{\text{ét}}_{n}(X^\bullet)} = 0$$

hold, where $\mathrm{Rev}_n^{\text{ét}}(X^\bullet)$ denotes the set of connected Galois étale coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$, and $\mathrm{Rev}_{n,g_X-1}^{\text{ét}}(X^\bullet) \subseteq \mathrm{Rev}_n^{\text{ét}}(X^\bullet)$ denotes the set of connected Galois étale coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ whose generalized Hasse-Witt invariants corresponding to the eigenspaces associated with eigenvalue ζ_n attain the maximum g_X-1, where ζ_n is a primitive n^{th} root. In particular, when $n = \ell \neq p$ is a prime number, there exists a *new-ordinary* connected Galois étale covering of $X^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ (cf. [R, Théorème 4.3.1]). Moreover, as a consequence, Raynaud obtained that $\Pi_{X^\bullet}$ is not a prime-to-p profinite group. This is the first deep result concerning the global structures of étale fundamental groups of projective curves over algebraically closed fields of characteristic $p > 0$.

Suppose that $n_X \geq 0$, and that $X^\bullet$ is smooth over k. The computations of generalized Hasse-invariants of admissible coverings of $X^\bullet$ (i.e., tame coverings of $X^\bullet$) are much more difficult than the case where $n_X = 0$. Note that the results of Nakajima, Zhang, and Ozman-Pries do not hold for tame coverings in general, and that the generalized Hasse-Witt invariants of each Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ are less than $g_X + n_X - 1$. In the remainder of the introduction, let t be an arbitrary positive natural number and $n = p^t - 1$. For each Galois admissible covering $Y^\bullet \to X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$, the Kummer theory implies that there exists a line bundle $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(-D)$, where D is an effective divisor on X whose support is contained in D_X, and whose degree is $\deg(D) = s(D)n$. Here we have

$$0 \leq s(D) \leq \begin{cases} 0, & \text{if } n_X = 0, \\ n_X - 1, & \text{if } n_X \geq 1. \end{cases}$$

A. Tamagawa observed that Raynaud's theory of theta divisors can be generalized to the case of tame coverings, and established a theory of ramified version of Raynaud's theta divisors when $s(D) \leq 1$. By applying the theory of theta divisors, Tamagawa proved that, if $n >> 0$, $n_X > 1$, and $s(D) = 1$, then the generalized Hasse-Witt invariants are equal to g_X for *almost* all of the Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Furthermore, he introduced a kind of group-theoretical invariant associated to $\Pi_{X^\bullet}$ called the limit of p-averages (see also Definition 2.4)

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) \overset{\text{def}}{=} \lim_{t \to \infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})},$$

where K_n denotes the kernel of the natural continuous surjective homomorphism $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$, and proved the following formula (cf. [T1, Theorem 0.5]).

Theorem 1.1. *Suppose that $X^\bullet$ is smooth over k. Then we have*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X > 1. \end{cases}$$

Remark 1.1.1. Theorem 1.1 implies immediately that (g_X, n_X) can be reconstructed group-theoretically from the isomorphism class of Π_X (cf. [T1, Theorem 0.1]). Moreover, as an application, Tamagawa proved that *the weak Isom-version of the Grothendieck conjecture for curves over algebraically closed fields of characteristic* $p > 0$ (=Weak Isom-version Conjecture) holds when $g = 0$ and $X^\bullet$ is smooth over an algebraic closure of $\mathbb{F}_p$

(cf. [T1, Theorem 0.2]). This means that the isomorphism class of U_X as a scheme can be determined group-theoretically from the isomorphism class of $\Pi_{X^\bullet}$ as a profinite group. The original anabelian conjectures of A. Grothendieck require the using of the highly non-trivial outer Galois actions induced by the fundamental exact sequences of étale (or tame) fundamental groups. Weak Isom-version Conjecture shows evidence for very strong anabelian phenomena for curves over algebraically closed fields of characteristic $p > 0$. In this situation, the Galois group of the base field is trivial, and étale (or tame) fundamental group coincides with the geometric fundamental group, thus in a total absence of a Galois action of the base field. This kind of anabelian phenomena *go beyond* Grothendieck's anabelian geometry, and shows that the tame fundamental group of a smooth pointed stable curve over an algebraically closed field must encode *moduli* of the curve. On the other hand, in the case of algebraically closed fields of characteristic 0, since the geometric fundamental groups of curves depend only on the types of curves, (g_X, n_X) cannot be reconstructed group-theoretically from the isomorphism class of Π_X, and the anabelian geometry of curves does not exist in this situation.

Let us return to the case where $X^\bullet$ is an arbitrary pointed stable curve over k. Furthermore, the following theorem was proved essentially by Tamagawa (cf. [T2, Theorem 3.10], Remark 5.2.1 and Remark 5.2.2 of the present paper), which is a generalized version of Theorem 1.1 to the case of pointed stable curves under certain assumptions of dual semi-graphs (see Definition 5.1 for the definitions of $V_{X^\bullet}^{\mathrm{tre}}$ and $E_{X^\bullet}^{\mathrm{tre}}$). This theorem is a key step toward proving a theorem concerning resolution of non-singularities (cf. [T2, Theorem 0.2]).

Theorem 1.2. *Suppose that* $\Gamma_{X^\bullet}^{\mathrm{cpt}}$ *is* 2*-connected (cf. Definition 2.1). Then we have*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#V_{X^\bullet}^{\mathrm{tre}} + \#E_{X^\bullet}^{\mathrm{tre}}.$$

Remark 1.2.1. Theorem 1.2 means that, if $n >> 0$, the generalized Hasse-Witt invariants are equal to $g_X - r_X - \#V_{X^\bullet}^{\mathrm{tre}} + \#E_{X^\bullet}^{\mathrm{tre}}$ for *almost* all of the Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. This means that

$$\lim_{n\to\infty} \frac{\#\mathrm{Rev}^{\mathrm{adm}}_{n,\mathrm{Avr}_p(\Pi_{X^\bullet})}(X^\bullet)}{\#\mathrm{Rev}_n^{\mathrm{adm}}(X^\bullet)} = 1 \text{ and } \lim_{n\to\infty} \frac{\#(\mathrm{Rev}_n^{\mathrm{adm}}(X^\bullet) \setminus \mathrm{Rev}^{\mathrm{adm}}_{n,\mathrm{Avr}_p(\Pi_{X^\bullet})}(X^\bullet))}{\#\mathrm{Rev}_n^{\mathrm{adm}}(X^\bullet)} = 0$$

hold, where $\mathrm{Rev}_n^{\mathrm{adm}}(X^\bullet)$ denotes the set of Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$, and $\mathrm{Rev}^{\mathrm{adm}}_{n,\mathrm{Avr}_p(\Pi_{X^\bullet})}(X^\bullet) \subseteq \mathrm{Rev}_n^{\mathrm{adm}}(X^\bullet)$ denotes the set of Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ whose generalized Hasse-Witt invariants corresponding to the eigenspaces associated with eigenvalue ζ_n are equal to $\mathrm{Avr}_p(\Pi_{X^\bullet})$, where ζ_n is a primitive n^{th} root.

Remark 1.2.2. Let $v \in v(\Gamma_{X^\bullet})$. Write X_v for the irreducible component of X corresponding to v and $\widetilde{X}_v$ for the smooth compactification of $U_{X_v} \stackrel{\mathrm{def}}{=} X_v \setminus X_v^{\mathrm{sing}}$, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$. We define a smooth pointed stable curve of type (g_v, n_v) to be

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \stackrel{\mathrm{def}}{=} (D_X \cap X_v) \cup (\widetilde{X}_v \setminus U_{X_v})).$$

We denote by Π_v the admissible fundamental group of $\widetilde{X}_v^\bullet$. Then we have a homomorphism $\phi_v : \Pi_v^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}}$ induced by the natural (outer) injective homomorphism $\Pi_v \hookrightarrow \Pi_{X^\bullet}$. Note that ϕ_v is not an injection in general. The key of the proof of Theorem 1.2 is to prove that ϕ_v is an injection for each $v \in v(\Gamma_{X^\bullet})$ when $\Gamma_{X^\bullet}^{\mathrm{cpt}}$ is 2-connected (cf. [T2, Proposition 3.4] or Corollary 3.5 of the present paper). This means that each Galois admissible covering of $\widetilde{X}_v^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ can be extended to a Galois admissible covering of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then Theorem 1.2 follows immediately from Theorem 1.1.

Remark 1.2.3. Let $H \subseteq \Pi_{X^\bullet}$ be an open normal subgroup. When the base field is an arithmetic field (e.g. a finite field, a p-adic field, and a number field among other things), the monodromy filtration (or the weight filtration) associated to H^{ab} can be reconstructed by using the theory of weight (e.g. Weil conjecture for abelian varieties, Hodge-Tate theory, weight-monodromy conjecture for curves). The reconstructions of monodromy filtrations plays a central role in the proofs of the main conjectures (=*combinatorial Grothendieck conjectures*) of the theory of *combinatorial anabelian geometry* introduced by S. Mochizuki. When the base field is an algebraically closed field of characteristic $p > 0$, the author observed that the following.

> The set of limits of p-averages
>
> $$\{\mathrm{Avr}_p(H) \mid H \subseteq \Pi_{X^\bullet} \text{ is an open normal subgroup for which } \mathrm{Avr}_p(H) \text{ exists}\}$$
>
> plays a role of (outer) Galois actions in the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (i.e., we can reconstruct the monodromy filtration (or the weight filtration) associated to H^{ab} by using the limit of p-average $\mathrm{Avr}_p(H)$).

Moreove, by applying Theorem 1.2, the author proved the combinatorial Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$ (cf. [Y1, Theorem 1.2], [Y3, Theorem 0.5]), and generalized Tamagawa's result concerning Weak Isom-version Conjecture to the case of (possibly singular) pointed stable curves (cf. [Y1, Theorem 1.3], [Y3, Corollary 0.6]).

Next, let us explain another motivation of the theory developed in the present paper. Since (g_X, n_X) can be reconstructed group-theoretically from the isomorphism class of $\Pi_{X^\bullet}$, Weak Isom-version Conjecture can be reformulated from the point of view of moduli spaces (cf. [Y2]). Then Weak Isom-version Conjecture means that the moduli spaces of curves can be reconstructed group-theoretically *as sets* from the isomorphism classes of admissible fundamental groups of curves. However, Weak Isom-version Conjecture cannot tell us any further information about moduli spaces (e.g. topological structure). In [Y2], the author posed a new conjecture which is called *the weak Hom-version of the Grothendieck conjecture for curves over algebraically closed fields of characteristic* $p > 0$ (=Weak Hom-version Conjecture). Roughly speaking, Weak Hom-version Conjecture means that the moduli spaces of curves can be reconstructed group-theoretically *as topological spaces* from the sets of continuous open homomorphisms of admissible fundamental groups of curves with a fixed type.

Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$ and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. The first step toward proving Weak Hom-version Conjecture is to prove that each continuous open surjective homomorphism $\phi : \Pi_{X_1} \to \Pi_{X_2}$ induces a morphism of semi-graphs of anabelioids (cf. [M3] for the definition of semi-graphs of anabelioids) associated to $X_i^\bullet$. In order to prove this, we have the following key observation.

> The set of inequalities of the limit of p-averages
>
> $$\{\mathrm{Avr}_p(\phi^{-1}(H_2)) \geq \mathrm{Avr}_p(H_2) \mid H_2 \subseteq \Pi_{X_2^\bullet} \text{ is an open normal subgroup}$$
>
> $$\text{for which the inequality is satisfied}\}$$
>
> induced by the surjection ϕ plays a role of the comparability of (outer) Galois actions in the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

Let H_2 be an arbitrary open normal subgroup of $\Pi_{X_2^\bullet}$, $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2)$, $X_{H_i}^\bullet$, $i \in \{1, 2\}$, the pointed stable curve over k_i corresponding to H_i, and $\Gamma_{X_{H_i}^\bullet}$ the dual semi-graph of $X_{H_i}^\bullet$. Since $\Gamma_{X_{H_i}^\bullet}^{\mathrm{cpt}}$, $i \in \{1, 2\}$, *is not* 2*-connected in general even in the case where* $\Gamma_{X_i^\bullet}^{\mathrm{cpt}}$ *is* 2*-connected*, we cannot use Theorem 1.2 to compute $\mathrm{Avr}_p(H_i)$. Thus, we need a generalized version of Theorem 1.2.

For each $v \in v(\Gamma_{X^\bullet})$, we introduce two sets $E_v^{>1}$ and $E_v^{=1}$ associated to v which only depend on $\Gamma_{X^\bullet}$ and v (cf. Definition 3.3). The first main theorem of the present paper is the following (cf. Theorem 5.2), which gives a lower bound and a upper bound of the generalized Hasse-Witt invariants for almost all of the Galois admissible coverings of an arbitrary pointed stable curve $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ when $n >> 0$ (see Definition 5.1 for the definition of $V_{X^\bullet}^{\mathrm{tre}, g_v=0}$).

Theorem 1.3. *We have*

$$g_X - r_X - \#V_{X^\bullet}^{\mathrm{tre}} + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}} - \sum_{v \in v(\Gamma_{X^\bullet}) \text{ s.t. } \#E_v^{>1} > 1} g_v$$

$$\leq \limsup_{t \to \infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} \leq g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}} + \sum_{v \in v(\Gamma_{X^\bullet})} \#E_v^{>1},$$

where $\limsup(-)$ *denotes the limit superior of* $(-)$*. In particular, if* $\#E_v^{>1} \leq 1$ *for each* $v \in v(\Gamma_{X^\bullet})$*, then we have*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#V_{X^\bullet}^{\mathrm{tre}} + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}} - \sum_{v \in v(\Gamma_{X^\bullet}) \text{ s.t. } \#E_v^{>1} > 1} g_v$$

$$= g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}} + \sum_{v \in v(\Gamma_{X^\bullet})} \#E_v^{>1}$$

$$= g_X - r_X - \#V_{X^\bullet}^{\mathrm{tre}} + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}}.$$

Remark 1.3.1. Since the condition that $\#E_v^{>1} \leq 1$ for each $v \in v(\Gamma_{X^\bullet})$ is weaker than the condition that $\Gamma_{X^\bullet}^{\mathrm{cpt}}$ is 2-connected, Theorem 1.3 is a generalized version of Theorem 1.2 (cf. Remark 5.2.1).

To verify Theorem 1.3, first, we give an explicit description of the image $\phi_v : \Pi_v^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}}$ for each $v \in v(\Gamma_{X^\bullet})$ (cf. Proposition 3.4). Then we obtain an explicit description of the set of the Galois admissible coverings of $\widetilde{X}_v^\bullet$, $v \in v(\Gamma_{X^\bullet})$, with Galois group $\mathbb{Z}/n\mathbb{Z}$ which can be extended to a Galois admissible covering of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, we can compute the generalized Hasse-Witt invariants of the Galois admissible coverings contained in the set, and obtain the lower bound and the upper bound of Theorem 1.3. On the other hand, we do not know whether $\mathrm{Avr}_p(\Pi_{X^\bullet})$ can attain the upper bound or not in general. The main difficulty is as follows. Let $v \in v(\Gamma_{X^\bullet})$ and $\mathcal{L}_v$ a line bundle on $\widetilde{X}_v$ such that $\mathcal{L}_v^{\otimes n} \cong \mathcal{O}_{\widetilde{X}_v}(-D_v)$, where D_v is an effective divisor on $\widetilde{X}_v$ of degree

$$\deg(D_v) = s(D_v)n$$

whose support is contained in $D_{\widetilde{X}_v}$. We do not know whether or not the Raynaud-Tamagawa theta divisor concerning D_v exists in general (if $s(D_v) = 0$ or $s(D_v) = 1$, the existence was proved by Raynaud and Tamagawa, respectively). In fact, there is an example that the Raynaud-Tamagawa theta divisor concerning to D_v does not exist when $s(D_v) \geq 2$ (cf. Remark 4.7.1). Thus, we cannot use the theory of theta divisors to compute the cardinality of the set of the Galois admissible coverings of $\widetilde{X}_v^\bullet$, $v \in v(\Gamma_{X^\bullet})$, with Galois group $\mathbb{Z}/n\mathbb{Z}$ whose generalized Hasse-Witt invariants are equal to $g_X + \#E_v^{>1} - 1$.

On the other hand, if $X^\bullet$ is a component-generic pointed stable curve over k (i.e., $\widetilde{X}_v^\bullet$, $v \in v(\Gamma_{X^\bullet})$, is a geometric generic curve of p-rank stratas of moduli space (cf. Definition 6.2)), we prove that the Raynaud-Tamagawa theta divisor concerning D_v exists under a certain assumption concerning D_v (cf. Proposition 6.4). Then we obtain the following formula of $\mathrm{Avr}_p(\Pi_{X^\bullet})$ for component-generic pointed stable curves without any assumptions of dual semi-graphs, which is the second main theorem of the present paper (cf. Theorem 6.6).

Theorem 1.4. *Suppose that $X^\bullet$ is a component-generic pointed stable curve over k. Then we have*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}} + \sum_{v \in v(\Gamma_{X^\bullet})} \#E_v^{>1}.$$

Remark 1.4.1. Theorem 1.4 means that, if $n >> 0$, the generalized Hasse-Witt invariants attain the upper bound for *almost* all of the Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Then Theorem 1.4 can be regarded as an averaged analogue of the results of Nakajima, Zhang, and Ozman-Pries for admissible coverings of pointed stable curves.

The present paper is organized as follows. In Section 2, we fix some notation and give some definitions which will be used in the present paper. In Section 3, we analyze images and kernels of homomorphisms between the abelianizations of admissible fundamental groups. In Section 4, we compute the limits of p-averages of images of homomorphisms between the abelianizations of admissible fundamental groups. In Section 5, we prove the first main theorem of the present paper. In Section 6, we prove the second main theorem of the present paper.

Acknowledgements

Many theories concerning curves in positive characteristic were first developed by Professor *Michel Raynaud*, the author is profoundly sad by his passing on March 10, 2018. The author would like to thank the referee very much for carefully reading the manuscript and for giving many comments which substantially helped improving the quality of the paper. This research was supported by JSPS KAKENHI Grant Numbers 18K03239 (Y. Hoshi), 16H06335 (A. Moriwaki), 15H03609 (A. Tamagawa), and 15K04781 (G. Yamashita). The author would like to thank Professors Yuichiro Hoshi, Atsushi Moriwaki, Akio Tamagawa, and Go Yamashita for providing economic support.

2 Preliminaries

In this section, we recall some definitions and results which will be used in the present paper.

Definition 2.1. Let $\mathbb{G} \stackrel{\text{def}}{=} (v(\mathbb{G}), e^{\text{cl}}(\mathbb{G}) \cup e^{\text{op}}(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G})})$ be a semi-graph (cf. [M3, Section 1]). Here, $v(\mathbb{G})$, $e^{\text{cl}}(\mathbb{G})$, $e^{\text{op}}(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of closed edges of $\mathbb{G}$, the set of open edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively.

We define an *one-point compactification* $\mathbb{G}^{\text{cpt}}$ of $\mathbb{G}$ as follows: if $e^{\text{op}}(\mathbb{G}) = \emptyset$, we set $\mathbb{G}^{\text{cpt}} = \mathbb{G}$; otherwise, the set of vertices of $\mathbb{G}^{\text{cpt}}$ is $v(\mathbb{G}^{\text{cpt}}) \stackrel{\text{def}}{=} v(\mathbb{G}) \coprod \{v_\infty\}$, the set of closed edges of $\mathbb{G}^{\text{cpt}}$ is $e^{\text{cl}}(\mathbb{G}^{\text{cpt}}) \stackrel{\text{def}}{=} e^{\text{cl}}(\mathbb{G}) \cup e^{\text{op}}(\mathbb{G})$, the set of open edges of $\mathbb{G}$ is empty, and each edge $e \in e^{\text{op}}(\mathbb{G}) \subseteq e(\mathbb{G}^{\text{cpt}})$ connects v_∞ with the vertex that is abutted by e.

Let $v \in v(\mathbb{G})$. We shall say that $\mathbb{G}$ is 2-*connected* at v if $\mathbb{G} \setminus \{v\}$ is either empty or connected. Moreover, we shall say that $\mathbb{G}$ is 2-connected if $\mathbb{G}$ is 2-connected at each $v \in v(\mathbb{G})$. Note that, if $\mathbb{G}$ is connected, then $\mathbb{G}^{\text{cpt}}$ is 2-connected at each $v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\text{cpt}})$ if and only if $\mathbb{G}^{\text{cpt}}$ is 2-connected.

Let k be an algebraically closed field and

$$X^\bullet = (X, D_X)$$

a pointed stable curve of type (g_X, n_X) over k. Here, X denotes the underlying curve of $X^\bullet$, and D_X denotes the set of marked points of $X^\bullet$. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$, $\Pi_{X^\bullet}^{\text{top}}$ for the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, and $r_X \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \cup e^{\text{op}}(\Gamma_{X^\bullet})$. We shall write X_v for the irreducible component of X corresponding to v, write x_e for the node corresponding to e of X if $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point corresponding to e of X if $e \in e^{\text{op}}(\Gamma_{X^\bullet})$.

Definition 2.2. Let $Y^\bullet = (Y, D_Y)$ be a pointed stable curve over k and $f^\bullet : Y^\bullet \to X^\bullet$ a morphism of pointed stable curves over k.

We shall say $f^\bullet$ a *Galois admissible covering* over k (or Galois admissible covering for short) if the following conditions are satisfied:

(i) there exists a finite group $G \subseteq \mathrm{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$;

(ii) for each $y \in Y^{\mathrm{sm}} \setminus D_Y$, $f^\bullet$ is étale at y, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$;

(iii) for any $y \in Y^{\mathrm{sing}}$, the image $f^\bullet(y)$ is contained in X^{sing}, where $(-)^{\mathrm{sing}}$ denotes the set of singular points of $(-)$;

(iv) for each $y \in Y^{\mathrm{sing}}$, the local morphism between two nodes induced by $f^\bullet$ may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f^\bullet(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$; moreover, write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y; then $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit, and $\#(-)$ denotes the cardinality of $(-)$;

(v) the local morphism between two marked points induced by $f^\bullet$ may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f^\bullet(y)} \cong k[[a]] & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$ (i.e., a tamely ramified extension).

Moreover, we shall say $f^\bullet$ an *admissible covering* if there exists a morphism of pointed stable curves $(f^\bullet)' : (Y^\bullet)' \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ (f^\bullet)' : (Y^\bullet)' \to X^\bullet$ is a Galois admissible covering over k. One can check easily that the definition of admissible covering coincides with the definition of [M1, §3.9 Definition] when the base scheme is k.

Let $Z^\bullet$ be a disjoint union of finitely many pointed stable curves over k. We shall say a morphism $f^\bullet_{Z^\bullet} : Z^\bullet \to X^\bullet$ over k *multi-admissible covering* if the restriction of $f^\bullet_{Z^\bullet}$ to each connected component of $Z^\bullet$ is admissible. For any category $\mathscr{C}$, we write $\mathrm{Ob}(\mathscr{C})$ for the class of objects of $\mathscr{C}$, and write $\mathrm{Hom}(\mathscr{C})$ for the class of morphisms of $\mathscr{C}$. We denote by

$$\mathrm{Cov}^{\mathrm{adm}}(X^\bullet) \stackrel{\mathrm{def}}{=} (\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)), \mathrm{Hom}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)))$$

the category which consists of the following data: (i) $\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$ consists of an empty object and all the pairs $(Z^\bullet, f^\bullet_{Z^\bullet} : Z^\bullet \to X^\bullet)$, where $Z^\bullet$ is a disjoint union of finitely many pointed stable curves over k, and $f^\bullet_{Z^\bullet}$ is a multi-admissible covering over k; (ii) for any $(Z^\bullet, f^\bullet_{Z^\bullet}), (Y^\bullet, f^\bullet_{Y^\bullet}) \in \mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$, we define

$$\mathrm{Hom}((Z^\bullet, f^\bullet_{Z^\bullet}), (Y^\bullet, f^\bullet_{Y^\bullet})) \stackrel{\mathrm{def}}{=} \{g^\bullet \in \mathrm{Hom}_k(Z^\bullet, Y^\bullet) \mid f^\bullet_{Y^\bullet} \circ g^\bullet = f^\bullet_{Z^\bullet}\},$$

where $\mathrm{Hom}_k(Z^\bullet, Y^\bullet)$ denotes the set of k-morphisms of pointed stable curves. By applying [M1, §3.11 Proposition] and the theory of Kummer log étale coverings, we may see that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\mathrm{sm}} \setminus D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the *admissible fundamental group* of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by $\Pi_{X^\bullet}$. Write $\Pi^{\text{ét}}_{X^\bullet}$ for the étale fundamental group of the underlying curve X of $X^\bullet$. Note that we have the following natural continuous surjective homomorphisms (for suitable choices of base points)

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{top}}_{X^\bullet}.$$

For more details on the theory of admissible coverings and admissible fundamental groups for pointed stable curves, see [M1], [M2].

Remark 2.2.1. Let $\overline{\mathcal{M}}_{g_X,n_X}$ be the moduli stack of pointed stable curves of type (g_X, n_X) over $\mathrm{Spec}\,\mathbb{Z}$ and $\mathcal{M}_{g_X,n_X}$ the open substack of $\overline{\mathcal{M}}_{g_X,n_X}$ parametrizing smooth pointed stable curves. Write $\overline{\mathcal{M}}^{\log}_{g_X,n_X}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g_X,n_X}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g_X,n_X} \setminus \mathcal{M}_{g_X,n_X} \subset \overline{\mathcal{M}}_{g_X,n_X}$ relative to $\mathrm{Spec}\,\mathbb{Z}$.

The pointed stable curve $X^\bullet$ over k induces a morphism $\mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_X,n_X}$. Write $s_X^{\log}$ for the log scheme whose underlying scheme is $\mathrm{Spec}\,k$, and whose log structure is the pulling-back log structure induced by the morphism $\mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_X,n_X}$. We obtain a natural morphism $s_X^{\log} \to \overline{\mathcal{M}}^{\log}_{g_X,n_X}$ induced by the morphism $\mathrm{Spec}\,k \to \overline{\mathcal{M}}_{g_X,n_X}$ and a stable log curve $X^{\log} \stackrel{\mathrm{def}}{=} s_X^{\log} \times_{\overline{\mathcal{M}}^{\log}_{g_X,n_X}} \overline{\mathcal{M}}^{\log}_{g_X,n_X+1}$ over $s_X^{\log}$ whose underlying scheme is X. Let $Y^{\log} \to X^{\log}$ be an arbitrary Kummer log étale covering. One can prove that there exists a Kummer log étale covering $t_X^{\log} \to s_X^{\log}$ such that $Y^{\log} \times_{s_X^{\log}} t_X^{\log} \to X^{\log} \times_{s_X^{\log}} t_X^{\log}$ is a log admissible covering (cf. [M1, §3.5 Definition]) over $t_X^{\log}$. Then the admissible fundamental group of $X^\bullet$ does not depend on the log structure of $X^{\log}$, and [M1, §3.11 Proposition] implies that the admissible fundamental group $\Pi_{X^\bullet}$ of $X^\bullet$ is naturally isomorphic to the geometric log étale fundamental group of $X^{\log}$ (i.e., $\ker(\pi_1(X^{\log}) \to \pi_1(s_X^{\log}))$).

Remark 2.2.2. If $X^\bullet$ is smooth over k, by the definition of admissible fundamental groups, then the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X \setminus D_X$.

In the remainder of the present paper, we suppose that the characteristic of k is $p > 0$.

Definition 2.3. We define the *p-rank* (or *Hasse-Witt invariant*) of $X^\bullet$ to be

$$\sigma(X^\bullet) \stackrel{\mathrm{def}}{=} \dim_{\mathbb{F}_p}(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{F}_p) = \dim_{\mathbb{F}_p}(\Pi^{\text{ét,ab}}_{X^\bullet} \otimes \mathbb{F}_p),$$

where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$.

Remark 2.3.1. For each $v \in v(\Gamma_{X^\bullet})$, write $\widetilde{X}_v$ for the smooth compactification of $X_v \setminus X_v^{\rm sing}$, where X_v denotes the irreducible component X corresponding to v. Then it is easy to see that

$$\sigma(X^\bullet) = \sigma(X) = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma(\widetilde{X}_v) + r_X.$$

Definition 2.4. Let t be an arbitrary positive natural number, $n \overset{\rm def}{=} p^t - 1$, and K_n the kernel of the natural surjective homomorphism $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\rm ab} \otimes \mathbb{Z}/n\mathbb{Z}$. For each n, we define the p-*average* of $\Pi_{X^\bullet}$ to be

$$\gamma_{p,n}^{\rm av}(\Pi_{X^\bullet}) \overset{\rm def}{=} \frac{\dim_{\mathbb{F}_p}(K_n^{\rm ab} \otimes \mathbb{F}_p)}{\#(\Pi_{X^\bullet}^{\rm ab} \otimes \mathbb{Z}/n\mathbb{Z})}.$$

Morever, we put

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) \overset{\rm def}{=} \lim_{t \to \infty} \gamma_{p,n}^{\rm av}(\Pi_{X^\bullet})$$

when the limit exists, and we shall say $\mathrm{Avr}_p(\Pi_{X^\bullet})$ *the limit of* p-*averages* of $\Pi_{X^\bullet}$.

Remark 2.4.1. We do not know whether $\mathrm{Avr}_p(\Pi_{X^\bullet})$ always exists or not in general. On the other hand, let ℓ be a prime number distinct from p, m an arbitrary positive natural number such that $(p, m) = 1$, and K_m the kernel of the natural surjective homomorphism $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\rm ab} \otimes \mathbb{Z}/m\mathbb{Z}$. Then the limit of ℓ-average

$$\lim_{m \to \infty} \frac{\dim_{\mathbb{F}_\ell}(K_m^{\rm ab} \otimes \mathbb{F}_\ell)}{\#(\Pi_{X^\bullet}^{\rm ab} \otimes \mathbb{Z}/m\mathbb{Z})}$$

of $\Pi_{X^\bullet}$ always exists. In fact, to compute $\lim_{m\to\infty} \gamma_{\ell,m}^{\rm av}(\Pi_{X^\bullet})$, by applying the specialization theorem of the maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]), we may assume that $X^\bullet$ is smooth over k. Thus, the Riemann-Hurwitz formula implies that

$$\lim_{m \to \infty} \frac{\dim_{\mathbb{F}_\ell}(K_m^{\rm ab} \otimes \mathbb{F}_\ell)}{\#(\Pi_{X^\bullet}^{\rm ab} \otimes \mathbb{Z}/m\mathbb{Z})} = 2g_X + n_X - 2 = \dim_{\mathbb{F}_\ell}(\Pi_{X^\bullet}^{\rm ab} \otimes \mathbb{F}_\ell) - 1.$$

Let $X_{v_\infty}^\bullet = (X_{v_\infty}, D_{X_{v_\infty}})$ be a smooth pointed stable curve of type $(g_{v_\infty}, n_{v_\infty})$ over k such that $g_{v_\infty} \geq 2$ and $n_{v_\infty} = n_X$. Write Γ_{v_∞} for the dual semi-graph of $X_{v_\infty}^\bullet$. If $n_X \neq 0$, we fix a bijection $D_{X_{v_\infty}} \overset{\sim}{\to} D_X$. Then we may glue $X^\bullet$ and $X_{v_\infty}^\bullet$ along the sets of marked points D_X and $D_{X_{v_\infty}}$ and obtain a stable curve X'_∞ of type $(g_X + g_{v_\infty} + n_X - 1, 0)$ over k. We define a stable curve X_∞ of type $(g_{X_\infty}, 0)$ over k to be

$$X_\infty \overset{\rm def}{=} \begin{cases} X, & \text{if } n_X = 0, \\ X'_\infty, & \text{if } n_X \neq 0. \end{cases}$$

Write Π_{X_∞} for the admissible fundamental group of X_∞ and Γ_{X_∞} for the dual graph of X_∞. Then we have a natural continuous (outer) injective homomorphism

$$\Pi_{X^\bullet} \hookrightarrow \Pi_{X_\infty},$$

and that, by the construction of X_∞, $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is naturally isomorphic to Γ_{X_∞}. Moreover, the natural (outer) injective homomorphism above induces a homomorphism of abelian profinite groups

$$\psi : \Pi^{\mathrm{ab}}_{X^\bullet} \to \Pi^{\mathrm{ab}}_{X_\infty}.$$

Let R be a complete discrete valuation ring of equal characteristic with residue field k, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. Let

$$L \subseteq e^{\mathrm{cl}}(\Gamma_{X_\infty})$$

be an arbitrary subset of closed edges. We claim that we may deform the pointed stable curve X_∞ along L to obtain a new pointed stable curve over $\overline{K}$ such that the set of edges of the dual graph of the new stable curve may be naturally identified with $e(\Gamma_{X_\infty}) \setminus L$. Suppose that

$$\phi_s : \mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_{X_\infty}R} \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g_{X_\infty}} \times_{\mathbb{Z}} R$$

is the classifying morphism determined by $X_\infty \to \mathrm{Spec}\, k$. Thus the completion of the local ring of the moduli stack at ϕ_s is isomorphic to $R[\![t_1, ..., t_{3g_{X_\infty}-3}]\!]$, where $t_1, ..., t_{3g_{X_\infty}-3}$ are indeterminates. Furthermore, the indeterminates $t_1, ..., t_m$ may be chosen so as to correspond to the deformations of the nodes of X_∞. Suppose that $\{t_1, ..., t_d\}$ is the subset of $\{t_1, ..., t_m\}$ corresponding to the subset $L \subseteq e^{\mathrm{cl}}(\Gamma_{X_\infty})$. Now fix a morphism $\mathrm{Spec}\, R \to \mathrm{Spec}\, R[\![t_1, ..., t_{3g_{X_\infty}-3}]\!]$ such that $t_{d+1}, ..., t_{3g_{X_\infty}-3} \mapsto 0 \in R$, but $t_1, ..., t_d$ map to nonzero elements of R. Then the composite morphism

$$\phi : \mathrm{Spec}\, R \to \mathrm{Spec}\, R[\![t_1, ..., t_{3g_{X_\infty}-3}]\!] \to \overline{\mathcal{M}}_{g_{X_\infty},R}$$

determines a pointed stable curve $\mathcal{X}_\infty \to \mathrm{Spec}\, R$. Moreover, the special fiber $\mathcal{X}_\infty \times_R k$ of $\mathcal{X}_\infty$ is naturally isomorphic to X_∞ over k. Write

$$X_\infty^{\setminus L}$$

for the geometric generic fiber $X_\infty \times_K \overline{K}$ of $\mathcal{X}_\infty$ over $\overline{K}$ and $\Gamma_{X_\infty^{\setminus L}}$ for the dual graph of $X_\infty^{\setminus L}$. It follows from the construction of $X_\infty^{\setminus L}$ that we have a natural bijective map

$$e(\Gamma_{X_\infty}) \setminus L \xrightarrow{\sim} e(\Gamma_{X_\infty^{\setminus L}}).$$

Let $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X_\infty})$ be an arbitrary vertex of $\Gamma_{X^\bullet}$ and

$$L_v \stackrel{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\Gamma_{X_\infty}) \mid e \text{ does not meet } v\}.$$

We shall denote by

$$X_v^{\mathrm{def}} \stackrel{\mathrm{def}}{=} X_\infty^{\setminus L_v}.$$

Write $\Pi_{X_v^{\mathrm{def}}}$ for the admissible fundamental group of X_v^{def} and $\Gamma_{X_v^{\mathrm{def}}}$ for the dual graph of X_v^{def}.

3 Images and kernels of homomorphisms of abelianizations of admissible fundamental groups

We maintain the notation introduced in Section 2. Let $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X_\infty})$ be an arbitrary vertex of $\Gamma_{X^\bullet}$. Write X_v for the irreducible component of X corresponding to v and $\widetilde{X}_v$ for the smooth compactification of $U_{X_v} \overset{\text{def}}{=} X_v \setminus X_v^{\text{sing}}$. We define a smooth pointed stable curve of type (g_v, n_v) to be

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \overset{\text{def}}{=} (D_X \cap X_v) \cup (\widetilde{X}_v \setminus U_{X_v})).$$

Moreover, we denote by Π_v the admissible fundamental group of $\widetilde{X}_v^\bullet$ and by Γ_v the dual semi-graph of $\widetilde{X}_v^\bullet$. Note that there is a natural map of semi-graphs $\rho_v : \Gamma_v \to \Gamma_{X^\bullet}$ induced by the natural morphism $U_{X_v} \hookrightarrow X$ and the natural map of sets $D_{\widetilde{X}_v} \to D_X \cup X^{\text{sing}}$. We have a natural homomorphism

$$\phi_v : \Pi_v^{\text{ab}} \to \Pi_{X^\bullet}^{\text{ab}}$$

induced by the natural (outer) injective homomorphism $\Pi_v \hookrightarrow \Pi_{X^\bullet}$. Note that ϕ_v is not an injection in general. We shall write

$$M_v$$

for the image of ϕ_v.

Let $X^{\bullet,*} = (X^*, D_{X^*}) \to X^\bullet$ be a universal admissible covering corresponding to $\Pi_{X^\bullet}$. For each $e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \cup e^{\text{op}}(\Gamma_{X^\bullet})$, write x_e for the marked point corresponding to e, and let x_{e^*} be a point of the inverse image of x_e in D_{X^*}. Write $I_{e^*} \subseteq \Pi_{X^\bullet}$ for the inertia subgroup of x_{e^*}. Note that I_{e^*} is isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Suppose that x_e is contained in X_v. Then we have an injection

$$\phi_{e^*} : I_{e^*} \hookrightarrow \Pi_{X^\bullet}^{\text{ab}}$$

induced by the composition of (outer) injective homomorphisms $I_{e^*} \hookrightarrow \Pi_v \hookrightarrow \Pi_{X^\bullet}$. Since the image of ϕ_{e^*} depends only on e, we may write I_e for the image $\phi_{e^*}(I_{e^*})$.

We denote by

$$\phi_v^{\text{ét}} : \Pi_v^{\text{ét,ab}} \to \Pi_{X^\bullet}^{\text{ét,ab}}, \ \psi^{\text{ét}} : \Pi_{X^\bullet}^{\text{ét,ab}} \to \Pi_{X_\infty}^{\text{ét,ab}}$$

the homomorphisms induced by ϕ_v and ψ, respectively. First, we have the following two lemmas.

Lemma 3.1. *The homomorphisms $\phi_v^{\text{ét}} : \Pi_v^{\text{ét,ab}} \to \Pi_{X^\bullet}^{\text{ét,ab}}$ and $\psi^{\text{ét}} : \Pi_{X^\bullet}^{\text{ét,ab}} \to \Pi_{X_\infty}^{\text{ét,ab}}$ are injections.*

Proof. The lemma follows immediately from the structures of the Picard schemes $\text{Pic}^0_{X/k}$ and $\text{Pic}^0_{X_\infty/k}$. □

Lemma 3.2. *The homomorphism*

$$\psi : \Pi_{X^\bullet}^{\text{ab}} \to \Pi_{X_\infty}^{\text{ab}}$$

is an injection.

Proof. Suppose that $n_X = 0$. Then the lemma follows immediately from the definition of X_∞ (i.e., $X^\bullet = X_\infty$).

Suppose that $n_X \neq 0$. Since each p-Galois admissible covering (i.e., a Galois admissible covering whose Galois group is isomorphic to a p-group) is a Galois étale covering, to verify the lemma, it is sufficient to prove that

$$\psi^{p'} : \Pi^{\mathrm{ab},p'}_{X^\bullet} \to \Pi^{\mathrm{ab},p'}_{X_\infty}$$

is an injection. The specialization theorem of the maximal prime-to-p quotients of admissible fundamental groups implies that we only need to treat the case where $X^\bullet$ is a smooth pointed stable curve over k. Write $I^{\mathrm{op}}_{X^\bullet}$ for the subgroup $\Pi^{\mathrm{ab}}_{X^\bullet}$ generated by I_e, $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. Note that $I^{\mathrm{op}}_{X^\bullet}$ is a free $\widehat{\mathbb{Z}}^{p'}$-module with rank $n_X - 1$. We have two exact sequences

$$0 \to I^{\mathrm{op}}_{X^\bullet} \to \Pi^{\mathrm{ab}}_{X^\bullet} \to \Pi^{\text{ét},\mathrm{ab}}_{X^\bullet} \to 0,$$

$$0 \to I^{\mathrm{op}}_{X^\bullet} \to \Pi^{\mathrm{ab},p'}_{X^\bullet} \to \Pi^{\text{ét},\mathrm{ab},p'}_{X^\bullet} \to 0,$$

and the following commutative diagram:

$$\begin{array}{ccc} \Pi^{\mathrm{ab},p'}_{X^\bullet} & \xrightarrow{\psi^{p'}} & \Pi^{\mathrm{ab},p'}_{X_\infty} \\ \downarrow & & \downarrow \\ \Pi^{\text{ét},\mathrm{ab},p'}_{X^\bullet} & \xrightarrow{\psi^{\text{ét},p'}} & \Pi^{\text{ét},\mathrm{ab},p'}_{X_\infty}. \end{array}$$

By Lemma 3.1, to verify the lemma, we only need to prove that the composition morphism

$$I^{\mathrm{op}}_{X^\bullet} \hookrightarrow \Pi^{\mathrm{ab},p'}_{X^\bullet} \to \Pi^{\mathrm{ab},p'}_{X_\infty}$$

is an injection. Write

$$S_{X_\infty} \subseteq \Pi^{\mathrm{ab},p'}_{X_\infty}$$

for the image of the homomorphism $I^{\mathrm{op}}_{X^\bullet} \hookrightarrow \Pi^{\mathrm{ab},p'}_{X^\bullet} \to \Pi^{\mathrm{ab},p'}_{X_\infty}$. Then S_{X_∞} is the subgroup generated by I_e, $e \in e^{\mathrm{cl}}(\Gamma_{X_\infty})$. The Poincaré duality for prime-to-p étale cohomology implies that

$$S_{X_\infty} \cong \mathrm{Hom}(\Pi^{\mathrm{top},p'}_{X_\infty}, \widehat{\mathbb{Z}}(1)^{p'}).$$

Then S_{X_∞} is a free $\widehat{\mathbb{Z}}^{p'}$-module with rank $n_X - 1$. Thus, the homomorphism $I^{\mathrm{op}}_{X^\bullet} \hookrightarrow \Pi^{\mathrm{ab},p'}_{X^\bullet} \to \Pi^{\mathrm{ab},p'}_{X_\infty}$ is an injection. This completes the proof of the lemma. □

Definition 3.3. For each $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma^{\mathrm{cpt}}_{X^\bullet})$, we denote by $\pi_0(v)$ the set of connected components of $\Gamma^{\mathrm{cpt}}_{X^\bullet} \setminus \{v\}$. For each $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma^{\mathrm{cpt}}_{X^\bullet})$ and each $C \in \pi_0(v)$, we put

$$E_{v,C} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_v) \mid \rho_v(e) \cap C \neq \emptyset\},$$

$$E_v^{>1} \overset{\mathrm{def}}{=} \{C \in \pi_0(v) \mid \#E_{v,C} > 1\},$$

$$E_v^{=1} \overset{\mathrm{def}}{=} \{C \in \pi_0(v) \mid \#E_{v,C} = 1\}.$$

Note that we have $e^{\mathrm{op}}(\Gamma_v) = \bigcup_{C \in \pi_0(v)} E_{v,C}$ and $\#\pi_0(v) = \#E_v^{=1} + \#E_v^{>1}$.

The structure of maximal prime-to-p quotients of admissible (or tame) fundamental groups of smooth pointed stable curves implies that, there exists a generator $[s_e]$ of I_e for each $e \in e^{\mathrm{op}}(\Gamma_v)$ for which the following holds

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_v)} [s_e] = 0$$

in Π_v^{ab}. Write I_v^{op} for the subgroup of Π_v^{ab} generated by I_e, $e \in e^{\mathrm{op}}(\Gamma_v)$. Note that, if $n_v \neq 0$, then I_v^{op} is a free $\widehat{\mathbb{Z}}^{p'}$-module with rank $n_v - 1$, and we have

$$0 \to I_v^{\mathrm{op}} \to \Pi_v^{\mathrm{ab}} \to \Pi_v^{\text{ét,ab}} \to 0.$$

Next, we have the following proposition.

Proposition 3.4. *Let $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X_\infty})$ be an arbitrary vertex of $\Gamma_{X^\bullet}$. Then the following holds:*

(i) Suppose that $n_v = 0$. We have $\Pi_v^{\mathrm{ab}} = \Pi_{X^\bullet}^{\mathrm{ab}}$.

(ii) Suppose that $n_v \neq 0$. We have that

$$K_v \overset{\mathrm{def}}{=} \langle \sum_{e \in E_{v,C}} [s_e],\ C \in \pi_0(v) \rangle \subseteq \Pi_v^{\mathrm{ab}}$$

is the kernel $\ker(\phi_v)$ of ϕ_v, where $\langle (-) \rangle$ denotes the subgroup generated by $(-)$. Moreover, $M_v^{p'}$ and K_v are free $\widehat{\mathbb{Z}}^{p'}$-modules with rank

$$2g_v + \sum_{C \in \pi_0(v)} (\#E_{v,C} - 1) \text{ and } \#\pi_0(v) - 1,$$

respectively.

Proof. (i) is trivial. We only prove (ii). Note that Lemma 3.1 implies that there is a natural surjection $M_v \twoheadrightarrow \Pi_v^{\text{ét,ab}}$. Then $K_v \subseteq I_v^{\mathrm{op}}$. To verify the proposition, we only need to prove that K_v is the kernel of the homomorphism

$$\phi_v^{p'} : \Pi_v^{\mathrm{ab},p'} \twoheadrightarrow M_v^{p'}$$

induced by ϕ_v, and that $M_v^{p'}$ is a free $\widehat{\mathbb{Z}}^{p'}$-module with rank

$$2g_v + \sum_{C \in \pi_0(v)} (\#E_{v,C} - 1).$$

On the other hand, Lemma 3.2 implies that M_v and $\ker(\phi_v)$ coincide with $\mathrm{Im}(\psi \circ \phi_v)$ and $\ker(\psi \circ \phi_v)$, respectively. Then we may assume that $X^\bullet = X_\infty$. By applying the specialization theorem of prime-to-p of admissible fundamental groups, we obtain that

$$\Pi_{X_v^{\mathrm{def}}}^{p'} \cong \Pi_{X_\infty}^{p'}.$$

To verify the proposition, we may assume that $X^\bullet = X_\infty = X_v^{\mathrm{def}}$. This means that there is a natural injection $v(\Gamma_{X^\bullet}) \setminus \{v\} \hookrightarrow \pi_0(v)$, and that, for each $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, the irreducible component X_C is smooth over k.

Moreover, in order to prove that K_v is the kernel of $\phi_v^{p'}$, it is sufficient to prove that, for each positive natural number n such that $(p, n) = 1$, $K_v \otimes \mathbb{Z}/n\mathbb{Z}$ is the kernel of the homomorphism

$$\phi_{v,n}^{p'} : \Pi_v^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow M_v^{p'} \otimes \mathbb{Z}/n\mathbb{Z}$$

induced by $\phi_v^{p'}$.

Let $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ be a surjection and α_v the composition of the morphisms

$$\Pi_v^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z} \overset{\phi_{v,n}^{p'}}{\twoheadrightarrow} M_v^{p'} \otimes \mathbb{Z}/n\mathbb{Z} \hookrightarrow \Pi_{X^\bullet}^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z} \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}.$$

Write $f_\alpha^\bullet : Y_\alpha^\bullet = (Y_\alpha, D_{Y_\alpha}) \to X^\bullet$ for the (connected) Galois admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ over k, and the restriction of $f_{\alpha,v}^\bullet$ to a connected component of $Y_{\alpha,v}^\bullet$ corresponds to α. Then by restricting $f_\alpha^\bullet$ to $\widetilde{X}_v^\bullet$, we obtain a morphism

$$f_{\alpha,v}^\bullet : Y_{\alpha,v}^\bullet = (Y_{\alpha,v}, D_{Y_{\alpha,v}}) \to \widetilde{X}_v^\bullet,$$

where $Y_{\alpha,v} = \widetilde{X}_v \times_X Y_\alpha$, and $D_{Y_{\alpha,v}}$ is the inverse image of $D_{\widetilde{X}_v}$ of the first projection $\widetilde{X}_v \times_X Y_\alpha \to \widetilde{X}_v$. Note that $f_{\alpha,v}^\bullet$ is a Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ of smooth pointed stable curves over k corresponding to α_v, and that the decomposition subgroup of a connected component of $Y_{\alpha,v}^\bullet$ under the action of $\mathbb{Z}/n\mathbb{Z}$ is the image $\alpha_v(\Pi_v^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z}) \subseteq \mathbb{Z}/n\mathbb{Z}$. On the other hand, for each $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, by restricting $f_\alpha^\bullet$ to $\widetilde{X}_C^\bullet$, we obtain a morphism

$$f_{\alpha,C}^\bullet : Y_{\alpha,C}^\bullet = (Y_{\alpha,C}, D_{Y_{\alpha,C}}) \to \widetilde{X}_C^\bullet,$$

where $Y_{\alpha,C} = \widetilde{X}_C \times_X Y_\alpha$, and $D_{Y_{\alpha,C}}$ is the inverse image of $D_{\widetilde{X}_C}$ of the first projection $\widetilde{X}_C \times_X Y_\alpha \to \widetilde{X}_C$. The morphism $f_{\alpha,C}^\bullet$ is a Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ of smooth pointed stable curves over k. The restriction of $f_{\alpha,C}^\bullet$ to a connected component of $Y_{\alpha,C}^\bullet$ corresponds to the composition morphism

$$\alpha_C : \Pi_C^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z} \to \Pi_{X^\bullet}^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z} \overset{\alpha}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}.$$

Note that α_C does not depend on the choices of the connected components of $Y_{\alpha,C}^\bullet$, and that the decomposition subgroup of a connected component of $Y_{\alpha,C}^\bullet$ under the action of $\mathbb{Z}/n\mathbb{Z}$ is the image $\alpha_C(\Pi_C^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z}) \subseteq \mathbb{Z}/n\mathbb{Z}$.

For each $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, we write $I_{E_{v,C}}^{\mathrm{op}} \subseteq I_v^{\mathrm{op}}$ for the subgroup $\langle [s_e], e \in E_{v,C} \rangle$. Note that $I_{E_{v,C}}^{\mathrm{op}}$ and I_C^{op} can be regarded as subgroups of $\Pi_{X^\bullet}^{\mathrm{ab},p'}$, and that $I_{E_{v,C}}^{\mathrm{op}} = I_C^{\mathrm{op}}$ in $\Pi_{X^\bullet}^{\mathrm{ab},p'}$. The definition of Galois admissible fundamental coverings implies that

$$\alpha_C|_{I_C^{\mathrm{op}}} = -\alpha_v|_{I_{E_{v,C}}^{\mathrm{op}}}, \; C \in v(\Gamma_{X^\bullet}) \setminus \{v\}.$$

Write $\overline{[s_e]}$, $e \in E_{v,C}$, for the image of $[s_e]$ in $\Pi_{X^\bullet}^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z}$. Then the structure of the maximal prime-to-p quotients of admissible (or tame) fundamental groups implies that

$$\sum_{e\in E_{v,C}} \alpha(\overline{[s_e]}) = 0, \ C \in v(\Gamma_{X^\bullet}) \setminus \{v\}.$$

On the other hand, if $C \in \pi_0(v) \setminus \{v(\Gamma_{X^\bullet}) \setminus \{v\}\}$, then C corresponds to a node of X_v. The structure of the maximal prime-to-p quotients of admissible (or tame) fundamental groups implies that

$$\sum_{e\in E_{v,C}} \alpha(\overline{[s_e]}) = 0, \ C \in \pi_0(v) \setminus \{v(\Gamma_{X^\bullet}) \setminus \{v\}\}.$$

This means that $K_v \otimes \mathbb{Z}/n\mathbb{Z} \subseteq \ker(\alpha)$ for each surjection $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. Since K_v, $M_v^{p'}$, $\Pi_{X^\bullet}^{\mathrm{ab},p'}$ are free $\widehat{\mathbb{Z}}^{p'}$-modules, we obtain that $K_v \otimes \mathbb{Z}/n\mathbb{Z} \subseteq \ker(\phi_{v,n}^{p'})$. Then $\phi_{v,n}^{p'}$ induces a surjection

$$(\Pi_v^{\mathrm{ab},p'}/K_v) \otimes \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow M_v \otimes \mathbb{Z}/n\mathbb{Z}.$$

Note that $\Pi_v^{\mathrm{ab},p'}/K_v$ is also a free $\widehat{\mathbb{Z}}^{p'}$-module.

To verify the proposition, we need to prove that the surjection $(\Pi_v^{\mathrm{ab},p'}/K_v) \otimes \mathbb{Z}/n\mathbb{Z} \twoheadrightarrow M_v \otimes \mathbb{Z}/n\mathbb{Z}$ above is also an injection (or, equivalently, for each surjection $\beta_v : \Pi_v^{\mathrm{ab},p'} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ such that $K_v \subseteq \ker(\beta_v)$, there exists $\beta : \Pi_{X^\bullet}^{\mathrm{ab},p'} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ such that the composite morphism

$$\Pi_v^{\mathrm{ab},p'} \overset{\phi_v^{p'}}{\twoheadrightarrow} M_v^{p'} \hookrightarrow \Pi_{X^\bullet}^{\mathrm{ab},p'} \overset{\beta}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}$$

is β_v). We write

$$g_v^\bullet : Z_v^\bullet = (Z_v, D_{Z_v}) \to \widetilde{X}_v^\bullet$$

for the (connected) Galois admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ over k corresponding to the surjection β_v. For each $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, we write $I_{E_{v,C}}^{\mathrm{op}} \subseteq I_v^{\mathrm{op}}$ for the subgroup $\langle [s_e], e \in E_{v,C}\rangle$. Then the definition of K_v and the structure of the maximal prime-to-p quotients of admissible (or tame) fundamental groups imply that, for each $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, we may construct a Galois multi-admissible covering

$$g_C^\bullet : Z_C^\bullet = (Z_C, D_{Z_C}) \to \widetilde{X}_C^\bullet$$

with Galois group $\mathbb{Z}/n\mathbb{Z}$ over k as follows. Let β_C, $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, be a surjection $\Pi_C^{\mathrm{ab},p'} \twoheadrightarrow \beta_v(I_{E_{v,C}}^{\mathrm{op}}) \subseteq \mathbb{Z}/n\mathbb{Z}$ such that

$$\beta_C|_{I_C^{\mathrm{op}}} = -\beta_v|_{I_{E_{v,C}}^{\mathrm{op}}}.$$

Write $h_C^\bullet : Y_C^\bullet \to \widetilde{X}_C^\bullet$, $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, for the (connected) Galois admissible covering with Galois group $\beta_v(I_{E_{v,C}}^{\mathrm{op}})$ induced by β_C and Q_C for the quotient group $(\mathbb{Z}/n\mathbb{Z})/\beta_v(I_{E_{v,C}}^{\mathrm{op}})$. Then we define

$$Z_C^\bullet \overset{\mathrm{def}}{=} \coprod_{\tau\in Q_C} Y_{C,\tau}^\bullet, \ C \in v(\Gamma_{X^\bullet}) \setminus \{v\},$$

where $Y^\bullet_{C,\tau}$ is a copy of $Y^\bullet_C$. The morphism $h^\bullet$ induces a morphism $g^\bullet_C : Z^\bullet_C \to \widetilde{X}^\bullet_C$. Moreover, the disjoint union of pointed stable curves $Z^\bullet_C$ admits a natural action of $\mathbb{Z}/n\mathbb{Z}$ as follows. Write $\{\widetilde{\tau}\}_{\tau\in Q_C}$ for a complete representation system of Q_C in $\mathbb{Z}/n\mathbb{Z}$. Then for each $\sigma \in \mathbb{Z}/n\mathbb{Z}$, we have $\sigma = \widetilde{\tau}\mu$ for some $\mu \in \beta_v(I^{\mathrm{op}}_{E_v,C})$. We define $\sigma(Y_{C,\tau}) = \mu(Y_{C,\widetilde{\tau}'\tau})$, where $\widetilde{\tau}'$ denotes the image of $\widetilde{\tau}$ in Q_C. This means that $g^\bullet_C : Z^\bullet_C \to \widetilde{X}^\bullet_C$ is a Galois multi-admissible coverings over k with Galois group $\mathbb{Z}/n\mathbb{Z}$.

By the definition of Galois admissible coverings, we may glue $g^\bullet_v : Z^\bullet_v = (Z_v, D_{Z_v}) \to \widetilde{X}^\bullet_v$ and $g^\bullet_C : Z^\bullet_C = (Z_C, D_{Z_C}) \to \widetilde{X}^\bullet_C$, $C \in v(\Gamma_{X^\bullet}) \setminus \{v\}$, along marked points that is compatible with the gluing of $\{\widetilde{X}^\bullet_v\}_{v\in v(\Gamma_{X^\bullet})}$ that gives rise to $X^\bullet$. Then we obtain a (connected) Galois admissible covering

$$g^\bullet_\beta : Z^\bullet_\beta \to X^\bullet$$

over k with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\beta \in \mathrm{Hom}(\Pi^{\mathrm{ab},p'}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ for the surjection corresponding to $g^\bullet_\beta$. Then by the construction above, the composition of the morphisms

$$\Pi^{\mathrm{ab},p'}_v \overset{\phi^{p'}_v}{\to} \Pi^{\mathrm{ab},p'}_{X^\bullet} \overset{\beta}{\twoheadrightarrow} \mathbb{Z}/n\mathbb{Z}$$

is equal to β_v.

Finally, let us compute the rank of $M^{p'}_v$. Since we assume that $X^\bullet = X^{\mathrm{def}}_v$, the kernel of the natural surjection $M^{p'}_v \twoheadrightarrow \Pi^{\text{ét},\mathrm{ab},p'}_v$ is the subgroup

$$S_{X^\bullet} \subseteq \Pi^{\mathrm{ab},p'}_{X^\bullet}$$

generated by I_e, $e \in e^{\mathrm{cl}}(X^\bullet)$. The Poincaré duality for prime-to-p étale cohomology implies that

$$S_{X^\bullet} \cong \mathrm{Hom}(\Pi^{\mathrm{top},p'}_{X^\bullet}, \widehat{\mathbb{Z}}(1)^{p'}).$$

Then we have $S_{X^\bullet}$ is a free $\widehat{\mathbb{Z}}^{p'}$-module with rank $r_X = \sum_{C\in\pi_0(v)}(\#E_{v,C} - 1)$. Thus, we obtain that $M^{p'}_v$ is a free $\widehat{\mathbb{Z}}^{p'}$-module with rank

$$2g_v + \sum_{C\in\pi_0(v)} (\#E_{v,C} - 1).$$

This completes the proof of the proposition. □

Corollary 3.5. *The following conditions are all equivalent.*

(i) The homomorphism $\phi_v : \Pi^{\mathrm{ab}}_v \to \Pi^{\mathrm{ab}}_{X^\bullet}$ is an injection.

(ii) $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected at v.

(iii) $\Gamma_{X^{\mathrm{def}}_v}$ is 2-connected at v.

Proof. If $n_v = 0$, the corollary is trivial. We may assume that $n_v \neq 0$. The constructions of $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ and $\Gamma_{X^{\mathrm{def}}_v}$ imply that $(ii) \Leftrightarrow (iii)$. We only prove that $(i) \Leftrightarrow (iii)$.

First, let us prove " $\Rightarrow$ ". Proposition 3.4 implies that $K_v = 0$. Then we have $\#\pi_0(v) = 1$ and $\#E_{v,C} = n_v$. Thus, $\Gamma_{X^{\mathrm{def}}_v}$ is 2-connected at v.

Next, let us prove " $\Leftarrow$ ". Since $\Gamma_{X_v^{\mathrm{def}}}$ is 2-connected at v, we have

$$n_v = \#E_{v,C} \text{ and } \#\pi_0(v) = 1.$$

Then Proposition 3.4 implies that $K_v = 0$. This means that the homomorphism $\phi_v : \Pi_v^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}}$ is an injection. This completes the proof of the corollary. □

Remark 3.5.1. Corollary 3.4 was also obtained by Tamagawa (cf. [T2, Proposition 3.4]) by using different methods.

4 Averages of generalized Hasse-Witt invariants

In this section, we compute the limits of averages of generalized Hasse-Witt invariants.

4.1 Generalized Hasse-Witt invariants and line bundles

In this subsection, we recall some notation concerning generalized Hasse-Witt invariants of cyclic tame coverings (see also [T1, Section 3]).

Let $X^\bullet \stackrel{\mathrm{def}}{=} (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over k, $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$, and $U_X \stackrel{\mathrm{def}}{=} X \setminus D_X$. Moreover, in this subsection, we assume that $X^\bullet$ is smooth over k. Let t be an arbitrary positive natural number, $n \stackrel{\mathrm{def}}{=} p^t - 1$, and $\mu_n \subseteq k^\times$ the group of n^{th} roots of unity. Fix a primitive n^{th} root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta^i \mapsto i$. For each $\alpha \in H^1_{\mathrm{\acute{e}t}}(U_X, \mu_n)$, we denote by U_{X_α} for the μ_n-torsor corresponding to α, and by X_α for the normalization of X in U_{X_α}. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition (cf. [S, Section 9])

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha}) = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}} \oplus H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{ni}}$. Moreover, we have

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}} = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k,$$

where $(-)^{F_{X_\alpha}}$ denotes the subspace of $(-)$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify

$$H_\alpha \stackrel{\mathrm{def}}{=} H^1_{\mathrm{\acute{e}t}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$$

with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-vector spaces H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α. We have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication. We define

$$\gamma_{\alpha,i} \stackrel{\mathrm{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z}.$$

These invariants are called *generalized Hasse-Witt invariants* (cf. [N]). Moreover, the decomposition above implies that

$$\dim_k(H_\alpha) = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.$$

Note that, if X_α is connected, then $\dim_k(H_\alpha) = \sigma(X_\alpha)$.

The generalized Hasse-Witt invariants can be also described in terms of line bundles and divisors. We denote by $\mathrm{Pic}(X)$ the Picard group of X and by $\mathbb{Z}[D_X]$ the group of divisors whose supports are contained in D_X. Note that $\mathbb{Z}[D_X]$ is a free $\mathbb{Z}$-module with basis D_X. Consider the following complex of abelian groups:

$$\mathbb{Z}[D_X] \stackrel{a_n}{\to} \mathrm{Pic}(X) \oplus \mathbb{Z}[D_X] \stackrel{b_n}{\to} \mathrm{Pic}(X),$$

where $a_n(D) = (\mathcal{O}_X(-D), nD)$, and $b_n(([\mathcal{L}], D)) = [\mathcal{L}^n \otimes \mathcal{O}_X(-D)]$. We denote by

$$\mathscr{P}_{X^\bullet,n} \stackrel{\mathrm{def}}{=} \ker(b_n)/\mathrm{Im}(a_n)$$

the homology group of the complex. Moreover, we have the following exact sequence

$$0 \to \mathrm{Pic}(X)[n] \stackrel{a'_n}{\to} \mathscr{P}_{X^\bullet,n} \stackrel{b'_n}{\to} \mathbb{Z}/n\mathbb{Z}[D_X] \stackrel{\mathrm{def}}{=} \mathbb{Z}[D_X]\otimes \mathbb{Z}/n\mathbb{Z} \stackrel{c'_n}{\to} \mathbb{Z}/n\mathbb{Z},$$

where $[n]$ means the n-torsion subgroup, and

$$a'_n([\mathcal{L}]) = ([\mathcal{L}], 0) \bmod \mathrm{Im}(a_n),$$

$$b'_n(([\mathcal{L}], D)) \bmod \mathrm{Im}(a_n)) = D \bmod n,$$

$$c'_n(D \bmod n) = \deg(D) \bmod n.$$

Then $\ker(c'_n)$ can be regarded as a subset of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$, where $(\mathbb{Z}/n\mathbb{Z})^\sim$ denotes the set $\{0, 1, \ldots, n-1\}$, and $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$ denotes the subset of $\mathbb{Z}[D_X]$ consisting of the elements whose coefficients are contained in $(\mathbb{Z}/n\mathbb{Z})^\sim$. We denote by $\mathbb{Z}/n\mathbb{Z}[D_X]^0$ the kernel of c'_n and by $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ the subset of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]$ corresponding to $\mathbb{Z}/n\mathbb{Z}[D_X]^0$ under the natural bijection $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X] \stackrel{\sim}{\to} \mathbb{Z}/n\mathbb{Z}[D_X]$. Note that, for each $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$, we have $n|\deg(D)$. Then

$$\deg(D) = s(D)n$$

for some integer $s(D)$ such that

$$s(D) \leq \begin{cases} 0, & \text{if } n_X = 0, \\ n_X - 1, & \text{if } n_X \geq 1. \end{cases}$$

We shall define

$$\widetilde{\mathscr{P}}_{X^\bullet,n}$$

to be the inverse image of $(\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/n\mathbb{Z})^\sim[D_X] \subseteq \mathbb{Z}[D_X]$ under the projection $\ker(b_n) \to \mathbb{Z}[D_X]$. It is easy to see that $\mathscr{P}_{X^\bullet,n}$ and $\widetilde{\mathscr{P}}_{X^\bullet,n}$ are free $\mathbb{Z}/n\mathbb{Z}$-groups with rank $2g_X + n_X - 1$ if $n_X \neq 0$ and with rank $2g_X$ if $n_X = 0$. Moreover, we have (cf. [T1, Proposition 3.5])

$$\widetilde{\mathscr{P}}_{X^\bullet,n} \cong \mathscr{P}_{X^\bullet,n} \cong H^1_{\text{ét}}(U_X, \mu_n).$$

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$. We fix an isomorphism $\mathcal{L}^n \cong \mathcal{O}_X(-D)$. Note that D is an effective divisor on X. We have the following composition of morphisms of line bundles

$$\mathcal{L} \xrightarrow{p^t} \mathcal{L}^{\otimes p^t} = \mathcal{L}^{\otimes n} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X(-D) \otimes \mathcal{L} \hookrightarrow \mathcal{L}.$$

The composite morphism induces a morphism

$$\phi_{([\mathcal{L}],D)} : H^1(X, \mathcal{L}) \to H^1(X, \mathcal{L}).$$

We denote by $\gamma_{([\mathcal{L}],D)} \stackrel{\text{def}}{=} \dim_k(\bigcap_{r\geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)}))$. Write $\alpha_{\mathcal{L}} \in H^1_{\text{ét}}(U_X, \mu_n)$ for the element corresponding to $([\mathcal{L}], D)$ and F_X for the absolute Frobenius morphism on X. Then [S, Section 9] implies that $\gamma_{\alpha_{\mathcal{L}},1}$ is equal to the dimension over k of the largest subspace of $H^1(X, \mathcal{L})$ on which F_X is a bijection. Moreover, we have

$$\gamma_{\alpha_{\mathcal{L}},1} = \dim_k(H^1(X, \mathcal{L})^{F_X} \otimes_{\mathbb{F}_p} k),$$

where $(-)^{F_X}$ denotes the subspace of $(-)$ on which F_X acts trivially. It is easy to check that

$$H^1(X, \mathcal{L})^{F_X} \otimes_{\mathbb{F}_p} k = \bigcap_{r\geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)}).$$

Then we obtain that $\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1}$.

On the other hand, the Riemann-Roch theorem implies that

$$\begin{aligned}\dim_k(H^1(X, \mathcal{L})) &= g_X - 1 - \deg(\mathcal{L}) + \dim_k(H^0(X, \mathcal{L})) \\ &= g_X - 1 + \frac{1}{n}\deg(D) + \dim_k(H^0(X, \mathcal{L})) \\ &\leq g_X - 1 + [\frac{n_X(n-1)}{n}] + \dim_k(H^0(X, \mathcal{L})) \\ &= g_X - 1 + n_X + [-\frac{n_X}{n}] + \dim_k(H^0(X, \mathcal{L})).\end{aligned}$$

Then we obtain the following rough estimate:

$$\gamma_{\alpha_{\mathcal{L}},1} \leq \dim_k(H^1(X, \mathcal{L})) \leq \begin{cases} g_X, & \text{if } ([\mathcal{L}], D) = ([\mathcal{O}_X], 0), \\ g_X - 1, & \text{if } n_X = 0, \\ g_X - 2 + n_X, & \text{if } n_X \neq 0. \end{cases}$$

4.2 Raynaud-Tamagawa theta divisors

In this subsection, we recall some notation and results concerning theta divisors defined by Raynaud and Tamagawa (see also [T1, Section 2]).

We maintain the notation introduced in Section 4.1. Let F_k be the absolute Frobenius morphism on $\operatorname{Spec} k$ and $F_{X/k}$ the relative Frobenius morphism $X \to X_1 \stackrel{\text{def}}{=} X \times_{k,F_k} k$ over k. We define

$$X_t \stackrel{\text{def}}{=} X \times_{k,F_k^t} k,$$

and define a morphism

$$F_{X/k}^t : X \to X_t$$

over k to be the composition of the t relative Frobenius morphism $F_{X/k}^t \stackrel{\text{def}}{=} F_{X_{t-1}/k} \circ \cdots \circ F_{X_1/k} \circ F_{X/k}$.

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$ and $\mathcal{L}_t$ the pull-back of $\mathcal{L}$ by the natural morphism $X_t \to X$. Note that $\mathcal{L}$ and $\mathcal{L}_t$ are line bundles of degree $-s(D)$. We put

$$\mathcal{B}_D^t \stackrel{\text{def}}{=} ((F_{X/k}^t)_* \mathcal{O}_X(D))/\mathcal{O}_{X_t},\ \mathcal{E}_D \stackrel{\text{def}}{=} \mathcal{B}_D^t \otimes \mathcal{L}_t.$$

Write $\mathrm{rk}(\mathcal{E}_D)$ for the rank of $\mathcal{E}_D$. Then we have

$$\chi(\mathcal{E}_D) = \deg(\det(\mathcal{E}_D)) - (g_X - 1)\mathrm{rk}(\mathcal{E}_D).$$

Moreover, $\chi(\mathcal{E}_D) = 0$ (cf. [T1, Lemma 2.3 (ii)]). In [R], Raynaud investigated the following property of the vector bundle $\mathcal{E}_D$ on X.

Condition 4.1. We shall say that $\mathcal{E}_D$ satisfies $(\star)$ if there exists a line bundle $\mathcal{L}'_t$ of degree 0 on X_t such that

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}'_t)), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}'_t))\}.$$

Let J_{X_t} be the Jacobian variety of X_t, and $\mathcal{L}_{X_t}$ a universal line bundle on $X_t \times J_{X_t}$. Let $\mathrm{pr}_{X_t} : X_t \times J_{X_t} \to X_t$ and $\mathrm{pr}_{J_{X_t}} : X_t \times J_{X_t} \to J_{X_t}$ be the natural projections. We denote by $\mathcal{F}$ the coherent $\mathcal{O}_{X_t}$-module $\mathrm{pr}_{X_t}^*(\mathcal{E}_D) \otimes \mathcal{L}_{X_t}$, and by

$$\chi_{\mathcal{F}} \stackrel{\text{def}}{=} \dim_k(H^0(X_t \times_k k(y), \mathcal{F} \otimes k(y))) - \dim_k(H^1(X_t \times_k k(y), \mathcal{F} \otimes k(y)))$$

for each $y \in J_{X_t}$, where $k(y)$ denotes the residue field of y. Note that since $\mathrm{pr}_{J_{X_t}}$ is flat, $\chi_{\mathcal{F}}$ is independent of $y \in J_{X_t}$. Write $(-\chi_{\mathcal{F}})^+$ for $\max\{0, -\chi_{\mathcal{F}}\}$. We denote by

$$\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$$

the closed subscheme of J_{X_t} defined by the $(-\chi_{\mathcal{F}})^+$-th Fitting ideal

$$\mathrm{Fitt}_{(-\chi_{\mathcal{F}})^+}(R^1(\mathrm{pr}_{J_{X_t}})_*(\mathcal{F}).$$

The definition of $\Theta_{\mathcal{E}_D}$ is independent of the choice of $\mathcal{L}_t$. Moreover, for each line bundle $\mathcal{L}''$ of degree 0 on X_t, we have that $[\mathcal{L}''] \notin \Theta_{\mathcal{E}_D}$ if and only if

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}'')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}''))\},$$

where $[\mathcal{L}'']$ denotes the point of J_{X_t} corresponding to $\mathcal{L}''$ (cf. [T1, Proposition 2.2 (i) (ii)]).

Suppose that $\mathcal{E}_D$ satisfies $(\star)$. [R, Proposition 1.8.1] implies that $\Theta_{\mathcal{E}_D}$ is algebraically equivalent to $\mathrm{rk}(\mathcal{E}_D)\Theta$, where Θ is the classical theta divisor (i.e., the image of $X_t^{g_X-1}$ in J_{X_t}). Then we have the following definition.

Definition 4.2. We shall say $\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$ the *Raynaud-Tamagawa theta divisor* associated to $\mathcal{E}_D$ if $\mathcal{E}_D$ satisfies $(\star)$.

Remark 4.2.1. The definition of $\mathcal{E}_D$ implies that the following natural exact sequence holds

$$0 \to \mathcal{L}_t \to (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \to \mathcal{E}_D \to 0.$$

Let $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$. Write $\mathcal{I}_t$ for the pull-back of $\mathcal{I}$ by the natural morphism $X_t \to X$. we obtain the following exact sequence

$$\ldots \to H^0(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \overset{\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}}{\to} H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t)$$

$$\to H^1(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to \ldots.$$

Note that we have that

$$H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}),$$

and that

$$H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{O}_X(D) \otimes (F^t_{X/k})^*(\mathcal{L}_t \otimes \mathcal{I}_t))$$

$$\cong H^1(X, \mathcal{O}_X(D) \otimes (\mathcal{L} \otimes \mathcal{I})^{\otimes p^t}) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}).$$

Moreover, it is easy to see that the homomorphism

$$H^1(X, \mathcal{L} \otimes \mathcal{I}) \to H^1(X, \mathcal{L} \otimes \mathcal{I})$$

induced by $\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}$ coincides with $\phi_{([\mathcal{L} \otimes \mathcal{I}], D)}$. Thus, we obtain that the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists (i.e., there exists a line bundle $\mathcal{I}_t$ on X_t of degree 0 such that $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$) if and only if

$$\gamma_{(([\mathcal{L} \otimes \mathcal{I}], D))} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I})).$$

The following theorem was proved by Raynaud and Tamagawa.

Theorem 4.3. *Suppose that $s(D) \in \{0, 1\}$. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists (i.e., $\mathcal{E}_D$ satisfies $(\star)$).*

Remark 4.3.1. Theorem 4.3 was proved by Raynaud if $s(D) = 0$ (cf. [R, Théorème 4.1.1]), and by Tamagawa if $s(D) \leq 1$ (cf. [T1, Theorem 2.5]).

Definition 4.4. Let D be an arbitrary effective divisor on X.

(i) For each natural number m, we put

$$[D/m] \overset{\text{def}}{=} \sum_{x \in X} [\mathrm{ord}_x(D)/m] x,$$

which is an effective divisor on X.

(ii) For $u \in \{0, 1, \ldots, n\}$, let $u = \sum_{j=0}^{t-1} u_j p^j$ be the p-adic expansion with $u_j \in \{0, 1, \ldots, p-1\}$. We identify $\{0, 1, \ldots, t-1\}$ with $\mathbb{Z}/t\mathbb{Z}$ naturally, and put

$$u^{(i)} \stackrel{\mathrm{def}}{=} \sum_{j=0}^{t-1} u_{i+j} p^j, \ i \in \{0, 1, \ldots, t-1\}.$$

Suppose that $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]$. Then, we put

$$D^{(i)} \stackrel{\mathrm{def}}{=} \sum_{x \in X} (\mathrm{ord}_x(D))^{(i)} x, \ i \in \{0, 1, \ldots, t-1\},$$

which is an effective divisor on X.

Lemma 4.5. *Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet, n}$. For each $i \in \{0, 1, \ldots, t-1\}$, we shall denote $\mathcal{L}^{\otimes p^{t-i}} \otimes \mathcal{O}_X([p^{t-i}D/n])$ and $p^{t-i}D - n[p^{t-i}D/n]$ by $\mathcal{L}(p^{t-i})$ and $D(p^{t-i})$, respectively. Then we have*

$$\gamma_{(\mathcal{L}, D)} = \gamma_{(\mathcal{L}(p^{t-i}), D(p^{t-i}))}, \ i \in \{0, 1, \ldots, t-1\}.$$

Proof. The lemma follows immediately from [T1, Claim (3.8)]. □

On the other hand, Tamagawa proved the following result (cf. [T1, Proposition 3.18]).)

Proposition 4.6. *Let $d \geq \log_p(n_X - 1)$ be an arbitrary positive natural number and $\epsilon < 1$ an arbitrary positive real number. We put*

$$\Lambda = \frac{d}{\epsilon}, \ \text{and} \ \lambda = (1 - \frac{1}{p^{d(n_X-1)}} \binom{n_X - 1}{p^d})^{\frac{(1-\epsilon)}{d}},$$

where $\binom{-}{-}$ denotes the binomial coefficient. Then if $n_X > 1$, we have

$$\#\{D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0 \mid s(D^{(i)}) = 1 \ \text{for some} \ i \in \{0, 1, \ldots, t-1\} \geq n^{n_X - 1}(1 - \lambda^t) - 1$$

for all $t \geq \Lambda$.

By applying [T1, Corollary 3.19] and similar arguments to the arguments given in the proofs of [T1, Theorem 3.12 and Corollary 3.16] imply that the following result holds.

Theorem 4.7. *We put*

$$C(g_X) \stackrel{\mathrm{def}}{=} \begin{cases} 0, & \text{if } g_X = 0, \\ 3^{g_X - 1} g_X!, & \text{if } g_X > 0. \end{cases}$$

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet, n}$. Suppose that the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists. Then the following statements hold.

(i) We have

$$\#\{[\mathcal{L}'] \in \mathrm{Pic}(X) \mid \phi_{([\mathcal{L} \otimes \mathcal{L}'], D)} \ \text{is bijective}\} \geq n^{2g_X} - C(g_X) n^{2g_X - 1}.$$

(ii) We have

$$\#\{[\mathcal{L}'] \in \mathrm{Pic}(X) \mid \gamma_{([\mathcal{L}\otimes\mathcal{L}'],D)} \geq g_X - 1 + s(D)\} \geq n^{2g_X} - C(g_X)n^{2g_X-1}$$

and

$$\#\{[\mathcal{L}'] \in \mathrm{Pic}(X) \mid \gamma_{([\mathcal{L}\otimes\mathcal{L}'],D)} = g_X - 1 + s(D)\}$$

$$\geq \begin{cases} n^{2g_X} - C(g_X)n^{2g_X-1} - 1, & \text{if } s(D) = 0, \\ n^{2g_X} - C(g_X)n^{2g_X-1}, & \text{if } s(D) \geq 1. \end{cases}$$

In particular, suppose that there exists $i \in \{0, 1, \dots, t-1\}$ *such that* $s(D^{(i)}) = 1$*. Then we have*

$$\#\{[\mathcal{L}'] \in \mathrm{Pic}(X) \mid \gamma_{([\mathcal{L}\otimes\mathcal{L}'],D)} = g_X\} \geq n^{2g_X} - C(g_X)n^{2g_X-1}.$$

Remark 4.7.1. Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. We may ask whether or not the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exists in general. Since the existence of $\Theta_{\mathcal{E}_D}$ implies that $\mathcal{E}_D$ is a semi-stable bundle, we obtain that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \dots, t-1\}$ (cf. [T1, Lemma 2.15]). Then we may consider the following problem.

> *Suppose that* $s(D) \geq 2$*, and that* $\deg(D^{(i)}) \geq \deg(D)$ *holds for each* $i \in \{0, 1, \dots, t-1\}$*. Does the Raynaud-Tamagawa theta divisor* $\Theta_{\mathcal{E}_D}$ *exist?*

In fact, the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ does not exist in general. Here, we have an example as follows. Let $X = \mathbb{P}^1_k$, $D_X = \{0, 1, \infty, \omega\}$, where $w \notin \{0, 1\}$, and

$$D = \sum_{x \in D_X} \frac{p-1}{2} x.$$

Then we have $s(D) = 2$. Let $([\mathcal{L}], D)$ be an arbitrary element of $\widetilde{\mathscr{P}}_{X^\bullet,n}$. We see immediately that $\mathcal{E}_D$ satisfies $(\star)$ if and only if the elliptic curve defined by the equation

$$y^2 = x(x-1)(x-\omega)$$

is ordinary. Thus, we cannot expect that $\Theta_{\mathcal{E}_D}$ exists in general. On the other hand, we have the following open problem posed by Tamagawa (cf. [T1, Question 2.20]).

Problem . *Let* $\overline{\mathbb{F}}_p$ *be the algebraic closure of* $\mathbb{F}_p$ *in* k *and* M_{g_X,n_X} *the coarse moduli space of the moduli stack* $\mathcal{M}_{g_X,n_X} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$*. Suppose that* $X^\bullet$ *is a geometric generic curve of* M_{g_X,n_X}*. Let* $([\mathcal{L}], D)$ *be an arbitrary element of* $\widetilde{\mathscr{P}}_{X^\bullet,n}$*. Moreover, suppose that* $\deg(D^{(i)}) \geq \deg(D)$ *holds for each* $i \in \{0, 1, \dots, t-1\}$*. Does the Raynaud-Tamagawa theta divisor* $\Theta_{\mathcal{E}_D}$ *associated to* $\mathcal{E}_D$ *exist?*

In Section 6, we will prove that Problem is true under a certain assumption of D.

4.3 Lower bounds and upper bounds of the limit of p-averages

Definition 4.8. Let G be an arbitrary cyclic group such that $(\#G, p) = 1$ and M a finitely generated $\mathbb{F}_p[G]$-module. For any given character $\chi : G \to k^\times$, we put

$$(M \otimes_{\mathbb{F}_p} k)[\chi] \overset{\text{def}}{=} \{m \in M \otimes_{\mathbb{F}_p} k \mid \tau(m) = \chi(\tau)m \text{ for all } \tau \in G\},$$

and define $\gamma_\chi(M) \overset{\text{def}}{=} \dim_k((M \otimes_{\mathbb{F}_p} k)[\chi])$. Moreover, we define the primitive part of M to be

$$M^{\text{pri}} \overset{\text{def}}{=} M/(\sum_{1 \neq \tau \in G} M^\tau),$$

where $M^\tau \overset{\text{def}}{=} \{m \in M \mid \tau(m) = m\}$ for each $\tau \in G$. We put $\gamma^{\text{pri}}(M) \overset{\text{def}}{=} \dim_{\mathbb{F}_p}(M)$.

Remark 4.8.1. We see immediately that

$$M^{\text{pri}} \otimes_{\mathbb{F}_p} k = \bigoplus_{\chi : G \hookrightarrow k^\times} (M \otimes_{\mathbb{F}_p} k)[\chi].$$

Then we have

$$\gamma^{\text{pri}}(M) = \sum_{\chi : G \hookrightarrow k^\times} \gamma_\chi(M).$$

On the other hand, we can define a $\mathbb{F}_p[G]$-module $M^\vee \overset{\text{def}}{=} \text{Hom}(M, \mathbb{F}_p)$ via $(\tau(a))(m) = a(\tau^{-1}(m))$ for each $\tau \in G$, $a \in M^\vee$, and $m \in M$. Then we have $\gamma_\chi(M) = \gamma_{\chi^{-1}}(M^\vee)$. Thus, we obtain

$$\gamma^{\text{pri}}(M) = \gamma^{\text{pri}}(M^\vee).$$

Let us return to the case where $X^\bullet$ is an arbitrary pointed stable curve and maintain the notation introduced in Section 3. Let t be an arbitrary positive natural number,

$$n \overset{\text{def}}{=} p^t - 1,$$

and $\mu_n \subseteq k^\times$ the group of n^{th} roots of unity. Fix a primitive n^{th} root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta^i \mapsto i$. Let $v \in v(\Gamma_{X^\bullet})$, $U_v \overset{\text{def}}{=} \widetilde{X}_v \setminus D_{\widetilde{X}_v}$, $\widetilde{\mathscr{P}}_{\widetilde{X}_v^\bullet, n}$ the abelian group associated to $\widetilde{X}_v^\bullet$ defined in Section 4.1, and

$$\mathscr{T}_{v,n} = \text{Hom}(\Pi_v^{\text{ab}}, \mu_n) \cong H^1_{\text{ét}}(U_v, \mu_n).$$

The structure of maximal prime-to-p quotients of admissible (or tame) fundamental groups of smooth pointed stable curves implies that, there exists a generator $[s_e]$ of I_e, $e \in e^{\text{op}}(\Gamma_v)$, for which the following holds

$$\sum_{e \in e^{\text{op}}(\Gamma_v)} [s_e] = 0$$

in Π_v^{ab}. Then for each $\alpha \in \mathscr{T}_{v,n}$, we have $\alpha([s_e]) = \zeta^{a_e}$ for each $e \in e^{\text{op}}(\Gamma_v)$ and

$$\prod_{e \in e^{\text{op}}(\Gamma_v)} \alpha([s_e]) = 1.$$

Note that the image $\mathrm{Im}(\alpha) = \langle \xi \stackrel{\mathrm{def}}{=} \zeta^{n/m} \rangle$ is a cyclic subgroup of μ_n with order m for some $m|n$, and $\Pi_v^{\mathrm{ab}}/\ker(\alpha) = \langle \tau \rangle$ is isomorphic to the image $\mathrm{Im}(\alpha)$ via $\tau \mapsto \xi$.

Let $\overline{a}_e \stackrel{\mathrm{def}}{=} a_e m/n$, and let $f_\alpha^\bullet : Y_{v,\alpha}^\bullet \to \widetilde{X}_v^\bullet$ be the μ_n-torsor induced by α and $Z^\bullet = (Z, D_Z)$ a connected component of $Y_{v,\alpha}^\bullet$. Then $f_\alpha^\bullet$ induces a connected Galois admissible covering

$$f^\bullet : Z^\bullet = (Z, D_Z) \to \widetilde{X}_v^\bullet$$

over k with Galois group $\Pi_v^{\mathrm{ab}}/\ker(\alpha)$. Write $f : Z \to \widetilde{X}_v$ for the underlying morphism induced by $f^\bullet$. Then we have

$$f_*(\mathcal{O}_Z) = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathcal{L}_i,$$

where locally $\mathcal{L}_i$ is the eigenspace of the natural action of τ with eigenvalue ξ^i. By considering the action of τ, we obtain a morphism of $\mathcal{O}_{\widetilde{X}_v}$-modules $\mathcal{L}_1^{\otimes m} \to \mathcal{O}_{\widetilde{X}_v}$. Moreover, since $\mathcal{L}_1^{\otimes m}|_{U_v} \cong \mathcal{O}_{\widetilde{X}_v}|_{U_v}$, the morphism $\mathcal{L}_1^{\otimes m} \hookrightarrow \mathcal{O}_{\widetilde{X}_v}$ is an injection. Then there is a unique effective divisor $D_{\overline{\alpha}}$ on $\widetilde{X}_v$ such that $\mathrm{Supp}(D_{\overline{\alpha}}) \subseteq D_{\widetilde{X}_v}$, and the image of the injection $\mathcal{L}_1^{\otimes m} \hookrightarrow \mathcal{O}_{\widetilde{X}_v}$ is $\mathcal{O}_{\widetilde{X}_v}(-D_{\overline{\alpha}}) \subseteq \mathcal{O}_{\widetilde{X}_v}$, where $\mathrm{Supp}(-)$ denotes the support of $(-)$. We fix the isomorphism $\mathcal{L}_1^{\otimes m} \stackrel{\sim}{\to} \mathcal{O}_{\widetilde{X}_v}(-D_{\overline{\alpha}})$ obtained above. Then we have the following lemma.

Lemma 4.9. *For each $e \in e^{\mathrm{op}}(\Gamma_v)$, we write $x_e \in D_{\widetilde{X}_v}$ for the marked point corresponding to e. Then we have*

$$D_{\overline{\alpha}} = \sum_{e \in e^{\mathrm{op}}(\Gamma_v)} \overline{a}_e x_e.$$

Proof. Let $e \in e^{\mathrm{op}}(\Gamma_v)$. We write $I_{x_e} \subseteq \Pi_v^{\mathrm{ab}}/\ker(\alpha)$ for the inertia subgroup of x_e, m_e for $\#I_{x_e}$, and q_e for m/m_e. Let $W_e \stackrel{\mathrm{def}}{=} Z/I_{x_e}$ and $f_1 : Z \to W_e$ the quotient morphism over k. We define a smooth pointed stable curve over k to be

$$W_e^\bullet \stackrel{\mathrm{def}}{=} (W_e, D_W \stackrel{\mathrm{def}}{=} f_1(D_Z)).$$

Then $f^\bullet$ and f_1 induce the following morphisms of smooth pointed stable curves

$$Z^\bullet \stackrel{f_1^\bullet}{\to} W_e^\bullet \stackrel{f_2^\bullet}{\to} \widetilde{X}_v^\bullet$$

over k such that $f_2^\bullet \circ f_1^\bullet = f^\bullet$. Write f_2 for the underlying morphism of $f_2^\bullet$. Moreover, we have

$$(f_1)_*(\mathcal{O}_Z) = \bigoplus_{j \in \mathbb{Z}/m_e\mathbb{Z}} \mathcal{L}_{W_e,j},$$

where locally $\mathcal{L}_{W_e,j}$ is the eigenspace of the natural action of τ^{q_e} with eigenvalue ξ^{jq_e}.

Let π_{x_e} be a uniformizer of the discrete valuation ring $\mathcal{O}_{\widetilde{X}_v,x_e}$ such that $\mathcal{L}_1^{\otimes m}$ is locally generated by $\pi_{x_e}^{\mathrm{ord}_{x_e}(D_{\overline{\alpha}})}$ at x_e via the fixed isomorphism $\mathcal{L}_1^{\otimes m} \stackrel{\sim}{\to} \mathcal{O}_{\widetilde{X}_v}(-D_{\overline{\alpha}})$, w_e a point of $f_2^{-1}(x_e) \stackrel{\mathrm{def}}{=} \{w_e, \tau(w_e), \ldots, \tau^{q_e-1}(w_e)\}$, z_e the point $f_2^{-1}(w_e)$. We may choose the generator $[s_e]$ of I_e such that the image of $[s_e]$ in $\widehat{\mathbb{Z}}(1)^{p'}/n = \mu_n$ via the identification $I_e \stackrel{\sim}{\to} \widehat{\mathbb{Z}}(1)^{p'}$ is equal to ζ. Then the Kummer theory implies that, there is a uniformizer π_{z_e} of the maximal ideal of the discrete valuation ring $\mathcal{O}_{Z,z_e}$ such that $\pi_{z_e}^{m_e} = \pi_{x_e}$ and

$$\tau^{q_e}(\pi_{z_e}) = \xi^{q_e r_e} \pi_{z_e},$$

where $r_e\overline{a}_e/q_e \equiv 1 \pmod{m_e}$. Then we obtain that

$$\tau^{q_e}(\pi_{z_e}^{\overline{a}_e/q_e}) = \xi^{q_e}\pi_{z_e}^{\overline{a}_e/q_e}.$$

This means that $\mathcal{L}_{W_e,1}$ is locally generated by $\pi_{z_e}^{\overline{a}_e/q_e}$ at w_e. Let $\pi_{w_e} \overset{\text{def}}{=} (\pi_{z_e})^{m_e} \in \mathcal{O}_{W_e,w_e}$ which is a uniformizer of the maximal ideal of the discrete valuation ring $\mathcal{O}_{W_e,w_e}$. Thus, $\mathcal{L}_{W_e,1}^{\otimes m_e}$ is locally isomorphic to $\mathcal{O}_{W_e}(-(\overline{a}_e/q_e)w_e)$ at w_e via $(\pi_{z_e}^{\overline{a}_e/q_e})^{\otimes m_e} \mapsto \pi_{w_e}^{\overline{a}_e/q_e}$.

We put

$$\pi \overset{\text{def}}{=} \prod_{i=0}^{q_e-1} \tau^i(\pi_{z_e}^{\overline{a}_e/q_e}).$$

Note that $\tau(\pi) = \xi^{q_e}\pi$. We obtain that $\mathcal{L}_1^{\otimes q_e}$ is locally generated by π at x_e. Since $\pi^m = \pi_{z_e}^{\overline{a}_e}$, the isomorphism $\mathcal{L}_{W_e,1}^{\otimes m_e} \cong \mathcal{O}_{W_e}(-(\overline{a}_e/q_e)w_e)$ locally at w_e obtained above induces

$$\mathcal{L}_1^{\otimes q_e m} \cong \det((f_2)_*(\mathcal{L}_{W_e,1}^{\otimes m_e}))^{\otimes q_e}$$

locally at x_e, where $\det(-)$ denotes the determinate of the sheaf $(-)$. On the other hand, by applying [H, Chapter IV Exercises 2.6], we obtain the following isomorphisms

$$\det((f_2)_*(\mathcal{L}_{W_e,1}^{\otimes m_e})) \cong \det((f_2)_*(\mathcal{O}_{W_e}(-\sum_{\tau^i(w_e)\in f_2^{-1}(x_e)} (\overline{a}_e/q_e)\tau^i(w_e)))$$

$$\cong \det((f_2)_*\mathcal{O}_{W_e}) \otimes \mathcal{O}_{\widetilde{X}_v}((f_2)_*(-\sum_{\tau^i(w_e)\in f_2^{-1}(x_e)} (\overline{a}_e/q_e)\tau^i(w_e)))$$

$$\cong \det((f_2)_*\mathcal{O}_{W_e}) \otimes \mathcal{O}_{\widetilde{X}_v}(-\overline{a}_e x_e)$$

locally at x_e. Moreover, since f_2 is étale over x_e, we have $\det((f_2)_*\mathcal{O}_{W_e})^{\otimes q_e} \cong \mathcal{O}_{\widetilde{X}_v}$ locally at x_e. Then we obtain

$$\mathcal{L}_1^{\otimes q_e m} \cong \mathcal{O}_{\widetilde{X}_v}(-q_e \sum_{e\in e^{\text{op}}(\Gamma_v)} \overline{a}_e x_e).$$

Since $\mathcal{L}_1^{\otimes m}$ is locally generated by $\pi_{x_e}^{\text{ord}_{x_e}(D_{\overline{\alpha}})}$ at x_e via the fixed isomorphism $\mathcal{L}_1^{\otimes m} \overset{\sim}{\to} \mathcal{O}_{\widetilde{X}_v}(-D_{\overline{\alpha}})$, we obtain that

$$q_e D_{\overline{\alpha}} = q_e \sum_{e\in e^{\text{op}}(\Gamma_v)} \overline{a}_e x_e.$$

Then $D_{\overline{\alpha}} = \sum_{e\in e^{\text{op}}(\Gamma_v)} \overline{a}_e x_e$. This completes the proof of the lemma. □

We denote by $\mathcal{L}_\alpha$ the line bundle $\mathcal{L}_1$ and by D_α the effective divisor $\sum_{e\in e^{\text{op}}(\Gamma_v)} a_e x_e$. Note that $\mathcal{L}_\alpha^{\otimes n} \cong \mathcal{O}_{\widetilde{X}_v}(-D_\alpha)$. Then we obtain a morphism

$$\mathscr{T}_{v,n} \to \widetilde{\mathscr{P}}_{\widetilde{X}_v^\bullet,n}$$

$$\alpha \mapsto ([\mathcal{L}_\alpha], D_\alpha).$$

It is easy to check that this morphism is an isomorphism.

We maintain the notation introduced in Section 3. Let $H_{v,n}$ be the kernel of the composition of surjective homomorphisms

$$\Pi_v \twoheadrightarrow \Pi_v^{\mathrm{ab}} \overset{\phi_v}{\twoheadrightarrow} M_v \otimes \mathbb{Z}/n\mathbb{Z}$$

and $X^\bullet_{H_{v,n}} \overset{\mathrm{def}}{=} (X_{H_{v,n}}, D_{X_{H_{v,n}}}) \to \widetilde{X}^\bullet_v$ the Galois admissible covering over k corresponding to $H_{v,n}$. For each $C \in \pi_0(v)$, we put

$$D'_{\widetilde{X}_{v,C}} \overset{\mathrm{def}}{=} \{x_e \in D_{\widetilde{X}_v} \mid e \in E_{v,C}\}.$$

We define a smooth pointed semi-stable curve of type $(g_v, n_{v,C} \overset{\mathrm{def}}{=} \#E_{v,C})$ over k to be

$$\widetilde{X}^\bullet_{v,C} = (\widetilde{X}_{v,C}, D_{\widetilde{X}_{v,C}}) \overset{\mathrm{def}}{=} (\widetilde{X}_v, D'_{\widetilde{X}_{v,C}}).$$

Then we have the following proposition.

Proposition 4.10. *(i) Suppose that* $(g_v, \#E_v^{>1}) = (0,0)$. *Then*

$$\lim_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = 0.$$

(ii) Suppose that $(g_v, \#E_v^{>1}) \neq (0,0)$. *Then we have*

$$0 \leq \limsup_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} \leq g_v + \#E_v^{>1} - 1,$$

where $\limsup(-)$ *denotes the limit superior of* $(-)$. *Moreover, we have*

$$\lim_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = \begin{cases} g_v - 1, & \text{if } \#E_v^{>1} = 0, \\ g_v, & \text{if } \#E_v^{>1} = 1. \end{cases}$$

Proof. Frist, we prove (i). Since $H_{v,n}$ is trivial, we have $\sigma(X^\bullet_{H_{v,n}}) = \sigma(\mathbb{P}^1_k) = 0$. Then we have

$$\lim_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = 0.$$

Next, we prove (ii). We put

$$\mathscr{N}_{v,n} \overset{\mathrm{def}}{=} \{H \subseteq \Pi_v \text{ an open normal} \mid H_{v,n} \subseteq H \text{ and } \Pi_v/H \text{ is cyclic}\}.$$

Note that the order of Π_v/H, $H \in \mathscr{N}_{v,n}$, is prime to p. Write $X^\bullet_H \overset{\mathrm{def}}{=} (X_H, D_{X_H})$ for the pointed stable curve over k corresponding to H. Since $M_v \otimes \mathbb{Z}/n\mathbb{Z}$ is an abelian group, we have the following canonical decomposition

$$H^1_{\text{ét}}(X_{H_{v,n}}, \mathbb{F}_p) = \bigoplus_{H \in \mathscr{N}_{v,n}} (H^1_{\text{ét}}(X_{H_{v,n}}, \mathbb{F}_p)^{H/H_{v,n}})^{(\Pi_v/H)\text{-pri}}$$

$$= \bigoplus_{H \in \mathscr{N}_{v,n}} H^1_{\text{ét}}(X_H, \mathbb{F}_p)^{(\Pi_v/H)\text{-pri}},$$

where $(-)$-pri means the primitive part as an $\mathbb{F}_p[(-)]$-module. Then we have

$$\sigma(X^\bullet_{H_{n,v}}) = \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(X_{H_{v,n}}, \mathbb{F}_p)^\vee) = \sum_{H \in \mathscr{N}_v} \sum_{\chi : \Pi_v/H \hookrightarrow k^\times} \gamma_\chi(H^1_{\text{ét}}(X_H, \mathbb{F}_p)).$$

Moreover, we put

$$\mathscr{Q}_{v,n} \overset{\text{def}}{=} \{(H, \chi) \mid H \in \mathscr{N}_{v,n} \text{ and } \chi : \Pi_v/H \hookrightarrow k^\times\}.$$

For each pair $(H, \chi) \in \mathscr{Q}_{v,n}$, the composition of the homomorphisms $\Pi_v \twoheadrightarrow \Pi_v/H \overset{\chi}{\hookrightarrow} \mu_n \subseteq k^\times$ induces an element

$$\alpha_{(H,\chi)} \in \mathscr{T}_{v,n}.$$

Moreover, [T1, 4.7] implies that $\gamma_{\alpha_{(H,\chi)},1} = \gamma_\chi(H^1_{\text{ét}}(X_H, \mathbb{F}_p))$. We obtain that

$$\sigma(X^\bullet_{H_{n,v}}) = \sum_{(H,\chi) \in \mathscr{Q}_{v,n}} \gamma_{\alpha_{(H,\chi)},1}.$$

We put

$$\mathscr{A}_{v,n} \overset{\text{def}}{=} \{\alpha \in \mathscr{T}_{v,n} \mid K_v \subseteq \ker(\alpha)\}.$$

Then we have $\#\mathscr{A}_{v,n} = \#M_v \otimes \mathbb{Z}/n\mathbb{Z}$. Moreover, Proposition 3.4 implies that

$$\prod_{e \in E_{v,C}} \alpha([s_e]) = 1, \ C \in \pi_0(v).$$

Let $(H, \chi) \in \mathscr{Q}_{v,n}$ and $\alpha_{(H,\chi)} \in \mathscr{T}_{v,n}$ induced by (H, χ). The definition of $\mathscr{N}_{v,n}$ implies that, the homomorphism $\Pi_v \twoheadrightarrow \Pi_v/H$ factors through the natural surjective homomorphism $\Pi_v \twoheadrightarrow M_v \otimes \mathbb{Z}/n\mathbb{Z}$. Then we obtain a map

$$\mathscr{Q}_{v,n} \to \mathscr{A}_{v,n}$$

defined by $(H, \chi) \mapsto \alpha_{(H,\chi)}$. Moreover, it is easy to check that this map is a bijection. Thus, we have

$$\sigma(X^\bullet_{H_{n,v}}) = \sum_{\alpha \in \mathscr{A}_{v,n}} \gamma_{([\mathcal{L}_\alpha], D_\alpha)}.$$

Note that we have $n | \deg(D_\alpha)$.

On the other hand, let $\gamma \in \mathscr{A}_{v,n}$ such that $s(D^{(i)}_\gamma)$=1 for some $i \in \{0, 1, \ldots, t-1\}$. We have the following claim.

Claim: There exists $\beta \in \mathscr{A}_{v,n}$ such that $D_\beta = D^{(i)}_\gamma$ (cf. Definition 4.4 (ii)).

Let us prove the claim. We see that

$$D^{(i)}_\gamma = D_\gamma(p^{t-i}),$$

where $D_\gamma(p^{t-i}) \stackrel{\text{def}}{=} p^{t-i}D_\gamma - n[p^{t-i}D_\gamma/n]$. Then we have that $s(D_\gamma(p^{t-i})) = 1$, and that

$$\mathrm{Supp}(D_\gamma(p^{t-i})) = \mathrm{Supp}(D_\gamma) \subseteq D_{\widetilde{X}_v, C_\gamma}$$

for a unique $C_\gamma \in \pi_0(v)$. For each $e \in e^{\mathrm{op}}(\Gamma_v)$, we write $\overline{[s_e]}$ for the image of $[s_e]$ under the natural surjection $I_e \twoheadrightarrow I_e \otimes \mathbb{Z}/n\mathbb{Z}$. Then the structure of the maximal prime-to-p quotients of admissible fundamental groups implies that

$$\Pi_v^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z} \cong$$

$$\cong (\langle a_1, \dots, a_{g_v}, b_1, \dots, b_{g_v}\rangle^{\mathrm{ab}} \oplus \langle \{[s_e]\}_{e \in e^{\mathrm{op}}(\Gamma_v)} \mid \sum_{e \in e^{\mathrm{op}}(\Gamma_v)} [s_e] = 0\rangle) \otimes \mathbb{Z}/n\mathbb{Z}.$$

Write $\Pi_{v,n}^{\mathrm{unr}}$ for the subgroup of $\Pi_v^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z}$ generated by $a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}$, and $\Pi_{v,n}^{\mathrm{ram}}$ for the subgroup $I_v^{\mathrm{op}} \otimes \mathbb{Z}/n\mathbb{Z} = \langle \{\overline{[s_e]}\}_{e \in e^{\mathrm{op}}(\Gamma_v)}\rangle$. Then we have

$$\Pi_v^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z} \cong \Pi_{v,n}^{\mathrm{unr}} \oplus \Pi_{v,n}^{\mathrm{ram}}.$$

Note that since $\mathscr{T}_{v,n}$ is naturally isomorphic to $\mathscr{T}'_{v,n} \stackrel{\text{def}}{=} \mathrm{Hom}(\Pi_v^{\mathrm{ab},p'} \otimes \mathbb{Z}/n\mathbb{Z}, \mu_n)$, γ can be regarded as an element of $\mathscr{T}'_{v,n}$. We define an element $\beta \in \mathscr{T}'_{v,n}$ to be

$$\beta|_{\Pi_{v,n}^{\mathrm{unr}}} \stackrel{\text{def}}{=} (\gamma|_{\Pi_{v,n}^{\mathrm{unr}}})^{p^{t-i}}, \ \beta(\overline{[s_e]}) = \zeta^{\mathrm{ord}_{x_e}(D_\gamma^{(i)})}.$$

Note that since $\prod_{e \in E_{v,C}} \gamma([s_e]) = 1$ for each $C \in \pi_0(v)$, we have $\beta \in \mathscr{A}_{v,n}$. Moreover, we have

$$\mathrm{Supp}(D_\beta) = \mathrm{Supp}(D_\gamma) \subseteq D_{\widetilde{X}_v, C_\gamma}, \ D_\beta = D_\gamma(p^{t-i}).$$

This completes the proof of the claim.

Write $\mathcal{L}_\gamma(p^{t-i})$ for $\mathcal{L}_\gamma^{\otimes p^{t-i}} \otimes \mathcal{O}_X([p^{t-i}D/n])$. Then we observer that

$$([\mathcal{L}_\gamma(p^{t-i})], D_\gamma(p^{t-i})) \in \widetilde{\mathscr{P}}_{X_v^\bullet, n}, \ ([\mathcal{L}_\beta], D_\beta) = ([\mathcal{L}_\gamma(p^{t-i})], D_\gamma(p^{t-i})), \ s(D_\beta) = 1.$$

Furthermore, Lemma 4.5 implies that

$$\gamma_{([\mathcal{L}_\beta], D_\beta)} = \gamma_{([\mathcal{L}_\gamma], D_\gamma)}.$$

Suppose that $\#E_v^{>1} = 0$. Since $(g_v, \#E_v^{>1}) \neq (0,0)$, we have $g_v > 0$. Then

$$\#\mathscr{A}_{v,n} = \#(M_v \otimes \mathbb{Z}/n\mathbb{Z}) = n^{2g_v}.$$

Moreover, for each $\alpha \in \mathscr{A}_{v,n}$, we have

$$\gamma_{([\mathcal{L}_\alpha], D_\alpha)} \leq \begin{cases} g_v, & \text{if } \mathcal{L}_a \cong \mathcal{O}_{\widetilde{X}_v}, \\ g_v - 1, & \text{otherwise.} \end{cases}$$

Thus, we obtain

$$\sigma(X_{H_{n,v}}^\bullet) \leq (g_v - 1)(n^{2g_v} - 1) + g_v.$$

On the other hand, note that for each $\alpha \in \mathscr{A}_{v,n}$, we have $D_\alpha = 0$. Then by applying Theorem 4.7 (i), we obtain that

$$\sigma(X^\bullet_{H_{n,v}}) \geq (g_v - 1)(n^{2g_v} - C(g_v)n^{2g_v-1}).$$

Thus,

$$\lim_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = g_v - 1.$$

Suppose that $\#E_v^{>1} \geq 1$. Let $d_{v,C} \geq \log_p(n_X - 1)$, $C \in E_v^{>1}$, be an arbitrary positive natural number and $\epsilon_{v,C} < 1$ an arbitrary positive real number. We put

$$\Lambda_{v,C} = \frac{d_{v,C}}{\epsilon_{v,C}}, \text{ and } \lambda_{v,C} = (1 - \frac{1}{p^{d_{v,C}(n_{v,C}-1)}}\binom{n_{v,C}-1}{p^{d_{v,C}}})^{\frac{(1-\epsilon_{v,C})}{d_{v,C}}},$$

and suppose that $t \geq \max\{\Lambda_{v,C}\}_{C\in\pi_0(v)}$. Dividing the sum

$$\sigma(X^\bullet_{H_{n,v}}) = \sum_{\alpha\in\mathscr{A}_{v,n}} \gamma_{([\mathcal{L}_\alpha],D_\alpha)} = S_1 + S_2 + S_3$$

into three parts, where S_l, $l \in \{1, 2, 3\}$, denotes the sum of $\gamma_{([\mathcal{L}_\alpha],D_\alpha)}$ that the D_α satisfies the condition (l): (1) $s(D_\alpha) = 0$; (2) $s(D_\alpha^{(i)}) = 1$ for some $i \in \{0, 1, \ldots, t-1\}$; (3) otherwise.

Suppose that $\#E_v^{>1} = 1$. Then we may assume that $E_v^{>1} = E_{v,C}$ for some $C \in \pi_0(v)$. By applying Corollary 3.5, we have

$$\#\mathscr{A}_v = \#(M_v \otimes \mathbb{Z}/n\mathbb{Z}) = n^{2g_v+n_{v,C}-1}.$$

Proposition 4.6 and Theorem 4.7 imply that

$$\sigma(X^\bullet_{H_{n,v}}) \leq g_v + (g_v - 1)(n^{2g_v} - 1) + g_v n^{2g_v}(n^{n_{v,C}-1}(1 - \lambda^t_{v,C}) - 1))$$
$$+(g_v + n_{v,C} - 2)n^{2g_v}(n^{n_{v,C}-1}\lambda^t_{v,C}).$$

On the other hand, we have

$$\sigma(X^\bullet_{H_{n,v}}) \geq S_1 + S_2$$
$$\geq (g_v - 1)(n^{2g_v} - C(g_v)n^{2g_v-1}) + g_v(n^{2g_v} - C(g_v)n^{2g_v-1})(n^{n_{v,C}-1}(1 - \lambda^t_{v,C}) - 1).$$

Thus, we obtain that

$$\lim_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = g_v.$$

Suppose that $\#E_v^{>1} > 1$. Let T_j, $j \in \{1, \ldots, \#E_v^{>1}-1\}$ (resp. $T'_{j'}$, $j' \in \{0, 1, \ldots, \#E_v^{>1}\}$), denotes the sum of $\gamma_{([\mathcal{L}_\alpha],D_\alpha)}$ that D_α satisfies the following conditions:

(i) $s(D_\alpha^{(i)}) > 1$ for each $i \in \{0, 1, \ldots, t-1\}$;

(ii) there exists a subset $E_\alpha \subseteq E_v^{>1}$ and a set of divisors

$$\{D_{\alpha,C} \overset{\text{def}}{=} \sum_{e\in E_{v,C}} \text{ord}_{x_e}(D_\alpha)x_e,\ C \in E_v^{>1}\}$$

such that $\#E_\alpha = j$ (resp. $\#E_\alpha = j'$), that $s(D^{(i_C)}_{\alpha,C}) \neq 0$ (resp. $s(D^{(i_C)}_{\alpha,C}) = 1$) for some $i_C \in \{0,1,\dots,t-1\}$ if $C \in E_\alpha$, and that $s(D^{(i)}_{\alpha,C}) = 0$ (resp. $s(D^{(i)}_{\alpha,C}) > 1$) holds for each $i \in \{0,1,\dots,t-1\}$ if $C \notin E_\alpha$.

Then we have

$$S_3 = \sum_{j=1}^{\#E_v^{>1}-1} T_j + \sum_{j'=0}^{\#E_v^{>1}} T'_{j'}.$$

Note that since $\deg(\mathcal{L}_\alpha) = -\deg(D_\alpha)/n = -s(D_\alpha)$, we have

$$\gamma_{([\mathcal{L}_\alpha],D_\alpha)} = \gamma_{\alpha_{\mathcal{L}_\alpha},1} \leq \dim_k(H^1(\widetilde{X}_v, \mathcal{L}_\alpha))$$

$$= g_v - 1 - \deg(\mathcal{L}_\alpha) + \dim_k(H^0(\widetilde{X}_v, \mathcal{L}_\alpha)) = g_v + s(D_\alpha) - 1.$$

For each $m \in \{0,1,\dots,E_v^{>1}\}$, we put

$$E_m \overset{\text{def}}{=} \{E \subseteq E_v^{>1} \text{ subset} \mid \#E = m\}.$$

When $j \in \{1,\dots,\#E_v^{>1}-1\}$, we have

$$T_j \leq \sum_{E \in E_j} (g_v + \sum_{C \in E} n_{v,C} - 1)(n^{2g_v + \sum_{C\in E}(n_{v,C}-1)}).$$

Moreover, since $s(D^{(i)}_\alpha) \leq n_v - 1$ for each $i \in \{0,1,\dots,t-1\}$, by applying Theorem 4.7 to $\widetilde{X}^\bullet_{v,C}$, $C \in E_v^{>1}$, we have and

$$T'_{j'} \leq (g_v + n_v - 2)n^{2g_v + \sum_{C\in E_v^{>1}}(n_{v,C}-1)}(\sum_{E\in E_{j'}} \prod_{C \in E_v^{>1}\setminus E} \lambda^t_{v,C})$$

when $j' \in \{0,1,\dots,\#E_v^{>1}-1\}$. When $j' = \#E_v^{>1}$, we have that, for each $C \in E_v^{>1}$, $s(D^{(i_C)}_{\alpha,C}) = 1$ for some $i_C \in \{0,1,\dots,t-1\}$. Then there exists an element $\delta \in \mathscr{A}_{v,n}$ such that

$$D_\delta = \sum_{C \in E_v^{>1}} D^{(i_C)}_{\alpha,C}.$$

Note that $s(D_\delta) = \#E_v^{>1}$. Then $\gamma_{([\mathcal{L}_\delta],D_\delta)} \leq \dim_k(H^1(\widetilde{X}_v, \mathcal{L}_\delta)) = g_v + \#E_v^{>1} - 1$. Thus, we have

$$T'_{\#E_v^{>1}} \leq (g_v + \#E_v^{>1} - 1)n^{2g_v + \sum_{C\in E_v^{>1}}(n_{v,C}-1)}.$$

Then we obtain that

$$\sigma(X^\bullet_{H_{n,v}}) = S_1 + S_2 + S_3 \leq g_v + (g_v - 1)(n^{2g_v} - 1) + g_v n^{2g_v}(\sum_{C\in E_v^{>1}} (n^{n_{v,C}-1}(1 - \lambda^t_{v,C}) - 1))$$

$$+ \sum_{j=1}^{\#E_v^{>1}-1} (\sum_{E\in E_j} (g_v + \sum_{C\in E} n_{v,C} - 1)(n^{2g_v + \sum_{C\in E}(n_{v,C}-1)}))$$

$$+\sum_{j'=0}^{\#E_v^{>1}-1}((g_v+n_v-1)n^{2g_v+\sum_{C\in E_v^{>1}}(n_{v,C}-1)}(\sum_{E\in E_{j'}}\prod_{C\in E_v^{>1}\setminus E}\lambda_{v,C}^t))$$
$$+(g_v+\#E_v^{>1}-1)n^{2g_v+\sum_{C\in E_v^{>1}}(n_{v,C}-1)}.$$

Note that

$$\sum_{C\in E_v^{>1}}(n_{v,C}-1)=\sum_{C\in\pi_0(v)}(n_{v,C}-1).$$

Proposition 3.4 implies that

$$\#\mathscr{A}_{v,n}=\#(M_v\otimes\mathbb{Z}/n\mathbb{Z})=n^{2g_v+\sum_{C\in\pi_0(v)}(n_{v,C}-1)}.$$

Then

$$0\leq\limsup_{t\to\infty}\frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v\otimes\mathbb{Z}/n\mathbb{Z})}\leq g_v+\#E_v^{>1}-1.$$

This completes the proof of the proposition. □

Remark 4.10.1. Suppose that $(g_v,\#E_v^{>1})\neq(0,0)$ and $\#E_v^{>1}>1$. We do not know whether

$$\lim_{t\to\infty}\frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v\otimes\mathbb{Z}/n\mathbb{Z})}$$

exists or not. Moreover, if the limit above exists, we do not know whether the limit can attain the upper bound $g_v+\#E_v^{>1}-1$ or not in general. The main difficulty is that we do not know whether or not the Raynaud-Tamagawa theta divisor exist if $s(D_\alpha)=\#E_v^{>1}$ and $\sum_{e\in E_{v,C}}\mathrm{ord}_{x_e}(D_\alpha)=n$, $C\in E_v^{>1}$ (cf. Remark 4.7.1).

Remark 4.10.2. Motivated by the theory of the combinatorial anabelian geometry of curves over algebraically closed fields of characteristic $p>0$, we may expect that

$$\lim_{t\to\infty}\frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v\otimes\mathbb{Z}/n\mathbb{Z})}$$

exists and admits a better lower bounder than 0. We pose the following question.

Question . *Does the limit*

$$\lim_{t\to\infty}\frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v\otimes\mathbb{Z}/n\mathbb{Z})}$$

exist? Moreover, if the limit exists, does

$$\lim_{t\to\infty}\frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v\otimes\mathbb{Z}/n\mathbb{Z})}\geq g_v-1$$

hold?

5 Lower bounds and upper bounds for the limits of p-averages of admissible fundamental groups

In this section, we prove the first main theorem of the present paper. We maintain the notation introduced in Section 3.

Let t be an arbitrary positive natural number and $n \overset{\text{def}}{=} p^t - 1$. We denote by K_n the kernel of the natural surjective homomorphism $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$ and denote by

$$X_{K_n}^\bullet = (X_{K_n}, D_{X_{K_n}})$$

the pointed stable curve over k corresponding to K_n. Write $\Gamma_{X_{K_n}^\bullet}$ for the dual semi-graph of $X_{K_n}^\bullet$ and $r_{X_{K_n}}$ for the Betti number of $\Gamma_{X_{K_n}^\bullet}$.

Definition 5.1. Let $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X_\infty})$ and $e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \subseteq e^{\text{cl}}(\Gamma_{X_\infty})$. We shall say that v is a tree-like vertex if $\Gamma_{X_v^{\text{def}}}$ is a tree (i.e., the Betti number of $\Gamma_{X_v^{\text{def}}}$ is 0), and call that e is a tree-like edge if there exists a vertex $w \in v(\Gamma_{X^\bullet})$ such that $E_{w,C} = \{e\}$ for some $C \in E_w^{=1}$. We put

$$V_{X^\bullet}^{\text{tre}} \overset{\text{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid v \text{ is tree-like}\},$$

$$V_{X^\bullet}^{\text{tre},g_v=0} \overset{\text{def}}{=} \{v \in V_{X^\bullet}^{\text{tre}} \mid g_v = 0\},$$

$$E_{X^\bullet}^{\text{tre}} \overset{\text{def}}{=} \{e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \mid e \text{ is tree-like}\} = \bigcup_{v \in v(\Gamma_{X^\bullet})} \bigcup_{C \in \pi_0(v) \text{ s.t. } C \in E_v^{=1}} E_{v,C}.$$

Remark 5.1.1. Note that the definition of tree-like vertices and tree-like edges does not depend on the choices of $X_{v_\infty}^\bullet$.

Then we have the following formula for the limits of the p-averages of admissible fundamental groups which generalizes Tamagawa's results (cf. [T1, Theorem 0.5] and [T2, Theorem 3.10]).

Theorem 5.2. *We have*

$$g_X - r_X - \#V_{X^\bullet}^{\text{tre}} + \#V_{X^\bullet}^{\text{tre},g_v=0} + \#E_{X^\bullet}^{\text{tre}} - \sum_{v \in v(\Gamma_{X^\bullet}) \text{ s.t. } \#E_v^{>1} > 1} g_v$$

$$\leq \limsup_{t \to \infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} \leq g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V_{X^\bullet}^{\text{tre},g_v=0} + \#E_{X^\bullet}^{\text{tre}} + \sum_{v \in v(\Gamma_{X^\bullet})} \#E_v^{>1}.$$

In particular, if $\#E_v^{>1} \leq 1$ for each $v \in v(\Gamma_{X^\bullet})$, then we have

$$\begin{aligned}\text{Avr}_p(\Pi_{X^\bullet}) &= g_X - r_X - \#V_{X^\bullet}^{\text{tre}} + \#V_{X^\bullet}^{\text{tre},g_v=0} + \#E_{X^\bullet}^{\text{tre}} - \sum_{v \in v(\Gamma_{X^\bullet}) \text{ s.t. } \#E_v^{>1} > 1} g_v \\ &= g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V_{X^\bullet}^{\text{tre},g_v=0} + \#E_{X^\bullet}^{\text{tre}} + \sum_{v \in v(\Gamma_{X^\bullet})} \#E_v^{>1} \\ &= g_X - r_X - \#V_{X^\bullet}^{\text{tre}} + \#V_{X^\bullet}^{\text{tre},g_v=0} + \#E_{X^\bullet}^{\text{tre}}.\end{aligned}$$

Proof. Remark 2.3.1 implies that

$$\sigma(X^\bullet_{K_n}) = \sum_{v\in v(\Gamma_{X^\bullet})} \frac{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})}{\#(M_v\otimes \mathbb{Z}/n\mathbb{Z})}\sigma(X^\bullet_{H_{v,n}}) + r_{X_{K_n}},$$

where $X^\bullet_{H_{v,n}}$, $v\in v(\Gamma_{X^\bullet})$, is a pointed stable curve over k defined in Section 4.2. Moreover, the Euler-Poincaré characteristic formula for dual semi-graphs implies that

$$\begin{aligned} r_{X_{K_n}} &= \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{K_n}}) - \#v(\Gamma_{X^\bullet_{K_n}}) + 1 \\ &= \sum_{e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet})} \frac{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})}{\#I_{e,n}} - \sum_{v\in v(\Gamma_{X^\bullet})} \frac{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})}{\#(M_v\otimes \mathbb{Z}/n\mathbb{Z})} + 1, \end{aligned}$$

where $I_{e,n}$, $e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, denotes the image of the inertia subgroup I_e of e in $\Pi^{\mathrm{ab}}_{X^\bullet}\otimes\mathbb{Z}/n\mathbb{Z}$. Moreover, the structure of the maximal prime-to-p quotients of admissible fundamental groups implies that

$$\#I_{e,n} = \begin{cases} 1, & \text{if } e\in E^{\mathrm{tre}}_{X^\bullet}, \\ n, & \text{otherwise.}\end{cases}$$

Then we obtain that

$$\frac{\sigma(X^\bullet_{K_n})}{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})} = \sum_{v\in v(\Gamma_{X^\bullet})} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v\otimes \mathbb{Z}/n\mathbb{Z})}$$

$$+\#E^{\mathrm{tre}}_{X^\bullet} + \sum_{e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet})\setminus \bigcup_{v\in v(\Gamma_{X^\bullet})} E_v^{=1}} \frac{1}{n} - \sum_{v\in v(\Gamma_{X^\bullet})} \frac{1}{\#(M_v\otimes \mathbb{Z}/n\mathbb{Z})} + \frac{1}{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})}.$$

Thus, by applying Proposition 4.10, we obtain that

$$\begin{aligned} g_X - r_X - \#V^{\mathrm{tre}}_{X^\bullet} + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet} - \sum_{v\in v(\Gamma_{X^\bullet}) \text{ s.t. } \#E_v^{>1}>1} g_v \\ = \sum_{v\in v(\Gamma_{X^\bullet}) \text{ s.t. } g_v\neq 0,\ \#E_v^{>1}\leq 1} (g_v + \#E_v^{>1} - 1) + \#E^{\mathrm{tre}}_{X^\bullet} \\ \leq \limsup_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\mathrm{ab}}\otimes \mathbb{F}_p)}{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})} = \limsup_{t\to\infty} \frac{\sigma(X^\bullet_{K_n})}{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})} \\ \leq \sum_{v\in v(\Gamma_{X^\bullet}) \text{ s.t. } (g_v,\#E_v^{>1})\neq(0,0)} (g_v + \#E_v^{>1} - 1) + \#E^{\mathrm{tre}}_{X^\bullet} \\ = \sum_{v\in v(\Gamma_{X^\bullet})} g_v + \sum_{v\in v(\Gamma_{X^\bullet})} \#E_v^{>1} - \#v(\Gamma_{X^\bullet}) + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet} \\ = g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet} + \sum_{v\in v(\Gamma_{X^\bullet})} \#E_v^{>1}. \end{aligned}$$

This completes the proof of the theorem. □

Remark 5.2.1. Suppose that $\Gamma^{\rm cpt}_{X^\bullet}$ is 2-connected. Then we have $\#E_v^{>1} \leq 1$ and $\#V^{{\rm tre}, g_v=0}_{X^\bullet} = 0$. Then we have

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#V^{\rm tre}_{X^\bullet} + \#E^{\rm tre}_{X^\bullet}.$$

This formula has been obtained essentially by Tamagawa (cf. [T2, Theorem 3.10]). Moreover, suppose that $X^\bullet$ is smooth over k. Then we have

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - \#V^{\rm tre}_{X^\bullet}.$$

Note that we have

$$\#V^{\rm tre}_{X^\bullet} = \begin{cases} 1 & \text{if } n_X \leq 1 \\ 0 & \text{if } n_X > 1. \end{cases}$$

This is the formula of Tamagawa obtained in [T1, Theorem 0.5].

Remark 5.2.2. For each $v \in v(\Gamma_{X^\bullet})$, we put

$$b(v) \overset{\rm def}{=} \sum_{e \in e^{\rm op}(\Gamma_{X^\bullet}) \cup e^{\rm cl}(\Gamma_{X^\bullet})} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that e meets v. Moreover, we put

$$v(\Gamma_{X^\bullet})^{b \leq 1} \overset{\rm def}{=} \{v \in v(\Gamma_{X^\bullet}) \mid b(v) \leq 1\}.$$

Note that if $\Gamma^{\rm cpt}_{X^\bullet}$ is 2-connected, then $\#v(\Gamma_{X^\bullet})^{b\leq 1} = \#V^{\rm tre}_{X^\bullet}$. Then the statement of [T2, Theorem 3.10] is as follows.

Suppose that $\Gamma^{\rm cpt}_{X^\bullet}$ is 2-connected. Then we have

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#v(\Gamma_{X^\bullet})^{b\leq 1}.$$

Since the computation of $r_{X_{K_n}}$ in the proof of [T2, Theorem 3.10] has an error, the statement of the formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ of [T2, Theorem 3.10] is not correct.

6 A formula for the limits of p-averages of admissible fundamental groups of component-generic pointed stable curves

In this section, we prove a formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ when each irreducible component is generic. We maintain the notation introduced in Section 3 and Section 4. Let t be an arbitrary positive natural number, $n \overset{\rm def}{=} p^t - 1$, and $\mu_n \subseteq k^\times$ the group of $n^{\rm th}$ roots of unity. Fix a primitive $n^{\rm th}$ root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the map $\zeta^i \mapsto i$.

6.1 Degeneration and existence of Raynaud-Tamagawa theta divisors

We introduce a condition concerning degeneration.

Condition 6.1. Let $v \in v(\Gamma_{X^\bullet})$. We shall say that the pair $(\widetilde{X}_v^\bullet, \pi_0(v))$ admits a (DEG) if there exist a complete discrete valuation ring R_v with an algebraically closed residue field k_{R_v} and a pointed stable curve $\mathcal{X}_v^\bullet = (\mathcal{X}_v, D_{\mathcal{X}_v})$ of type (g_v, n_v) over R_v satisfying the following conditions:

(i) There is an algebraically closed field $k' \supseteq k$ such that k' contains the quotient field K_{R_v} of R_v, and that, by replacing k by k' and $\widetilde{X}_v^\bullet$ by $\widetilde{X}_v^\bullet \times_k k'$, $\widetilde{X}_v^\bullet$ is k-isomorphic to $\mathcal{X}_v^\bullet \times_{R_v} k$. Moreover, the k-isomorphism induces a bijection $\iota_v : D_{\mathcal{X}_v} \overset{\sim}{\to} D_{\widetilde{X}_v}$. For each $C \in \pi_0(v)$, write $D_{v,C}$ for $\iota_v^{-1}(\{x_e\}_{e \in E_{v,C}})$. Then we have

$$D_{\mathcal{X}_v} = \bigcup_{C \in \pi_0(v)} D_{v,C}.$$

(ii) By replacing k and $\widetilde{X}_v^\bullet$ by k' and $\widetilde{X}_v^\bullet \times_k k'$, respectively, we may assume that $k' = k$. We write $\overline{K}_{R_v}$ for the algebraic closure of K_{R_v} in k. Write $\mathcal{X}_{v,\overline{\eta}}^\bullet = (\mathcal{X}_{v,\overline{\eta}}, D_{\mathcal{X}_{v,\overline{\eta}}})$ for the geometric generic fiber $\mathcal{X}_v^\bullet \times_{R_v} \overline{K}_{R_v}$ of $\mathcal{X}_v$ and $\mathcal{X}_{v,s}^\bullet = (\mathcal{X}_{v,s}, D_{\mathcal{X}_{v,s}})$ for the special fiber $\mathcal{X}_v^\bullet \times_{R_v} k_{R_v}$ of $\mathcal{X}_v^\bullet$. For each $C \in \pi_0(v)$, write $D_{v,C}^{\overline{\eta}}$ for $D_{v,C} \times_{R_v} \overline{K}_{R_v}$ and $D_{v,C}^s$ for $D_{v,C} \times_{R_v} k_{R_v}$. Then we have

$$D_{\mathcal{X}_{v,\overline{\eta}}} = \bigcup_{C \in \pi_0(v)} D_{v,C}^{\overline{\eta}}, \; D_{\mathcal{X}_{v,s}} = \bigcup_{C \in \pi_0(v)} D_{v,C}^s.$$

Moreover, we have

$$\mathcal{X}_{v,s} = (\bigcup_{C \in E_v^{>1}} P_{v,C}) \cup Z_v$$

such that the following conditions hold: (1) $D_{v,C}^s$ is contained in $P_{v,C}$ for each $C \in E_v^{>1}$; (2) $P_{v,C} \cong \mathbb{P}^1_{k_{R_v}}$; (3) the dual semi-graph of $\mathcal{X}_{v,s}^\bullet$ is a tree; (4) if $\#E_v^{>1} \neq 0$, then Z_v is either a smooth projective curve over k_{R_v} of genus g_v when $g_v \neq 0$ or an empty set when $g_v = 0$; (5) if $\#E_v^{>1} = 0$, then Z_v is a smooth projective curve over k_{R_v} of genus g_v.

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$ in k. For each $v \in v(\Gamma_{X^\bullet})$, write $\overline{\mathcal{M}}_{g_v,n_v}$ for the moduli stack $\overline{\mathcal{M}}_{g_v,n_v,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$. For each $0 \leq \sigma \leq g_v$, we denote by

$$\overline{\mathcal{M}}_{g_v,n_v}^\sigma$$

the p-rank strata of $\overline{\mathcal{M}}_{g_v,n_v}$ with p-rank σ (i.e., the locally closed reduced substack of $\overline{\mathcal{M}}_{g_v,n_v}$ whose geometric points corresponding to pointed stable curves with p-rank σ). Note that $\overline{\mathcal{M}}_{g_v,n_v}^\sigma$ is not irreducible in general.

Definition 6.2. For each $v \in v(\Gamma_{X^\bullet})$, write $\overline{M}^{\sigma}_{g_v,n_v}$ for the coarse moduli space of the substack $\overline{\mathcal{M}}^{\sigma}_{g_v,n_v}$. Let $q_v^{\sigma\text{-gen}}$ be a generic point of $\overline{M}^{\sigma}_{g_v,n_v}$ and $k(q_v^{\sigma\text{-gen}})$ the residue field of $q_v^{\sigma\text{-gen}}$. Suppose that $k(q_v^{\sigma\text{-gen}}) \subseteq k$ for each $v \in v(\Gamma_{X^\bullet})$. Let $k_{q_v^{\sigma\text{-gen}}}$ be the algebraic closure of the residue field of $k(q_v^{\sigma\text{-gen}})$ in k and $X^\bullet_{q_v^{\sigma\text{-gen}}}$ the geometric generic curve corresponding to the geometric generic point $\mathrm{Spec}\, k_{q_v^{\sigma\text{-gen}}} \to \mathrm{Spec}\, k(q_v^{\sigma\text{-gen}}) \to \overline{M}^{\sigma}_{g_v,n_v}$. We shall say that $X^\bullet$ is a *component-generic pointed stable curve* over k if $\widetilde{X}^\bullet_v$ is k-isomorphic to $X^\bullet_{q_v^{\sigma\text{-gen}}} \times_{k_{q_v^{\sigma\text{-gen}}}} k$ for each $v \in v(\Gamma_{X^\bullet})$.

We have the following proposition.

Proposition 6.3. *Suppose that $X^\bullet$ is a component-generic pointed stable curve over k. Then $(\widetilde{X}^\bullet_v, \pi_0(v))$ admits a* (DEG) *for each $v \in v(\Gamma_{X^\bullet})$.*

Proof. If $E_v^{>1} = \emptyset$, then the proposition is trivial. We may assume that $E_v^{>1} \neq \emptyset$, and let $E_v^{>1} \overset{\mathrm{def}}{=} \{C_1, \dots, C_q\}$. Then we have $n_v \geq 2$. For each $C_i \in E_v^{>1}$, we put

$$E_{v,C_i} = \{e_{(\sum_{j<i} n_{v,C_j})+1}, \dots, e_{\sum_{j\leq i} n_{v,C_j}}\}.$$

Moreover, we put

$$\bigcup_{C \in E_v^{=1}} E_{v,C} = \{e_{(\sum_{C\in E_v^{>1}} n_{v,C})+1}, \dots, e_{n_v}\}.$$

Then the order of $e^{\mathrm{op}}(\Gamma_v)$ defined above induces an order of the set of marked points $D_{\widetilde{X}_v}$.

Suppose that $g_v = 0$. Then the definition of component-generic pointed stable curves implies that $\widetilde{X}^\bullet_v$ is a geometric generic curve of $\overline{M}_{0,n_v}$. Then $(\widetilde{X}^\bullet_v, \pi_0(v))$ admits a (DEG).

Suppose that $g_v \geq 1$, and let $\sigma \overset{\mathrm{def}}{=} \sigma(\widetilde{X}^\bullet_v)$. Write $\pi_{g_v,n_v,1} : \overline{\mathcal{M}}_{g_v,n_v} \to \overline{\mathcal{M}}_{g_v,1}$ for the morphism induced by forgetting the marked points except the first marked point and $c_v : \mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_v,n_v}$ for the classifying morphism determined by $\widetilde{X}^\bullet_v \to \mathrm{Spec}\, k$. Then the composite morphism

$$\pi_{g_v,n_v,1} \circ c_v : \mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_v,1}$$

determines a smooth pointed stable curve

$$Z_v^{*,\bullet} = (Z_v^*, D_{Z_v^*} = \{z_v\})$$

over k. Note that $\sigma(Z_v^{*,\bullet}) = \sigma$. Let $P_{v,C_i} \cong \mathbb{P}^1_k$ for each $i \in \{1, \dots, q\}$, $D_{P_{v,C_i}}$ a set of distinct closed points $\{x_{1,C_i}, x_{2,C_i}\} \cup \{x_{(\sum_{j<i} n_{v,C_j})+1}, \dots, x_{\sum_{j\leq i} n_{v,C_j}}\}$ of P_{v,C_i} if $i \in \{1, \dots, q-1\}$, $D_{P_{v,C_q}}$ a set of distinct closed points $\{x_{1,C_1}\} \cup \{x_{(\sum_{j<q} n_{v,C_j})+1}, \dots, x_{\sum_{j\leq q} n_{v,C_j}}\}$ of P_{v,C_q}. Then we obtain a pointed stable curve

$$P^\bullet_{v,C_i} = (P_{v,C_i}, D_{P_{v,C_i}}),\ i \in \{1, \dots, q\},$$

and a pointed stable curve

$$Z^\bullet_v = (Z_v \overset{\mathrm{def}}{=} Z_v^*, D_{Z_v} \overset{\mathrm{def}}{=} \{z_v\} \cup \{x_{(\sum_{C\in E_v^{>1}} n_{v,C})+1}, \dots, x_{n_v}\})$$

over k. Note that $z_v \notin \{x_{(\sum_{C\in E_v^{>1}} n_{v,C})+1}, \dots, x_{n_v}\}$. We glue $\{P^\bullet_{v,C_i}\}_{i\in\{1,\dots,q\}}$ and $Z^\bullet_v$ by identifying z_v, x_{2,C_i}, $i \in \{1,\dots,q-1\}$, with x_{1,C_1}, $x_{1,C_{i+1}}$, $i\in\{1,\dots,q-1\}$, respectively. Thus, we obtain a pointed stable curve

$$\mathcal{X}^\bullet_{v,s} = (\mathcal{X}_{v,s}, D_{\mathcal{X}_{v,s}})$$

of type (g_v, n_v) over k which determines a classifying morphism $c_{v,s} : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_v,n_v}$. Moreover, we write $q_{v,s}$ for the image of the composite morphism

$$\operatorname{Spec} k \stackrel{c_{v,s}}{\to} \overline{\mathcal{M}}_{g_v,n_v} \to \overline{M}_{g_v,n_v}.$$

The construction of $\mathcal{X}^\bullet_{v,s}$ implies that the curve corresponding to the composite morphism

$$\pi_{g_v,n_v,1} \circ c_{v,s} : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_v,1}$$

is k-isomorphic to $Z^\bullet_v$. This means that $q_{v,s}$ is contained in the closure of $q_v^{\text{0-gen}}$ in $\overline{M}_{g_v,n_v}$. This completes the proof of the proposition. $\square$

In the remainder of the present paper, we assume that $X^\bullet$ is a component-generic pointed stable curve over k. Then Proposition 6.3 implies that, for each $v \in v(\Gamma_{X^\bullet})$, $(\widetilde{X}^\bullet_v, \pi_0(v))$ admits a (DEG). Moreover, we denote by $\Pi_{v,\overline{\eta}}$ and $\Pi_{v,s}$ the admissible fundamental groups of $\mathcal{X}^\bullet_{v,\overline{\eta}}$ and $\mathcal{X}^\bullet_{v,s}$, respectively. Then $\Pi_{v,\overline{\eta}}$ is naturally isomorphic to Π_v. By Remark 2.2.1 and [V, Théorème 2.2 (b)], there is a specialization map

$$\mathrm{sp}_{R_v} : \Pi_{v,\overline{\eta}} \twoheadrightarrow \Pi_{v,s}.$$

Moreover, we obtain a continuous surjective homomorphism of maximal pro-p quotients

$$\mathrm{sp}^p_{R_v} : \Pi^p_{v,\overline{\eta}} \twoheadrightarrow \Pi^p_{v,s},$$

where $(-)^p$ denotes the maximal pro-p quotient of $(-)$. On the other hand, the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]) implies that

$$\mathrm{sp}^{p'}_{R_v} : \Pi^{p'}_{v,\overline{\eta}} \stackrel{\sim}{\to} \Pi^{p'}_{v,s}.$$

Let Q_v be an effective divisor on $\mathcal{X}_v$ of degree $(\#E_v^{>1})n$ such that $\mathrm{Supp}(Q_v) \subseteq D_{\mathcal{X}_v}$ and

$$\sum_{x\in D_{v,C}} \mathrm{ord}_x(Q_v) = \begin{cases} n, & \text{if } C \in E_v^{>1}, \\ 0, & \text{if } C \in E_v^{=1}. \end{cases}$$

Write $Q_v^{\overline{\eta}}$ for $Q_v \times_{R_v} \overline{K}_{R_v}$, Q_v^s for $Q_v \times_{R_v} k_{R_v}$, and $Q^s_{v,C}$, $C \in E_v^{>1}$, for $Q_v^s \cap P_{v,C}$. Then we have $\deg(Q^s_{v,C}) = n,\ C \in E_v^{>1}$. This means that

$$s(Q^s_{v,C}) = 1,\ C \in E_v^{>1}.$$

Let $\mathcal{L}_{v,\overline{\eta}}$ be a line bundle on $\mathcal{X}_{v,\overline{\eta}}$ such that $\mathcal{L}^{\otimes n}_{v,\overline{\eta}} \cong \mathcal{O}_{\mathcal{X}_{v,\overline{\eta}}}(-Q_v^{\overline{\eta}})$. We put

$$\mathcal{E}_{Q_v^{\overline{\eta}}} \stackrel{\text{def}}{=} \mathcal{B}^t_{Q_v^{\overline{\eta}}} \otimes \mathcal{L}_{v,\overline{\eta}}.$$

Then we have the following proposition.

Proposition 6.4. *The scheme $\Theta_{\mathcal{E}_{Q_v^{\overline{\eta}}}}$ is the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_{Q_v^{\overline{\eta}}}$.*

Proof. If $\#E_v^{>1} \leq 1$, then the proposition follows immediately from Theorem 4.3. We may assume that $\#E_v^{>1} \geq 2$. By Remark 4.2.1, $\Theta_{\mathcal{E}_{Q_v^{\overline{\eta}}}}$ is the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_{Q_v^{\overline{\eta}}}$ (or $\mathcal{E}_{Q_v^{\overline{\eta}}}$ satisfies $(\star)$) if and only if there exists a line bundle $\mathcal{I}_{v,\overline{\eta}}$ on $\mathcal{X}_{v,\overline{\eta}}$ of degree 0 such that $[\mathcal{I}_{v,\overline{\eta}}] \notin \Theta_{\mathcal{E}_{Q_v^{\overline{\eta}}}}$. This is equivalent to prove that there exists a line bundle $\mathcal{I}_{v,\overline{\eta}}$ on $\mathcal{X}_{v,\overline{\eta}}$ of degree 0 such that

$$\gamma_{([\mathcal{L}_{v,\overline{\eta}} \otimes \mathcal{I}_{v,\overline{\eta}}], Q_v^{\overline{\eta}})} = \dim_{\overline{K}_{R_v}}(H^1(\mathcal{X}_{v,\overline{\eta}}, \mathcal{L}_{v,\overline{\eta}} \otimes \mathcal{I}_{v,\overline{\eta}})) = g_v + \#E_v^{>1} - 1.$$

For each $C \in E_v^{>1}$, let $\mathcal{L}_{v,C}$ be a line bundle on $P_{v,C}$ such that $\mathcal{L}_{v,C}^{\otimes n} \cong \mathcal{O}_{P_{v,C}}(-Q_{v,C}^s)$, and let

$$f_{v,C}^{\bullet} : Y_{v,C}^{\bullet} = (Y_{v,C}, D_{Y_{v,C}}) \to P_{v,C}^{\bullet} = (P_{v,C}, D_{P_{v,C}}),$$

be the connected Galois admissible covering corresponding to the pair $([\mathcal{L}_{v,C}], Q_{v,C}^s)$ over k_{R_v} with Galois group $\mathbb{Z}/n\mathbb{Z}$, where $D_{P_{v,C}} \overset{\text{def}}{=} D_{\mathcal{X}_{v,s}} \cap P_{v,C}$. Then the $k_{R_v}[\mu_n]$-module $H^1_{\text{ét}}(Y_{v,C}, \mathbb{F}_p) \otimes k_{R_v}$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Y_{v,C}, \mathbb{F}_p) \otimes k_{R_v} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M_{v,C}(i),$$

where $\zeta \in \mu_n$ acts on $M_{v,C}(i)$ as the ζ^i-multiplication. By applying Theorem 4.3 and Theorem 4.7, we may choose $\mathcal{L}_{v,C},\ C \in E_v^{>1}$, such that

$$\dim_{k_{R_v}}(M_{v,C}(1)) = 0.$$

If $g_v \neq 0$, let $\mathcal{L}_{Z_v}$ be a non-trivial line bundle on Z_v of degree 0 such that $\mathcal{L}_{Z_v}^{\otimes n} \cong \mathcal{O}_{Z_v}$. We denote by

$$f_{Z_v}^{\bullet} : Y_{Z_v}^{\bullet} = (Y_{Z_v}, D_{Y_{Z_v}}) \to Z_v^{\bullet} = (Z_v, D_{Z_v})$$

the connected Galois étale covering corresponding to the pair $([\mathcal{L}_{Z_v}], 0)$ over k_{R_v} with Galois group $\mathbb{Z}/n\mathbb{Z}$, where $D_{Z_v} \overset{\text{def}}{=} D_{\mathcal{X}_{v,s}} \cap Z_v$. Then the $k_{R_v}[\mu_n]$-module $H^1_{\text{ét}}(Y_{Z_v}, \mathbb{F}_p) \otimes k_{R_v}$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Y_{Z_v}, \mathbb{F}_p) \otimes k_{R_v} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M_{Z_v}(i),$$

where $\zeta \in \mu_n$ acts on $M_{Z_v}(i)$ as the ζ^i-multiplication. By applying Theorem 4.3 and Theorem 4.7, we may choose $\mathcal{L}_{Z_v}$ such that

$$\dim_{k_{R_v}}(M_{Z_v}(1)) = g_v - 1.$$

We denote $S_{P_{v,C}}$ by $P_{v,C} \cap \mathcal{X}_{v,s}^{\text{sing}}$ for each $C \in E_v^{>1}$, and S_{Z_v} by $Z_v \cap \mathcal{X}_{v,s}^{\text{sing}}$. Then $f_{v,C}^{\bullet}$, $C \in E_v^{>1}$, and $f_{Z_v}^{\bullet}$ induce the following Galois admissible coverings

$$f_{v,C}^{*,\bullet} : Y_{v,C}^{*,\bullet} = (Y_{v,C}, f_{v,C}^{-1}(D_{P_{v,C}^*})) \to P_{v,C}^{*,\bullet} = (P_{v,C}, D_{P_{v,C}^*} \overset{\text{def}}{=} S_{P_{v,C}} \cup D_{P_{v,C}}),\ C \in E_v^{>1},$$

and
$$f^{*,\bullet}_{Z_v} : Y^{*,\bullet}_{Z_v} = (Y_{Z_v}, f^{-1}_{Z_v}(D_{Z^*_v})) \to Z^{*,\bullet}_v = (Z_v, D_{Z^*_v} \overset{\text{def}}{=} S_{Z_v} \cup D_{Z_v})$$
over k_{R_v} with Galois group $\mathbb{Z}/n\mathbb{Z}$, respectively, where $f_{v,C}$ and f_{Z_v} denote the underlying morphism induced by $f^\bullet_{v,C}$ and $f^\bullet_{Z_v}$, respectively. Note that the actions of $\mathbb{Z}/n\mathbb{Z}$ on $f^{-1}_{v,C}(S_{P_{v,C}})$, $C \in E^{>1}_v$, and $f^{-1}_{Z_v}(S_{Z_v})$ are transitive, respectively. Then we may glue $\{Y^{*,\bullet}_{v,C}\}_{C\in E^{>1}_v}$ along $\{f^{-1}_{v,C}(S_{P_{v,C}})\}_{C\in E^{>1}_v}$ that is compatible with the gluing of $\{P^{*,\bullet}_{v,C}\}_{C\in E^{>1}_v}$ that gives rise to $\mathcal{X}^\bullet_{v,s}$ if $g_v = 0$, and glue $\{Y^{*,\bullet}_{v,C}\}_{C\in E^{>1}_v}$ and $Y^{*,\bullet}_{Z_v}$ along $\{f^{-1}_{v,C}(S_{P_{v,C}})\}_{C\in E^{>1}_v}$ and $f^{-1}_{Z_v}(S_{Z_v})$ that is compatible with the gluing of $\{P^{*,\bullet}_{v,C}\}_{C\in E^{>1}_v} \cup \{Z^{*,\bullet}_v\}$ that gives rise to $\mathcal{X}^\bullet_{v,s}$ if $g_v \neq 0$. Then we obtain a connected Galois admissible covering
$$f^\bullet_{v,s} : \mathcal{Y}^\bullet_{v,s} = (\mathcal{Y}_{v,s}, D_{\mathcal{Y}_{v,s}}) \to \mathcal{X}^\bullet_{v,s}$$
over k_{R_v} with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\Gamma_{\mathcal{Y}^\bullet_{v,s}}$ for the dual semi-graph of $\mathcal{Y}^\bullet_{v,s}$ and $r_{\mathcal{Y}_{v,s}}$ for the Betti number of $\Gamma_{\mathcal{Y}^\bullet_{v,s}}$. The construction of $\mathcal{Y}^\bullet_{v,s}$ implies that
$$r_{\mathcal{Y}_{v,s}} = \begin{cases} (\#E^{>1}_v - 1)(n-1), & \text{if } g_v = 0, \\ \#E^{>1}_v(n-1), & \text{if } g_v \neq 0. \end{cases}$$
The $k[\mu_n]$-module $H^1_{\text{ét}}(\mathcal{Y}_{v,s}, \mathbb{F}_p) \otimes k_{R_v}$ admits the following canonical decomposition
$$H^1_{\text{ét}}(\mathcal{Y}_{v,s}, \mathbb{F}_p) \otimes k_{R_v} = \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} M_{v,s}(i),$$
where $\zeta \in \mu_n$ acts on $M_{v,s}(i)$ as the ζ^i-multiplication. Moreover, we have a natural $k[\mu_n]$-submodule
$$H^1(\Gamma_{\mathcal{Y}^\bullet_{v,s}}, \mathbb{F}_p) \otimes k_{R_v} \subseteq H^1_{\text{ét}}(\mathcal{Y}_{v,s}, \mathbb{F}_p) \otimes k_{R_v}$$
which admits a canonical decomposition
$$H^1(\Gamma_{\mathcal{Y}^\bullet_{v,s}}, \mathbb{F}_p) \otimes k_{R_v} = \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{\mathcal{Y}^\bullet_{v,s}}}(i),$$
where $\zeta \in \mu_n$ acts on $M_{\Gamma_{\mathcal{Y}^\bullet_{v,s}}}(i)$ as the ζ^i-multiplication. Then we have
$$M_{v,s}(1) = M_{Z_v}(1) \oplus M_{\Gamma_{\mathcal{Y}^\bullet_{v,s}}}(1).$$
We see immediately that
$$\dim_{k_{R_v}} M_{\Gamma_{\mathcal{Y}^\bullet_{v,s}}}(i) = \begin{cases} 0, & \text{if } i = 0, \\ \#E^{>1}_v - 1, & \text{if } i \neq 0 \text{ and } g_v = 0, \\ \#E^{>1}_v, & \text{if } i \neq 0 \text{ and } g_v \neq 0. \end{cases}$$
Thus, we obtain that
$$\dim_{k_{R_v}}(M_{v,s}(1)) = g_v + \#E^{>1}_v - 1.$$

On the other hand, since $(p, n) = 1$, the isomorphism $\mathrm{sp}^{p'}_{R_v} : \Pi^{p'}_{v,\overline{\eta}} \xrightarrow{\sim} \Pi^{p'}_{v,s}$ implies that, by replacing R_v by a finite extension of R_v, there exists a finite morphism of pointed stable curves
$$f^\bullet_v : \mathcal{Y}^\bullet_v = (\mathcal{Y}_v, D_{\mathcal{Y}_v}) \to \mathcal{X}^\bullet_v$$

over R_v such that the restriction of $f_v^\bullet$ on the special fibers is k_{R_v}-isomorphic to $f_{v,s}^\bullet$, and that the restriction of $f_v^\bullet$ on the geometric generic fibers is a connected Galois admissible covering

$$f_{v,\overline{\eta}}^\bullet : \mathcal{Y}_{v,\overline{\eta}}^\bullet = (\mathcal{Y}_{v,\overline{\eta}}, D_{\mathcal{Y}_{v,\overline{\eta}}}) \overset{\text{def}}{=} \mathcal{Y}_v^\bullet \times_{R_v} \overline{K}_{R_v} \to \mathcal{X}_{v,\overline{\eta}}^\bullet$$

withe Galois group $\mathbb{Z}/n\mathbb{Z}$ over $\overline{K}_{R_v}$. The $k_{R_v}[\mu_n]$-module $H^1_{\text{ét}}(\mathcal{Y}_{v,\overline{\eta}}, \mathbb{F}_p) \otimes k_{R_v}$ admits the following canonical decomposition

$$H^1_{\text{ét}}(\mathcal{Y}_{v,\overline{\eta}}, \mathbb{F}_p) \otimes k_{R_v} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M_{v,\overline{\eta}}(i),$$

where $\zeta \in \mu_n$ acts on $M_{v,\overline{\eta}}(i)$ as the ζ^i-multiplication. Write $\Pi_{\mathcal{Y}_{v,\overline{\eta}}^\bullet} \subseteq \Pi_{v,\overline{\eta}}$ and $\Pi_{\mathcal{Y}_{v,s}^\bullet} \subseteq \Pi_{v,s}$ for the open normal subgroups corresponding to $\mathcal{Y}_{v,\overline{\eta}}^\bullet$ and $\mathcal{Y}_{v,s}^\bullet$, respectively. Then the surjection $\text{sp}_v : \Pi_{v,\overline{\eta}} \twoheadrightarrow \Pi_{v,s}$ induces a surjection $\text{sp}_{v,\mathcal{Y}} : \Pi_{\mathcal{Y}_{v,\overline{\eta}}^\bullet} \twoheadrightarrow \Pi_{\mathcal{Y}_{v,s}^\bullet}$. Thus, we obtain a surjection

$$\text{sp}_{v,\mathcal{Y}}^p : \Pi^p_{\mathcal{Y}_{v,\overline{\eta}}^\bullet} \twoheadrightarrow \Pi^p_{\mathcal{Y}_{v,s}^\bullet}.$$

Since $H^1_{\text{ét}}(\mathcal{Y}_{v,\overline{\eta}}, \mathbb{F}_p) \otimes k_{R_v}$ and $H^1_{\text{ét}}(\mathcal{Y}_{v,s}, \mathbb{F}_p) \otimes k_{R_v}$ are semi-simple $k_{R_v}[\mu_n]$-modules, the surjection $\text{sp}_{v,\mathcal{Y}}^p$ induces an injection $M_{v,s}(1) \hookrightarrow M_{v,\overline{\eta}}(1)$. This implies that

$$\dim_{k_{R_v}}(M_{v,\overline{\eta}}(1)) \geq g_v + \#E_v^{>1} - 1.$$

Write $\mathcal{L}'_{v,\overline{\eta}}$ for the line bundle on $\mathcal{X}_{v,\overline{\eta}}$ corresponding to $\mathcal{Y}_{v,\overline{\eta}}$. Then Lemma 4.9 implies that $(\mathcal{L}'_{v,\overline{\eta}})^{\otimes n} \cong \mathcal{O}_{\mathcal{X}_{v,\overline{\eta}}}(-Q_v^{\overline{\eta}})$. Moreover, we have

$$\dim_{k_{R_v}}(M_{v,\overline{\eta}}(1)) = \gamma_{([\mathcal{L}'_{v,\overline{\eta}}], Q_v^{\overline{\eta}})} \leq \dim_{\overline{K}_{R_v}}(H^1(\mathcal{X}_{v,\overline{\eta}}, \mathcal{L}'_{v,\overline{\eta}})) = g_v + \#E_v^{>1} - 1.$$

Then we obtain that

$$\dim_{k_{R_v}}(M_{v,\overline{\eta}}(1)) = \gamma_{([\mathcal{L}'_{v,\overline{\eta}}], Q_v^{\overline{\eta}})} = \dim_{\overline{K}_{R_v}}(H^1(\mathcal{X}_{v,\overline{\eta}}, \mathcal{L}'_{v,\overline{\eta}})) = g_v + \#E_v^{>1} - 1.$$

Let $\mathcal{I}_{v,\overline{\eta}} \overset{\text{def}}{=} \mathcal{L}_{v,\overline{\eta}}^{-1} \otimes \mathcal{L}'_{v,\overline{\eta}}$. Note that $\mathcal{I}_{v,\overline{\eta}}$ is a line bundle on $\mathcal{X}_{v,\overline{\eta}}$ of degree 0. Then we have

$$\gamma_{([\mathcal{L}_{v,\overline{\eta}} \otimes \mathcal{I}_{v,\overline{\eta}}], Q_v^{\overline{\eta}})} = \gamma_{([\mathcal{L}'_{v,\overline{\eta}}], Q_v^{\overline{\eta}})}$$
$$= \dim_{\overline{K}_{R_v}}(H^1(\mathcal{X}_{v,\overline{\eta}}, \mathcal{L}'_{v,\overline{\eta}})) = \dim_{\overline{K}_{R_v}}(H^1(\mathcal{X}_{v,\overline{\eta}}, \mathcal{L}_{v,\overline{\eta}} \otimes \mathcal{I}_{v,\overline{\eta}})) = g_v + \#E_v^{>1} - 1.$$

This completes the proof of the proposition. □

Remark 6.4.1. Proposition 6.4 gives a positive answer of Problem of Remark 4.7.1 under certain assumptions of divisors. On the other hand, we may pose a generalized version of Tamagawa's problem as follows.

Problem . *We maintain the notation introduced in Remark 4.7.1. Suppose that $X^\bullet$ is a component-generic smooth pointed stable curve over k. Let $([\mathcal{L}], D)$ be an arbitrary element of $\widetilde{\mathscr{P}}_{X^\bullet,n}$. Suppose that $\deg(D^{(i)}) \geq \deg(D)$ holds for each $i \in \{0, 1, \ldots, t-1\}$. Does the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exist?*

6.2 A formula for the limits of p-averages

In this subsection, we prove the second main theorem of the present paper. First, we have the following proposition.

Proposition 6.5. *Suppose that $X^\bullet$ is a component-generic pointed stable curve over k. Then we have*

$$\lim_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = \begin{cases} 0, & \text{if } v \in V^{\mathrm{tre},g_v=0}_{X^\bullet}, \\ g_v + \#E^{>1}_v - 1, & \text{if } v \in v(\Gamma_{X^\bullet}) \setminus V^{\mathrm{tre},g_v=0}_{X^\bullet}. \end{cases}$$

Proof. We maintain the notation introduced in the proof of Proposition 4.10. Moreover, Proposition 4.10 implies that we may assume that $\#E^{>1}_v \geq 2$. Then

$$\#(M_v \otimes \mathbb{Z}/n\mathbb{Z}) = 2g_v + \sum_{C\in\pi_0(v)} (n_{v,C} - 1).$$

Suppose that $v \in V^{\mathrm{tre},g_v=0}_{X^\bullet}$. Then the proposition follows from Proposition 4.10 (i). Suppose that $v \in v(\Gamma_{X^\bullet}) \setminus V^{\mathrm{tre},g_v=0}_{X^\bullet}$. Theorem 4.7 and Proposition 6.4 imply that

$$\sigma(X^\bullet_{H_{n,v}}) \geq (g_v + \#E^{>1}_v - 1)(n^{2g_v} - C(g_v)n^{2g_v-1}) \prod_{C\in E^{>1}_v} (n^{n_{v,C}-1}(1-\lambda^t_{v,C}) - 1)$$

$$\geq (g_v + \#E^{>1}_v - 1)(n^{2g_v+\sum_{C\in E^{>1}_v}(n_{v,C}-1)} - C(g_v)n^{2g_v+\sum_{C\in E^{>1}_v}(n_{v,C}-1)-1})$$
$$= (g_v + \#E^{>1}_v - 1)(n^{2g_v+\sum_{C\in \pi_0(v)}(n_{v,C}-1)} - C(g_v)n^{2g_v+\sum_{C\in \pi_0(v)}(n_{v,C}-1)-1}).$$

Then Proposition 4.10 (ii) implies that

$$\lim_{t\to\infty} \frac{\sigma(X^\bullet_{H_{v,n}})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = g_v + \#E^{>1}_v - 1.$$

This completes the proof of the proposition. □

The second main theorem of the present paper is as follows, which is a formula of the limits of p-averages without any assumptions of dual semi-graphs.

Theorem 6.6. *Suppose that $X^\bullet$ is a component-generic pointed stable curve over k. Then we have*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet} + \sum_{v\in v(\Gamma_{X^\bullet})} \#E^{>1}_v.$$

Proof. We denote by K_n the kernel of the natural surjective homomorphism $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}$. By applying similar arguments to the arguments given in the proof of Theorem 5.2 imply that

$$\frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_n \otimes \mathbb{F}_p)}{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})} = \sum_{v\in v(\Gamma_{X^\bullet})} \frac{\dim_{\mathbb{F}_p}(H^{\mathrm{ab}}_{v,n} \otimes \mathbb{F}_p)}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})}$$

$$+\#E^{\mathrm{tre}}_{X^\bullet} + \sum_{e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet})\setminus \bigcup_{v\in v(\Gamma_{X^\bullet})} E_v^{=1}} \frac{1}{n} - \sum_{v\in v(\Gamma_{X^\bullet})} \frac{1}{\#(M_v\otimes \mathbb{Z}/n\mathbb{Z})} + \frac{1}{\#(\Pi^{\mathrm{ab}}_{X^\bullet}\otimes \mathbb{Z}/n\mathbb{Z})}.$$

Thus, Proposition 6.5 implies that

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = \sum_{v\in v(\Gamma_{X^\bullet})\ \text{s.t.}\ (g_v,\#E_v^{>1})\neq(0,0)} (g_v + \#E_v^{>1} - 1) + \#E^{\mathrm{tre}}_{X^\bullet}$$

$$= \sum_{v\in v(\Gamma_{X^\bullet})} g_v + \sum_{v\in v(\Gamma_{X^\bullet})} \#E_v^{>1} - \#v(\Gamma_{X^\bullet}) + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet}$$

$$= g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet} + \sum_{v\in v(\Gamma_{X^\bullet})} \#E_v^{>1}.$$

This completes the proof of the theorem. □

Remark 6.6.1. We can also prove Theorem 6.6 by applying Theorem 5.2 directly (i.e., without using the existence of Raynaud-Tamagawa theta divisors). Let us explain the arguments in this remark.

Suppose that $X^\bullet$ is a component-generic pointed stable curve over k. Then there exists a complete discrete valuation ring R with algebraically closed residue field k_R of characteristic $p>0$ and a pointed stable curve $\mathcal{X}^\bullet = (\mathcal{X}, D_{\mathcal{X}})$ over R such that the following conditions are satisfied:

(i) Write $\mathcal{X}^\bullet_\eta = (\mathcal{X}_\eta, D_{\mathcal{X}_\eta})$ for the generic fiber $\mathcal{X}^\bullet \times_R K_R$. Then each point of $\mathcal{X}^{\mathrm{sing}}$ is K_R-rational.

(ii) There is an algebraically closed field $k' \supseteq k$ such that k' contains the quotient field K_R of R, and that, by replacing k by k' and $X^\bullet$ by $X^\bullet \times_k k'$, $X^\bullet$ is k-isomorphic to $\mathcal{X}^\bullet_\eta \times_{K_R} k$.

(iii) By replacing k by k' and $X^\bullet$ by $X^\bullet\times_k k'$, we assume that $k'=k$. Write $\Gamma_{\mathcal{X}^\bullet_\eta}$ for the dual semi-graph of $\mathcal{X}^\bullet_\eta$. Note that $\Gamma_{\mathcal{X}^\bullet_\eta}$ can be naturally identified with $\Gamma_{X^\bullet}$ via the k-isomorphism $X^\bullet \cong_k \mathcal{X}^\bullet_\eta \times_{K_R} k$. For each $v\in v(\Gamma_{\mathcal{X}^\bullet_\eta})$, write $\mathcal{X}_{v,\eta}$ for the irreducible component of $\mathcal{X}_\eta$ corresponding to v and $\widetilde{\mathcal{X}}_{v,\eta}$ for the smooth compactification of $U_{\mathcal{X}_{v,\eta}} \overset{\mathrm{def}}{=} \mathcal{X}_{v,\eta}\setminus \mathcal{X}^{\mathrm{sing}}_{v,\eta}$. We define a smooth pointed stable curve of type (g_v,n_v) to be

$$\widetilde{\mathcal{X}}^\bullet_{v,\eta} = (\widetilde{\mathcal{X}}_{v,\eta}, D_{\widetilde{\mathcal{X}}_{v,\eta}} \overset{\mathrm{def}}{=} (D_{\mathcal{X}_\eta}\cap \mathcal{X}_{v,\eta}) \cup (\widetilde{\mathcal{X}}_{v,\eta}\setminus U_{\mathcal{X}_{v,\eta}}))$$

over K_R. Then we have that $\widetilde{\mathcal{X}}^\bullet_{v,\eta}\times_{K_R} k$ is k-isomorphic to $\widetilde{X}^\bullet_v$, and that the reduction of $\widetilde{\mathcal{X}}^\bullet_{v,\eta}$ over R satisfies the (DEG) (iii) defined in Section 6.1.

(iv) Write $\overline{K}_R$ for the algebraic closure of K_R in k and $\mathcal{X}^\bullet_{\overline{\eta}} = (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}})$ for the geometric generic fiber $\mathcal{X}^\bullet\times_R \overline{K}_R$ of $\mathcal{X}^\bullet$. Then $X^\bullet$ is k-isomorphic to $\mathcal{X}^\bullet \times_{\overline{K}_R} k$. Moreover, we write $\Gamma_{\mathcal{X}^\bullet_{\overline{\eta}}}$ for the dual semi-graph of $\mathcal{X}^\bullet_{\overline{\eta}}$. Note that $\Gamma_{\mathcal{X}^\bullet_{\overline{\eta}}}$ can be naturally identified with $\Gamma_{\mathcal{X}^\bullet_\eta}$.

Write $\Pi_{\overline{\eta}}$ and Π_s for the admissible fundamental groups of $\mathcal{X}_{\overline{\eta}}^\bullet$ and $\mathcal{X}_s^\bullet$, respectively. Let us compute $\mathrm{Avr}_p(\Pi_s)$. Note that the construction of $\mathcal{X}_s^\bullet$ implies that, for each $w \in v(\Gamma_{\mathcal{X}_s^\bullet})$, we have $\#E_w^{>1} \leq 1$. Thus, by applying Theorem 5.2, we obtain that

$$\mathrm{Avr}_p(\Pi_s) = g_X - r_X - \#v(\Gamma_{\mathcal{X}_s^\bullet}) + \#V_{\mathcal{X}_s^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{\mathcal{X}_s^\bullet}^{\mathrm{tre}} + \sum_{w \in v(\Gamma_{\mathcal{X}_s^\bullet})} \#E_w^{>1}.$$

Moreover, the construction of $\mathcal{X}_s^\bullet$ implies that

$$\#v(\Gamma_{\mathcal{X}_s^\bullet}) = \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v \neq 0} (\#E_v^{>1}+1) + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v = 0, \#E_v^{>1} \neq 0} \#E_v^{>1} + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v = 0, \#E_v^{>1} = 0} 1$$

$$= \#v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v \neq 0} \#E_v^{>1} + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v = 0, \#E_v^{>1} \neq 0} (\#E_v^{>1} - 1),$$

$$\#V_{\mathcal{X}_s^\bullet}^{\mathrm{tre}, g_v=0} = \#V_{\mathcal{X}_{\overline{\eta}}^\bullet}^{\mathrm{tre}, g_v=0},$$

$$\sum_{w \in v(\Gamma_{\mathcal{X}_s^\bullet})} \#E_w^{>1} = \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet})} \#E_v^{>1},$$

$$\#E_{\mathcal{X}_s^\bullet}^{\mathrm{tre}} = \#E_{\mathcal{X}_{\overline{\eta}}^\bullet}^{\mathrm{tre}} + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v \neq 0} \#E_v^{>1} + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v = 0, \#E_v^{>1} \neq 0} (\#E_v^{>1} - 1).$$

Thus, we have

$$\mathrm{Avr}_p(\Pi_s) = g_X - r_X - \#v(\Gamma_{\mathcal{X}_s^\bullet}) + \#V_{\mathcal{X}_s^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{\mathcal{X}_s^\bullet}^{\mathrm{tre}} + \sum_{w \in v(\Gamma_{\mathcal{X}_s^\bullet})} \#E_w^{>1}$$

$$= g_X - r_X - \#v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) - \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v \neq 0} \#E_v^{>1} - \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v = 0, \#E_v^{>1} \neq 0} (\#E_v^{>1} - 1)$$

$$+ \#V_{\mathcal{X}_{\overline{\eta}}^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{\mathcal{X}_{\overline{\eta}}^\bullet}^{\mathrm{tre}}$$

$$+ \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v \neq 0} \#E_v^{>1} + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) \text{ s.t. } g_v = 0, \#E_v^{>1} \neq 0} (\#E_v^{>1} - 1) + \sum_{w \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet})} \#E_w^{>1}$$

$$= g_X - r_X - \#v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet}) + \#V_{\mathcal{X}_{\overline{\eta}}^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{\mathcal{X}_{\overline{\eta}}^\bullet}^{\mathrm{tre}} + \sum_{v \in v(\Gamma_{\mathcal{X}_{\overline{\eta}}^\bullet})} \#E_v^{>1}$$

$$= g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}} + \sum_{v \in v(\Gamma_{X^\bullet})} \#E_v^{>1}.$$

On the other hand, since $\Pi_{\overline{\eta}}$ is naturally (outer) isomorphic to $\Pi_{X^\bullet}$, we obtain $\gamma_{p,n}^{\mathrm{av}}(\Pi_{X^\bullet}) = \gamma_{p,n}^{\mathrm{av}}(\Pi_{\overline{\eta}})$. Moreover, Since $\mathcal{X}_{\overline{\eta}}^\bullet$ and $\mathcal{X}_s^\bullet$ are same types, we have a specialization map

$$\mathrm{sp}_R : \Pi_{\overline{\eta}} \twoheadrightarrow \Pi_s.$$

Then we have
$$\gamma^{\mathrm{av}}_{p,n}(\Pi_{X^\bullet}) = \gamma^{\mathrm{av}}_{p,n}(\Pi_{\overline{\eta}}) \geq \gamma^{\mathrm{av}}_{p,n}(\Pi_s).$$
This means that
$$\liminf_{t\to\infty} \gamma^{\mathrm{av}}_{p,n}(\Pi_{\overline{\eta}}) \geq g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet} + \sum_{v\in v(\Gamma_{X^\bullet})} \#E^{>1}_v,$$
where $\liminf(-)$ denotes the limit infimum of $(-)$. Thus, Theorem 5.2 implies that $\mathrm{Avr}_p(\Pi_{X^\bullet})$ exists, and that
$$\mathrm{Avr}_p(\Pi_{X^\bullet}) = g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V^{\mathrm{tre},g_v=0}_{X^\bullet} + \#E^{\mathrm{tre}}_{X^\bullet} + \sum_{v\in v(\Gamma_{X^\bullet})} \#E^{>1}_v.$$
This completes the proof of Theorem 6.6.

References

[C] R. Crew, Étale p-covers in characteristic p, *Compositio Math.* **52** (1984), 31–45.

[FJ] M. D. Fried, M. Jarden, Field arithmetic. Third edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete.* 3. Folge. A Series of Modern Surveys in Mathematics **11**. *Springer-Verlag*, Berlin, 2008.

[G] A. Grothendieck, Revêtements Étales ét Groupe Fondamental (SGA1), Séminaire de Géométrie Algébrique de Bois-Marie 1960/61., *Lecture Notes in Math.* **224**, Springer-Verlag, 1971.

[H] R. Hartshorne, Algebraic geometry. *Graduate Texts in Mathematics*, **52**. *Springer-Verlag, New York-Heidelberg*, 1977.

[M1] S. Mochizuki, The Geometry of the Compactification of the Hurwitz Scheme. *Publ. Res. Inst. Math. Sci.* **31** (1995), 355–441.

[M2] S. Mochizuki, The profinite Grothendieck conjecture for closed hyperbolic curves over number fields. *J. Math. Sci. Univ. Tokyo* **3** (1996), 571–627.

[M3] S. Mochizuki, Semi-graphs of anabelioids. *Publ. Res. Inst. Math. Sci.* **42** (2006), 221–322.

[N] S. Nakajima, On generalized Hasse-Witt invariants of an algebraic curve, *Galois groups and their representations* (Nagoya 1981) (Y. Ihara, ed.), *Adv. Stud. Pure Math*, **2**, North-Holland Publishing Company, Amsterdam, 1983, 69-88.

[OP] E. Ozman, R. Pries, On the existence of ordinary and almost ordinary Prym varieties, to appear in *Asian Journal of Math.*

[R] M. Raynaud, Sections des fibrés vectoriels sur une courbe. *Bull. Soc. math. France* **110** (1982), 103–125.

[S] J-P. Serre, Sur la topologie des variétés algébriques en caractéristique p. *Symp. Int. Top. Alg., Mexico* (1958), 24-53.

[T1] A. Tamagawa, On the tame fundamental groups of curves over algebraically closed fields of characteristic > 0. *Galois groups and fundamental groups*, 47-105, *Math. Sci. Res. Inst. Publ.*, **41**, *Cambridge Univ. Press, Cambridge*, 2003.

[T2] A. Tamagawa, Resolution of nonsingularities of families of curves. *Publ. Res. Inst. Math. Sci.* **40** (2004), 1291–1336.

[V] I. Vidal, Contributions à la cohomologie étale des schémas et des log-schémas, Thèse, U. Paris-Sud (2001).

[Y1] Y. Yang, On the admissible fundamental groups of curves over algebraically closed fields of characteristic $p > 0$, *Publ. Res. Inst. Math. Sci.* **54** (2018), 649–678.

[Y2] Y. Yang, Tame anabelian geometry and moduli spaces of curves over algebraically closed fields of characteristic $p > 0$, RIMS Preprint 1879. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/

[Y3] Y. Yang, The combinatorial mono-anabelian geometry of curves over algebraically closed fields of positive characteristic I: combinatorial Grothendieck conjecture. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/

[Z] B. Zhang, Revêtements étales abeliens de courbes génériques et ordinarité, *Ann. Fac. Sci. Toulouse Math.* (5) **6** (1992), 133–138.

Yu Yang

Address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail: yuyang@kurims.kyoto-u.ac.jp