RAYNAUD-TAMAGAWA THETA DIVISORS AND NEW-ORDINARINESS OF RAMIFIED COVERINGS OF CURVES

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ABSTRACT. Let (X, D_X) be a smooth pointed stable curve over an algebraically closed field k of characteristic p > 0. Suppose that (X, D_X) is generic. We give a necessary and sufficient condition for new-ordinariness of prime-to-p cyclic tame coverings of (X, D_X) . This result generalizes a result of S. Nakajima concerning the ordinariness of prime-to-p cyclic étale coverings of generic curves to the case of tamely ramified coverings.

Keywords: pointed stable curve, admissible covering, generalized Hasse-Witt invariant, new-ordinary, Raynaud-Tamagawa theta divisor, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14F35, 14G32.

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1. INTRODUCTION

1.1. Fundamental groups in positive characteristic.

1.1.1. Let $X^{\bullet} = (X, D)$ be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic p > 0, where X denotes the underlying curve, D_X denotes a finite set of marked points satisfying [K, Definition 1.1 (iv)], g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X .

By choosing a suitable base point of $X \setminus D_X$, we have the tame fundamental group $\Pi_{X^{\bullet}}$ of X^{\bullet} . Note that since all the tame coverings in positive characteristic can be lifted to characteristic 0, $\Pi_{X^{\bullet}}$ is topologically finitely generated. Moreover, A. Grothendieck ([G]) showed that the structure of maximal prime-to-p quotient $\Pi_{X^{\bullet}}^{p'}$ of $\Pi_{X^{\bullet}}$ is isomorphic to the pro-prime-to-p completion of the following group

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle.$$

1.1.2. On the other hand, the structure of Π_X is very mysterious. Some developments of F. Pop-M. Saïdi ([PoSa]), M. Raynaud ([R2]), A. Tamagawa ([T1], [T2], [T3], [T4]), and the author ([Y1], [Y3], [Y4]) showed evidence for very strong anabelian phenomena for curves over algebraically closed fields of characteristic p > 0. In this situation, the Galois group of the base field is trivial, and the étale (or tame) fundamental group coincides with the geometric fundamental group, thus in a total absence of a Galois action of the base field. This kind of anabelian phenomenon goes beyond Grothendieck's anabelian geometry, and shows that the tame fundamental group of a smooth pointed stable curve over an algebraically closed field must encode "moduli" of the curve. This is the reason that we do not have an explicit description of the tame fundamental group of any smooth pointed stable curve in positive characteristic.

Furthermore, the theories developed in [T3] and [Y4] imply that the isomorphism class of X^{\bullet} as a scheme can possibly be determined by not only the isomorphism class of $\Pi_{X^{\bullet}}$ as a profinite group but also the isomorphism class of the maximal pro-solvable quotient $\Pi_{X^{\bullet}}^{\text{sol}}$ of $\Pi_{X^{\bullet}}$. Since the isomorphism class of $\Pi_{X^{\bullet}}^{\text{sol}}$ is determined by the set of finite quotients of $\Pi_{X^{\bullet}}^{\text{sol}}$ ([FJ, Proposition 16.10.6]), we may ask the following question: Which finite solvable groups can appear as quotients of $\Pi_{X^{\bullet}}^{\text{sol}}$?

1.2. *p*-rank of coverings.

1.2.1. Let $N \subseteq \Pi_{X^{\bullet}}$ be an arbitrary open normal subgroup and $X_N^{\bullet} = (X_N, D_{X_N})$ the smooth pointed stable curve of type (g_{X_N}, n_{X_N}) over k corresponding to N. We have an important invariant σ_{X_N} associated to X_N^{\bullet} (or N) which is called *p*-rank (see 2.2.2). Roughly speaking, σ_{X_N} controls the finite quotients of $\Pi_{X^{\bullet}}$ which are extensions of the group $\Pi_{X^{\bullet}}/N$ by *p*-groups. Moreover, if we can compute the *p*-rank σ_{X_N} when $\Pi_{X^{\bullet}}/N$ is abelian, together with the structure theorem of maximal prime-to-*p* quotients of tame fundamental groups mentioned in 1.1.1, we can answer the above question for an arbitrary solvable group step-by-step.

Suppose that $\Pi_{X^{\bullet}}/N$ is abelian. If $\Pi_{X^{\bullet}}/N$ is a *p*-group, then σ_{X_N} can be computed by using the Deuring-Shafarevich formula ([C], [Su]). Moreover, by applying the Deuring-Shafarevich formula, to compute σ_{X_N} , we may assume that $\Pi_{X^{\bullet}}/N$ is a prime-to-*p* abelian group. Furthermore, since a Galois tame covering of X^{\bullet} with Galois group $\Pi_{X^{\bullet}}/N$ is a tower of prime-to-*p* cyclic tame coverings, we obtain σ_{X_N} if we can compute *p*-rank for *prime-to-p cyclic* tame coverings. Thus, in the remainder of the introduction, we suppose that $\Pi_{X^{\bullet}}/N \cong \mathbb{Z}/m\mathbb{Z}$ is a prime-to-*p* cyclic group. 1.2.2. The situation of σ_{X_N} is very complicated when $\Pi_{X^{\bullet}}/N$ is not a *p*-group. In fact, if X^{\bullet} is an *arbitrary* pointed stable curve over k, then σ_{X_N} cannot be explicitly computed in general ([R1], [T3], [Y2]). On the other hand, when X^{\bullet} is generic (i.e., a curve corresponding to a geometric generic point of the moduli space \mathcal{M}_{g_X,n_X}), the following interesting result was proved by S. Nakajima:

Theorem 1.1. ([N, Proposition 4]) Suppose that $\Pi_{X^{\bullet}}/N$ is a prime-to-p cyclic group, that $n_X = 0$, and that X^{\bullet} is generic. Then we have $\sigma_{X_N} = g_{X_N}$ (i.e., X_N^{\bullet} is ordinary).

Nakajima's result was generalized by B. Zhang to the case where $\Pi_{X^{\bullet}}/N$ is an arbitrary prime-to-p abelian group ([Z]). Moreover, recently, E. Ozman and R. Pries generalized Nakajima's result to the case where X^{\bullet} is curve corresponding to a geometric generic point of the p-rank strata of the moduli space \mathcal{M}_{g_X,n_X} (see [OP] or 3.6.2 of the present paper).

1.2.3. Suppose that $n_X \neq 0$. The computations of σ_{X_N} are much more difficult than the case of $n_X = 0$. Let D be the ramification divisor (see Definition 2.2) associated to the Galois tame covering $X^{\bullet}_N \to X^{\bullet}$ over k with Galois group $\prod_{X^{\bullet}}/N \cong \mathbb{Z}/m\mathbb{Z}$. Firstly, we note that there exists an upper bound $B(\sigma_X, D, m)$ for σ_{X_N} depending on the *p*-rank σ_X of X^{\bullet} , D, and m such that the following holds (e.g. [B, Section 3]):

$$0 \le \sigma_{X_N} \le B(\sigma_X, D, m) \le g_{X_N}.$$

Note that $B(\sigma_X, D, m)$ is not equal to g_{X_N} in general. This means that Nakajima's result mentioned above does not hold for tame coverings in general. Then we have the following natural question: Can σ_{X_N} attain the upper bound $B(\sigma_X, D, m)$?

If X^{\bullet} is generic, I. Bouw proved that $\sigma_{X_N} = B(\sigma_X, D, m)$ if *m* satisfies certain conditions and *p* is sufficiently large ([B]). In general, the above question is still open. Moreover, by applying the theory of theta divisors developed by Raynaud ([R1]) and Tamagawa ([T3]), the above question is equivalent to the following open problem posed by Tamagawa ([T3, Question 2.18]): Does the Raynaud-Tamagawa theta divisor (see 2.4.4) associated to D exist when X^{\bullet} is generic?

1.3. Main result.

1.3.1. In the present paper, we study the problem mentioned in 1.2.3 without making any assumptions about m and p. More precisely, we prove that the Raynaud-Tamagawa theta divisor associated to certain D exists, and obtain the following *necessary and sufficient* condition for the ordinariness of X_N^{\bullet} which generalizes Nakajima's result to the case of tamely ramified coverings.

Theorem 1.2. (Theorem 3.5) Suppose that $\Pi_{X^{\bullet}}/N$ is a prime-to-p cyclic group, and that X^{\bullet} is generic. Then we have that $\sigma_{X_N} = B(\sigma_X, D, m) = g_{X_N}$ (i.e., X_N^{\bullet} is ordinary) if and only if $D(j), j \in \{1, \ldots, m-1\}$, is Frobenius stable (cf. Definition 2.3 and Definition 3.3 for the definitions of D(j) and Frobenius stable, respectively).

Remark 1.2.1. By applying the result of Ozman-Pries mentioned above, we also obtain a slightly stronger version of Theorem 1.2 for certain m (see Corollary 3.6).

Remark 1.2.2. As an application (Proposition 4.2), we generalize a result of Pacheco-Stevenson concerning inverse Galois problems for étale coverings of projective generic curves to the case of tame coverings.

1.3.2. Suppose that $g_X = 0$. Let us explain some relationships between Theorem 1.2 and the ordinary Newton polygon strata of the Torelli locus in PEL-type Shimura varieties. If $\sigma_{X_N} = B(\sigma_X, D, m)$ holds for every D, then the intersection of the open Torelli locus with all μ -ordinary Newton polygon strata of certain PEL-Shimura varieties is non-empty (see [LMPT, Section 4]). Note that if the Newton polygon of the p-divisible group associated to an abelian variety is μ -ordinary, the abelian variety is not ordinary in general. On the other hand, we call the Newton polygon of the p-divisible group associated to an abelian variety is "classic" μ -ordinary if the abelian variety is ordinary. Then Theorem 1.2 gives a criterion for determining whether or not the intersection of the open Torelli locus with classic μ -ordinary Newton polygon strata of certain PEL-Shimura varieties is non-empty.

1.4. Structure of the present paper. The present paper is organized as follows. In Section 2, we recall some definitions and properties of pointed stable curves, admissible coverings, generalized Hasse-Witt invariants, and Raynaud-Tamagawa theta divisors. In Section 3, we study the new-ordinariness of prime-to-p cyclic tame coverings of generic curves by using the theory of Raynaud-Tamagawa theta divisors and prove our main theorem. In Section 4, we give two applications of the main theorem.

1.5. Acknowledgements. The author would like to thank the referee very much for carefully reading to the former version of the present paper and for giving various comments on it, which were very useful in improving the presentation of the present paper. This work was supported by JSPS KAKENHI Grant Number 20K14283, and by the Research Institute for Mathematical Sciences (RIMS), an International Joint Usage/Research Center located in Kyoto University.

2. Preliminaries

2.1. Pointed stable curves and admissible fundamental groups. In this subsection, we recall some notation concerning admissible fundamental groups.

2.1.1. Let $X^{\bullet} = (X, D_X)$ be a pointed stable curve over an algebraically closed field k of characteristic p > 0, where X denotes the underlying curve and D_X denotes a finite set of marked points satisfying [K, Definition 1.1 (iv)]. Write g_X for the genus of X and n_X for the cardinality $\#D_X$ of D_X . We shall call (g_X, n_X) the type of X^{\bullet} .

Write $\Gamma_{X^{\bullet}}$ for the dual semi-graph of X^{\bullet} which is defined as follows: (i) the set of vertices $v(\Gamma_{X^{\bullet}})$ of $\Gamma_{X^{\bullet}}$ is the set of irreducible components of X; (ii) the set of open edges $e^{\mathrm{op}}(\Gamma_{X^{\bullet}})$ of $\Gamma_{X^{\bullet}}$ is the set of marked points D_X ; (iii) the set of closed edges $e^{\mathrm{cl}}(\Gamma_{X^{\bullet}})$ of $\Gamma_{X^{\bullet}}$ is the set of nodes of X. Moreover, we write $r_X \stackrel{\mathrm{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^{\bullet}},\mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^{\bullet}}$.

Example 2.1. We give an example to explain dual semi-graphs of pointed stable curves. Let X^{\bullet} be a pointed stable curve over k whose irreducible components are X_{v_1} and X_{v_2} , whose node is x_{e_1} , and whose marked point is $x_{e_2} \in X_{v_2}$. We use the notation " \bullet " and " \circ " to denote a node and a marked point, respectively. Then X^{\bullet} is as follows:



We write v_1 and v_2 for the vertices of Γ_X • corresponding to X_{v_1} and X_{v_2} , respectively, e_1 for the closed edge corresponding to x_{e_1} , and e_2 for the open edge corresponding to x_{e_2} . Moreover, we use the notation "•" and "o with a line segment" to denote a vertex and an open edge, respectively. Then the dual semi-graph Γ_X • of X^{\bullet} is as follows:

$$\Gamma_X \bullet : \qquad v_1 \bullet \underbrace{e_1 \quad v_2}_{\circ \circ e_2} \circ e_2$$

2.1.2. Let $v \in v(\Gamma_{X^{\bullet}})$ and $e \in e^{\operatorname{op}}(\Gamma_{X^{\bullet}}) \cup e^{\operatorname{cl}}(\Gamma_{X^{\bullet}})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\operatorname{cl}}(\Gamma_{X^{\bullet}})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\operatorname{op}}(\Gamma_{X^{\bullet}})$. Moreover, write $\operatorname{nor}_v : \widetilde{X}_v \to X_v$ for the normalization of X_v . We define a smooth pointed stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}_v^{\bullet} = (\widetilde{X}_v, D_{\widetilde{X}_v} \stackrel{\text{def}}{=} \operatorname{nor}_v^{-1}((X^{\operatorname{sing}} \cap X_v) \cup (D_X \cap X_v))),$$

where X^{sing} denotes the singular locus of X. We shall call $\widetilde{X}^{\bullet}_{v}$ the smooth pointed stable curve associated to v.

2.1.3. Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack parameterizing pointed stable curves of type (g, n)over Spec \mathbb{Z} , $\overline{\mathbb{F}}_p$ the algebraic closure of \mathbb{F}_p in k, $\overline{\mathcal{M}}_{g,n} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, and $\overline{\mathcal{M}}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. Then $X^{\bullet} \to \text{Spec } k$ determines a morphism $c_X : \text{Spec } k \to \overline{\mathcal{M}}_{g_X,n_X}$ and $\widetilde{X}_v^{\bullet} \to \text{Spec } k, v \in v(\Gamma_{X^{\bullet}})$, determines a morphism $c_v : \text{Spec } k \to \overline{\mathcal{M}}_{g_v,n_v}$. Moreover, we have a clutching morphism of moduli stacks ([K, Definition 3.8])

$$c:\prod_{v\in v(\Gamma_X\bullet)}\overline{\mathcal{M}}_{g_v,n_v}\to\overline{\mathcal{M}}_{g_X,n_X}$$

such that $c \circ (\prod_{v \in v(\Gamma_X \bullet)} c_v) = c_X$. We shall call X^{\bullet} a *component-generic* pointed stable curve over k if the image of

$$\prod_{v \in v(\Gamma_X \bullet)} c_v : \operatorname{Spec} k \to \prod_{v \in v(\Gamma_X \bullet)} \overline{\mathcal{M}}_{g_v, n_v}$$

v

is a generic point in $\prod_{v \in v(\Gamma_X \bullet)} \overline{M}_{g_v, n_v}$. In particular, we shall call X^{\bullet} generic if X^{\bullet} is non-singular component-generic.

2.1.4. By choosing a smooth point $x \in X \setminus D_X$, we obtain a fundamental group $\pi_1^{\text{adm}}(X^{\bullet}, x)$ which is called the *admissible fundamental group* of X^{\bullet} (see [Y1, Definition 2.2] or [Y3, Section 2.1] for the definitions of admissible coverings and admissible fundamental groups). The admissible fundamental group of X^{\bullet} is naturally isomorphic to the tame fundamental group of X^{\bullet} when X^{\bullet} is smooth over k. For simplicity of notation, we omit the base point and denote the admissible fundamental group by

$\Pi_{X^{\bullet}}$.

The structure of the maximal prime-to-p quotient of Π_X • is well-known, and is isomorphic to the prime-to-p completion of the following group ([V, Théorème 2.2 (c)])

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle.$$

2.2. Hasse-Witt invariants and generalized Hasse-Witt invariants. In this subsection, we recall some notation concerning Hasse-Witt invariants and generalized Hasse-Witt invariants. On the other hand, in the case of *smooth* pointed stable curves, the generalized Hasse-Witt invariants of cyclic tame coverings were discussed in [B, Section 2] and [T3, Section 3].

2.2.1. Settings. We maintain the notation introduced in 2.1.1. Let $X^{\bullet} = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over k and Π_X^{\bullet} the admissible fundamental group of X^{\bullet} .

2.2.2. Let Z^{\bullet} be a disjoint union of finitely many pointed stable curves over k. We define the *p*-rank (or Hasse-Witt invariant) of Z^{\bullet} to be

$$\sigma_Z \stackrel{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{\'et}}(Z, \mathbb{F}_p)).$$

We shall call Z^{\bullet} ordinary if $g_Z = \sigma_Z$, where $g_Z \stackrel{\text{def}}{=} \dim_k(H^1(Z, \mathcal{O}_Z))$. Moreover, let $Z^{\bullet} \to X^{\bullet}$ be a multi-admissible covering ([Y1, Definition 2.2]) over k. We shall call $Z^{\bullet} \to X^{\bullet}$ new-ordinary if $g_Z - g_X = \sigma_Z - \sigma_X$, where σ_X denotes the p-rank of X^{\bullet} . Note that if X^{\bullet} is ordinary, then $Z^{\bullet} \to X^{\bullet}$ is new-ordinary if and only if Z^{\bullet} is ordinary.

On the other hand, the structure of $\operatorname{Pic}^{0}_{X/k}$ ([BLR, §9.2 Example 8]) implies

$$\sigma_X = \sum_{v \in v(\Gamma_X \bullet)} \sigma_{\widetilde{X}_v} + r_X$$

Then X^{\bullet} is ordinary if and only if $\widetilde{X}_{v}^{\bullet}$, $v \in v(\Gamma_{X^{\bullet}})$, is ordinary. Moreover, let $g^{\bullet} : Z^{\bullet} \to X^{\bullet}$ be a multi-admissible covering over k and $\widetilde{g}_{v}^{\bullet} : \widetilde{Z}_{v}^{\bullet} \to \widetilde{X}_{v}^{\bullet}$, $v \in v(\Gamma_{X^{\bullet}})$, the admissible covering over k induced by g^{\bullet} , where the underlying curve of $\widetilde{Z}_{v}^{\bullet}$ is the normalization of $g^{-1}(X_{v})$. Then g^{\bullet} is new-ordinary if and only if $\widetilde{g}_{v}^{\bullet}$ is new-ordinary for each $v \in v(\Gamma_{X^{\bullet}})$.

2.2.3. Let m be an arbitrary positive natural number prime to p and $\mu_m \subseteq k^{\times}$ the group of mth roots of unity. Fix a primitive mth root ζ , we may identify μ_m with $\mathbb{Z}/m\mathbb{Z}$ via the homomorphism $\zeta^i \mapsto i$. Let $\alpha \in \text{Hom}(\Pi_{X^{\bullet}}^{\text{ab}}, \mathbb{Z}/m\mathbb{Z})$. We denote by $X_{\alpha}^{\bullet} = (X_{\alpha}, D_{X_{\alpha}}) \to X^{\bullet}$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/m\mathbb{Z}$ corresponding to α . Write $F_{X_{\alpha}}$ for the absolute Frobenius morphism on X_{α} . Then there exists a decomposition ([Se, Section 9])

$$H^{1}(X_{\alpha}, \mathcal{O}_{X_{\alpha}}) = H^{1}(X_{\alpha}, \mathcal{O}_{X_{\alpha}})^{\mathrm{st}} \oplus H^{1}(X_{\alpha}, \mathcal{O}_{X_{\alpha}})^{\mathrm{ni}},$$

where $F_{X_{\alpha}}$ is a bijection on $H^1(X_{\alpha}, \mathcal{O}_{X_{\alpha}})^{\text{st}}$ and is nilpotent on $H^1(X_{\alpha}, \mathcal{O}_{X_{\alpha}})^{\text{ni}}$. Moreover, we have $H^1(X_{\alpha}, \mathcal{O}_{X_{\alpha}})^{\text{st}} = H^1(X_{\alpha}, \mathcal{O}_{X_{\alpha}})^{F_{X_{\alpha}}} \otimes_{\mathbb{F}_p} k$, where $H^1(X_{\alpha}, \mathcal{O}_{X_{\alpha}})^{F_{X_{\alpha}}}$ denotes the subspace of $H^1(X_{\alpha}, \mathcal{O}_{X_{\alpha}})$ on which $F_{X_{\alpha}}$ acts trivially. Then Artin-Schreier theory implies that we may identify

$$H_{\alpha} \stackrel{\text{def}}{=} H^{1}_{\text{\'et}}(X_{\alpha}, \mathbb{F}_{p}) \otimes_{\mathbb{F}_{p}} k$$

with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-linear space H_{α} has the structure of a finitely generated $k[\mu_m]$ module induced by the natural action of μ_m on X_{α} . Then we have the following canonical decomposition

$$H_{\alpha} = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_m$ acts on $H_{\alpha,i}$ as the ζ^i -multiplication.

2.2.4. We call

$$\gamma_{\alpha,i} \stackrel{\text{def}}{=} \dim_k(H_{\alpha,i}), \ i \in \mathbb{Z}/m\mathbb{Z},$$

a generalized Hasse-Witt invariant (see [B], [N], [T3] for the case of étale or tame coverings of smooth pointed stable curves) of the cyclic multi-admissible covering $X^{\bullet}_{\alpha} \to X^{\bullet}$. In particular, we call

$$\gamma_{\alpha,z}$$

the first generalized Hasse-Witt invariant of the cyclic multi-admissible covering $X^{\bullet}_{\alpha} \to X^{\bullet}$. Note that the above decomposition implies that

$$\dim_k(H_\alpha) = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \gamma_{\alpha,i}$$

In particular, if X_{α} is connected, then $\dim_k(H_{\alpha}) = \sigma_{X_{\alpha}}$.

2.2.5. We write $\mathbb{Z}[D_X]$ for the group of divisors whose supports are contained in D_X . Note that $\mathbb{Z}[D_X]$ is a free \mathbb{Z} -module with basis D_X . We put $\mathbb{Z}/m\mathbb{Z}[D_X] \stackrel{\text{def}}{=} \mathbb{Z}[D_X] \otimes \mathbb{Z}/m\mathbb{Z}$ and define the following

$$c'_m : \mathbb{Z}/m\mathbb{Z}[D_X] \to \mathbb{Z}/m\mathbb{Z}, D \mod m \mapsto \deg(D) \mod m.$$

Write $(\mathbb{Z}/m\mathbb{Z})^{\sim}$ for the set $\{0, 1, \ldots, m-1\}$ and $(\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]$ for the subset of $\mathbb{Z}[D_X]$ consisting of the elements whose coefficients are contained in $(\mathbb{Z}/m\mathbb{Z})^{\sim}$. Then we have a natural bijection $\iota_m : (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X] \xrightarrow{\sim} \mathbb{Z}/m\mathbb{Z}[D_X]$.

We put

 $(\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0 \stackrel{\text{def}}{=} \iota_m^{-1}(\ker(c'_m)).$

Note that we have $m|\deg(D)$ for all $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$. Moreover, we put

$$s(D) \stackrel{\text{def}}{=} \frac{\deg(D)}{m} \in \mathbb{Z}_{\geq 0}.$$

Since every $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$ can be regarded as a ramification divisor associated to some cyclic admissible covering, the structure of the maximal prime-to-p quotient of Π_X • (2.1.4) implies the following:

$$0 \le s(D) \le \begin{cases} 0, & \text{if } n_X \le 1, \\ n_X - 1, & \text{if } n_X \ge 2. \end{cases}$$

2.2.6. We put

$$\widehat{X} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^{\bullet}} \text{ open}} X_{H}, \ D_{\widehat{X}} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^{\bullet}} \text{ open}} D_{X_{H}}, \ \Gamma_{\widehat{X}^{\bullet}} \stackrel{\text{def}}{=} \varprojlim_{H \subseteq \Pi_{X^{\bullet}} \text{ open}} \Gamma_{X_{H}^{\bullet}}.$$

We call $\widehat{X}^{\bullet} = (\widehat{X}, D_{\widehat{X}})$ the universal admissible covering of X^{\bullet} corresponding to $\Pi_{X^{\bullet}}$, and $\Gamma_{\widehat{X}^{\bullet}}$ the dual semi-graph of \widehat{X}^{\bullet} . Note that $\operatorname{Aut}(\widehat{X}^{\bullet}/X^{\bullet}) = \Pi_{X^{\bullet}}$, and that $\Gamma_{\widehat{X}^{\bullet}}$ admits a natural action of $\Pi_{X^{\bullet}}$. For every $e \in e^{\operatorname{op}}(\Gamma_{X^{\bullet}})$, write $\widehat{e} \in e^{\operatorname{op}}(\Gamma_{\widehat{X}^{\bullet}})$ for an open edge over e and x_e for the marked point corresponding to e.

We denote by $I_{\widehat{e}} \subseteq \Pi_{X^{\bullet}}$ the stabilizer of \widehat{e} . The definition of admissible coverings implies that $I_{\widehat{e}}$ is isomorphic to the Galois group $\operatorname{Gal}(K_{x_e}^t/K_{x_e}) \cong \widehat{\mathbb{Z}}(1)^{p'}$, where K_{x_e} denotes the quotient field of \mathcal{O}_{X,x_e} , $K_{x_e}^t$ denotes a maximal tamely ramified extension, and $\widehat{\mathbb{Z}}(1)^{p'}$ denotes the maximal prime-to-p quotient of $\widehat{\mathbb{Z}}(1)$. Suppose that x_e is contained in X_v . Then we have an injection

$$\phi_{\widehat{e}}: I_{\widehat{e}} \hookrightarrow \Pi^{\mathrm{ab}}_{X^{\bullet}}$$

which factors through $I_{\widehat{e}} \hookrightarrow \Pi_{\widetilde{X}_{v}^{\bullet}}^{\mathrm{ab}}$ induced by the composition of (outer) injective homomorphisms $I_{\widehat{e}} \hookrightarrow \Pi_{\widetilde{X}_{v}^{\bullet}} \hookrightarrow \Pi_{X^{\bullet}}$, where $\Pi_{\widetilde{X}_{v}^{\bullet}}$ denotes the admissible fundamental group of the smooth pointed stable curve $\widetilde{X}_{v}^{\bullet}$ associated to v (2.1.2). Since the image of $\phi_{\widehat{e}}$ depends only on e, we may write I_{e} for the image $\phi_{\widehat{e}}(I_{\widehat{e}})$. Moreover, the structure of maximal prime-to-p quotients of admissible fundamental groups of pointed stable curves (2.1.4) implies that the following holds: There exists a generator $[s_{e}]$ of I_{e} for each $e \in e^{\mathrm{op}}(\Gamma_{X^{\bullet}})$ such that

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_X \bullet)} [s_e] = 0$$

in $\Pi_{X^{\bullet}}^{ab}$. In the remainder of the present paper, we fix a set of generators $\{[s_e]\}_{e \in e^{op}(\Gamma_X^{\bullet})}$ of I_e satisfying the above condition.

Definition 2.2. We maintain the notation introduced above.

(i) We put

$$D_{\alpha} \stackrel{\text{def}}{=} \sum_{e \in e^{\text{op}}(\Gamma_X \bullet)} \alpha([s_e]) x_e, \ \alpha \in \text{Hom}(\Pi_{X \bullet}^{\text{ab}}, \mathbb{Z}/m\mathbb{Z}).$$

Note that we have $D_{\alpha} \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$. On the other hand, for each $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$, we denote by

$$\operatorname{Rev}_{D}^{\operatorname{adm}}(X^{\bullet}) \stackrel{\text{def}}{=} \{ \alpha \in \operatorname{Hom}(\Pi_{X^{\bullet}}^{\operatorname{ab}}, \mathbb{Z}/m\mathbb{Z}) \mid D_{\alpha} = D \}.$$

Moreover, we put

(1)

$$\gamma_{(\alpha,D)} \stackrel{\text{def}}{=} \gamma_{\alpha,1}.$$

(ii) Let $Q \in \mathbb{Z}[D_X]$ be an arbitrary effective divisor on X and m an arbitrary natural number. We put

$$\left[\frac{Q}{m}\right] \stackrel{\text{def}}{=} \sum_{x \in D_X} \left[\frac{\operatorname{ord}_x(Q)}{m}\right] x,$$

which is an effective divisor on X. Here [(-)] denotes the maximum integer which is less than or equal to (-).

2.3. Generalized Hasse-Witt invariants via line bundles. The generalized Hasse-Witt invariants can be also described in terms of line bundles and divisors.

2.3.1. Settings. We maintain the settings introduced in 2.2.1. Moreover, we suppose that X^{\bullet} is *smooth* over k.

2.3.2. Let $m \in \mathbb{N}$ be an arbitrary natural number prime to p. We denote by $\operatorname{Pic}(X)$ the Picard group of X. Consider the following complex of abelian groups:

$$\mathbb{Z}[D_X] \xrightarrow{a_m} \operatorname{Pic}(X) \oplus \mathbb{Z}[D_X] \xrightarrow{b_m} \operatorname{Pic}(X),$$

where $a_m(D) = ([\mathcal{O}_X(-D)], mD), b_m(([\mathcal{L}], D)) = [\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(D)]$. We denote by

$$\mathscr{P}_{X^{\bullet},m} \stackrel{\mathrm{def}}{=} \ker(b_m) / \mathrm{Im}(a_m)$$

the homology group of the complex. Moreover, we have the following exact sequence

$$0 \to \operatorname{Pic}(X)[m] \xrightarrow{a'_m} \mathscr{P}_{X^{\bullet},m} \xrightarrow{b'_m} \mathbb{Z}/m\mathbb{Z}[D_X] \xrightarrow{c'_m} \mathbb{Z}/m\mathbb{Z},$$

where $\operatorname{Pic}(X)[m]$ denotes the *m*-torsion subgroup of $\operatorname{Pic}(X)$, and

$$a'_m([\mathcal{L}]) = ([\mathcal{L}], 0) \mod \operatorname{Im}(a_m), \ b'_m(([\mathcal{L}], D)) \mod \operatorname{Im}(a_m)) = D \mod m,$$

 $c'_m(D \mod m) = \deg(D) \mod m.$

We shall define

$$\widetilde{\mathscr{P}}_{X^{\bullet},m} \subseteq \ker(b_m) \subseteq \operatorname{Pic}(X) \oplus \mathbb{Z}[D_X]$$

to be the inverse image of $(\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0 \subseteq (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X] \subseteq \mathbb{Z}[D_X]$ under the projection $\ker(b_m) \to \mathbb{Z}[D_X]$. It is easy to see that $\mathscr{P}_{X^{\bullet},m}$ and $\widetilde{\mathscr{P}}_{X^{\bullet},m}$ are free $\mathbb{Z}/m\mathbb{Z}$ -modules with rank $2g_X + n_X - 1$ if $n_X \neq 0$ and with rank $2g_X$ if $n_X = 0$. Note that we have $\widetilde{\mathscr{P}}_{X^{\bullet},m} \xrightarrow{\sim} \widetilde{\mathscr{P}}_{X^{\bullet},m}/\operatorname{Im}(a_m) \xrightarrow{\sim} \mathscr{P}_{X^{\bullet},m}$.

On the other hand, let $\alpha \in \operatorname{Hom}(\Pi_{X^{\bullet}}^{\operatorname{ab}}, \mathbb{Z}/m\mathbb{Z})$ and $f_{\alpha}^{\bullet} : X_{\alpha}^{\bullet} \to X^{\bullet}$ the Galois multiadmissible covering over k with Galois group $\mathbb{Z}/m\mathbb{Z}$ corresponding to α . Then we see

$$f_{\alpha,*}\mathcal{O}_{X_{\alpha}} \cong \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathcal{L}_{\alpha,i},$$

where locally $\mathcal{L}_{\alpha,i}$ is the eigenspace of the natural action of *i* with eigenvalue ζ^i . Moreover, we have the following natural isomorphism ([T3, Proposition 3.5]):

$$\operatorname{Hom}(\Pi^{\operatorname{ab}}_{X^{\bullet}}, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathscr{P}}_{X^{\bullet}, m}, \ \alpha \mapsto ([\mathcal{L}_{\alpha, 1}], D_{\alpha}).$$

Then every element of $\widetilde{\mathscr{P}}_{X^{\bullet},m}$ induces a Galois multi-admissible covering of X^{\bullet} over k with Galois group $\mathbb{Z}/m\mathbb{Z}$.

2.3.3. Further assumption. In the remainder of the present paper, we may assume that

$$n \stackrel{\text{def}}{=} p^t - 1$$

. .

for some positive natural number $t \in \mathbb{N}$ unless indicated otherwise.

2.3.4. We introduce the following notation concerning an effective divisor D on X.

Definition 2.3. For $u \in \{0, ..., n\}$, write its *p*-adic expansion as

$$u = \sum_{r=0}^{t-1} u_r p^r$$

with $u_r \in \{0, \ldots, p-1\}$. We identify $\{0, \ldots, t-1\}$ with $\mathbb{Z}/t\mathbb{Z}$ naturally. Then $\{0, \ldots, t-1\}$ admits an additional structure induced by the natural additional structure of $\mathbb{Z}/t\mathbb{Z}$. We put

$$u^{(i)} \stackrel{\text{def}}{=} \sum_{r=0}^{t-1} u_{i+r} p^r, \ i \in \{0, \dots, t-1\}.$$

Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5). We put

$$D^{(i)} \stackrel{\text{def}}{=} \sum_{x \in D_X} (\operatorname{ord}_x(D))^{(i)} x, \ i \in \{0, 1, \dots, t-1\},\$$

which is an effective divisor on X. Moreover, for each $j \in \{0, \ldots, n-1\}$, we put

$$D(j) \stackrel{\text{def}}{=} jD - n\left[\frac{jD}{n}\right].$$

Note that $D(p^{t-i}) = D^{(i)}, i \in \{0, \dots, t-1\}.$

By the various definitions, we have the following lemma.

Lemma 2.4. We maintain the notation introduced above. Suppose $n \stackrel{\text{def}}{=} p^t - 1$. Then we have the following holds (see 2.2.4 for the definition of $\gamma_{\alpha,j}$)

$$\gamma_{\alpha,j} = \gamma_{j\alpha,1} = \gamma_{(j\alpha,D(j))}.$$

In particular, by using Definition 2.2 (i)-(1), we have

$$\gamma_{(\alpha,D)} = \gamma_{\alpha,1} = \gamma_{\alpha,p^{t-i}} = \gamma_{p^{t-i}\alpha,1} = \gamma_{(p^{t-i}\alpha,D(p^{t-i}))} = \gamma_{(p^{t-i}\alpha,D^{(i)})}, \ i \in \{0,\dots,t-1\}$$

2.3.5. We explain that $D(j), j \in \{0, ..., n-1\}$, naturally arises from a Galois multiadmissible covering of X^{\bullet} with Galois group $\mathbb{Z}/n\mathbb{Z}$ whose ramification divisor is D. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet}, n}$ and $\alpha \in \operatorname{Hom}(\Pi_{X^{\bullet}}^{\operatorname{ab}}, \mathbb{Z}/n\mathbb{Z})$ the element such that $([\mathcal{L}], D) = ([\mathcal{L}_{\alpha,1}], D_{\alpha})$ via the isomorphism $\operatorname{Hom}(\Pi_{X^{\bullet}}^{\operatorname{ab}}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathscr{P}}_{X^{\bullet}, n}$ explained in 2.3.2. We fix an isomorphism $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(-D) \subseteq \mathcal{O}_X$ and put

$$\mathcal{L}(j) \stackrel{\text{def}}{=} \mathcal{L}^{\otimes j} \otimes \mathcal{O}_X(\left[\frac{jD}{n}\right]), \ j \in \{1, \dots, n-1\}.$$

Then we have $\mathcal{L}(j)^{\otimes n} \cong \mathcal{O}_X(-D(j))$ and $([\mathcal{L}(j)], D(j)) = ([\mathcal{L}_{\alpha,j}], D_\alpha(j)) = ([\mathcal{L}_{\alpha,j}], D(j)) \in \widetilde{\mathscr{P}}_{X^{\bullet}, n}$. Moreover, the action of $j \in \mathbb{Z}/n\mathbb{Z}$ on $\widetilde{\mathscr{P}}_{X^{\bullet}, n}$ is given by

$$([\mathcal{L}], D) \mapsto ([\mathcal{L}(j)], D(j)).$$

When j = p, the action of j is induced by the Frobenius action $F_{X_{\alpha}}$. In particular, we shall denote $\mathcal{L}(j)$ and D(j) by $\mathcal{L}^{(i)}$ and $D^{(i)}$, respectively, if $j = p^{t-i}, i \in \{0, \ldots, t-1\}$.

2.3.6. On the other hand, we have the following composition of morphisms of line bundles

$$\mathcal{L} \xrightarrow{p^t} \mathcal{L}^{\otimes p^t} = \mathcal{L}^{\otimes n} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X(-D) \otimes \mathcal{L} \hookrightarrow \mathcal{L}.$$

The composite morphism induces a morphism $\phi_{([\mathcal{L}],D)} : H^1(X,\mathcal{L}) \to H^1(X,\mathcal{L})$. We denote by

$$\gamma_{([\mathcal{L}],D)} \stackrel{\text{def}}{=} \dim_k(\bigcap_{r \ge 1} \operatorname{Im}(\phi_{([\mathcal{L}],D)}^r))$$

Write $\alpha_{\mathcal{L}} \in \operatorname{Hom}(\Pi_{X^{\bullet}}^{ab}, \mathbb{Z}/n\mathbb{Z})$ for the element corresponding to $([\mathcal{L}], D)$ and F_X for the absolute Frobenius morphism on X. Then we see that $\gamma_{\alpha_{\mathcal{L}},1}$ (2.2.4) is equal to the dimension over k of the largest subspace of $H^1(X, \mathcal{L})$ on which $F_X^t \stackrel{\text{def}}{=} F_X \circ \cdots \circ F_X$ is a bijection. Then we obtain $\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1}$. Moreover, since $D_{\alpha_{\mathcal{L}}} = D$, we have

$$\gamma_{([\mathcal{L}],D)} = \gamma_{(\alpha_{\mathcal{L}},D)} (\stackrel{\text{def}}{=} \gamma_{\alpha_{\mathcal{L}},1}).$$

We have the following lemma.

Lemma 2.5. We maintain the notation introduced above. Suppose that X^{\bullet} is smooth over k. Then we have

$$\gamma_{(\alpha_{\mathcal{L}},D)} \le \dim_{k}(H^{1}(X,\mathcal{L})) = \begin{cases} g_{X}, & \text{if } ([\mathcal{L}],D) = ([\mathcal{O}_{X}],0), \\ g_{X}-1, & \text{if } s(D) = 0, \ [\mathcal{L}] \neq [\mathcal{O}_{X}], \\ g_{X}+s(D)-1, & \text{if } s(D) \ge 1, \end{cases}$$

where s(D) is the integer defined in 2.2.5.

Proof. The first inequality follows from the definition of generalized Hasse-Witt invariants. On the other hand, the Riemann-Roch theorem implies that

$$\dim_k(H^1(X,\mathcal{L})) = g_X - 1 - \deg(\mathcal{L}) + \dim_k(H^0(X,\mathcal{L}))$$
$$= g_X - 1 + \frac{1}{n}\deg(D) + \dim_k(H^0(X,\mathcal{L})) = g_X - 1 + s(D) + \dim_k(H^0(X,\mathcal{L})).$$
completes the proof of the lemma.

This completes the proof of the lemma.

2.4. Raynaud-Tamagawa theta divisors. In this subsection, we recall the theory of theta divisors which was introduced by Raynaud in the case of étale coverings ([R1]), and which was generalized by Tamagawa in the case of tame coverings ([T3]).

2.4.1. Settings. We maintain the notation introduced in 2.3.1.

2.4.2. Let F_k be the absolute Frobenius morphism on Spec k, $F_{X/k}$ the relative Frobenius morphism $X \to X_1 \stackrel{\text{def}}{=} X \times_{k, F_k} k$ over k, and $F_k^t \stackrel{\text{def}}{=} F_k \circ \cdots \circ F_k$. We put $X_t \stackrel{\text{def}}{=} X \times_{k, F_k^t} k$, and define a morphism

$$F_{X/k}^t: X \to X_t$$

over k to be $F_{X/k}^t \stackrel{\text{def}}{=} F_{X_{t-1}/k} \circ \cdots \circ F_{X_1/k} \circ F_{X/k}.$

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet}, n}$, and let \mathcal{L}_t be the pulling back of \mathcal{L} by the natural morphism $X_t \to X$. Note that \mathcal{L} and \mathcal{L}_t are line bundles of degree -s(D) (2.2.5). We put

$$\mathcal{B}_D^t \stackrel{\text{def}}{=} (F_{X/k}^t)_* \big(\mathcal{O}_X(D) \big) / \mathcal{O}_{X_t}, \ \mathcal{E}_D \stackrel{\text{def}}{=} \mathcal{B}_D^t \otimes \mathcal{L}_t.$$

Write $\operatorname{rk}(\mathcal{E}_D)$ for the rank of \mathcal{E}_D . Then we obtain

$$\chi(\mathcal{E}_D) = \deg(\det(\mathcal{E}_D)) - (g_X - 1)\operatorname{rk}(\mathcal{E}_D).$$

Moreover, we have $\chi(\mathcal{E}_D) = 0$ ([T3, Lemma 2.3 (ii)]).

2.4.3. Let J_{X_t} be the Jacobian variety of X_t and \mathcal{L}_{X_t} a universal line bundle on $X_t \times J_{X_t}$. Let $\operatorname{pr}_{X_t} : X_t \times J_{X_t} \to X_t$ and $\operatorname{pr}_{J_{X_t}} : X_t \times J_{X_t} \to J_{X_t}$ be the natural projections. We denote by \mathcal{F} the coherent \mathcal{O}_{X_t} -module $\operatorname{pr}^*_{X_t}(\mathcal{E}_D) \otimes \mathcal{L}_{X_t}$, and by

$$\chi_{\mathcal{F}} \stackrel{\text{def}}{=} \dim_k(H^0(X_t \times_k k(y), \mathcal{F} \otimes k(y))) - \dim_k(H^1(X_t \times_k k(y), \mathcal{F} \otimes k(y)))$$

for each $y \in J_{X_t}$, where k(y) denotes the residue field of y. Note that since $\operatorname{pr}_{J_{X_t}}$ is flat, $\chi_{\mathcal{F}}$ is independent of $y \in J_{X_t}$. Write $(-\chi_{\mathcal{F}})^+$ for $\max\{0, -\chi_{\mathcal{F}}\}$. We denote by

$$\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$$

the closed subscheme of J_{X_t} defined by the $(-\chi_{\mathcal{F}})^+$ th Fitting ideal $\operatorname{Fitt}_{(-\chi_{\mathcal{F}})^+}(R^1(\operatorname{pr}_{J_{X_t}})_*(\mathcal{F}))$. The definition of $\Theta_{\mathcal{E}_D}$ is independent of the choice of \mathcal{L}_t . Moreover, we have $\operatorname{codim}(\Theta_{\mathcal{E}_D}) \leq 1$.

2.4.4. In [R1], Raynaud investigated the following property of the vector bundle \mathcal{E}_D on X.

Condition 2.6. We shall say that \mathcal{E}_D satisfies (\star) if there exists a line bundle \mathcal{L}'_t of degree 0 on X_t such that

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}'_t)), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}'_t))\}\}$$

Moreover, [T3, Proposition 2.2 (i) (ii)] implies that $[\mathcal{L}'] \notin \Theta_{\mathcal{E}_D}$ if and only if \mathcal{E}_D satisfies (\star) for \mathcal{L}' , where $[\mathcal{L}']$ denotes the point of J_{X_t} corresponding to \mathcal{L}' . Namely, $\Theta_{\mathcal{E}_D}$ is a *divisor* of J_{X_t} when \mathcal{E}_D satisfies (\star) . Then we have the following definition:

Definition 2.7. We shall call that the *Raynaud-Tamagawa theta divisor* $\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$ associated to \mathcal{E}_D exists if \mathcal{E}_D satisfies (*).

Remark 2.7.1. Suppose that \mathcal{E}_D satisfies (*) (i.e., Condition 2.6). [R1, Proposition 1.8.1] implies that $\Theta_{\mathcal{E}_D}$ is algebraically equivalent to $\operatorname{rk}(\mathcal{E}_D)\Theta$, where Θ is the classical theta divisor (i.e., the image of $X_t^{g_X-1}$ in J_{X_t}).

Lemma 2.8. We maintain the notation introduced above. Let $[\mathcal{I}] \in Pic(X)[n]$ and \mathcal{I}_t the pulling back of \mathcal{I} by the natural morphism $X_t \to X$. Suppose

$$\gamma_{([\mathcal{L}\otimes\mathcal{I}],D)} = \dim_k(H^1(X,\mathcal{L}\otimes\mathcal{I})).$$

Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to \mathcal{E}_D exists (i.e., $[\mathcal{I}_t] \notin \Theta_{\mathcal{E}_D}$).

Proof. The definition of \mathcal{E}_D implies the following natural exact sequence

$$0 \to \mathcal{L}_t \to (F_{X/k}^t)_* \big(\mathcal{O}_X(D) \big) \otimes \mathcal{L}_t \to \mathcal{E}_D \to 0.$$

Then the following natural sequence is exact

$$\dots \to H^0(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \xrightarrow{\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}} H^1(X_t, (F_{X/k}^t)_* (\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t) \\ \to H^1(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to \dots$$

Note that we have

$$H^{1}(X_{t}, \mathcal{L}_{t} \otimes \mathcal{I}_{t}) \cong H^{1}(X, \mathcal{L} \otimes \mathcal{I}),$$

$$H^{1}(X_{t}, (F_{X/k}^{t})_{*} (\mathcal{O}_{X}(D)) \otimes \mathcal{L}_{t} \otimes \mathcal{I}_{t}) \cong H^{1}(X, \mathcal{O}_{X}(D) \otimes (F_{X/k}^{t})^{*} (\mathcal{L}_{t} \otimes \mathcal{I}_{t}))$$

$$\cong H^{1}(X, \mathcal{O}_{X}(D) \otimes (\mathcal{L} \otimes \mathcal{I})^{\otimes p^{t}}) \cong H^{1}(X, \mathcal{L} \otimes \mathcal{I}).$$

Moreover, it is easy to see that the homomorphism $H^1(X, \mathcal{L} \otimes \mathcal{I}) \to H^1(X, \mathcal{L} \otimes \mathcal{I})$ induced by $\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}$ coincides with $\phi_{([\mathcal{L} \otimes \mathcal{I}], D)}$. Then Condition 2.6 implies that the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to \mathcal{E}_D exists if $\gamma_{([\mathcal{L} \otimes \mathcal{I}], D)} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$.

2.4.5. The following fundamental theorem of theta divisors was proved by Raynaud and Tamagawa.

Theorem 2.9. Suppose that $s(D) \in \{0, 1\}$. Then the Raynaud-Tamagawa theta divisor associated to \mathcal{E}_D exists (i.e., \mathcal{E}_D satisfies (\star)).

Remark 2.9.1. Theorem 2.9 was proved by Raynaud if s(D) = 0 ([R1, Théorème 4.1.1]), and by Tamagawa if $s(D) \leq 1$ ([T3, Theorem 2.5]). On the other hand, the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to \mathcal{E}_D does not exist in general ([T3, Example 2.19]).

Moreover, by applying Theorem 2.9, we have the following lemma (see [T3, Corollary 2.6, Lemma 2.12 (ii)]).

Lemma 2.10. (i) Let $Q \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree s(Q)n such that $\operatorname{ord}_x(Q) \leq n$ for each $x \in \operatorname{Supp}(Q)$, \mathcal{L}_Q a line bundle on X of degree -s(Q), and $\mathcal{L}_{Q,t}$ the pulling back of \mathcal{L}_Q by the natural morphism $X_t \to X$. Let $S_Q \stackrel{\text{def}}{=} \{x \in X \mid \operatorname{ord}_x(Q) = n\}$,

$$Q' \stackrel{\text{def}}{=} Q - \sum_{x \in S_Q} nx.$$

an effective divisor on X of degree s(Q')n, $\mathcal{L}_{Q'}$ a line bundle on X of degree -s(Q'), and $\mathcal{L}_{Q',t}$ the pulling back of $\mathcal{L}_{Q'}$ by the natural morphism $X_t \to X$. Suppose that the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_Q^t \otimes \mathcal{L}_{Q,t}$ exists. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_{Q'}^t \otimes \mathcal{L}_{Q',t}$ exists.

(ii) Let t_i , $i \in \{1, 2\}$, be an arbitrary positive natural number and $n_i \stackrel{\text{def}}{=} p^{t_i} - 1$. Let $Q_i \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree $\deg(Q_i) = s(Q_i)n_i$, \mathcal{L}_{Q_i} a line bundle on X of degree $-s(Q_i)$, and \mathcal{L}_{Q_i,t_i} the pulling back of \mathcal{L}_{Q_i} by the natural morphism $X_{t_i} \to X$. Suppose that $s \stackrel{\text{def}}{=} s(Q_1) = s(Q_2)$. Let $t \stackrel{\text{def}}{=} t_1 + t_2$, $n \stackrel{\text{def}}{=} n_1 + p^{t_1}n_2$,

$$Q \stackrel{\text{def}}{=} Q_1 + p^{t_1} Q_2 \in \mathbb{Z}[D_X]$$

an effective divisor on X of degree $\deg(Q) = sn$, \mathcal{L}_Q a line bundle on X of degree -s, and $\mathcal{L}_{Q,t}$ the pulling back of \mathcal{L}_Q by the natural morphism $X_t \to X$. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_Q^t \otimes \mathcal{L}_{Q,t}$ exists if and only if the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}_{Q,i}^t \otimes \mathcal{L}_{Q,t}$ exists for each $i \in \{1, 2\}$.

3. New-ordinariness of cyclic admissible coverings of generic curves

In this section, we prove our main theorem of the present paper (see Theorem 3.5), namely, a sufficient and necessary condition for ordinariness of prime-to-p cyclic admissible coverings of generic curves.

3.1. Idea. In this subsection, we briefly explain the idea of our proof of Theorem 3.5.

3.1.1. Settings. We maintain the notation introduced in 2.2.1 and suppose that X^{\bullet} is generic (2.1.3). Moreover, for simplicity, we assume $n = p^t - 1$.

3.1.2. Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5) be an effective divisor on $X, \alpha \in \operatorname{Rev}_D^{\operatorname{adm}}(X^{\bullet}) \setminus \{0\}$ (Definition 2.2 (i)), and $f^{\bullet} : Y^{\bullet} \to X^{\bullet}$ the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α .

We observe that if Y^{\bullet} is ordinary, then D must satisfy a certain condition which we call *Frobenius stable* (see Definition 3.3). The goal of this section is to prove its converse, namely, if D is Frobenius stable then Y^{\bullet} is ordinary. By 2.3 and 2.4, in other words, we need to prove that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_{D(j)}$ (2.4.2) exists for every D(j) (Definition 2.3), $j \in \{1, \ldots, n-1\}$, when D is Frobenius stable.

3.1.3. Special case (Section 3.3). Suppose that $g_X = 0$. Firstly, we assume n = p - 1. If $n_X = 3$, then the result has been essentially obtained by Bouw. If $n_X > 3$, since X^{\bullet} is generic, we consider a suitable degeneration (i.e., (DEG-A) defined in 3.3.2) \mathcal{X}_s^{\bullet} of X^{\bullet} such that the smooth pointed stable curves associated to vertices (2.1.2) of the dual semigraph \mathcal{X}_s^{\bullet} are of type (0,3). Then our goal follows from specialization maps of admissible fundamental groups. Second, we assume that $n = p^t - 1$ for an arbitrary $t \in \mathbb{N}$. We observe that D(j) can be constructed by certain effective divisors whose degree is equal to s(D(j))(p-1) when D is Frobenius stable. Then by Lemma 2.10, we may prove that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_{D(j)}$ exists.

3.1.4. General case (Section 3.5). Suppose that $g_X \ge 0$. In order to reduce the general case to the special case (i.e., $g_X = 0$), we consider a suitable degeneration (i.e., (DEG-B) defined in 3.5.2) \mathcal{X}_s^{\bullet} of X^{\bullet} such that the smooth pointed stable curves associated to vertices of the dual semi-graph \mathcal{X}_s^{\bullet} are either of type $(0, n_X)$ or of type $(g_X, 1)$. Then by applying specialization maps of admissible fundamental groups and Nakajima's result concerning ordinariness of cyclic étale coverings of generic curves, we may prove that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_{D(i)}$ exists.

3.1.5. In the cases mentioned above, since we compute generalized Hasse-Witt invariants of prime-to-p cyclic admissible coverings of *singular* pointed stable curves, we need to compute not only the generalized Hasse-Witt invariants arising from tame coverings of irreducible components but also arising from coverings of dual semi-graphs.

3.2. **Degeneration settings.** Let R be a discrete valuation ring with algebraically closed residue field k_R , K_R the quotient field of R, and \overline{K}_R an algebraic closure of K_R . Suppose that $k \subseteq K_R$. Let

$$\mathcal{X}^{\bullet} = (\mathcal{X}, D_{\mathcal{X}} \stackrel{\text{def}}{=} \{e_1, \dots, e_{n_X}\})$$

be a pointed stable curve of type (g_X, n_X) over R, where $e_i, i \in \{1, \ldots, n_X\}$, is a Rpoint of \mathcal{X} . We shall write $\mathcal{X}^{\bullet}_{\eta} = (\mathcal{X}_{\eta}, D_{\mathcal{X}_{\eta}} \stackrel{\text{def}}{=} \{e_{\eta,1}, \ldots, e_{\eta,n_X}\}), \ \mathcal{X}^{\bullet}_{\overline{\eta}} = (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \{e_{\eta,1}, \ldots, e_{\eta,n_X}\}), \ \mathcal{X}^{\bullet}_{\overline{\eta}} = (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \{e_{\eta,1}, \ldots, e_{\eta,n_X}\}), \ \mathcal{X}^{\bullet}_{\overline{\eta}} = (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \{e_{s,1}, \ldots, e_{s,n_X}\})$ for the generic fiber $\mathcal{X}^{\bullet} \times_R K_R$ of \mathcal{X}^{\bullet} , the geometric generic fiber $\mathcal{X}^{\bullet} \times_R \overline{K}_R$ of \mathcal{X}^{\bullet} , and the special fiber $\mathcal{X}^{\bullet} \times_R k_R$ of \mathcal{X}^{\bullet} , respectively. Write $\Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}$ and $\Pi_{\mathcal{X}^{\bullet}_{s}}$ for the admissible fundamental groups of $\mathcal{X}^{\bullet}_{\overline{\eta}}$ and $\mathcal{X}^{\bullet}_{s}$, respectively. Then we have a surjective specialization map ([V, Théorème 2.2 (b)])

$$sp_R: \Pi_{\mathcal{X}^{\bullet}_{\overline{n}}} \twoheadrightarrow \Pi_{\mathcal{X}^{\bullet}_s}.$$

Moreover, we shall suppose that the geometric generic fiber $\mathcal{X}^{\bullet}_{\overline{\eta}}$ of \mathcal{X}^{\bullet} is \overline{K}_R -isomorphic to $X^{\bullet} \times_k \overline{K}_R$. Then without loss of generality, we may identify $e_{\overline{\eta},i}$, $i \in \{1, \ldots, n_X\}$, with

 $x_i \times_k \overline{K}_R$ via this isomorphism. Note that since the admissible fundamental groups do not depend on the base fields, $\Pi_{\mathcal{X}^{\bullet}_{\overline{n}}}$ is naturally isomorphic to $\Pi_{X^{\bullet}}$.

3.3. **Basic case.** In this subsection, we treat the case where $g_X = 0$ and n = p - 1.

3.3.1. Settings. We maintain the notation introduced in 2.2.1 and suppose that X^{\bullet} is a generic curve (2.1.3) over k of type $(0, n_X)$. Moreover, we assume $n \stackrel{\text{def}}{=} p - 1$.

3.3.2. We maintain the notation introduced in 3.2. We shall say that X^{\bullet} admits a (DEG-A) if the following conditions hold, where "(DEG)" means "degeneration": (i) $n_X \geq 4$. (ii) \mathcal{X}_s^{\bullet} is a component-generic pointed stable curve (2.1.3) over k_R . (iii) The underlying curve \mathcal{X}_s of \mathcal{X}_s^{\bullet} is a chain of projective lines $\{P_u \cong \mathbb{P}_{k_R}^1\}_{u=1,\dots,n_X-2}$ over k_R such that $D_{\mathcal{X}_s} \cap P_1 = \{e_{s,1}, e_{s,2}\}, D_{\mathcal{X}_s} \cap P_{n_X-2} = \{e_{s,n_X-1}, e_{s,n_X}\}$, and $D_{\mathcal{X}_s} \cap P_u = \{e_{s,u+1}\}, u \notin \{1, n_X - 2\}$.

3.3.3. We have the following lemma.

Lemma 3.1. Let $g_X = 0$, n = p - 1, $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$, and $\alpha \in \operatorname{Rev}_D^{\operatorname{adm}}(X^{\bullet}) \setminus \{0\}$. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ (2.4.3) associated to \mathcal{E}_D exists. Moreover, we have (see 2.3.2 for $\widetilde{\mathscr{P}}_{X^{\bullet},n}$)

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X,\mathcal{L})), \ ([\mathcal{L}],D) \in \widetilde{\mathscr{P}}_{X^{\bullet},n}.$$

Proof. Let $f^{\bullet}: Y^{\bullet} = (Y, D_Y) \to X^{\bullet}$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α . Suppose that $n_X = 3$. Then [B, Corollary 6.8] and Lemma 2.5 imply $\gamma_{(\alpha,D)} = \dim_k(H^1(X,\mathcal{L})) = s(D) - 1$ for every $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet},n}$. Then Lemma 2.8 (i.e., $\mathcal{I} \cong \mathcal{O}_X$) implies that $[\mathcal{O}_X] \notin \Theta_{\mathcal{E}_D}$ (i.e., $\Theta_{\mathcal{E}_D} = \emptyset$).

Suppose that $n_X \geq 4$. Since X^{\bullet} is a generic curve, X^{\bullet} admits a (DEG-A) (3.3.2). Furthermore, we write $Q_{\overline{\eta}}$ (resp. Q_s) for the effective divisor on $\mathcal{X}_{\overline{\eta}}$ (resp. \mathcal{X}_s) induced by D and $\alpha_{\overline{\eta}} \in \operatorname{Rev}_{Q_{\overline{\eta}}}^{\operatorname{adm}}(\mathcal{X}_{\overline{\eta}}^{\bullet})$ for the element induced by α . Then we have $\gamma_{(\alpha,D)} = \gamma_{(\alpha_{\overline{\eta}},Q_{\overline{\eta}})}$.

Since \mathcal{X}_s is a chain, for each $u \in \{1, \ldots, n_X - 3\}$, we may write y_u and z_{u+1} for the inverse image of $P_u \cap P_{u+1}$ of the natural closed immersion $P_v \hookrightarrow \mathcal{X}_s$ and the inverse image of $P_u \cap P_{u+1}$ of the natural closed immersion $P_{u+1} \hookrightarrow \mathcal{X}_s$, respectively. We define

$$P_1^{\bullet} = (P_1, D_{P_1} \stackrel{\text{def}}{=} \{e_{s,1}, e_{s,2}, y_1\}),$$

$$P_{n_X-2}^{\bullet} = (P_{n_X-2}, D_{P_{n_X-2}} \stackrel{\text{def}}{=} \{z_{n_X-2}, e_{s,n_X-1}, e_{s,n_X}\}),$$

$$P_u^{\bullet} = (P_u, D_{P_u} \stackrel{\text{def}}{=} \{z_u, e_{s,u+1}, y_u\}), \ u \notin \{1, n_X - 2\},$$

to be smooth pointed stable curves of types (0,3) over k_R , respectively. Let

$$f_{\overline{\eta}}^{\bullet} \stackrel{\text{def}}{=} f^{\bullet} \times_k \overline{K}_R : \mathcal{Y}_{\overline{\eta}}^{\bullet} = (\mathcal{Y}_{\overline{\eta}}, D_{\mathcal{Y}_{\overline{\eta}}}) \stackrel{\text{def}}{=} Y^{\bullet} \times_k \overline{K}_R \to \mathcal{X}_{\overline{\eta}}^{\bullet}$$

be the Galois multi-admissible covering over \overline{K}_R with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by f^{\bullet} , and $\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}} \subseteq \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}$ the admissible fundamental group of an arbitrary connected component of $\mathcal{Y}^{\bullet}_{\overline{\eta}}$. By the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]), we have $sp_R^{p'} : \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}^{p'} \xrightarrow{\sim} \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of (-). Then we obtain a normal open subgroup $\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}}^{p'} \stackrel{\text{def}}{=} sp_R^{p'}(\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}}^{p'}) \subseteq \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}^{p'}$. Write $\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}} \subseteq \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}$ for the inverse image of $\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}}^{p'}$ of the natural surjection $\Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}} \twoheadrightarrow \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}^{p'}$. Then $\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}}$ and $f^{\bullet}_{\overline{\eta}}$ determine a Galois multi-admissible covering

 $f_s^{\bullet}: \mathcal{Y}_s^{\bullet} = (\mathcal{Y}_s, D_{\mathcal{Y}_s}) \to \mathcal{X}_s^{\bullet}$ over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_s \in \operatorname{Rev}_{Q_s}^{\operatorname{adm}}(\mathcal{X}_s^{\bullet})$ for an element induced by f_s^{\bullet} .

Let $Y_u \stackrel{\text{def}}{=} f_s^{-1}(P_u), u \in \{1, \dots, n_X - 2\}$. We put

$$Y_u^{\bullet} \stackrel{\text{def}}{=} (Y_u, D_{Y_u} \stackrel{\text{def}}{=} f_s^{-1}(D_{P_u})), \ u \in \{1, \dots, n_X - 2\}.$$

Then f_s^{\bullet} induces a Galois multi-admissible covering $f_u^{\bullet}: Y_u^{\bullet} \to P_u^{\bullet}, u \in \{1, \ldots, n_X - 2\}$, over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $([\mathcal{L}_u], Q_u) \in \widetilde{\mathscr{P}}_{P_u^{\bullet}, n}$ for the element induced by f_u^{\bullet} for every $u \in \{1, \ldots, n_X - 2\}$. Note that we have $\operatorname{Supp}(Q_u) \subseteq D_{P_u}$. Moreover, the $k_R[\mu_n]$ -module $H^1_{\text{\acute{e}t}}(Y_u, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\mathrm{\acute{e}t}}(Y_u, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Y_u}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Y_u}(j)$ as the ζ^j -multiplication. Then the case of $n_X = 3$ of the lemma implies $\gamma_{([\mathcal{L}_u],Q_u)} = \dim_{k_R}(M_{Y_u}(1)) = \dim_{k_R}(H^1(P_u,\mathcal{L}_u)).$

Write $\Gamma_{\mathcal{Y}_s^{\bullet}}$ for the dual semi-graph of \mathcal{Y}_s^{\bullet} . The natural $k[\mu_n]$ -submodule $H^1(\Gamma_{\mathcal{Y}_s^{\bullet}}, \mathbb{F}_p) \otimes k \subseteq H^1_{\text{\'et}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k$ admits the following canonical decomposition

$$H^1(\Gamma_{\mathcal{Y}^{\bullet}_s}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{\mathcal{Y}^{\bullet}_s}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{\mathcal{Y}_s^{\bullet}}}(j)$ as the ζ^j -multiplication. Then we see

$$\gamma_{(\alpha_s,Q_s)} = \sum_{u=1}^{n_X-2} \gamma_{([\mathcal{L}_u],Q_u)} + \dim_k(M_{\Gamma_{\mathcal{Y}_s^{\bullet}}}(1)) = s(D_{\alpha_s}) - 1.$$

On the other hand, the $k_R[\mu_n]$ -modules $H^1_{\text{\acute{e}t}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ and $H^1_{\text{\acute{e}t}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R$ admit the following canonical decompositions

$$H^{1}_{\text{\'et}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_{p}) \otimes k_{R} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{\overline{\eta}}}(j), \ H^{1}_{\text{\'et}}(\mathcal{Y}_{s}, \mathbb{F}_{p}) \otimes k_{R} = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{s}}(j),$$

respectively, where $\zeta \in \mu_n$ acts on $M_{\mathcal{Y}_{\overline{\eta}}}(j)$ and $M_{\mathcal{Y}_s}(j)$ as the ζ^j -multiplication. Moreover, we have an injection as $k_R[\mu_n]$ -modules $H^1_{\text{\acute{e}t}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R \hookrightarrow H^1_{\text{\acute{e}t}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ induced by the surjective specialization map $\Pi_{\mathcal{Y}_{\overline{\eta}}^{\bullet}} \twoheadrightarrow \Pi_{\mathcal{Y}_s^{\bullet}}$. Thus, we have

$$s(D_{\alpha_s}) - 1 = \gamma_{(\alpha_s, Q_s)} = \dim_{k_R}(M_{\mathcal{Y}_s}(1)) \le \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \dim_{k_R}(M_{\mathcal{Y}_{\overline{\eta}}}(1)) \le s(D_{\alpha_{\overline{\eta}}}) - 1.$$

Since $s(D) = s(D_{\alpha_{\overline{\eta}}}) = s(D_{\alpha_s})$, we obtain

$$\gamma_{([\mathcal{L}],D)} = s(D) - 1 = \dim_k(H^1(X,\mathcal{L})).$$

Then Lemma 2.8 implies that $[\mathcal{O}_X] \notin \Theta_{\mathcal{E}_D}$ (i.e., $\Theta_{\mathcal{E}_D} = \emptyset$). We complete the proof of the lemma.

3.4. Frobenius stable effective divisors. We introduce Frobenius stable effective divisors.

3.4.1. Settings. We maintain the notation introduced in 2.2.1 and suppose that X^{\bullet} is a generic curve (2.1.3) over k of type (g_X, n_X) . Moreover, we assume $n \stackrel{\text{def}}{=} p^t - 1$.

3.4.2. Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5) be an effective divisor on X of degree s(D)n (2.2.5) and $x \in D_X$. For each $i \in \{0, \ldots, t-1\}$, we put $d_x^{(i)} \stackrel{\text{def}}{=} \operatorname{ord}_x(D^{(i)})$ (see Definition 2.3 for $D^{(i)}$), and write

$$d_x^{(i)} = \sum_{r=0}^{t-1} d_{x,r}^{(i)} p^r$$

for the *p*-adic expansion. In particular, if i = 0, we write D, d_x , and $d_{x,r}$ for $D^{(0)}$, $d_x^{(0)}$, and $d_{x,r}^{(0)}$, respectively. Then we have the following lemma.

Lemma 3.2. Let $n \stackrel{\text{def}}{=} p^t - 1$. The following statements are equivalent: (i)

$$s(D)n = \deg(D) = \deg(D^{(i)})$$

holds for each $i \in \{0, 1, \dots, t-1\}$. (ii)

$$\sum_{x \in D_X} d_{x,r} = s(D)(p-1)$$

holds for each $r \in \{0, \dots, t-1\}$. (iii)

$$\sum_{x \in D_X} d_{x,r}^{(i)} = s(D)(p-1)$$

holds for each $i \in \{0, ..., t-1\}$ and each $r \in \{0, ..., t-1\}$.

 $\langle \rangle$

Proof. We see that (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) follows immediately from the definition of $D^{(i)}$. Let us prove (i) \Rightarrow (ii).

Let $r \in \{0, ..., t - 1\}$. We have

$$d_x^{(r+1)} = d_{x,r}p^{t-1} + \frac{d_x^{(r)} - d_{x,r}}{p} = \frac{1}{p}d_x^{(r)} + \frac{p^t - 1}{p}d_{x,r} = \frac{1}{p}d_x^{(r)} + \frac{n}{p}d_{x,r}.$$

Note that (i) implies that

$$s(D)n = \sum_{x \in D_X} d_x^{(r+1)} = \sum_{x \in D_X} d_x^{(r)}.$$

Then we have

$$s(D)n = \sum_{x \in D_X} d_x^{(r+1)} = \frac{1}{p} \sum_{x \in D_X} d_x^{(r)} + \frac{n}{p} \sum_{x \in D_X} d_{x,r}$$
$$= \frac{1}{p} s(D)n + \frac{n}{p} \sum_{x \in D_X} d_{x,r}.$$

This means that

$$\sum_{x \in D_X} d_{x,r} = s(D)(p-1).$$

We complete the proof of the lemma.

3.4.3. We introduce the following condition concerning effective divisors on X.

Definition 3.3. Let m be a natural number prime to $p, t \in \mathbb{N}$ the order of p in the finite group $(\mathbb{Z}/m\mathbb{Z})^{\times}$, and $n \stackrel{\text{def}}{=} p^t - 1$. Let $Q \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$ be an effective divisor on X and $m' \stackrel{\text{def}}{=} n/m$. We shall call Q Frobenius stable if $Q' \stackrel{\text{def}}{=} m'Q \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ satisfies one of the statements mentioned in Lemma 3.2.

Remark 3.3.1. Let $Y^{\bullet} \to X^{\bullet}$ be a Galois tame covering over k whose Galois group is $\mathbb{Z}/m\mathbb{Z}$, and whose ramification divisor is Q. If Q is Frobenius stable, then the eigenspaces in each Frobenius orbit of $H^1(Y, \mathcal{O}_Y)$ have the same dimension (see the "moreover" part of Proposition 3.4). See Proposition 4.1 (ii), (iii) for some examples of Frobenius stable divisors.

3.5. General case. In this subsection, we generalize Lemma 3.1 to arbitrary pointed stable curves.

3.5.1. Settings. We maintain the notation introduced in 2.2.1 and suppose that X^{\bullet} is a generic curve (2.1.3) over k of type (g_X, n_X) . Moreover, we assume $n \stackrel{\text{def}}{=} p^t - 1$ for some positive natural number $t \in \mathbb{N}$.

3.5.2. We maintain the notation introduced in 3.2. We shall say that X^{\bullet} admits a (DEG-B) if the following conditions hold: (i) $n_X \geq 2$. (ii) \mathcal{X}_s^{\bullet} is a component-generic pointed stable curve (2.1.3) over k_R . (iii) The underlying curve \mathcal{X}_s of \mathcal{X}_s^{\bullet} is a chain consisting of a projective line $P \cong \mathbb{P}^1_{k_R}$ over k_R and a smooth projective curve C over k_R of genus g_X . (iv) $D_{\mathcal{X}_s} \subseteq P$.

3.5.3. We have the following proposition.

Proposition 3.4. Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ and $\alpha \in \operatorname{Rev}_D^{\operatorname{adm}}(X^{\bullet}) \setminus \{0\}$. Suppose that D is Frobenius stable. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to \mathcal{E}_D exists. Moreover, for every $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet}, n}$, we have

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X,\mathcal{L})) = \dim_k(H^1(X,\mathcal{L}^{(i)})) = \gamma_{([\mathcal{L}^{(i)}],D^{(i)})}, \ i \in \{0,\ldots,t-1\}.$$

Proof. Since D is Frobenius stable, we have

$$\dim_k(H^1(X,\mathcal{L})) = \dim_k(H^1(X,\mathcal{L}^{(i)})) = g_X + s(D) - 1$$

for each $i \in \{0, \ldots, t-1\}$. Then to verify the proposition, it is sufficient to prove that $\gamma_{([\mathcal{L}^{(i)}],D^{(i)})} = \dim_k(H^1(X,\mathcal{L}^{(i)}))$ holds for each $i \in \{0,\ldots, t-1\}$. Furthermore, we see immediately that it is sufficient to prove

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X,\mathcal{L})).$$

Suppose that $g_X = 0$. We maintain the notation introduced in Lemma 3.2. We put

$$D_r \stackrel{\text{def}}{=} \sum_{x \in D_X} d_{x,r} x, \ r \in \{0, \dots, t-1\},$$

which is an effective divisor on X. Since D is Frobenius stable, we have $\deg(D_r) = s(D)(p-1)$ for each $r \in \{0, \ldots, t-1\}$. Moreover, we have

$$D = \sum_{r=0}^{t-1} D_r p^r.$$

Since deg $(D_r) = s(D)(p-1)$, by applying Lemma 3.1 (i.e., by replacing D, n, by D_r , deg $(D_r)/s(D)$, respectively), the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_{D_r}}$, $r \in \{0, \ldots, t-1\}$, exists. Furthermore, by using Lemma 2.10 repeatedly (e.g. by replacing Q, Q_1 , and Q_2 by D, D_0 , and $D_1 + pD_1 + \cdots + p^{r-1}D_r$, respectively), we obtain that the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ exists.

Suppose that $g_X \ge 1$ and $n_X \le 1$. Then D = 0 and every cyclic admissible covering of X^{\bullet} is étale. The proposition follows immediately from [N, Proposition 4].

Suppose that $g_X \geq 1$ and $n_X \geq 2$. Since X^{\bullet} is generic, we see that X^{\bullet} admits a (DEG-B) (3.5.2). Let $f^{\bullet}: Y^{\bullet} = (Y, D_Y) \to X^{\bullet}$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α . Furthermore, we write $Q_{\overline{\eta}}$ (resp. Q_s) for the effective divisor on $\mathcal{X}_{\overline{\eta}}$ (resp. \mathcal{X}_s) induced by D and $\alpha_{\overline{\eta}} \in \operatorname{Rev}_{Q_{\overline{\eta}}}^{\operatorname{adm}}(\mathcal{X}_{\overline{\eta}}^{\bullet})$ for the element induced by α . Then we have $\gamma_{(\alpha,D)} = \gamma_{(\alpha_{\overline{\eta}},Q_{\overline{\eta}})}$.

We define

$$P^{\bullet} \stackrel{\text{def}}{=} (P, D_P \stackrel{\text{def}}{=} D_{\mathcal{X}_s} \cup (C \cap P)),$$
$$C^{\bullet} = (C, D_C \stackrel{\text{def}}{=} C \cap P)$$

to be smooth pointed stable curves over k of types $(0, n_X + 1)$ and $(g_X, 1)$, respectively. Let

$$f_{\overline{\eta}}^{\bullet} \stackrel{\text{def}}{=} f^{\bullet} \times_k \overline{K}_R : \mathcal{Y}_{\overline{\eta}}^{\bullet} = (\mathcal{Y}_{\overline{\eta}}, D_{\mathcal{Y}_{\overline{\eta}}}) \stackrel{\text{def}}{=} Y^{\bullet} \times_k \overline{K}_R \to \mathcal{X}_{\overline{\eta}}^{\bullet}$$

be the Galois multi-admissible covering over \overline{K}_R with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by f^{\bullet} , and $\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}} \subseteq \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}$ the admissible fundamental group of an arbitrary connected component of $\mathcal{Y}^{\bullet}_{\overline{\eta}}$. By the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]), we have $sp_R^{p'} : \Pi_{\mathcal{X}^{\bullet}_{\overline{\eta}}}^{p'} \xrightarrow{\sim} \Pi_{\mathcal{X}^{\bullet}_{s}}^{p'}$. Then we obtain a normal open subgroup $\Pi_{\mathcal{Y}^{\bullet}_{s}}^{p'} \stackrel{\text{def}}{=} sp_R^{p'}(\Pi_{\mathcal{Y}^{\bullet}_{\overline{\eta}}}^{p'}) \subseteq \Pi_{\mathcal{X}^{\bullet}_{s}}^{p'}$. Write $\Pi_{\mathcal{Y}^{\bullet}_{s}} \subseteq \Pi_{\mathcal{X}^{\bullet}_{s}}$ for the inverse image of $\Pi_{\mathcal{Y}^{\bullet}_{s}}^{p'}$ of the natural surjection $\Pi_{\mathcal{X}^{\bullet}_{s}} \to \Pi_{\mathcal{X}^{\bullet}_{s}}^{p'}$. Then $\Pi_{\mathcal{Y}^{\bullet}_{s}}$ and $f^{\bullet}_{\overline{\eta}}$ determine a Galois multi-admissible covering

$$f_s^{\bullet}: \mathcal{Y}_s^{\bullet} = (\mathcal{Y}_s, D_{\mathcal{Y}_s}) \to \mathcal{X}_s^{\bullet}$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_s \in \operatorname{Rev}_{Q_s}^{\operatorname{adm}}(\mathcal{X}^{\bullet}_s)$ for an element induced by f_s^{\bullet} .

The structures of the maximal prime-to-p quotients of admissible fundamental groups (2.1.4) imply that f_s is étale over C. Then we obtain that f_s is étale over $C \cap P$. Thus, f_s is étale over D_C . Let $Y_P \stackrel{\text{def}}{=} f_s^{-1}(P)$. We put $Y_P^{\bullet} \stackrel{\text{def}}{=} (Y_P, D_{Y_P} \stackrel{\text{def}}{=} f_s^{-1}(D_P))$. Then f_s^{\bullet} induces a Galois multi-admissible covering $f_P^{\bullet} : Y_P^{\bullet} \to P^{\bullet}$ over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $([\mathcal{L}_P], Q_P) \in \widetilde{\mathscr{P}}_{P^{\bullet,n}}$ for the element induced by f_P^{\bullet} . Note that since f_s is étale over $C \cap P$, we have $\operatorname{Supp}(Q_P) \subseteq D_P$. Moreover, the $k_R[\mu_n]$ -module $H^1_{\text{ét}}(Y_P, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\mathrm{\acute{e}t}}(Y_P,\mathbb{F}_p)\otimes k_R=\bigoplus_{j\in\mathbb{Z}/n\mathbb{Z}}M_{Y_P}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Y_P}(j)$ as the ζ^j -multiplication. Then Lemma 3.1 implies that

$$\gamma_{([\mathcal{L}_P],Q_P)} = \dim_{k_R}(M_{Y_P}(1)) = \dim_{k_R}(H^1(P,\mathcal{L}_P)) = s(D_{\alpha_s}) - 1$$

We put $Z \stackrel{\text{def}}{=} f_s^{-1}(C)$, and denote by $\pi_0(Z)$ the set of connected components of Z. Then f_s^{\bullet} induces a Galois étale covering (not necessarily connected) $f_C^{\bullet} : Z^{\bullet} = (Z, D_Z \stackrel{\text{def}}{=} f_s^{-1}(D_C)) \to C^{\bullet}$ over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, f_C^{\bullet} induces an element

 $\alpha_C \in \operatorname{Rev}_0^{\operatorname{adm}}(C^{\bullet})$. Suppose that $\#\pi_0(Z) \neq n$. Then we have $\alpha_C \neq 0$. The $k_R[\mu_n]$ -module $H^1_{\operatorname{\acute{e}t}}(Z, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\mathrm{\acute{e}t}}(Z,\mathbb{F}_p)\otimes k_R = \bigoplus_{j\in\mathbb{Z}/n\mathbb{Z}} M_Z(j),$$

where $\zeta \in \mu_n$ acts on $M_Z(j)$ as the ζ^j -multiplication. By applying [N, Proposition 4], we obtain $\gamma_{(\alpha_C,0)} = \dim_{k_R}(M_Z(1)) = g_X - 1$. Suppose that $\#\pi_0(Z) = n$. Then we have $\alpha_C = 0$. Since C is ordinary, we obtain $\gamma_{(\alpha_C,0)} = g_X$.

Write $\Gamma_{\mathcal{Y}^{\bullet}_{s}}$ for the dual semi-graph of $\mathcal{Y}^{\bullet}_{s}$. The natural $k[\mu_{n}]$ -submodule $H^{1}(\Gamma_{\mathcal{Y}^{\bullet}_{s}}, \mathbb{F}_{p}) \otimes k \subseteq H^{1}_{\text{\acute{e}t}}(\mathcal{Y}_{s}, \mathbb{F}_{p}) \otimes k$ admits the following canonical decomposition

$$H^1(\Gamma_{\mathcal{Y}^{\bullet}_s}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{\mathcal{Y}^{\bullet}_s}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{\mathcal{Y}_s^{\bullet}}}(j)$ as the ζ^j -multiplication. Then we have (e.g. [Y3, Lemma 3.2])

$$\dim_k(M_{\Gamma_{\mathcal{Y}_s^{\bullet}}}(1)) = \begin{cases} 0, & \text{if } \#\pi_0(Z) = n, \\ 1, & \text{if } \#\pi_0(Z) \neq n. \end{cases}$$

Thus, we have $\gamma_{(\alpha_s, Q_s)} = \gamma_{(\alpha_c, 0)} + \gamma_{([\mathcal{L}_P], Q_P)} + \dim_k(M_{\Gamma_{\mathcal{Y}_{\bullet}}}(1)) = g_X + s(D_{\alpha_s}) - 1.$

On the other hand, the $k_R[\mu_n]$ -modules $H^1_{\text{\acute{e}t}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ and $H^1_{\text{\acute{e}t}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R$ admit the following canonical decompositions

$$\begin{aligned} H^{1}_{\text{\acute{e}t}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_{p}) \otimes k_{R} &= \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{\overline{\eta}}}(j), \\ H^{1}_{\text{\acute{e}t}}(\mathcal{Y}_{s}, \mathbb{F}_{p}) \otimes k_{R} &= \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{s}}(j), \end{aligned}$$

respectively, where $\zeta \in \mu_n$ acts on $M_{\mathcal{Y}_{\overline{\eta}}}(j)$ and $M_{\mathcal{Y}_s}(j)$ as the ζ^j -multiplication. Moreover, we have an injection as $k_R[\mu_n]$ -modules $H^1_{\text{\acute{e}t}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R \hookrightarrow H^1_{\text{\acute{e}t}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ induced by the surjective specialization map $\Pi_{\mathcal{Y}_{\overline{\eta}}^{\bullet}} \twoheadrightarrow \Pi_{\mathcal{Y}_s^{\bullet}}$. Thus, we have

$$g_X + s(D_{\alpha_s}) - 1 = \gamma_{(\alpha_s, Q_s)} = \dim_{k_R}(M_{\mathcal{Y}_s}(1)) \le \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \dim_{k_R}(M_{\mathcal{Y}_{\overline{\eta}}}(1)) \le g_X + s(D_{\alpha_{\overline{\eta}}}) - 1.$$

Since $s(D) = s(D_{\alpha_{\overline{\eta}}}) = s(D_{\alpha_s})$, we obtain

$$\gamma_{([\mathcal{L}],D)} = g_X + s(D) - 1 = \dim_k(H^1(X,\mathcal{L})).$$

This completes the proof of the proposition.

3.6. Main result.

3.6.1. Now, we are going to prove the main result of the present paper.

Theorem 3.5. Let X^{\bullet} be a generic pointed stable curve (2.1.3) over an algebraically closed field k of characteristic p > 0. Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to $p, f^{\bullet}: Y^{\bullet} \to X^{\bullet}$ an arbitrary Galois multi-admissible covering over k with Galois group $\mathbb{Z}/m\mathbb{Z}$, and $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5) the ramification divisor associated to f^{\bullet} . Then f^{\bullet} is new-ordinary (2.2.2) if and only if $D(j), j \in \{1, \ldots, m-1\}$, is Frobenius stable (Definition 3.3).

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Proof. Let $t \in \mathbb{N}$ be the order of p in the finite group $(\mathbb{Z}/m\mathbb{Z})^{\times}$ and $n \stackrel{\text{def}}{=} p^t - 1$. We put $m' \stackrel{\text{def}}{=} n/m$ and write D' for the effective divisor $m'D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order m. We see immediately that, to verify the proposition, it is sufficient to prove the theorem when D = D'. This means that we may assume m = n.

Since we have the following canonical isomorphism

$$f_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \bigoplus_{j=1}^{n-1} \mathcal{L}(j),$$

we obtain that

$$g_Y = g_X + \sum_{j=1}^{n-1} \dim_k(H^1(X, \mathcal{L}(j))).$$

On the other hand, we have

$$\sigma_Y = \sigma_X + \sum_{j=1}^{n-1} \gamma_{([\mathcal{L}(j)], D(j))}$$

and $\gamma_{([\mathcal{L}(j)],D(j))} \leq \dim_k(H^1(X,\mathcal{L}(j)))$ for each $j \in \{1,\ldots,n-1\}$. Then f^{\bullet} is new-ordinary if and only if

$$\sum_{j=1}^{n-1} \gamma_{([\mathcal{L}(j)], D(j))} = \sum_{j=1}^{n-1} \dim_k(H^1(X, \mathcal{L}(j))).$$

Suppose that $D(j), j \in \{1, ..., n-1\}$, is Frobenius stable. Then Proposition 3.4 implies that f^{\bullet} is new-ordinary. Conversely, suppose that f^{\bullet} is new-ordinary. Let $j \in \{1, ..., n-1\}$ and $i \in \{0, ..., t-1\}$. Then we have $\gamma_{([\mathcal{L}^{(i)}(j)], D^{(i)}(j))} = \dim_k(H^1(X, \mathcal{L}^{(i)}(j)))$. Moreover, we have

$$\gamma_{([\mathcal{L}^{(i)}(j)], D^{(i)}(j))} \leq \dim_k(H^1(X, \mathcal{L}^{(i)}(j)))$$

holds for every $i' \in \{0, \ldots, t-1\}$. This implies that

$$g_X + s(D^{(i)}(j)) - 1 = \dim_k(H^1(X, \mathcal{L}^{(i)}(j))) = \dim_k(H^1(X, \mathcal{L}^{(i')}(j))) = g_X + s(D^{(i')}(j)) - 1$$

holds for every $i' \in \{0, \ldots, t-1\}$. This means that the statement of Lemma 3.2 (i) holds. Then $D(j), j \in \{1, \ldots, n-1\}$, is Frobenius stable. We complete the proof of the theorem.

3.6.2. Let $0 \leq \sigma \leq g_X$ be an integer, $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ the moduli stack parameterizing pointed stable curves of type (g_X, n_X) over \mathbb{Z} , $\overline{\mathcal{M}}_{g_X,n_X} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{Z}} k$, and $\overline{\mathcal{M}}_{g_X,n_X}^{\sigma}$ the locally closed reduced substack of $\overline{\mathcal{M}}_{g_X,n_X}$ whose points represent pointed stable curves with *p*-rank σ . Suppose that X^{\bullet} is a pointed stable curve corresponding to a geometric point over a generic point of $\overline{\mathcal{M}}_{g_X,n_X}^{\sigma}$. E. Ozman and R. Pries proved the following interesting result ([OP, Theorem 1.1 (1)]) which is a generalized version of [N, Proposition 4]:

Let $\ell \neq p$ be a prime number and $n_X = 0$. Suppose that $g_X \geq 2$, and that $0 \leq \sigma \leq g_X$ with $\sigma \neq 0$ if $g_X = 2$. Then every Galois admissible covering of X^{\bullet} (i.e., Galois étale covering of X) with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ is new-ordinary.

Note that this result gives a new result concerning torsion points of Raynaud-Tamagawa theta divisors ([OP, Theorem 1.1 (2)]). The proof of the above result depends on a deep result of J. Achter and Pries concerning monodromy of the *p*-rank strata of moduli spaces of curves ([AP]). One may ask whether or not Ozman-Pries' result hold for arbitrary prime-to-p cyclic coverings (or abelian coverings in general).

On the other hand, by applying Ozman-Pries' result mentioned above and similar arguments to the arguments given in the proofs of Proposition 3.4 and Theorem 3.5, we obtain the following generalized version of Theorem 3.5 if $m = \ell$:

Corollary 3.6. We maintain the notation introduced above. Let X^{\bullet} be a pointed stable curve corresponding to a geometric point over a generic point of $\overline{\mathcal{M}}_{g_X,n_X}^{\sigma}$ such that $0 \leq \sigma \leq g_X$ with $\sigma \neq 0$ if $g_X = 2$. Let ℓ be an arbitrary prime number prime to $p, f^{\bullet}: Y^{\bullet} \rightarrow X^{\bullet}$ an arbitrary Galois multi-admissible covering over k with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, and $D \in (\mathbb{Z}/\ell\mathbb{Z})^{\sim}[D_X]^0$ the ramification divisor associated to f^{\bullet} . Then f^{\bullet} is new-ordinary if and only if $D(j), j \in \{1, \ldots, \ell - 1\}$, is Frobenius stable.

4. Applications

In this section, we give some applications of Theorem 3.5.

4.1. Application 1. By applying Theorem 3.5, one may construct new-ordinary ramified coverings easily for generic curves. For instance, we have the following proposition.

Proposition 4.1. Let X^{\bullet} be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic p > 0. Suppose that X^{\bullet} is generic (2.1.3). Then the following statements hold.

(i) Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. Suppose that $n_X \leq 1$ (note that since we assume that X^{\bullet} is pointed stable, we have $g_X \geq 1$). Then there exists a new-ordinary Galois admissible covering

$$f^{\bullet}: Y^{\bullet} \to X^{\bullet}$$

with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that f is étale.

(ii) Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. Suppose that $n_X \ge 2$ is an even number, and we put $d \stackrel{\text{def}}{=} n_X/2$. Let $D_X \stackrel{\text{def}}{=} \{x_1, x_2, \ldots, x_{2d-1}, x_{2d}\}$ be a generic set of 2d points of X and

$$D \stackrel{\text{def}}{=} \sum_{r=1}^d (a_r x_{2r-1} + b_r x_{2r})$$

an effective divisor on X such that $1 \leq a_r, b_r \leq m-1$ and $a_r + b_r = m$ for each $r \in \{1, \ldots, d\}$. Note that we have $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$ (2.2.5). Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet},m}$ (2.3.2) and $f^{\bullet}: Y^{\bullet} \to X^{\bullet}$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/m\mathbb{Z}$ induced by $([\mathcal{L}], D)$. Then f^{\bullet} is new-ordinary. In particular, there exists a new-ordinary Galois admissible covering whose Galois group is $\mathbb{Z}/m\mathbb{Z}$, and whose branch locus of f is equal to D_X .

(iii) Let t be an arbitrary positive natural number and $n \stackrel{\text{def}}{=} p^t - 1$. Suppose that $p \ge 5$, and that $n_X \ge 3$ is an odd number. We put $n_X = 2d + 1$ and $c \stackrel{\text{def}}{=} n/(p-1)$. Let $D_X \stackrel{\text{def}}{=} \{x_1, \dots, x_{2d-1}, x_{2d}, x_{2d+1}\}$ be a generic set of 2d + 1 points of X and

$$Q \stackrel{\text{def}}{=} cx_{2d-1} + cx_{2d} + c(p-3)x_{2d+1}, \ Q^* \stackrel{\text{def}}{=} \sum_{r=1}^{d-1} (x_{2r-1} + (n-1)x_{2r})$$

effective divisors on X. We put

$$D \stackrel{\text{def}}{=} Q + Q^* \in (\mathbb{Z}/n\mathbb{Z})^{\sim} [D_X]^0.$$

Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet}, n}$ and $f^{\bullet} : Y^{\bullet} \to X^{\bullet}$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/m\mathbb{Z}$ induced by $([\mathcal{L}], D)$. Then f^{\bullet} is new-ordinary. In particular, there exists a new-ordinary Galois multi-admissible covering whose Galois group is $\mathbb{Z}/n\mathbb{Z}$, and whose branch locus of f is equal to D_X .

Proof. (i) follows immediately from [N, Proposition 4]. Let us prove (ii). In order to verify (ii), we only need to prove that the restriction of f^{\bullet} on an arbitrary connected component of Y^{\bullet} is new-ordinary. Let $t' \in \mathbb{N}$ be the order of p in $(\mathbb{Z}/m\mathbb{Z})^{\times}$. We put

$$D' \stackrel{\text{def}}{=} m'D = \sum_{r=1}^{d} (m'a_r x_{2r-1} + m'b_r x_{2r}) \in (\mathbb{Z}/n'\mathbb{Z})^{\sim} [D_X]^0$$

when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n'\mathbb{Z}$ of order m, where $n' \stackrel{\text{def}}{=} p^{t'} - 1$ and $m' \stackrel{\text{def}}{=} n'/m$. Note that we have $n' = m'a_r + m'b_r$ for each $r \in \{1, \ldots, d\}$. Moreover, we see immediately

$$\deg(D'(j)^{(i)}) = dn', \ i \in \{0, \dots, t'-1\}, \ j \in \{1, \dots, n'-1\}.$$

This means that $D'(j), j \in \{1, \ldots, n'\}$, is Frobenius stable.

Since $\mathcal{L}^{\otimes n'} \cong \mathcal{O}_X(-D)^{\otimes m'} \cong \mathcal{O}_X(-D')$, we have $([\mathcal{L}], D') \in \widetilde{\mathscr{P}}_{X^{\bullet}, n'}$. Let $g^{\bullet} : Z^{\bullet} \to X^{\bullet}$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n'\mathbb{Z}$ corresponding to $([\mathcal{L}], D')$. Then Theorem 3.5 implies that g^{\bullet} is new-ordinary. Let W^{\bullet} be an arbitrary connected component of Z^{\bullet} and

$$g^{\bullet}|_{W^{\bullet}}: W^{\bullet} \to X^{\bullet}$$

the Galois admissible covering over k with Galois group $\mathbb{Z}/m\mathbb{Z}$ induced by g^{\bullet} . Then $g^{\bullet}|_{W^{\bullet}}$ is new-ordinary. This completes the proof of (ii).

Next, let us prove (iii). Let $j \in \{1, \ldots, n-1\}$. We put $\alpha_j \stackrel{\text{def}}{=} j - (p-1)[j/(p-1)]$ and $\beta_j \stackrel{\text{def}}{=} j(p-3) - [j(p-3)/(p-1)]$. Note that $(p-1)|(2\alpha_j + \beta_j)$ and $2\alpha_j + \beta_j \in \{p-1, 2(p-1)\}$. Then we have

$$jQ = cjx_{2d-1} + cjx_{2d} + cj(p-3)x_{2d+1}$$

 $= c(\alpha_j x_{2d-1} + \alpha_j x_{2d} + \beta_j x_{2d+1}) + (n[j/(p-1)]x_{2d-1} + n[j/(p-1)]x_{2d} + n[j(p-3)/(p-1)]x_{2d+1}).$ Thus, we have

$$Q(j) = c\alpha_j x_{2d-1} + c\alpha_j x_{2d} + c\beta_j x_{2d+1}$$

Then we obtain

$$\deg(Q(j)^{(i)}) = (2\alpha_j + \beta_j)c = (2\alpha_j + \beta_j)n/(p-1), \ i \in \{0, \dots, t-1\}.$$

Thus, we see immediately that $Q(j), j \in \{1, ..., n-1\}$, is Frobenius stable (Definition 3.3).

On the other hand, we see immediately that

 $\deg(Q^*(j)^{(i)}) = dn, \ i \in \{0, \dots, t-1\}, \ j \in \{1, \dots, n-1\}.$

This means that $Q^*(j), j \in \{1, \ldots, n-1\}$, is Frobenius stable. Thus, we have that $D(j), j \in \{1, \ldots, n-1\}$, is Frobenius stable. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet}, n}$ and $f^{\bullet} : Y^{\bullet} \to X^{\bullet}$ the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to $([\mathcal{L}], D)$. Then Theorem 3.5 implies that Y^{\bullet} is ordinary. This completes the proof of (iii).

Remark 4.1.1. The referee kindly pointed out to me that Proposition 4.1 (ii), (iii) can also be deduced from [LMPT, Corollary 4.8 and Corollary 4.9], respectively.

4.2. Application 2. Next, we consider an inverse Galois problem for X^{\bullet} . Let m be an arbitrary positive natural number prime to $p, t \in \mathbb{N}$ the order of p in $(\mathbb{Z}/m\mathbb{Z})^{\times}$, $n \stackrel{\text{def}}{=} p^t - 1$, and $m' \stackrel{\text{def}}{=} n/m$. Let G be a finite group which is an extension of a group $H \stackrel{\text{def}}{=} \mathbb{Z}/m\mathbb{Z}$ by a p-group P. Then the Schur-Zassenhaus theorem implies that G is a semi-direct product of the form

 $P \rtimes H$.

Write $\Phi(P) \stackrel{\text{def}}{=} P^p[P, P]$ for the Frattini subgroup of P and $\overline{P} \stackrel{\text{def}}{=} P/\Phi(P)$. We put $\overline{G} \stackrel{\text{def}}{=} G/\Phi(P)$. Then we obtain an \mathbb{F}_p -linear representation

$$\rho: H \to \operatorname{Aut}(\overline{P}).$$

Let Z(H) be the set of irreducible characters of H with values in k and ζ_m a primitive mth root and χ_j , $j \in \mathbb{Z}/m\mathbb{Z}$, the irreducible character such that $\chi_j(1) = \zeta_m^j$. Then we see $Z(H) = {\chi_j}_{j \in \mathbb{Z}/m\mathbb{Z}}$. Let $\rho_{\chi_j} : H \to \operatorname{GL}(\overline{P}_{\chi_j})$ be an irreducible k-representation of H of character χ_j of degree 1. The canonical decomposition of $\overline{P} \otimes_{\mathbb{F}_p} k$ as a k[H]-module is given by

$$\overline{P} \otimes_{\mathbb{F}_p} k = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \overline{P}_{\chi_j}^{m_{\chi_j}},$$

where m_{χ_j} is the multiplicity of the character in the representation ρ . Then we have the following result.

Proposition 4.2. Let X^{\bullet} be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic p > 0. Suppose that X^{\bullet} is generic. Let m be a positive natural number prime to p and $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$. Moreover, let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^{\bullet}, m}$ and $\alpha \in \operatorname{Hom}(\Pi_{X^{\bullet}}^{ab}, \mathbb{Z}/m\mathbb{Z})$ the element induced by $([\mathcal{L}], D)$. We put

$$\phi: \Pi_{X^{\bullet}} \twoheadrightarrow \Pi_{X^{\bullet}}^{\mathrm{ab}} \xrightarrow{\alpha} H \stackrel{\mathrm{def}}{=} \mathbb{Z}/m\mathbb{Z}.$$

Suppose that D(j), $j \in \{1, ..., m-1\}$, is Frobenius stable (Definition 3.3). Then an embedding problem $(\phi : \Pi_{X^{\bullet}} \to H, G \twoheadrightarrow H)$ has a solution if and only if

$$m_{\chi_j} \leq \begin{cases} g_X, & \text{if } j = 0, \\ g_X + s(D(j)) - 1, & \text{if } j \in \{1, \dots, m - 1\}. \end{cases}$$

Proof. Note that the dimensions of irreducible k-representations of prime-to-p cyclic groups are 1. Then "only if" part of the proposition follows from [B, Proposition 2.4]. On the other hand, the "if part" of the proposition follows immediately from Theorem 3.5 and [B, Proposition 2.4 and Proposition 2.5].

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Remark 4.2.1. When $n_X = 0$, A. Pacheco and K. Stevenson obtained a necessary and sufficient for the existence of a solution of an embedding problem $(\Pi_X \bullet \to G', G' \to H')$ when G' is a finite group which is an extension of an abelian group H' with a prime-to-p order by a p-group P' ([PaSt, Theorem 7.4]). Proposition 4.2 generalizes their result to the case of prime-to-p cyclic tamely ramified coverings.

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