p-Rank and Semi-stable Reduction of Coverings of Curves

Yu Yang

Abstract

In the present paper, we investigate the p-rank of coverings of curves. Let G be a finite p-group, $f : \mathscr{Y} \to \mathscr{X}$ a morphism of pointed semi-stable curves over a complete discrete valuation ring R with algebraically closed residue field of characteristic $p > 0$, and x a closed point of $\mathscr{X}$. Write η for the generic point of $S \overset{\text{def}}{=} \operatorname{Spec} R$, and s for the closed point of S. Suppose that the generic fiber $\mathscr{X}_\eta$ of $\mathscr{X}$ is a smooth pointed stable curve over η, and that the morphism $f_\eta : \mathscr{Y}_\eta \to \mathscr{X}_\eta$ induced by f on generic fibers is a Galois covering whose Galois group is isomorphic to G, and whose branch locus is contained in the set of marked points of $\mathscr{X}_\eta$. We give an explicit formula for the p-rank $\sigma(\mathscr{Y}_s)$ of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ and an explicit formula for the p-rank $\sigma(f^{-1}(x))$ of the vertical fiber $f^{-1}(x)$ associated to x. In particular, the formula for the p-rank $\sigma(f^{-1}(x))$ generalizes results of M. Raynaud and M. Saïdi concerning $\sigma(f^{-1}(x))$ to the case of arbitrary p-groups and arbitrary non-marked closed point $x \in \mathscr{X}$. Moreover, as an application, if G is an abelian p-group, we give an affirmative answer to a problem concerning bounds of $\sigma(f^{-1}(x))$ which was asked by Saïdi.

Keywords: p-rank, semi-stable reduction, pointed semi-stable curve, pointed semi-stable covering, Deuring-Shafarevich formula.

Mathematics Subject Classification: Primary 14H30; Secondary 11G20.

Contents

1 Introduction

Let C be a smooth projective curve over an algebraically closed field of characteristic $p > 0$. There are two natural invariants associated to C: the genus $g(C)$ and the p-rank

$$\sigma(C) \stackrel{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(C, \mathbb{F}_p)).$$

These two invariants determine, respectively, the isomorphism classes (as profinite groups) of the maximal pro-Σ and pro-p quotients of the étale fundamental group $\pi_1(C)$ of C, for Σ a set of prime numbers which does not contain p. The genus and p-rank have some similar properties. Let G be a finite group and $h : C' \to C$ a G-Galois covering of smooth projective curves (i.e., the extension of function fields $K(C')/K(C)$ induced by h is a Galois extension with Galois group G). The genus $g(C')$ of C' can be calculated by the Riemann-Hurwitz formula. In particular, if $(\#G, p) = 1$, then the Riemann-Hurwitz formula has the following form:

$$g(C') = \#G(g(C) - 1) + \sum_{c' \in (C')^{\text{cl}}} (e_{c'} - 1)/2 + 1,$$

where $(C')^{\text{cl}}$ denotes the set of the closed points of C', $e_{c'}$ denotes the ramification index at c', and $\#G$ denotes the order of G. If G is a p-group, then we have the Deuring-Shafarevich formula (cf. [C]) for the p-rank $\sigma(C')$, as follows:

$$\sigma(C') = \#G(\sigma(C) - 1) + \sum_{c' \in (C')^{\text{cl}}} (e_{c'} - 1) + 1,$$

where $(C')^{\text{cl}}$ denotes the set of the closed points of C', $e_{c'}$ denotes the ramification index at c', and $\#G$ denotes the order of G. In the present paper, we study the geometry of coverings of curves over a complete discrete valuation ring and calculate the global and local p-rank of coverings.

Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$. Write K for the quotient field of R, $S \stackrel{\text{def}}{=} \operatorname{Spec} R$, $\eta : \operatorname{Spec} K \to S$ and $s : \operatorname{Spec} k \to S$ for the natural morphisms. Let G be a finite group and

$$\mathscr{X} = (X, D_X)$$

a pointed semi-stable curve of genus g_X over S. Here, X denotes the underlying curve of $\mathscr{X}$ and D_X denotes the set of marked points of $\mathscr{X}$. Write $\mathscr{X}_\eta = (X_\eta, D_{X_\eta})$ and

$\mathscr{X}_s = (X_s, D_{X_s})$ for the result of base-changing $\mathscr{X}$ by η and s, respectively. Moreover, we suppose that $\mathscr{X}_\eta$ is a smooth pointed stable curve over η.

Let $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ be a smooth pointed stable curve over η and $f_\eta : \mathscr{Y}_\eta \to \mathscr{X}_\eta$ a morphism of pointed stable curves over η. Suppose that f_η is a Galois covering whose Galois group is isomorphic to G, that $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$, and that the branch locus of f_η is contained in D_{X_η}. By replacing S by a finite extension of S (i.e., the spectrum of the normalization of R in a finite extension of K), f_η extends to a *G-pointed semi-stable covering*

$$f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$$

over S (cf. Definition 3.3 and Proposition 3.4). We are interested in understanding the structure of the special fiber $\mathscr{Y}_s = (Y_s, D_{Y_s})$ of $\mathscr{Y}$. If the order $\#G$ of G is prime to p, then by the specialization theorem for log étale fundamental groups, the morphism $f_s : \mathscr{Y}_s \to \mathscr{X}_s$ on special fibers induced by f is an admissible covering (cf. [V]); thus, $\mathscr{Y}_s$ may be obtained by gluing together tame coverings of the irreducible components of $\mathscr{X}_s$. On the other hand, if $p | \#G$, then f_s is not a finite morphism in general. If $f^{-1}(x)$ is not finite, then we shall say that x is a *vertical point associated to f*, and that $f^{-1}(x)$ is the *vertical fiber associated to x* (cf. Definition 3.6).

In order to investigate the properties of $\mathscr{Y}_s$, we focus on the p-rank $\sigma(\mathscr{Y}_s)$ of $\mathscr{Y}_s$ (cf. Definition 3.1 (b)). By the definition of the p-rank of a pointed semi-stable curve, to calculate $\sigma(\mathscr{Y}_s)$, it suffices to calculate $\dim_{\mathbf{C}}(H^1(\Gamma_{\mathscr{Y}_s}, \mathbf{C}))$ (where $\Gamma_{\mathscr{Y}_s}$ denotes the dual semi-graph of $\mathscr{Y}_s$ (cf. Definition 3.1 (a)), and $H^1(\Gamma_{\mathscr{Y}_s}, \mathbf{C})$ denotes the first singular cohomology group of $\Gamma_{\mathscr{Y}_s}$ with coefficients in the field of complex numbers $\mathbf{C}$), the p-rank of the irreducible components of $\mathscr{Y}_s$ which are finite over $\mathscr{X}_s$, and the p-rank of the vertical fibers of f. We consider the following question:

Question 1.1. *Let G be a finite p-group and $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semi-stable covering over S.*

(Global Version): Does there exist an explicit formula for the p-rank $\sigma(\mathscr{Y}_s)$ of $\mathscr{Y}_s$ in terms of the dual semi-graph of $\mathscr{X}_s$ and the inertia subgroups of the irreducible components and marked points of $\mathscr{Y}_s$?

(Local Version): Let x be a vertical point associated to f. Then does there exist an explicit formula for the p-rank $\sigma(f^{-1}(x))$ of $f^{-1}(x)$ in terms of the inertia subgroups of the irreducible components and marked points of $f^{-1}(x)$?

The main theorems of the present paper give an answer to Question 1.1. For (Global Version), we have the following theorem which can be regarded as *a relative version of the Deuring-Shafarevich formula* (cf. Theorem 4.5 for a more precise statement):

Theorem 1.2. *We have*

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \Big(\#G/\#I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v) \setminus e^{\mathrm{lp}}(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1 \Big)$$

$$+ \sum_{e \in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}) \setminus e^{\mathrm{lp}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\#G/\#I_e - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \mathbf{C})).$$

In particular, if $f : \mathscr{Y} \to \mathscr{X}$ is a G-semi-stable covering, then we have

$$\sigma(\mathscr{Y}_s) = \sum_{v\in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \Big(\#G/\#I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e\in e(v)\setminus e^{\mathrm{lp}}(v)} \#G/\#I_e^{\mathrm{m}}(\#I_e^{\mathrm{m}}/\#I_v - 1) + 1\Big)$$

$$+ \sum_{e\in e^{\mathrm{cl}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})\setminus e^{\mathrm{lp}}(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} (\#G/\#I_e^{\mathrm{m}} - 1) + \sum_{v\in v(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}})} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{\mathscr{X}_s^{\mathrm{sst}}}, \mathbf{C})).$$

For (Local Version), we may assume that $f^{-1}(x)$ is connected. Then we have the following theorem (cf. Theorem 4.6 and Theorem 4.8 for more precise statements):

Theorem 1.3. *Suppose that $f^{-1}(x)$ is connected. Then we have*

$$\sigma(f^{-1}(x)) = \sum_{v\in v(\Gamma_{\mathscr{E}_X})} \Big(-\#G/\#I_v + \sum_{e\in e(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1\Big)$$

$$+ \sum_{e\in e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})} (\#G/\#I_e - 1).$$

Moreover, if x is a singular point of $\mathscr{X}_s$, we have

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \#G/\#I_{V_i} - \sum_{i=1}^{n+1} \#G/\#\langle I_{V_{i-1}}, I_{V_i}\rangle + 1$$

$$= \sum_{I\in\mathcal{I}} \#G/\#I - \sum_{J\in\mathcal{J}} \#G/\#J + 1,$$

where $\langle I_{V_{i-1}}, I_{V_i}\rangle$ denotes the subgroup of G generated by $I_{V_{i-1}}$ and I_{V_i}.

Remark 1.3.1. If the vertical point x is not contained in D_{X_s}, (Local Version) had been studied by M. Raynaud and M. Saïdi. If x is a smooth point of $\mathscr{X}_s$ which is not contained in D_{X_s}, then Raynaud proved that $\sigma(f^{-1}(x)) = 0$ (cf. [R, Théorème and Proposition 2 (ii)]). If G is a cyclic p-group, and x is a singular point of $\mathscr{X}_s$, then an explicit formula has been obtained by Saïdi (cf. [S, Proposition 1]).

Remark 1.3.2. Theorem 1.3 generalizes Saïdi's formula to the case where G is an arbitrary p-group (cf. Corollary 4.10).

Remark 1.3.3. By applying his formula for $\sigma(f^{-1}(x))$, Saïdi obtained a bound of $\sigma(f^{-1}(x))$ when G is a cyclic p-group (cf. [S, Theorem 1]). In Section 5 of the present paper, by applying Theorem 1.3, we generalize Saïdi's result to the case where G is an arbitrary *abelian* p-group (cf. Theorem 5.4), which gives an affirmative answer to a problem asked by Saïdi (cf. [S, Question]).

The present paper is organized as follows. In Section 2, we introduce a kind of purely combinatorial object called a *semi-graph with p-rank*. We define p-rank, coverings, and G-coverings for semi-graphs with p-rank; then we calculate the p-rank of G-coverings of semi-graphs with p-rank. In Section 3, by using the theory of semi-stable curves, we prove that, for any G-pointed semi-stable covering, one may construct a G-covering of semi-graphs with p-rank associated to the G-pointed semi-stable covering (resp. the vertical fibers of the G-pointed semi-stable covering) in a natural way. In Section 4, we study the relationship between the inertia subgroups of irreducible components and the inertia subgroups of nodes of the special fiber of a G-pointed semi-stable covering; then together with the results obtained in Section 2 and Section 3, we obtain our main theorems. In Section 5, we give a bound of local p-rank of a G-pointed semi-stable covering by using the results obtained in Section 4 when G is an abelian p-group.

ACKNOWLEDGEMENTS

The main results of the present paper were obtained in April 2016. The author would like to thank Professor *Michel Raynaud* for his interest in this work and encouraging me to publish this paper. It is with deep regret and sadness to hear of his passing. The author would like to thank *the referee* very much for carefully reading the manuscript and for giving me many comments which substantially helped improving the quality of the paper. This research was supported by JSPS KAKENHI Grant Number 16J08847.

2 Semi-graphs with p-rank

In this section, we develop the theory of semi-graphs with p-rank. We always assume that G is a finite p-group with order p^r.

2.1 Definitions

We begin with some general remarks concerning semi-graphs (cf. [M]). A *semi-graph* $\mathbb{G}$ consists of the following data:

(i) A set $v(\mathbb{G})$ whose elements we refer to as vertices;

(ii) A set $e(\mathbb{G})$ whose elements we refer to as edges; moreover, any element $e \in e(\mathbb{G})$ is a set of cardinality 2 satisfying the following property: for each $e \neq e' \in e(\mathbb{G})$, we have $e \cap e' = \emptyset$;

(iii) A set of maps $\{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})}$ such that $\zeta_e^{\mathbb{G}} : e \to v(\mathbb{G}) \cup \{v(\mathbb{G})\}$ is a map from the set e to the set $v(\mathbb{G}) \cup \{v(\mathbb{G})\}$, and that $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) \in \{0, 1\}$, where $\#(-)$ denotes the cardinality of $(-)$.

Let $e \in e(\mathbb{G})$ be an edge of $\mathbb{G}$. We shall refer to an element $b \in e$ as a *branch* of the edge e. We shall say that $e \in e(\mathbb{G})$ is *closed* (resp. *open*) if $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) = 0$ (resp. $\#((\zeta_e^{\mathbb{G}})^{-1}(\{v(\mathbb{G})\})) = 1$). Moreover, write $v(\mathbb{G})$ for the set of vertices of $\mathbb{G}$, $e(\mathbb{G})$ for the set of edges, $e^{\mathrm{cl}}(\mathbb{G})$ for the set of closed edges of $\mathbb{G}$, and $e^{\mathrm{op}}(\mathbb{G})$ for the set of open edges of $\mathbb{G}$. Note that $e(\mathbb{G}) = e^{\mathrm{cl}}(\mathbb{G}) \cup e^{\mathrm{op}}(\mathbb{G})$. Let $v \in v(\mathbb{G})$ be a vertex of $\mathbb{G}$. Write $b(v)$ for the set of branches $\bigcup_{e\in e(\mathbb{G})}(\zeta_e^{\mathbb{G}})^{-1}(v)$ and $e(v)$ for the set of edges which abut to v. Write $v(e)$ for the set of vertices which are abutted by e. We shall say an edge $e \in e^{\mathrm{cl}}(\mathbb{G})$ *loop* if $\#v(e) = 1$. Moreover, we use the notation $e^{\mathrm{lp}}(v)$ to denote the set of loops which abut to v.

A semi-graph $\mathbb{G}$ will be called *finite* if $v(\mathbb{G})$ and $e(\mathbb{G})$ are finite. In the present paper, we only consider finite semi-graphs. Since a semi-graph can be regarded as a topological space (i.e., a subspace of $\mathbf{R}^2$), we shall say $\mathbb{G}$ a *connected* semi-graph if $\mathbb{G}$ is connected as a topological space.

Let $\mathbb{G}$ and $\mathbb{H}$ be two semi-graphs. A morphism between semi-graphs $\mathbb{G} \to \mathbb{H}$ is a collection of maps $v(\mathbb{G}) \to v(\mathbb{H})$, $e^{\mathrm{cl}}(\mathbb{G}) \to e^{\mathrm{cl}}(\mathbb{H})$, and $e^{\mathrm{op}}(\mathbb{G}) \to e^{\mathrm{op}}(\mathbb{H})$ satisfying the following: for each $e_{\mathbb{G}} \in e(\mathbb{G})$, write $e_{\mathbb{H}} \in e(\mathbb{H})$ for the image of $e_{\mathbb{G}}$; then the map $e_{\mathbb{G}} \overset{\sim}{\to} e_{\mathbb{H}}$ is a bijection, and is compatible with the $\{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G})}$ and $\{\zeta_e^{\mathbb{H}}\}_{e \in e(\mathbb{H})}$.

Definition 2.1. Let $\mathbb{G}$ be a semi-graph.

(a) We shall say $\mathbb{G}'$ a *sub-semi-graph* of $\mathbb{G}$ if $\mathbb{G}'$ is a semi-graph satisfying the following conditions:

(i) $v(\mathbb{G}')$ (resp. $e(\mathbb{G}')$) is a subset of $v(\mathbb{G})$ (resp. $e(\mathbb{G})$);

(ii) if $e \in e^{\mathrm{cl}}(\mathbb{G}')$, then $\zeta_e^{\mathbb{G}'}(e) \overset{\mathrm{def}}{=} \zeta_e^{\mathbb{G}}(e)$;

(iii) if $e = \{b_1, b_2\} \in e^{\mathrm{op}}(\mathbb{G}')$ such that $\zeta_e^{\mathbb{G}}(b_1) \in v(\mathbb{G}')$ and $\zeta_e^{\mathbb{G}}(b_2) \notin v(\mathbb{G}')$, then $\zeta_e^{\mathbb{G}'}(b_1) = \zeta_e^{\mathbb{G}}(b_1)$ and $\zeta_e^{\mathbb{G}'}(b_2) = \{v(\mathbb{G}')\}$.

(b) Let $\mathbb{G}'$ be a sub-semi-graph of $\mathbb{G}$. We define a semi-graph $\mathbb{G} \setminus \mathbb{G}'$ as follows:

(i) $v(\mathbb{G} \setminus \mathbb{G}') \overset{\mathrm{def}}{=} v(\mathbb{G}) \setminus v(\mathbb{G}')$;

(ii) $e^{\mathrm{cl}}(\mathbb{G} \setminus \mathbb{G}') \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\mathbb{G}) \mid v(e) \subseteq v(\mathbb{G} \setminus \mathbb{G}') \text{ in } \mathbb{G}\}$;

(iii) $e^{\mathrm{op}}(\mathbb{G} \setminus \mathbb{G}') \overset{\mathrm{def}}{=} \{e \in e^{\mathrm{cl}}(\mathbb{G}) \mid v(e) \cap v(\mathbb{G}') \neq \emptyset \text{ in } \mathbb{G} \text{ and } v(e) \cap v(\mathbb{G} \setminus \mathbb{G}') \neq \emptyset \text{ in } \mathbb{G}\} \cup \{e \in e^{\mathrm{op}}(\mathbb{G}) \mid v(e) \cap v(\mathbb{G} \setminus \mathbb{G}') \neq \emptyset \text{ in } \mathbb{G}\}$;

(iv) for each $e = \{b_i\}_{i \in \{1,2\}} \in e^{\mathrm{cl}}(\mathbb{G} \setminus \mathbb{G}') \cup e^{\mathrm{op}}(\mathbb{G} \setminus \mathbb{G}')$, we put $\zeta_e^{\mathbb{G} \setminus \mathbb{G}'}(b_i) \overset{\mathrm{def}}{=} \zeta_e^{\mathbb{G}}(b_i)$ (resp. $\zeta_e^{\mathbb{G} \setminus \mathbb{G}'}(b_i) \overset{\mathrm{def}}{=} \{v(\mathbb{G} \setminus \mathbb{G}')\}$) if $\zeta_e^{\mathbb{G}}(b_i) \notin v(\mathbb{G}')$ and $\zeta_e^{\mathbb{G}}(b_i) \neq \{v(\mathbb{G})\}$ (resp. $\zeta_e^{\mathbb{G}}(b_i) \in v(\mathbb{G}')$ or $\zeta_e^{\mathbb{G}}(b_i) = \{v(\mathbb{G})\}$).

Definition 2.2. (a) Let $\mathbb{G}$ be a semi-graph and $\sigma_{\mathfrak{G}} : v(\mathbb{G}) \to \mathbb{Z}$ a map. We shall say that the pair $\mathfrak{G} \overset{\mathrm{def}}{=} (\mathbb{G}, \sigma_{\mathfrak{G}})$ is a *semi-graph with p-rank*. We shall say that the semi-graph $\mathbb{G}$ is the underlying semi-graph of $\mathfrak{G}$, and that the map $\sigma_{\mathfrak{G}}$ is the p-rank map of $\mathfrak{G}$.

(b) Let $\mathfrak{G} \overset{\mathrm{def}}{=} (\mathbb{G}, \sigma_{\mathfrak{G}})$ be a semi-graph with p-rank. We define the *p-rank* of $\mathfrak{G}$ to be

$$\sigma_{\mathfrak{G}}(\mathfrak{G}) \overset{\mathrm{def}}{=} \sum_{v \in v(\mathbb{G})} \sigma_{\mathfrak{G}}(v) + \sum_{\mathbb{G}_i \in \pi_0(\mathbb{G})} \dim_{\mathbf{C}}(H^1(\mathbb{G}_i, \mathbf{C})),$$

where $\pi_0(-)$ denotes the set of the connected components of $(-)$, and $\mathbf{C}$ denotes the field of complex numbers.

(c) A semi-graph with p-rank is called *connected* if the underlying semi-graph $\mathbb{G}$ is a connected semi-graph.

(d) A morphism between two semi-graphs with p-rank

$$\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$$

is defined by a morphism of the underlying semi-graphs $\beta : \mathbb{G}^1 \to \mathbb{G}^2$. We shall refer to the morphism β as the underlying morphism of $\mathfrak{b}$.

From now on, we only consider connected semi-graphs with p-rank.

Definition 2.3. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a morphism of semi-graphs with p-rank and $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$.

(a) We shall say that $\mathfrak{b}$ is *p-étale* (resp. *purely inseparable*) at an edge $e \in e(\mathbb{G}^1)$ if $\#\beta^{-1}(\beta(e)) = p$ (resp. $\#\beta^{-1}(\beta(e)) = 1$). We shall say that $\mathfrak{b}$ is *p-generically étale* at $v \in v(\mathbb{G}^1)$ if one of the following holds.

(Type-I): $\#\beta^{-1}(\beta(v)) = p$ and $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$;

(Type-II): $\#\beta^{-1}(\beta(v)) = 1$ and

$$\sigma_{\mathfrak{G}^1}(v) - 1 = p(\sigma_{\mathfrak{G}^2}(\beta(v)) - 1) + \sum_{e \in e(v)} (r_e - 1),$$

where r_e is equal to $p/\#\beta^{-1}(\beta(e))$.

(b) We shall say that $\mathfrak{b}$ is *purely inseparable* at $v \in v(\mathbb{G}^1)$ if $\#\beta^{-1}(\beta(v)) = 1$, $\mathfrak{b}$ is purely inseparable at each element of $e(v)$, and $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^2}(\beta(v))$.

(c) We shall say that $\mathfrak{b}$ is a *p-covering* if the following conditions hold:

(i) there exists a $\mathbb{Z}/p\mathbb{Z}$-action (which may be trivial) on $\mathbb{G}^1$ (resp. a trivial $\mathbb{Z}/p\mathbb{Z}$-action on $\mathbb{G}^2$), and the underlying morphism β of $\mathfrak{b}$ is compatible with the $\mathbb{Z}/p\mathbb{Z}$-actions;

(ii) the natural morphism $\mathbb{G}^1/(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{G}^2$ induced by β is an isomorphism, where $\mathbb{G}^1/(\mathbb{Z}/p\mathbb{Z})$ denotes the quotient semi-graph;

(iii) for each $v \in v(\mathbb{G}^1)$, $\mathfrak{b}$ is either p-generically étale or purely inseparable at v;

(iv) let $e \in e^{\mathrm{cl}}(\mathbb{G}^1)$ and $v(e) = \{v, v'\}$ (note that $v = v'$ if e is a loop); if $\mathfrak{b}$ is p-generically étale at v and v', then $\mathfrak{b}$ is p-étale at e;

(v) for each $v \in v(\mathbb{G}^1)$, then $\sigma_{\mathfrak{G}^1}(v) = \sigma_{\mathfrak{G}^1}(\tau(v))$ holds for each $\tau \in \mathbb{Z}/p\mathbb{Z}$.

Note that the definition of p-coverings implies that the identity morphism of a semi-graph with p-rank is a p-covering.

(d) We shall say that $\mathfrak{b}$ is a *covering* if $\mathfrak{b}$ is a composite of p-coverings.

(e) Let G be finite p-group. We shall say

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

an *maximal normal filtration* of G if G_j is a normal subgroup of G and $G_j/G_{j+1} \cong \mathbb{Z}/p\mathbb{Z}$ for each $j = 0, \ldots, r-1$. Suppose that $\mathbb{G}^1$ admits a G-action (which may be trivial), that $\mathbb{G}^2$ admits a trivial G-action, and that the underlying morphism β of $\mathfrak{b}$ is compatible with the G-actions. We obtain a maximal normal filtration Φ of G induces a sequence of semi-graphs:

$$\mathbb{G}^1 = \mathbb{G}_r \xrightarrow{\beta_r} \mathbb{G}_{r-1} \xrightarrow{\beta_{r-1}} \dots \xrightarrow{\beta_1} \mathbb{G}_0,$$

where $\mathbb{G}_j, j = 0, \dots, r$, denotes the quotient semi-graph $\mathbb{G}^1/G_j$. We shall say $\mathfrak{b}$ a *G-covering* if there exist a maximal normal filtration Φ of G and a set of p-coverings $\{\mathfrak{b}_j : \mathfrak{G}_j \to \mathfrak{G}_{j-1},\ j = 1, \dots, r\}$ such that the following conditions hold:

(i) the underlying semi-graph of $\mathfrak{G}_j$ is equal to $\mathbb{G}_j$ for each $j = 0, \dots, r$;

(ii) the underlying morphism of $\mathfrak{b}_j$ is equal to β_j for each $j = 1, \dots, r$;

(iii) The composite morphism $\mathfrak{b}_1 \circ \dots \circ \mathfrak{b}_r$ is equal to $\mathfrak{b}$.

(f) Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering. By the definition of G-coverings, we obtain a maximal normal filtration Φ of G and a sequence of p-coverings:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

We shall say $\Phi_{\mathfrak{G}^1/\mathfrak{G}^2}$ *a sequence of p-coverings induced by* Φ.

Definition 2.4. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering, $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$, $v^1 \in v(\mathbb{G}^1)$, and $e^1 \in e(\mathbb{G}^1)$. By the definition of G-coverings, we have a maximal normal filtration Φ of G and a sequence of p-coverings induced by Φ:

$$\Phi_{\mathfrak{G}^1/\mathfrak{G}^2} : \mathfrak{G}^1 = \mathfrak{G}_r \xrightarrow{\mathfrak{b}_r} \mathfrak{G}_{r-1} \xrightarrow{\mathfrak{b}_{r-1}} \dots \xrightarrow{\mathfrak{b}_1} \mathfrak{G}_0 = \mathfrak{G}^2.$$

Write $\beta_j : \mathbb{G}_j \to \mathbb{G}_{j-1}, j = 1, \dots, r$, for the underlying morphism of $\mathfrak{b}_j$. Write v_j (resp. e_j) for the image $\beta_{j+1} \circ ... \circ \beta_r(v^1)$ (resp. $\beta_{j+1} \circ ... \circ \beta_r(e^1)$), $j = 0, \dots, r-1$, and v_r for v^1. We put

$$I_{e^1} \stackrel{\text{def}}{=} \{\tau \in G \mid \tau(e^1) = e^1\},$$

$$I_{v^1} \stackrel{\text{def}}{=} \{\tau \in G \mid \tau(v^1) = v^1 \text{ and } \tau(e) = e \text{ for every } e \in e(v^1)\},$$

$$D_{v^1} \stackrel{\text{def}}{=} \{\tau \in G \mid \tau(v^1) = v^1\}.$$

Note that we have that $\#I_{v^1}$ and $\#I_{e^1}$ do not depend on the choice of Φ, and that

$$\#I_{v^1} = p^{\#\{j \in \{1,\dots,r\} \ \mid \ \mathfrak{b}_j \text{ is purely inseparable at } v_j\}},$$

$$\#I_{e^1} = p^{\#\{j \in \{1,\dots,r\} \ \mid \ \mathfrak{b}_j \text{ is purely inseparable at } e_j\}}.$$

Remark 2.4.1. We maintain the notation introduced in Definition 2.4. Note that if $e^1 \in e(v^1)$, then we have $\#I_{v^1} | \#I_{e^1}$. In particular, if e^1 is a loop, then Definition 2.3 (c-iv) implies that $\#I_{v^1} = \#I_{e^1}$. Moreover, Definition 2.3 (c-iv) also implies that $\#I_{e^1} | \#D_{v^1}$. Write v^2 (resp. e^2) for $\beta(v^1)$ (resp. $\beta(e^1)$). Let $(v^1)'$ (resp. $(e^1)'$) be an arbitrary element of $\beta^{-1}(v^2)$ (resp. $\beta^{-1}(e^2)$). By the action of G on $\mathbb{G}^1$, we have $\#I_{v^1} = \#I_{(v^1)'}$, $\#I_{e^1} = \#I_{(e^1)'}$, and $\#D_{v^1} = \#D_{(v^1)'}$. Thus, we may use the notation $\#I_{v^2}$ (resp. $\#I_{e^2}$, $\#D_{v^2}$) to denote $\#I_{v^1}$ (resp. $\#I_{e^1}$, $\#D_{v^1}$). Then we obtain that $\#I_{v^2} | \#I_{e^2} | \#D_{v^2}$.

Remark 2.4.2. We maintain the notation introduced in Definition 2.4. It is easy to compute the p-rank $\sigma_{\mathfrak{G}^1}(v^1)$ by using Definition 2.3 (a). Then we have the following Deuring-Shafarevich type formula for the p-rank of G-coverings (cf. Proposition 3.2 for the Deuring-Shafarevich formula for curves)

$$\sigma_{\mathfrak{G}^1}(v^1) - 1 = \#D_{v^2}/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2)} (\#D_{v^2}/\#I_{e^2})(\#I_{e^2}/\#I_{v^2} - 1)$$

$$= \#D_{v^2}/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e^2 \in e(v^2) \setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_{e^2})(\#I_{e^2}/\#I_{v^2} - 1).$$

Here, the second equality follows from Definition 2.3 (c-iv).

2.2 An operator on quasi-G-coverings of semi-graphs with p-rank

Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a covering and $\beta : \mathbb{G}^1 \to \mathbb{G}^2$ the underlying morphism of $\mathfrak{b}$. Let V^1 be the subset of $v(\mathbb{G}^1)$ consisting of all vertices $v^1 \in v(\mathbb{G}^1)$ satisfying $\#\beta^{-1}(\beta(v^1)) = 1$. Write V^2 for the image $\beta(V^1) \subseteq v(\mathbb{G}^2)$. Moreover, we suppose that $\mathbb{G}^1$ (resp. $\mathbb{G}^2$) admits an action (resp. a trivial action) of G such that β is a G-equivariant, and that $\mathbb{G}^1/G = \mathbb{G}^2$.

We shall say $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ a *quasi-G-covering* if the covering $\mathfrak{G}^1 \setminus \{V^1\} \to \mathfrak{G}^2 \setminus \{V^2\}$ induced by $\mathfrak{b}$ is a G-covering, where $\mathfrak{G}^i \setminus \{V^i\}$, $i = 1, 2$, denotes the semi-graph with p-rank whose underlying semi-graph is $\mathbb{G}^i \setminus \{V^i\}$ (in the sense of Definition 2.1 (b)), and whose p-rank map is $\sigma_{\mathfrak{G}^i}|_{v(\mathbb{G}^i \setminus \{V^i\})}$.

In this subsection, we introduce an operator for $\mathfrak{b}$. Let $v^2 \in v(\mathbb{G}^2)$ and $v^1 \in \beta^{-1}(v^2)$. First, let us define a new semi-graph associated to v^2. We define a semi-graph $(\mathbb{G}^1)^*[v^2]$ as follows:

(a) Suppose that $v^2 \in V^2$. We put $(\mathbb{G}^1)^*[v^2] \stackrel{\mathrm{def}}{=} \mathbb{G}^1$.

(b) Suppose that $v^2 \notin V^2$. We have the following:

(i) $v((\mathbb{G}^1)^*[v^2]) \stackrel{\mathrm{def}}{=} (v(\mathbb{G}^1) \setminus \beta^{-1}(v^2)) \sqcup \{v^*[v^2]\}$, $e^{\mathrm{cl}}((\mathbb{G}^1)^*[v^2]) \stackrel{\mathrm{def}}{=} e^{\mathrm{cl}}(\mathbb{G}^1)$, and $e^{\mathrm{op}}((\mathbb{G}^1)^*[v^2]) \stackrel{\mathrm{def}}{=} e^{\mathrm{op}}(\mathbb{G}^1)$;

(ii) the collection of maps $\{\zeta_e^{(\mathbb{G}^1)^*[v^2]}\}_e$ is as follows:

(1) for each $b \in \bigcup_{e \in e^{\mathrm{op}}(\mathbb{G}^1)} e$, we put $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = \{v((\mathbb{G}^1)^*[v^2])\}$ if $\zeta_e^{\mathbb{G}^1}(b) = \{v(\mathbb{G}^1)\}$, put $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = v^*[v^2]$ if $\zeta_e^{\mathbb{G}^1}(b) \in \beta^{-1}(v^2)$, and put $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = \zeta_e^{\mathbb{G}^1}(b)$ if $\zeta_e^{\mathbb{G}^1}(b) \neq \{v(\mathbb{G}^1)\}$ and $\zeta_e^{\mathbb{G}^1}(b) \notin \beta^{-1}(v^2)$;

(2) for each $b \in \bigcup_{e \in e^{\mathrm{cl}}(\mathbb{G}^1)} e$, we put $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = \zeta_e^{\mathbb{G}^1}(b)$ if $\zeta_e^{\mathbb{G}^1}(b) \notin \beta^{-1}(v^2)$, and put $\zeta_e^{(\mathbb{G}^1)^*[v^2]}(b) = v^*[v^2]$ if $\zeta_e^{\mathbb{G}^1}(b) \in \beta^{-1}(v^2)$.

Second, we define a map $\sigma_{(\mathfrak{G}^1)^*[v^2]} : v((\mathbb{G}^1)^*[v^2]) \to \mathbb{Z}$ as follows:

(a) Suppose that $v^2 \in V^2$. We put $\sigma_{(\mathfrak{G}^1)^*[v^2]} \stackrel{\text{def}}{=} \sigma_{(\mathfrak{G}^1)}$.

(b) Suppose that $v^2 \notin V^2$. We have the following:

(i) if $v \neq v^*[v^2]$, we put $\sigma_{(\mathfrak{G}^1)^*[v^2]}(v) \stackrel{\text{def}}{=} \sigma_{\mathfrak{G}^1}(v)$;

(ii) if $v = v^*[v^2]$, we put

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}(v^*[v^2]) \stackrel{\text{def}}{=} \#G/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)} \#G/\#I_e(\#I_e/\#I_{v^2}-1)+1.$$

We define a semi-graph with p-rank $(\mathfrak{G}^1)^*[v^2]$ to be the pair $((\mathbb{G}^1)^*[v^2], \sigma_{(\mathfrak{G}^1)^*[v^2]})$.

Next, we define a morphism of semi-graphs $\beta^*[v^2] : (\mathbb{G}^1)^*[v^2] \to \mathbb{G}^2$ as follows:

(i) for each $v \in v((\mathbb{G}^1)^*[v^2])$, we put $\beta^*[v^2](v) \stackrel{\text{def}}{=} v^2$ if $v = v^*[v^2]$ and $\beta^*[v^2](v) \stackrel{\text{def}}{=} \beta(v)$ if $v \neq v^*[v^2]$;

(ii) for each $e \in e((\mathbb{G}^1)^*[v^2])$, we put $\beta^*[v^2](e) \stackrel{\text{def}}{=} \beta(e)$.

Then we obtain a morphism of semi-graphs with p-rank

$$\mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \to \mathfrak{G}^2$$

induced by $\beta^*[v^2]$.

Definition 2.5. Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a quasi-G-covering and $v^2 \in v(\mathbb{G}^2)$. We define an operator $\rightleftharpoons^{I}_{II} [v^2]$ on $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ to be

$$\rightleftharpoons^{I}_{II} [v^2](\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2) \stackrel{\text{def}}{=} \mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \to \mathfrak{G}^2.$$

Remark 2.5.1. The semi-graph with p-rank $(\mathbb{G}^1)^*[v^2]$ admits a natural G-action as follows:

(i) the action of G on $v((\mathbb{G}^1)^*[v^2]) \setminus \{v^*[v^2]\}$ (resp. $e((\mathbb{G}^1)^*[v^2])$) is the action of G on $v(\mathbb{G}) \setminus \beta^{-1}(v^2)$ (resp. $e(\mathbb{G})$);

(ii) the action of G on $v^*[v^2]$ is a trivial action.

We see that $\mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \to \mathfrak{G}^2$ is a quasi-G-covering.

On the other hand, suppose that $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ is a G-covering. If G is an abelian p-group, then together with the G-action defined above, it is easy to check that $\mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \to \mathfrak{G}^2$ is a G-covering. However, $\mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \to \mathfrak{G}^2$ is not a G-covering in general if G is not abelian. The reason is the following. With the natural action of G on $(\mathfrak{G}^1)^*[v^2]$, the order $\#I_w$ of the inertia subgroup I_w of the vertex $w \stackrel{\text{def}}{=} (\beta^*[v^2])^{-1}(v^2)$ over v^2 is not equal to $\#I_{v^2}$ in general when I_{v^2} is not a normal subgroup of G. If $\mathfrak{b}^*[v^2]$ is a G-covering, then the p-rank $\sigma_{(\mathfrak{G}^1)^*[v^2]}(w)$ is equal to

$$\#G/\#I_w(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} \#G/\#I_e(\#I_e/\#I_w - 1) + 1$$

which is not equal to $\#G/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} \#G/\#I_e(\#I_e/\#I_{v^2} - 1) + 1$ in general if $\#I_w \neq \#I_{v^2}$. This contradicts the definition of $(\mathfrak{G}^1)^*[v^2]$. Thus, $\mathfrak{b}^*[v^2]$ is not a G-covering in general.

2.3 Formula for p-rank of G-coverings of semi-graphs with p-rank

In this subsection, we give an explicit formula for the p-rank of G-coverings of semi-graphs with p-rank.

Lemma 2.6. *Let $\mathbb{G}$ be a connected semi-graph, $\{\mathbb{G}_i\}_{i=1,\dots,n}$ a set of connected sub-semi-graph of $\mathbb{G}$, and $v_i \in v(\mathbb{G}_i)$, $i = 1,\dots,n$. Suppose that $\mathbb{G}_s \cap \mathbb{G}_t = \emptyset$ for each $s,t \in \{1,\dots,n\}$ when $s \neq t$. Let $\mathbb{G}^{\mathrm{c}}$ be a semi-graph defined as follows: (i) $v(\mathbb{G}^{\mathrm{c}}) = v(\mathbb{G}) \sqcup \{v^{\mathrm{c}}\}$; (ii) $e^{\mathrm{op}}(\mathbb{G}^{\mathrm{c}}) = e^{\mathrm{op}}(\mathbb{G})$; (iii) $e^{\mathrm{cl}}(\mathbb{G}^{\mathrm{c}}) = e^{\mathrm{cl}}(\mathbb{G}) \sqcup \{e^{\mathrm{c}}_i\}_{i=1,\dots,n}$; (iv) if $e \notin \{e^{\mathrm{c}}_i\}_{i=1,\dots,n}$, for each $b \in e$, we put $\zeta^{\mathbb{G}^{\mathrm{c}}}_e(b) = \zeta^{\mathbb{G}}_e(b)$ if $\zeta^{\mathbb{G}}_e(b) \neq \{v(\mathbb{G})\}$, and put $\zeta^{\mathbb{G}^{\mathrm{c}}}_e(b) = \{v(\mathbb{G}^{\mathrm{c}})\}$ if $\zeta^{\mathbb{G}}_e(b) = \{v(\mathbb{G})\}$; (v) let $e^{\mathrm{c}}_i = \{b^1_{e^{\mathrm{c}}_i}, b^2_{e^{\mathrm{c}}_i}\}$, $i = 1,\dots,n$; we put $\zeta^{\mathbb{G}^{\mathrm{c}}}_{e^{\mathrm{c}}_i}(b^1_{e^{\mathrm{c}}_i}) = v_i$ and $\zeta^{\mathbb{G}^{\mathrm{c}}}_{e^{\mathrm{c}}_i}(b^2_{e^{\mathrm{c}}_i}) = v^{\mathrm{c}}$ for each $i = 1,\dots,n$. Then we have*

$$\dim_{\mathbf{C}}(H^1(\mathbb{G}, \mathbf{C})) = \dim_{\mathbf{C}}(H^1(\mathbb{G}^{\mathrm{c}}, \mathbf{C})) - n + 1.$$

Proof. The lemma follows from the construction of $\mathbb{G}^{\mathrm{c}}$. □

The following proposition is a key for calculating the p-rank of G-coverings.

Proposition 2.7. *Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a quasi-G-covering of semi-graphs with p-rank, $v^2 \in v(\mathbb{G}^2)$, and $\rightleftharpoons^I_{II} [v^2](\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2) = \mathfrak{b}^*[v^2] : (\mathfrak{G}^1)^*[v^2] \to \mathfrak{G}^2$. Then we have*

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2]).$$

Proof. Suppose that $\#\beta^{-1}(v^2) = 1$. Then the proposition is trivial. Thus, we may assume that $\#\beta^{-1}(v^2) \neq 1$.

Write β (resp. $\beta^*[v^2]$) for the underlying morphism of $\mathfrak{b}$ (resp. $\mathfrak{b}^*[v^2]$). Write r (resp. $r_{\setminus\{v^2\}}, r^*, r^*_{\setminus\{v^2\}}$) for the Betti number $\dim_{\mathbf{C}}(H^1(\mathbb{G}^1, \mathbf{C}))$ (resp. $\dim_{\mathbf{C}}(H^1(\mathbb{G}^1 \setminus \beta^{-1}(v^2), \mathbf{C}))$, $\dim_{\mathbf{C}}(H^1((\mathbb{G}^1)^*[v^2], \mathbf{C}))$, $\dim_{\mathbf{C}}(H^1((\mathbb{G}^1)^*[v^2] \setminus (\beta^*[v^2])^{-1}(v^2), \mathbf{C}))$), where $\mathbb{G}^1 \setminus \beta^{-1}(v^2)$ and $(\mathbb{G}^1)^*[v^2] \setminus (\beta^*[v^2])^{-1}(v^2)$ denote the semi-graph defined in Definition 2.1 (b).

Then we have

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sum_{v \in v(\mathbb{G}^1 \setminus \beta^{-1}(v^2))} \sigma_{\mathfrak{G}^1}(v) + \sum_{v \in \beta^{-1}(v^2)} \sigma_{\mathfrak{G}^1}(v) + r_{\setminus\{v^2\}} + r - r_{\setminus\{v^2\}}$$

and

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2]) = \sum_{v \in v((\mathbb{G}^1)^*[v^2] \setminus (\beta^*[v^2])^{-1}(v^2))} \sigma_{(\mathfrak{G}^1)^*[v^2]}(v) + \sigma_{(\mathfrak{G}^1)^*[v^2]}(v^*[v^2]) + r^*_{\setminus\{v^2\}} + r^* - r^*_{\setminus\{v^2\}}.$$

Note that, by the construction of $(\mathfrak{G}^1)^*[v^2]$, we have

$$A \stackrel{\text{def}}{=} \sum_{v \in v(\mathbb{G}^1 \setminus \beta^{-1}(v^2))} \sigma_{\mathfrak{G}^1}(v) = \sum_{v \in v((\mathbb{G}^1)^*[v^2] \setminus (\beta^*[v^2])^{-1}(v^2))} \sigma_{(\mathfrak{G}^1)^*[v^2]}(v)$$

and

$$B \stackrel{\text{def}}{=} r_{\setminus\{v^2\}} = r^*_{\setminus\{v^2\}}.$$

Let us calculate $r - r_{\setminus\{v^2\}}$ and $r^* - r^*_{\setminus\{v^2\}}$. Follows from Lemma 2.6, it is sufficient to treat the case where $\mathbb{G}^1 \setminus \beta^{-1}(v^2) = (\mathbb{G}^1)^*[v^2] \setminus (\beta^*[v^2])^{-1}(v^2)$ is connected. Then we obtain

$$r - r_{\setminus\{v^2\}} = \#G/\#D_{v^2}\Big(\big(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#D_{v^2}/\#I_e\big) - 1\Big) + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

and

$$r^* - r^*_{\setminus\{v^2\}} = \big(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e\big) - 1 + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}).$$

Remark 2.4.2 implies that, for each $v \in \beta^{-1}(v^2)$, we have

$$\sigma_{\mathfrak{G}^1}(v) = \#D_{v^2}/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1$$

$$= \#D_{v^2}/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2} - 1) + 1.$$

On the other hand, the construction of $(\mathfrak{G}^1)^*[v^2]$ implies that

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}(v^*[v^2]) = \#G/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)} \#G/\#I_e(\#I_e/\#I_{v^2} - 1) + 1$$

$$= \#G/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2) - 1) + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e(\#I_e/\#I_{v^2} - 1) + 1$$

Thus, we obtain

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = A+B+\sum_{v\in\beta^{-1}(v^2)} (\#D_{v^2}/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2}-1)+1)$$

$$+\#G/\#D_{v^2}\Big(\big(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#D_{v^2}/\#I_e\big) - 1\Big) + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A+B+\#G/\#D_{v^2}(\#D_{v^2}/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+\sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} (\#D_{v^2}/\#I_e)(\#I_e/\#I_{v^2}-1)+1)$$

$$+\#G/\#D_{v^2}\Big(\big(\sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#D_{v^2}/\#I_e\big) - 1\Big) + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A + B + (\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \#G/\#I_{v^2} + \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_{v^2} - \sum_{e\in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e$$

$$+ \sum_{e\in(e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A + B + (\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \#G/\#I_{v^2} + \sum_{e \in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_{v^2}$$

$$- \sum_{e \in (e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

and

$$\sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2]) = A+B+\#G/\#I_{v^2}(\sigma_{\mathfrak{G}^2}(v^2)-1)+ \sum_{e \in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e(\#I_e/\#I_{v^2}-1)+1$$

$$+(\sum_{e \in (e(v^2)\cap e^{\mathrm{cl}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e) - 1 + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2})$$

$$= A + B + (\#G/\#I_{v^2})\sigma_{\mathfrak{G}^2}(v^2) - \#G/\#I_{v^2} + \sum_{e \in e(v^2)\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_{v^2}$$

$$- \sum_{e \in (e(v^2)\cap e^{\mathrm{op}}(\mathbb{G}^2))\setminus e^{\mathrm{lp}}(v^2)} \#G/\#I_e + \#e^{\mathrm{lp}}(v^2)(\#G/\#I_{v^2}).$$

This means that

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sigma_{(\mathfrak{G}^1)^*[v^2]}((\mathfrak{G}^1)^*[v^2]).$$

We complete the proof of the proposition. □

Theorem 2.8. *Let $\mathfrak{b} : \mathfrak{G}^1 \to \mathfrak{G}^2$ be a G-covering of semi-graphs with p-rank. Then we have*

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sum_{v \in v(\mathbb{G}^2)} \Big(\#G/\#I_v(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v)\setminus e^{\mathrm{lp}}(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1\Big)$$

$$+ \sum_{e \in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \sum_{v \in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \dim_{\mathbf{C}}(H^1(\mathbb{G}^2, \mathbf{C})).$$

Proof. By applying Proposition 2.7 and the operator $\rightleftharpoons^{I}_{II}$ defined in the previous section, we may construct a quasi-G-covering $\mathfrak{b}^* : \mathfrak{G}^{1,*} \to \mathfrak{G}^2$ such that the following conditions hold:

(i) for each $v \in v(\mathbb{G}^2)$, $\#(\beta^*)^{-1}(v) = 1$, where β^* denotes the underlying morphism of $\mathfrak{b}^*$;

(ii) for each $v \in v(\mathbb{G}^2)$, we have

$$\sigma_{\mathfrak{G}^{1,*}}((\beta^*)^{-1}(v)) = \#G/\#I_v(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1$$

$$= \#G/\#I_v(\sigma_{\mathfrak{G}^2}(v) - 1) + \sum_{e \in e(v)\setminus e^{\mathrm{lp}}(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1;$$

(iii) $\sigma_{\mathfrak{G}^{1,*}}(\mathfrak{G}^{1,*}) = \sigma_{\mathfrak{G}^1}(\mathfrak{G}^1)$.

Write $\mathbb{G}^{1,*}$ for the underlying semi-graph of $\mathfrak{G}^{1,*}$. We observe that

$$\dim_{\mathbf{C}}(H^1(\mathbb{G}^{1,*},\mathbf{C})) = \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \dim_{\mathbf{C}}(H^1(\mathbb{G}^2,\mathbf{C})) - \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)$$

$$+ \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v)$$

$$= \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \dim_{\mathbf{C}}(H^1(\mathbb{G}^2,\mathbf{C})) + \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1).$$

Thus, we obtain

$$\sigma_{\mathfrak{G}^1}(\mathfrak{G}^1) = \sigma_{\mathfrak{G}^{1,*}}(\mathfrak{G}^{1,*}) = \sum_{v\in v(\mathbb{G}^2)} \Big(\#G/\#I_v(\sigma_{\mathfrak{G}^2}(v)-1) + \sum_{e\in e(v)\setminus e^{\mathrm{lp}}(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1\Big)$$

$$+ \sum_{e\in e^{\mathrm{cl}}(\mathbb{G}^2)\setminus e^{\mathrm{lp}}(\mathbb{G}^2)} (\#G/\#I_e - 1) + \sum_{v\in v(\mathbb{G}^2)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1) + \dim_{\mathbf{C}}(H^1(\mathbb{G}^2,\mathbf{C})).$$

This completes the proof of the theorem. □

Definition 2.9. Let n be a positive natural number and $\mathbb{P}_n$ a semi-graph such that the following conditions hold:

(i) $v(\mathbb{P}_n) = \{p_1, \ldots, p_n\}$;

(ii) $e^{\mathrm{cl}}(\mathbb{P}_n) = \{e_{1,2}, \ldots, e_{n-1,n}\}$ and $e^{\mathrm{op}}(\mathbb{P}_n) = \{e_{0,1}, e_{n,n+1}\}$;

(iii) $v(e_{0,1}) = \{p_1\}$ and $v(e_{n,n+1}) = \{p_n\}$;

(iv) $v(e_{i,i+1}) = \{p_i, p_{i+1}\}$.

Let $\sigma_{\mathfrak{P}_n}$ be a map from $v(\mathbb{P}_n)$ to $\mathbb{Z}$ such that $\sigma_{\mathfrak{P}_n}(p_i)$ is equal to 0 for each $i = 1, \ldots, n$. We define a semi-graph with p-rank $\mathfrak{P}_n$ to be $(\mathbb{P}_n, \sigma_{\mathfrak{P}_n})$. We shall say $\mathfrak{P}_n$ an *n-chain*.

Theorem 2.8 implies the following important corollary.

Corollary 2.10. *Let $\mathfrak{b} : \mathfrak{G} \to \mathfrak{P}_n$ be a G-covering of semi-graphs with p-rank. Then we have*

$$\sigma_{\mathfrak{G}}(\mathfrak{G}) = \sum_{i=1}^{n} \#G/\#I_{p_i} - \sum_{i=1}^{n+1} \#G/\#I_{e_{i-1,i}} + 1.$$

Proof. Since $\sum_{v\in v(\mathbb{P}_n)} \#e^{\mathrm{lp}}(v)(\#G/\#I_v - 1)$ and $\dim_{\mathbf{C}}(H^1(\mathbb{P}_n,\mathbf{C}))$ are equal to 0, the corollary follows from Theorem 2.8. □

In the next two sections, we will use Theorem 2.8 and Corollary 2.10 to calculate the global and local p-rank of coverings of curves over a complete discrete valuation ring.

3 Semi-graphs with p-rank associated to pointed semi-stable coverings

3.1 p-rank and pointed semi-stable coverings

Definition 3.1. (a) Let $\mathscr{C} \overset{\text{def}}{=} (C, D_C)$ be a pointed semi-stable curve over a scheme A (i.e., a marked curve over A such that each geometric fiber is a semi-stable curve). We shall say C the underlying curve of $\mathscr{C}$ and D_C the set of marked points of $\mathscr{C}$.

Suppose that A is the spectrum of a field, we write $\mathrm{Irr}(C)$ for the set of the irreducible components of C and C^{sing} for the set of singular points of C. We define the dual semi-graph $\Gamma_{\mathscr{C}}$ of the pointed semi-stable curve $\mathscr{C}$ to be a semi-graph as follows:

(i) $v(\Gamma_{\mathscr{C}}) \overset{\text{def}}{=} \{v_E\}_{E \in \mathrm{Irr}(C)}$;

(ii) $e^{\mathrm{cl}}(\Gamma_{\mathscr{C}}) \overset{\text{def}}{=} \{e_s\}_{s \in C^{\mathrm{sing}}}$ and $e^{\mathrm{op}}(\Gamma_{\mathscr{C}}) \overset{\text{def}}{=} \{e_m\}_{m \in D_C}$;

(iii) for each $e_s = \{b_s^1, b_s^2\} \in e^{\mathrm{cl}}(\Gamma_{\mathscr{C}})$, we put $\zeta_{e_s}^{\Gamma_{\mathscr{C}}}(e_s) \overset{\text{def}}{=} \{v_E \in v(\Gamma_{\mathscr{C}}) \mid s \in E\}$;

(iv) for each $e_m = \{b_m^1, b_m^2\} \in e^{\mathrm{op}}(\Gamma_{\mathscr{C}})$, we put $\zeta_{e_m}^{\Gamma_{\mathscr{C}}}(b_m^1) \overset{\text{def}}{=} v_E$ where E is the irreducible component of C satisfying $m \in E$, and put $\zeta_{e_m}^{\Gamma_{\mathscr{C}}}(b_m^2) \overset{\text{def}}{=} \{v(\Gamma_{\mathscr{C}})\}$.

(b) Let C be a disjoint union of projective curves over an algebraically closed field of characteristic $p > 0$. We define the *p-rank* of C to be

$$\sigma(C) \overset{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(C, \mathbb{F}_p)).$$

Suppose that C is a semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write Γ_C for the dual semi-graph of C and $v(\Gamma_C)$ for the set of vertices of Γ_C. Then we have

$$\sigma(C) = \sum_{v \in v(\Gamma_C)} \sigma(\widetilde{C_v}) + \dim_{\mathbf{C}}(H^1(\Gamma_C, \mathbf{C})),$$

where $\widetilde{C}_v$ denotes the normalization of the irreducible component C_v of C corresponding to $v \in v(\Gamma_C)$. Let $\mathscr{C} \overset{\text{def}}{=} (C, D_C)$ be a pointed semi-stable curve over an algebraically closed field of characteristic $p > 0$. We define the p-rank $\sigma(\mathscr{C})$ of $\mathscr{C}$ to be the p-rank $\sigma(C)$ of the underlying curve C.

Let G be a finite group. The p-rank of a G-Galois covering (i.e., Galois covering whose Galois group is isomorphic to G) of a smooth projective curve over an algebraically closed field of characteristic $p > 0$ is difficult to calculate in general. On the other hand, if G is a p-group, then the p-rank of a G-Galois covering can be calculated by the Deuring-Shafarevich formula as follows (cf. [C]):

Proposition 3.2. *Let $h : C' \to C$ be a Galois covering (possibly ramified) of smooth projective curves over an algebraically closed field of characteristic $p > 0$, whose Galois group is a finite p-group G. Then we have*

$$\sigma(C') - 1 = \#G(\sigma(C) - 1) + \sum_{c' \in (C')^{\mathrm{cl}}} (e_{c'} - 1),$$

where $(C')^{\mathrm{cl}}$ *denotes the set of closed points of* C' *and* $e_{c'}$ *denotes the ramification index at* c'.

From now on, let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$ and K the quotient field. We use the notation S to denote the spectrum of R. Write η and s for the generic point and the closed point corresponding to the natural morphisms $\mathrm{Spec}\, K \to S$ and $\mathrm{Spec}\, k \to S$, respectively. Let $\mathscr{X} \overset{\mathrm{def}}{=} (X, D_X)$ be a pointed semi-stable curve over S. Write $\mathscr{X}_\eta \overset{\mathrm{def}}{=} (X_\eta, D_{X_\eta})$ and $\mathscr{X}_s \overset{\mathrm{def}}{=} (X_s, D_{X_s})$ for the generic fiber of $\mathscr{X}$ and the special fiber of $\mathscr{X}$, respectively. We suppose that $\mathscr{X}_\eta$ is a *smooth pointed stable curve* over η, and write $\Gamma_{\mathscr{X}_s}$ for the dual semi-graph of $\mathscr{X}_s$.

Definition 3.3. Let $l : \mathscr{W} \overset{\mathrm{def}}{=} (W, D_W) \to \mathscr{X}$ be a morphism of pointed semi-stable curves over S and G a finite group. The morphism l is called a *pointed semi-stable covering* (resp. *G-pointed semi-stable covering*) over S if the morphism

$$l_\eta : \mathscr{W}_\eta \overset{\mathrm{def}}{=} (W_\eta, D_{W_\eta}) \to \mathscr{X}_\eta = (X_\eta, D_{X_\eta})$$

over η induced by l on generic fibers is a finite generically étale morphism (resp. a Galois covering whose Galois group is isomorphic to G) such that the following conditions hold:

(i) the branch locus of l_η is contained in D_{X_η};

(ii) $l_\eta^{-1}(D_{X_\eta}) = D_{W_\eta}$;

(iii) the following universal property holds: if $g : \mathscr{W}' \to \mathscr{X}$ is a morphism of pointed semi-stable curves over S such that the generic fiber $\mathscr{W}'_\eta$ of $\mathscr{W}'$ and the morphism $g_\eta : \mathscr{W}'_\eta \to \mathscr{X}_\eta$ induced by g on generic fibers are equal to $\mathscr{W}_\eta$ and l_η, respectively, then there exists a unique morphism $h : \mathscr{W}' \to \mathscr{W}$ such that $g = l \circ h$.

We shall say l a *pointed stable covering* (resp. *G-pointed stable covering*) over S if l is a pointed semi-stable covering (resp. G-pointed semi-stable covering) over S, and $\mathscr{X}$ is a pointed stable curve over S. We shall call l a *semi-stable covering* (resp. *stable covering, G-semi-stable covering, G-stable covering*) over S if l is a pointed semi-stable covering (resp. pointed stable covering, G-pointed semi-stable covering, G-pointed stable covering) over S, and D_X is empty.

Proposition 3.4. *Let* $f_\eta : \mathscr{Y}_\eta \overset{\mathrm{def}}{=} (Y_\eta, D_{Y_\eta}) \to \mathscr{X}_\eta$ *be a finite morphism of pointed smooth curves over* η. *Suppose that the branch locus of* f_η *is contained in* D_{X_η} *and that* $f_\eta^{-1}(D_{X_\eta}) = D_{Y_\eta}$. *Then, by replacing* S *by a finite extension of* S, f_η *extends to a pointed semi-stable covering* $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ *over* S *such that the restriction of* f *to the generic fibers is* f_η.

Proof. The proposition follows from [L, Theorem 0.2 and Remark 4.13]. □

Remark 3.4.1. We maintain the notation introduced in Proposition 3.4. In fact, we have that f_η extends *uniquely* to a pointed semi-stable covering f. Let us explain roughly in this remark.

By adding some marked points, we may obtain a pointed stable curve $\mathscr{X}^{\mathrm{add}} \stackrel{\mathrm{def}}{=} (X^{\mathrm{add}}, D_{X^{\mathrm{add}}})$ whose underlying curve X^{add} is X, and whose set of marked points contains D_X. Write $D_{X^{\mathrm{add}}_\eta}$ for $D_{X^{\mathrm{add}}}|_\eta$, and $D_{Y^{\mathrm{add}}_\eta}$ for $f_\eta^{-1}(D_{X^{\mathrm{add}}_\eta})$. Then $D_{Y^{\mathrm{add}}_\eta}$ contains D_{Y_η}. Moreover, we have a finite morphism of pointed smooth curves

$$f^{\mathrm{add}}_\eta : \mathscr{Y}^{\mathrm{add}}_\eta \to \mathscr{X}^{\mathrm{add}}_\eta$$

over η induced by f_η.

By applying Proposition 3.4 and by replacing S by a finite extension of S, f^{add}_η extends to a pointed semi-stable covering

$$f^{\mathrm{add}} : \mathscr{Y}^{\mathrm{add}} \stackrel{\mathrm{def}}{=} (Y^{\mathrm{add}}, D_{Y^{\mathrm{add}}}) \to \mathscr{X}^{\mathrm{add}}$$

over S. Since $\mathscr{X}^{\mathrm{add}}$ is a pointed stable curve over S, we see that $\mathscr{Y}^{\mathrm{add}}$ is a pointed stable model of $\mathscr{Y}^{\mathrm{add}}_\eta$. Then the uniqueness of f^{add} follows from the uniqueness of the pointed stable model $\mathscr{Y}^{\mathrm{add}}$.

We put $D^{\mathrm{ss}}_Y \stackrel{\mathrm{def}}{=} D^{\mathrm{add}}_Y \setminus D_Y$ and $D^{\mathrm{ss}}_{Y_s} \stackrel{\mathrm{def}}{=} D^{\mathrm{ss}}_Y|_s$. Let $\mathrm{Con}(Y^{\mathrm{add}}_s)$ be the subset of the set of irreducible components of Y^{add}_s consisting of all irreducible components E of Y^{add}_s satisfying the following conditions: (i) E is isomorphic to $\mathbb{P}^1_k$; (ii) $E \cap D^{\mathrm{ss}}_{Y_s} \neq \emptyset$ and $E \cap D_Y = \emptyset$; (iii) $f^{\mathrm{add}}(E)$ is a closed point of $\mathscr{X}^{\mathrm{add}}$. Note that $\mathrm{Con}(Y^{\mathrm{add}}_s)$ may be an empty set. Then by forgetting the marked points D^{ss}_Y and by contracting the irreducible components of $\mathrm{Con}(Y^{\mathrm{add}}_s)$, we obtain a pointed semi-stable curve $\mathscr{Y}$ and a morphism of pointed semi-stable curves $f : \mathscr{Y} \to \mathscr{X}$ over S induced by f^{add}. We see that f is a pointed semi-stable covering over S, and that f does not depend on the choices of $D_{X^{\mathrm{add}}}$. Moreover, the uniqueness follows from the uniqueness of f^{add}.

Proposition 3.5. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a* **finite** *G-pointed semi-stable covering over S, and $\Gamma_{\mathscr{Y}_s}$ the dual semi-graph of $\mathscr{Y}_s$. Then the images of nodes (resp. smooth points) of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ are nodes (resp. smooth points) of $\mathscr{X}_s$. In particular, the map of dual semi-graphs $\Gamma_{\mathscr{Y}_s} \to \Gamma_{\mathscr{X}_s}$ induced by the morphism of the special fibers $f_s : \mathscr{Y}_s \to \mathscr{X}_s$ over s induced by f is a morphism of semi-graphs.*

Proof. Let y be a closed point of $\mathscr{Y}$. Write $I_y \subseteq G$ for the inertia subgroup of y. Thus, the natural morphism $\mathscr{Y}/I_y \to \mathscr{X}$ induced by f is étale at the image of y of the quotient morphism $\mathscr{Y} \to \mathscr{Y}/I_y$. Then to verify the proposition, we may assume that $G = I_y$.

If y is a smooth point, then x is a smooth point (cf. [R, Proposition 5]). If y is a node, let Y_1 and Y_2 be the irreducible components (which may be equal) of the underlying curve of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ which contain y. Write $D_1 \subseteq G$ and $D_2 \subseteq G$ for the decomposition subgroups of Y_1 and Y_2, respectively. The proof of [R, Proposition 5] implies that the following: (i) if D_1 and D_2 are not equal to $I_y = G$, then x is a smooth point; (ii) if $D_1 = D_2 = G$, then x is a node.

Next, we prove that the case (i) will not occur. If D_1 and D_2 are not equal to G, then, for each $\tau \in G \setminus D_1$ (or $\tau \in G \setminus D_2$), we have $\tau(Y_1) = Y_2$ and $\tau(Y_2) = Y_1$. Thus, we obtain that $D \stackrel{\mathrm{def}}{=} D_1 = D_2$. Moreover, D is a normal subgroup of G. By replacing I_y by I_y/D and $\mathscr{Y}$ by $\mathscr{Y}/D$ and applying the case (ii), we may assume that D is trivial. Then f_s is étale at the generic points of Y_1 and Y_2. Consider the local morphism $f_y : \operatorname{Spec} \mathcal{O}_{\mathscr{Y},y} \to \operatorname{Spec} \mathcal{O}_{\mathscr{X},f(y)}$ induced by f. Since f_y is étale at all the points of $\operatorname{Spec} \mathcal{O}_{\mathscr{Y},y}$ corresponding to the prime ideals of $\mathcal{O}_{\mathscr{Y},y}$ of height 1, the Zariski-Nagata purity theorem implies that f_y is étale. This means that if $f(y)$ is a smooth point, y is a smooth point too. This contradicts our assumption. We complete the proof of the proposition. □

Definition 3.6. Let $f : \mathscr{Y} \to \mathscr{X}$ be a pointed semi-stable covering over S. A closed point $x \in \mathscr{X}$ is called a *vertical point associated to* f, or for simplicity, a *vertical point* when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the *vertical fiber associated to* x.

Remark 3.6.1. We fix the pointed semi-stable curve $\mathscr{X}$ over S. First, we have that the specialization homomorphism of admissible fundamental groups of generic fiber and special fiber of $\mathscr{X}$ is not an isomorphism in general. When $\operatorname{char}(K) = 0$, this result follows from $\sigma(\mathscr{X}_s) \leq g_X$, where g_X denotes the genus of $\mathscr{X}$. On the other hand, when $\operatorname{char}(K) = p > 0$, this result is highly nontrivial (cf. [T1, Theorem 0.3] and [Y2, Theorem 5.2 and Remark 5.2.1]). Then we may ask the following problem:

> By replacing S by a finite extension of S, does there exist a pointed semi-stable covering $f : \mathscr{Y} \to \mathscr{X}$ over S such that the set of vertical points associated to f is not empty?

Suppose that $\operatorname{char}(K) = 0$. Then the problem was solved by A. Tamagawa (cf. [T2, Theorem 0.2]). In fact, Tamagawa proved a very strong result as following:

> Suppose that $\operatorname{char}(K) = 0$, k is an algebraic closure of $\mathbb{F}_p$, and that $\mathscr{X}$ is a pointed stable curve over S. Let $x \in \mathscr{X}$ be a closed point of $\mathscr{X}$. Then there exists a pointed stable covering $f : \mathscr{Y} \to \mathscr{X}$ over S such that x is a vertical point associated to f.

Moreover, the author generalized this result to the case where k is an arbitrary algebraically closed field (cf. [Y1, Theorem 3.2]). On the other hand, suppose that $\operatorname{char}(K) = p > 0$. The problem was solved by the author when $\mathscr{X}_s$ is irreducible (cf. [Y1, Theorem 0.2]).

If a vertical point x is a smooth point of $\mathscr{X}_s$ and $x \notin D_{X_s}$, then the following result was proved by Raynaud (cf. [R, Théorème 1, Proposition 1, and Proposition 2]).

Proposition 3.7. *Let G be a finite p-group, $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semi-stable covering over S, and x a vertical point associated to f. If x is a smooth point of $\mathscr{X}_s$ and $x \notin D_{X_s}$, then the p-rank of each connected component of the vertical fiber $f^{-1}(x)$ associated to x is equal to 0. On the other hand, by contracting the vertical fibers $f^{-1}(x)$, we obtain a curve $\mathscr{Y}^{\mathrm{c}}$ over S and a contracting morphism $c : \mathscr{Y} \to \mathscr{Y}^{\mathrm{c}}$. Then $\mathscr{Y}^{\mathrm{c}}_s \overset{\mathrm{def}}{=} \mathscr{Y}^{\mathrm{c}} \times_S s$ is geometrically unibranch at $c(f^{-1}(x))$.*

3.2 Global cases

In the remainder of the present subsection, we always assume that G is a finite p-group of order p^r. Let $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X} = (X, D_X)$ be a G-pointed semi-stable covering over S and

$$\Phi : \{1\} = G_r \subset G_{r-1} \subset \cdots \subset G_1 \subset G_0 = G$$

a maximal normal filtration of G. By applying [R, Appendice, Corollaire], we have that $\mathscr{Y}_j \stackrel{\text{def}}{=} \mathscr{Y}/G_j$, $j = 0, \ldots, r$, is a pointed semi-stable curve over S. Write $\mathscr{X}^{\text{sst}} \stackrel{\text{def}}{=} (X^{\text{sst}}, D_{X^{\text{sst}}})$ for $\mathscr{Y}_0$. We obtain two natural morphisms of pointed semi-stable curves $h : \mathscr{Y} \to \mathscr{X}^{\text{sst}}$ and $g : \mathscr{X}^{\text{sst}} \to \mathscr{X}$ induced by f such that $g \circ h = f$. The maximal normal filtration Φ of G induces a sequence of morphism of pointed semi-stable curves

$$\Phi_{\mathscr{Y}/\mathscr{X}^{\text{sst}}} : \mathscr{Y} = \mathscr{Y}_r \stackrel{\phi_r}{\to} \mathscr{Y}_{r-1} \stackrel{\phi_{r-1}}{\to} \ldots \stackrel{\phi_1}{\to} \mathscr{Y}_0 = \mathscr{X}^{\text{sst}}$$

over S such that $\phi_1 \circ ... \circ \phi_r = h$. Note that $\phi_j, j = 1, \ldots, r$, is a *finite* $\mathbb{Z}/p\mathbb{Z}$-pointed semi-stable covering over S. Write $\Gamma_{\mathscr{Y}_j}$, $j = 0, \ldots, r$, for the dual semi-graph of the special fiber $(\mathscr{Y}_j)_s$ of $\mathscr{Y}_j$. Then, for each $j \in \{1, \ldots, r\}$, the morphism of the special fibers $(\phi_j)_s : (\mathscr{Y}_j)_s \to (\mathscr{Y}_{j-1})_s$ induces a map of semi-graphs $\beta_j : \Gamma_{\mathscr{Y}_j} \to \Gamma_{\mathscr{Y}_{j-1}}$. Moreover, Proposition 3.5 implies that β_j, $j = 1, \ldots, r$, is a morphism of semi-graphs.

For each $v \in v(\Gamma_{\mathscr{Y}_j})$, write $\widetilde{Y}_{j,v}$ for the normalization of the irreducible component $Y_{j,v} \subseteq (\mathscr{Y}_j)_s$ corresponding to v. We define a semi-graph with p-rank

$$\mathfrak{G}_{\mathscr{Y}_j} \stackrel{\text{def}}{=} (\mathbb{G}_{\mathscr{Y}_j}, \sigma_{\mathfrak{G}_{\mathscr{Y}_j}}),\ j = 0, \ldots, r,$$

associated to $(\mathscr{Y}_j)_s$ as follows:

(i) $\mathbb{G}_{\mathscr{Y}_j} \stackrel{\text{def}}{=} \Gamma_{\mathscr{Y}_j}$;

(ii) for each $v \in v(\mathbb{G}_{\mathscr{Y}_j})$, we put $\sigma_{\mathfrak{G}_{\mathscr{Y}_j}}(v) \stackrel{\text{def}}{=} \sigma(\widetilde{Y}_{j,v})$.

Then $\Phi_{\mathscr{Y}/\mathscr{X}^{\text{sst}}}$ induces a sequence of morphisms of semi-graphs with p-rank

$$\Phi_{\mathfrak{G}_{\mathscr{Y}}/\mathfrak{G}_{\mathscr{X}^{\text{sst}}}} : \mathfrak{G}_{\mathscr{Y}} \stackrel{\text{def}}{=} \mathfrak{G}_{\mathscr{Y}_r} \stackrel{\mathfrak{b}_r}{\to} \mathfrak{G}_{\mathscr{Y}_{r-1}} \stackrel{\mathfrak{b}_{r-1}}{\to} \ldots \stackrel{\mathfrak{b}_1}{\to} \mathfrak{G}_{\mathscr{X}^{\text{sst}}} \stackrel{\text{def}}{=} \mathfrak{G}_{\mathscr{Y}_0},$$

where $\mathfrak{b}_j : \mathfrak{G}_{\mathscr{Y}_j} \to \mathfrak{G}_{\mathscr{Y}_{j-1}}$, $j = 1, \ldots, r$, is induced by $\beta_j : \Gamma_{\mathscr{Y}_j} \to \Gamma_{\mathscr{Y}_{j-1}}$, $j = 1, \ldots, r$. By using the Deuring-Shafarevich formula and the Zariski-Nagata purity theorem, we have that $\mathfrak{b}_j$, $j = 1, \ldots, r$, is a p-covering. Moreover, $\mathfrak{b} \stackrel{\text{def}}{=} \mathfrak{b}_1 \circ \cdots \circ \mathfrak{b}_r$ is a G-covering. Then we have

$$\sigma_{\mathfrak{G}_{\mathscr{Y}}}(\mathfrak{G}_{\mathscr{Y}}) = \sigma(\mathscr{Y}_s).$$

Summarizing the discussions above, we obtain the following proposition.

Proposition 3.8. *Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and $\mathscr{Y}_s$ the special fiber of $\mathscr{Y}$ over s. Then there exists a G-covering of semi-graphs with p-rank $\mathfrak{b} : \mathfrak{G}_{\mathscr{Y}} \to \mathfrak{G}_{\mathscr{X}^{\text{sst}}}$ associated to f (which is constructed above) such that $\sigma(\mathscr{Y}_s) = \sigma_{\mathfrak{G}_{\mathscr{Y}}}(\mathfrak{G}_{\mathscr{Y}})$.*

3.3 Local cases

We maintain the notation introduced in Section 3.2. Let x be a vertical point associated to f. Write Y' for the normalization of X in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f, and write ψ for the normalization morphism $Y' \to X$. Then Y' admits a natural action of G induced by the action of G on the generic fiber of Y. Let $y' \in \psi^{-1}(x)$. Write $I_{y'} \subseteq G$ for the inertia subgroup of y'. Proposition 3.4 implies that the morphism of pointed smooth curves $(Y_\eta/I_{y'}, D_{Y_\eta}/I_{y'}) \to \mathscr{X}_\eta$ over η induced by f extends to a pointed semi-stable covering $\mathscr{Y}_{I_{y'}} \to \mathscr{X}$ over S. In order to calculate the p-rank of $f^{-1}(x)$, since the morphism $\mathscr{Y}_{I_{y'}} \to \mathscr{X}$ is finite étale over x, by replacing $\mathscr{X}$ by $\mathscr{Y}_{I_{y'}}$, we may assume that G is equal to $I_{y'}$. In the remainder of this subsection, we shall assume that $G = I_{y'}$. Then $f^{-1}(x)$ is connected.

Write $\mathscr{X}_s^{\mathrm{sst}} = (X_s^{\mathrm{sst}}, D_{X_s^{\mathrm{sst}}})$ (resp. $\mathscr{Y}_s = (Y_s, D_{Y_s})$) for the special fiber of $\mathscr{X}^{\mathrm{sst}}$ (resp. $\mathscr{Y}$) over s, and $(-)_{\mathrm{red}}$ for the reduced induced closed subscheme of $(-)$. By the general theory of semi-stable curves, $g^{-1}(x)_{\mathrm{red}} \subset X_s^{\mathrm{sst}}$ (resp. $f^{-1}(x)_{\mathrm{red}} = h^{-1}(g^{-1}(x))_{\mathrm{red}} \subset Y_s$) is a semi-stable curve over s. In particular, the irreducible components of $g^{-1}(x)_{\mathrm{red}}$ are isomorphic to $\mathbb{P}^1_k$. Write V_X for the set of closed points

$$g^{-1}(x)_{\mathrm{red}} \cap \overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}},$$

where $\overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}}$ denotes the topological closure of $X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}$ in X_s^{sst}. Write $V_Y \subset \mathscr{Y}_s$ for the set of closed points $\{h^{-1}(p)_{\mathrm{red}}\}_{p \in V_X}$. Note that $\#V_X = 1$ if x is a smooth point of $\mathscr{X}_s$, and $\#V_X = 2$ if x is a node of $\mathscr{X}_s$. We define two pointed semi-stable curves over s to be

$$\mathscr{E}_X \stackrel{\mathrm{def}}{=} (g^{-1}(x)_{\mathrm{red}}, (D_{X^{\mathrm{sst}}} \cap g^{-1}(x)_{\mathrm{red}}) \cup V_X)$$

and

$$\mathscr{E}_Y \stackrel{\mathrm{def}}{=} (f^{-1}(x)_{\mathrm{red}}, (D_Y \cap f^{-1}(x)_{\mathrm{red}}) \cup V_Y).$$

Then we obtain a morphism of pointed semi-stable curves

$$\rho_{\mathscr{E}_Y/\mathscr{E}_X} : \mathscr{E}_Y \to \mathscr{E}_X$$

induced by h. Since $f^{-1}(x)$ is connected, $\mathscr{E}_Y$ admits a natural action of G induced by the action of G on the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$. Write $\Gamma_{\mathscr{E}_Y}$ (resp. $\Gamma_{\mathscr{E}_X}$) for the dual semi-graph of $\mathscr{E}_Y$ (resp. $\mathscr{E}_X$). Note that $\Gamma_{\mathscr{E}_X}$ is a tree. We obtain a map of semi-graphs

$$\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$$

induced by $\rho_{\mathscr{E}_Y/\mathscr{E}_X}$. Moreover, Proposition 3.5 implies that the map $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$ is a morphism of semi-graphs.

For each $v \in v(\Gamma_{\mathscr{E}_Y})$, write $\widetilde{Y}_v$ for the normalization of the irreducible component $Y_v \subseteq \mathscr{E}_Y$ corresponding to v. Furthermore, we define two semi-graphs with p-rank

$$\mathfrak{E}_Y \stackrel{\mathrm{def}}{=} (\mathbb{E}_Y, \sigma_{\mathfrak{E}_Y})$$

and

$$\mathfrak{E}_X \stackrel{\mathrm{def}}{=} (\mathbb{E}_X, \sigma_{\mathfrak{E}_X})$$

associated to $\mathscr{E}_Y$ and $\mathscr{E}_X$, respectively, as follows:

(i) $\mathbb{E}_Y \overset{\text{def}}{=} \Gamma_{\mathscr{E}_Y}$ and $\mathbb{E}_X \overset{\text{def}}{=} \Gamma_{\mathscr{E}_X}$;

(ii) for each $v \in v(\mathbb{E}_Y)$ and each $w \in v(\mathbb{E}_X)$, we put $\sigma_{\mathfrak{E}_Y}(v) \overset{\text{def}}{=} \sigma(\widetilde{Y}_v)$ and $\sigma_{\mathfrak{E}_X}(w) \overset{\text{def}}{=} 0$, respectively.

The morphism of dual semi-graphs $\delta_{\mathscr{E}_Y/\mathscr{E}_X} : \Gamma_{\mathscr{E}_Y} \to \Gamma_{\mathscr{E}_X}$ induces a morphism of semi-graphs with p-rank

$$\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \to \mathfrak{E}_X.$$

Moreover, we see that $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X}$ is a G-covering. Then we have

$$\sigma_{\mathfrak{E}_Y}(\mathfrak{E}_Y) = \sigma(f^{-1}(x)_{\text{red}}) = \sigma(f^{-1}(x)).$$

Summarizing the discussions above, we obtain the following proposition.

Proposition 3.9. *Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and x a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected. Then there exists a G-covering of semi-graphs with p-rank $\mathfrak{d}_{\mathfrak{E}_Y/\mathfrak{E}_X} : \mathfrak{E}_Y \to \mathfrak{E}_X$ associated to f and x which is constructed above such that $\sigma_{\mathfrak{E}_Y}(\mathfrak{E}_Y) = \sigma(f^{-1}(x))$.*

In the remainder of this subsection, we suppose that the vertical point x is a node of $\mathscr{X}_s$. We will prove that, when the vertical point x is a node of $\mathscr{X}_s$, Proposition 3.9 can be simplified by applying Raynaud's result (cf. Proposition 3.7).

Write X_1' and X_2' (which may be equal) for the irreducible components of $\mathscr{X}_s$ which contain x. Write X_1 and X_2 for the strict transforms of X_1' and X_2' under the birational morphism $g : \mathscr{X}^{\text{sst}} \to \mathscr{X}$, respectively. By the general theory of semi-stable curves, $g^{-1}(x)_{\text{red}} \subseteq X_s^{\text{sst}}$ is a semi-stable curve over s and $g^{-1}(x)_{\text{red}} \cap D_{X_s^{\text{sst}}} = \emptyset$. Moreover, the irreducible components of $g^{-1}(x)_{\text{red}}$ are isomorphic to $\mathbb{P}^1_k$. Let C be the semi-stable subcurve of $g^{-1}(x)_{\text{red}}$ which is a chain of projective lines $\cup_{i=1}^n P_i$ such that the following conditions hold:

(i) for any $w, t \in \{1, \ldots, n\}$, $P_w \cap P_t = \emptyset$ if $|w-t| \geq 2$, and $P_w \cap P_t$ is reduced to a point if $|w-t| = 1$;

(ii) $P_1 \cap X_1$ (resp. $P_n \cap X_2$) is reduced to a point;

(iii) $C \cap \overline{\{X_s^{\text{sst}} \setminus g^{-1}(x)_{\text{red}}\}} = (P_1 \cap X_1) \cup (P_n \cap X_2)$, where $\overline{\{X_s^{\text{sst}} \setminus g^{-1}(x)_{\text{red}}\}}$ denotes the topological closure of $X_s^{\text{sst}} \setminus g^{-1}(x)_{\text{red}}$ in X_s^{sst}.

Then we have

$$g^{-1}(x)_{\text{red}} = C \cup B,$$

where B denotes the topological closure of $g^{-1}(x)_{\text{red}} \setminus C$ in $g^{-1}(x)_{\text{red}}$. Note that $B \cap C$ are smooth points of C. Then Proposition 3.7 implies that the p-rank of the connected components of $h^{-1}(B)$ are equal to 0. Thus, we have $\sigma(f^{-1}(x)) = \sigma(h^{-1}(C))$.

Let $\{V_i\}_{i=1}^n$ be a set of irreducible components of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ such that the following conditions hold:

(i) $h(V_i) = P_i$ for $i = 1, \ldots, n$;

(ii) the union $\bigcup_{i=1}^n V_i \subseteq \mathscr{Y}_s$ is a connected semi-stable curve over s and $(\bigcup_{i=1}^n V_i) \cap D_{Y_s} = \emptyset$.

Write $I_{V_i} \subseteq G$, $i = 1, \ldots, n$, for the inertia subgroup of V_i, and for any closed point $y_i \in V_i$, write $I_{y_i} \subseteq G$ for the inertia subgroup of y_i. Then we have the following lemma.

Lemma 3.10. *Write* Ray_{V_i}, $i = 1, \ldots, n$, *for the set of the closed points* $h^{-1}(C \cap B)_{\mathrm{red}} \cap V_i$. *Then, for any* $y_i \in \mathrm{Ray}_{V_i}$, *we have* $I_{y_i} = I_{V_i}$.

Proof. Since $I_{y_i} \supseteq I_{V_i}$, we only need to prove that $I_{y_i} \subseteq I_{V_i}$. Note that I_{V_i} is a normal subgroup of I_{y_i}. By replacing G and $\mathscr{X}^{\mathrm{sst}}$ by I_{y_i} and $\mathscr{Y}/I_{y_i}$, respectively, we may assume that $G = I_{y_i}$. Then we have $\# h^{-1}(h(y_i))_{\mathrm{red}} = 1$.

Consider the quotient curve $\mathscr{Y}/I_{V_i}$. By [R, Appendice Corollaire], we have that $\mathscr{Y}/I_{V_i}$ is a pointed semi-stable curve over S. Write $h_{I_{V_i}}$ for the quotient morphism $\mathscr{Y} \to \mathscr{Y}/I_{V_i}$ and $g_{I_{V_i}}$ for the $\mathscr{Y}/I_{V_i} \to \mathscr{X}^{\mathrm{sst}}$ induced by h such that $h = g_{I_{V_i}} \circ h_{I_{V_i}}$. Write E_{y_i} for the connected component of $h^{-1}(B)_{\mathrm{red}}$ which contains y_i. By contracting $h_{I_{V_i}}(E_{y_i}) \subset \mathscr{Y}/I_{V_i} \times_S s$ (resp. $h(E_{y_i}) \subset \mathscr{X}^{\mathrm{sst}}_s$), we obtain a fiber surface $(\mathscr{Y}/I_{V_i})^{\mathrm{c}}$ and a semi-stable curve $(\mathscr{X}^{\mathrm{sst}})^c$ over S. Moreover, we have two contracting morphisms

$$c_{h_{I_{V_i}}(E_{y_i})} : \mathscr{Y}/I_{V_i} \to (\mathscr{Y}/I_{V_i})^{\mathrm{c}}$$

and

$$c_{h(E_{y_i})} : \mathscr{X}^{\mathrm{sst}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}.$$

Furthermore, we obtain a morphism of fiber surfaces

$$g^{\mathrm{c}}_{I_{V_i}} : (\mathscr{Y}/I_{V_i})^{\mathrm{c}} \to (\mathscr{X}^{\mathrm{sst}})^c$$

induced by $g_{I_{V_i}}$ such that $c_{h(E_{y_i})} \circ g_{I_{V_i}} = g^{\mathrm{c}}_{I_{V_i}} \circ c_{h_{I_{V_i}}(E_{y_i})}$. Note that $(c_{h(E_{y_i})} \circ h)(y_i)$ is a smooth point of the special fiber of $(\mathscr{X}^{\mathrm{sst}})^c$, and $g^{\mathrm{c}}_{I_{V_i}}$ is étale at the generic point of $(c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}})(V_i)$.

Write $y_i^{\mathrm{c}} \in (\mathscr{Y}/I_{V_i})^{\mathrm{c}}$ and $x_i^{\mathrm{c}} \in (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ for $(c_{h_{I_{V_i}}(E_{y_i})} \circ h_{I_{V_i}})(y_i)$ and $(c_{h(E_{y_i})} \circ h)(y_i)$, respectively. Consider the local morphism

$$g_{y_i^{\mathrm{c}}} : \mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \to \mathrm{Spec}\, \mathcal{O}_{(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}, x_i^{\mathrm{c}}}$$

induced by $g^{\mathrm{c}}_{I_{V_i}}$. Note that Proposition 3.7 implies that $\mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \times_S s$ is irreducible. Then $g_{y_i^{\mathrm{c}}}$ is generically étale at the generic point of $\mathrm{Spec}\, \mathcal{O}_{(\mathscr{Y}/I_{V_i})^{\mathrm{c}}, y_i^{\mathrm{c}}} \times_S s$. Thus, the Zariski-Nagata purity theorem implies that $g_{y_i^{\mathrm{c}}}$ is étale.

If $I_{V_i} \neq I_{y_i}$, then we obtain that $g_{y_i^{\mathrm{c}}}$ is not an identity. Thus, we have $\# h^{-1}(h(y_i))_{\mathrm{red}} \neq 1$. This contradicts our assumption. Then we obtain $I_{V_i} = I_{y_i}$. We complete the proof of the lemma. □

We define two pointed semi-stable curves to be

$$\mathscr{C}_Y \overset{\mathrm{def}}{=} (h^{-1}(C)_{\mathrm{red}}, h^{-1}((C \cap X_1) \cup (C \cap X_2)))$$

and

$$\mathscr{C}_X \overset{\mathrm{def}}{=} (C, (C \cap X_1) \cup (C \cap X_2)).$$

Moreover, we have a natural morphism of the pointed semi-stable curves

$$\rho_{\mathscr{C}_Y/\mathscr{C}_X} : \mathscr{C}_Y \to \mathscr{C}_X$$

over s induced by $h : \mathscr{Y} \to \mathscr{X}^{\mathrm{sst}}$. Since $f^{-1}(x)_{\mathrm{red}}$ is connected, $\mathscr{C}_Y$ admits a natural action of G induced by the action of G on $f^{-1}(x)_{\mathrm{red}}$. Write $\Gamma_{\mathscr{C}_Y}$ (resp. $\Gamma_{\mathscr{C}_X}$) for the dual semi-graph of $\mathscr{C}_Y$ (resp. $\mathscr{C}_X$). Proposition 3.5 implies that the map of semi-graphs

$$\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \to \Gamma_{\mathscr{C}_X}$$

induced by $\rho_{\mathscr{C}_Y/\mathscr{C}_X}$ is a morphism of semi-graphs.

For each $v \in v(\Gamma_{\mathscr{C}_Y})$, write $\widetilde{Y}_v$ for the normalization of the irreducible component $Y_v \subseteq \mathscr{C}_Y$ corresponding to v. We define two semi-graphs with p-rank

$$\mathfrak{C}_Y \overset{\mathrm{def}}{=} (\mathbb{C}_Y, \sigma_{\mathfrak{C}_Y})$$

and

$$\mathfrak{C}_X \overset{\mathrm{def}}{=} (\mathbb{C}_X, \sigma_{\mathfrak{C}_X})$$

associated to $\mathscr{C}_Y$ and $\mathscr{C}_X$, respectively, as follows:

(i) $\mathbb{C}_Y \overset{\mathrm{def}}{=} \Gamma_{\mathscr{C}_Y}$ and $\mathbb{C}_X \overset{\mathrm{def}}{=} \Gamma_{\mathscr{C}_X}$;

(ii) for each $v \in v(\mathbb{C}_Y)$ and each $w \in v(\mathbb{C}_X)$, we put $\sigma_{\mathfrak{C}_Y}(v) \overset{\mathrm{def}}{=} \sigma(\widetilde{Y}_v)$ and $\sigma_{\mathfrak{C}_X}(w) \overset{\mathrm{def}}{=} 0$, respectively.

The morphism of dual semi-graphs $\delta_{\mathscr{C}_Y/\mathscr{C}_X} : \Gamma_{\mathscr{C}_Y} \to \Gamma_{\mathscr{C}_X}$ induces a morphism of semi-graphs with p-rank

$$\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \to \mathfrak{C}_X.$$

Moreover, by Lemma 3.10, we see that $\sigma_{\mathfrak{C}_Y}(v)$ satisfies the Deuring-Shafarevich type formula for each $v \in v(\mathbb{C}_Y)$. This implies that $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \to \mathfrak{C}_X$ is a G-covering of semi-graphs with p-rank. Note that by the construction, $\mathfrak{C}_X$ is an n-chain (cf. Definition 2.9). Furthermore, we have that

$$\sigma_{\mathfrak{C}_Y}(\mathfrak{C}_Y) = \sigma(h^{-1}(C)) = \sigma(f^{-1}(x)).$$

Summarizing the discussion above, we obtain the following proposition.

Proposition 3.11. *Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-pointed semi-stable covering over S and $x \in \mathscr{X}_s$ a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected, and that x is a node of $\mathscr{X}_s$. Then there exists a G-covering of semi-graphs with p-rank $\mathfrak{d}_{\mathfrak{C}_Y/\mathfrak{C}_X} : \mathfrak{C}_Y \to \mathfrak{C}_X$ associated to f and x (which is constructed above) such that $\mathfrak{C}_X$ is an n-chain and $\sigma_{\mathfrak{C}_Y}(\mathfrak{C}_Y) = \sigma(f^{-1}(x))$.*

4 Formulas for local and global p-rank of coverings of curves

4.1 Inertia subgroups and a criterion for the existence of vertical fibers

In this subsection, we study the relationship between the inertia subgroups of nodes and the inertia subgroups of irreducible components of special fibers of G-pointed semi-stable coverings.

Lemma 4.1. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a* **finite** *G-pointed semi-stable covering over S, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$, and $y \in \mathscr{Y}_s$ a node. Let Y_1 and Y_2 (which may be equal) be the irreducible components of $\mathscr{Y}_s$ which contain y. Write $I_y \subseteq G$ (resp. $I_{Y_1} \subseteq G$, $I_{Y_2} \subseteq G$) for the inertia subgroup of y (resp. Y_1, Y_2). If G is a p-group, then the inertia subgroup I_y is generated by I_{Y_1} and I_{Y_2}.*

Proof. Write I for the group generated by I_{Y_1} and I_{Y_2}. Then we have $I \subseteq I_y$. Consider the quotient $\mathscr{Y}/I$. We obtain two morphisms of pointed semi-stable curves $\mu_1 : \mathscr{Y} \to \mathscr{Y}/I$ and $\mu_2 : \mathscr{Y}/I \to \mathscr{X}$ over S such that $\mu_2 \circ \mu_1 = f$. Note that $\mathscr{Y}/I$ is a pointed semi-stable curve over S, and $\mu_1(y)$ is a node of the special fiber $(\mathscr{Y}/I)_s$ of $\mathscr{Y}/I$ (cf. [R, Appendice, Corollaire] and Proposition 3.5). Moreover, μ_2 is generically étale at the generic points of $\mu_1(Y_1)$ and $\mu_1(Y_2)$. Then, by applying [T2, Lemma 2.1 (iii)] to the local morphism $\mathrm{Spec}\, \mathcal{O}_{\mathscr{Y}/I, \mu_1(y)} \to \mathrm{Spec}\, \mathcal{O}_{\mathscr{X}, f(y)}$ induced by μ_2, we obtain that μ_2 is tamely ramified at $\mu_1(y)$. Moreover, since G is a p-group, μ_2 is étale at $\mu_1(y)$. This means that $I_y \subseteq I$. Thus, $I_y = I$. □

The following criterion for the existence of vertical fibers due to A. Tamagawa (cf. [T2, Propoisiton 4.3 (ii)]).

Proposition 4.2. *Let G be a finite group, $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semi-stable covering over S and x a node of $\mathscr{X}_s$. Suppose that, for each irreducible component $Z \overset{\mathrm{def}}{=} \overline{\{z\}}$ of $\mathrm{Spec}\, \widehat{\mathcal{O}}_{\mathscr{X}_s, x}$ and each point w of the fiber $\mathscr{Y} \times_{\mathscr{X}} z$, the natural morphism from the integral closure W^s of Z in $k(w)^s$ to Z is wildly ramified, where $k(w)^s$ denotes the maximal separable subextension of $k(w)$ in $k(z)$. Then x is a vertical point associated to f.*

Remark 4.2.1. In [T2], Tamagawa only treated the case where f is a stable covering. It is easy to see that Tamagawa's proof also holds for pointed semi-stable coverings.

Next, we prove a criterion of existence of vertical fibers over nodes as follows:

Proposition 4.3. *Let G be a finite group, $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ a G-pointed semi-stable covering over S, $\mathscr{Y}_\eta = (Y_\eta, D_{Y_\eta})$ the generic fiber of $\mathscr{Y}$ over η, $\mathscr{Y}_s = (Y_s, D_{Y_s})$ the special fiber of $\mathscr{Y}$ over s, and x a node of $\mathscr{X}_s$. Write $\mathscr{Y}'$ for the normalization of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f, and write ψ_2 for the resulting normalization morphism $\mathscr{Y}' \to \mathscr{X}$. There is a natural morphism of fiber surfaces $\psi_1 : \mathscr{Y} \to \mathscr{Y}'$ induced by f such that $\psi_2 \circ \psi_1 = f$. Write X_1 and X_2 (which may be equal) for the irreducible components of $\mathscr{X}_s$ which contain x. Let*

$y' \in \psi_2^{-1}(x)_{\mathrm{red}}$, Y_1 *and* Y_2 *the irreducible components of* $\mathscr{Y}_s$ *such that* $y' \in \psi_1(Y_1) \cap \psi_1(Y_2)$. *Write* $I_{Y_1} \subseteq G$ *and* $I_{Y_2} \subseteq G$ *for the inertia subgroups of* Y_1 *and* Y_2*, respectively. Then, if neither* $I_{Y_1} \subseteq I_{Y_2}$ *nor* $I_{Y_1} \supseteq I_{Y_2}$ *holds, we have that* x *is a vertical point associated to* f.

Proof. To verify the proposition, we may assume that x is not a vertical point associated to f. Then $f^{-1}(x)$ is a finite set. Let $a \in \psi_2^{-1}(x)$ and $b \in \psi_1^{-1}(a)$. Thus, ψ_1 induces an isomorphism $\mathrm{Spec}\,\mathcal{O}_{\mathscr{Y},b} \to \mathrm{Spec}\,\mathcal{O}_{\mathscr{Y}',a}$. Write y for $\psi_1^{-1}(y')_{\mathrm{red}}$. By replacing $\mathscr{X}$ by the quotient $\mathscr{Y}/D_y$ and G by $D_y \subseteq G$, respectively, where D_y denotes the decomposition group of y, we may assume that $f^{-1}(x)_{\mathrm{red}} = \{y\} \subseteq Y_1 \cap Y_2$. Consider the quotient curve $\mathscr{Y}/I_{Y_1}$ (resp. $\mathscr{Y}/I_{Y_2}$). Note that $\mathscr{Y}/I_{Y_1}$ (resp. $\mathscr{Y}/I_{Y_2}$) is a pointed semi-stable curve over S. We obtain two morphisms of pointed semi-stable curves

$$\lambda_1^1 : \mathscr{Y} \to \mathscr{Y}/I_{Y_1} \text{ (resp. } \lambda_1^2 : \mathscr{Y} \to \mathscr{Y}/I_{Y_2})$$

and

$$\lambda_2^1 : \mathscr{Y}/I_{Y_1} \to \mathscr{X} \text{ (resp. } \lambda_2^2 : \mathscr{Y}/I_{Y_2} \to \mathscr{X})$$

over S such that $\lambda_2^1 \circ \lambda_1^1 = f$ (resp. $\lambda_2^2 \circ \lambda_1^2 = f$). Note that λ_2^1 (resp. λ_2^2) is étale at the generic point of $\lambda_1^1(Y_1)$ (resp. $\lambda_1^2(Y_2)$) of degree $\#G/\#I_{Y_1}$ (resp. $\#G/\#I_{Y_2}$).

If λ_2^1 (resp. λ_2^2) is generically étale at the generic point of $\lambda_1^1(Y_2)$ (resp. $\lambda_1^2(Y_1)$), then, by applying [T2, Lemma 2.1 (iii)] to

$$\mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_1},\lambda_1^1(y)} \to \mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathscr{X},x} \text{ (resp. } \mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathscr{Y}/I_{Y_2},\lambda_1^2(y)} \to \mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathscr{X},x}),$$

we obtain $\mathrm{Spec}\,\widehat{\mathcal{O}}_{\lambda_1^1(Y_1),\lambda_1^1(y)} \to \mathrm{Spec}\,\widehat{\mathcal{O}}_{X_1,x}$ (resp. $\mathrm{Spec}\,\widehat{\mathcal{O}}_{\lambda_1^2(Y_2),\lambda_1^2(y)} \to \mathrm{Spec}\,\widehat{\mathcal{O}}_{X_2,x}$) induced by λ_2^1 (resp. λ_2^2) is tamely ramified with ramification index t_1 (resp. t_2). Thus, we have $(t_1, p) = 1$ (resp. $(t_2, p) = 1$). On the other hand, since I_{Y_1} (resp. I_{Y_2}) does not contain I_{Y_2} (resp. I_{Y_1}), and I_{Y_2} (resp. I_{Y_1}) is a p-group, we have $p|t_1$ (resp. $p|t_2$). This is a contradiction. Thus, λ_2^1 (resp. λ_2^2) is not generically étale at the generic point of $\lambda_1^1(Y_2)$ (resp. $\lambda_1^2(Y_1)$).

Moreover, the morphism $\mathrm{Spec}\,\widehat{\mathcal{O}}_{\lambda_1^1(Y_1),\lambda_1^1(y)} \to \mathrm{Spec}\,\widehat{\mathcal{O}}_{X_1,x}$ (resp. $\mathrm{Spec}\,\widehat{\mathcal{O}}_{\lambda_1^2(Y_2),\lambda_1^2(y)} \to \mathrm{Spec}\,\widehat{\mathcal{O}}_{X_2,x}$) induced by λ_2^1 (resp. λ_2^2) is wildly ramified. Thus, Proposition 4.2 implies that x is a vertical point associated to f. This is a contradiction. We complete the proof of the proposition. □

Corollary 4.4. *Let* G *be a finite group,* $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X}$ *a* G*-pointed semi-stable covering over* S*,* $\mathscr{Y}_s = (Y_s, D_{Y_s})$ *the special fiber of* $\mathscr{Y}$*,* $y \in \mathscr{Y}_s$ *a node. Let* Y_1 *and* Y_2 *(which may be equal) be the irreducible components of* $\mathscr{Y}_s$ *which contain* y*. Write* $I_y \subseteq G$ *(resp.* $I_{Y_1} \subseteq G$*,* $I_{Y_2} \subseteq G$*) for the inertia subgroup of* y *(resp.* Y_1*,* Y_2*). Suppose that* f *is a* **finite** *morphism. Then either* $I_{Y_1} \subseteq I_{Y_2}$ *or* $I_{Y_1} \supseteq I_{Y_2}$ *holds. Moreover, if* G *is a* p*-group, then the inertia subgroup* I_y *is equal to either* I_{Y_1} *or* I_{Y_2}*.*

Proof. The corollary follows from Lemma 4.1 and Proposition 4.3. □

4.2 Global version

In this subsection, we assume that G is a finite p-group. Let $f : \mathscr{Y} = (Y, D_Y) \to \mathscr{X} = (X, D_X)$ be a G-pointed semi-stable covering over S, $h : \mathscr{Y} \to \mathscr{Y}/G \stackrel{\mathrm{def}}{=} \mathscr{X}^{\mathrm{sst}} =$

$(X^{\rm sst}, D_{X^{\rm sst}})$ the quotient morphism, and $\Gamma_{\mathscr{X}_s^{\rm sst}}$ the dual semi-graph of the special fiber $\mathscr{X}_s^{\rm sst} = (X_s^{\rm sst}, D_{X_s^{\rm sst}})$ of $\mathscr{X}^{\rm sst}$.

For each $v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})$ (resp. $e \in e(\Gamma_{\mathscr{X}_s^{\rm sst}})$), write X_v (resp. x_e) for the irreducible component of $\mathscr{X}_s^{\rm sst}$ corresponding to v (resp. for the node of $\mathscr{X}_s^{\rm sst}$ corresponding to e if $e \in e^{\rm cl}(\Gamma_{\mathscr{X}_s^{\rm sst}})$ or the marked point of $\mathscr{X}_s^{\rm sst}$ corresponding to e if $e \in e^{\rm op}(\Gamma_{\mathscr{X}_s^{\rm sst}})$) and $\widetilde{X}_v$ for the normalization of X_v. For each $v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})$ (resp. $e \in e(\Gamma_{\mathscr{X}_s^{\rm sst}})$), let Y_v (resp. y_e) be an irreducible component of $h^{-1}(X_v)_{\rm red}$ (resp. a closed point of $h^{-1}(x_e)_{\rm red}$). Write $I_{Y_v} \subseteq G$ (resp. $I_{y_e} \subseteq G$) for the inertia subgroup of Y_v (resp. y_e). Since $\#I_{Y_v}$ (resp. $\#I_{y_e}$) does not depend on the choices of Y_v (resp. y_e), we may use the notation $\#I_v$ (resp. $\#I_e$) to denote $\#I_{Y_v}$ (resp. $\#I_{y_e}$). We put

$$\#I_e^{\rm m} \overset{\rm def}{=} \max_{v \in v(e)}\{\#I_v\},\ e \in e^{\rm cl}(\Gamma_{\mathscr{X}_s^{\rm sst}}).$$

Note that Corollary 4.4 implies that $\#I_e = \#I_e^{\rm m}$. Then we have the following theorem.

Theorem 4.5. *We maintain the notations introduced above. Then we have*

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})} \Big(\#G/\#I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v)\setminus e^{\rm lp}(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1\Big)$$

$$+ \sum_{e \in e^{\rm cl}(\Gamma_{\mathscr{X}_s^{\rm sst}})\setminus e^{\rm lp}(\Gamma_{\mathscr{X}_s^{\rm sst}})} (\#G/\#I_e - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})} \#e^{\rm lp}(v)(\#G/\#I_v - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{\mathscr{X}_s^{\rm sst}}, \mathbf{C})).$$

In particular, if $f : \mathscr{Y} \to \mathscr{X}$ is a G-semi-stable covering, then we have

$$\sigma(\mathscr{Y}_s) = \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})} \Big(\#G/\#I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v)\setminus e^{\rm lp}(v)} \#G/\#I_e^{\rm m}(\#I_e^{\rm m}/\#I_v - 1) + 1\Big)$$

$$+ \sum_{e \in e^{\rm cl}(\Gamma_{\mathscr{X}_s^{\rm sst}})\setminus e^{\rm lp}(\Gamma_{\mathscr{X}_s^{\rm sst}})} (\#G/\#I_e^{\rm m} - 1) + \sum_{v \in v(\Gamma_{\mathscr{X}_s^{\rm sst}})} \#e^{\rm lp}(v)(\#G/\#I_v - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{\mathscr{X}_s^{\rm sst}}, \mathbf{C})).$$

Proof. The theorem follows from Theorem 2.8 and Proposition 3.8. □

Remark 4.5.1. Let G be a finite p-group, $f : \mathscr{Y} \to \mathscr{X}$ a G-pointed semi-stable covering over S, and $\mathscr{Z}$ a pointed semi-stable curve over S such that the generic fiber $\mathscr{Z}_\eta$ is equal to $\mathscr{X}_\eta$. By applying Proposition 3.4 and by replacing S by a finite extension of S, the morphism $f_\eta : \mathscr{Y}_\eta \to \mathscr{X}_\eta$ over η induced by f on generic fibers can be extended to a G-pointed semi-stable covering $g : \mathscr{W} \to \mathscr{Z}$ over S such that $g_\eta = f_\eta$. In this remark, let us explain that the formula of Theorem 4.5 only depends on the G-pointed stable coverings. This means that we should prove that $\sigma(\mathscr{Y}_s) = \sigma(\mathscr{W}_s)$ by applying the formula of Theorem 4.5.

For simplicity, we only treat the case of semi-stable coverings. Write X, Y, Z, and W for $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$, and $\mathscr{W}$, respectively. To verify $\sigma(Y_s) = \sigma(W_s)$ by using the formula of Theorem 4.5, it is sufficient to treat the case where $\mathscr{X}$ is a stable curve over S. Let $c_X : Z \to X$ be the natural contracting morphism over S. Moreover, for simplicity, we suppose that all the irreducible components of X_s are smooth, and that Z^* is the unique irreducible component of the special fiber Z_s of Z such that $c_X(Z^*)$ is a closed point of X.

Since $g : W \to Z$ is a G-semi-stable coverings such that $f_\eta = g_\eta$, we obtain a contracting morphism $c_Y : W \to Y$ over S. If c_Y is an isomorphism, then $\sigma(Y_s) = \sigma(W_s)$. Thus, we may assume that c_Y is not an isomorphism. Then c_Y is not a finite morphism.

Write X^{sst} and Z^{sst} for the quotient curves Y/G and W/G, $h_Y : Y \to X^{\text{sst}}$ and $h_W : W \to Z^{\text{sst}}$ for the quotient morphisms, respectively. Then c_Y induces a natural morphism $c_{X^{\text{sst}}} : Z^{\text{sst}} \to X^{\text{sst}}$ over S. We still use that notation Z^* to denote the irreducible component of Z^{sst}_s whose image (under the natural morphism $Z^{\text{sst}} \to Z$) is $Z^* \subset Z_s$. Write x for $c_{X^{\text{sst}}}(Z^*)$. Write Z_1 and Z_2 for the irreducible components of Z^{sst}_s such that $z_1 \stackrel{\text{def}}{=} Z^* \cap Z_1 \neq \emptyset$ and $z_2 \stackrel{\text{def}}{=} Z^* \cap Z_2 \neq \emptyset$. Let $y \in h_Y^{-1}(x)_{\text{red}}$ be a closed point of Y. Write W^* for the irreducible component of W_s such that $c_Y(W^*) = y$ (thus, $h_W(W^*) = Z^*$), W_1 for the irreducible component of W_s such that $h_W(W_1) = Z_1$ and $W_1 \cap W^* \neq \emptyset$, W_2 for the irreducible component of W_s such that $h_W(W_2) = Z_2$ and $W_2 \cap W^* \neq \emptyset$, Y_1 for $c_Y(W_1)$, and Y_2 for $c_Y(W_2)$. Let $w_1 \in W_1 \cap W^*$ and $w_2 \in W_2 \cap W^*$. Write I_y, I_{w_1}, I_{w_2}, I_{W_1}, I_{W_2}, I_{Y_1}, I_{Y_2}, and I_{W^*} for the inertia subgroups of y, w_1, w_2, W_1, W_2, Y_1, Y_2, and W^*, respectively. Note that we have $I_y = I_{W^*}$, $I_{Y_1} = I_{W_1}$, and $I_{Y_2} = I_{W_2}$. By applying Corollary 4.4, we may assume that $I_{Y_1} \subseteq I_{Y_2}$. Thus, Lemma 4.1 implies that $I_y = I_{Y_2} = I_{W_2} = I_{w_1} = I_{w_2} = I_{W^*}$.

Write v^*, v_{Z_1}, and v_{Z_2} for the vertices of $\Gamma_{Z^{\text{sst}}}$ corresponding to Z^*, Z_1, and Z_2, respectively. Write e_x, e_{z_1}, and e_{z_2} for the edges corresponding to x, z_1, and z_2, respectively. Since $\#I_y$, $\#I_{w_1}$, $\#I_{w_2}$, $\#I_{W_1}$, $\#I_{W_2}$, and $\#I_{W^*}$ do not depend on the choices of y, w_1, w_2, W_1, W_2, and W^*, respectively, we use the notations $\#I_{e_x}$, $\#I_{e_{z_1}}$, $\#I_{e_{z_2}}$, $\#I_{v_{Z_1}}$, $\#I_{v_{Z_2}}$, and $\#I_{v^*}$ to denote $\#I_y$, $\#I_{w_1}$, $\#I_{w_2}$, $\#I_{W_1}$, $\#I_{W_2}$, and $\#I_{W^*}$, respectively. Thus, we obtain

$$\sum_{e \in e(v^*)} \#G/\#I_e^{\text{m}}(\#I_e^{\text{m}}/\#I_{v^*} - 1) = 0.$$

Moreover, we have

$$\begin{aligned}
\sigma(W_s) = &\sum_{v \in v(\Gamma_{Z_s^{\text{sst}}})} \Big(\#G/\#I_v(\sigma(\widetilde{Z}_v) - 1) + \sum_{e \in e(v)} \#G/\#I_e^{\text{m}}(\#I_e^{\text{m}}/\#I_v - 1) + 1\Big) \\
&+ \sum_{e \in e(\Gamma_{Z_s^{\text{sst}}})} (\#G/\#I_e^{\text{m}} - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{Z_s^{\text{sst}}}, \mathbf{C})) \\
= &\sum_{v \in v(\Gamma_{Z_s^{\text{sst}}}) \setminus \{v^*\}} \Big(\#G/\#I_v(\sigma(\widetilde{Z}_v) - 1) + \sum_{e \in e(v)} \#G/\#I_e^{\text{m}}(\#I_e^{\text{m}}/\#I_v - 1) + 1\Big) \\
&+ \sum_{e \in e(\Gamma_{Z_s^{\text{sst}}}) \setminus \{e_{z_1}, e_{z_2}\}} (\#G/\#I_e^{\text{m}} - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{Z_s^{\text{sst}}}, \mathbf{C})) - \#G/\#I_{v^*} + 2(\#G/\#I_{v^*} - 1) + 1 \\
= &\sum_{v \in v(\Gamma_{Z_s^{\text{sst}}}) \setminus \{v^*\}} \Big(\#G/\#I_v(\sigma(\widetilde{Z}_v) - 1) + \sum_{e \in e(v)} \#G/\#I_e^{\text{m}}(\#I_e^{\text{m}}/\#I_v - 1) + 1\Big) \\
&+ \sum_{e \in e(\Gamma_{Z_s^{\text{sst}}}) \setminus \{e_{z_1}, e_{z_2}\}} (\#G/\#I_e^{\text{m}} - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{Z_s^{\text{sst}}}, \mathbf{C})) + \#G/\#I_{v^*} - 1.
\end{aligned}$$

On the other hand, we have

$$\sigma(Y_s) = \sum_{v \in v(\Gamma_{X_s^{\mathrm{sst}}})} \Big(\#G/\#I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v)} \#G/\#I_e^{\mathrm{m}}(\#I_e^{\mathrm{m}}/\#I_v - 1) + 1 \Big)$$

$$+ \sum_{e \in e(\Gamma_{X_s^{\mathrm{sst}}})} (\#G/\#I_e^{\mathrm{m}} - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{X_s^{\mathrm{sst}}}, \mathbf{C}))$$

$$= \sum_{v \in v(\Gamma_{X_s^{\mathrm{sst}}})} \Big(\#G/\#I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v)} \#G/\#I_e^{\mathrm{m}}(\#I_e^{\mathrm{m}}/\#I_v - 1) + 1 \Big)$$

$$+ \sum_{e \in e(\Gamma_{X_s^{\mathrm{sst}}}) \setminus \{e_x\}} (\#G/\#I_e^{\mathrm{m}} - 1) + \dim_{\mathbf{C}}(H^1(\Gamma_{X_s^{\mathrm{sst}}}, \mathbf{C})) + \#G/\#I_{e_x} - 1.$$

Since

$$\sum_{v \in v(\Gamma_{Z_s^{\mathrm{sst}}}) \setminus \{v^*\}} \Big(\#G/\#I_v(\sigma(\widetilde{Z}_v) - 1) + \sum_{e \in e(v)} \#G/\#I_e^{\mathrm{m}}(\#I_e^{\mathrm{m}}/\#I_v - 1) + 1 \Big)$$

$$= \sum_{v \in v(\Gamma_{X_s^{\mathrm{sst}}})} \Big(\#G/\#I_v(\sigma(\widetilde{X}_v) - 1) + \sum_{e \in e(v)} \#G/\#I_e^{\mathrm{m}}(\#I_e^{\mathrm{m}}/\#I_v - 1) + 1 \Big),$$

$$\sum_{e \in e(\Gamma_{Z_s^{\mathrm{sst}}}) \setminus \{e_{z_1}, e_{z_2}\}} (\#G/\#I_e^{\mathrm{m}} - 1) = \sum_{e \in e(\Gamma_{X_s^{\mathrm{sst}}}) \setminus \{e_x\}} (\#G/\#I_e^{\mathrm{m}} - 1),$$

$$\dim_{\mathbf{C}}(H^1(\Gamma_{Z_s^{\mathrm{sst}}}, \mathbf{C})) = \dim_{\mathbf{C}}(H^1(\Gamma_{X_s^{\mathrm{sst}}}, \mathbf{C})),$$

and $\#I_{e_x} = \#I_{v^*}$, we obtain $\sigma(Y_s) = \sigma(W_s)$.

4.3 Local version

We maintain the notation introduced in Section 4.2. Moreover, in this subsection, we suppose that x is a vertical point associated to f, and that $f^{-1}(x)$ is connected. Write g for the natural morphism $\mathscr{X}^{\mathrm{sst}} \stackrel{\mathrm{def}}{=} (X^{\mathrm{sst}}, D_{X^{\mathrm{sst}}}) \to \mathscr{X}$ over S induced by f such that $f = g \circ h$. Write V_X for the set of closed points

$$g^{-1}(x)_{\mathrm{red}} \cap \overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}},$$

where $\overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}}$ denotes the topological closure of $X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}$ in X_s^{sst}. We define a pointed semi-stable curve over s to be

$$\mathscr{E}_X \stackrel{\mathrm{def}}{=} (g^{-1}(x)_{\mathrm{red}}, (D_{X_s^{\mathrm{sst}}} \cap g^{-1}(x)_{\mathrm{red}}) \cup V_X).$$

Write $\Gamma_{\mathscr{E}_X}$ for the dual semi-graph of $\mathscr{E}_X$. Note that $\Gamma_{\mathscr{E}_X}$ is a tree. Then we have the following theorem.

Theorem 4.6. *We maintain the notation introduced above. Then we have*

$$\sigma(f^{-1}(x)) = \sum_{v\in v(\Gamma_{\mathscr{E}_X})} \Big(-\#G/\#I_v + \sum_{e\in e(v)} \#G/\#I_e(\#I_e/\#I_v - 1) + 1\Big)$$
$$+ \sum_{e\in e^{\mathrm{cl}}(\Gamma_{\mathscr{E}_X})} (\#G/\#I_e - 1).$$

Proof. The theorem follows from Theorem 2.8 and Proposition 3.9. □

Remark 4.6.1. We maintain the notation introduced in Theorem 4.6. In this remark, we explain that Raynaud's result (i.e., Proposition 3.7) can be deduced from Theorem 4.6 if $x \in X_s \setminus (X_s^{\mathrm{sing}} \cup D_{X_s})$. Note that, since $x \notin D_{X_s}$, we have $g^{-1}(x)_{\mathrm{red}} \cap D_{X_s^{\mathrm{sst}}} = \emptyset$.

Let X'_0 be the irreducible component of X_s which contains x. Moreover, we write X_0 for the strict transform of X'_0 under the birational morphism $g : \mathscr{X}^{\mathrm{sst}} \to \mathscr{X}$. Then there exists a unique irreducible component $X_1 \subseteq g^{-1}(x)_{\mathrm{red}} \subseteq X_s^{\mathrm{sst}}$ such that $X_0 \cap X_1 \neq \emptyset$. Note that $\#(X_0 \cap X_1) = 1$. Write v_1 for the vertex of $v(\Gamma_{\mathscr{E}_X})$ corresponding to X_1. Since $\Gamma_{\mathscr{E}_X}$ is connected, for each $v \in v(\Gamma_{\mathscr{E}_X})$, there exists a path $p(v_1, v)$ connecting v_1 and v. We define

$$\mathrm{leng}(p(v_1, v)) \overset{\mathrm{def}}{=} \#\{p(v_1, v) \cap v(\Gamma_{\mathscr{E}_X})\}$$

to be the length of the path $p(v_1, v)$. Moreover, for each $v \in v(\Gamma_{\mathscr{E}_X})$, we write

$$p_{v_1,v}$$

for the path such that $\mathrm{leng}(p_{v_1,v}) = \min\{\mathrm{leng}(p(v_1, v))\}_{p(v_1,v)}$.

By applying the general theory of semi-stable curves and Corollary 4.4, one may prove the following:

> Let $v, v' \in v(\Gamma_{\mathscr{E}_X})$ and $X_v, X_{v'}$ the irreducible components of $g^{-1}(x)_{\mathrm{red}}$ corresponding to v, v', respectively. Suppose that $X_v \cap X_{v'} \neq \emptyset$, and that $\mathrm{leng}(p_{v_1,v}) < \mathrm{leng}(p_{v_1,v'})$. Then we have $I_v \supseteq I_{v'}$.

Note that the inertia subgroup of the unique open edge of $\Gamma_{\mathscr{E}_X}$ (which abuts to v_1) is equal to G. Then Theorem 4.6 implies that $\sigma(f^{-1}(x)) = 0$.

In the remainder of this subsection, we suppose that x is a node of $\mathscr{X}_s$. Write X'_1 and X'_2 (which may be equal) for the irreducible components of $\mathscr{X}_s$ which contain x. Write X_1 and X_2 for the strict transforms of X'_1 and X'_2 under the birational morphism $g : \mathscr{X}^{\mathrm{sst}} \to \mathscr{X}$, respectively. By the general theory of semi-stable curves, $g^{-1}(x)_{\mathrm{red}} \subseteq X_s^{\mathrm{sst}}$ is a semi-stable curve over s whose irreducible components are isomorphic to $\mathbb{P}^1_k$. Let C be the semi-stable subcurve of $g^{-1}(x)_{\mathrm{red}}$ which is a chain of projective lines $\bigcup_{i=1}^n P_i$ such that the following conditions hold:

> (i) for any $w, t \in \{1, \dots, n\}$, $P_w \cap P_t = \emptyset$ if $|w - t| \geq 2$ and $P_w \cap P_t$ is reduced to a point if $|w - t| = 1$;
>
> (ii) $P_1 \cap X_1$ (resp. $P_n \cap X_2$) is reduced to a point;
>
> (iii) $C \cap \overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}} = (P_1 \cap X_1) \cup (P_n \cap X_2)$, where $\overline{\{X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}\}}$ denotes the closure of $X_s^{\mathrm{sst}} \setminus g^{-1}(x)_{\mathrm{red}}$ in X_s^{sst}.

Then we have
$$g^{-1}(x)_{\text{red}} = C \cup B,$$
where B denotes the topological closure of $g^{-1}(x)_{\text{red}} \setminus C$ in $g^{-1}(x)_{\text{red}}$. Let $\{V_i\}_{i=0}^{n+1}$ be a set of irreducible components of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ such that the following conditions hold:

(i) $h(V_i) = P_i$ for $i = 1, \ldots, n$;

(ii) $h(V_0) = X_1$ and $h(V_{n+1}) = X_2$;

(iii) the union $\bigcup_{i=0}^{n+1} V_i \subseteq Y_s$ is a connected semi-stable curve over s.

Write $I_{V_i} \subseteq G,\ i = 0, \ldots, n+1$ for the inertia subgroup of V_i. We have the following lemma.

Lemma 4.7. *We maintain the notation introduced in above. Then we have*
$$G = \langle I_{V_0}, I_{V_{n+1}} \rangle,$$
where $\langle I_{V_0}, I_{V_{n+1}} \rangle$ denotes the subgroup of G generated by I_{V_0} and $I_{V_{n+1}}$.

Proof. Suppose that $G \neq \langle I_{V_0}, I_{V_{n+1}} \rangle$. Since G is a p-group, there exists a normal subgroup $H \subseteq G$ of index p such that $\langle I_{V_0}, I_{V_{n+1}} \rangle \subseteq H$. Write $\mathscr{Y}'$ for the normalization of $\mathscr{X}$ in the function field $K(Y)$ induced by the natural injection $K(X) \hookrightarrow K(Y)$ induced by f. The normalization $\mathscr{Y}'$ admits an action of G induced by the action of G on $\mathscr{Y}$. Consider the quotient $\mathscr{Y}'/H$. Then we obtain a morphism of fiber surfaces $f_H : \mathscr{Y}'/H \to \mathscr{X}$ over S induced by f. Moreover, $\mathscr{Y}'/H$ admits an action of $G/H \cong \mathbb{Z}/p\mathbb{Z}$ induced by the action of G on $\mathscr{Y}'$. Then f_H is generically étale over X_1' and X_2'. Thus, [T2, Lemma 2.1 (iii)] implies that f_H is étale above x. Then $f^{-1}(x)$ is not connected. This is a contradiction. We complete the proof of the lemma. $\square$

Let $(u, w) \in \{0, \ldots, n+1\} \times \{0, \ldots, n+1\}$ be a pair such that $u \leq w$. We shall say that a group $I_{u,w}^{\min}$ is a minimal element of $\{I_{V_i}\}_{i=0}^{n+1}$ if one of the following conditions holds:

(i) $u = 0, w \neq n+1$, and $I_{0,w}^{\min} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \subset I_{V_{w+1}}$;

(ii) $u \neq 0, w = n+1$, and $I_{V_{u-1}} \supset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = I_{u,n+1}^{\min}$;

(iii) $u \neq 0,\ w \neq n+1$, and $I_{V_{u-1}} \supset I_{u,w}^{\min} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \subset I_{V_{w+1}}$.

We shall say that a group $J_{u,w}^{\max}$ is a maximal element of $\{I_{V_i}\}_{i=0}^{n+1}$ if one of the following conditions holds:

(i) $(u, w) = (0, n+1)$ and for any $I_{V_i}, i = 0, \ldots, n+1,\ J_{0,n+1}^{\max} = I_{V_i}$;

(ii) $u = 0, w \neq n+1$, and $J_{0,w}^{\max} = I_{V_0} = I_{V_1} = \cdots = I_{V_w} \supset I_{V_{w+1}}$;

(iii) $u \neq 0, w = n+1$, and $I_{V_{u-1}} \subset I_{V_u} = I_{V_{u+1}} \cdots = I_{V_{n+1}} = J_{u,n+1}^{\max}$;

(iv) $u \neq 0,\ w \neq n+1$, and $I_{V_{u-1}} \subset J_{u,w}^{\max} = I_{V_u} = I_{V_{u+1}} \cdots = I_{V_w} \supset I_{V_{w+1}}$.

Moreover, we put

$$\mathcal{I} \overset{\text{def}}{=} \{\text{minimal element of } \{I_{V_i}\}_{i=0}^{n+1}\} \setminus \{I_{0,0}^{\min}\}$$

and

$$\mathcal{J} \overset{\text{def}}{=} \{\text{maximal element of } \{I_{V_i}\}_{i=0}^{n+1}\}.$$

Note that the set $\mathcal{I}$ may be an empty set (e.g. if $I_{V_0} \subset I_{V_1} \subset \cdots \subset I_{V_{n+1}}$, then $\mathcal{I}$ is empty). Then we have the following theorem.

Theorem 4.8. *We maintain the notation introduced above. Then we have*

$$\begin{aligned}\sigma(f^{-1}(x)) &= \sum_{i=1}^{n} \#G/\#I_{V_i} - \sum_{i=1}^{n+1} \#G/\#\langle I_{V_{i-1}}, I_{V_i}\rangle + 1 \\ &= \sum_{I \in \mathcal{I}} \#G/\#I - \sum_{J \in \mathcal{J}} \#G/\#J + 1,\end{aligned}$$

where $\langle I_{V_{i-1}}, I_{V_i}\rangle$ denotes the subgroup of G generated by $I_{V_{i-1}}$ and I_{V_i}.

Proof. The theorem follows from Corollary 2.10, Proposition 3.11, Lemma 4.1, Corollary 4.4, and Lemma 4.7. □

Finally, suppose that G is a cyclic p-group. We will show that the formula of Theorem 4.8 coincides with the formula of Saïdi (cf. [S, Proposition 1]). Since G is an abelian group, $I_{V_i},\ i = 0, \ldots, n+1$, does not depend on the choices of V_i. Then we may use the notation $I_{P_i},\ i = 0, \ldots, n+1$, to denote I_{V_i}.

Lemma 4.9. *We maintain the notation introduced above. If G is a cyclic p-group, then there exists $0 \leq u \leq n+1$ such that*

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_u} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$

Proof. If the lemma is not true, then there exist w, t and v such that $I_{P_v} \neq I_{P_w}$, $I_{P_v} \neq I_{P_t}$ and $I_{P_w} \subset I_{P_{w+1}} = \cdots = I_{P_v} = \cdots = I_{P_{t-1}} \supset I_{P_t}$. Since G is a cyclic group, we may assume $I_{P_w} \supseteq I_{P_t}$. Consider the quotient of $\mathscr{Y}$ by I_{P_w}, we obtain a natural morphism of pointed semi-stable curves $h_w : \mathscr{Y}/I_{P_w} \to \mathscr{X}^{\text{sst}}$ over S.

We define B_j, $j = 0, \ldots, n+1$, to be the union of the connected components of B which intersect with P_j non-trivially. By contracting $P_{w+1}, P_{w+2}, \ldots, P_{t-1}, B_{w+1}, \ldots, B_{t-1}$ and $(h_w)^{-1}(P_{w+1})_{\text{red}}, (h_w)^{-1}(P_{w+2})_{\text{red}}, \ldots, (h_w)^{-1}(P_{t-1})_{\text{red}}, (h_w)^{-1}(B_{w+1})_{\text{red}}, \ldots, (h_w)^{-1}(B_{t-1})_{\text{red}}$, respectively, we obtain a pointed semi-stable curve $(\mathscr{X}^{\text{sst}})^{\text{c}}$ and a fiber surface $(\mathscr{Y}/I_{P_w})^{\text{c}}$. Write

$$c_{\mathscr{X}^{\text{sst}}} : \mathscr{X}^{\text{sst}} \to (\mathscr{X}^{\text{sst}})^{\text{c}}$$

and

$$c_{\mathscr{Y}/I_{P_w}} : \mathscr{Y}/I_{P_w} \to (\mathscr{Y}/I_{P_w})^{\text{c}}$$

for the resulting contracting morphisms, respectively. The morphism h_w induces a morphism of fiber surfaces $h_w^{\text{c}} : (\mathscr{Y}/I_{P_w})^{\text{c}} \to (\mathscr{X}^{\text{sst}})^{\text{c}}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{Y}/I_{P_w} & \xrightarrow{c_{\mathscr{Y}/I_{P_w}}} & (\mathscr{Y}/I_{P_w})^{\text{c}} \\ {\scriptstyle h_w}\downarrow & & \downarrow{\scriptstyle h_w^{\text{c}}} \\ \mathscr{X}^{\text{sst}} & \xrightarrow{c_{\mathscr{X}^{\text{sst}}}} & (\mathscr{X}^{\text{sst}})^{\text{c}}. \end{array}$$

Write $P^{\rm c}_w$ and $P^{\rm c}_t$ for the images $c_{\mathscr{X}^{\rm sst}}(P_w)$ and $c_{\mathscr{X}^{\rm sst}}(P_t)$, respectively, and $x^{\rm c}_{wt}$ for the closed point $P^{\rm c}_w \cap P^{\rm c}_t$. Since $h^{\rm c}_w$ is generically étale above $P^{\rm c}_w$ and $P^{\rm c}_t$, [T2, Lemma 2.1 (iii)] implies that $(h^{\rm c}_w)^{-1}(x^{\rm c}_{wt})_{\rm red}$ are nodes. Thus, $(\mathscr{Y}/I_{P_w})^{\rm c}$ is a semi-stable curve over S; moreover, we have $h^{\rm c}_w$ is étale over $x^{\rm c}_{wt}$. Then the inertia subgroups of the closed points $(h^{\rm c}_w)^{-1}(x^{\rm c}_{wt})_{\rm red}$ of the special fiber $(\mathscr{Y}/I_{P_w})^{\rm c}_s$ of $(\mathscr{Y}/I_{P_w})^{\rm c}$ are trivial.

On the other hand, since I_{P_w} is a proper subgroup of I_{P_v}, we have that the inertia subgroups of the irreducible components of $h_w^{-1}(\bigcup_{j=w+1}^{t-1} P_j)_{\rm red}$ is I_{P_v}/I_{P_w}. Thus, the inertia subgroups of the closed points $c_{\mathscr{Y}/I_{P_w}}(h_w^{-1}(\bigcup_{j=w+1}^{t-1} P_j)_{\rm red}) = (h^{\rm c}_w)^{-1}(x^{\rm c}_{wt})_{\rm red}$ of the special fiber $(\mathscr{Y}/I_{P_w})^{\rm c}_s$ of $(\mathscr{Y}/I_{P_w})^{\rm c}$ are not trivial. This is a contradiction. Then we complete the proof of the lemma. □

Then we have the following corollary.

Corollary 4.10. *Suppose that G is a cyclic p-group, and that I_{P_0} is equal to G. Then we have*

$$\sigma(f^{-1}(x)) = \#G/\#I_{\min} - \#G/\#I_{P_{n+1}},$$

where $I_{\min}$ denotes the group $\bigcap_{i=0}^{n+1} I_{P_i}$.

Proof. The corollary follows from Theorem 4.8 and Lemma 4.9. □

Remark 4.10.1. The formula in Corollary 4.10 had been obtained by Saïdi (cf. [S, Proposition 1]). On the other hand, Corollary 4.10 implies that

$$\sigma(f^{-1}(x)) \leq \#G - 1$$

when G is a cyclic p-group, which is the main theorem of [S] (cf. [S, Theorem 1]). Moreover, Saïdi asked whether or not there exists a bound of $\sigma(f^{-1}(x))$ which only depends on $\#G$ if G is an arbitrary p-group (cf. [S, Question]). In the next section, we will give a bound of $\sigma(f^{-1}(x))$ which only depends on $\#G$ when G is an arbitrary *abelian* p-group.

5 Bounds of p-rank of vertical fibers of abelian G-semi-stable coverings

In this section, we maintain the notation introduced in Section 4.2 and Section 4.3. Moreover, we assume that G is an *abelian* p-group, and that $f^{-1}(x)$ is connected. We fix a set of irreducible components $\{V_i\}_{i=0}^{n+1}$ of the special fiber $\mathscr{Y}_s$ of $\mathscr{Y}$ such that the following conditions hold:

(i) $h(V_i) = P_i$ for $i = 1, \dots, n$;

(ii) $h(V_0) = X_1$ and $h(V_{n+1}) = X_2$;

(iii) the union $\bigcup_{i=0}^{n+1} V_i \subseteq Y_s$ is a connected semi-stable curve over s.

Write $I_{V_i} \subseteq G$, $i = 0, \dots, n+1$ for the inertia subgroup of V_i. Since G is abelian, I_{V_i}, $i = 0, \dots, n+1$, does not depend on the choices of V_i. Then we use the notation I_{P_i} to denote I_{V_i} for each $i = 0, \dots, n+1$. First, we have the following key proposition.

Proposition 5.1. *Suppose that $\#\mathcal{I} \geq 2$. Let I' and I'' be two different elements of $\mathcal{I}$. Then neither $I' \subseteq I''$ nor $I' \supseteq I''$ holds.*

Proof. Without loss of generality, we may assume that $I' = I_{P_a}$ and $I'' = I_{P_b}$ such that $0 \leq a < b \leq n+1$, $I_{P_a} \neq I_{P_{a+1}}$, and $I_{P_{b-1}} \neq I_{P_b}$. Note that by the definition of Min, $I_{P_{a+1}}$ (resp. $I_{P_{b-1}}$) contains I_{P_a} (resp. I_{P_b}).

If $I' \subseteq I''$, we consider the quotient curve $\mathscr{Y}/I''$. Then we obtain two morphisms of semi-stable curves $\xi_1 : \mathscr{Y} \to \mathscr{Y}/I''$ and $\xi_2 : \mathscr{Y}/I'' \to \mathscr{X}^{\mathrm{sst}}$ such that $\xi_2 \circ \xi_1 = h$. Note that $h(V_a) = P_a$ and $h(V_b) = P_b$, respectively. By contracting $\bigcup_{i=a+1}^{b-1} P_i$ and $\xi_2^{-1}(\bigcup_{i=a+1}^{b-1} P_i)_{\mathrm{red}}$ (cf. [BLR, 6.7 Proposition 4]), we obtain two contracting morphisms $c_{\mathscr{X}^{\mathrm{sst}}} : \mathscr{X}^{\mathrm{sst}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ and $c_{\mathscr{Y}/I''} : \mathscr{Y}/I'' \to (\mathscr{Y}/I'')^{\mathrm{c}}$. Moreover, ξ_2 induces a morphism $\xi_2^{\mathrm{c}} : (\mathscr{Y}/I'')^{\mathrm{c}} \to (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{Y}/I'' & \xrightarrow{c_{\mathscr{Y}/I''}} & (\mathscr{Y}/I'')^{\mathrm{c}} \\ {\scriptstyle \xi_2}\downarrow & & {\scriptstyle \xi_2^{\mathrm{c}}}\downarrow \\ \mathscr{X}^{\mathrm{sst}} & \xrightarrow{c_{\mathscr{X}^{\mathrm{sst}}}} & (\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}. \end{array}$$

Note that $(\mathscr{X}^{\mathrm{sst}})^{\mathrm{c}}$ is a semi-stable curve over S.

Since $I' = I_{P_a} \subseteq I'' = I_{P_b}$, ξ_2^{c} is étale at the generic points of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a)$ and $c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$. Thus, by applying the Zariski-Nagata purity theorem and [T2, Lemma 2.1 (iii)], we obtain that ξ_2^{c} is étale at $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ (i.e., the inertia group of each point of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ is trivial). On the other hand, since $I_{P_{b-1}}$ contains I_{P_b}, we have the inertia group of each point of $c_{\mathscr{Y}/I''} \circ \xi_1(V_a) \cap c_{\mathscr{Y}/I''} \circ \xi_1(V_b)$ is $I_{P_{b-1}}/I''$. Then we obtain $I_{P_{b-1}} = I''$. This is a contradiction. Then I'' does not contain I'.

Similar arguments to the arguments given in the proof above imply that I' does not contain I''. We complete the proof of the proposition. □

Definition 5.2. Let N be a finite p-group and H a subgroup of N. Write $\mathrm{Sub}(-)$ for the set of the subgroups of $(-)$. We put

$$\#I(H) \overset{\mathrm{def}}{=} \max\{\#J \mid J \subseteq \mathrm{Sub}(N), H \in J, \text{ for any } H', H'' \in J \text{ such that } H' \neq H'',$$

$$\text{neither } H' \subseteq H'' \text{ nor } H' \supseteq H'' \text{ holds}\}.$$

Moreover, we put

$$M(N) \overset{\mathrm{def}}{=} \max\{\#I(N')\}_{N' \in \mathrm{Sub}(N)}.$$

For any $1 \leq d \leq \#N$, write $S_d(N)$ for the set of the subgroups of N with order d. Let A be an elementary abelian p-group such that $\#A = \#N$. We put

$$B(\#N) \overset{\mathrm{def}}{=} \#\mathrm{Sub}(A).$$

Note that $B(\#N)$ depends only on $\#N$.

Lemma 5.3. *Let N be a finite p-group, A an elementary abelian p-group with order $\#N$, and $1 \leq d \leq \#N$ an integer number. Then we have*

$$\#S_d(N) \leq \#S_d(A).$$

In particular, we have

$$M(N) \leq B(\#N).$$

Proof. Since N is a p-group, N has a non-trivial central subgroup. Fix a central subgroup Z of order p in N. Write $S_d^Z(N)$ (resp. $S_d^{\setminus Z}(N)$) for the set of subgroups of N of order d which contain Z (resp. do not contain Z). If H is a subgroup of N/Z, let $S_d^{(Z,H)}(N)$ be the set of $L \in S_d^{\setminus Z}(N)$ whose projection on N/Z is H. Let $S_d[N/Z]$ be the set of $H \in S_d(N/Z)$ for which $S_d^{(Z,H)}(N) \neq \emptyset$.

Let $H \in S_d[N/Z]$. Then we obtain that

$$\#S_d^{(Z,H)}(N) \leq \#H^1(H, Z) = \#\mathrm{Hom}(H^{\mathrm{ab},p}, Z),$$

where $(-)^{\mathrm{ab},p}$ denotes $(-)/((-)^p[(-),(-)])$. Moreover, let H' be a subgroup of A of order d and $Z' \cong \mathbb{Z}/p\mathbb{Z}$ a subgroup of A of order p. Then we have that

$$\#\mathrm{Hom}(H^{\mathrm{ab},p}, Z) \leq \#\mathrm{Hom}((H')^{\mathrm{ab},p}, Z')$$

If $d = 1$, the lemma is trivial. Then we may assume that p divides d. We have

$$\begin{aligned}
\#S_d(N) &= \#S_d^Z(N) + \#S_d^{\setminus Z}(N) = \#S_{d/p}(N/Z) + \#S_d^{\setminus Z}(N) \\
&= \#S_{d/p}(N/Z) + \sum_{H \in S_d[N/Z]} \#S_d^{(Z,H)}(N) \\
&\leq \#S_{d/p}(N/Z) + \sum_{H \in S_d[N/Z]} \#(\mathrm{Hom}(H^{\mathrm{ab},p}, Z)) \\
&\leq \#S_{d/p}(N/Z) + \#S_d(N/Z)\#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z'))
\end{aligned}$$

By induction, we have $\#S_{d/p}(N/Z) \leq \#S_{d/p}(A/Z')$ and $\#S_d(N/Z) \leq \#S_d(A/Z')$. Moreover, we have that

$$\begin{aligned}
\#S_d(A) &= \#S_{d/p}(A/Z') + \sum_{H' \in S_d[A/Z']} \#S_d^{(Z',H')}(A) \\
&= \#S_{d/p}(A/Z') + \sum_{H' \in S_d[A/Z']} \#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z')) \\
&= \#S_{d/p}(A/Z') + \#S_d(A/Z')\#(\mathrm{Hom}((H')^{\mathrm{ab},p}, Z')).
\end{aligned}$$

Thus, we obtain

$$\#S_d(N) \leq \#S_d(A).$$

This completes the proof of the lemma. □

The following theorem gives an affirmative answer to a problem posed by Saïdi (cf. [S, Question]) when G is an arbitrary abelian p-group.

Theorem 5.4. *Let $f : \mathscr{Y} \to \mathscr{X}$ be a G-semi-stable covering over S, and x a vertical point associated to f. Suppose that $f^{-1}(x)$ is connected, and that G is an abelian p-group. Then we have*

$$\sigma(f^{-1}(x)) \leq M(G)\#G - 1 \leq B(\#G)\#G - 1.$$

Proof. If x is a smooth point of the special fiber $\mathscr{X}_s$ of $\mathscr{X}$, then $\sigma(f^{-1}(x)) = 0$ (cf. Proposition 3.7). Thus, we may assume that x is a singular point of $\mathscr{X}_s$.

If $\mathcal{I} = \emptyset$, then Theorem 4.8 implies that $\sigma(f^{-1}(x)) = 0$. Then the theorem follows. If $\mathcal{I} \neq \emptyset$, then we have $\#\mathcal{J} \geq 2$. Thus, by applying Theorem 4.8, we obtain

$$\sigma(f^{-1}(x)) = \sum_{I \in \mathcal{I}} \#G/\#I - \sum_{J \in \mathcal{J}} \#G/\#J + 1$$

$$\leq \#\mathcal{I}\#G - 1 \leq M(G)\#G - 1 \leq B(\#G)\#G - 1.$$

□

Remark 5.4.1. If G is a cyclic p-group, then by the definition of $M(G)$, we have $M(G) = 1$. Thus, if G is a cyclic p-group, we have

$$\sigma(f^{-1}(x)) \leq \#G - 1.$$

This is the main theorem of [S, Theorem 1].

References

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, Néron Models, *Ergeb. Math. Grenz.,* **21**. Springer, New York-Heidelberg-Berlin. 1990.

[C] R. Crew, Étale p-covers in characteristic p, *Compositio Math.* **52** (1984), 31–45.

[L] Q. Liu, Stable reduction of finite covers of curves, *Compositio Math.* **142** (2006), 101–118.

[M] S. Mochizuki, Semi-graphs of anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), 221–322.

[R] M. Raynaud, p-groupes et réduction semi-stable des courbes, *The Grothendieck Festschrift, Vol. III,* 179-197, *Progr. Math.,* **88**, Birkhäuser Boston, Boston, MA, 1990.

[S] M. Saïdi, p-rank and semi-stable reduction of curves, *C. R. Acad. Sci. Paris, t.* **326***, Série I*, 63–68, 1998.

[T1] A. Tamagawa, Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups. *J. Algebraic Geom.* **13** (2004), 675–724.

[T2] A. Tamagawa, Resolution of nonsingularities of families of curves, *Publ. Res. Inst. Math. Sci.* **40** (2004), 1291–1336.

[V] I. Vidal, *Contributions à la cohomologie étale des schémas et des log-schémas,* Thèse, U. Paris-Sud (2001).

[Y1] Y. Yang, On the existence of non-finite coverings of stable curves over complete discrete valuation rings, *Math. J. Okayama Univ.* **61** (2019), 1–18.

[Y2] Y. Yang, On the averages of generalized Hasse-Witt invariants of pointed stable curves in positive characteristic, *Math. Z.* **295** (2020), 1-45.

Yu Yang

Address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail: yuyang@kurims.kyoto-u.ac.jp