p-GROUPS, p-RANK, AND SEMI-STABLE REDUCTION OF COVERINGS OF CURVES

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ABSTRACT. In the present paper, we prove various explicit formulas concerning p-rank of p-coverings of pointed semi-stable curves over discrete valuation rings. In particular, we obtain a full generalization of Raynaud's formula for p-rank of fibers over *non-marked smooth* closed points in the case of *arbitrary* closed points. As an application, for abelian p-coverings, we give an affirmative answer to an open problem concerning boundedness of p-rank asked by Saïdi more than twenty years ago.

Keywords: p-rank, semi-stable reduction, pointed semi-stable curve, pointed semi-stable covering.

Mathematics Subject Classification: Primary 14E20, 14G17; Secondary 14G20, 14H30.

CONTENTS

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Introduction

Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$ and $S \stackrel{\text{def}}{=} \operatorname{Spec} R$. Write K for the quotient field of R, $\eta : \operatorname{Spec} K \to S$ for the generic point of S, and $s : \operatorname{Spec} k \to S$ for the closed point of S. Let $\mathscr{X} = (X, D_X)$ be a pointed semi-stable curve of genus g_X over S. Here, X denotes the underlying semi-stable curve of $\mathscr{X}$, and D_X denotes the finite (ordered) set of marked points of $\mathscr{X}$. Write $\mathscr{X}_\eta = (X_\eta, D_{X_\eta})$ and $\mathscr{X}_s = (X_s, D_{X_s})$ for the generic fiber and the special fiber of $\mathscr{X}$, respectively. Moreover, we suppose that $\mathscr{X}_\eta$ is a smooth pointed stable curve over η (i.e. D_X satisfies [K, Definition 1.1 (iv)]).

0.1. Raynaud's formula for p-rank of non-finite fibers.

0.1.1. Let G be a finite group, and let $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ be a smooth pointed stable curve over η and $f_\eta : \mathscr{Y}_\eta \to \mathscr{X}_\eta$ a morphism of pointed stable curves over η. Suppose that f_η is a Galois covering whose Galois group is isomorphic to G, that $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$, and that the branch locus of f_η is contained in D_{X_η}. By replacing S by a finite extension of S (i.e. the spectrum of the normalization of R in a finite extension of K), f_η extends to a *G-pointed semi-stable covering*

$$f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$$

over S (see Definition 1.5 and Proposition 1.6). We write $\mathscr{Y}_s = (Y_s, D_{Y_s})$ for the special fiber of $\mathscr{Y}$ and $f_s : \mathscr{Y}_s \to \mathscr{X}_s$ for the morphism of pointed semi-stable curves over s induced by f.

Suppose that the order of G is prime to p. Then f_s is a finite, generically étale morphism ([SGA1], [V]). On the other hand, suppose that $p | \#G$. Then the situation is quite different from that in the case of prime-to-p coverings. The geometry of $\mathscr{Y}_s$ is very complicated and the morphism f_s is not generically étale, and moreover, is *not finite* in general. This kind of phenomenon is called "resolution of non-singularities" ([T2]) which has many important applications in the theory of arithmetic fundamental groups and anabelian geometry (e.g. [M1], [Le], [PoSt], [St]).

0.1.2. In [R], M. Raynaud investigated the geometry of reduction of étale p-group schemes over $\mathscr{X}_\eta$ (i.e. G is a p-group), and proved an explicit formula for the p-rank (see 1.2.3 for the definition of p-rank) of non-finite fibers of f_s. More precisely, we have the following famous result which is the main theorem of [R]:

Theorem 0.1. ([R, Théorème 1, Théorème 2]) *Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and x a closed point of $\mathscr{X}_s$. Suppose that x is a* **non-marked smooth** *point (i.e. $x \notin X_s^{\text{sing}} \cup D_{X_s}$, where X_s^{sing} denote the singular locus of X_s) of $\mathscr{X}_s$. Then we have the following formula for the p-rank of $f^{-1}(x)$:*

$$\sigma(f^{-1}(x)) = 0.$$

In particular, suppose that $\mathscr{X}$ is a smooth pointed stable curve (i.e. X is stable and $D_X = \emptyset$) over S. As a direct consequence of the above formula, the following statements hold: (i) The Jacobian of $\mathscr{Y}_\eta$ has potentially good reduction. (ii) The dual semi-graph (1.2.2) of $\mathscr{Y}_s$ is a tree (1.1.3). (iii) The slopes of the crystalline cohomology of connected components of vertical fibers of f are in $(0, 1)$.

Remark 0.1.1. If x is *not* a non-marked smooth point of $\mathscr{X}_s$, $\sigma(f^{-1}(x))$ is not equal to 0 in general. For instance, if x is a singular point of $\mathscr{X}_s$, the dual semi-graph of $f^{-1}(x)$ is no longer to be a tree even the simplest case where $G = \mathbf{Z}/p\mathbf{Z}$.

On the other hand, if G is not a p-group, the p-rank of irreducible components of $\mathscr{Y}_s$ cannot be calculated explicitly in general (see Remark 1.4.1).

0.2. **Main result.** We maintain the notation introduced in 0.1. In the present paper, we give a full generalization of Raynaud's formula. Namely, we will prove various formulas for $\sigma(f^{-1}(x))$ where x is an *arbitrary* closed point of $\mathscr{X}_s$. Note that if $f^{-1}(x)$ is finite, then $\sigma(f^{-1}(x)) = 0$ by the definition of p-rank. Moreover, since f is a Galois covering, to calculate $\sigma(f^{-1}(x)) = 0$, we only need to calculate the p-rank of a connected component of $f^{-1}(x)$. Thus, to calculate $\sigma(f^{-1}(x))$, we may assume that $f^{-1}(x)$ is *non-finite* and *connected.*

0.2.1. Our main result is the following formulas for $\sigma(f^{-1}(x))$ in terms of the orders of inertia subgroups of irreducible components of $f^{-1}(x)$ which depend only on the action of G on $f^{-1}(x)$ (in the introduction, we do not give the list of definitions of the notation appeared in the main theorem, see Theorem 3.4 and Theorem 3.9 for more precise forms):

Theorem 0.2. *Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and x an* **arbitrary** *closed point of $\mathscr{X}_s$. Suppose that $f^{-1}(x)$ is non-finite and connected. Then we have (see 3.2.3 for $\Gamma_{\mathscr{E}_X}$, 3.1.5 for $\#I_v$, $\#I_e$, and 1.1.1 for $v(\Gamma_{\mathscr{E}_X})$, $e(v)$, $e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})$)*

$$\sigma(f^{-1}(x)) = \sum_{v\in v(\Gamma_{\mathscr{E}_X})} \Big(1 - \#G/\#I_v + \sum_{e\in e(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1)\Big) + \sum_{e\in e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})} (\#G/\#I_e - 1).$$

Moreover, suppose that x is a **singular** *point of $\mathscr{X}_s$. Then we have a more simple form as follows:*

$$\sigma(f^{-1}(x)) = \sum_{\#I\in\mathcal{I}(x)} \#G/\#I - \sum_{\#J\in\mathcal{J}(x)} \#G/\#J + 1,$$

where $\mathcal{I}(x)$ and $\mathcal{J}(x)$ are the sets of minimal and maximal orders of inertia subgroups associated to x and f (see Definition 3.5 (b)), respectively.

0.2.2. If x is a non-marked smooth closed point of $\mathscr{X}_s$, Raynaud's formula (i.e. Theorem 0.1) can be deduced by the "non-moreover" part of Theorem 0.2 (see 3.2.7). If x is a singular closed point of $\mathscr{X}_s$, the p-rank $\sigma(f^{-1}(x))$ had been studied by M. Saïdi under the assumption where G is a *cyclic* p-group ([S1], [S2]), and his result can be deduced by the "moreover" part of Theorem 0.2 (see Corollary 3.11). Moreover, as an application, in Section 4 of the present paper, by applying the "moreover" part of Theorem 0.2, we give an affirmative answer to an open problem posed by Saïdi (4.0.1) when G is an abelian p-group (see Theorem 4.3).

On the other hand, our approach to proving the formulas for $\sigma(f^{-1}(x))$ is *completely different* from that of Raynaud and Saïdi (Saïdi's method is close to the method of Raynaud), and we calculate $\sigma(f^{-1}(x))$ by introducing a kind of new object which we call *semi-graphs with p-rank* (Section 2). Moreover, our method can be used not only for calculating the p-rank of a fiber $f^{-1}(x)$ of a closed point x, but also for *calculating the p-rank $\sigma(\mathscr{Y}_s)$ of the special fiber $\mathscr{Y}_s$* of $\mathscr{Y}$ (see Theorem 3.2 for a formula for $\sigma(\mathscr{Y}_s)$).

0.3. **Strategy of proof.** We briefly explain the method of proving Theorem 0.2.

0.3.1. We maintain the notation introduced in 0.2. To calculate the p-rank $\sigma(f^{-1}(x))$ of $f^{-1}(x)$, we need to calculate (i) the p-rank of the normalizations of irreducible components of $f^{-1}(x)$, and (ii) the Betti number γ_x (1.1.3) of the dual semi-graph Γ_x (1.2.2) of $f^{-1}(x)$. By using the general theory of semi-stable curves, (i) can be obtained by using the Deuring-Shafarevich formula (Proposition 1.4).

The major difficulty is (ii). In the cases treated by Raynaud and Saïdi, the geometry of the fiber $f^{-1}(x)$ is well-managed (in fact, Γ_x is a tree when x is a non-marked smooth point). On the other hand, in the general case (i.e. x is an arbitrary closed point and G is an arbitrary p-group), the geometry of $f^{-1}(x)$ is very complicated, and its dual semi-graph is *far from being tree-like.*

0.3.2. The author of the present paper observed that we can "avoid" to compute directly the Betti number γ_x of Γ_x if $f^{-1}(x)$ admits a good "deformation" such that the decomposition groups of irreducible components of the deformation are G, and that $\sigma(f^{-1}(x))$ is equal to the p-rank of the deformation. However, in general, such deformations *do not exist* in the theory of algebraic geometry (i.e. we cannot find such deformations in moduli spaces of curves, see Remark 2.4.1).

To overcome this difficulty, we introduce the so-called *semi-graphs with p-rank* (Section 2), and define p-rank, coverings, and G-coverings for semi-graphs with p-rank. Moreover, we can deform semi-graphs with p-rank in a natural way, and prove that the deformations do not change the p-rank of semi-graphs with p-rank (Proposition 2.6). Then we may obtain an explicit formula for the p-rank of G-coverings of semi-graphs with p-rank (Theorem 2.7). Furthermore, by using the theory of semi-stable curves, we construct semi-graphs with p-rank (Section 3) from G-pointed semi-stable coverings (in particular, we construct a semi-graph with p-rank from $f^{-1}(x)$). Together with some precise analyzations of inertia groups (Section 1) of singular points and irreducible components of G-pointed semi-stable coverings, we obtain Theorem 0.2.

0.4. **Structure of the present paper.** The present paper is organized as follows. In Section 1, we introduce some notation concerning semi-graphs, pointed semi-stable curves, and pointed semi-stable coverings. Moreover, we prove some results concerning inertia subgroups of singular points and irreducible components of pointed semi-stable coverings. In Section 2, we introduce semi-graphs with p-rank, and study the p-rank of G-coverings of semi-graphs with p-rank. In Section 3, we construct various G-coverings of semi-graphs with p-rank from G-pointed semi-stable coverings. Moreover, by applying the results obtained in Section 2, we obtain various formulas for p-rank concerning G-pointed semi-stable coverings. In Section 4, we study bounds of p-rank of vertical fibers of G-pointed semi-stable coverings by using formulas obtained in Section 3.

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1. Pointed semi-stable coverings

In this section, we introduce pointed semi-stable coverings of pointed semi-stable curves over discrete valuation rings.

1.1. **Semi-graphs.** We begin with some general remarks concerning semi-graphs (see also [M2, Section 1]).

1.1.1. A *semi-graph* $\mathbb{G}$ consists of the following data:

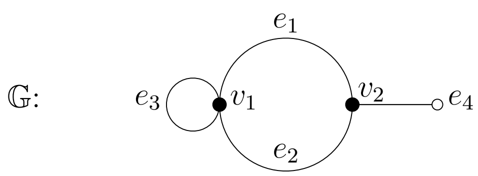
(i) A set $v(\mathbb{G})$ whose elements we refer to as vertices.
(ii) A set $e(\mathbb{G})$ whose elements we refer to as edges. Moreover, any element $e \in e(\mathbb{G})$ is a set of cardinality 2 satisfying the following property: for each $e \neq e' \in e(\mathbb{G})$, we have $e \cap e' = \emptyset$.
(iii) A set of maps $\{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G})}$ such that $\zeta_e^{\mathbb{G}} : e \to v(\mathbb{G}) \cup \{v(\mathbb{G})\}$ is a map from the set e to the set $v(\mathbb{G}) \cup \{v(\mathbb{G})\}$, and that $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) \in \{0, 1\}$, where $\#(-)$ denotes the cardinality of $(-)$.

Let $e \in e(\mathbb{G})$ be an edge of $\mathbb{G}$. We shall refer to an element $b \in e$ as a *branch* of the edge e. We shall call that $e \in e(\mathbb{G})$ is *closed* (resp. *open*) if $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) = 0$ (resp. $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) = 1$). Moreover, write $e^{\mathrm{cl}}(\mathbb{G})$ for the set of closed edges of $\mathbb{G}$ and $e^{\mathrm{op}}(\mathbb{G})$ for the set of open edges of $\mathbb{G}$. Note that we have $e(\mathbb{G}) = e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G})$.

Let $v \in v(\mathbb{G})$ be a vertex of $\mathbb{G}$. Write $b(v)$ for the set of branches $\bigcup_{e \in e(\mathbb{G})} (\zeta_e^{\mathbb{G}})^{-1}(v)$, $e(v)$ for the set of edges which abut to v, and $v(e)$ for the set of vertices which are abutted by e. Note that we have $\#(v(e)) \leq 2$. We shall call a closed edge $e \in e^{\mathrm{cl}}(\mathbb{G})$ *loop* if $\#v(e) = 1$ (i.e. $\#(\zeta_e^{\mathbb{G}}(e)) = 1$). Moreover, we use the notation $e^{\mathrm{lp}}(v)$ to denote the set of loops which abut to v.

Example 1.1. Let us give an example of semi-graph to explain the above definitions. We use the notation "•" and "◦ with a line segment" to denote a vertex and an open edge, respectively.

Let $\mathbb{G}$ be a semi-graph as follows:



Then we have $v(\mathbb{G}) = \{v_1, v_2\}$, $e(\mathbb{G}) = \{e_1, e_2, e_3, e_4\}$, $e^{\mathrm{cl}}(\mathbb{G}) = \{e_1, e_2, e_3\}$, $e^{\mathrm{op}}(\mathbb{G}) = \{e_4\}$, $\zeta_{e_1}^{\mathbb{G}}(e_1) = \zeta_{e_2}^{\mathbb{G}}(e_2) = \{v_1, v_2\}$, $\zeta_{e_3}^{\mathbb{G}}(e_3) = \{v_1\}$, and $\zeta_{e_4}^{\mathbb{G}}(e_4) = \{v_2, \{v(\mathbb{G})\}\}$. Moreover, we have $e^{\mathrm{lp}}(\mathbb{G}) = e^{\mathrm{lp}}(v_1) = \{e_3\}$, $v(e_1) = v(e_2) = \{v_1, v_2\}$, $v(e_3) = \{v_1\}$, $v(e_4) = \{v_2\}$, $e(v_1) = \{e_1, e_2, e_3\}$, and $e(v_2) = \{e_1, e_2, e_4\}$.

1.1.2. Let $\mathbb{G}$ be a semi-graph. We shall call $\mathbb{G}'$ a *sub-semi-graph* of $\mathbb{G}$ if $\mathbb{G}'$ is a semi-graph satisfying the following conditions:

(i) $v(\mathbb{G}')$ (resp. $e(\mathbb{G}')$) is a subset of $v(\mathbb{G})$ (resp. $e(\mathbb{G})$).
(ii) If $e \in e^{\mathrm{cl}}(\mathbb{G}')$, then $\zeta_e^{\mathbb{G}'}(e) \overset{\mathrm{def}}{=} \zeta_e^{\mathbb{G}}(e)$.
(iii) If $e = \{b_1, b_2\} \in e^{\mathrm{op}}(\mathbb{G}')$ such that $\zeta_e^{\mathbb{G}}(b_1) \in v(\mathbb{G}')$ and $\zeta_e^{\mathbb{G}}(b_2) \notin v(\mathbb{G}')$, then $\zeta_e^{\mathbb{G}'}(b_1) \overset{\mathrm{def}}{=} \zeta_e^{\mathbb{G}}(b_1)$ and $\zeta_e^{\mathbb{G}'}(b_2) \overset{\mathrm{def}}{=} \{v(\mathbb{G}')\}$.

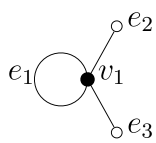
Moreover, we define a semi-graph $\mathbb{G} \setminus \mathbb{G}'$ as follows:

(i) $v(\mathbb{G} \setminus \mathbb{G}') \stackrel{\text{def}}{=} v(\mathbb{G}) \setminus v(\mathbb{G}')$.
(ii) $e^{\text{cl}}(\mathbb{G} \setminus \mathbb{G}') \stackrel{\text{def}}{=} \{e \in e^{\text{cl}}(\mathbb{G}) \mid v(e) \subseteq v(\mathbb{G} \setminus \mathbb{G}') \text{ in } \mathbb{G}\}$.
(iii) $e^{\text{op}}(\mathbb{G} \setminus \mathbb{G}') \stackrel{\text{def}}{=} \{e \in e^{\text{cl}}(\mathbb{G}) \mid v(e) \cap v(\mathbb{G}') \neq \emptyset \text{ in } \mathbb{G} \text{ and } v(e) \cap v(\mathbb{G} \setminus \mathbb{G}') \neq \emptyset \text{ in } \mathbb{G}\} \cup \{e \in e^{\text{op}}(\mathbb{G}) \mid v(e) \cap v(\mathbb{G} \setminus \mathbb{G}') \neq \emptyset \text{ in } \mathbb{G}\}$.
(iv) For each $e = \{b_i\}_{i \in \{1,2\}} \in e^{\text{cl}}(\mathbb{G} \setminus \mathbb{G}') \cup e^{\text{op}}(\mathbb{G} \setminus \mathbb{G}')$, we put

$$\zeta_e^{\mathbb{G} \setminus \mathbb{G}'}(b_i) \stackrel{\text{def}}{=} \begin{cases} \zeta_e^{\mathbb{G}}(b_i), & \text{if } \zeta_e^{\mathbb{G}}(b_i) \notin v(\mathbb{G}') \text{ and } \zeta_e^{\mathbb{G}}(b_i) \neq \{v(\mathbb{G})\}, \\ \{v(\mathbb{G} \setminus \mathbb{G}')\}, & \text{otherwise.} \end{cases}$$

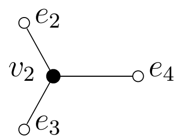
Example 1.2. We give some examples to explain the above definition. Let $\mathbb{G}$ be the semi-graph of Example 1.1 and $\mathbb{G}'$ be a sub-semi-graph as follows:

$\mathbb{G}'$:



Moreover, the semi-graph $\mathbb{G} \setminus \mathbb{G}'$ is the following:

$\mathbb{G} \setminus \mathbb{G}'$:



Remark 1.2.1. We explain the motivation of the constructions of $\mathbb{G}'$ and $\mathbb{G} \setminus \mathbb{G}'$. Let $\mathscr{X} = (X, D_X)$ be a pointed semi-stable curve (1.2.1) over an algebraically closed field such that the dual semi-graph $\Gamma_{\mathscr{X}}$ (1.2.1) is equal to $\mathbb{G}$ defined in Example 1.1. Write X_{v_1} and X_{v_2} for the irreducible components corresponding to v_1 and v_2, respectively. Then we have the following natural pointed semi-stable curves:

$$(X_{v_1}, D_{X_{v_1}} \stackrel{\text{def}}{=} X_{v_1} \cap X_{v_2}),\ (X_{v_2}, D_{X_{v_2}} \stackrel{\text{def}}{=} (X_{v_1} \cap X_{v_2}) \cup D_X)$$

whose dual semi-graphs are equal to $\mathbb{G}'$ and $\mathbb{G} \setminus \mathbb{G}'$ defined in Example 1.2, respectively.

1.1.3. A semi-graph $\mathbb{G}$ will be called *finite* if $v(\mathbb{G})$ and $e(\mathbb{G})$ are finite. In the present paper, *we only consider finite semi-graphs.* Since a semi-graph can be regarded as a topological space (i.e. a subspace of $\mathbf{R}^2$), we shall call $\mathbb{G}$ *connected* if $\mathbb{G}$ is connected as a topological space. Moreover, we write

$$\gamma_{\mathbb{G}} \stackrel{\text{def}}{=} \dim_{\mathbf{C}}(H^1(\mathbb{G}, \mathbf{C}))$$

for the Betti number of $\mathbb{G}$, where $\mathbf{C}$ denotes the field of complex numbers. In particular, we shall call $\mathbb{G}$ a *tree* (or $\mathbb{G}$ *tree-like*) if $\gamma_{\mathbb{G}} = 0$.

Let $\mathbb{G}$ and $\mathbb{H}$ be two semi-graphs. A *morphism* between semi-graphs $\mathbb{G} \to \mathbb{H}$ is a collection of maps $v(\mathbb{G}) \to v(\mathbb{H})$, $e^{\text{cl}}(\mathbb{G}) \to e^{\text{cl}}(\mathbb{H})$, and $e^{\text{op}}(\mathbb{G}) \to e^{\text{op}}(\mathbb{H})$ satisfying the following: for each $e_{\mathbb{G}} \in e(\mathbb{G})$, write $e_{\mathbb{H}} \in e(\mathbb{H})$ for the image of $e_{\mathbb{G}}$; then the map $e_{\mathbb{G}} \stackrel{\sim}{\to} e_{\mathbb{H}}$ is a bijection, and is compatible with the $\{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G})}$ and $\{\zeta_e^{\mathbb{H}}\}_{e \in e(\mathbb{H})}$.

1.2. **Pointed semi-stable curves.**

1.2.1. Let $\mathscr{C} \overset{\text{def}}{=} (C, D_C)$ be a *pointed semi-stable curve* over a scheme A, namely, a marked curve over A such that every geometric fiber $C_{\overline{a}}$, $a \in A$, is a semi-stable curve, and that $D_{C_{\overline{a}}} \subseteq C_{\overline{a}}^{\mathrm{sm}}$, where $C_{\overline{a}}^{\mathrm{sm}}$ denotes the smooth locus of $C_{\overline{a}}$. We shall call C the underlying curve of $\mathscr{C}$ and the finite (ordered) set D_C the set of marked points of $\mathscr{C}$. In particular, we shall call that $\mathscr{C}$ is a *pointed stable curve* if D_C satisfies [K, Definition 1.1 (iv)].

1.2.2. Suppose that A is the spectrum of an algebraically closed field. We write $\mathrm{Irr}(C)$ for the set of the irreducible components of C and C^{sing} for the set of singular points (or nodes) of C. We define the *dual semi-graph* $\Gamma_{\mathscr{C}}$ of the pointed semi-stable curve $\mathscr{C}$ to be the following semi-graph:

(i) $v(\Gamma_{\mathscr{C}}) \overset{\text{def}}{=} \{v_E\}_{E \in \mathrm{Irr}(C)}$.
(ii) $e^{\mathrm{cl}}(\Gamma_{\mathscr{C}}) \overset{\text{def}}{=} \{e_s\}_{s \in C^{\mathrm{sing}}}$ and $e^{\mathrm{op}}(\Gamma_{\mathscr{C}}) \overset{\text{def}}{=} \{e_m\}_{m \in D_C}$.
(iii) For each $e_s = \{b_s^1, b_s^2\} \in e^{\mathrm{cl}}(\Gamma_{\mathscr{C}})$, $s \in C^{\mathrm{sing}}$, we put

$$\zeta_{e_s}^{\Gamma_{\mathscr{C}}}(e_s) \overset{\text{def}}{=} \{v_E \in v(\Gamma_{\mathscr{C}}) \mid s \in E\}.$$

(iv) For each $e_m = \{b_m^1, b_m^2\} \in e^{\mathrm{op}}(\Gamma_{\mathscr{C}})$, $m \in D_C$, we put

$$\zeta_{e_m}^{\Gamma_{\mathscr{C}}}(b_m^1) \overset{\text{def}}{=} v_E,\ \zeta_{e_m}^{\Gamma_{\mathscr{C}}}(b_m^2) \overset{\text{def}}{=} \{v(\Gamma_{\mathscr{C}})\},$$

where E is the irreducible component of C satisfying $m \in E$.

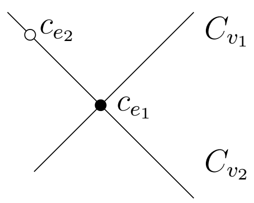
Moreover, we put

$$\gamma_{\mathscr{C}} \overset{\text{def}}{=} \gamma_{\Gamma_{\mathscr{C}}} = \dim_{\mathbf{C}}(H^1(\Gamma_{\mathscr{C}}, \mathbf{C}))\ (1.1.3).$$

Let $v \in v(\Gamma_{\mathscr{C}})$ (resp. $e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{C}})$, $e \in e^{\mathrm{op}}(\Gamma_{\mathscr{C}})$). We write C_v (resp. c_e, c_e) for the irreducible component of C corresponding to v (resp. the singular point of C corresponding to e, the marked point of $\mathscr{C}$ corresponding to e) and $\widetilde{C}_v$ for the normalization of C_v.

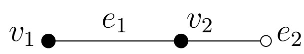
Example 1.3. We give an example to explain dual semi-graphs of pointed semi-stable curves. Let $\mathscr{C} \overset{\text{def}}{=} (C, D_C)$ be a pointed semi-stable curve over k whose irreducible components are C_{v_1} and C_{v_2}, whose node is c_{e_1}, and whose marked point is $c_{e_2} \in C_{v_2}$. We use the notation "•" and "∘" to denote a node and a marked point, respectively. Then $\mathscr{C}$ is as follows:

$\mathscr{C}$:



We write v_1 and v_2 for the vertices of $\Gamma_{\mathscr{C}}$ corresponding to C_{v_1} and C_{v_2}, respectively, e_1 for the closed edge corresponding to c_{e_1}, and e_2 for the open edge corresponding to c_{e_2}. Moreover, we use the notation "•" and "∘ with a line segment" to denote a vertex and an open edge, respectively. Then the dual semi-graph $\Gamma_{\mathscr{C}}$ of $\mathscr{C}$ is as follows:

$\Gamma_{\mathscr{C}}$:



1.2.3. Let C be a disjoint union of projective curves over an algebraically closed field of characteristic $p > 0$. We define the *p-rank* (or *Hasse-Witt invariant*) $\sigma(C)$ of C to be

$$\sigma(C) \stackrel{\text{def}}{=} \dim_{\mathbf{F}_p}(H^1_{\text{ét}}(C, \mathbf{F}_p)).$$

Moreover, let $\mathscr{C} \stackrel{\text{def}}{=} (C, D_C)$ be a pointed semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write $\Gamma_{\mathscr{C}}$ for the dual semi-graph of $\mathscr{C}$. Then we put

$$\sigma(\mathscr{C}) \stackrel{\text{def}}{=} \sigma(C) = \gamma_{\mathscr{C}} + \sum_{v \in v(\Gamma_C)} \sigma(\widetilde{C}_v).$$

1.2.4. Let G be a finite p-group. The p-rank of a Galois covering whose Galois group is isomorphic to G can be calculated by the Deuring-Shafarevich formula (or Crew's formula) as follows:

Proposition 1.4. ([C, Corollary 1.8]) *Let $h : C' \to C$ be a (possibly ramified) Galois covering of smooth projective curves over an algebraically closed field of characteristic $p > 0$ whose Galois group is a finite p-group G. Then we have*

$$\sigma(C') - 1 = \#G(\sigma(C) - 1) + \sum_{c' \in (C')^{\text{cl}}} (e_{c'} - 1),$$

where $(C')^{\text{cl}}$ denotes the set of closed points of C' and $e_{c'}$ denotes the ramification index at c'.

Remark 1.4.1. We maintain the notation introduced in Proposition 1.4. Suppose that G is *not* a p-group. Then $\sigma(C')$ cannot be calculated explicitly in general. In fact, the p-rank (or more precisely, generalized Hasse-Witt invariants) of prime-to-p étale coverings can almost determine the isomorphism class of C (e.g. [T1], [Y1]).

1.3. **Pointed semi-stable coverings.**

1.3.1. **Settings.** We fix some notation of the present subsection. Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$ and K the quotient field. We put $S \stackrel{\text{def}}{=} \operatorname{Spec} R$. Write η and s for the generic point and the closed point corresponding to the natural morphisms $\operatorname{Spec} K \to S$ and $\operatorname{Spec} k \to S$, respectively. Let $\mathscr{X} \stackrel{\text{def}}{=} (X, D_X)$ be a pointed semi-stable curve over S. Write $\mathscr{X}_\eta \stackrel{\text{def}}{=} (X_\eta, D_{X_\eta})$ for the generic fiber of $\mathscr{X}$, $\mathscr{X}_s \stackrel{\text{def}}{=} (X_s, D_{X_s})$ for the special fiber of $\mathscr{X}$, and $\Gamma_{\mathscr{X}_s}$ for the dual semi-graph of $\mathscr{X}_s$. Moreover, we suppose that $\mathscr{X}_\eta$ is a *smooth pointed stable curve* over η (note that $\mathscr{X}_s$ is not a pointed stable curve in general).

1.3.2. Let $l : \mathscr{W} \overset{\text{def}}{=} (W, D_W) \to \mathscr{X}$ be a morphism of pointed semi-stable curves over S and G a finite group. We define pointed semi-stable coverings as follows:

Definition 1.5. The morphism l is called a *pointed semi-stable covering* (resp. *G-pointed semi-stable covering*) over S if the morphism

$$l_\eta : \mathscr{W}_\eta \overset{\text{def}}{=} (W_\eta, D_{W_\eta}) \to \mathscr{X}_\eta = (X_\eta, D_{X_\eta})$$

over η induced by l on generic fibers is a finite generically étale morphism (resp. a Galois covering whose Galois group is isomorphic to G) such that the following conditions hold:

(i) The branch locus of l_η is contained in D_{X_η}.
(ii) $l_\eta^{-1}(D_{X_\eta}) = D_{W_\eta}$.
(iii) The following universal property holds: if $g : \mathscr{W}' \to \mathscr{X}$ is a morphism of pointed semi-stable curves over S such that the generic fiber $\mathscr{W}'_\eta$ of $\mathscr{W}'$ and the morphism $g_\eta : \mathscr{W}'_\eta \to \mathscr{X}_\eta$ induced by g on generic fibers are equal to $\mathscr{W}_\eta$ and l_η, respectively, then there exists a unique morphism $h : \mathscr{W}' \to \mathscr{W}$ such that $g = l \circ h$.

We shall call l a *pointed stable covering* (resp. *G-pointed stable covering*) over S if l is a pointed semi-stable covering (resp. G-pointed semi-stable covering) over S, and $\mathscr{X}$ is a pointed stable curve over S. We shall call l a *semi-stable covering* (resp. *stable covering, G-semi-stable covering, G-stable covering*) over S if l is a pointed semi-stable covering (resp. pointed stable covering, G-pointed semi-stable covering, G-pointed stable covering) over S, and D_X is empty.

1.3.3. We have the following proposition.

Proposition 1.6. *Let $f_\eta : \mathscr{Y}_\eta \overset{\text{def}}{=} (Y_\eta, D_{Y_\eta}) \to \mathscr{X}_\eta$ be a finite morphism of pointed smooth curves over η. Suppose that the branch locus of f_η is contained in D_{X_η} and that $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$. Then, by replacing S by a finite extension of S, f_η extends to a pointed semi-stable covering $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ over S such that the restriction of f to the generic fibers is f_η.*

Proof. The proposition follows from [Liu, Theorem 0.2 and Remark 4.13]. □

Remark 1.6.1. We maintain the notation introduced in Proposition 1.6. In fact, we have that f_η extends *uniquely* to a pointed semi-stable covering f. Let us explain roughly in this remark.

By adding some marked points, we may obtain a pointed stable curve $\mathscr{X}^{\text{add}} \overset{\text{def}}{=} (X^{\text{add}}, D_{X^{\text{add}}})$ whose underlying curve X^{add} is X, and whose set of marked points contains D_X. Write $D_{X_\eta^{\text{add}}}$ for $D_{X^{\text{add}}}|_\eta$, and $D_{Y_\eta^{\text{add}}}$ for $f_\eta^{-1}(D_{X_\eta^{\text{add}}})$. Then $D_{Y_\eta^{\text{add}}}$ contains D_{Y_η}. Moreover, we have a finite morphism of pointed smooth curves

$$f_\eta^{\text{add}} : \mathscr{Y}_\eta^{\text{add}} \to \mathscr{X}_\eta^{\text{add}}$$

over η induced by f_η.

By applying Proposition 1.6 and by replacing S by a finite extension of S, f_η^{add} extends to a pointed semi-stable covering

$$f^{\text{add}} : \mathscr{Y}^{\text{add}} \overset{\text{def}}{=} (Y^{\text{add}}, D_{Y^{\text{add}}}) \to \mathscr{X}^{\text{add}}$$

over S. Since $\mathscr{X}^{\text{add}}$ is a pointed stable curve over S, we see that $\mathscr{Y}^{\text{add}}$ is a pointed stable model of $\mathscr{Y}_\eta^{\text{add}}$. Then the uniqueness of f^{add} follows from the uniqueness of the pointed stable model $\mathscr{Y}^{\text{add}}$.

We put $D_Y^{\mathrm{ss}} \overset{\mathrm{def}}{=} D_Y^{\mathrm{add}} \setminus D_Y$ and $D_{Y_s}^{\mathrm{ss}} \overset{\mathrm{def}}{=} D_Y^{\mathrm{ss}}|_s$. Let $\mathrm{Con}(Y_s^{\mathrm{add}})$ be the subset of the set of irreducible components of Y_s^{add} consisting of all irreducible components E of Y_s^{add} satisfying the following conditions: (i) E is isomorphic to $\mathbb{P}_k^1$; (ii) $E \cap D_{Y_s}^{\mathrm{ss}} \neq \emptyset$ and $E \cap D_Y = \emptyset$; (iii) $f^{\mathrm{add}}(E)$ is a closed point of $\mathscr{X}^{\mathrm{add}}$. Note that $\mathrm{Con}(Y_s^{\mathrm{add}})$ may be an empty set. Then by forgetting the marked points D_Y^{ss} and by contracting the irreducible components of $\mathrm{Con}(Y_s^{\mathrm{add}})$ ([BLR, 6.7 Proposition 4]), we obtain a pointed semi-stable curve $\mathscr{Y}$ and a morphism of pointed semi-stable curves $f : \mathscr{Y} \to \mathscr{X}$ over S induced by f^{add}. We see that f is a pointed semi-stable covering over S, and that f does not depend on the choices of $D_{X^{\mathrm{add}}}$. Moreover, the uniqueness follows from the uniqueness of f^{add}.

1.3.4. If a G-pointed semi-stable covering over S is finite, then it induces a morphism of dual semi-graphs of special fibers. More precisely, we have the following result:

Proposition 1.7. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a* **finite** *G-pointed semi-stable covering over S, and $\Gamma_{\mathscr{Y}_s}$ the dual semi-graph of $\mathscr{Y}_s$. Then the images of nodes (resp. smooth points) of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ are nodes (resp. smooth points) of $\mathscr{X}_s$. In particular, the map of dual semi-graphs $\Gamma_{\mathscr{Y}_s} \to \Gamma_{\mathscr{X}_s}$ induced by the morphism of the special fibers $f_s : \mathscr{Y}_s \to \mathscr{X}_s$ over s induced by f is a morphism of semi-graphs (1.1.3).*

Proof. Let y be a closed point of $\mathscr{Y}$. Write $I_y \subseteq G$ for the inertia subgroup of y. Thus, the natural morphism $\mathscr{Y}/I_y \to \mathscr{X}$ induced by f is étale at the image of y of the quotient morphism $\mathscr{Y} \to \mathscr{Y}/I_y$. Then to verify the proposition, we may assume that $G = I_y$.

If y is a smooth point, then x is a smooth point ([R, Proposition 5]). If y is a node, let Y_1 and Y_2 be the irreducible components (which may be equal) of the underlying curve of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ containing y. Write $D_1 \subseteq G$ and $D_2 \subseteq G$ for the decomposition subgroups of Y_1 and Y_2, respectively. The proof of [R, Proposition 5] implies the following: (i) If D_1 and D_2 are not equal to $I_y = G$, then x is a smooth point. (ii) If $D_1 = D_2 = G$, then x is a node.

Next, we prove that the case (i) will not occur. If D_1 and D_2 are not equal to G, then, for each $\tau \in G \setminus D_1$ (or $\tau \in G \setminus D_2$), we have $\tau(Y_1) = Y_2$ and $\tau(Y_2) = Y_1$. Thus, we obtain $D \overset{\mathrm{def}}{=} D_1 = D_2$. Moreover, D is a normal subgroup of G. By replacing I_y by I_y/D and $\mathscr{Y}$ by $\mathscr{Y}/D$, and by applying the case (ii), we may assume that D is trivial. Then f_s is étale at the generic points of Y_1 and Y_2. Consider the local morphism $f_y : \mathrm{Spec}\, \mathcal{O}_{\mathscr{Y},y} \to \mathrm{Spec}\, \mathcal{O}_{\mathscr{X},f(y)}$ induced by f. Since f_y is étale at all the points of $\mathrm{Spec}\, \mathcal{O}_{\mathscr{Y},y}$ corresponding to the prime ideals of $\mathcal{O}_{\mathscr{Y},y}$ of height 1, the Zariski-Nagata purity theorem implies that f_y is étale. This means that if $f(y)$ is a smooth point, y is a smooth point too. This contradicts our assumption. We complete the proof of the proposition. □

1.3.5. On the other hand, pointed semi-stable coverings are not finite morphisms in general.

Definition 1.8. Let $f : \mathscr{Y} \to \mathscr{X}$ be a pointed semi-stable covering over S. A closed point $x \in \mathscr{X}$ is called a *vertical point associated to f*, or for simplicity, a *vertical point* when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the *vertical fiber associated to f and x*.

Remark 1.8.1. We maintain the notation introduced above. Then the specialization homomorphism of admissible fundamental groups of generic fiber and special fiber of $\mathscr{X}$ is not an isomorphism in general. When $\mathrm{char}(K) = 0$, this result follows from $\sigma(\mathscr{X}_s) \leq g_X$, where g_X denotes the genus of $\mathscr{X}$. On the other hand, when $\mathrm{char}(K) = p > 0$, this result is highly nontrivial ([T1, Theorem 0.3] and [Y3, Theorem 5.2 and Remark 5.2.1]). Then we may ask the following problem:

> By replacing S by a finite extension of S, does there exist a pointed semi-stable covering $f : \mathscr{Y} \to \mathscr{X}$ over S such that the set of vertical points associated to f is not empty?

Suppose $\mathrm{char}(K) = 0$. The problem was solved by A. Tamagawa ([T2, Theorem 0.2]). In fact, Tamagawa proved a very strong result as following:

> Suppose that $\mathrm{char}(K) = 0$, that k is an algebraic closure of a finite field, and that $\mathscr{X}$ is a pointed *stable* curve over S. Let $x \in \mathscr{X}$ be a closed point of $\mathscr{X}$. Then there exists a pointed stable covering $f : \mathscr{Y} \to \mathscr{X}$ over S such that x is a vertical point associated to f.

Moreover, the author generalized this result to the case where k is an arbitrary algebraically closed field ([Y2, Theorem 3.2]). On the other hand, suppose that $\mathrm{char}(K) = p > 0$. The problem was solved by the author when $\mathscr{X}_s$ is irreducible ([Y2, Theorem 0.2]).

1.3.6. For the p-rank of vertical fibers of pointed semi-stable coverings, we have the following famous result proved by Raynaud, which is the main theorem of [R].

Theorem 1.9. ([R, Théorème 2]) *Let G be a finite p-group, $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semi-stable covering over S, and x a vertical point associated to f. If x is a* **non-marked smooth** *point of $\mathscr{X}_s$ (i.e. $x \notin X^{\mathrm{sing}} \cup D_{X_s}$), then we have $\sigma(f^{-1}(x)) = 0$.*

1.3.7. In the remainder of the present paper, we will generalize Theorem 1.9 to the case where x is an *arbitrary* (possibly singular) closed point of $\mathscr{X}$. Namely, we will give an explicit formula for p-rank of vertical fibers associated to arbitrary vertical points of G-pointed semi-stable coverings, where G is a finite p-group.

1.4. **Inertia subgroups and a criterion for vertical fibers.** In this subsection, we study the relationship between the inertia subgroups of nodes and the inertia subgroups of irreducible components of special fibers of G-pointed semi-stable coverings. The main result of the present subsection is Proposition 1.12.

1.4.1. **Settings.** We maintain the settings introduced in 1.3.1.

1.4.2. Firstly, we have the following lemmas.

Lemma 1.10. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a* **finite** *G-pointed semi-stable covering over S, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, and $y \in \mathscr{Y}_s$ a node. Let Y_1 and Y_2 (which may be equal) be the irreducible components of $\mathscr{Y}_s$ containing y. Write $I_y \subseteq G$ (resp. $I_{Y_1} \subseteq G$, $I_{Y_2} \subseteq G$) for the inertia subgroup of y (resp. Y_1, Y_2). Suppose that G is a p-group. Then the inertia subgroup I_y is generated by I_{Y_1} and I_{Y_2}.*

Proof. Write I for the group generated by I_{Y_1} and I_{Y_2}. Then we have $I \subseteq I_y$. Consider the quotient $\mathscr{Y}/I$. We obtain morphisms of pointed semi-stable curves $\mu_1 : \mathscr{Y} \to \mathscr{Y}/I$ and $\mu_2 : \mathscr{Y}/I \to \mathscr{X}$ over S such that $\mu_2 \circ \mu_1 = f$. Note that $\mathscr{Y}/I$ is a pointed semi-stable curve over S ([R, Appendice, Corollaire]), and that $\mu_1(y)$ is a node of the special fiber $(\mathscr{Y}/I)_s$ of $\mathscr{Y}/I$ (Proposition 1.7). Moreover, μ_2 is generically étale at the generic points of $\mu_1(Y_1)$ and $\mu_1(Y_2)$. Then by applying the well-known result concerning the structures of étale fundamental groups of nodes of pointed stable curves (e.g. [T2, Lemma 2.1 (iii)]) to the local morphism $\mathrm{Spec}\, \mathcal{O}_{\mathscr{Y}/I, \mu_1(y)} \to \mathrm{Spec}\, \mathcal{O}_{\mathscr{X}, f(y)}$ induced by μ_2, we obtain that μ_2 is tamely ramified at $\mu_1(y)$. Moreover, since G is a p-group, μ_2 is étale at $\mu_1(y)$. This means $I_y \subseteq I$. Namely, we have $I_y = I$. We complete the proof of the lemma. □

Lemma 1.11. ([T2, Propoisiton 4.3 (ii)]) *Let G be a finite group, $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semi-stable covering over S, and x a node of $\mathscr{X}_s$. Suppose that, for each irreducible component $Z \stackrel{\text{def}}{=} \overline{\{z\}}$ of $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X}_s,x}$ and each point w of the fiber $\mathscr{Y} \times_{\mathscr{X}} z$, the natural morphism from the integral closure W^s of Z in $k(w)^s$ to Z is wildly ramified, where $k(w)^s$ denotes the maximal separable subextension of $k(z)$ in $k(w)$. Then x is a vertical point associated to f (i.e. $f^{-1}(x)$ is not finite).*

Remark 1.11.1. In [T2], Tamagawa only treated the case where f is a stable covering. It is easy to see that Tamagawa's proof also holds for pointed semi-stable coverings.

1.4.3. Next, we prove a criterion for existence of vertical fibers over nodes as follows:

Proposition 1.12. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a G-pointed semi-stable covering over S, $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ the generic fiber of $\mathscr{Y}$ over η, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$ over s, and x a node of $\mathscr{X}_s$. Write $\psi_2 : \mathscr{Y}' \to \mathscr{X}$ for the normalization morphism of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f. We obtain a natural morphism of fiber surfaces $\psi_1 : \mathscr{Y} \to \mathscr{Y}'$ induced by f such that $\psi_2 \circ \psi_1 = f$. Write X_1 and X_2 (which may be equal) for the irreducible components of $\mathscr{X}_s$ containing x. Let $y' \in \psi_2^{-1}(x)_{\mathrm{red}}$, and let Y_1 and Y_2 be the irreducible components of $\mathscr{Y}_s$ such that $y' \in \psi_1(Y_1) \cap \psi_1(Y_2)$. Write $I_{Y_1} \subseteq G$ and $I_{Y_2} \subseteq G$ for the inertia subgroups of Y_1 and Y_2, respectively. Suppose that neither $I_{Y_1} \subseteq I_{Y_2}$ nor $I_{Y_1} \supseteq I_{Y_2}$ holds. Then x is a vertical point associated to f (i.e. $f^{-1}(x)$ is not finite).*

Proof. To verify the proposition, we may assume that x is not a vertical point associated to f. Then $f^{-1}(x)$ is a finite set. Let $a \in \psi_2^{-1}(x)$ and $b \in \psi_1^{-1}(a)$. Thus, ψ_1 induces an isomorphism $\operatorname{Spec} \mathcal{O}_{\mathscr{Y},b} \to \operatorname{Spec} \mathcal{O}_{\mathscr{Y}',a}$. Write y for $\psi_1^{-1}(y')_{\mathrm{red}}$. By replacing $\mathscr{X}$ by the quotient $\mathscr{Y}/D_y$ and G by $D_y \subseteq G$, respectively, where $D_y \subseteq G$ denotes the decomposition group of y, we may assume $f^{-1}(x)_{\mathrm{red}} = \{y\} \subseteq Y_1 \cap Y_2$.

Consider the quotient curve $\mathscr{Y}/I_{Y_1}$ (resp. $\mathscr{Y}/I_{Y_2}$) over S. Note that $\mathscr{Y}/I_{Y_1}$ (resp. $\mathscr{Y}/I_{Y_2}$) is a pointed semi-stable curve over S. We obtain the following morphisms of pointed semi-stable curves

$$\lambda_1 : \mathscr{Y} \to \mathscr{Y}/I_{Y_1} \text{ (resp. } \lambda_2 : \mathscr{Y} \to \mathscr{Y}/I_{Y_2}),$$

$$\mu_1 : \mathscr{Y}/I_{Y_1} \to \mathscr{X} \text{ (resp. } \mu_2 : \mathscr{Y}/I_{Y_2} \to \mathscr{X})$$

over S such that $\mu_1 \circ \lambda_1 = f$ (resp. $\mu_2 \circ \lambda_2 = f$). Note that μ_1 (resp. μ_2) is étale at the generic point of $\lambda_1(Y_1)$ (resp. $\lambda_2(Y_2)$) of degree $\#G/\#I_{Y_1}$ (resp. $\#G/\#I_{Y_2}$).

If μ_1 (resp. μ_2) is also generically étale at the generic point of $\lambda_1(Y_2)$ (resp. $\lambda_2(Y_1)$), then, by applying [T2, Lemma 2.1 (iii)] to

$$\operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_1},\lambda_1(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X},x} \text{ (resp. } \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_2},\lambda_2(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{\mathscr{X},x}),$$

we obtain that $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1(Y_1),\lambda_1(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x}$ (resp. $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_2(Y_2),\lambda_2(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_2,x}$) induced by μ_1 (resp. μ_2) is tamely ramified with ramification index t_1 (resp. t_2). Thus, we have $(t_1, p) = 1$ (resp. $(t_2, p) = 1$). On the other hand, since I_{Y_1} (resp. I_{Y_2}) does not contain I_{Y_2} (resp. I_{Y_1}), and I_{Y_2} (resp. I_{Y_1}) is a p-group, we have $p|t_1$ (resp. $p|t_2$). This is a contradiction. Thus, μ_1 (resp. μ_2) is not generically étale at the generic point of $\lambda_1(Y_2)$ (resp. $\lambda_2(Y_1)$). Thus, the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_1(Y_1),\lambda_1(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_1,x}$ (resp. $\operatorname{Spec} \widehat{\mathcal{O}}_{\lambda_2(Y_2),\lambda_2(y)} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_2,x}$) induced by μ_1 (resp. μ_2) is wildly ramified. Lemma 1.11 implies that x is a vertical point associated to f. This contradicts our assumptions. We complete the proof of the proposition. □

The following corollary follows immediately from Lemma 1.10 and Proposition 1.12.

Corollary 1.13. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a G-pointed semi-stable covering over S, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, and $y \in \mathscr{Y}_s$ a node. Let Y_1 and Y_2 (which may be equal) be the irreducible components of $\mathscr{Y}_s$ containing y. Write $I_y \subseteq G$ (resp. $I_{Y_1} \subseteq G$, $I_{Y_2} \subseteq G$) for the inertia subgroup of y (resp. Y_1, Y_2). Suppose that f is a* **finite** *morphism. Then either $I_{Y_1} \subseteq I_{Y_2}$ or $I_{Y_1} \supseteq I_{Y_2}$ holds. Moreover, if G is a p-group, then the inertia subgroup I_y is equal to either I_{Y_1} or I_{Y_2}.*

2. Semi-graphs with p-rank

In this section, we develop the theory of semi-graphs with p-rank. The main result of the present section is Theorem 2.7.

2.1. Semi-graphs with p-rank and their coverings.

2.1.1. We define semi-graphs with p-rank as follows:

Definition 2.1. Let $\mathbb{G}$ be a semi-graph (1.1.1) and $\sigma_{\mathfrak{G}} : v(\mathbb{G}) \to \mathbf{Z}$ a map. We shall call the pair $\mathfrak{G} \overset{\text{def}}{=} (\mathbb{G}, \sigma_{\mathfrak{G}})$ a *semi-graph with p-rank*. Moreover, we call that the semi-graph $\mathbb{G}$ is the underlying semi-graph of $\mathfrak{G}$, and that the map $\sigma_{\mathfrak{G}}$ is the p-rank map of $\mathfrak{G}$. We define the *p-rank* $\sigma(\mathfrak{G})$ of $\mathfrak{G}$ to be

$$\sigma(\mathfrak{G}) \overset{\text{def}}{=} \sum_{v \in v(\mathbb{G})} \sigma_{\mathfrak{G}}(v) + \gamma_{\mathbb{G}}.$$

A *morphism* of semi-graphs with p-rank $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ is defined by a morphism of the underlying semi-graphs $\beta : \mathbb{G}^1 \to \mathbb{G}^2$. We shall refer to the morphism β as the underlying morphism of $\mathfrak{b}$.

A semi-graph with p-rank is called connected if the underlying semi-graph $\mathbb{G}$ is a connected semi-graph.

Remark 2.1.1. We explain the geometric motivation of the above definitions. Let $\mathscr{X} \overset{\text{def}}{=} (X, D_X)$ be a pointed semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write $\Gamma_{\mathscr{X}}$ for the dual semi-graph (1.2.2) of $\mathscr{X}$ and we define $\sigma_{\Gamma_{\mathscr{X}}}(v)$, $v \in v(\Gamma_{\mathscr{X}})$, to be the p-rank (1.2.3) of the normalization of the irreducible component X_v corresponding to v. Then $(\Gamma_{\mathscr{X}}, \sigma_{\Gamma_{\mathscr{X}}})$ is a semi-graph with p-rank. On the other hand, a semi-graph with p-rank $\mathfrak{G} \overset{\text{def}}{=} (\mathbb{G}, \sigma_{\mathfrak{G}})$ is not arose from a pointed semi-stable curve in positive characteristic in general since $\sigma_{\mathfrak{G}}$ can attain *negative* integers.

2.1.2. **Settings.** Let G be a finite p-group of order p^r.

2.1.3. Let $\mathfrak{b} : \mathfrak{G}^1 \overset{\text{def}}{=} (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}) \to \mathfrak{G}^2 \overset{\text{def}}{=} (\mathbb{G}^2, \sigma_{\mathfrak{G}^2})$ be a morphism of semi-graphs with p-rank and $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$.

Definition 2.2. (a) We shall call that $\mathfrak{b}$ is *p-étale* (resp. *purely inseparable*) at an edge $e \in e(\mathbb{G}^1)$ if $\#\beta^{-1}(\beta(e)) = p$ (resp. $\#\beta^{-1}(\beta(e)) = 1$). We shall call that $\mathfrak{b}$ is *p-generically étale* at $v \in v(\mathbb{G}^1)$ if one of the following conditions holds (see 1.1.1 for $e(v)$):

(Type-I): $\#\beta^{-1}(\beta(v)) = p$ and $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$.
(Type-II): $\#\beta^{-1}(\beta(v)) = 1$ and

$$\sigma_{\mathfrak{G}^1}(v) - 1 = p(\sigma_{\mathfrak{G}^2}(\beta(v)) - 1) + \sum_{e \in e(v)} (\frac{p}{\#\beta^{-1}(\beta(e))} - 1).$$

(b) We shall call that $\mathfrak{b}$ is *purely inseparable* at $v \in v(\mathbb{G}^1)$ if $\#\beta^{-1}(\beta(v)) = 1$, $\mathfrak{b}$ is purely inseparable at each element of $e(v)$, and $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$.

(c) We shall call that $\mathfrak{b}$ is a *p-covering* if the following conditions hold (see 1.1.1 for $v(e)$):

(i) There exists a $\mathbf{Z}/p\mathbf{Z}$-action (which may be trivial) on $\mathbb{G}^1$ and a trivial $\mathbf{Z}/p\mathbf{Z}$-action on $\mathbb{G}^2$ such that the underlying morphism β of $\mathfrak{b}$ is compatible with the $\mathbf{Z}/p\mathbf{Z}$-actions.
(ii) The natural morphism $\mathbb{G}^1/(\mathbf{Z}/p\mathbf{Z}) \to \mathbb{G}^2$ induced by β is an isomorphism, where $\mathbb{G}^1/(\mathbf{Z}/p\mathbf{Z})$ denotes the quotient semi-graph.
(iii) For each $v \in v(\mathbb{G}^1)$, $\mathfrak{b}$ is either p-generically étale or purely inseparable at v.
(iv) Let $e \in e^{\mathrm{cl}}(\mathbb{G}^1)$ and $v(e) = \{v, v'\}$ (note that $v = v'$ if and only if e is a loop (1.1.1)). Suppose that $\mathfrak{b}$ is p-generically étale at v and v'. Then $\mathfrak{b}$ is p-étale at e.
(v) for each $v \in v(\mathbb{G}^1)$, then $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^1}(\tau(v))$ for each $\tau \in \mathbf{Z}/p\mathbf{Z}$.

Note that the definition of p-coverings implies that the identity morphism of a semi-graph with p-rank is a p-covering.

(d) We shall call that $\mathfrak{b}$ is a *covering* if $\mathfrak{b}$ is a composite of p-coverings.

(e) We maintain the notation introduced in 2.1.2. We shall call

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

a *maximal normal filtration* of G if G_j is a normal subgroup of G and $G_j/G_{j+1} \cong \mathbf{Z}/p\mathbf{Z}$ for $j \in \{0, \ldots, r-1\}$. Note that since G is a p-group, a maximal normal filtration of G exists.

Suppose that $\mathbb{G}^1$ admits a G-action (which may be trivial), that $\mathbb{G}^2$ admits a trivial G-action, and that the underlying morphism β of $\mathfrak{b}$ is compatible with the G-actions. A maximal normal filtration Φ of G induces a sequence of semi-graphs:

$$\mathbb{G}^1 = \mathbb{G}_r \xrightarrow{\beta_r} \mathbb{G}_{r-1} \xrightarrow{\beta_{r-1}} \ldots \xrightarrow{\beta_1} \mathbb{G}_0,$$

where $\mathbb{G}_j$, $j \in \{0, \ldots, r\}$, denotes the quotient semi-graph $\mathbb{G}^1/G_j$. We shall call that $\mathfrak{b}$ is a *G-covering* if there exist a maximal normal filtration Φ of G and a set of p-coverings $\{\mathfrak{b}_j : \mathfrak{G}_j \to \mathfrak{G}_{j-1},\ j = 1, \ldots, r\}$ such that the following conditions are satisfied:

(i) The underlying semi-graph of $\mathfrak{G}_j$ is equal to $\mathbb{G}_j$ for $j \in \{0, \ldots, r\}$ such that $\mathbb{G}_0 = \mathbb{G}^2$.
(ii) The underlying morphism of $\mathfrak{b}_j$ is equal to β_j for $j \in \{1, \ldots, r\}$.
(iii) The composite morphism $\mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is equal to $\mathfrak{b}$.

(f) Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering. By the above definition of G-coverings, we obtain a maximal normal filtration Φ of G and a sequence of p-coverings:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \ldots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

We shall call $\Phi_{\mathfrak{G}^1/\mathfrak{G}^2}$ *a sequence of p-coverings induced by* Φ.

Remark 2.2.1. We explain the geometric motivation of the above definitions. Let R be a discrete valuation ring with algebraically closed residue field of characteristic $p > 0$, and let $f : \mathscr{Y} \stackrel{\mathrm{def}}{=} (Y, D_Y) \to \mathscr{X} \stackrel{\mathrm{def}}{=} (X, D_X)$ be a *finite* G-pointed semi-stable covering over R (Definition 1.5). Write $(\Gamma_{\mathscr{Y}_s}, \sigma_{\Gamma_{\mathscr{Y}_s}})$ and $(\Gamma_{\mathscr{X}_s}, \sigma_{\Gamma_{\mathscr{X}_s}})$ for the semi-graphs with p-rank associated to the special fibers $\mathscr{Y}_s$ and $\mathcal{X}_s$ of $\mathscr{Y}$ and $\mathscr{X}$ (see Remark 2.1.1), respectively.

Then the morphism of special fibers induced by f induces a G-covering $(\Gamma_{\mathscr{Y}_s}, \sigma_{\Gamma_{\mathscr{Y}_s}}) \to (\Gamma_{\mathscr{X}_s}, \sigma_{\Gamma_{\mathscr{X}_s}})$ (see Section 3.1).

On the other hand, the definitions of p-étale, purely inseparable, p-generically étale, purely inseparable, and p-coverings of semi-graphs with p-rank are motivated by p-étale, purely inseparable, p-generically étale, purely inseparable, and p-coverings of special fibers of finite $\mathbf{Z}/p\mathbf{Z}$-pointed semi-stable coverings over R. In particular, Definition 2.2 (a-Type-II) is motivated by the Deuring-Shafarevich formula (see Proposition 1.4), and Definition 2.2 (c-iv) is motivated by the Zariski-Nagata purity theorem of finite $\mathbf{Z}/p\mathbf{Z}$-pointed semi-stable coverings over R.

2.1.4. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering, $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$, $v^1 \in v(\mathbb{G}^1)$, and $e^1 \in e(\mathbb{G}^1)$. By the definition of G-coverings, we have a maximal normal filtration Φ of G and a sequence of p-coverings induced by Φ:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

Write $\beta_j : \mathbb{G}_j \to \mathbb{G}_{j-1}$, $j \in \{1, \dots, r\}$, for the underlying morphism of $\mathfrak{b}_j$. Write v_j (resp. e_j) for the image $\beta_{j+1} \circ \dots \circ \beta_r(v^1)$ (resp. $\beta_{j+1} \circ \dots \circ \beta_r(e^1)$), $j \in \{0, \dots, r-1\}$, and v_r for v^1. We put

$$\#I_{v^1} = p^{\#\{j \in \{1,\dots,r\} \ | \ \mathfrak{b}_j \text{ is purely inseparable at } v_j\}},$$
$$\#I_{e^1} = p^{\#\{j \in \{1,\dots,r\} \ | \ \mathfrak{b}_j \text{ is purely inseparable at } e_j\}}.$$

Note that $\#I_{v^1}$ and $\#I_{e^1}$ do not depend on the choice of Φ. Moreover, we put $D_{v^1} \overset{\text{def}}{=} \{\tau \in G \mid \tau(v^1) = v^1\}$, and

$$\#D_{v^1}$$

the cardinality of D_{v^1}.

2.1.5. We maintain the notation introduced in 2.1.4. If $e^1 \in e(v^1)$, then we have $\#I_{v^1} | \#I_{e^1}$. In particular, if e^1 is a loop, then Definition 2.2 (c-iv) implies that $\#I_{v^1} = \#I_{e^1}$. Moreover, Definition 2.2 (c-iv) also implies that $\#I_{e^1} | \#D_{v^1}$. Write v^2 (resp. e^2) for $\beta(v^1)$ (resp. $\beta(e^1)$). Let $(v^1)'$ (resp. $(e^1)'$) be an arbitrary element of $\beta^{-1}(v^2)$ (resp. $\beta^{-1}(e^2)$). By the action of G on $\mathbb{G}^1$, we have $\#I_{v^1} = \#I_{(v^1)'}$, $\#I_{e^1} = \#I_{(e^1)'}$, and $\#D_{v^1} = \#D_{(v^1)'}$. Thus, we may use the notation $\#I_{v^2}$ (resp. $\#I_{e^2}$, $\#D_{v^2}$) to denote $\#I_{v^1}$ (resp. $\#I_{e^1}$, $\#D_{v^1}$). Namely, $\#I_{v^1}$ (resp. $\#I_{e^1}$, $\#D_{v^1}$) does not depend on the choice of $v^1 \in \beta^{-1}(\beta(v^1))$. Then we have $\#I_{v^2} | \#I_{e^2} | \#D_{v^2}$.

2.1.6. We maintain the notation introduced in 2.1.4 and 2.1.5. One may compute the p-rank $\sigma_{\mathfrak{G}^1}(v^1)$ by using Definition 2.2 (a). Then we have the following Deuring-Shafarevich type formula for the p-rank of G-coverings (see Proposition 1.4 for the Deuring-Shafarevich formula for curves)

$$\sigma_{\mathfrak{G}^1}(v^1) - 1 = (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2)} (\#D_{v^2}/\#I_{e^2})(\#I_{e^2}/\#I_{v^2} - 1)$$
$$= (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2) \setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_{e^2})(\#I_{e^2}/\#I_{v^2} - 1).$$

Here, the second equality follows from Definition 2.2 (c-iv) .

2.2. **An operator concerning coverings.** In this subsection, we introduce an operator (or a deformation) concerning coverings of semi-graphs with p-rank which is a key in our computations of p-rank.

2.2.1. **Settings.** We fix some notation. Let G be a finite p-group of order p^r, and let $\mathfrak{b} : \mathfrak{G}^1 \overset{\text{def}}{=} (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}) \to \mathfrak{G}^2 \overset{\text{def}}{=} (\mathbb{G}^2, \sigma_{\mathfrak{G}^2})$ be a covering of semi-graphs with p-rank (Definition 2.2 (d)) and $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$ (Definition 2.1). We put

$$V^1 \overset{\text{def}}{=} \{v \in v(\mathbb{G}^1) \mid \#\beta^{-1}(\beta(v^1)) = 1\} \subseteq v(\mathbb{G}^1),$$
$$V^2 \overset{\text{def}}{=} \beta(V^1) \subseteq v(\mathbb{G}^2).$$

Moreover, we suppose that $\mathbb{G}^1$, $\mathbb{G}^2$ are *connected*, that $\mathbb{G}^1$ (resp. $\mathbb{G}^2$) admits an action (resp. a trivial action) of G such that β is a G-equivariant, and that $\mathbb{G}^1/G = \mathbb{G}^2$.

2.2.2. Let $v^2 \in v(\mathbb{G}^2)$ and $v^1 \in \beta^{-1}(v^2)$. Firstly, we define a new semi-graph $\mathbb{G}^1_{v^2}$ associated to v^2 as follows (see Example 2.3 below): (a) Suppose $v^2 \in V^2$. We put $\mathbb{G}^1_{v^2} \overset{\text{def}}{=} \mathbb{G}^1$. (b) Suppose $v^2 \notin V^2$. We have the following:

(i) $v(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} (v(\mathbb{G}^1) \setminus \beta^{-1}(v^2)) \sqcup \{v^2_\star\}$, $e^{\text{cl}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{cl}}(\mathbb{G}^1)$, and $e^{\text{op}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{op}}(\mathbb{G}^1)$, where $v^2_\star$ is a new vertex and $\sqcup$ means disjoint union.

(ii) The collection of maps $\{\zeta_e^{\mathbb{G}^1_{v^2}}\}_e$ is as follows:

(1) For each $e \in e^{\text{op}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{op}}(\mathbb{G}^1)$ and $b \in e$ (i.e. a branch of e, see 1.1.1), we put

$$\zeta_e^{\mathbb{G}^1_{v^2}}(b) = \begin{cases} \{v(\mathbb{G}^1_{v^2})\}, & \text{if } \zeta_e^{\mathbb{G}^1}(b) = \{v(\mathbb{G}^1)\}, \\ v^2_\star, & \text{if } \zeta_e^{\mathbb{G}^1}(b) \in \beta^{-1}(v^2), \\ \zeta_e^{\mathbb{G}^1}(b), & \text{otherwise.} \end{cases}$$

(2) For each $e \in e^{\text{cl}}(\mathbb{G}^1_{v^2}) \overset{\text{def}}{=} e^{\text{cl}}(\mathbb{G}^1)$ and $b \in e$, we put

$$\zeta_e^{\mathbb{G}^1_{v^2}}(b) = \begin{cases} v^2_\star, & \text{if } \zeta_e^{\mathbb{G}^1}(b) \in \beta^{-1}(v^2), \\ \zeta_e^{\mathbb{G}^1}(b), & \text{otherwise.} \end{cases}$$

Next, we define a morphism of semi-graphs $\beta_{v^2} : \mathbb{G}^1_{v^2} \to \mathbb{G}^2$ as follows (see Example 2.3 below):

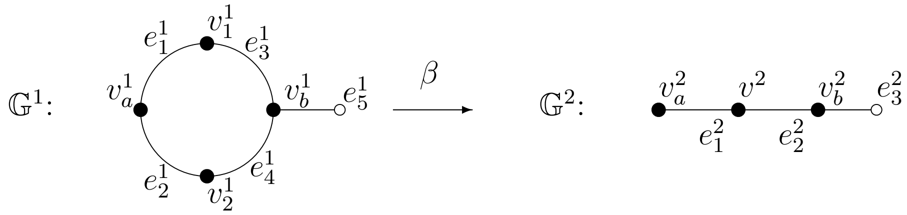
(i) For each $v \in v(\mathbb{G}^1_{v^2})$, we put

$$\beta_{v^2}(v) = \begin{cases} v^2, & \text{if } v = v^2_\star, \\ \beta(v), & \text{otherwise.} \end{cases}$$

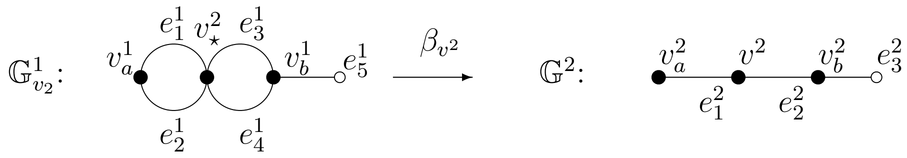
(ii) For each $e \in e(\mathbb{G}^1_{v^2}) = e^{\text{cl}}(\mathbb{G}^1_{v^2}) \cup e^{\text{op}}(\mathbb{G}^1_{v^2})$, we put $\beta_{v^2}(e) \overset{\text{def}}{=} \beta(e)$.

Example 2.3. We give an example to explain the above constructions. We use the notation "•" and "∘ with a line segment" to denote a vertex and an open edge, respectively.

Let $p = 2$, and let $\mathbb{G}^1$, $\mathbb{G}^2$ be the semi-graphs below. Moreover, let $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ be a morphism of semi-graphs such that $\beta(v^1_a) = v^2_a$, $\beta(v^1_b) = v^2_b$, $\beta(v^1_1) = \beta(v^1_2) = v^2$, $\beta(e^1_1) = \beta(e^1_2) = e^2_1$, $\beta(e^1_3) = \beta(e^1_4) = e^2_2$, and $\beta(e^1_5) = e^2_3$. Note that $\mathbb{G}^1$ admits an action of $\mathbf{Z}/2\mathbf{Z}$ such that $\mathbb{G}^1/(\mathbf{Z}/p\mathbf{Z}) = \mathbb{G}^2$. Then we have the following:



By the definitions of $\mathbb{G}^1_{v^2}$ and β_{v^2}, we have the following:



2.2.3. We maintain the notation introduced in 2.2.2. Next, we define a p-rank map $\sigma_{\mathfrak{G}^1_{v^2}} : v(\mathbb{G}^1_{v^2}) \to \mathbf{Z}$ for $\mathbb{G}^1_{v^2}$ as follows: (a) Suppose $v^2 \in V^2$. We put $\sigma_{\mathfrak{G}^1_{v^2}} \overset{\text{def}}{=} \sigma_{\mathfrak{G}^1}$. (b) Suppose $v^2 \notin V^2$. Let $v \in v(\mathbb{G}^1_{v^2})$. We have the following:

(i) If $v \neq v^2_\star$, we put $\sigma_{\mathfrak{G}^1_{v^2}}(v) \overset{\text{def}}{=} \sigma_{\mathfrak{G}^1}(v)$.
(ii) If $v = v^2_\star$, we put (see 1.1.1 for $e(v^2)$ and 2.1.6 for $\#I_{v^2}$, $\#I_e$)

$$\sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star) \overset{\text{def}}{=} (\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e \in e(v^2)} (\#G/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1.$$

2.2.4. We maintain the notation introduced in 2.2.2 and 2.2.3. Let $v^2 \in v(\mathbb{G}^2)$. We define a semi-graph with p-rank and a morphism of semi-graphs with p-rank associated to $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ and v^2, respectively, to be

$$\mathfrak{G}^1_{v^2} \overset{\text{def}}{=} (\mathbb{G}^1_{v^2}, \sigma_{\mathfrak{G}^1_{v^2}}), \ \mathfrak{b}_{v^2} : \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2,$$

where the underlying morphism of $\mathfrak{b}_{v^2}$ is β_{v^2}.

2.2.5. We maintain the settings introduced in 2.2.1. Let $\mathfrak{G}^i \setminus \{V^i\}$, $i \in \{1, 2\}$, be the (possibly non-connected) semi-graph with p-rank whose underlying semi-graph is $\mathbb{G}^i \setminus \{V^i\}$ (in the sense of Definition 1.1.2 (b)), and whose p-rank map is $\sigma_{\mathfrak{G}^i}|_{v(\mathbb{G}^i \setminus \{V^i\})}$. We shall call $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ a *quasi-G-covering* if the covering $\mathfrak{G}^1 \setminus \{V^1\} \to \mathfrak{G}^2 \setminus \{V^2\}$ induced by $\mathfrak{b}$ is a G-covering.

Definition 2.4. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a quasi-G-covering of connected semi-graphs with p-rank and $v^2 \in v(\mathbb{G}^2)$. We define an operator $\rightleftharpoons^I_{II} [v^2]$ on $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ to be

$$\rightleftharpoons^I_{II} [v^2](\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2) \overset{\text{def}}{=} \mathfrak{b}_{v^2} : \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2.$$

Here $\rightleftharpoons^I_{II}$ means that "from (Type-I) to (Type-II)" in the sense of Definition 2.2 (a).

Remark 2.4.1. Suppose that $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ is a G-covering of semi-graphs with p-rank. Then $\sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star)$ is not contained in $\mathbb{Z}_{\geq 0}$ in general. Thus, $\mathfrak{b}_{v^2} : \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2$ cannot be arose from a G-pointed semi-stable covering in general (see also Remark 2.2.1). On the other hand, in the next subsection, we will see (Proposition 2.6 below) that the operator defined above *does not change* global p-rank (i.e. $\sigma(\mathfrak{G}^1_{v^2}) = \sigma(\mathfrak{G}^1)$).

2.2.6. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a quasi-G-covering and $v^2 \in v(\mathbb{G}^2)$. Then the semi-graph with p-rank $\mathbb{G}^1_{v^2}$ admits a natural G-action as follows:

(i) The action of G on $v(\mathbb{G}^1_{v^2} \setminus \{v^2_\star\}) = v(\mathbb{G}^1) \setminus \beta^{-1}(v^2)$ (resp. $e(\mathbb{G}^1_{v^2}) = e(\mathbb{G}^1)$) is the action of G on $v(\mathbb{G}^1) \setminus \beta^{-1}(v^2)$ (resp. $e(\mathbb{G}^1)$) induced by the action of G on $\mathbb{G}^1$.
(ii) The action of G on $v^2_\star$ is a trivial action.

We see immediately that $\mathfrak{b}_{v^2}: \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2$ is a quasi-G-covering.

Let $\mathfrak{b}: \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering. Suppose that G is an *abelian* p-group. Then together with the G-action defined above, it is easy to check that $\mathfrak{b}_{v^2}: \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2$ is a G-covering.

On the other hand, if G is *not abelian*, then $\mathfrak{b}_{v^2}: \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2$ is *not* a G-covering in general for the following reason. Let $w \overset{\text{def}}{=} v^2_\star = \beta^{-1}_{v^2}(v^2)$. With the action of G on $\mathfrak{G}^1_{v^2}$ defined above, if I_{v^1}, $v^1 \in \beta^{-1}(v^2)$, is not a normal subgroup of G, then the order $\#I_w$ of the inertia subgroup I_w of w is not equal to $\#I_{v^2} \overset{\text{def}}{=} \#I_{v^1}$ (2.1.5) in general. If $\mathfrak{b}_{v^2}$ is a G-covering, we have (2.1.6)

$$\sigma_{\mathfrak{G}^1_{v^2}}(w) = (\#G/\#I_w)(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e \in e(v^2)} (\#G/\#I_e)(\#I_e/\#I_w - 1) + 1$$

which is not equal to (2.2.3 (b-ii))

$$\#G/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e \in e(v^2)} \#G/\#I_e(\#I_e/\#I_{v^2} - 1) + 1$$

in general if $\#I_w \neq \#I_{v^2}$. This contradicts the definition of $\mathfrak{G}^1_{v^2}$. Thus, $\mathfrak{b}_{v^2}: \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2$ is not a G-covering in general.

2.3. **Formula for p-rank of coverings.** In this subsection, we give an explicit formula (i.e. Theorem 2.7) for the p-rank of G-coverings of semi-graphs with p-rank.

2.3.1. **Settings.** We maintain the settings introduced in 2.2.1. Moreover, we assume that $\mathfrak{b}: \mathfrak{G}^1 \overset{\text{def}}{=} (\mathbb{G}^1, \sigma_{\mathfrak{G}^1}) \to \mathfrak{G}^2 \overset{\text{def}}{=} (\mathbb{G}^2, \sigma_{\mathfrak{G}^2})$ is a quasi-G-covering (2.2.5).

2.3.2. Firstly, we have the following lemma.

Lemma 2.5. *Let $i \in \{1, \dots, n\}$, and let $\mathbb{G}$ be a connected semi-graph, $\mathbb{G}_i$ a connected sub-semi-graph of $\mathbb{G}$ (1.1.2), and $v_i \in v(\mathbb{G}_i)$ a vertex of $\mathbb{G}_i$. Suppose $\mathbb{G}_s \cap \mathbb{G}_t = \emptyset$ for each $s, t \in \{1, \dots, n\}$ if $s \neq t$. Let $\mathbb{G}^{\mathrm{c}}$ be a semi-graph defined as follows:*

(i) $v(\mathbb{G}^{\mathrm{c}}) = v(\mathbb{G}) \sqcup \{v^{\mathrm{c}}\}$, $e^{\mathrm{op}}(\mathbb{G}^{\mathrm{c}}) = e^{\mathrm{op}}(\mathbb{G})$, $e^{\mathrm{cl}}(\mathbb{G}^{\mathrm{c}}) = e^{\mathrm{cl}}(\mathbb{G}) \sqcup \{e^{\mathrm{c}}_i\}_{i \in \{1,\dots,n\}}$.

(ii) Let $e \in e(\mathbb{G}^{\mathrm{c}}) \setminus \{e^{\mathrm{c}}_i\}_{i \in \{1,\dots,n\}} = e(\mathbb{G})$ and $b \in e$ a branch of e (1.1.1). We put

$$\zeta^{\mathbb{G}^{\mathrm{c}}}_e(b) = \begin{cases} \zeta^{\mathbb{G}}_e(b), & \text{if } \zeta^{\mathbb{G}}_e(b) \neq \{v(\mathbb{G})\}, \\ \{v(\mathbb{G}^{\mathrm{c}})\}, & \text{if } \zeta^{\mathbb{G}}_e(b) = \{v(\mathbb{G})\}. \end{cases}$$

(iii) Let $e^{\mathrm{c}}_i = \{b^1_{e^{\mathrm{c}}_i}, b^2_{e^{\mathrm{c}}_i}\}$. We put $\zeta^{\mathbb{G}^{\mathrm{c}}}_{e^{\mathrm{c}}_i}(b^1_{e^{\mathrm{c}}_i}) = v_i$, $\zeta^{\mathbb{G}^{\mathrm{c}}}_{e^{\mathrm{c}}_i}(b^2_{e^{\mathrm{c}}_i}) = v^{\mathrm{c}}$.

Then we have (see 1.1.3 for $\gamma_{\mathbb{G}}$, $\gamma_{\mathbb{G}^{\mathrm{c}}}$)

$$\gamma_{\mathbb{G}} = \gamma_{\mathbb{G}^{\mathrm{c}}} - n + 1.$$

Proof. The lemma follows from the construction of $\mathbb{G}^{\mathrm{c}}$. □

2.3.3. We have the following key proposition which says that the operator introduced in Definition 2.4 does not change the p-rank of semi-graphs with p-rank.

Proposition 2.6. *We maintain the settings introduced in 2.3.1. Let $v^2 \in v(\mathbb{G}^2)$ be an arbitrary vertex of $\mathbb{G}^2$ and $\rightleftharpoons^I_{II} [v^2](\mathfrak{b}: \mathfrak{G}^1 \to \mathfrak{G}^2) = \mathfrak{b}_{v^2}: \mathfrak{G}^1_{v^2} \to \mathfrak{G}^2$ (Definition 2.4). Then we have*

$$\sigma(\mathfrak{G}^1) = \sigma(\mathfrak{G}^1_{v^2}),$$

where $\sigma(\mathfrak{G}^1)$ and $\sigma(\mathfrak{G}^1_{v^2})$ are the p-rank of $\mathfrak{G}^1$ and $\mathfrak{G}^1_{v^2}$, respectively, defined in Definition 2.1.

Proof. Suppose $\#\beta^{-1}(v^2) = 1$ (i.e. $v^2 \in V^2$). Then the proposition is trivial since $\mathfrak{G}^1 = \mathfrak{G}^1_{v^2}$. Thus, we may assume $\#\beta^{-1}(v^2) \neq 1$ (i.e. $v^2 \notin V^2$).

Write β_{v^2} for the underlying morphism of $\mathfrak{b}_{v^2}$. Moreover, we put

$$W \overset{\text{def}}{=} \beta^{-1}(v^2),\ W^* \overset{\text{def}}{=} \beta_{v^2}^{-1}(v^2) = \{v^2_\star\}.$$

For simplicity, we shall write γ (resp. $\gamma_{\backslash\{v^2\}}, \gamma^*, \gamma^*_{\backslash\{v^2\}}$) for the Betti number (1.1.3) of $\mathbb{G}^1$ (resp. $\mathbb{G}^1 \setminus W$, $\mathbb{G}^1_{v^2}$, $\mathbb{G}^1_{v^2} \setminus W^*$), where $\mathbb{G}^1 \setminus W$ and $\mathbb{G}^1_{v^2} \setminus W^*$ are semi-graphs defined in Definition 1.1.2.

Then we have

$$\sigma(\mathfrak{G}^1) = \gamma_{\backslash\{v^2\}} + \gamma - \gamma_{\backslash\{v^2\}} + \sum_{v\in v(\mathbb{G}^1\setminus W)} \sigma_{\mathfrak{G}^1}(v) + \sum_{v\in W}\sigma_{\mathfrak{G}^1}(v),$$

$$\sigma(\mathfrak{G}^1_{v^2}) = \sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star) + \gamma^*_{\backslash\{v^2\}} + \gamma^* - \gamma^*_{\backslash\{v^2\}} + \sum_{v\in v(\mathbb{G}^1_{v^2}\setminus W^*)} \sigma_{\mathfrak{G}^1_{v^2}}(v).$$

Note that the construction of $\mathfrak{G}^1_{v^2}$ (2.2.2, 2.2.3) implies

$$A \overset{\text{def}}{=} \sum_{v\in v(\mathbb{G}^1\setminus W)} \sigma_{\mathfrak{G}^1}(v) = \sum_{v\in v(\mathbb{G}^1_{v^2}\setminus W^*)} \sigma_{\mathfrak{G}^1_{v^2}}(v),\ B \overset{\text{def}}{=} \gamma_{\backslash\{v^2\}} = \gamma^*_{\backslash\{v^2\}}.$$

We calculate $\gamma - \gamma_{\backslash\{v^2\}}$ and $\gamma^* - \gamma^*_{\backslash\{v^2\}}$. By applying Lemma 2.5, it is sufficient to treat the case where $\mathbb{G}^1 \setminus W = \mathbb{G}^1_{v^2} \setminus W^*$ is connected. Then we obtain (see 1.1.1 for $e(v^2)$, $e^{\text{lp}}(v^2)$ and 2.1.4, 2.1.5 for $\#D_{v^2}$, $\#I_{v^2}$, $\#I_e$)

$$\gamma - \gamma_{\backslash\{v^2\}} = (\#G/\#D_{v^2})\Big(\big(\sum_{e\in(e(v^2)\cap e^{\text{cl}}(\mathbb{G}^2))\setminus e^{\text{lp}}(v^2)} \#D_{v^2}/\#I_e\big) - 1\Big) + \#e^{\text{lp}}(v^2)(\#G/\#I_{v^2}),$$

$$\gamma^* - \gamma^*_{\backslash\{v^2\}} = \big(\sum_{e\in(e(v^2)\cap e^{\text{cl}}(\mathbb{G}^2))\setminus e^{\text{lp}}(v^2)} \#G/\#I_e\big) - 1 + \#e^{\text{lp}}(v^2)(\#G/\#I_{v^2}).$$

On the other hand, for each $v \in W \overset{\text{def}}{=} \beta^{-1}(v^2)$, we have (2.1.6)

$$\sigma_{\mathfrak{G}^1}(v) = (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1$$

$$= (\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\text{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1.$$

Moreover, the construction of $\mathfrak{G}^1_{v^2}$ (2.2.3) implies that

$$\sigma_{\mathfrak{G}^1_{v^2}}(v^2_\star) = (\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} (\#G/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1$$

$$= (\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\text{lp}}(v^2)} (\#G/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1$$

We obtain

$$\sigma(\mathfrak{G}^1) = A + B + \sum_{v\in W}\sigma_{\mathfrak{G}^1}(v) + \gamma - \gamma_{\backslash\{v^2\}}$$

$$= A + B + \sum_{v\in W}\Big((\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\text{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1\Big)$$

$$+(\#G/\#D_{v^2})\Big(\big(\sum_{e\in(e(v^2)\cap e^{\text{cl}}(\mathbb{G}^2))\setminus e^{\text{lp}}(v^2)} \#D_{v^2}/\#I_e\big) - 1\Big) + \#e^{\text{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A+B+(\#G/\#D_{v^2})\Big((\#D_{v^2}/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)}(\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2}-1)+1\Big)$$

$$+(\#G/\#D_{v^2})\Big((\sum_{e\in (e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)}\#D_{v^2}/\#I_e)-1\Big)+\#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A+B+(\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2)-\#G/\#I_{v^2}+\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_{v^2}-\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_e$$

$$+\sum_{e\in (e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_e+\#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A+B+(\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2)-\#G/\#I_{v^2}+\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_{v^2}$$

$$-\sum_{e\in (e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_e+\#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}).$$

Note that the last equality holds since we have

$$e(v^2)\setminus e^{\mathrm{lp}}(v^2)=((e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2))\sqcup((e(v^2)\cap e^{\mathrm{lp}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)).$$

On the other hand, we obtain

$$\sigma(\mathfrak{G}^1_{v^2})=A+B+\sigma_{\mathfrak{G}^1}(v^2_\star)+\gamma^*-\gamma^*_{\setminus\{v^2\}}$$

$$= A+B+(\#G/\#I_{v^2})(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)}(\#G/\#I_e)(\#I_e/\#I_{v^2}-1)+1$$

$$+(\sum_{e\in (e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_e)-1+\#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A+B+(\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2)-\#G/\#I_{v^2}+\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_{v^2}$$

$$-\sum_{e\in (e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)}\#G/\#I_e+\#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}).$$

Namely, we have

$$\sigma(\mathfrak{G}^1)=\sigma(\mathfrak{G}^1_{v^2}).$$

We complete the proof of the proposition. □

2.3.4. The main result of the present section is as follows:

Theorem 2.7. *Let $\mathfrak{b}:\mathfrak{G}^1\to\mathfrak{G}^2$ be a G-covering of connected semi-graphs with p-rank (Definition 2.2 (e)). Then we have (see 1.1.1 for $e(v)$, $e^{\mathrm{lp}}(v)$)*

$$\sigma(\mathfrak{G}^1)=\sum_{v\in v(\mathbb{G}^2)}\Big((\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v)-1)+\sum_{e\in e(v)\setminus e^{\mathrm{lp}}(v)}(\#G/\#I_e)(\#I_e/\#I_v-1)+1\Big)$$

$$+\sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)}(\#G/\#I_e-1)+\sum_{v\in v(\mathbb{G}^2)}\#e^{\mathrm{lp}}(v)(\#G/\#I_v-1)+\gamma_{\mathbb{G}^2}.$$

Proof. By applying Proposition 2.6 and the operator $\rightleftharpoons^{I}_{II}$ (Definition 2.4), we may construct a quasi-G-covering $\mathfrak{b}^*:\mathfrak{G}^{1,*}\to\mathfrak{G}^2$ from $\mathfrak{b}$ such that the following conditions are satisfied:

(i) We have $\#(\beta^*)^{-1}(v) = 1$ for each $v \in v(\mathbb{G}^2)$, where β^* denotes the underlying morphism of $\mathfrak{b}^*$.
(ii) For each $v \in v(\mathbb{G}^2)$ and $v^* \in (\beta^*)^{-1}(v)$, we have

$$\sigma_{\mathfrak{G}^{1,*}}(v^*) = (\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1) + 1$$

$$= (\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1) + 1.$$

(iii) $\sigma(\mathfrak{G}^{1,*}) = \sigma(\mathfrak{G}^1)$.

Write $\mathbb{G}^{1,*}$ for the underlying semi-graph of $\mathfrak{G}^{1,*}$. We observe that

$$\gamma_{\mathbb{G}^{1,*}} = \gamma_{\mathbb{G}^2} + \sum_{e \in e^{\mathrm{cl}}(\mathbb{G}^2) \setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) - \sum_{v \in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v) + \sum_{v \in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v)$$

$$= \gamma_{\mathbb{G}^2} + \sum_{e \in e^{\mathrm{cl}}(\mathbb{G}^2) \setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \sum_{v \in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1).$$

Thus, we obtain

$$\sigma(\mathfrak{G}^1) = \sigma(\mathfrak{G}^{1,*}) = \sum_{v \in v(\mathbb{G}^2)} \Big((\#G/\#I_v)(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1) + 1 \Big)$$

$$+ \sum_{e \in e^{\mathrm{cl}}(\mathbb{G}^2) \setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \sum_{v \in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \gamma_{\mathbb{G}^2}.$$

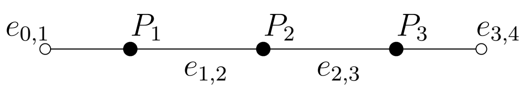
This completes the proof of the theorem. □

2.3.5. We introduce a kind of special semi-graph. Let n be a positive natural number and $\mathbb{P}_n$ a semi-graph (see Example 2.8 below) such that the following conditions are satisfied:

(i) $v(\mathbb{P}_n) = \{P_1, \dots, P_n\}$, $e^{\mathrm{cl}}(\mathbb{P}_n) = \{e_{1,2}, \dots, e_{n-1,n}\}$, and $e^{\mathrm{op}}(\mathbb{P}_n) = \{e_{0,1}, e_{n,n+1}\}$.
(ii) $\zeta^{\mathbb{P}_n}_{e_{0,1}}(e_{0,1}) = \{P_1, \{v(\mathbb{P}_n)\}\}$, $\zeta^{\mathbb{P}_n}_{e_{n,n+1}}(e_{n,n+1}) = \{P_n, \{v(\mathbb{P}_n)\}\}$, and $\zeta^{\mathbb{P}_n}_{e_{i,i+1}}(e_{i,i+1}) = \{P_i, P_{i+1}\}$, $i \in \{1, \dots, n-1\}$.

Example 2.8. We give an example to explain the notion defined above. If $n = 3$, then $\mathbb{P}_3$ is as follows:

$\mathbb{P}_3$:



Definition 2.9. Let $\mathbb{P}_n$ be a semi-graph defined above and $\sigma_{\mathfrak{P}_n} : v(\mathbb{P}_n) \to \mathbf{Z}$ a map such that $\sigma_{\mathfrak{P}_n}(P_i) = 0$ for each $i = \{1, \dots, n\}$. We define a semi-graph with p-rank $\mathfrak{P}_n$ to be

$$\mathfrak{P}_n \overset{\mathrm{def}}{=} (\mathbb{P}_n, \sigma_{\mathfrak{P}_n}),$$

and shall call $\mathfrak{P}_n$ an *n-chain*.

Remark 2.9.1. In Section 3.3, we will see that n-chains can be naturally arose from quotients of the vertical fibers associated to *singular* vertical points (Definition 1.8) of G-pointed semi-stable coverings.

2.3.6. When $\mathfrak{G}^2 = \mathfrak{P}_n$ is a n-chain, Theorem 2.7 has the following important consequence.

Corollary 2.10. *Let $\mathfrak{b} : \mathfrak{G} \to \mathfrak{P}_n$ be a G-covering of connected semi-graphs with p-rank. Then we have*

$$\sigma(\mathfrak{G}) = \sum_{i=1}^{n} \#G/\#I_{P_i} - \sum_{i=1}^{n+1} \#G/\#I_{e_{i-1,i}} + 1.$$

Proof. The construction of $\mathbb{P}_n$ implies

$$\sum_{v \in v(\mathbb{P}_n)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) = \gamma_{\mathbb{P}_n} = 0.$$

Then the corollary follows immediately from Theorem 2.7. □

3. Formulas for p-rank of coverings of curves

In this section, we construct various semi-graphs with p-rank from G-pointed semi-stable coverings. Moreover, we prove various formulas for p-rank concerning G-pointed semi-stable coverings when G is a finite p-group. More precisely, we prove a formula for p-rank of special fibers (see Theorem 3.2), a formula for p-rank of vertical fibers over vertical points (see Theorem 3.4), and a simpler form of Theorem 3.4 when the vertical points are singular (see Theorem 3.9 which plays a key in Section 4). In particular, Theorem 3.4 and Theorem 3.9 generalize Raynaud's result (Theorem 1.9) to the case of *arbitrary closed points*.

3.1. p-rank of special fibers.

3.1.1. **Settings.** We maintain the settings introduced in 1.3.1. Let G be a finite p-group of order p^r, and let $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X} = (X, D_X)$ be a G-pointed semi-stable covering (Definition 1.5) over S. Moreover, let

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

be a maximal normal filtration (Definition 2.2) of G. By applying [R, Appendice, Corollaire], we have that $\mathscr{X}^{\mathrm{sst}} = (X^{\mathrm{sst}}, D_{X^{\mathrm{sst}}}) \stackrel{\mathrm{def}}{=} \mathscr{Y}/G$ is a pointed semi-stable curve over S. Write $h : \mathscr{Y} \to \mathscr{X}^{\mathrm{sst}}$ and $g : \mathscr{X}^{\mathrm{sst}} \to \mathscr{X}$ for the natural morphisms of pointed semi-stable curves over S induced by f such that $f = g \circ h : \mathscr{Y} \xrightarrow{h} \mathscr{X}^{\mathrm{sst}} \xrightarrow{g} \mathscr{X}$.

3.1.2. Let $j \in \{0, \ldots, r\}$. [R, Appendice, Corollaire] implies that $\mathscr{Y}_j \stackrel{\mathrm{def}}{=} \mathscr{Y}/G_j$ is a pointed semi-stable curve over S. Then the maximal normal filtration Φ of G induces a sequence of morphism of pointed semi-stable curves

$$\Phi_{\mathscr{Y}/\mathscr{X}^{\mathrm{sst}}} : \mathscr{Y}_r \stackrel{\mathrm{def}}{=} \mathscr{Y} \xrightarrow{\phi_r} \mathscr{Y}_{r-1} \xrightarrow{\phi_{r-1}} \ldots \xrightarrow{\phi_1} \mathscr{Y}_0 \stackrel{\mathrm{def}}{=} \mathscr{X}^{\mathrm{sst}}$$

over S such that $\phi_1 \circ \ldots \circ \phi_r = h$. Note that ϕ_j is a *finite* $\mathbf{Z}/p\mathbf{Z}$-pointed semi-stable covering over S.

Write $\Gamma_{\mathscr{Y}_j}$ for the dual semi-graph (1.2.2) of the special fiber $(\mathscr{Y}_j)_s$ of $\mathscr{Y}_j$. Then, for each $j \in \{1, \ldots, r\}$, the morphism of the special fibers $(\phi_j)_s : (\mathscr{Y}_j)_s \to (\mathscr{Y}_{j-1})_s$ induces a map of semi-graphs $\beta_j : \Gamma_{\mathscr{Y}_j} \to \Gamma_{\mathscr{Y}_{j-1}}$. Moreover, Proposition 1.7 implies that β_j, $j \in \{1, \ldots, r\}$, is a *morphism* of semi-graphs.

3.1.3. *Semi-graph with p-rank associated to* $(\mathscr{Y}_j)_s$. Let $v \in v(\Gamma_{\mathscr{Y}_j})$ and $j \in \{0, \ldots, r\}$. We write $\widetilde{Y}_{j,v}$ for the normalization of the irreducible component $Y_{j,v} \subseteq (\mathscr{Y}_j)_s$ corresponding to v. We define a semi-graph with p-rank associated to $(\mathscr{Y}_j)_s$ to be

$$\mathfrak{G}_{\mathscr{Y}_j} \stackrel{\text{def}}{=} (\mathbb{G}_{\mathscr{Y}_j}, \sigma_{\mathfrak{G}_{\mathscr{Y}_j}}),\ j \in \{0, \ldots, r\},$$

where $\mathbb{G}_{\mathscr{Y}_j} \stackrel{\text{def}}{=} \Gamma_{\mathscr{Y}_j}$ and $\sigma_{\mathfrak{G}_{\mathscr{Y}_j}}(v) \stackrel{\text{def}}{=} \sigma(\widetilde{Y}_{j,v})$ for $v \in v(\mathbb{G}_{\mathscr{Y}_j})$.

3.1.4. *G-covering of semi-graphs with p-rank associated to f*. The sequence of pointed semi-stable coverings $\Phi_{\mathscr{Y}/\mathscr{X}^{\text{sst}}}$ induces a sequence of morphisms of semi-graphs with p-rank

$$\Phi_{\mathfrak{G}_{\mathscr{Y}}/\mathfrak{G}_{\mathscr{X}^{\text{sst}}}} : \mathfrak{G}_{\mathscr{Y}} \stackrel{\text{def}}{=} \mathfrak{G}_{\mathscr{Y}_r} \stackrel{\mathfrak{b}_r}{\to} \mathfrak{G}_{\mathscr{Y}_{r-1}} \stackrel{\mathfrak{b}_{r-1}}{\to} \ldots \stackrel{\mathfrak{b}_1}{\to} \mathfrak{G}_{\mathscr{X}^{\text{sst}}} \stackrel{\text{def}}{=} \mathfrak{G}_{\mathscr{Y}_0},$$

where $\mathfrak{b}_j : \mathfrak{G}_{\mathscr{Y}_j} \to \mathfrak{G}_{\mathscr{Y}_{j-1}}$, $j \in \{1, \ldots, r\}$, is induced by $\beta_j : \Gamma_{\mathscr{Y}_j} \to \Gamma_{\mathscr{Y}_{j-1}}$. By using the Deuring-Shafarevich formula (Proposition 1.4) and the Zariski-Nagata purity theorem ([SGA1, Exposé X, Théorème de pureté 3.1]), we see that $\mathfrak{b}_j$, $j \in \{1, \ldots, r\}$, is a p-covering (Definition 2.2 (c)). Moreover, $\mathfrak{b} \stackrel{\text{def}}{=} \mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is a G-covering (Definition 2.2 (e)). Then we have

$$\sigma(\mathfrak{G}_{\mathscr{Y}}) = \sigma(\mathscr{Y}_s).$$

Summarizing the discussions above, we obtain the following proposition.

Proposition 3.1. *We maintain the notation introduced above. Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and $\mathscr{Y}_s$ the special fiber of $\mathscr{Y}$ over s. Then there exists a G-covering of semi-graphs with p-rank $\mathfrak{b} : \mathfrak{G}_{\mathscr{Y}} \to \mathfrak{G}_{\mathscr{X}^{\text{sst}}}$ associated to f (which is constructed above) such that $\sigma(\mathscr{Y}_s) = \sigma_{\mathfrak{G}_{\mathscr{Y}}}(\mathfrak{G}_{\mathscr{Y}})$.*

3.1.5. We maintain the notation introduced in 3.1.1 and write $\Gamma_{\mathscr{X}_s^{\text{sst}}}$ for the dual semi-graph of the special fiber $\mathscr{X}_s^{\text{sst}} = (X_s^{\text{sst}}, D_{X_s^{\text{sst}}})$ of $\mathscr{X}^{\text{sst}}$. Let $v \in v(\Gamma_{\mathscr{X}_s^{\text{sst}}})$ and $e \in e(\Gamma_{\mathscr{X}_s^{\text{sst}}})$ (1.1.1). We write Y_v and y_e for an irreducible component of $h^{-1}(X_v)_{\text{red}}$ and a closed point of $h^{-1}(x_e)_{\text{red}}$, respectively, where X_v and x_e denote the irreducible component and the closed point of $\mathscr{X}_s^{\text{sst}}$ corresponding to v and e (1.2.2), respectively. Write $I_{Y_v} \subseteq G$ and $I_{y_e} \subseteq G$ for the inertia subgroup of Y_v and y_e, respectively. Note that since $\#I_{Y_v}$ and $\#I_{y_e}$ do not depend on the choices of Y_v and y_e, respectively, we may denote $\#I_{Y_v}$ and $\#I_{y_e}$ by $\#I_v$ and $\#I_e$, respectively. We put (see 1.1.1 for $v(e)$)

$$\#I_e^{\text{m}} \stackrel{\text{def}}{=} \max_{v \in v(e)}\{\#I_v\},\ e \in e^{\text{cl}}(\Gamma_{\mathscr{X}_s^{\text{sst}}}).$$

Note that Corollary 1.13 implies that $\#I_e = \#I_e^{\text{m}}$.

We have the following formula for p-rank of special fibers of G-pointed stable coverings when G is a finite p-group.

Theorem 3.2. *We maintain the settings introduced above. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S. Then we have (see 1.2.2 for $\widetilde{X}_v$, 1.1.1 for $e^{\text{cl}}(\Gamma_{\mathscr{X}_s^{\text{sst}}})$, $e^{\text{lp}}(\Gamma_{\mathscr{X}_s^{\text{sst}}})$, $e(v)$, $e^{\text{lp}}(v)$, and 1.1.3 for $\gamma_{\Gamma_{\mathscr{X}_s^{\text{sst}}}}$)*

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\text{sst}}})} \Big(1 + (\#G/\#I_v)(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\text{lp}}(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1)\Big)$$

$$+ \sum_{e \in e^{\text{cl}}(\Gamma_{\mathscr{X}_s^{\text{sst}}}) \setminus e^{\text{lp}}(\Gamma_{\mathscr{X}_s^{\text{sst}}})} (\#G/\#I_e - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\text{sst}}})} \#e^{\text{lp}}(v)(\#G/\#I_v - 1) + \gamma_{\Gamma_{\mathscr{X}_s^{\text{sst}}}}.$$

In particular, if $f : \mathscr{Y} \to \mathscr{X}$ is a G-semi-stable covering (i.e. $D_X = \emptyset$), then we have

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})} \Big(1 + (\#G/\#I_v)(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\rm lp}(v)} (\#G/\#I_e^{\rm m})(\#I_e^{\rm m}/\#I_v - 1)\Big)$$

$$+ \sum_{e \in e^{\rm cl}(\Gamma_{\mathscr{X}_s^{\rm sst}}) \setminus e^{\rm lp}(\Gamma_{\mathscr{X}_s^{\rm sst}})} (\#G/\#I_e^{\rm m} - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})} \#e^{\rm lp}(v)(\#G/\#I_v - 1) + \gamma_{\Gamma_{\mathscr{X}_s^{\rm sst}}}.$$

Proof. The theorem follows from Theorem 2.7 and Proposition 3.1. □

Remark 3.2.1. Note that it is easy to check that *the formula of Theorem 3.2 depends only on the G-pointed stable coverings.*

3.2. p-rank of vertical fibers.

3.2.1. **Settings.** We maintain the settings introduced in 3.1.1. Let x be a *vertical point* (Definition 1.8) associated to f. Write $\psi : Y' \to X$ for the normalization of X in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f. Then Y' admits a natural action of G induced by the action of G on the generic fiber of Y.

Let $y' \in \psi^{-1}(x)$. Write $I_{y'} \subseteq G$ for the inertia subgroup of y'. Proposition 1.6 implies that the morphism of pointed smooth curves $(Y_\eta/I_{y'}, D_{Y_\eta}/I_{y'}) \to \mathscr{X}_\eta$ over η induced by f extends to a pointed semi-stable covering $\mathscr{Y}_{I_{y'}} \to \mathscr{X}$ over S. In order to calculate the p-rank of $f^{-1}(x)$, since the morphism $\mathscr{Y}_{I_{y'}} \to \mathscr{X}$ is finite étale over x, by replacing $\mathscr{X}$ by $\mathscr{Y}_{I_{y'}}$, we may assume that G is equal to $I_{y'}$. In the remainder of this subsection, we shall assume $G = I_{y'}$ (note that $G = I_{y'}$ if and only if $f^{-1}(x)$ is *connected*).

3.2.2. Write $\mathscr{X}_s^{\rm sst} = (X_s^{\rm sst}, D_{X_s^{\rm sst}})$ and $\mathscr{Y}_s = (Y_s, D_{Y_s})$ for the special fibers of $\mathscr{X}^{\rm sst}$ and $\mathscr{Y}$ over s, respectively. By the general theory of semi-stable curves, $g^{-1}(x)_{\rm red} \subset X_s^{\rm sst}$ and $f^{-1}(x)_{\rm red} = h^{-1}(g^{-1}(x))_{\rm red} \subset Y_s$ are semi-stable curves over s, where $(-)_{\rm red}$ denotes the reduced induced closed subscheme of $(-)$. In particular, the irreducible components of $g^{-1}(x)_{\rm red}$ are isomorphic to $\mathbb{P}^1_k$.

Write V_X for the set of closed points

$$g^{-1}(x)_{\rm red} \cap \overline{\{X_s^{\rm sst} \setminus g^{-1}(x)_{\rm red}\}},$$

where $\overline{\{X_s^{\rm sst} \setminus g^{-1}(x)_{\rm red}\}}$ denotes the topological closure of $X_s^{\rm sst} \setminus g^{-1}(x)_{\rm red}$ in $X_s^{\rm sst}$. Write $V_Y \subset \mathscr{Y}_s$ for the set of closed points $\{h^{-1}(q)_{\rm red}\}_{q \in V_X}$. We have $\#V_X = 1$ if x is a *smooth point* of $\mathscr{X}_s$, and $\#V_X = 2$ if x is a *node* of $\mathscr{X}_s$.

3.2.3. We define two pointed semi-stable curves over s to be

$$\mathscr{E}_X \stackrel{\rm def}{=} (g^{-1}(x)_{\rm red}, (D_{X^{\rm sst}} \cap g^{-1}(x)_{\rm red}) \cup V_X),$$

$$\mathscr{E}_Y \stackrel{\rm def}{=} (f^{-1}(x)_{\rm red}, (D_Y \cap f^{-1}(x)_{\rm red}) \cup V_Y).$$

Then we obtain a *finite* morphism of pointed semi-stable curves $\rho_{\mathscr{E}_Y/\mathscr{E}_X} : \mathscr{E}_Y \to \mathscr{E}_X$ induced by h. Since $f^{-1}(x)$ is connected, $\mathscr{E}_Y$ admits a natural action of G induced by the action of G on the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$. Write $\Gamma_{\mathscr{E}_Y}$ and $\Gamma_{\mathscr{E}_X}$ for the dual semi-graphs of $\mathscr{E}_Y$ and $\mathscr{E}_X$, respectively. Note that $\Gamma_{\mathscr{E}_X}$ is a tree, and is *not* a n-chain (Definition 2.9) in general if x is not a node. We obtain a map of semi-graphs

$$\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$$

induced by $\rho_{\mathscr{E}_Y/\mathscr{E}_X}$. Moreover, Proposition 1.7 implies that the map $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$ is a morphism of semi-graphs.

3.2.4. *Semi-graphs with p-rank associated to $\mathscr{E}_Y$ and $\mathscr{E}_X$.* Let $v \in v(\Gamma_{\mathscr{E}_Y})$. Write $\widetilde{Y}_v$ for the normalization of the irreducible component $Y_v \subseteq \mathscr{E}_Y$ corresponding to v. We define semi-graphs with p-rank associated to $\mathscr{E}_Y$ and $\mathscr{E}_X$, respectively, as follows:

$$\mathfrak{E}_Y \stackrel{\text{def}}{=} (\mathbb{E}_Y, \sigma_{\mathfrak{E}_Y}),\ \mathfrak{E}_X \stackrel{\text{def}}{=} (\mathbb{E}_X, \sigma_{\mathfrak{E}_X}),$$

where $\mathbb{E}_Y \stackrel{\text{def}}{=} \Gamma_{\mathscr{E}_Y}$, $\mathbb{E}_X \stackrel{\text{def}}{=} \Gamma_{\mathscr{E}_X}$, $\sigma_{\mathfrak{E}_Y}(v) \stackrel{\text{def}}{=} \sigma(\widetilde{Y}_v)$ for $v \in v(\mathbb{E}_Y)$, and $\sigma_{\mathfrak{E}_X}(w) \stackrel{\text{def}}{=} 0$ for $w \in v(\mathbb{E}_X)$.

3.2.5. *G-coverings of semi-graphs with p-rank associated to vertical fibers.* The morphism of dual semi-graphs $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$ induces a morphism of semi-graphs with p-rank

$$\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \to \mathfrak{E}_X.$$

Moreover, we see that $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X}$ is a G-covering. Then we have

$$\sigma(\mathfrak{E}_Y) = \sigma(f^{-1}(x)_{\mathrm{red}}) = \sigma(f^{-1}(x)).$$

Summarizing the discussions above, we obtain the following proposition.

Proposition 3.3. *We maintain the notation introduced above. Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and x a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected. Then there exists a G-covering of semi-graphs with p-rank $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \to \mathfrak{E}_X$ associated to f and x (which is constructed above) such that $\sigma(\mathfrak{E}_Y) = \sigma(f^{-1}(x))$.*

3.2.6. Then we have the following formula for p-rank of vertical fibers.

Theorem 3.4. *We maintain the settings introduced in 1.3.1. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering (Definition 1.5) over S and x a vertical point (Definition 1.8) associated to f. We maintain the notation introduced in 3.2.2 and 3.2.3. Suppose that $f^{-1}(x)$ is connected. Then we have (see 3.1.5 for $\#I_v$, $\#I_e$, and 1.1.1 for $v(\Gamma_{\mathscr{E}_X})$, $e(v)$, $e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})$)*

$$\sigma(f^{-1}(x)) = \sum_{v \in v(\Gamma_{\mathscr{E}_X})} \Big(1 - \#G/\#I_v + \sum_{e \in e(v)} (\#G/\#I_e)(\#I_e/\#I_v - 1)\Big) + \sum_{e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})} (\#G/\#I_e - 1).$$

Proof. The theorem follows from Theorem 2.7 and Proposition 3.3. □

3.2.7. We maintain the notation introduced in Theorem 3.4. We explain that Raynaud's result (i.e. Theorem 1.9) can be directly calculated by using Theorem 3.4 if $x \in X_s \setminus (X_s^{\mathrm{sing}} \cup D_{X_s})$. Note that, since $x \notin D_{X_s}$, we have $g^{-1}(x)_{\mathrm{red}} \cap D_{X_s^{\mathrm{sst}}} = \emptyset$.

Let X_0' be the irreducible component of X_s which contains x. Moreover, we write X_0 for the strict transform of X_0' under the birational morphism $g : \mathscr{X}^{\mathrm{sst}} \to \mathscr{X}$. Then there exists a unique irreducible component $X_1 \subseteq g^{-1}(x)_{\mathrm{red}} \subseteq X_s^{\mathrm{sst}}$ such that $X_0 \cap X_1 \neq \emptyset$. Note that $\#(X_0 \cap X_1) = 1$. Write v_1 for the vertex of $v(\Gamma_{\mathscr{E}_X})$ corresponding to X_1. Since $\Gamma_{\mathscr{E}_X}$ is a connected tree, for each $v \in v(\Gamma_{\mathscr{E}_X})$, there exists a path $l(v_1, v)$ connecting v_1 and v. We define

$$\mathrm{leng}(l(v_1, v)) \stackrel{\text{def}}{=} \#\{l(v_1, v) \cap v(\Gamma_{\mathscr{E}_X})\}$$

to be the length of the path $l(v_1, v)$. Moreover, for each $v \in v(\Gamma_{\mathscr{E}_X})$, we write

$$l_{v_1, v}$$

for the path such that $\mathrm{leng}(l_{v_1,v}) = \min\{\mathrm{leng}(l(v_1, v))\}_{l(v_1,v)}$.

By applying the general theory of semi-stable curves, Lemma 1.10, and Corollary 1.13, one may prove the following:

Let $v, v' \in v(\Gamma_{\mathscr{E}_X})$ and $X_v, X_{v'}$ the irreducible components of $g^{-1}(x)_{\rm red}$ corresponding to v, v', respectively. Suppose that $\{x_e\} \overset{\rm def}{=} X_v \cap X_{v'} \neq \emptyset$, and that $\mathrm{leng}(l_{v_1,v}) < \mathrm{leng}(l_{v_1,v'})$. Write $e \in e^{\rm cl}(\Gamma_{\mathscr{E}_X})$ for the closed edge corresponding to x_e. Then we have $\#I_v = \#I_e$ and $\#I_{v'} | \#I_v$.

Note that the inertia subgroup of the unique open edge of $\Gamma_{\mathscr{E}_X}$ (which abuts to v_1) is equal to G. Then Theorem 3.4 implies that $\sigma(f^{-1}(x)) = 0$.

3.3. **p-rank of vertical fibers associated to singular vertical points.** In this subsection, we will see that Theorem 3.4 has a very simple form if x is a *singular vertical point* which plays a central role in Section 4.

3.3.1. **Settings.** We maintain the settings introduced in 3.2.1. Moreover, we suppose that the vertical point x is a *node* of $\mathscr{X}_s$. Write X_1' and X_2' (which may be equal) for the irreducible components of $\mathscr{X}_s$ containing x. Write X_1 and X_2 for the strict transforms of X_1' and X_2' under the birational morphism $g : \mathscr{X}^{\rm sst} \to \mathscr{X}$, respectively.

By the general theory of semi-stable curves, $g^{-1}(x)_{\rm red} \subseteq X_s^{\rm sst}$ is a semi-stable curve over s and $g^{-1}(x)_{\rm red} \cap D_{X_s^{\rm sst}} = \emptyset$. Moreover, the irreducible components of $g^{-1}(x)_{\rm red}$ are isomorphic to $\mathbb{P}^1_k$. Let C be the semi-stable subcurve of $g^{-1}(x)_{\rm red}$ which is a chain of projective lines $\bigcup_{i=1}^n P_i$ such that the following conditions are satisfied:

(i) For any $w, t \in \{1, \dots, n\}, P_w \cap P_t = \emptyset$ if $|w - t| \geq 2$, and $P_w \cap P_t$ is reduced to a point if $|w - t| = 1$;
(ii) $P_1 \cap X_1$ (resp. $P_n \cap X_2$) is reduced to a point.
(iii) $C \cap \overline{\{X_s^{\rm sst} \setminus g^{-1}(x)_{\rm red}\}} = (P_1 \cap X_1) \cup (P_n \cap X_2)$, where $\overline{\{X_s^{\rm sst} \setminus g^{-1}(x)_{\rm red}\}}$ denotes the topological closure of $X_s^{\rm sst} \setminus g^{-1}(x)_{\rm red}$ in $X_s^{\rm sst}$.

Then we have
$$g^{-1}(x)_{\rm red} = C \cup B,$$
where B denotes the topological closure of $g^{-1}(x)_{\rm red} \setminus C$ in $g^{-1}(x)_{\rm red}$. Note that $B \cap C$ are smooth points of C. Then Theorem 1.9 (or 3.2.7) implies that the p-rank of the connected components of $h^{-1}(B)$ are equal to 0. Thus, we have $\sigma(f^{-1}(x)) = \sigma(h^{-1}(C))$.

3.3.2. We introduce the following notation concerning inertia subgroups of irreducible components of vertical fibers.

Definition 3.5. We maintain the notation introduced above.

(a) Let $\mathcal{V}_x \overset{\rm def}{=} \{V_0, V_1, \dots, V_n, V_{n+1}\}$ be a set of irreducible components of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$. We shall call $\mathcal{V}_x$ a *collection of vertical fibers* associated to x if the following conditions are satisfied:

(i) $h(V_i) = P_i$ for $i \in \{1, \dots, n\}$.
(ii) $h(V_0) = X_1$ and $h(V_{n+1}) = X_2$.
(iii) The union $\bigcup_{i=0}^{n+1} V_i \subseteq \mathscr{Y}_s$ is a connected semi-stable subcurve of $\mathscr{Y}_s$ over s. Note that we have $(\bigcup_{i=1}^{n} V_i) \cap D_{Y_s} = \emptyset$.

Moreover, we write $I_{V_i} \subseteq G$, $i \in \{0, \dots, n+1\}$, for the inertia subgroup of V_i, and put
$$\mathcal{I}_{\mathcal{V}_x} \overset{\rm def}{=} \{I_{V_0}, \dots, I_{V_{n+1}}\}.$$
Note that Corollary 1.13 implies that either $I_{V_i} \subseteq I_{V_{i+1}}$ or $I_{V_i} \supseteq I_{V_{i+1}}$ holds for $i \in \{0, \dots, n\}$.

(b) Let $(u, w) \in \{0, \dots, n+1\} \times \{0, \dots, n+1\}$ be a pair such that $u \leq w$. We shall call that a group $I_{u,w}^{\rm min}$ is a *minimal element* of $\mathcal{I}_{\mathcal{V}_x}$ if one of the following conditions are satisfied, where "$\subset$" means that "is a subset which is not equal":

(i) $u = 0$, $w \neq 0$, $w \neq n+1$, and $I^{\min}_{0,w} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \subset I_{V_{w+1}}$.
(ii) $u \neq 0$, $w = n+1$, and $I_{V_{u-1}} \supset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = I^{\min}_{u,n+1}$.
(iii) $u \neq 0$, $w \neq n+1$, and $I_{V_{u-1}} \supset I^{\min}_{u,w} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \subset I_{V_{w+1}}$.

Note that we *do not* define $I^{\min}_{0,0}$. We shall call that a group $J^{\max}_{u,w}$ is a *maximal element* of $\mathcal{I}_{\mathcal{V}_x}$ if one of the following conditions are satisfied:

(i) $(u, w) = (0, n+1)$ and $J^{\max}_{0,n+1} = I_{V_i}$ for all $i \in \{0, \ldots, n+1\}$.
(ii) $u = 0, w \neq n+1$, and $J^{\max}_{0,w} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \supset I_{V_{w+1}}$.
(iii) $u \neq 0, w = n+1$, and $I_{V_{u-1}} \subset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = J^{\max}_{u,n+1}$.
(iv) $u \neq 0$, $w \neq n+1$, and $I_{V_{u-1}} \subset J^{\max}_{u,w} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \supset I_{V_{w+1}}$.

Moreover, we put

$$\mathcal{I}(x) \overset{\text{def}}{=} \bigsqcup_{I^{\min}_{u,w}\ :\ \text{a minimal element of } \mathcal{I}_{\mathcal{V}_x}} \{\# I^{\min}_{u,w}\},$$

$$\mathcal{J}(x) \overset{\text{def}}{=} \bigsqcup_{J^{\max}_{u,w}\ :\ \text{a maximal element of } \mathcal{I}_{\mathcal{V}_x}} \{\# J^{\max}_{u,w}\},$$

where $\sqcup$ means disjoint union.

Note that the set $\mathcal{I}(x)$ may be empty (e.g. if $I_{V_0} \subset I_{V_1} \subset \cdots \subset I_{V_{n+1}}$, then $\mathcal{I}(x)$ is empty). On the other hand, since $\# I_{V_i}$, $i \in \{0, \ldots, n+1\}$, does not depend on the choice of V_i (i.e. if $h(V_i) = h(V'_i)$ for irreducible components V_i, V'_i of $\mathscr{Y}_s$, then $\# I_{V_i} = \# I_{V'_i}$), $\mathcal{I}(x)$ and $\mathcal{J}(x)$ *do not depend on* the choice of $\mathcal{V}_x$.

We shall call $\mathcal{I}(x)$ *the set of minimal orders of inertia subgroups associated to x and f*, and $\mathcal{J}(x)$ *the set of maximal orders of inertia subgroups associated to x and f*, respectively.

3.3.3. We have the following lemmas.

Lemma 3.6. *We maintain the notation introduced above. Let $y_i \in V_i$ be a closed point and $I_{y_i} \subseteq G$, $i \in \{1, \ldots, n\}$ the inertia subgroup of y_i. Write* Ray_{V_i}, *$i = 1, \ldots, n$, for the set of the closed points $h^{-1}(C \cap B)_{\mathrm{red}} \cap V_i$. Then we have $I_{y_i} = I_{V_i}$ for any $y_i \in$* Ray_{V_i}.

Proof. Since $I_{y_i} \supseteq I_{V_i}$, we only need to prove that $I_{y_i} \subseteq I_{V_i}$. Note that I_{V_i} is a normal subgroup of I_{y_i}. To verify the lemma, by replacing G and $\mathscr{X}^{\mathrm{sst}}$ by I_{y_i} and $\mathscr{Y}/I_{y_i}$, respectively, we may assume $G = I_{y_i}$. Then we have $\# h^{-1}(h(y_i))_{\mathrm{red}} = 1$.

We consider the quotient $\mathscr{Y}/I_{V_i}$. By [R, Appendice Corollaire], we have that $\mathscr{Y}/I_{V_i}$ is a pointed semi-stable curve over S. Write $h_{I_{V_i}}$ for the quotient morphism $\mathscr{Y} \to \mathscr{Y}/I_{V_i}$ and $g_{I_{V_i}}$ for the morphism $\mathscr{Y}/I_{V_i} \to \mathscr{X}^{\mathrm{sst}}$ induced by h such that $h = g_{I_{V_i}} \circ h_{I_{V_i}}$. Write E_{y_i} for the connected component of $h^{-1}(B)_{\mathrm{red}}$ which contains y_i. By contracting $h_{I_{V_i}}(E_{y_i}) \subset \mathscr{Y}/I_{V_i} \times_S s$ (resp. $h(E_{y_i}) \subset \mathscr{X}^{\mathrm{sst}}_s$) ([BLR, 6.7 Proposition 4]), we obtain a fiber surface $(\mathscr{Y}/I_{V_i})^{\mathrm{c}}$ and a semi-stable curve $(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ over S. Moreover, we have contracting morphisms as follows:

$$c_{h_{I_{V_i}}(E_{y_i})} : \mathscr{Y}/I_{V_i} \to (\mathscr{Y}/I_{V_i})^{\mathrm{c}},\ c_{h(E_{y_i})} : \mathscr{X}^{\mathrm{sst}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}.$$

Furthermore, we obtain a morphism of fiber surfaces

$$g^{\mathrm{c}}_{I_{V_i}} : (\mathscr{Y}/I_{V_i})^{\mathrm{c}} \to (\mathscr{X}^{\mathrm{sst}})^{c}$$

induced by $g_{I_{V_i}}$ such that $c_{h(E_{y_i})} \circ g_{I_{V_i}} = g^{\mathrm{c}}_{I_{V_i}} \circ c_{h_{I_{V_i}}(E_{y_i})}$. Note that $(c_{h(E_{y_i})} \circ h)(y_i)$ is a smooth point of the special fiber of $(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$, and $g^{\mathrm{c}}_{I_{V_i}}$ is étale at the generic point of $(c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}})(V_i)$.

We put $y_i^{\mathrm{c}} \overset{\mathrm{def}}{=} (c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}})(y_i) \in (\mathscr{Y}/I_{V_i})^{\mathrm{c}}$ and $x_i^{\mathrm{c}} \overset{\mathrm{def}}{=} (c_{h(E_{y_i})} \circ h)(y_i) \in (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$. Consider the local morphism

$$g_{y_i^{\mathrm{c}}} : \operatorname{Spec} \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \to \operatorname{Spec} \mathcal{O}_{(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}, x_i^{\mathrm{c}}}$$

induced by $g_{I_{V_i}}^{\mathrm{c}}$. Note that [R, Proposition 1] implies that $\operatorname{Spec} \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \times_S s$ is irreducible. Then $g_{y_i^{\mathrm{c}}}$ is generically étale at the generic point of $\operatorname{Spec} \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \times_S s$. Thus, the Zariski-Nagata purity theorem implies that $g_{y_i^{\mathrm{c}}}$ is étale.

If $I_{V_i} \neq I_{y_i}$, then $g_{y_i^{\mathrm{c}}}$ is not an identity. Namely, we have $\#h^{-1}(h(y_i))_{\mathrm{red}} \neq 1$. This contradicts our assumption. Then we obtain $I_{V_i} = I_{y_i}$. We complete the proof of the lemma. □

Lemma 3.7. *We maintain the notation introduced in above. Then we have*

$$G = \langle I_{V_0}, I_{V_{n+1}} \rangle,$$

where $\langle I_{V_0}, I_{V_{n+1}} \rangle$ denotes the subgroup of G generated by I_{V_0} and $I_{V_{n+1}}$.

Proof. Suppose that $G \neq \langle I_{V_0}, I_{V_{n+1}} \rangle$. Since G is a p-group, there exists a normal subgroup $H \subseteq G$ of index p such that $\langle I_{V_0}, I_{V_{n+1}} \rangle \subseteq H$. Write $\mathscr{Y}'$ for the normalization of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f. The normalization $\mathscr{Y}'$ admits an action of G induced by the action of G on $\mathscr{Y}$. Consider the quotient $\mathscr{Y}'/H$. Then we obtain a morphism of fiber surfaces $f_H : \mathscr{Y}'/H \to \mathscr{X}$ over S induced by f. Moreover, $\mathscr{Y}'/H$ admits an action of $G/H \cong \mathbf{Z}/p\mathbf{Z}$ induced by the action of G on $\mathscr{Y}'$. Then f_H is generically étale over X_1' and X_2'. Thus, [T2, Lemma 2.1 (iii)] implies that f_H is étale above x. Then $f^{-1}(x)$ is not connected. This contradicts our assumptions. We complete the proof of the lemma. □

3.3.4. We define pointed semi-stable curves over s as follows:

$$\mathscr{C}_Y \overset{\mathrm{def}}{=} (h^{-1}(C)_{\mathrm{red}}, h^{-1}((C \cap X_1) \cup (C \cap X_2))),$$

$$\mathscr{C}_X \overset{\mathrm{def}}{=} (C, (C \cap X_1) \cup (C \cap X_2)).$$

Moreover, we have a natural morphism of pointed semi-stable curves

$$\rho_{\mathscr{C}_Y/\mathscr{C}_X} : \mathscr{C}_Y \to \mathscr{C}_X$$

over s induced by $h : \mathscr{Y} \to \mathscr{X}^{\mathrm{sst}}$. Since $f^{-1}(x)_{\mathrm{red}}$ is connected, $\mathscr{C}_Y$ admits a natural action of G induced by the action of G on $f^{-1}(x)_{\mathrm{red}}$. Write $\Gamma_{\mathscr{C}_Y}$ and $\Gamma_{\mathscr{C}_X}$ for the dual semi-graphs of $\mathscr{C}_Y$ and $\mathscr{C}_X$, respectively. Proposition 1.7 implies that the map of semi-graphs

$$\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \to \Gamma_{\mathscr{C}_X}$$

induced by $\rho_{\mathscr{C}_Y/\mathscr{C}_X}$ is a morphism of semi-graphs.

3.3.5. *Semi-graphs with p-rank associated to vertical fibers over singular vertical points.* Let $v \in v(\Gamma_{\mathscr{C}_Y})$ and $\widetilde{Y}_v$ the normalization of the irreducible component $Y_v \subseteq \mathscr{C}_Y$ corresponding to v. We define semi-graphs with p-rank associated to $\mathscr{C}_Y$ and $\mathscr{C}_X$, respectively, as follows:

$$\mathfrak{C}_Y \overset{\mathrm{def}}{=} (\mathbb{C}_Y, \sigma_{\mathfrak{C}_Y}),\ \mathfrak{C}_X \overset{\mathrm{def}}{=} (\mathbb{C}_X, \sigma_{\mathfrak{C}_X}),$$

where $\mathbb{C}_Y \overset{\mathrm{def}}{=} \Gamma_{\mathscr{C}_Y}$, $\mathbb{C}_X \overset{\mathrm{def}}{=} \Gamma_{\mathscr{C}_X}$, $\sigma_{\mathfrak{C}_Y}(v) \overset{\mathrm{def}}{=} \sigma(\widetilde{Y}_v)$ for $v \in v(\mathbb{C}_Y)$, and $\sigma_{\mathfrak{C}_X}(w) \overset{\mathrm{def}}{=} 0$ for $w \in v(\mathbb{C}_X)$.

3.3.6. *G-coverings of semi-graphs with p-rank associated to vertical fibers over singular vertical points.* The morphism of dual semi-graphs $\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \to \Gamma_{\mathscr{C}_X}$ induces a morphism of semi-graphs with p-rank

$$\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \to \mathfrak{C}_X.$$

Moreover, by Lemma 3.6, we see that $\sigma_{\mathfrak{C}_Y}(v)$ satisfies the Deuring-Shafarevich type formula for $v \in v(\mathbb{C}_Y)$. This implies that $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X}$ is a G-covering of semi-graphs with p-rank. Note that by the above construction, $\mathfrak{C}_X$ is an *n-chain* (Definition 2.9). Furthermore, we have

$$\sigma(\mathfrak{C}_Y) = \sigma(h^{-1}(C)) = \sigma(f^{-1}(x)).$$

Summarizing the discussion above, we obtain the following proposition.

Proposition 3.8. *We maintain the notation introduced above. Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and $x \in \mathscr{X}_s$ a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected, and that x is a node of $\mathscr{X}_s$. Then there exists a G-covering of semi-graphs with p-rank $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \to \mathfrak{C}_X$ associated to f and x (which is constructed above) such that $\mathfrak{C}_X$ is an n-chain and $\sigma(\mathfrak{C}_Y) = \sigma(f^{-1}(x))$.*

3.3.7. Then we have the following theorem.

Theorem 3.9. *We maintain the settings introduced in 1.3.1. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering (Definition 1.5) over S and $x \in \mathscr{X}_s$ a vertical point (Definition 1.8) associated to f. Suppose that $f^{-1}(x)$ is connected, and that x is a node of $\mathscr{X}_s$. Let $\mathcal{I}(x)$ and $\mathcal{J}(x)$ be the sets of minimal and maximal orders of inertia subgroups associated to x and f (Definition 3.5 (b)), respectively. Then we have*

$$\sigma(f^{-1}(x)) = \sum_{\#I \in \mathcal{I}(x)} \#G/\#I - \sum_{\#J \in \mathcal{J}(x)} \#G/\#J + 1.$$

Proof. Let $\mathcal{V}_x$ be a collection of vertical fibers associated to x (Definition 3.5 (a)). By Proposition 3.8, Corollary 2.10, and Lemma 1.10, we have

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \#G/\#I_{V_i} - \sum_{i=1}^{n+1} \#G/\#\langle I_{V_{i-1}}, I_{V_i}\rangle + 1,$$

where $\langle I_{V_{i-1}}, I_{V_i}\rangle$ denotes the subgroup of G generated by $I_{V_{i-1}}$ and I_{V_i}. The theorem follows from Corollary 1.13 and Lemma 3.7. □

3.3.8. In the remainder of the present subsection, we suppose that G is a cyclic p-group. We show that the formula of Theorem 3.9 coincides with the formula of Saïdi ([S1, Proposition 1]). Since G is an abelian group, I_{V_i}, $i = \{0, \dots, n+1\}$, does not depend on the choice of V_i. Then we may use the notation I_{P_i}, $i \in \{0, \dots, n+1\}$, to denote I_{V_i}.

Lemma 3.10. *We maintain the notation introduced above. If G is a cyclic p-group, then there exists $0 \leq u \leq n+1$ such that*

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_u} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$

Proof. If the lemma is not true, then there exist w, t and v such that $I_{P_v} \neq I_{P_w}$, $I_{P_v} \neq I_{P_t}$ and $I_{P_w} \subset I_{P_{w+1}} = \cdots = I_{P_v} = \cdots = I_{P_{t-1}} \supset I_{P_t}$. Since G is a cyclic group, we may assume $I_{P_w} \supseteq I_{P_t}$. Consider the quotient of $\mathscr{Y}$ by I_{P_w}, we obtain a natural morphism of pointed semi-stable curves $h_w : \mathscr{Y}/I_{P_w} \to \mathscr{X}^{\mathrm{sst}}$ over S.

We define B_j, $j = \{0, \dots, n+1\}$, to be the union of the connected components of B (3.3.1) which intersect with P_j non-trivially. By contracting ([BLR, 6.7 Proposition 4])

$$P_{w+1}, \dots, P_{t-1}, B_{w+1}, \dots, B_{t-1},$$

$$(h_w)^{-1}(P_{w+1})_{\rm red}, \dots, (h_w)^{-1}(P_{t-1})_{\rm red}, (h_w)^{-1}(B_{w+1})_{\rm red}, \dots, (h_w)^{-1}(B_{t-1})_{\rm red},$$

respectively, we obtain a pointed semi-stable curve $(\mathscr{X}^{\rm sst})^{\rm c}$ and a fiber surface $(\mathscr{Y}/I_{P_w})^{\rm c}$ over S. Write

$$c_{\mathscr{X}^{\rm sst}} : \mathscr{X}^{\rm sst} \to (\mathscr{X}^{\rm sst})^{\rm c}, \ c_{\mathscr{Y}/I_{P_w}} : \mathscr{Y}/I_{P_w} \to (\mathscr{Y}/I_{P_w})^{\rm c}$$

for the resulting contracting morphisms, respectively. The morphism h_w induces a morphism of fiber surfaces $h_w^{\rm c} : (\mathscr{Y}/I_{P_w})^{\rm c} \to (\mathscr{X}^{\rm sst})^{\rm c}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{Y}/I_{P_w} & \xrightarrow{c_{\mathscr{Y}/I_{P_w}}} & (\mathscr{Y}/I_{P_w})^{\rm c} \\ {\scriptstyle h_w}\downarrow & & {\scriptstyle h_w^{\rm c}}\downarrow \\ \mathscr{X}^{\rm sst} & \xrightarrow{c_{\mathscr{X}^{\rm sst}}} & (\mathscr{X}^{\rm sst})^{\rm c}. \end{array}$$

Write $P_w^{\rm c}$ and $P_t^{\rm c}$ for the images $c_{\mathscr{X}^{\rm sst}}(P_w)$ and $c_{\mathscr{X}^{\rm sst}}(P_t)$, respectively, and $x_{wt}^{\rm c}$ for the closed point $P_w^{\rm c} \cap P_t^{\rm c}$. Since $h_w^{\rm c}$ is generically étale above $P_w^{\rm c}$ and $P_t^{\rm c}$, [T2, Lemma 2.1 (iii)] implies that $(h_w^{\rm c})^{-1}(x_{wt}^{\rm c})_{\rm red}$ are nodes. Thus, $(\mathscr{Y}/I_{P_w})^{\rm c}$ is a semi-stable curve over S, and moreover, $h_w^{\rm c}$ is étale over $x_{wt}^{\rm c}$. Then the inertia subgroups of the closed points $(h_w^{\rm c})^{-1}(x_{wt}^{\rm c})_{\rm red}$ of the special fiber $(\mathscr{Y}/I_{P_w})_s^{\rm c}$ of $(\mathscr{Y}/I_{P_w})^{\rm c}$ are trivial.

On the other hand, since I_{P_w} is a proper subgroup of I_{P_v}, we have that the inertia subgroups of the irreducible components of $h_w^{-1}(\bigcup_{j=w+1}^{t-1} P_j)_{\rm red}$ is I_{P_v}/I_{P_w}. Thus, the inertia subgroups of the closed points $c_{\mathscr{Y}/I_{P_w}}(h_w^{-1}(\bigcup_{j=w+1}^{t-1} P_j)_{\rm red}) = (h_w^{\rm c})^{-1}(x_{wt}^{\rm c})_{\rm red} \subseteq (\mathscr{Y}/I_{P_w})_s^{\rm c}$ are not trivial. This is a contradiction. Then we complete the proof of the lemma. □

The above lemma implies the following corollary.

Corollary 3.11. *We maintain the settings introduced in Theorem 3.9. Suppose that G is a cyclic p-group, and that I_{P_0} is equal to G. Then we have*

$$\sigma(f^{-1}(x)) = \#G/\#I_{\rm min} - \#G/\#I_{P_{n+1}},$$

where $I_{\rm min}$ denotes the group $\bigcap_{i=0}^{n+1} I_{P_i}$.

Proof. The corollary follows immediately from Theorem 3.9 and Lemma 3.10. □

Remark 3.11.1. The formula in Corollary 3.11 had been obtained by Saïdi ([S1, Proposition 1]). On the other hand, Corollary 3.11 implies immediately that

$$\sigma(f^{-1}(x)) \leq \#G - 1$$

when G is a cyclic p-group, which is the main theorem of [S1] (i.e. [S1, Theorem 1]).

4. Bounds of p-rank of vertical fibers

In this section, we gives an affirmative answer to an open problem posed by Saïdi concerning bounds of p-rank of vertical fibers posed by Saïdi if G is an arbitrary finite *abelian* p-group. The main result of the present section is Theorem 4.3.

4.0.1. The following was asked by Saïdi ([S1, Question]):

Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a *G-semi-stable covering* (i.e. $D_X = \emptyset$, see Definition 1.5) over S and $x \in \mathscr{X}_s$ a vertical point (Definition 1.8) associated to f. Suppose that $f^{-1}(x)$ is connected. Whether or not $\sigma(f^{-1}(x))$ can be bounded by a constant which depends only on $\#G$?

The above problem was solved by Saïdi when G is a cyclic p-group (Remark 3.11.1).

4.0.2. **Settings.** We maintain the settings introduced in 1.3.1 and assume that $\mathscr{X}$ is a *stable curve* over S (i.e. $D_X = \emptyset$). Moreover, when x is a *node* of $\mathscr{X}_s^{\mathrm{sst}}$, let $\mathcal{V}_x$ be a collection of vertical fibers (Definition 3.5) and $\mathcal{I}_{\mathcal{V}_x} \overset{\mathrm{def}}{=} \{I_{V_i} \subseteq G\}_{i=\{0,\dots,n+1\}}$ the set of inertia subgroups of V_i (Definition 3.5). Furthermore, in the remainder of the present section, *we assume that G is a finite abelian p-group.*

4.0.3. Since G is abelian, I_{V_i}, $\{i \in \{0,\dots,n+1\}$, does not depend on the choice of V_i. We use the notation I_{P_i} to denote I_{V_i} for $i \in \{0,\dots,n+1\}$. Then we have the following proposition.

Proposition 4.1. *Let I' and I'' be minimal elements of $\mathcal{I}_{\mathcal{V}_x}$ (Definition 3.5 (b)) distinct from each other. Then neither $I' \subseteq I''$ nor $I' \supseteq I''$ holds.*

Proof. Without loss of generality, we may assume that $I' = I_{P_a}$ and $I'' = I_{P_b}$ such that $0 \leq a < b \leq n+1$, $I_{P_a} \neq I_{P_{a+1}}$, and $I_{P_{b-1}} \neq I_{P_b}$. Note that by the definition of minimal elements (Definition 3.5 (b)), $I_{P_{a+1}}$ (resp. $I_{P_{b-1}}$) contains I_{P_a} (resp. I_{P_b}).

If $I' \subseteq I''$, we consider the quotient curve $\mathscr{Y}/I''$. Then we obtain morphisms of semi-stable curves $\xi_1 : \mathscr{Y} \to \mathscr{Y}/I''$ and $\xi_2 : \mathscr{Y}/I'' \to \mathscr{X}^{\mathrm{sst}}$ such that $\xi_2 \circ \xi_1 = h$. Note that $h(V_a) = P_a$ and $h(V_b) = P_b$, respectively. By contracting $\bigcup_{i=a+1}^{b-1} P_i$ and $\xi_2^{-1}(\bigcup_{i=a+1}^{b-1} P_i)_{\mathrm{red}}$ ([BLR, 6.7 Proposition 4]), respectively, we obtain contracting morphisms $c_{\mathscr{X}^{\mathrm{sst}}} : \mathscr{X}^{\mathrm{sst}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ and $c_{\mathscr{Y}/I''} : \mathscr{Y}/I'' \to (\mathscr{Y}/I'')^{\mathrm{c}}$, respectively. Moreover, ξ_2 induces a morphism $\xi_2^{\mathrm{c}} : (\mathscr{Y}/I'')^{\mathrm{c}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{Y}/I'' & \xrightarrow{c_{\mathscr{Y}/I''}} & (\mathscr{Y}/I'')^{\mathrm{c}} \\ \xi_2 \downarrow & & \xi_2^{\mathrm{c}} \downarrow \\ \mathscr{X}^{\mathrm{sst}} & \xrightarrow{c_{\mathscr{X}^{\mathrm{sst}}}} & (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}. \end{array}$$

Note that $(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ is a semi-stable curve over S.

Since $I' = I_{P_a} \subseteq I'' = I_{P_b}$, ξ_2^{c} is étale at the generic points of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a)$ and $c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$. Thus, by applying the Zariski-Nagata purity theorem and [T2, Lemma 2.1 (iii)], we obtain that ξ_2^{c} is étale at $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ (i.e. the inertia group of each point of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ is trivial). On the other hand, since $I_{P_{b-1}}$ contains I_{P_b}, the inertia group of each point of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ is $I_{P_{b-1}}/I''$. Then we obtain $I_{P_{b-1}} = I''$. This is a contradiction. Then I'' does not contain I'.

Similar arguments to the arguments given in the proof above imply that I' does not contain I''. We complete the proof of the proposition. □

4.0.4. Let N be a finite p-group and H a subgroup of N. Write $\mathrm{Sub}(-)$ for the set of the subgroups of $(-)$. We put

$$\#I(H) \overset{\mathrm{def}}{=} \max\{\#\mathcal{S} \mid \mathcal{S} \subseteq \mathrm{Sub}(N), H \in \mathcal{S}, \text{ for any } H', H'' \in \mathcal{S} \text{ such that } H' \neq H'', \\ \text{neither } H' \subseteq H'' \text{ nor } H' \supseteq H'' \text{ holds}\}.$$

Moreover, we put

$$M(N) \stackrel{\text{def}}{=} \max\{\#I(N')\}_{N'\in \mathrm{Sub}(N)}.$$

For any $1 \leq d \leq \#N$, write $S_d(N)$ for the set of the subgroups of N with order d. Let A be an elementary abelian p-group (i.e. $pA = 0$) such that $\#A = \#N$. We put

$$B(\#N) \stackrel{\text{def}}{=} \#\mathrm{Sub}(A).$$

Note that $B(\#N)$ depends only on $\#N$.

4.0.5. We need a lemma of finite groups.

Lemma 4.2. *Let N be a finite p-group, A an elementary abelian p-group with order $\#N$, and $1 \leq d \leq \#N$ an integer number. Then we have $\#S_d(N) \leq \#S_d(A)$. In particular, we have $M(N) \leq B(\#N)$.*

Proof. Since N is a p-group, N has a non-trivial central subgroup. Fix a central subgroup Z of order p in N. Write $S_d^Z(N)$ (resp. $S_d^{\setminus Z}(N)$) for the set of subgroups of N of order d which contain Z (resp. do not contain Z). If H is a subgroup of N/Z, let $S_d^{(Z,H)}(N)$ be the set of $L \in S_d^{\setminus Z}(N)$ whose projection on N/Z is H. Let $S_d[N/Z]$ be the set of $H \in S_d(N/Z)$ for which $S_d^{(Z,H)}(N) \neq \emptyset$.

Let $H \in S_d[N/Z]$. Then we obtain $\#S_d^{(Z,H)}(N) \leq \#H^1(H, Z) = \#\mathrm{Hom}(H^{\mathrm{ab},p}, Z)$, where $(-)^{\mathrm{ab},p}$ denotes $(-)/((-)^p[(-),(-)])$. Moreover, let H' be a subgroup of A of order d and $Z' \cong \mathbf{Z}/p\mathbf{Z}$ a subgroup of A of order p. Then we have

$$\#\mathrm{Hom}(H^{\mathrm{ab},p}, Z) \leq \#\mathrm{Hom}((H')^{\mathrm{ab},p}, Z').$$

If $d = 1$, the lemma is trivial. Then we may assume that p divides d. We have

$$\begin{aligned}
\#S_d(N) &= \#S_d^Z(N) + \#S_d^{\setminus Z}(N) = \#S_{d/p}(N/Z) + \#S_d^{\setminus Z}(N) \\
&= \#S_{d/p}(N/Z) + \sum_{H\in S_d[N/Z]} \#S_d^{(Z,H)}(N) \\
&\leq \#S_{d/p}(N/Z) + \sum_{H\in S_d[N/Z]} \#(\mathrm{Hom}(H^{\mathrm{ab},p}, Z)) \\
&\leq \#S_{d/p}(N/Z) + \#S_d(N/Z)\#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z'))
\end{aligned}$$

By induction, we have $\#S_{d/p}(N/Z) \leq \#S_{d/p}(A/Z')$ and $\#S_d(N/Z) \leq \#S_d(A/Z')$. Moreover, we have

$$\begin{aligned}
\#S_d(A) &= \#S_{d/p}(A/Z') + \sum_{H'\in S_d[A/Z']} \#S_d^{(Z',H')}(A) \\
&= \#S_{d/p}(A/Z') + \sum_{H'\in S_d[A/Z']} \#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z')) \\
&= \#S_{d/p}(A/Z') + \#S_d(A/Z')\#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z')).
\end{aligned}$$

Thus, we obtain

$$\#S_d(N) \leq \#S_d(A).$$

This completes the proof of the lemma. □

4.0.6. We have the following result.

Theorem 4.3. *We maintain the settings introduced in 1.3.1. Let G be a finite p-group, and let $f : \mathscr{Y} \to \mathscr{X}$ be a G-semi-stable covering (i.e. $D_X = \emptyset$, see Definition 1.5) over S and $x \in \mathscr{X}_s$ a vertical point (Definition 1.8) associated to f. Suppose that $f^{-1}(x)$ is connected, and that G is an abelian p-group. Then we have (see Definition 4.0.4 for $M(G)$, $B(\#G)$)*

$$\sigma(f^{-1}(x)) \leq M(G)\#G - 1 \leq B(\#G)\#G - 1.$$

In particular, if G is an abelian p-group, then the p-rank $\sigma(f^{-1}(x))$ can be bounded by a constant $B(\#G)$ which depends only on $\#G$.

Proof. If x is a smooth point of the special fiber $\mathscr{X}_s$ of $\mathscr{X}$, then $\sigma(f^{-1}(x)) = 0$ (Theorem 1.9). Thus, we may assume that x is a singular point of $\mathscr{X}_s$.

If $\mathcal{I}(x) = \emptyset$ (Definition 3.5 (b)), then Theorem 3.9 implies that $\sigma(f^{-1}(x)) = 0$. If $\mathcal{I}(x) \neq \emptyset$, by applying Theorem 3.9 and Proposition 4.1, we obtain

$$\sigma(f^{-1}(x)) = \sum_{I \in \#I \in \mathcal{I}(x)} \#G/\#I - \sum_{\#J \in \mathcal{J}(x)} \#G/\#J + 1$$
$$\leq \#\mathcal{I}\#G - 1 \leq M(G)\#G - 1 \leq B(\#G)\#G - 1.$$

□

Remark 4.3.1. If G is a cyclic p-group, then by the definition of $M(G)$, we have $M(G) = 1$. Thus, if G is a cyclic p-group, we have $\sigma(f^{-1}(x)) \leq \#G - 1$. This is the main theorem of [S1, Theorem 1].

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