ON THE AVERAGES OF p-RANK OF GENERIC CURVES IN POSITIVE CHARACTERISTIC

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Abstract. Let $X^\bullet \stackrel{\text{def}}{=} (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$. Under a certain generic condition concerning $X^\bullet$, we prove a formula concerning the averages of p-rank of prime-to-p cyclic admissible coverings of $X^\bullet$. Roughly speaking, this formula says that the p-rank of prime-to-p cyclic admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ can be determined by n, (g_X, n_X), and the dual semi-graph of $X^\bullet$ when $n \to \infty$. In particular, this formula gives an affirmative answer (in the case of generic curves) to an open problem concerning p-averages of tame fundamental groups of smooth pointed stable curves asked by A. Tamagawa.

Keywords: pointed stable curve, admissible fundamental group, p-rank, positive characteristic.

Mathematics Subject Classification: Primary 14H30; 14G17; Secondary 14G32.

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1. Introduction

Let $X^\bullet = (X, D_X)$ be a pointed stable curve over an algebraically closed field k of characteristic $\mathrm{char}(k) = p \geq 0$, where X denotes the underlying curve, and D_X denotes the (finite) set of marked points satisfying [K, Definition 1.1 (iv)]. Write g_X for the arithmetic genus of X and n_X for the cardinality $\#(D_X)$ of D_X. We call (g_X, n_X) the topological type (or type for short) of $X^\bullet$. By choosing a suitable base point of $X^\bullet$, we have the admissible fundamental group (see 2.3.1)

$$\Pi_{X^\bullet}$$

of $X^\bullet$. The admissible fundamental groups of pointed stable curves are natural generalizations of the tame fundamental groups of smooth pointed stable curves (i.e., $\Pi_{X^\bullet}$ is isomorphic to the tame fundamental group of $X^\bullet$ if $X^\bullet$ is smooth over k).

1.1. Motivation and Tamagawa's question.

We explain some backgrounds concerning anabelian geometry that motivated the theory developed in the present paper.

1.1.1. When $\mathrm{char}(k) = 0$, the structure of admissible fundamental group $\Pi_{X^\bullet}$ is well-known which is isomorphic to the profinite completion of the topological fundamental group of a Riemann surface of type (g_X, n_X). In the remainder of the introduction, we assume $\mathrm{char}(k) = p > 0$.

Unlike the case of characteristic 0, the situation is quite different when $\mathrm{char}(k) = p > 0$, and the structure of $\Pi_{X^\bullet}$ is no longer known. At present, we *do not* have an explicit description of the admissible (or tame) fundamental group of *any* pointed stable curve in positive characteristic. In fact, we *cannot* expect that the structures of admissible fundamental groups in positive characteristic can be described explicitly in general since there exist *anabelian phenomena* (i.e., the isomorphism class of $X^\bullet$ can be completely determined by the isomorphism class of $\Pi_{X^\bullet}$).

1.1.2. The original anabelian geometry suggested by A. Grothendieck in 1980s is a theory over *arithmetic fields* (e.g. number fields). Roughly speaking, it means that

$$\text{scheme theory} = \text{Galois actions} + \text{geometric fundamental groups},$$

and the Galois actions play a central role in the theory of anabelian geometry over arithmetic fields (i.e., Galois actions determines scheme structures).

On the other hand, since the late 1990s, some results of M. Raynaud ([R2]), F. Pop-M. Saïdi ([PS]), A. Tamagawa ([T1], [T2], [T3]), and the author of the present paper ([Y2], [Y4], [Y5]) showed evidence for very strong *anabelian phenomena for curves over algebraically closed fields of positive characteristic*. This kinds of anabelian phenomena go beyond Grothendieck's anabelian geometry, and it means that, in positive characteristic,

$$\text{scheme theory} = \text{geometric fundamental groups}.$$

We denote by $\Pi^{p'}_{X^\bullet}$ the maximal prime-to-p quotient of $\Pi_{X^\bullet}$. The specialization theorem of admissible fundamental groups implies that $\Pi^{p'}_{X^\bullet}$ is isomorphic to the prime-to-p completion of the topological fundamental group of a Riemann surface of type (g_X, n_X) (see 2.3.1). In particular, $\Pi^{p'}_{X^\bullet}$ depends only on g_X if $n_X = 0$, and $2g_X + n_X - 1$ if $n_X \neq 0$. This fact means that the anabelian phenomena of curves over algebraically closed fields of positive characteristic *are arose from the complex behaviors of p-parts* of open subgroups of $\Pi_{X^\bullet}$.

1.1.3. *p-rank and its averages.* Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open normal subgroup and $X_H^\bullet \to X^\bullet$ the Galois admissible covering corresponding to H. To analyze the p-part of H, we have an important invariant σ_{X_H} associated to $X_H^\bullet$ (or H) which is called *p-rank* (or *Hasse-Witt invariant*, see 2.4.1). When $\Pi_{X^\bullet}/H$ is a p-group, σ_{X_H} can be explicitly calculated by using the Deuring-Shafarevich formula ([C], [Su]). Then to calculate σ_{X_H}, it sufficient to treat the case where $\#(\Pi_{X^\bullet}/H)$ is *prime to p* (which is the most mysterious part of the structures of admissible fundamental groups of curves in positive characteristic). Furthermore, for anabelian geometry, we need to reconstruct the geometric information of $X^\bullet$ group-theoretically from its admissible fundamental group. However, the geometric information of $X^\bullet$ (e.g. (g_X, n_X)) cannot be carried out directly from σ_{X_H} in general since $\sigma_{X_H} \to \infty$ when $\#(\Pi_{X^\bullet}/H) \to \infty$.

To overcome the gaps between the geometric information of $X^\bullet$ and the p-rank of admissible coverings of $X^\bullet$, in [T2], Tamagawa introduced the following important group-theoretical invariant (see also Definition 2.1) concerning the p-parts of open subgroups of $\Pi_{X^\bullet}$:

$$\gamma^{\mathrm{av}}_{p,n}(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \frac{\dim_{\mathbb{F}_p}(K_n^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})},$$

where n is an arbitrary natural number prime to p, $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$, and K_n denotes the kernel of the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$. Note that $\dim_{\mathbb{F}_p}(K_n^{\mathrm{ab}} \otimes \mathbb{F}_p) = \sigma_{X_{K_n}}$, where $X_{K_n}^\bullet$ denotes the Galois admissible covering of $X^\bullet$ corresponding to K_n.

1.1.4. *Tamagawa's p-average theorem for tame fundamental groups.* Suppose that $X^\bullet$ is *smooth* over k (in this situation, $\Pi_{X^\bullet}$ is isomorphic to the tame fundamental group of $X^\bullet$). By developing a tamely ramified version of Raynaud's theory of theta divisors, Tamagawa obtained the following highly non-trivial result (see [T2, Theorem 0.5]) which is very important in the theory of anabelian geometry of curves in positive characteristic:

Theorem 1.1. *Let $t \in \mathbb{N}$ be a natural number. Then we have (i.e., $n \overset{\mathrm{def}}{=} p^t - 1$)*

$$\mathrm{Avr}_p(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \lim_{t\to\infty} \gamma^{\mathrm{av}}_{p,p^t-1}(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X > 1. \end{cases}$$

As applications, Tamagawa obtained that (g_X, n_X) is a group-theoretical invariant ([T2, Theorem 0.1]), and proved a weak Isom-version of the Grothendieck conjecture for smooth pointed stable curves of type $(0, n_X)$ over $\overline{\mathbb{F}}_p$ ([T2, Theorem 0.2]).

1.1.5. *A question of Tamagawa.* We maintain the notation introduced in 1.1.4. In other words, Theorem 1.1 says that, if $p^t - 1 >> 0$, then the *generalized Hasse-Witt invariants* (i.e., refined invariants of p-rank, see 2.4.2) are equal to $\mathrm{Avr}_p(\Pi_{X^\bullet})$ for *almost all* of the Galois tame coverings of $X^\bullet$ with Galois group $\mathbb{Z}/(p^t-1)\mathbb{Z}$.

On the other hand, we do not know what will happen for $\gamma^{\mathrm{av}}_p(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \lim_{n\to\infty} \gamma^{\mathrm{av}}_{p,n}(\Pi_{X^\bullet})$ if n is an *arbitrary* natural number prime to p. In [T2, Remark 4.15], Tamagawa asked the following question:

Question 1.2. *Let n be an* **arbitrary** *natural number n prime to p, and let $X^\bullet$ be a* **smooth** *pointed stable curve over k and $\Pi_{X^\bullet}$ the tame fundamental group of $X^\bullet$. What* *is* $\gamma_p^{\rm av}(\Pi_{X^\bullet}) \overset{\rm def}{=} \lim_{n\to\infty}\gamma_{p,n}^{\rm av}(\Pi_{X^\bullet})$*? Does the formula*

$$\gamma_p^{\rm av}(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \textit{if } n_X \leq 1, \\ g_X, & \textit{if } n_X > 1, \end{cases}$$

hold?

1.2. **A generalized version of Tamagawa's question.** Let us return to the general case where $X^\bullet$ is an arbitrary pointed stable curve.

1.2.1. In [Y3], under certain conditions concerning dual semi-graphs, the author generalized Tamagawa's result (i.e., Theorem 1.1) to the case of admissible fundamental groups of pointed stable curves (see [Y3, Theorem 5.2] or Remark 4.6.2 of the present paper). As an application, the author proved the so-called *combinatorial Grothendieck conjecture in positive characteristic* ([Y2], [Y5]), and generalized [T2, Theorem 0.2] to the case of pointed stable curves ([Y2]). Furthermore, recently, the author introduced the so-called *moduli spaces of admissible fundamental groups* ([Y6]) which gives a general formulation for describing anabelian phenomena of curves over algebraically closed fields of positive characteristics. The generalized version of Theorem 1.1 ([Y3, Theorem 5.2]) plays one of the central roles to established the theory of the moduli spaces of admissible fundamental groups ([Y6, Section 5]).

1.2.2. [Y3, Theorem 5.2] says that, under certain conditions of dual semi-graph of $X^\bullet$, if $p^t - 1 >> 0$, then the generalized Hasse-Witt invariants can be completely determined by (g_X, n_X) and the dual semi-graph of $X^\bullet$ for *almost all* of the Galois admissible coverings of $X^\bullet$ with Galois group $\mathbb{Z}/(p^t-1)\mathbb{Z}$. Moreover, we may ask the following generalized version of Tamagawa's question (=Question 1.2):

Question 1.3. *Let n be an* **arbitrary** *natural number n prime to p, and let $X^\bullet$ be an* **arbitrary** *pointed stable curve over k and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. What is $\gamma_p^{\rm av}(\Pi_{X^\bullet})$? Does the following formula (see 2.2.1 for $\Gamma_{X^\bullet}$, 2.1 for $v(\Gamma_{X^\bullet})$, Definition 4.2 for $E_{X^\bullet}^{\rm tre}$, and Definition 4.1 for $E_v^{>1}$)*

$$\gamma_p^{\rm av}(\Pi_{X^\bullet}) = g_X - r_X - \#(v(\Gamma_{X^\bullet})) + \#(E_{X^\bullet}^{\rm tre}) + \sum_{v\in v(\Gamma_{X^\bullet})} \#(E_v^{>1})$$

hold?

Note that Question 1.3 coincides with Question 1.2 if $X^\bullet$ is smooth over k. Question 1.3 is very important for the following reason. If the formula mentioned in Question 1.3 holds for arbitrary pointed stable curves, then the main result of [Y6, Section 5] can be extended to the case of arbitrary pointed stable curves, in particular, to the case of stable curves (i.e., $D_X = \emptyset$). This is one of main steps to prove the main conjecture (=the Homeomorphism Conjecture, see [Y6, Section 3.3]) of the theory of moduli spaces of admissible fundamental groups for higher-dimensional moduli spaces.

1.3. **Main result.** In the present paper, we solve Question 1.3 under a "generic" condition. Our main theorem of the present paper is as follows (see also Theorem 4.6):

Theorem 1.4. *Let $X^\bullet$ be a component-generic pointed stable curve (2.2.3) of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph, r_X*

the Betti number of $\Gamma_{X^\bullet}$ *(2.2.1), and* $\Pi_{X^\bullet}$ *the admissible fundamental group of* $X^\bullet$*. Then we have the following formula:*

$$\gamma_p^{\mathrm{av}}(\Pi_{X^\bullet}) = g_X - r_X - \#(v(\Gamma_{X^\bullet})) + \#(E_{X^\bullet}^{\mathrm{tre}}) + \sum_{v\in v(\Gamma_{X^\bullet})} \#(E_v^{>1}).$$

1.4. **Structure of the present paper.** The present paper is organized as follows. In Section 2, we recall some notation concerning semi-graphs, pointed stable curves, admissible fundamental groups, p-rank, and generalized Hasse-Witt invariants. In Section 3, we prove Theorem 1.4 in the case of smooth component-generic pointed stable curves. In Section 4, we prove Theorem 1.4 in general.

1.5. **Acknowledgments.** This work was supported by JSPS KAKENHI Grant Number 20K14283, and by the Research Institute for Mathematical Sciences (RIMS), an International Joint Usage/Research Center located in Kyoto University.

2. Preliminaries

In this section, we set up notation and terminology concerning semi-graphs, pointed stable curves, admissible coverings and admissible fundamental groups.

2.1. **Semi-graphs.** Let Γ be a semi-graph ([M, Section 1]). Roughly speaking, a semi-graph consists of the following data: a set of vertices, a set of open edges, a set of closed edges, and a set of coincidence maps between the sets of (open and closed) edges and the set of vertices.

(a) We shall denote by $v(\Gamma)$, $e^{\mathrm{op}}(\Gamma)$, and $e^{\mathrm{cl}}(\Gamma)$ the set of vertices of Γ, the set of open edges of Γ, and the set of closed edges of Γ, respectively.

(b) The semi-graph Γ can be regarded as a topological space with natural topology induced by $\mathbb{R}^2$, where $\mathbb{R}$ denotes the field of real number. We define an *one-point compactification* Γ^{cpt} of Γ as follows: if $e^{\mathrm{op}}(\Gamma) = \emptyset$, we put $\Gamma^{\mathrm{cpt}} = \Gamma$; otherwise, the set of vertices of Γ^{cpt} is the disjoint union $v(\Gamma^{\mathrm{cpt}}) \overset{\mathrm{def}}{=} v(\Gamma) \sqcup \{v_\infty\}$, the set of closed edges of Γ^{cpt} is $e^{\mathrm{cl}}(\Gamma^{\mathrm{cpt}}) \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma) \cup e^{\mathrm{cl}}(\Gamma)$, the set of open edges of Γ is empty, and every edge $e \in e^{\mathrm{op}}(\Gamma) \subseteq e^{\mathrm{cl}}(\Gamma^{\mathrm{cpt}})$ connects v_∞ with the vertex of Γ that is abutted by e.

(c) Let $v \in v(\Gamma)$. We shall say that Γ is 2-*connected* at v if $\Gamma \setminus \{v\}$ is either empty or connected. Moreover, we shall say that Γ is 2-connected if Γ is 2-connected at each $v \in v(\Gamma)$. Note that, if Γ is connected, then Γ^{cpt} is 2-connected at each $v \in v(\Gamma) \subseteq v(\Gamma^{\mathrm{cpt}})$ if and only if Γ^{cpt} is 2-connected.

2.2. **Pointed stable curves.**

2.2.1. **Settings.** In the remainder of this section, we maintain the following notation. Let k be an algebraically closed field of characteristic $p > 0$ and

$$X^\bullet = (X, D_X)$$

a pointed stable curve of type (g_X, n_X) over k. Here, X denotes the underlying curve of $X^\bullet$, and D_X denotes the (finite) set of marked points of $X^\bullet$ satisfying [K, Definition 1.1 (iv)]. In particular, if $D_X = 0$, we shall call $X^\bullet = X$ stable. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ (e.g. [Y1, Definition 3.1]) and $r_X \overset{\mathrm{def}}{=} \dim_{\mathbb{Q}}(H^1_{\mathrm{sing}}(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$, where $\mathbb{Q}$ denotes the field of rational number.

2.2.2. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the singular point of $X^\bullet$ (or X) corresponding to e if $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of $X^\bullet$ corresponding to e if $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$. Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \overset{\mathrm{def}}{=} X_v \setminus X_v^{\mathrm{sing}}$, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$. We put

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \overset{\mathrm{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v))$$

a smooth pointed stable curve of type (g_v, n_v) over k. We shall call $\widetilde{X}_v^\bullet$ *the smooth pointed stable curve of type* (g_v, n_v) *associated to* v, or the smooth pointed semi-stable curve associated to v for short.

2.2.3. Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack parameterizing pointed stable curves of type (g, n) over $\operatorname{Spec} \mathbb{Z}$, $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$ in k, $\overline{\mathcal{M}}_{g,n} \overset{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. Then $X^\bullet \to \operatorname{Spec} k$ determines a morphism $c_X : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_X,n_X}$ and $\widetilde{X}_v^\bullet \to \operatorname{Spec} k$, $v \in v(\Gamma_{X^\bullet})$, determines a morphism $c_v : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_v,n_v}$. Moreover, we have a clutching morphism of moduli stacks ([K, Definition 3.8])

$$c : \prod_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v} \to \overline{\mathcal{M}}_{g_X,n_X}$$

such that $c \circ (\prod_{v \in v(\Gamma_{X^\bullet})} c_v) = c_X$. We shall say that $X^\bullet$ is a *component-generic* pointed stable curve over k if the image of

$$\prod_{v \in v(\Gamma_{X^\bullet})} c_v : \operatorname{Spec} k \to \prod_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v}$$

is a generic point in $\prod_{v \in v(\Gamma_{X^\bullet})} \overline{M}_{g_v,n_v}$. Note that, if $X^\bullet$ is *smooth* component-generic, then c_X is a geometric point over the generic point of $\overline{M}_{g_X,n_X}$.

2.3. **Admissible fundamental groups.** We maintain the settings introduced in 2.2.1.

2.3.1. By choosing a base point $x \in X \setminus X^{\mathrm{sing}}$, we have the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ of $X^\bullet$ (see [Y5, 2.1.5] and [Y6, 1.2.2] for the definitions of admissible coverings, multi-admissible coverings, Galois admissible coverings, Galois multi-admissible coverings, and admissible fundamental groups). Since we only focus on the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ in the present paper, for simplicity of notation, we omit the base point x and denote by

$$\Pi_{X^\bullet}$$

the admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$. Note that, by the definition of admissible coverings, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X^\bullet$ when $X^\bullet$ is smooth over k. Moreover, the structure of the maximal prime-to-p quotient of $\Pi_{X^\bullet}$ is well-known, and is isomorphic to the prime-to-p completion of the following group

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle.$$

2.3.2. We denote by $\Pi^{\text{ét}}_{X^\bullet}$ and $\Pi^{\text{top}}_{X^\bullet}$ the étale fundamental group of the underlying curve X of $X^\bullet$ and the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, respectively. We have the following natural surjective open continuous homomorphisms (for suitable choices of base points)

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{ét}}_{X^\bullet} \twoheadrightarrow \Pi^{\text{top}}_{X^\bullet}.$$

Moreover, for each $v \in v(\Gamma_{X^\bullet})$, we denote by

$$\Pi_{\widetilde{X}^\bullet_v}$$

the admissible fundamental group of $\widetilde{X}^\bullet_v$ (i.e., the tame fundamental group of the smooth pointed stable curve associated to v). Then we have a natural outer injective homomorphism $\Pi_{\widetilde{X}^\bullet_v} \hookrightarrow \Pi_{X^\bullet}$ (i.e., up to inner automorphisms of $\Pi_{X^\bullet}$).

2.3.3. We put

$$\widehat{X} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} X_H, \ D_{\widehat{X}} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} D_{X_H}, \ \Gamma_{\widehat{X}^\bullet} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^\bullet} \text{ open}} \Gamma_{X^\bullet_H}.$$

We call

$$\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}}) \to X^\bullet$$

the universal admissible covering of $X^\bullet$ corresponding to $\Pi_{X^\bullet}$, and $\Gamma_{\widehat{X}^\bullet}$ the dual semi-graph of $\widehat{X}^\bullet$. Note that $\text{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi_{X^\bullet}$, and that $\Gamma_{\widehat{X}^\bullet}$ admits a natural action of $\Pi_{X^\bullet}$.

Write $\pi_X : \Gamma_{\widehat{X}^\bullet} \to \Gamma_{X^\bullet}$ for the map of dual semi-graphs induced by the universal admissible covering. For every $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$, write $\widehat{e} \in \pi_X^{-1}(e) \subseteq e^{\text{op}}(\Gamma_{\widehat{X}^\bullet}) \cup e^{\text{cl}}(\Gamma_{\widehat{X}^\bullet})$ for an edge over e and write

$$I_{\widehat{e}} \subseteq \Pi_{X^\bullet}$$

for the stabilizer of $\widehat{e}$. Note that $I_{\widehat{e}}$ is isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, where $\widehat{\mathbb{Z}}(1)^{p'}$ denotes the maximal prime-to-p quotient of $\widehat{\mathbb{Z}}(1)$.

2.4. ***p*-rank, generalized Hasse-Witt invariants, and their averages.** We maintain the settings introduced in 2.2.1.

2.4.1. The *p-rank* (or *Hasse-Witt invariant*) of $X^\bullet$ is defined to be

$$\sigma_X \stackrel{\text{def}}{=} \dim_{\mathbb{F}_p}(\text{Pic}^0_{X/k}(k)[p]),$$

where $(-)[p]$ denotes the subgroup of p-torsion points of $(-)$. Note that we have

$$\sigma_X = \dim_{\mathbb{F}_p}(\Pi^{\text{ab}}_{X^\bullet} \otimes \mathbb{F}_p) = \dim_{\mathbb{F}_p}(\Pi^{\text{ét,ab}}_{X^\bullet} \otimes \mathbb{F}_p),$$

where $(-)^{\text{ab}}$ denotes the abelianization of $(-)$. Moreover, we have the following well-known fact

$$\sigma_X = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma_{\widetilde{X}_v} + r_X.$$

2.4.2. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Let $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/n\mathbb{Z})$. We denote by $X_\alpha^\bullet = (X_\alpha, D_{X_\alpha}) \to X^\bullet$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. We put

$$H_\alpha \overset{\mathrm{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k.$$

The finite dimensional k-linear space H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α. Then we have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication.

We call

$$\gamma_{\alpha,i} \overset{\mathrm{def}}{=} \dim_k(H_{\alpha,i}),\ \ i \in \mathbb{Z}/n\mathbb{Z},$$

a *generalized Hasse-Witt invariant* (see [N], [T2] for the case of étale or tame coverings of smooth pointed stable curves) of the cyclic multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. In particular, we call

$$\gamma_{\alpha,1}$$

the *first* generalized Hasse-Witt invariant of the cyclic multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. Note that the above decomposition implies

$$\dim_k(H_\alpha) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \gamma_{\alpha,i}.$$

In particular, if X_α is connected, then $\dim_k(H_\alpha) = \sigma_{X_\alpha}$.

2.4.3. Next, we introduce the main object of the present paper.

Definition 2.1. Let n be an arbitrary positive natural number prime to p and Π an arbitrary profinite group. We put $K_n \overset{\mathrm{def}}{=} \ker(\Pi \twoheadrightarrow \Pi^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})$ and

$$\gamma_{p,n}^{\mathrm{av}}(\Pi) \overset{\mathrm{def}}{=} \frac{\dim_{\mathbb{F}_p}(K_n^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(\Pi^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}.$$

Morever, we put

$$\gamma_p^{\mathrm{av}}(\Pi) \overset{\mathrm{def}}{=} \lim_{n \to \infty} \gamma_{p,n}^{\mathrm{av}}(\Pi)$$

when the limit exists, and we shall call $\gamma_p^{\mathrm{av}}(\Pi)$ *the prime-to-p limit of p-averages* of Π. In particular, if $\Pi = \Pi_{X^\bullet}$ and $X_{K_n}^\bullet$ denotes the Galois admissible covering of $X^\bullet$ corresponding to $K_n \subseteq \Pi_{X^\bullet}$, we have

$$\gamma_p^{\mathrm{av}}(\Pi_{X^\bullet}) = \lim_{n \to \infty} \frac{\sigma_{X_{K_n}}}{\#(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}.$$

3. p-averages for smooth component-generic curves

In this section, we calculate the prime-to-p limit of p-averages for *smooth* component-generic pointed stable curves. The main result of the present section is Proposition 3.5.

3.1. Étale fundamental group case.

In this subsection, we compute the p-averages for *étale* fundamental groups of *arbitrary* smooth stable curves.

3.1.1. **Settings.** We maintain the settings introduced in 2.2.1. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Moreover, we suppose the following conditions hold:

- ◇ $X^\bullet$ is an *arbitrary smooth* pointed stable curve.
- ◇ $n_X = 0$ (i.e., $X^\bullet = (X, \emptyset)$).

Thus, we have that $\Pi_{X^\bullet}$ is the étale fundamental group of $X^\bullet$. Note that since $X^\bullet$ is pointed stable, we have $g_X \geq 2$.

3.1.2. Let n be an arbitrary positive natural number prime to p, t the order of p in $(\mathbb{Z}/n\mathbb{Z})^\times$, and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$.

We put (see 2.4.2 for $\gamma_{\alpha,1}$)

$$\mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})^{\mathrm{ord}} \overset{\mathrm{def}}{=} \{\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \mid \gamma_{\alpha,1} = g_X - 1\},$$

where "ord" means "ordinary". Then we have the following result.

Lemma 3.1. *We maintain the notation introduced above. Then we have*

$$\#(\mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})^{\mathrm{ord}}) \geq n^{2g_X} - 3^{g_X-1} g_X!(p-1)tn^{2g_X-2} - 1.$$

In particular, we have

$$\#(\mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})^{\mathrm{ord}}) \geq n^{2g_X} - 3^{g_X-1} g_X!(p-1)n^{2g_X-1} - 1.$$

Proof. Let $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$ be an arbitrary element and $f_\alpha : X_\alpha \to X$ the étale covering corresponding to α. Then we have

$$f_{\alpha,*}(\mathcal{O}_{X_\alpha}) \cong \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \mathcal{L}_\alpha^{\otimes i}$$

for some line bundle $\mathcal{L}_\alpha$ on X such that $\zeta \in \mu_n$ acts locally on $\mathcal{L}_\alpha^{\otimes i}$ as ζ^i-multiplication.

Let F_k be the absolute Frobenius morphism on $\mathrm{Spec}\, k$ and $F_{X/k} : X \to X_1 \overset{\mathrm{def}}{=} X \times_{k, F_k} k$ the relative Frobenius morphism over k. Let J_{X_1} be the Jacobian of X_1 and

$$\Theta_{\mathrm{RT}} \subseteq J_{X_1}$$

the Raynaud-Tamagawa theta divisor associated to the vector bundle $F_{X/k,*}(\mathcal{O}_X)/\mathcal{O}_{X_1}$ (see [R1, Section 4]). Write $\mathcal{L}_{\alpha,1}$ for the line bundle on X_1 induced by $\mathcal{L}_\alpha$ via the natural morphism $X_1 \to X$ and $[\mathcal{L}_{\alpha,1}]$ for the point of J_{X_1} corresponding to $\mathcal{L}_{\alpha,1}$. Then the definition of Θ_{RT} implies that $[\mathcal{L}_{\alpha,1}] \in \Theta_{\mathrm{RT}}$ if and only if the homomorphism

$$\phi_{\mathcal{L}_{\alpha,1}} : H^1(X_1, \mathcal{L}_{\alpha,1}) \to H^1(X_1, \mathcal{L}_{\alpha,1}^{\otimes p})$$

induced by the absolute Frobenius morphism F_{X_1} on X_1 is an injection. By [T2, Corollary 3.10 (iii)], we have

$$\#\{\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \setminus \{0\} \mid \phi_{\mathcal{L}_{\alpha,1}^{\otimes p^j}} \text{ is injective for all } j \in \{0, 1, \ldots, t-1\}\}$$
$$\geq n^{2g_X} - 3^{g_X-1} g_X!(p-1)tn^{2g_X-2} - 1.$$

Then the lemma follows immediately from the following observation

$$\{\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \setminus \{0\} \mid \phi_{\mathcal{L}_{\alpha,1}^{\otimes p^j}} \text{ is injective for all } j \in \{0, 1, \ldots, t-1\}\}$$
$$\subseteq \mathrm{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})^{\mathrm{ord}}.$$

This completes the proof of the lemma. □

3.1.3. Let G be a finite cyclic group and M a finite $k[G]$-module. Suppose that $\#(G)$ is prime to p. For any $\tau \in G$, we put $M^{\tau} \stackrel{\text{def}}{=} \{m \in M \mid \tau \cdot m = m\} \subseteq M$ and

$$M^{G\text{-prim}} \stackrel{\text{def}}{=} M/(\sum_{\sigma \neq 1} M^{\tau}) = \sum_{\chi: G \to k^{\times} \ \text{non-trivial}} M_{\chi},$$

where $(-)_{\chi}$ denotes the subspace of $(-)$ associated to the character χ, Then we have the following proposition.

Proposition 3.2. *We maintain the settings introduced in 3.1.1. Then we have*

$$\gamma_p^{\text{av}}(\Pi_{X^\bullet}) = g_X - 1.$$

Proof. Let n be an arbitrary natural number prime to p, K_n the kernel of the natural homomorphism $\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$, and $X^\bullet_{K_n}$ the Galois admissible covering of $X^\bullet$ (=Galois étale covering of X since $n_X = 0$) corresponding to K_n.

We put

$$\mathscr{C}_{K_n} \stackrel{\text{def}}{=} \{H \subseteq \Pi_{X^\bullet} \text{ an open normal subgroup} \mid K_n \subseteq H,\ \Pi_{X^\bullet}/H \text{ is cyclic}\}.$$

Since n is prime to p, we have the following canonical decomposition as $k[\Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}]$-modules

$$H^1_{\text{ét}}(X_{K_n}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k = \bigoplus_{\chi: \Pi_{X^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z} \to k^{\times}} (H^1_{\text{ét}}(X_{K_n}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k)_{\chi}$$

$$= \bigoplus_{H \in \mathscr{C}_{K_n}} ((H^1_{\text{ét}}(X_{K_n}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k)^{H/K_n})^{(\Pi_{X^\bullet}/H)\text{-prim}}$$

$$= \bigoplus_{H \in \mathscr{C}_{K_n}} (H^1_{\text{ét}}(X_H, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k)^{(\Pi_{X^\bullet}/H)\text{-prim}},$$

where X_H denotes the underlying curve of the pointed stable curve $X^\bullet_H$ corresponding to $H \subseteq \Pi_{X^\bullet}$. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Thus, we obtain

$$\sigma_{X_{K_n}} = \dim_k(H^1_{\text{ét}}(X_{K_n}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k) = \sum_{\alpha \in \text{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})} \gamma_{\alpha,1}.$$

Note that $0 \leq \gamma_{\alpha,1} \leq g_X - 1 = \dim_k(H^1(X, \mathcal{L}_{\alpha}))$ for all $\alpha \in \text{Hom}(\Pi_{X^\bullet}, \mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$. By applying Lemma 3.1, we have

$$(n^{2g_X} - 3^{g_X-1} g_X!(p-1)n^{2g_X-1} - 1)(g_X - 1) \leq \sigma_{X_{K_n}} \leq (n^{2g} - 1)(g_X - 1) + g_X.$$

Then the proposition follows immediately from $\#(\Pi_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}) = n^{2g_X}$. □

3.2. **Tame fundamental group case.** In this subsection, by using Proposition 3.2, we compute the p-averages for *tame* fundamental groups of *smooth component-generic* pointed stable curves.

3.2.1. **Settings.** We maintain the notation introduced in 2.2.1. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Moreover, we suppose the following condition holds:

- ⋄ $X^\bullet$ is a *smooth component-generic* pointed stable curve (2.2.3).

Thus, we have that $\Pi_{X^\bullet}$ is the tame fundamental group of $X^\bullet$.

3.2.2. We introduce a singular pointed stable curve. Let $X_s^\bullet = (X_s, D_{X_s})$ be a pointed stable curve of type (g_s, n_s) over an algebraically closed field k_s of characteristic $p > 0$ satisfying the following conditions:

- ⋄ $g_s \geq 1$ and $n_s \geq 2$.
- ⋄ $\mathrm{Irr}(X_s) = \{X_{s,1}, X_{s,2}\}$ and $X_{s,1}$, $X_{s,2}$ are smooth over k_s, where $\mathrm{Irr}(-)$ denotes the set of irreducible components of $(-)$.
- ⋄ The genus of $X_{s,1}$, $X_{s,2}$ are g_s, 0, respectively.
- ⋄ $X_s^{\mathrm{sing}} = \{x_s\}$ (i.e., $X_{s,1} \cap X_{s,2} = \{x_s\}$), and D_{X_s} is contained in $X_{s,2}$.

Then we obtain the following pointed stable curves (2.2.2)

$$X_{s,1}^\bullet \overset{\mathrm{def}}{=} (X_{s,1}, D_{X_{s,1}} \overset{\mathrm{def}}{=} \{x_s\}),\ X_{s,2}^\bullet \overset{\mathrm{def}}{=} (X_{s,2}, D_{X_{s,2}} \overset{\mathrm{def}}{=} \{x_s\} \cup \{D_{X_s}\})$$

of types $(g_s, 1)$ and $(0, n_s + 1)$, respectively.

Let $\Pi_{X_s^\bullet}$ and $\Pi_{X_{s,i}^\bullet}$, $i \in \{1, 2\}$, be the admissible fundamental groups of $X_s^\bullet$ and $X_{s,i}^\bullet$, respectively. Then we have a natural outer injection $\phi_i : \Pi_{X_{s,i}^\bullet} \hookrightarrow \Pi_{X_s^\bullet}$ (2.3.2). Then we have the following result:

Lemma 3.3. *We maintain the notation introduced above. Then we have*

$$\gamma_p^{\mathrm{av}}(\Pi_{X_s^\bullet}) = g_s.$$

Proof. Let n be an arbitrary natural number prime to p, $K_{s,n}$ the kernel of the natural homomorphism $\Pi_{X_s^\bullet} \twoheadrightarrow \Pi_{X_s^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$, and $f_{s,n}^\bullet : X_{s,K_{s,n}}^\bullet \to X_s^\bullet$ the Galois admissible covering over k_s corresponding to $K_{s,n} \subseteq \Pi_{X_s^\bullet}$. We put $K_{s,i,n} \overset{\mathrm{def}}{=} \phi_i^{-1}(K_{s,n})$.

Write $\Gamma_{X_s^\bullet}$ for the dual semi-graph of $X_s^\bullet$. We see that $\Gamma_{X_s^\bullet}^{\mathrm{cpt}}$ is 2-connected (2.1 (b), (c)). By applying [Y3, Corollary 3.5], we obtain

$$K_{s,i,n} = \ker(\Pi_{X_{s,i}^\bullet} \to \Pi_{X_{s,i}^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z}).$$

Then we have (see 2.2.1 for $r_{X_{s,K_{s,n}}}$ and 2.1 (a) for $e^{\mathrm{cl}}(\Gamma_{X_{s,K_{s,n}}^\bullet})$ and $v(\Gamma_{X_{s,K_{s,n}}^\bullet})$)

$$\sigma_{X_{s,K_{s,n}}} = \dim_{\mathbb{F}_p}(K_{s,n}^{\mathrm{ab}} \otimes \mathbb{F}_p)$$

$$= r_{X_{s,K_{s,n}}} + \sum_{i\in\{1,2\}} \frac{\#(\Pi_{X_s^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}{\#(\Pi_{X_{s,i}^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} \cdot \dim_{\mathbb{F}_p}(K_{s,i,n}^{\mathrm{ab}} \otimes \mathbb{F}_p)$$

$$= \#(e^{\mathrm{cl}}(\Gamma_{X_{s,K_{s,n}}^\bullet})) - \#(v(\Gamma_{X_{s,K_{s,n}}^\bullet})) + 1 + \sum_{i\in\{1,2\}} \frac{\#(\Pi_{X_s^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}{\#(\Pi_{X_{s,i}^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} \cdot \dim_{\mathbb{F}_p}(K_{s,i,n}^{\mathrm{ab}} \otimes \mathbb{F}_p).$$

Note that

$$\#(v(\Gamma_{X_{s,K_{s,n}}^\bullet})) = \sum_{i\in\{1,2\}} \frac{\#(\Pi_{X_s^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}{\#(\Pi_{X_{s,i}^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}.$$

On the other hand, since the type of $X^{\bullet}_{s,1}$ is $(g_s, 1)$, we have that $f^{\bullet}_{s,n}$ is étale over the singular point $x_s \in X_s$, and that $\Pi^{\mathrm{ab}}_{X^{\bullet}_{s,1}} = \Pi^{\text{ét},\mathrm{ab}}_{X^{\bullet}_{s,1}}$. This implies

$$\#(e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{s,K_{s,n}}})) = \#(\Pi^{\mathrm{ab}}_{X^{\bullet}_s} \otimes \mathbb{Z}/n\mathbb{Z}).$$

Thus, we have

$$\begin{aligned}\gamma_p^{\mathrm{av}}(\Pi_{X^{\bullet}_s}) &= 1 + \gamma_p^{\mathrm{av}}(\Pi_{X^{\bullet}_{s,1}}) + \gamma_p^{\mathrm{av}}(\Pi_{X^{\bullet}_{s,2}})\\ &= 1 + \gamma_p^{\mathrm{av}}(\Pi^{\text{ét}}_{X^{\bullet}_{s,1}}) + \gamma_p^{\mathrm{av}}(\Pi_{X^{\bullet}_{s,2}}).\end{aligned}$$

Proposition 3.2 implies $\gamma_p^{\mathrm{av}}(\Pi^{\text{ét}}_{X^{\bullet}_{s,1}}) = g_s - 1$. Furthermore, [T2, Appendix, Theorem A.1] implies $0 = \gamma_p^{\mathrm{av}}(\Pi_{X^{\bullet}_{s,2}}) \leq 0$. Then we obtain

$$\gamma_p^{\mathrm{av}}(\Pi_{X^{\bullet}_s}) = g_s.$$

This completes the proof of the lemma. $\square$

3.2.3. We maintain the settings introduced in 3.1.1. Moreover, we suppose $g_X \geq 1$ and $n_X \geq 2$. Since we assume that $X^{\bullet}$ is a component-generic pointed stable curve over k, there exist a discrete valuation ring R of equal characteristic with algebraically closed residue field k_R and a pointed stable curve $\mathcal{X}^{\bullet}$ of type (g_X, n_X) over R satisfying the following conditions:

- ◇ Write $\eta \overset{\mathrm{def}}{=} \operatorname{Spec} K_R$ and $s \overset{\mathrm{def}}{=} \operatorname{Spec} k_R$ for the generic point and the closed point of $\operatorname{Spec} R$, respectively, where K_R denotes the quotient field of R. Then we have
  - (i) There exists an algebraically closed field k' containing K_R and k such that $\mathcal{X}^{\bullet} \times_R k'$ is k'-isomorphic to $X^{\bullet} \times_k k'$.
  - (ii) The special fiber $\mathcal{X}^{\bullet}_s \overset{\mathrm{def}}{=} \mathcal{X}^{\bullet} \times_R k_R$ satisfying the conditions which were mentioned at the beginning of 3.2.2.

We write $\overline{K}_R$ for the algebraic closure of K_R in k' and put $\mathcal{X}^{\bullet}_{\overline{\eta}} \overset{\mathrm{def}}{=} \mathcal{X}^{\bullet} \times_R \overline{K}_R$. Then we obtain the following specialization surjective homomorphism of admissible fundamental groups (which is not an isomorphism)

$$sp_R : \Pi_{X^{\bullet}} \cong \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}} \twoheadrightarrow \Pi_{\mathcal{X}^{\bullet}_s}.$$

We have the following lemma.

Lemma 3.4. *We maintain the notation introduced above. Then we have*

$$\gamma_p^{\mathrm{av}}(\Pi_{X^{\bullet}}) = \gamma_p^{\mathrm{av}}(\Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}) \geq \gamma_p^{\mathrm{av}}(\Pi_{\mathcal{X}^{\bullet}_s}).$$

Proof. Note that sp_R induces an isomorphism

$$sp^{p'} : \Pi^{p'}_{\mathcal{X}^{\bullet}_{\overline{\eta}}} \twoheadrightarrow \Pi^{p'}_{\mathcal{X}^{\bullet}_s},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Then the lemma follows immediately from the definition of the prime-to-p limits of p-averages. $\square$

Remark 3.4.1. Note that Lemma 3.4 holds for an arbitrary pointed stable curve $\mathcal{X}^{\bullet}$ over an arbitrary discrete valuation ring R.

3.2.4. We have the following result.

Proposition 3.5. *We maintain the settings introduced in 3.1.1. Then we have*

$$\gamma_p^{\mathrm{av}}(\Pi_{X^\bullet}) = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X > 1. \end{cases}$$

Proof. Suppose $g_X = 0$. Then the proposition follows immediately from [T2, Appendix, Theorem A.1].

Suppose $n_X \leq 1$. Then all abelian admissible coverings of $X^\bullet$ are étale. This implies $\gamma_p^{\mathrm{av}}(\Pi_{X^\bullet}) = \gamma_p^{\mathrm{av}}(\Pi_{X^\bullet}^{\text{ét}})$. Thus, the proposition follows from Proposition 3.2.

Suppose $g_X \geq 1$ and $n_X \geq 2$. Then the proposition follows from [T2, Appendix, Theorem A.1], Lemma 3.3, and Lemma 3.4. □

4. p-AVERAGES FOR ARBITRARY COMPONENT-GENERIC CURVES

In this section, we generalize Proposition 3.5 to the case of *arbitrary* (possibly singular) component-generic pointed stable curves. The main result of the present section is Theorem 4.6.

4.1. **Notation.** We introduced some notation.

4.1.1. **Settings.** We maintain the notation introduced in 2.2.1.

4.1.2. Let $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X^\bullet}^{\mathrm{cpt}})$ be an arbitrary vertex of $\Gamma_{X^\bullet}$ (see 2.1 (b) for $\Gamma_{X^\bullet}^{\mathrm{cpt}}$) and $\widetilde{X}_v^\bullet$ the smooth pointed stable curve of type (g_v, n_v) associated to v (2.2.2). Write Γ_v for the dual semi-graph of $\widetilde{X}_v^\bullet$. Then we obtain a map of semi-graphs $\rho'_v : \Gamma_v \to \Gamma_{X^\bullet}$ induced by the natural morphism $U_{X_v} \hookrightarrow X$ and the natural map of sets of closed points $D_{\widetilde{X}_v} \to D_X \cup X^{\mathrm{sing}}$. We put

$$\rho_v : \Gamma_v \overset{\rho'_v}{\to} \Gamma_{X^\bullet} \to \Gamma_{X^\bullet}^{\mathrm{cpt}},$$

where $\Gamma_{X^\bullet} \to \Gamma_{X^\bullet}^{\mathrm{cpt}}$ is the natural map of semi-graphs induced by the definition of $\Gamma_{X^\bullet}^{\mathrm{cpt}}$.

Definition 4.1. We maintain the notation introduced above. Let $\pi_0(v)$ be the set of connected components of $\Gamma_{X^\bullet}^{\mathrm{cpt}} \setminus \{v\}$. We put

- ◇ $E_{v,C} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_v) \mid \rho_v(e) \cap C \neq \emptyset\},\ C \in \pi_0(v)$,
- ◇ $E_v^{>1} \overset{\mathrm{def}}{=} \{C \in \pi_0(v) \mid \#(E_{v,C}) > 1\}$,
- ◇ $E_v^{=1} \overset{\mathrm{def}}{=} \{C \in \pi_0(v) \mid \#(E_{v,C}) = 1\}$.

Note that the definitions imply

$$e^{\mathrm{op}}(\Gamma_v) = \bigcup_{C \in \pi_0(v)} E_{v,C},\ \#(\pi_0(v)) = \#(E_v^{=1}) + \#(E_v^{>1}).$$

4.1.3. Let $X_{v_\infty}^\bullet = (X_{v_\infty}, D_{X_{v_\infty}})$ be a smooth pointed stable curve of type $(g_{v_\infty}, n_{v_\infty})$ over k such that $g_{v_\infty} \geq 2$ and $n_{v_\infty} = n_X$. Write Γ_{v_∞} for the dual semi-graph of $X_{v_\infty}^\bullet$. If $n_X \neq 0$, we fix a bijection $D_{X_{v_\infty}} \overset{\sim}{\to} D_X$. Then we may glue $X^\bullet$ and $X_{v_\infty}^\bullet$ along the sets of marked points D_X and $D_{X_{v_\infty}}$, and obtain a stable curve X'_∞ of type $(g_X + g_{v_\infty} + n_X - 1, 0)$ over k. We define a stable curve X_∞ of type $(g_{X_\infty}, 0)$ over k to be

$$X_\infty \overset{\mathrm{def}}{=} \begin{cases} X, & \text{if } n_X = 0, \\ X'_\infty, & \text{if } n_X \neq 0. \end{cases}$$

Write Γ_{X_∞} for the dual semi-graph of X_∞. Note that by the construction of X_∞, $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is naturally isomorphic to Γ_{X_∞}. Then we may identify $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ with Γ_{X_∞}.

Let R be a complete discrete valuation ring of equal characteristic with residue field k, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. Let $L \subseteq e^{\mathrm{cl}}(\Gamma_{X_\infty})$ be an arbitrary subset of closed edges. We may deform the pointed stable curve X_∞ along L to obtain a new pointed stable curve over $\overline{K}$ such that the set of edges of the dual semi-graph of the new stable curve may be naturally identified with $e(\Gamma_{X_\infty}) \setminus L$. Suppose that

$$c_s : \mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_{X_\infty} R} \overset{\mathrm{def}}{=} \overline{\mathcal{M}}_{g_{X_\infty}} \times_{\mathbb{Z}} R$$

is the classifying morphism determined by $X_\infty \to \mathrm{Spec}\, k$. Thus the completion of the local ring of the moduli stack at c_s is isomorphic to $R[\![t_1, ..., t_{3g_{X_\infty}-3}]\!]$, where $t_1, ..., t_{3g_{X_\infty}-3}$ are indeterminates. Furthermore, the indeterminates $t_1, ..., t_m$ may be chosen so as to correspond to the deformations of the nodes of X_∞. Suppose that $\{t_1, ..., t_d\}$ is the subset of $\{t_1, ..., t_m\}$ corresponding to the subset $L \subseteq e^{\mathrm{cl}}(\Gamma_{X_\infty})$. Now fix a morphism $\mathrm{Spec}\, R \to \mathrm{Spec}\, R[\![t_1, ..., t_{3g_{X_\infty}-3}]\!]$ such that $t_{d+1}, ..., t_{3g_{X_\infty}-3} \mapsto 0 \in R$, but $t_1, ..., t_d$ map to nonzero elements of R. Then the composite morphism

$$c : \mathrm{Spec}\, R \to \mathrm{Spec}\, R[\![t_1, ..., t_{3g_{X_\infty}-3}]\!] \to \overline{\mathcal{M}}_{g_{X_\infty}, R}$$

determines a stable curve $\mathcal{X}_\infty \to \mathrm{Spec}\, R$. Moreover, the special fiber $\mathcal{X}_\infty \times_R k$ of $\mathcal{X}_\infty$ is naturally isomorphic to X_∞ over k. Write

$$X_\infty^{\setminus L}$$

for the geometric generic fiber $X_\infty \times_K \overline{K}$ of $\mathcal{X}_\infty$ over $\overline{K}$ and $\Gamma_{X_\infty^{\setminus L}}$ for the dual semi-graph of $X_\infty^{\setminus L}$. It follows from the construction of $X_\infty^{\setminus L}$ that we have a natural bijective map

$$e(\Gamma_{X_\infty}) \setminus L \overset{\sim}{\to} e(\Gamma_{X_\infty^{\setminus L}}).$$

Let $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X_\infty}) = v(\Gamma^{\mathrm{cpt}}_{X^\bullet})$ be an arbitrary vertex of $\Gamma_{X^\bullet}$ and

$$L_v \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\Gamma_{X_\infty}) \mid e \text{ does not meet } v\}.$$

We put

$$X_v^{\mathrm{def}} \overset{\mathrm{def}}{=} X_\infty^{\setminus L_v},$$

and $\Gamma_{X_v^{\mathrm{def}}}$ the dual semi-graph of X_v^{def}. Then we have the following definition.

Definition 4.2. Let $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma_{X_\infty}) = v(\Gamma^{\mathrm{cpt}}_{X^\bullet})$ and $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \subseteq e^{\mathrm{cl}}(\Gamma_{X_\infty}) = e^{\mathrm{cl}}(\Gamma^{\mathrm{cpt}}_{X^\bullet})$. We shall say that v is a *tree-like vertex* if $\Gamma_{X_v^{\mathrm{def}}}$ is a tree (i.e., the Betti number of $\Gamma_{X_v^{\mathrm{def}}}$ is 0), and that e is a *tree-like edge* if there exists a vertex $w \in v(\Gamma_{X^\bullet})$ such that $E_{w,C} = \{e\}$ for some $C \in E_w^{=1}$. We put

- ⋄ $V^{\mathrm{tre}}_{X^\bullet} \overset{\mathrm{def}}{=} \{v \in v(\Gamma_{X^\bullet}) \mid v \text{ is tree-like}\}$,
- ⋄ $V^{\mathrm{tre}, g_v=0}_{X^\bullet} \overset{\mathrm{def}}{=} \{v \in V^{\mathrm{tre}}_{X^\bullet} \mid g_v = 0\}$,
- ⋄ $E^{\mathrm{tre}}_{X^\bullet} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid e \text{ is tree-like}\}$.

Note that we have

$$E^{\mathrm{tre}}_{X^\bullet} = \bigcup_{v \in v(\Gamma_{X^\bullet})} \bigcup_{C \in \pi_0(v) \text{ s.t. } C \in E_v^{=1}} E_{v,C}.$$

4.2. **Upper bounds of the p-averages of irreducible components.** In this subsection, we compute upper bounds of the p-averages concerning irreducible components.

4.2.1. **Settings.** We maintain the notation introduced in 2.2.1. Let $X^\bullet$ be an arbitrary pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Let $v \in v(\Gamma_{X^\bullet})$ be a vertex of $\Gamma_{X^\bullet}$, $\widetilde{X}_v^\bullet$ the smooth pointed stable curve of type (g_v, n_v) associated to v, and $\Pi_{\widetilde{X}_v^\bullet}$ the admissible fundamental group of $\widetilde{X}_v^\bullet$. We denote by

$$\phi_v^{\mathrm{ab}} : \Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}}$$

the homomorphism induced by the natural outer injection $\Pi_{\widetilde{X}_v^\bullet} \to \Pi_{X^\bullet}$. Note that ϕ_v^{ab} is not an injection if $\Gamma_{X^\bullet}$ is not 2-connected ([Y3, Corollary 3.5]). We put

$$M_v \overset{\mathrm{def}}{=} \mathrm{Im}(\phi_v^{\mathrm{ab}}).$$

4.2.2. Let n be a natural number prime to p,

$$H_{v,n} \overset{\mathrm{def}}{=} \ker(\Pi_v \twoheadrightarrow \Pi_v^{\mathrm{ab}} \overset{\phi_v^{\mathrm{ab}}}{\twoheadrightarrow} M_v \otimes \mathbb{Z}/n\mathbb{Z}),$$

and $X_{H_{v,n}}^\bullet \to \widetilde{X}_v^\bullet$ the Galois admissible covering over k corresponding to $H_{v,n}$. For each $C \in \pi_0(v)$, we put $D'_{\widetilde{X}_{v,C}} \overset{\mathrm{def}}{=} \{x_e \in D_{\widetilde{X}_v} \mid e \in E_{v,C}\}$ (see Definition 4.1). We define a smooth pointed semi-stable curve of type $(g_v, n_{v,C} \overset{\mathrm{def}}{=} (\#E_{v,C}))$ over k to be

$$\widetilde{X}_{v,C}^\bullet = (\widetilde{X}_{v,C}, D_{\widetilde{X}_{v,C}}) \overset{\mathrm{def}}{=} (\widetilde{X}_v, D'_{\widetilde{X}_{v,C}}).$$

Then we have the following result.

Proposition 4.3. *We maintain the notation introduced above. Then the following statements hold (see Definition 4.1 for $E_v^{>1}$):*

(i) Suppose $(g_v, \#(E_v^{>1})) = (0,0)$. Then we have

$$\lim_{n\to\infty} \frac{\sigma_{X_{H_{v,n}}}}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = \lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}(H_{v,n}^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = 0.$$

(ii) Suppose $(g_v, \#(E_v^{>1})) \neq (0,0)$. Then we have

$$\limsup_{n\to\infty} \frac{\sigma_{X_{H_{v,n}}}}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = \limsup_{n\to\infty} \frac{\dim_{\mathbb{F}_p}(H_{v,n}^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} \leq g_v + \#(E_v^{>1}) - 1,$$

where $\limsup(-)$ *denotes the limit superior of* $(-)$.

Proof. (i) Since $X_{H_{v,n}}$ is isomorphic to $\mathbb{P}_k^1$ for all natural numbers prime to p, (i) follows immediately from that $\sigma_{X_{H_{v,n}}} = 0$.

(ii) We put

$$\mathscr{S}_{H_{v,n}} \overset{\mathrm{def}}{=} \{H \subseteq \Pi_{\widetilde{X}_v^\bullet} \text{ an open normal subgroup} \mid H_{v,n} \subseteq H,\ \Pi_{\widetilde{X}_v^\bullet}/H \text{ is cyclic}\}.$$

Note that $\#(\Pi_{\widetilde{X}_v^\bullet}/H)$, $H \in \mathscr{S}_{H_{v,n}}$, is prime to p. Write $X_H^\bullet \overset{\mathrm{def}}{=} (X_H, D_{X_H})$ for the pointed stable curve over k corresponding to H. Since $M_v \otimes \mathbb{Z}/n\mathbb{Z}$ is an abelian group, we have the following canonical decomposition as $k[M_v \otimes \mathbb{Z}/n\mathbb{Z}]$-modules (see 3.1.3 for $(-)^{(\Pi_{\widetilde{X}_v^\bullet}/H)\text{-prim}}$)

$$H^1_{\text{ét}}(X_{H_{v,n}}, \mathbb{F}_p) \otimes k = \bigoplus_{\chi: M_v \otimes \mathbb{Z}/n\mathbb{Z} \to k^\times} (H^1_{\text{ét}}(X_{H_{v,n}}, \mathbb{F}_p) \otimes k)_\chi$$

$$= \bigoplus_{H\in\mathscr{S}_{H_{v,n}}} (H^1_{\text{ét}}(X_{H_{v,n}}, \mathbb{F}_p)^{H/H_{v,n}} \otimes k)^{(\Pi_{\widetilde{X}^\bullet_v}/H)\text{-prim}}$$
$$= \bigoplus_{H\in\mathscr{S}_{H_{v,n}}} (H^1_{\text{ét}}(X_H, \mathbb{F}_p) \otimes k)^{(\Pi_{\widetilde{X}^\bullet_v}/H)\text{-prim}}.$$

On the other hand, we put (i.e., the subset of $\mathrm{Hom}(\Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_v}, \mathbb{Z}/n\mathbb{Z})$ corresponding to $\mathscr{S}_{H_{v,n}}$)

$$\mathscr{T}_{H_{v,n}} \stackrel{\text{def}}{=} \{\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_v}, \mathbb{Z}/n\mathbb{Z}) \mid H_{v,n} \subseteq \ker(\Pi_{\widetilde{X}^\bullet_v} \twoheadrightarrow \Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_v} \stackrel{\alpha}{\to} \mathbb{Z}/n\mathbb{Z})\}.$$

Then we have

$$\sigma_{X_{H_{v,n}}} = \dim_k(H^1_{\text{ét}}(X_{H_{v,n}}, \mathbb{F}_p) \otimes k) = \bigoplus_{\alpha\in\mathscr{T}_{H_{v,n}}} \gamma_{\alpha,1}.$$

Let $f^\bullet_{v,\alpha} : X^\bullet_{v,\alpha} \to \widetilde{X}^\bullet_v$ be the Galois multi-admissible covering with Galois group $\mathbb{Z}/n\mathbb{Z}$. Fix a primitive nth root ζ, we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Then we have

$$f_{v,\alpha,*}\mathcal{O}_{X_{v,\alpha}} \cong \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \mathcal{L}_{\alpha,i},$$

where $\mathcal{L}_{\alpha,0} \cong \mathcal{O}_{\widetilde{X}_v}$, and $\zeta \in \mu_n$ acts locally on $\mathcal{L}_{\alpha,i}$ as ζ^i-multiplication. Moreover, we have $\mathcal{L}^{\otimes n}_{\alpha,1} \cong \mathcal{O}_{\widetilde{X}_v}(-D_\alpha)$ for some effective divisor D_α on $\widetilde{X}_v$ whose support is contained in $D_{\widetilde{X}_v} \setminus (\bigcup_{C\in E^{=1}_v} D_{\widetilde{X}_{v,C}})$ (see Definition 4.1 for $E^{=1}_v$). Note that $\deg(D_\alpha)$ is divided by n. We put

$$s(D_\alpha) \stackrel{\text{def}}{=} \frac{\deg(D_\alpha)}{n}.$$

Then we have that $s(D_\alpha) \leq \#(E^{>1}_v)$, and that the Riemann-Roch theorem implies $\dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\alpha,1})) = g_v + s(D_\alpha) - 1$. Write t for the order of p in $(\mathbb{Z}/n\mathbb{Z})^\times$ and $t_\alpha \in \{0, 1, \ldots, t-1\}$ for an integer such that

$$s(p^{t_\alpha}D_\alpha) = \min_{j\in\{0,1,\ldots,t-1\}}\{s(p^jD_\alpha)\}.$$

Then we obtain

$$\gamma_{\alpha,1} \leq \dim_k(H^1(\widetilde{X}_v, \mathcal{L}_{\alpha,p^{t_\alpha}})) = g_v + s(p^{t_\alpha}D_\alpha) - 1.$$

Note that we have $D_{\widetilde{X}_{v,C}} \subseteq D_{\widetilde{X}_v}$. We put

$$D_{\alpha,C} \stackrel{\text{def}}{=} D_\alpha|_{D_{\widetilde{X}_{v,C}}},\ C \in \pi_0(v).$$

Since $\alpha \in \mathscr{T}_{H_{v,n}}$, [Y3, Proposition 3.4 (ii)] implies that $\deg(D_{\alpha,C})$ is divided by n. Then we put

$$s(D_{\alpha,C}) \stackrel{\text{def}}{=} \frac{\deg(D_{\alpha,C})}{n}.$$

Moreover, we put

$$\mathscr{A}_{H_{v,n},C} \stackrel{\text{def}}{=} \{\alpha \in \mathscr{T}_{H_{v,n}} \mid s(D_{\alpha,C}) = 1\},\ C \in \pi_0(v),$$
$$\mathscr{A}_{H_{v,n}} \stackrel{\text{def}}{=} \bigcap_{C\in\pi_0(v)} \mathscr{A}_{H_{v,n},C}.$$

Then we have $s(D_\alpha) = \#(E^{>1}_v)$ for all $\alpha \in \mathscr{A}_{H_{v,n}}$. Thus, we obtain

$$\gamma_{\alpha,1} \leq g_v + \#(E^{>1}_v) - 1,\ \alpha \in \mathscr{A}_{H_{v,n}}.$$

By applying [T2, p99 Appendix, A.3], we obtain

$$\lim_{n\to\infty} \frac{\#(\mathscr{A}_{H_{v,n},C})}{n^{2g_v+\#(E_{v,C})-1}} = 1.$$

Furthermore, we see

$$\lim_{n\to\infty} \frac{\#(\mathscr{A}_{H_{v,n}})}{n^{2g_v+\sum_{C\in\pi_0(v)}(\#(E_{v,C})-1)}} = 1 \ (\text{or } \lim_{n\to\infty} \frac{\#(\mathscr{T}_{H_{v,n}} \setminus \mathscr{A}_{H_{v,n}})}{n^{2g_v+\sum_{C\in\pi_0(v)}(\#(E_{v,C})-1)}} = 0).$$

Note that $\gamma_{\alpha,1} \leq g_v + \#(E_v^{>1}) - 1$ for all $\alpha \in \mathscr{A}_{H_{v,n}}$. We obtain

$$\begin{aligned}\sigma_{X_{H_{v,n}}} &\leq \#(\mathscr{A}_{H_{v,n}})(g_v + \#(E_v^{>1}) - 1) + \#(\mathscr{T}_{H_{v,n}} \setminus \mathscr{A}_{H_{v,n}})(g_v + n_v - 2)\\ &\leq \#(\mathscr{T}_{H_{v,n}})(g_v + \#(E_v^{>1}) - 1) + \#(\mathscr{T}_{H_{v,n}} \setminus \mathscr{A}_{H_{v,n}})(g_v + n_v - 2).\end{aligned}$$

By applying [Y3, Proposition 3.4 (ii)], we obtain (see Definition 4.1 for $E_{v,C}$)

$$\#(M_v \otimes \mathbb{Z}/n\mathbb{Z}) = n^{2g_v+\sum_{C\in\pi_0(v)}(\#(E_{v,C})-1)}.$$

Thus, we have

$$\limsup_{n\to\infty} \frac{\sigma_{X_{H_{v,n}}}}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} \leq g_v + \#(E_v^{>1}) - 1.$$

This completes the proof of the proposition. □

4.3. **The p-averages of irreducible components.** In this subsection, we compute the p-averages concerning irreducible components of component-generic pointed stable curves.

4.3.1. **Settings.** We maintain the settings introduced in 4.2.1. Moreover, we suppose the following holds:

- ◇ $X^\bullet$ is an arbitrary *component-generic* pointed stable curve (2.2.3).

4.3.2. Let $v \in v(\Gamma_{X^\bullet})$ and (g_v, n_v) the type of the smooth pointed stable curve $\widetilde{X}_v^\bullet$ associated to v. Let $X_{v,s}^\bullet = (X_{v,s}, D_{X_{v,s}})$ be a pointed stable curve of type (g_v, n_v) over an algebraically closed field $k_{v,s}$ of characteristic $p > 0$ satisfying the following conditions:

- ◇ Suppose $\#(E_v^{>1}) \leq 1$. Then we have $k_{v,s} = k$ and
$$X_{v,s}^\bullet \overset{\text{def}}{=} \widetilde{X}_v^\bullet.$$
- ◇ Suppose $g_v = 0$ and $\#(E_v^{>1}) = 2$. We put $E_v^{>1} = \{C_1, C_2\}$. Then we have
$$\mathrm{Irr}(X_{v,s}) \overset{\text{def}}{=} \{P_{C_1}, P_{C_2}\}$$
such that
  - (i) P_{C_i}, $i \in \{1, 2\}$, is isomorphic to $\mathbb{P}^1_{k_{v,s}}$;
  - (ii) $\#(P_{C_1} \cap P_{C_2}) = 1$ and $\#(X_{v,s}^{\mathrm{sing}}) = 1$;
  - (iii) $\#(D_{X_{v,s}} \cap P_{C_1}) = \#(E_{v,C_1}) + \#(E_v^{=1})$ and $\#(D_{X_{v,s}} \cap P_{C_2}) = \#(E_{v,C_2})$;
  - (iv) $P_{C_i}^\bullet \overset{\text{def}}{=} (P_{C_i}, D_{P_{C_i}} \overset{\text{def}}{=} (D_{X_{v,s}} \cap P_{C_i}) \cup (P_{C_1} \cup P_{C_2}))$, $i \in \{1, 2\}$, is a smooth component-generic pointed stable curve of type $(0, \#(E_{v,C_i}) + 1)$.
- ◇ Suppose that either $g_v \geq 1$ or $\#(E_v^{>1}) > 2$ holds. Then we have
$$\mathrm{Irr}(X_{v,s}) \overset{\text{def}}{=} \{Z_v\} \cup \{P_C\}_{C\in E_v^{>1}}$$
such that

(i) Z_v is a smooth projective curve over $k_{v,s}$ of genus g_v;
(ii) P_C, $C \in E_v^{>1}$, is isomorphic to $\mathbb{P}^1_{k_{v,s}}$ over $k_{v,s}$;
(iii) $\#(P_C \cap Z_v) = 1$ for all $C \in E_v^{>1}$ and $\#(X_{v,s}^{\mathrm{sing}}) = \#(E_v^{>1})$;
(iv) $\#(D_{X_{v,s}} \cap P_C) = \#(E_{v,C})$, $C \in E_v^{>1}$;
(v) $\#(D_{X_{v,s}} \cap Z_v) = \#(E_v^{=1})$;
(vi) $P_C^\bullet \overset{\mathrm{def}}{=} (P_C, D_{P_C} \overset{\mathrm{def}}{=} (D_{X_{v,s}} \cap P_C) \cup (Z_v \cap P_C))$, $C \in E_v^{>1}$, is a smooth component-generic pointed stable curve over $k_{v,s}$ of type $(0, \#(E_{v,C})+1)$;
(vii) $Z_v^\bullet \overset{\mathrm{def}}{=} (Z_v, D_{Z_v} \overset{\mathrm{def}}{=} (Z_v \cap D_{X_{v,s}}) \cup (Z_v \cap (\bigcup_{C \in E_v^{>1}} P_C)))$ is a smooth component-generic pointed stable curve over $k_{v,s}$ of type $(g_v, \#(\pi_0(v)))$.

Let $\Pi_{X_{v,s}^\bullet}$, $\Pi_{Z_v^\bullet}$, and $\Pi_{P_C^\bullet}$, $C \in E_v^{>1}$, be the admissible fundamental groups of $X_{v,s}^\bullet$, $Z_v^\bullet$, and $P_C^\bullet$, respectively. We have natural outer injections $\phi_{Z_v} : \Pi_{Z_v^\bullet} \hookrightarrow \Pi_{X_{v,s}^\bullet}$ and $\phi_C : \Pi_{P_C^\bullet} \hookrightarrow \Pi_{X_{v,s}^\bullet}$, $C \in E_v^{>1}$. Write $\Gamma_{X_{v,s}^\bullet}$, $\Gamma_{Z_v^\bullet}$, and $\Gamma_{P_C^\bullet}$, $C \in E_v^{>1}$, for the dual semi-graphs of $X_{v,s}^\bullet$, $Z_v^\bullet$, and $P_C^\bullet$, respectively.

4.3.3. We maintain the notation introduced in 4.3.2. We put

$$B_v \overset{\mathrm{def}}{=} \{E_{v,C}\}_{C \in E_v^{=1}} \cup e^{\mathrm{cl}}(\Gamma_{X_{v,s}^\bullet}),$$

$$S_v \overset{\mathrm{def}}{=} \{x_e \text{ is a closed point of } X_{v,s} \text{ corresponding to } e \in B_v\},$$

and put

$$B_{Z_v} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_{Z_v^\bullet}) \mid x_e \in S_v\},$$

$$B_{v,C} \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{op}}(\Gamma_{P_C^\bullet}) \mid x_e \in S_v\}, \ C \in E_v^{>1}.$$

Note that by the above constructions, we have

- ◇ $\#(B_v) = \#(B_{v,C_1})$ and $\#(B_{v,C_2}) = 1$ if $g_v = 0$ and $\#(E_v^{>1}) = 2$.
- ◇ $\#(B_v) = \#(B_{Z_v})$ and $\#(B_{v,C}) = 1$ if either $g_v \geq 1$ or $\#(E_v^{>1}) > 2$ holds.

We put (see 2.3.3 for notation concerning universal admissible coverings and their dual semi-graphs)

$$\widehat{B}_v \overset{\mathrm{def}}{=} \pi_{X_{v,s}}^{-1}(B_v) \subseteq \Gamma_{\widehat{X}_{v,s}^\bullet},$$

$$\widehat{B}_{Z_v} \overset{\mathrm{def}}{=} \pi_{Z_v}^{-1}(B_{Z_v}) \subseteq \Gamma_{\widehat{Z}_v^\bullet},$$

$$\widehat{B}_{v,C} \overset{\mathrm{def}}{=} \pi_{P_C}^{-1}(B_{v,C}) \subseteq \Gamma_{\widehat{P}_C^\bullet}, \ C \in E_v^{>1}.$$

Furthermore, we put

$$I_{B_v} \subseteq \Pi_{X_{v,s}^\bullet}, \ I_{B_{Z_v}} \subseteq \Pi_{Z_v^\bullet}, \ I_{B_{v,C}} \subseteq \Pi_{P_C^\bullet}, \ C \in E_v^{>1},$$

the closed normal subgroup generated by $\{I_{\widehat{e}}\}_{\widehat{e} \in \widehat{B}_v}$, $\{I_{\widehat{e}}\}_{\widehat{e} \in \widehat{B}_{Z_v}}$, $\{I_{\widehat{e}}\}_{\widehat{e} \in \widehat{B}_{v,C}}$, respectively. Then the theory of admissible fundamental groups implies immediately

$$\phi_{Z_v}^{-1}(I_{B_v}) = I_{B_{Z_v}}, \ \phi_C^{-1}(I_{B_v}) = I_{B_{v,C}}, \ C \in E_v^{>1}.$$

Moreover, we have the following lemma.

Lemma 4.4. *We maintain the notation introduced above. Then we have*

$$\gamma_p^{\mathrm{av}}(\Pi_{X_{v,s}^\bullet}/I_{B_v}) = g_v + \#(E_v^{>1}) - 1.$$

Proof. Suppose $\#(E_v^{>1}) \leq 1$. Then the lemma follows immediately from Proposition 3.5. Thus, to verify the lemma, we may assume $\#(E_v^{>1}) \geq 2$.

Let n be an arbitrary natural number prime to p and $C \in E_v^{>1}$. We put

$$K_{v,s,n} \overset{\text{def}}{=} \ker(\Pi_{X_{v,s}^\bullet} \twoheadrightarrow \Pi_{X_{v,s}^\bullet}^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}),\ K_{Z_v,n} \overset{\text{def}}{=} \phi_{Z_v}^{-1}(K_{v,s,n}),\ K_{v,C,n} \overset{\text{def}}{=} \phi_C^{-1}(K_{v,s,n}),$$

and

$$I_{B_v,n} \overset{\text{def}}{=} K_{v,s,n} \cap I_{B_v},\ I_{B_{Z_v},n} \overset{\text{def}}{=} K_{Z_v,n} \cap I_{B_{Z_v}},\ I_{B_{v,C},n} \overset{\text{def}}{=} K_{v,C,n} \cap I_{B_{v,C}}.$$

Since $\Gamma^{\text{cpt}}_{X^\bullet_{K_{v,s,n}}}$ is 2-connected (2.1), where $X^\bullet_{K_{v,s,n}}$ denotes the Galois admissible covering of $X_{v,s}^\bullet$ corresponding to $K_{v,s,n} \subseteq \Pi_{X_{v,s}^\bullet}$, [Y3, Corollary 3.5] implies that the homomorphisms

$$K_{Z_v,n}^{\text{ab}} \hookrightarrow K_{v,s,n}^{\text{ab}},\ K_{v,C,n}^{\text{ab}} \hookrightarrow K_{v,s,n}^{\text{ab}}$$

induced by the natural injections $\phi_{Z_v}|_{K_{Z_v,n}} : K_{Z_v,n} \hookrightarrow K_{v,s,n}$ and $\phi_C|_{K_{v,C,n}} : K_{v,C,n} \hookrightarrow K_{v,s,n}$ are injections.

We denote by

$$\overline{I}_{B_v,n} \overset{\text{def}}{=} \text{Im}(I_{B_v,n} \hookrightarrow K_{v,s,n} \twoheadrightarrow K_{v,s,n}^{\text{ab}}),$$
$$\overline{I}_{B_{Z_v},n} \overset{\text{def}}{=} \text{Im}(I_{B_{Z_v},n} \hookrightarrow K_{Z_v,n} \twoheadrightarrow K_{Z_v,n}^{\text{ab}}),$$
$$\overline{I}_{B_{v,C},n} \overset{\text{def}}{=} \text{Im}(I_{B_{v,C},n} \hookrightarrow K_{v,C,n} \twoheadrightarrow K_{v,C,n}^{\text{ab}}).$$

Then we have

$$(K_{Z_v,n}/I_{B_{Z_v},n})^{\text{ab}} \cong K_{Z_v,n}^{\text{ab}}/\overline{I}_{B_{Z_v},n} \hookrightarrow K_{v,s,b}^{\text{ab}}/\overline{I}_{B_v,n} \cong (K_{v,s,b}/I_{B_v,n})^{\text{ab}},$$
$$(K_{v,C,n}/I_{B_{v,C},n})^{\text{ab}} \cong K_{v,C,n}^{\text{ab}}/\overline{I}_{B_{v,C},n} \hookrightarrow K_{v,s,b}^{\text{ab}}/\overline{I}_{B_v,n} \cong (K_{v,s,b}/I_{B_v,n})^{\text{ab}}.$$

Write $Y_{v,n}^\bullet$ for the Galois admissible covering of $X_{v,s}^\bullet$ with Galois group $(\Pi_{X_{v,s}^\bullet}/I_{B_v})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z}$, $\Gamma_{Y_{v,n}^\bullet}$ for the dual semi-graph of $Y_{v,n}^\bullet$, and $r_{Y_{v,n}}$ for the Betti number of $\Gamma_{Y_{v,n}^\bullet}$. Thus, we obtain

$$\gamma_p^{\text{av}}(\Pi_{X_{v,s}^\bullet}/I_{B_v}) = \lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}((K_{v,s,b}/I_{B_v,n})^{\text{ab}} \otimes \mathbb{F}_p)}{\#((\Pi_{X_{v,s}^\bullet}/I_{B_v})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}$$
$$= \lim_{n\to\infty} \frac{r_{Y_{v,n}}}{\#((\Pi_{X_{v,s}^\bullet}/I_{B_v})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} + \lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}((K_{Z_v,n}/I_{B_{Z_v},n})^{\text{ab}} \otimes \mathbb{F}_p)}{\#((\Pi_{Z_v^\bullet}/I_{B_{Z_v}})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}$$
$$+ \sum_{C \in E_v^{>1}} \lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}((K_{v,C,n}/I_{B_{v,C},n})^{\text{ab}} \otimes \mathbb{F}_p)}{\#((\Pi_{P_C^\bullet}/I_{B_{v,C}})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})}$$
$$= \lim_{n\to\infty} \frac{r_{Y_{v,n}}}{\#((\Pi_{X_{v,s}^\bullet}/I_{B_v})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} + \gamma_p^{\text{av}}(\Pi_{Z_v^\bullet}/I_{B_{Z_v}}) + \sum_{C \in E_v^{>1}} \gamma_p^{\text{av}}(\Pi_{P_C^\bullet}/I_{B_{v,C}}).$$

Note that $Y_{v,n}^\bullet \to X_{v,s}^\bullet$ is étale over S_v (4.3.3). Then we have

$$\lim_{n\to\infty} \frac{\#(e^{\text{cl}}(\Gamma_{Y_{v,n}^\bullet}))}{\#((\Pi_{X_{v,s}^\bullet}/I_{B_v})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} = \#(e^{\text{cl}}(\Gamma_{X_{v,s}^\bullet})) = \#(E_v^{>1}).$$

Suppose $g_v = 0$ and $\#(E_v^{>1}) \geq 3$. We have that $\Pi_{Z_v^\bullet}/I_{B_v}$ is trivial, and that $\Pi_{P_C^\bullet}/I_{B_{v,C}}$ is non-trivial. Then we obtain

$$\lim_{n\to\infty} \frac{\#(v(\Gamma_{Y_{v,n}^\bullet}))}{\#((\Pi_{X_{v,s}^\bullet}/I_{B_v})^{\text{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} = 1,\ \gamma_p^{\text{av}}(\Pi_{Z_v^\bullet}/I_{B_v}) = 0.$$

On the other hand, $\Pi_{P_C^\bullet}/I_{B_{v,C}}$ is naturally isomorphic to the admissible fundamental group (=tame fundamental group since P_C is non-singular) of $(P_C, D_{X_{v,s}} \cap P_C)$. Then we have $\gamma_p^{\mathrm{av}}(\Pi_{P_C^\bullet}/I_{B_{v,C}}) = 0$. Thus, we obtain

$$\gamma_p^{\mathrm{av}}(\Pi_{X_{v,s}^\bullet}/I_{B_v}) = \#(E_v^{>1}) - 1.$$

Suppose that either $g_v \geq 1$ or $g_v = 0$ and $\#(E_v^{>1}) = 2$ hold. Then $\Pi_{Z_v^\bullet}/I_{B_v}$ and $\Pi_{P_C^\bullet}/I_{B_{v,C}}$ are non-trivial. This means

$$\lim_{n\to\infty} \frac{\#(v(\Gamma_{Y_{v,n}^\bullet}))}{\#((\Pi_{X_{v,s}^\bullet}/I_{B_v})^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})} = 0.$$

On the other hand, since $\Pi_{Z_v^\bullet}/I_{B_{Z_v}}$ is naturally isomorphic to the étale fundamental group of Z_v and $\Pi_{P_C^\bullet}/I_{B_{v,C}}$ is naturally isomorphic to the admissible fundamental group (=tame fundamental group since P_C is non-singular) of $(P_C, D_{X_{v,s}} \cap P_C)$, Proposition 3.2 and Proposition 3.5 imply

$$\gamma_p^{\mathrm{av}}(\Pi_{Z_v^\bullet}/I_{B_v}) = \begin{cases} 0, & \text{if } g_v = 0, \\ g_v - 1, & \text{if } g_v \geq 1, \end{cases}$$

$$\gamma_p^{\mathrm{av}}(\Pi_{P_C^\bullet}/I_{B_{v,C}}) = 0.$$

Then we obtain

$$\gamma_p^{\mathrm{av}}(\Pi_{X_{v,s}^\bullet}/I_{B_v}) = g_v + \#(E_v^{>1}) - 1.$$

This completes the proof of the lemma. □

4.3.4. We have the following result.

Proposition 4.5. *We maintain the settings introduced in 4.3.1 and maintain the notation introduced in Proposition 4.3. Let $v \in v(\Gamma_{X^\bullet})$. Then we have (see 4.2.2 for $H_{v,n}$)*

$$\lim_{n\to\infty} \frac{\sigma_{X_{H_{v,n}}}}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} =$$

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}(H_{v,n}^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} = \begin{cases} 0, & \textit{if } (g_v, \#(E_v^{>1})) = (0,0), \\ g_v + \#(E_v^{>1}) - 1, & \textit{if } (g_v, \#(E_v^{>1})) \neq (0,0). \end{cases}$$

Proof. If $(g_v, \#(E_v^{>1})) = (0,0)$, then the proposition follows from Proposition 4.3 (a). To verify the proposition, we may assume $(g_v, \#(E_v^{>1})) \neq (0,0)$.

Since we assume that $X^\bullet$ is a component-generic pointed stable curve, for each $v \in v(\Gamma_{X^\bullet})$, there exist a discrete valuation ring R_v of equal characteristic with algebraically closed residue field k_{R_v} and a pointed stable curve $\mathcal{X}_v^\bullet$ of type (g_v, n_v) over R_v satisfying the following conditions:

- ◇ Write $\eta_v \overset{\mathrm{def}}{=} \operatorname{Spec} K_{R_v}$ and $s_v \overset{\mathrm{def}}{=} \operatorname{Spec} k_{R_v}$ for the generic point and the closed point of $\operatorname{Spec} R_v$, respectively, where K_{R_v} denotes the quotient field of R_v. Then we have
  - (i) There exists an algebraically closed field k'_v containing K_{R_v} and k such that $\mathcal{X}_v^\bullet \times_{R_v} k'_v$ is k'_v-isomorphic to $\widetilde{X}_v^\bullet \times_k k'_v$.
  - (ii) The special fiber $\mathcal{X}_{v,s}^\bullet \overset{\mathrm{def}}{=} \mathcal{X}_v^\bullet \times_{R_v} k_{R_v}$ satisfying the conditions defined in 4.3.2.

We write $\overline{K}_{R_v}$ for the algebraic closure of K_{R_v} in k'_v and put $\mathcal{X}^\bullet_{\overline{\eta}_v} \stackrel{\mathrm{def}}{=} \mathcal{X}^\bullet_v \times_{R_v} \overline{K}_{R_v}$. Then we obtain the following specialization surjective homomorphism of admissible fundamental groups (which is not an isomorphism)

$$sp_{R_v} : \Pi_{\widetilde{X}^\bullet_v} \cong \Pi_{\mathcal{X}^\bullet_{\overline{\eta}_v}} \twoheadrightarrow \Pi_{\mathcal{X}^\bullet_{v,s}}.$$

Moreover, sp_{R_v} induces an isomorphism of maximal prime-to-p quotients

$$sp^{p'}_{R_v} : \Pi^{p'}_{\widetilde{X}^\bullet_v} \cong \Pi^{p'}_{\mathcal{X}^\bullet_{\overline{\eta}_v}} \twoheadrightarrow \Pi^{p'}_{\mathcal{X}^\bullet_{v,s}}.$$

On the other hand, let $H \subseteq \Pi_{\widetilde{X}^\bullet_v}$ be an arbitrary open normal subgroup such that $\#(\Pi_{\widetilde{X}^\bullet_v}/H)$ is prime to p, and let $H_s \stackrel{\mathrm{def}}{=} sp_{R_v}(H) \subseteq \Pi_{\mathcal{X}^\bullet_{v,s}}$. Write $f^\bullet_{H_s} : \mathcal{X}^\bullet_{H_s} \to \mathcal{X}^\bullet_{v,s}$ for the Galois admissible covering corresponding to H_s. Write $D_{E^{=1}_v} \subseteq D_{\mathcal{X}^\bullet_v}$ for the subset of the marked points of $\mathcal{X}^\bullet_v$ such that $\{x_e \times_k k'_v\}_{e \in E^{=1}_v} \subseteq D_{\widetilde{X}^\bullet_v} \times_k k'_v$ is equal to $D_{E^{=1}_v} \times_{R_v} k'_v$ via the isomorphism $\mathcal{X}^\bullet_v \times_{R_v} k'_v \cong \widetilde{X}^\bullet_v \times_k k'_v$. Since $\#(\Pi_{\widetilde{X}^\bullet_v}/H) = \#(\Pi_{\mathcal{X}^\bullet_{v,s}}/H_s)$ is prime to p, the isomorphism $sp^{p'}_{R_v}$ and [Y3, Proposition 3.4 (ii)] imply that H contains $H_{v,n}$ if and only if $f^\bullet_{H_s}$ is étale over $D_{E^{=1}_v} \times_{R_v} k_{R_v}$ and $\mathcal{X}^{\mathrm{sing}}_{v,s}$. This means that H contains $H_{v,n}$ if and only if H_s contains I_{B_v} (see 4.3.3 for I_{B_v}). Then the surjection sp_{R_v} implies

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p}(H^{\mathrm{ab}}_{v,n} \otimes \mathbb{F}_p)}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} \geq \gamma^{\mathrm{av}}_p(\Pi_{\mathcal{X}^\bullet_{v,s}}/I_{B_v}).$$

Thus, the proposition follows immediately from Proposition 4.3 (ii) and Lemma 4.4. We complete the proof of the proposition. □

4.4. **Admissible fundamental group case.** In this subsection, we generalize Proposition 3.5 to the case of arbitrary component-generic pointed stable curves.

4.4.1. The main result of the present paper is as follows.

Theorem 4.6. *Let $X^\bullet$ be a component-generic pointed stable curve (2.2.3) of type (g_X, n_X) over an algebraically closed field of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph, r_X the Betti number of $\Gamma_{X^\bullet}$, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Then we have the following formula (see Definition 2.1 for $\gamma^{\mathrm{av}}_p(\Pi_{X^\bullet})$, 2.1 for $v(\Gamma_{X^\bullet})$, Definition 4.2 for $E^{\mathrm{tre}}_{X^\bullet}$, and Definition 4.1 for $E^{>1}_v$):*

$$\gamma^{\mathrm{av}}_p(\Pi_{X^\bullet}) = g_X - r_X - \#(v(\Gamma_{X^\bullet})) + \#(E^{\mathrm{tre}}_{X^\bullet}) + \sum_{v \in v(\Gamma_{X^\bullet})} \#(E^{>1}_v).$$

Proof. Let n be an arbitrary number prime to p, K_n the kernel of $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{ab}}_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}$, and $X^\bullet_{K_n}$ the Galois admissible covering of $X^\bullet$ corresponding to $K_n \subseteq \Pi_{X^\bullet}$. Then we have

$$\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_n \otimes \mathbb{F}_p) = r_{X_{K_n}} + \sum_{v \in v(\Gamma_{X^\bullet})} \frac{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} \cdot \dim_{\mathbb{F}_p}(H^{\mathrm{ab}}_{v,n} \otimes \mathbb{F}_p),$$

where $H_{v,n}$ is the profinite group defined in 4.2.2, M_v is the profinite group defined in 4.2.1, and $r_{X_{K_n}}$ denotes the Betti number of the dual semi-graph of $X^\bullet_{K_n}$.

Let $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ be a closed edge and $\widehat{e} \in \pi^{-1}_X(e) \subseteq e^{\mathrm{cl}}(\widehat{\Gamma}_{X^\bullet})$ (2.3.3). We put

$$I_{e,n} \stackrel{\mathrm{def}}{=} \mathrm{Im}(I_{\widehat{e}} \hookrightarrow \Pi_{X^\bullet} \twoheadrightarrow \Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z}).$$

Note that $I_{e,n}$ depends only on $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. Then we have

$$r_{X_{K_n}} = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{K_n}}) - \#v(\Gamma_{X^\bullet_{K_n}}) + 1$$

$$= \sum_{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})} \frac{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})}{\#(I_{e,n})} - \sum_{v \in v(\Gamma_{X^\bullet})} \frac{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} + 1.$$

Moreover, we see immediately

$$\#I_{e,n} = \begin{cases} 1, & \text{if } e \in E^{\mathrm{tre}}_{X^\bullet}, \\ n, & \text{otherwise.} \end{cases}$$

On the other hand, [Y3, Proposition 3.4 (ii)] implies that $M_v \otimes \mathbb{Z}/n\mathbb{Z}$ is trivial if and only if $(g_v, \#(E^{>1}_v)) = (0,0)$ (or equivalently, $v \in V^{\mathrm{tre}, g_v=0}_{X^\bullet}$ (Definition 4.2)). Then we obtain

$$\#(M_v \otimes \mathbb{Z}/n\mathbb{Z}) = 1, \ v \in V^{\mathrm{tre}, g_v=0}_{X^\bullet}.$$

Then we obtain

$$\gamma^{\mathrm{av}}_{p,n}(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_n \otimes \mathbb{F}_p)}{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})} = \sum_{v \in v(\Gamma_{X^\bullet})} \frac{\dim_{\mathbb{F}_p}(H^{\mathrm{ab}}_{v,n} \otimes \mathbb{F}_p)}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})}$$

$$+\#(E^{\mathrm{tre}}_{X^\bullet}) + \sum_{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \setminus \bigcup_{v \in v(\Gamma_{X^\bullet})} E^{=1}_v} \frac{1}{n}$$

$$- \sum_{v \in v(\Gamma_{X^\bullet}) \setminus V^{\mathrm{tre}, g_v=0}_{X^\bullet}} \frac{1}{\#(M_v \otimes \mathbb{Z}/n\mathbb{Z})} - \#(V^{\mathrm{tre}, g_v=0}_{X^\bullet}) + \frac{1}{\#(\Pi^{\mathrm{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})}.$$

Thus, by applying Proposition 4.5, we obtain

$$\gamma^{\mathrm{av}}_p(\Pi_{X^\bullet}) \overset{\mathrm{def}}{=} \lim_{n \to \infty} \gamma^{\mathrm{av}}_{p,n}(\Pi_{X^\bullet}) = \sum_{v \in v(\Gamma_{X^\bullet}) \text{ s.t. } (g_v, \#(E^{>1}_v)) \neq (0,0)} (g_v + \#(E^{>1}_v) - 1) + \#(E^{\mathrm{tre}}_{X^\bullet}) - \#(V^{\mathrm{tre}, g_v=0}_{X^\bullet})$$

$$= \sum_{v \in v(\Gamma_{X^\bullet})} g_v + \sum_{v \in v(\Gamma_{X^\bullet})} \#(E^{>1}_v) - \#(v(\Gamma_{X^\bullet})) + \#(V^{\mathrm{tre}, g_v=0}_{X^\bullet}) + \#(E^{\mathrm{tre}}_{X^\bullet}) - \#(V^{\mathrm{tre}, g_v=0}_{X^\bullet}).$$

$$= g_X - r_X - \#(v(\Gamma_{X^\bullet})) + \#(E^{\mathrm{tre}}_{X^\bullet}) + \sum_{v \in v(\Gamma_{X^\bullet})} \#(E^{>1}_v).$$

This completes the proof of the theorem. □

Remark 4.6.1. We maintain the settings of Theorem 4.6. Suppose that $X^\bullet$ is smooth over k. It is easy to check that the formula of Theorem 4.6 coincides with the formula of Proposition 3.5.

Remark 4.6.2. In this remark, we take the opportunity to correct an unfortunate error in [Y3, Theorem 5.2 and Theorem 6.6]. Since $\#(M_v \otimes \mathbb{Z}/n\mathbb{Z}) = 1$, $v \in V^{\mathrm{tre}, g_v=0}_{X^\bullet}$, the correct forms of [Y3, Theorem 5.2 and Theorem 6.6] are as follows:

[Y3, Theorem 5.2]. *Let* $n \overset{\text{def}}{=} p^t - 1$*, and let* $X^\bullet$ *be an arbitrary pointed stable curve over an algebraically closed field of characteristic* $p > 0$ *of type* (g_X, n_X)*. Then we have*

$$g_X - r_X - \#(V^{\text{tre}}_{X^\bullet}) + \#(E^{\text{tre}}_{X^\bullet}) - \sum_{v \in v(\Gamma_{X^\bullet}) \text{ s.t. } \#(E_v^{>1})>1} g_v$$

$$\leq \limsup_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi^{\text{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})} \leq g_X - r_X - \#(v(\Gamma_{X^\bullet})) + \#(E^{\text{tre}}_{X^\bullet}) + \sum_{v\in v(\Gamma_{X^\bullet})} \#(E_v^{>1}).$$

In particular, if $\#(E_v^{>1}) \leq 1$ *for each* $v \in v(\Gamma_{X^\bullet})$*, then we have*

$$\begin{aligned}\text{Avr}_p(\Pi_{X^\bullet}) &= g_X - r_X - \#(V^{\text{tre}}_{X^\bullet}) + \#E^{\text{tre}}_{X^\bullet} - \sum_{v \in v(\Gamma_{X^\bullet}) \text{ s.t. } \#(E_v^{>1})>1} g_v \\ &= g_X - r_X - \#(v(\Gamma_{X^\bullet})) + \#(E^{\text{tre}}_{X^\bullet}) + \sum_{v\in v(\Gamma_{X^\bullet})} \#(E_v^{>1}) \\ &= g_X - r_X - \#(V^{\text{tre}}_{X^\bullet}) + \#(E^{\text{tre}}_{X^\bullet}).\end{aligned}$$

[Y3, Theorem 6.6]. *Let* $n \overset{\text{def}}{=} p^t - 1$*, and let* $X^\bullet$ *be an arbitrary component-generic pointed stable curve over an algebraically closed field of characteristic* $p > 0$ *of type* (g_X, n_X)*. Then we have*

$$\text{Avr}_p(\Pi_{X^\bullet}) \overset{\text{def}}{=} \lim_{t\to\infty} \frac{\dim_{\mathbb{F}_p}(K_n^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\Pi^{\text{ab}}_{X^\bullet} \otimes \mathbb{Z}/n\mathbb{Z})} = g_X - r_X - \#(v(\Gamma_{X^\bullet})) + \#(E^{\text{tre}}_{X^\bullet}) + \sum_{v\in v(\Gamma_{X^\bullet})} \#(E_v^{>1}).$$

On the other hand, the applications of [Y3, Theorem 5.2 and Theorem 6.6] (e.g. [Y5], [Y6]) still hold since we only use the formulas when $\Gamma^{\text{cpt}}_{X^\bullet}$ is 2-connected.

Remark 4.6.3. Since we assume $n \overset{\text{def}}{=} p^t - 1$ in [Y3, Theorem 6.6], Theorem 4.6 is a generalization of [Y3, Theorem 6.6].

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