Raynaud-Tamagawa Theta Divisors and New-ordinariness of Ramified Coverings of Curves

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Abstract

In this paper, we study coverings of curves in positive characteristic. Let $X^\bullet = (X, D_X)$ and $Y^\bullet = (Y, D_Y)$ be pointed stable curves over an algebraically closed field k of characteristic $p > 0$. Suppose that $X^\bullet$ is a *generic* pointed stable curve of type (g_X, n_X). Let m be a natural number and $Y \setminus D_Y \to X \setminus D_X$ a cyclic tame covering (or equivalently, $Y^\bullet \to X^\bullet$ a cyclic admissible covering) with Galois group $\mathbb{Z}/m\mathbb{Z}$. We give a *necessary and sufficient* condition for the ordinariness of Y. This result generalizes a result of Nakajima concerning the ordinariness of cyclic étale covering of generic stable curve to the case of tamely ramified coverings.

Keywords: pointed stable curve, admissible covering, generalized Hasse-Witt invariant, new-ordinary, Raynaud-Tamagawa theta divisor, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14F35, 14G32.

Contents

1 Introduction

In the present paper, we study coverings of curves in positive characteristic. Let

$$X^\bullet = (X, D)$$

be a smooth pointed stable curve of topological type or type, for short, (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. We put $U_X \stackrel{\mathrm{def}}{=} X \setminus D_X$. Moreover, by choosing a suitable base point of U_X, we have the tame fundamental group $\pi_1^{\mathrm{t}}(U_X)$ of U_X.

The structure of maximal prime-to-p quotient $\pi_1^{\mathrm{t}}(U_X)^{p'}$ of $\pi_1^{\mathrm{t}}(U_X)$ is well-known, which is isomorphic to the pro-prime-to-p completion of the following group (cf. [G])

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X}[a_i, b_i]\prod_{j=1}^{n_X} c_j = 1\rangle.$$

On the other hand, the structure of $\pi_1^{\mathrm{t}}(U_X)$ is very mysterious. In fact, some developments of F. Pop-M. Saïdi, M. Raynaud, A. Sarashina, A. Tamagawa, and the author (cf. [PoSa], [R2], [Sa], [T1], [T2], [T3], [T4], [Y1], [Y2], [Y3], [Y5], [Y6]) showed evidence for very strong *anabelian* phenomena for curves over *algebraically closed fields of characteristic* $p > 0$. In this situation, the Galois group of the base field is trivial, and the étale (or tame) fundamental group coincides with the geometric fundamental group, thus in a total absence of a Galois action of the base field. This kind of anabelian phenomena go beyond Grothendieck's anabelian geometry, and shows that the tame fundamental group of a smooth pointed stable curve over an algebraically closed field must encode"*moduli*" of the curve. This is the reason that we do not have an explicit description of the tame fundamental group of any smooth pointed stable curve in positive characteristic. Note that since all the tame coverings in positive characteristic can be lifted to characteristic 0, we obtain that $\pi_1^{\mathrm{t}}(U_X)$ is topologically finitely generated. Then the isomorphism class of $\pi_1^{\mathrm{t}}(U_X)$ is determined by the set of finite quotients of $\pi_1^{\mathrm{t}}(U_X)$ (cf. [FJ, Proposition 16.10.6]).

Furthermore, the theory developed in [T3] and [Y2] implies that the isomorphism class of U_X as a scheme can possibly be determined by not only the isomorphism class of $\pi_1^{\mathrm{t}}(U_X)$ as a profinite group but also the isomorphism class of the maximal pro-solvable quotient of $\pi_1^{\mathrm{t}}(U_X)$. Then we may ask the following question:

Which finite solvable group can appear as a quotient of $\pi_1^{\mathrm{t}}(U_X)$*?*

Let $N \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an arbitrary open normal subgroup and $X_N^\bullet = (X_N, D_{X_N})$ the pointed stable curve of type (g_{X_N}, n_{X_N}) over k corresponding to N. We have an important invariant σ_{X_N} associated to X_N (or N) which is called *p-rank* (or *Hasse-Witt invariant*, see Definition 2.3). Roughly speaking, σ_{X_N} controls the finite quotients of $\pi_1^{\mathrm{t}}(U_X)$ which are extensions of the group $\pi_1^{\mathrm{t}}(U_X)/N$ by p-groups. Since the structures of maximal prime-to-p quotients of tame fundamental groups have been known, in order to solve the question mentioned above, we need compute the p-rank σ_{X_H} when $\pi_1^{\mathrm{t}}(U_X)/N$ is abelian. If $\pi_1^{\mathrm{t}}(U_X)/N$ is a p-group, then σ_{X_N} can be computed by applying the Deuring-Shafarevich formula (cf. [C]). Moreover, the Deuring-Shafarevich formula implies that, to compute σ_{X_N}, we only need to assume that $\pi_1^{\mathrm{t}}(U_X)/N$ is a prime-to-p group.

The situation of σ_{X_N} is very complicated if $\pi_1^{\mathrm{t}}(U_X)/N$ is not a p-group. If $X^\bullet$ is arbitrary smooth pointed stable curve over k, σ_{X_N} cannot be computed explicitly in general (cf. [R1], [T3], [Y4]). Suppose that $n_X = 0$, and that $X^\bullet$ is a curve corresponding to a geometric generic point of the moduli space (i.e., a generic pointed stable curve). The following result proved by S. Nakajima (cf. [N]).

Theorem 1.1. *We maintain the notation introduced above. Suppose that $\pi_1^{\mathrm{t}}(U_X)/N$ is a cyclic group, that $n_X = 0$ (i.e., $X = U_X$), and that $X^\bullet$ is generic. Then we have that $\sigma_{X_N} = g_{X_N}$ (i.e., $X_N^\bullet$ is ordinary).*

Suppose that $n_X \neq 0$. The computations of σ_{X_N} are much more difficult than the case where $n_X = 0$. Note that Nakajima's result mentioned above does not hold in general when $U_{X_N} \to U_X$ is a tamely ramified covering, where $U_{X_N} \stackrel{\mathrm{def}}{=} X_N \setminus D_{X_N}$. Then main result of the present paper is as follows, which generalizes Nakajima's result to the case of tamely ramified coverings, and which gives a *necessary and sufficient* condition for the ordinariness of $X_N^\bullet$ when $\pi_1^{\mathrm{t}}(U_X)/N$ is a cyclic group (see Theorem 3.6 for precise form).

Theorem 1.2. *We maintain the notation introduced above. Suppose that $\pi_1^{\mathrm{t}}(U_X)/N \cong \mathbb{Z}/p^r m\mathbb{Z}$ is a cyclic group, and that $X^\bullet$ is generic, where $(m, p) = 1$. Let $t \in \mathbb{N}$ be a natural number such that $m|(p^t - 1)$, $m' \stackrel{\mathrm{def}}{=} (p^t - 1)/m$, D the ramification divisor associated to the cyclic tame covering $U_{X_N} \to U_X$ induced by $N \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$, and $D' \stackrel{\mathrm{def}}{=} m'D$. Then we have that $\sigma_{X_N} = g_{X_N}$ if and only if $D'(im')$, $i \in \{1, \dots, m-1\}$, is Frobenius-stable (cf. Definition 2.4 and Definition 3.4 for the definitions of $D'(im')$ and Frobenius-stable, respectively).*

By applying Theorem 1.2, it is easy to construct prime-to-p cyclic new-ordinary (cf. Definition 2.3) tame covering of generic curves (cf. Proposition 4.1). On the other hand, we apply Theorem 1.2 to a certain inverse Galois problem for $X^\bullet$. Let G' be a finite group which is an extension of a group H' with prime-to-p order and a p-group P'. If $n_X = 0$, A. Pacheco and K. Stevenson gave a necessary and sufficient condition for whether or not an embedding problem $(\pi_1^{\mathrm{t}}(U_X) \to G', G' \twoheadrightarrow H')$ has a solution (cf. [PaSt, Theorem 1.3]). Moreover, I. Bouw generalized this result to the case where $n_X \geq 0$ (cf. [B, Proposition 2.4 and Proposition 2.5]). Roughly speaking, the criterion obtained of Pacheco-Stevenson and Bouw is that the product of the multiplicity and the degree of every irreducible character of H' is less than the generalized Hasse-Witt invariant associated to the irreducible character of H'. But, as we mentioned above, the generalized Hasse-Witt invariants cannot be computed explicitly in general. On the other hand, by applying Nakajima (cf. Theorem 1.1) and B. Zhang's (cf. [Z]) results concerning the ordinariness of abelian étale coverings of projective generic curves, Pacheco and Stevenson obtained a numerical criterion for whether or not an embedding problem $(\pi_1^{\mathrm{t}}(U_X) \to G', G' \twoheadrightarrow H')$ has a solution when H' is abelian (cf. [PaSt, Theorem 7.4]). When H' is cyclic, by applying Theorem 1.2, we generalize the numerical criterion of Pacheco-Stevenson to the case of ramified coverings (cf. Proposition 4.2).

The present paper is organized as follows. In Section 2, we recall some definitions and properties of admissible coverings, generalized Hasse-Witt invariants, and Raynaud-Tamagawa theta divisors. In Section 3, we study the new-ordinariness of prime-to-p cyclic admissible coverings of generic curves by using the theory of Raynaud-Tamagawa theta divisors and prove our main theorem. In Section 4, we give two applications of the main theorem.

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2 Preliminaries

2.1 Admissible coverings and admissible fundamental groups

In this subsection, we recall some definitions and results concerning admissible fundamental groups which will be used in the present paper.

Definition 2.1. Let $\mathbb{G} \stackrel{\text{def}}{=} (v(\mathbb{G}), e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})})$ be a semi-graph (cf. [Y4, Definition 2.1]). Here, $v(\mathbb{G})$, $e^{\text{op}}(\mathbb{G})$, $e^{\text{cl}}(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of closed edges of $\mathbb{G}$, the set of open edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively. Note that, for each $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$, $e \stackrel{\text{def}}{=} \{b_e^1, b_e^2\}$ is a set of cardinality 2. Then e is a closed edge if $\zeta_e^{\mathbb{G}}(e) \subseteq v(\mathbb{G})$, and e is an open edge if $\zeta_e^{\mathbb{G}}(e) = \{\zeta_e^{\mathbb{G}}(e) \cap v(\mathbb{G}), \{v(\mathbb{G})\}\}$.

In the present paper, let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve over an algebraically closed field k of characteristic $p > 0$, where X denotes the underlying curve, D_X denotes the set of marked points, g_X denotes the genus of X, and n_X denotes the cardinality $\#D_X$ of D_X. We shall say that (g_X, n_X) is the topological type (or type for short) of $X^\bullet$. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ and $r_X \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\text{op}}(\Gamma_{X^\bullet})$. Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \stackrel{\text{def}}{=} X_v \setminus X_v^{\text{sing}}$, where $(-)^{\text{sing}}$ denotes the singular locus of $(-)$. We define a smooth pointed stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \stackrel{\text{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v)).$$

Let $\overline{\mathcal{M}}_{g_X, n_X, \mathbb{Z}}$ be the moduli stack of pointed stable curves of type (g_X, n_X) over $\operatorname{Spec} \mathbb{Z}$. We shall say that $X^\bullet$ is *generic* if $X^\bullet$ is a curve corresponding to a geometric generic point of $\overline{\mathcal{M}}_{g_X, n_X, \mathbb{Z}} \times k$. In particular, $X^\bullet$ is smooth over k. Moreover, we shall say that $X^\bullet$ is a *component-generic* pointed stable curve over k if $\widetilde{X}_v^\bullet$, $v \in v(\Gamma_{X^\bullet})$, is a generic pointed stable curve of type (g_v, n_v) over k.

Definition 2.2. Let $Y^\bullet = (Y, D_Y)$ be a pointed stable curve over k, $f^\bullet : Y^\bullet \to X^\bullet$ a morphism of pointed stable curves over k, and $f : Y \to X$ the morphism of underlying curves induced by $f^\bullet$.

We shall say $f^\bullet$ a *Galois admissible covering* over k (or Galois admissible covering for short) if the following conditions are satisfied: (i) There exists a finite group $G \subseteq \operatorname{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$. (ii) For each $y \in Y^{\text{sm}} \setminus D_Y$, f is étale at y, where $(-)^{\text{sm}}$ denotes the smooth locus of $(-)$. (iii)

For any $y \in Y^{\mathrm{sing}}$, the image $f(y)$ is contained in X^{sing}, where $(-)^{\mathrm{sing}}$ denotes the set of singular points of $(-)$. (iv) For each $y \in Y^{\mathrm{sing}}$, we write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y. Then we have that $(\#D_y, \mathrm{char}(k)) = 1$, and that the local morphism between two nodes induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^{\#D_y} \\ v & \mapsto & t^{\#D_y}. \end{array}$$

Moreover, we have that $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit, and $\#(-)$ denotes the cardinality of $(-)$. (v) The local morphism between two marked points induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[a]] & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m, \mathrm{char}(k)) = 1$ (i.e., a tamely ramified extension).

Moreover, we shall say $f^\bullet$ an *admissible covering* if there exists a morphism of pointed stable curves $h^\bullet : W^\bullet \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ h^\bullet : W^\bullet \to X^\bullet$ is a Galois admissible covering over k. We shall say an admissible covering $f^\bullet$ *étale* if f is an étale morphism.

Let $Z^\bullet$ be a disjoint union of finitely many pointed stable curves over k. We shall say a morphism $f_Z^\bullet : Z^\bullet \to X^\bullet$ over k *multi-admissible covering* if the restriction of $f_Z^\bullet$ to each connected component of $Z^\bullet$ is admissible. For any category $\mathscr{C}$, we write $\mathrm{Ob}(\mathscr{C})$ for the class of objects of $\mathscr{C}$, and write $\mathrm{Hom}(\mathscr{C})$ for the class of morphisms of $\mathscr{C}$. We denote by

$$\mathrm{Cov}^{\mathrm{adm}}(X^\bullet) \stackrel{\mathrm{def}}{=} (\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)), \mathrm{Hom}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)))$$

the category which consists of the following data: (i) $\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$ consists of an empty object and all the pairs $(Z^\bullet, f_Z^\bullet : Z^\bullet \to X^\bullet)$, where $Z^\bullet$ is a disjoint union of finitely many pointed stable curves over k, and $f_Z^\bullet$ is a multi-admissible covering over k; (ii) for any $(Z^\bullet, f_Z^\bullet)$, $(Y^\bullet, f_Y^\bullet) \in \mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$, we define

$$\mathrm{Hom}((Z^\bullet, f_Z^\bullet), (Y^\bullet, f_Y^\bullet)) \stackrel{\mathrm{def}}{=} \{g^\bullet \in \mathrm{Hom}_k(Z^\bullet, Y^\bullet) \mid f_Y^\bullet \circ g^\bullet = f_Z^\bullet\},$$

where $\mathrm{Hom}_k(Z^\bullet, Y^\bullet)$ denotes the set of k-morphisms of pointed stable curves. It is well known that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\mathrm{sm}} \backslash D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the *admissible fundamental group* of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by

$$\Pi_{X^\bullet}.$$

Remark 2.2.1. Suppose that $X^\bullet$ is smooth over k. By the definition of admissible fundamental groups, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X \setminus D_X$.

Definition 2.3. Let $Z^\bullet$ be a disjoint union of finitely many pointed stable curves over k. We define the *p-rank* (or *Hasse-Witt invariant*) of $Z^\bullet$ to be

$$\sigma_Z \overset{\text{def}}{=} \dim_{\mathbb{F}_p}(H^1_{\text{ét}}(Z, \mathbb{F}_p)).$$

We shall say that $Z^\bullet$ is *ordinary* if

$$g_Z = \sigma_Z,$$

where $g_Z \overset{\text{def}}{=} \dim_k(H^1(Z, \mathcal{O}_Z))$. Moreover, let $Z^\bullet \to X^\bullet$ is a multi-admissible covering over k. We shall say that the admissible covering $Z^\bullet \to X^\bullet$ is *new-ordinary* if

$$g_Z - g_X = \sigma_Z - \sigma_X,$$

where σ_X denotes the p-rank of $X^\bullet$. Note that if $X^\bullet$ is ordinary, then $Z^\bullet \to X^\bullet$ is new-ordinary if and only if $Z^\bullet$ is ordinary.

Remark 2.3.1. Note that we have

$$\sigma_X = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma_{\widetilde{X}_v} + r_X.$$

Then $X^\bullet$ is ordinary if and only if $\widetilde{X}^\bullet_v$, $v \in v(\Gamma_{X^\bullet})$, is ordinary. Let $g^\bullet : Z^\bullet \to X^\bullet$ is a multi-admissible covering over k and $\widetilde{g}^\bullet_v : \widetilde{Z}^\bullet_v \to \widetilde{X}^\bullet_v$, $v \in v(\Gamma_{X^\bullet})$, the admissible covering over k induced by $g^\bullet$, where the underlying curve of $\widetilde{Z}^\bullet_v$ is the normalization of $g^{-1}(X_v)$. Then $g^\bullet$ is new-ordinary if and only if $\widetilde{g}^\bullet_v$ is new-ordinary for each $v \in v(\Gamma_{X^\bullet})$.

2.2 Generalized Hasse-Witt invariants of cyclic admissible coverings

In this subsection, we recall some notation concerning generalized Hasse-Witt invariants of cyclic admissible coverings.

We maintain the notation introduced in Section 2.1, and let $X^\bullet = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over k, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Let m be an arbitrary positive natural number prime to p and $\mu_m \subseteq k^\times$ the group of mth roots of unity. Fix a primitive mth root ζ, we may identify μ_m with $\mathbb{Z}/m\mathbb{Z}$ via the map $\zeta^i \mapsto i$. Let $\alpha \in \text{Hom}(\Pi^{\text{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z})$. We denote by $X^\bullet_\alpha = (X_\alpha, D_{X_\alpha})$ the Galois multi-admissible covering with Galois group $\mathbb{Z}/m\mathbb{Z}$ corresponding to α. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition (cf. [Se, Section 9])

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha}) = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} \oplus H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{ni}}$. Moreover, we have

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\text{st}} = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k,$$

where $(-)^{F_{X_\alpha}}$ denotes the subspace of $(-)$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify

$$H_\alpha \overset{\text{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$$

with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-vector spaces H_α is a finitely generated $k[\mu_m]$-module induced by the natural action of μ_m on X_α. We have the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_m$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication. We define

$$\gamma_{\alpha,i} \overset{\text{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/m\mathbb{Z}.$$

These invariants are called *generalized Hasse-Witt invariants* (cf. [N]) of the cyclic multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. Moreover, we shall say that $\gamma_{\alpha,1}$ is the *first* generalized Hasse-Witt invariant of the cyclic multi-admissible covering $X_\alpha^\bullet \to X^\bullet$. Note that the decomposition above implies that

$$\dim_k(H_\alpha) = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \gamma_{\alpha,i}.$$

In particular, if X_α is connected, then $\dim_k(H_\alpha) = \sigma_{X_\alpha}$.

We write $\mathbb{Z}[D_X]$ for the group of divisors whose supports are contained in D_X. Note that $\mathbb{Z}[D_X]$ is a free $\mathbb{Z}$-module with basis D_X. We define

$$c'_m : \mathbb{Z}/m\mathbb{Z}[D_X] \overset{\text{def}}{=} \mathbb{Z}[D_X] \otimes \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z},\ D \bmod m \mapsto \deg(D) \bmod m.$$

Then $\ker(c'_m)$ can be regarded as a subset of $(\mathbb{Z}/m\mathbb{Z})^\sim[D_X]$, where $(\mathbb{Z}/m\mathbb{Z})^\sim$ denotes the set $\{0, 1, \ldots, m-1\}$, and $(\mathbb{Z}/m\mathbb{Z})^\sim[D_X]$ denotes the subset of $\mathbb{Z}[D_X]$ consisting of the elements whose coefficients are contained in $(\mathbb{Z}/m\mathbb{Z})^\sim$. We denote by $\mathbb{Z}/m\mathbb{Z}[D_X]^0$ the kernel of c'_m and by $(\mathbb{Z}/m\mathbb{Z})^\sim[D_X]^0$ the subset of $(\mathbb{Z}/m\mathbb{Z})^\sim[D_X]$ corresponding to $\mathbb{Z}/m\mathbb{Z}[D_X]^0$ under the natural bijection $(\mathbb{Z}/m\mathbb{Z})^\sim[D_X] \overset{\sim}{\to} \mathbb{Z}/m\mathbb{Z}[D_X]$. Note that, for each $D \in (\mathbb{Z}/m\mathbb{Z})^\sim[D_X]^0$, we have $m|\deg(D)$. Then

$$\deg(D) = s(D)m$$

for some integer $s(D)$ such that

$$0 \leq s(D) \leq \begin{cases} 0, & \text{if } n_X \leq 1, \\ n_X - 1, & \text{if } n_X \geq 2. \end{cases}$$

Let $X^{\bullet,*} = (X^*, D_{X^*}) \to X^\bullet$ be a universal admissible covering corresponding to $\Pi_{X^\bullet}$. For each $e \in e^{\text{cl}}(\Gamma_{X^\bullet}) \cup e^{\text{op}}(\Gamma_{X^\bullet})$, write x_e for the marked point corresponding to e, and let x_{e^*} be a point of the inverse image of x_e in D_{X^*}. Write $I_{e^*} \subseteq \Pi_{X^\bullet}$ for the inertia subgroup of x_{e^*}. Note that I_{e^*} is isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Suppose that x_e is contained in X_v. Then we have an injection

$$\phi_{e^*} : I_{e^*} \hookrightarrow \Pi_{X^\bullet}^{\text{ab}}$$

induced by the composition of (outer) injective homomorphisms $I_{e^*} \hookrightarrow \Pi_{\widetilde{X}_v^\bullet} \hookrightarrow \Pi_{X^\bullet}$, where $\Pi_{\widetilde{X}_v^\bullet}$ denotes the admissible fundamental group of $\widetilde{X}_v^\bullet$. Since the image of ϕ_{e^*}

depends only on e, we may write I_e for the image $\phi_{e^*}(I_{e^*})$. Moreover, the specialization theorem of the maximal prime-to-p quotients of admissible fundamental groups of pointed stable curves (cf. [V, Théorème 2.2 (c)]) implies that, there exists a generator $[s_e]$ of I_e for each $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ such that the following holds

$$\sum_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} [s_e] = 0$$

in $\Pi^{\mathrm{ab}}_{X^\bullet}$.

Definition 2.4. We maintain the notation introduced above.

(i) We put

$$D_\alpha \stackrel{\mathrm{def}}{=} \sum_{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})} \alpha([s_e])x_e, \ \alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z}).$$

Note that we have $D_\alpha \in (\mathbb{Z}/m\mathbb{Z})^\sim[D_X]^0$. On the other hand, for each $D \in (\mathbb{Z}/m\mathbb{Z})^\sim[D_X]^0$, we denote by

$$\mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet) \stackrel{\mathrm{def}}{=} \{\alpha \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z}) \mid D_\alpha = D\}.$$

Moreover, we put

$$\gamma_{(\alpha, D)} \stackrel{\mathrm{def}}{=} \gamma_{\alpha,1}.$$

(ii) Let $Q \in \mathbb{Z}[D_X]$ be an arbitrary effective divisor on X and m an arbitrary natural number. We put

$$[Q/m] \stackrel{\mathrm{def}}{=} \sum_{x \in D_X} [\mathrm{ord}_x(Q)/m]x,$$

which is an effective divisor on X. Here $[(-)]$ denotes the maximum integer which is less than or equal to $(-)$.

(iii) Let $t \in \mathbb{N}$ be an arbitrary positive natural number, $n \stackrel{\mathrm{def}}{=} p^t - 1$, and

$$u = \sum_{j=0}^{t-1} u_j p^j, \ u \in \{0, \dots, n\},$$

the p-adic expansion with $u_j \in \{0, \dots, p-1\}$. We identify $\{0, \dots, t-1\}$ with $\mathbb{Z}/t\mathbb{Z}$ naturally. Then $\{0, \dots, t-1\}$ admits an additional structure induced by the natural additional structure of $\mathbb{Z}/t\mathbb{Z}$. We put

$$u^{(i)} \stackrel{\mathrm{def}}{=} \sum_{j=0}^{t-1} u_{i+j} p^j, \ i \in \{0, \dots, t-1\}.$$

Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$. We put

$$D^{(i)} \stackrel{\mathrm{def}}{=} \sum_{x \in D_X} (\mathrm{ord}_x(D))^{(i)}x, \ i \in \{0, 1, \dots, t-1\},$$

which is an effective divisor on X. Moreover, for each $j \in \mathbb{Z}$, we put

$$D(j) \stackrel{\mathrm{def}}{=} jD - n[jD/n].$$

Remark 2.4.1. We maintain the notation introduced in Definition 2.4 (iii). Then we have

$$D(p^{t-i}) = D^{(i)}, \ i \in \{0, \dots, t-1\}.$$

2.3 Raynaud-Tamagawa theta divisors

In this subsection, we recall some notation and results concerning theta divisors introduced by Raynaud and Tamagawa (see also [R1] and [T3, Section 2]).

We maintain the notation introduced in Section 2.2. Moreover, in the present subsection, we suppose that $X^\bullet$ is *smooth* over k. The generalized Hasse-Witt invariants can be also described in terms of line bundles and divisors. Let $m \in \mathbb{N}$ be an arbitrary natural number prime to p. We denote by $\mathrm{Pic}(X)$ the Picard group of X. Consider the following complex of abelian groups:

$$\mathbb{Z}[D_X] \overset{a_m}{\to} \mathrm{Pic}(X) \oplus \mathbb{Z}[D_X] \overset{b_m}{\to} \mathrm{Pic}(X),$$

where $a_m(D) = ([\mathcal{O}_X(-D)], mD)$, $b_m(([\mathcal{L}], D)) = [\mathcal{L}^m \otimes \mathcal{O}_X(-D)]$. We denote by

$$\mathscr{P}_{X^\bullet,m} \overset{\mathrm{def}}{=} \ker(b_m)/\mathrm{Im}(a_m)$$

the homology group of the complex. Moreover, we have the following exact sequence

$$0 \to \mathrm{Pic}(X)[m] \overset{a'_m}{\to} \mathscr{P}_{X^\bullet,m} \overset{b'_m}{\to} \mathbb{Z}/m\mathbb{Z}[D_X] \overset{c'_m}{\to} \mathbb{Z}/m\mathbb{Z},$$

where $(-)[m]$ means the m-torsion subgroup of $(-)$, and

$$a'_m([\mathcal{L}]) = ([\mathcal{L}], 0) \text{ mod } \mathrm{Im}(a_m),$$

$$b'_m(([\mathcal{L}], D)) \text{ mod } \mathrm{Im}(a_m)) = D \text{ mod } m,$$

$$c'_m(D \text{ mod } m) = \deg(D) \text{ mod } m.$$

We shall define

$$\widetilde{\mathscr{P}}_{X^\bullet,m}$$

to be the inverse image of $(\mathbb{Z}/m\mathbb{Z})^\sim[D_X]^0 \subseteq (\mathbb{Z}/m\mathbb{Z})^\sim[D_X] \subseteq \mathbb{Z}[D_X]$ under the projection $\ker(b_m) \to \mathbb{Z}[D_X]$. It is easy to see that $\mathscr{P}_{X^\bullet,m}$ and $\widetilde{\mathscr{P}}_{X^\bullet,m}$ are free $\mathbb{Z}/m\mathbb{Z}$-groups with rank $2g_X + n_X - 1$ if $n_X \neq 0$ and with rank $2g_X$ if $n_X = 0$. Moreover, [T3, Proposition 3.5] implies that

$$\widetilde{\mathscr{P}}_{X^\bullet,m} \cong \mathscr{P}_{X^\bullet,m} \cong \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/m\mathbb{Z}).$$

Then every element of $\widetilde{\mathscr{P}}_{X^\bullet,m}$ induces a Galois multi-admissible covering of $X^\bullet$ over k with Galois group $\mathbb{Z}/m\mathbb{Z}$.

In the remainder of the present paper, we put

$$n \overset{\mathrm{def}}{=} p^t - 1,$$

where $t \in \mathbb{N}$ is a positive natural number. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$. We put

$$\mathcal{L}(j) \overset{\mathrm{def}}{=} \mathcal{L}^{\otimes j} \otimes \mathcal{O}_X([jD/n]),\ j \in \{1, \dots, n-1\}.$$

Then we have that $([\mathcal{L}(j)], D(j)) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$, and that the j-action on $\widetilde{\mathscr{P}}_{X^\bullet,n}$ is given by

$$([\mathcal{L}], D) \mapsto ([\mathcal{L}(j)], D(j)).$$

Moreover, we shall denote $\mathcal{L}(j)$ and $D(j)$ by $\mathcal{L}^{(i)}$ and $D^{(i)}$, respectively, when $j = p^{t-i}$, $i \in \{0, \ldots, t-1\}$.

On the other hand, we fix an isomorphism $\mathcal{L}^n \cong \mathcal{O}_X(-D)$. Note that D is an effective divisor on X. We have the following composition of morphisms of line bundles

$$\mathcal{L} \xrightarrow{p^t} \mathcal{L}^{\otimes p^t} = \mathcal{L}^{\otimes n} \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X(-D) \otimes \mathcal{L} \hookrightarrow \mathcal{L}.$$

The composite morphism induces a morphism $\phi_{([\mathcal{L}],D)} : H^1(X, \mathcal{L}) \to H^1(X, \mathcal{L})$. We denote by

$$\gamma_{([\mathcal{L}],D)} \stackrel{\text{def}}{=} \dim_k(\bigcap_{r \geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)})).$$

Write $\alpha_{\mathcal{L}} \in \mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{Z}/n\mathbb{Z})$ for the element corresponding to $([\mathcal{L}], D)$ and F_X for the absolute Frobenius morphism on X. Then we see immediately that $\gamma_{\alpha_{\mathcal{L}},1}$ is equal to the dimension over k of the largest subspace of $H^1(X, \mathcal{L})$ on which $F_X^t \stackrel{\text{def}}{=} F_X \circ \cdots \circ F_X$ is a bijection. Moreover, we have

$$\gamma_{\alpha_{\mathcal{L}},1} = \dim_k(H^1(X, \mathcal{L})^{F_X^t} \otimes_{\mathbb{F}_p} k),$$

where $(-)^{F_X^t}$ denotes the subspace of $(-)$ on which F_X^t acts trivially. It is easy to check that

$$H^1(X, \mathcal{L})^{F_X^t} \otimes_{\mathbb{F}_p} k = \bigcap_{r \geq 1} \mathrm{Im}(\phi^r_{([\mathcal{L}],D)}).$$

Then we obtain that $\gamma_{([\mathcal{L}],D)} = \gamma_{\alpha_{\mathcal{L}},1}$. Moreover, we observer that $D_{\alpha_{\mathcal{L}}} = D$. Then we obtain that

$$\gamma_{([\mathcal{L}],D)} = \gamma_{(\alpha_{\mathcal{L}},D)} (\stackrel{\text{def}}{=} \gamma_{\alpha_{\mathcal{L}},1}).$$

Lemma 2.5. *We maintain the notation introduced above. Suppose that $X^\bullet$ is smooth over k. Then we have*

$$\gamma_{(\alpha_{\mathcal{L}},D)} \leq \dim_k(H^1(X, \mathcal{L})) = \begin{cases} g_X, & \text{if } ([\mathcal{L}], D) = ([\mathcal{O}_X], 0), \\ g_X - 1, & \text{if } s(D) = 0, \\ g_X + s(D) - 1, & \text{if } s(D) \geq 1. \end{cases}$$

Proof. The first inequality follows from the definition of generalized Hasse-Witt invariants. The Riemann-Roch theorem implies that

$$\dim_k(H^1(X, \mathcal{L})) = g_X - 1 - \deg(\mathcal{L}) + \dim_k(H^0(X, \mathcal{L}))$$

$$= g_X - 1 + \frac{1}{n}\deg(D) + \dim_k(H^0(X, \mathcal{L})) = g_X - 1 + s(D) + \dim_k(H^0(X, \mathcal{L})).$$

This completes the proof of the lemma. □

Next, let us explain the Raynaud-Tamagawa theta divisors. Let F_k be the absolute Frobenius morphism on $\mathrm{Spec}\, k$ and $F_{X/k}$ the relative Frobenius morphism $X \to X_1 \stackrel{\text{def}}{=} X \times_{k,F_k} k$ over k and $F_k^t \stackrel{\text{def}}{=} F_k \circ \cdots \circ F_k$. We define

$$X_t \stackrel{\text{def}}{=} X \times_{k,F_k^t} k,$$

and define a morphism

$$F_{X/k}^t : X \to X_t$$

over k to be $F_{X/k}^t \stackrel{\text{def}}{=} F_{X_{t-1}/k} \circ \cdots \circ F_{X_1/k} \circ F_{X/k}$.

Let $\mathcal{L}$ be a line bundle on X of degree $-s(D)$ and $\mathcal{L}_t$ the pulling-back of $\mathcal{L}$ by the natural morphism $X_t \to X$. We put

$$\mathcal{B}_D^t \stackrel{\text{def}}{=} (F_{X/k}^t)_*(\mathcal{O}_X(D))/\mathcal{O}_{X_t},\ \mathcal{E}_D \stackrel{\text{def}}{=} \mathcal{B}_D^t \otimes \mathcal{L}_t.$$

Write $\mathrm{rk}(\mathcal{E}_D)$ for the rank of $\mathcal{E}_D$. Then we have

$$\chi(\mathcal{E}_D) = \deg(\det(\mathcal{E}_D)) - (g_X - 1)\mathrm{rk}(\mathcal{E}_D).$$

Moreover, $\chi(\mathcal{E}_D) = 0$ (cf. [T3, Lemma 2.3 (ii)]). In [R1], Raynaud investigated the following property of the vector bundle $\mathcal{E}_D$ on X.

Condition 2.6. We shall say that $\mathcal{E}_D$ satisfies $(\star)$ if there exists a line bundle $\mathcal{L}_t'$ of degree 0 on X_t such that

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}_t')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}_t'))\}.$$

Let J_{X_t} be the Jacobian variety of X_t, and $\mathcal{L}_{X_t}$ a universal line bundle on $X_t \times J_{X_t}$. Let $\mathrm{pr}_{X_t} : X_t \times J_{X_t} \to X_t$ and $\mathrm{pr}_{J_{X_t}} : X_t \times J_{X_t} \to J_{X_t}$ be the natural projections. We denote by $\mathcal{F}$ the coherent $\mathcal{O}_{X_t}$-module $\mathrm{pr}_{X_t}^*(\mathcal{E}_D) \otimes \mathcal{L}_{X_t}$, and by

$$\chi_{\mathcal{F}} \stackrel{\text{def}}{=} \dim_k(H^0(X_t \times_k k(y), \mathcal{F} \otimes k(y))) - \dim_k(H^1(X_t \times_k k(y), \mathcal{F} \otimes k(y)))$$

for each $y \in J_{X_t}$, where $k(y)$ denotes the residue field of y. Note that since $\mathrm{pr}_{J_{X_t}}$ is flat, $\chi_{\mathcal{F}}$ is independent of $y \in J_{X_t}$. Write $(-\chi_{\mathcal{F}})^+$ for $\max\{0, -\chi_{\mathcal{F}}\}$. We denote by

$$\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$$

the closed subscheme of J_{X_t} defined by the $(-\chi_{\mathcal{F}})^+$-th Fitting ideal

$$\mathrm{Fitt}_{(-\chi_{\mathcal{F}})^+}(R^1(\mathrm{pr}_{J_{X_t}})_*(\mathcal{F})).$$

The definition of $\Theta_{\mathcal{E}_D}$ is independent of the choice of $\mathcal{L}_t$. Moreover, for each line bundle $\mathcal{L}''$ of degree 0 on X_t, we have that $[\mathcal{L}''] \notin \Theta_{\mathcal{E}_D}$ if and only if

$$0 = \min\{\dim_k(H^0(X_t, \mathcal{E}_D \otimes \mathcal{L}'')), \dim_k(H^1(X_t, \mathcal{E}_D \otimes \mathcal{L}''))\},$$

where $[\mathcal{L}'']$ denotes the point of J_{X_t} corresponding to $\mathcal{L}''$ (cf. [T3, Proposition 2.2 (i) (ii)]).

Suppose that $\mathcal{E}_D$ satisfies $(\star)$. [R1, Proposition 1.8.1] implies that $\Theta_{\mathcal{E}_D}$ is algebraically equivalent to $\mathrm{rk}(\mathcal{E}_D)\Theta$, where Θ is the classical theta divisor (i.e., the image of $X_t^{g_X-1}$ in J_{X_t}). Then we have the following definition.

Definition 2.7. We shall say $\Theta_{\mathcal{E}_D} \subseteq J_{X_t}$ the *Raynaud-Tamagawa theta divisor* associated to $\mathcal{E}_D$ if $\mathcal{E}_D$ satisfies $(\star)$.

Remark 2.7.1. We maintain the notation introduced above. Moreover, we suppose that $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$. The definition of $\mathcal{E}_D$ implies that the following natural exact sequence holds

$$0 \to \mathcal{L}_t \to (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \to \mathcal{E}_D \to 0.$$

Let $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$. Write $\mathcal{I}_t$ for the pulling-back of $\mathcal{I}$ by the natural morphism $X_t \to X$. we obtain the following exact sequence

$$\ldots \to H^0(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \overset{\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}}{\to} H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t)$$

$$\to H^1(X_t, \mathcal{E}_D \otimes \mathcal{I}_t) \to \ldots.$$

Note that we have that

$$H^1(X_t, \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}),$$

and that

$$H^1(X_t, (F^t_{X/k})_*(\mathcal{O}_X(D)) \otimes \mathcal{L}_t \otimes \mathcal{I}_t) \cong H^1(X, \mathcal{O}_X(D) \otimes (F^t_{X/k})^*(\mathcal{L}_t \otimes \mathcal{I}_t))$$

$$\cong H^1(X, \mathcal{O}_X(D) \otimes (\mathcal{L} \otimes \mathcal{I})^{\otimes p^t}) \cong H^1(X, \mathcal{L} \otimes \mathcal{I}).$$

Moreover, it is easy to see that the homomorphism

$$H^1(X, \mathcal{L} \otimes \mathcal{I}) \to H^1(X, \mathcal{L} \otimes \mathcal{I})$$

induced by $\phi_{\mathcal{L}_t \otimes \mathcal{I}_t}$ coincides with $\phi_{([\mathcal{L} \otimes \mathcal{I}], D)}$. Thus, we obtain that if

$$\gamma_{(([\mathcal{L} \otimes \mathcal{I}], D))} = \dim_k(H^1(X, \mathcal{L} \otimes \mathcal{I}))$$

for some line bundle $[\mathcal{I}] \in \mathrm{Pic}(X)[n]$, then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists (i.e., $[\mathcal{I}_t] \not\in \Theta_{\mathcal{E}_D}$).

Remark 2.7.2. Suppose that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists. We see immediately that the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_{D^{(i)}}$, $i \in \{0. \ldots, t-1\}$, exists.

The following fundamental theorem of theta divisors was proved by Raynaud and Tamagawa.

Theorem 2.8. *Suppose that $s(D) \in \{0, 1\}$. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{E}_D$ exists (i.e., $\mathcal{E}_D$ satisfies $(\star)$).*

Remark 2.8.1. Theorem 2.8 was proved by Raynaud if $s(D) = 0$ (cf. [R1, Théorème 4.1.1]), and by Tamagawa if $s(D) \leq 1$ (cf. [T3, Theorem 2.5]). On the other hand, the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ does not exist in general (cf. [T3, Example 2.19]).

By applying Theorem 2.8, the following lemma was proved by Tamagawa (cf. [T3, Corollary 2.6, Lemma 2.12 (ii)]).

Lemma 2.9. *Let $X^\bullet$ be a smooth pointed stable curve over k.*

(i) Let $Q \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree $s(Q)n$ such that $\mathrm{ord}_x(Q) \leq n$ for each $x \in \mathrm{Supp}(Q)$, $\mathcal{L}_Q$ a line bundle on X of degree $-s(Q)$, and $\mathcal{L}_{Q,t}$ the pulling-back of $\mathcal{L}_Q$ by the natural morphism $X_t \to X$. Let $S_Q \stackrel{\mathrm{def}}{=} \{x \in X \mid \mathrm{ord}_x(Q) = n\}$,

$$Q' \stackrel{\mathrm{def}}{=} Q - (\sum_{x \in S_Q} nx).$$

an effective divisor on X of degree $s(Q')n$, $\mathcal{L}_{Q'}$ a line bundle on X of degree $-s(Q')$, and $\mathcal{L}_{Q',t}$ the pulling-back of $\mathcal{L}_{Q'}$ by the natural morphism $X_t \to X$. Suppose that the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^t_Q \otimes \mathcal{L}_{Q,t}$ exists. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^t_{Q'} \otimes \mathcal{L}_{Q',t}$ exists.

(ii) Let t_i, $i \in \{1, 2\}$, be an arbitrary positive natural number and $n_i \stackrel{\mathrm{def}}{=} p^{t_i} - 1$. Let $Q_i \in \mathbb{Z}[D_X]$ be an effective divisor on X of degree $\deg(Q_i) = s(Q_i)n_i$, $\mathcal{L}_{Q_i}$ a line bundle on X of degree $-s(Q_i)$, and $\mathcal{L}_{Q_i,t_i}$ the pulling-back of $\mathcal{L}_{Q_i}$ by the natural morphism $X_{t_i} \to X$. Suppose that $s \stackrel{\mathrm{def}}{=} s(Q_1) = s(Q_2)$. Let $t \stackrel{\mathrm{def}}{=} t_1 + t_2$, $n \stackrel{\mathrm{def}}{=} n_1 + p^{t_1} n_2$,

$$Q \stackrel{\mathrm{def}}{=} Q_1 + p^{t_1} Q_2 \in \mathbb{Z}[D_X]$$

an effective divisor on X of degree $\deg(Q) = sn$, $\mathcal{L}_Q$ a line bundle on X of degree $-s$, and $\mathcal{L}_{Q,t}$ the pulling-back of $\mathcal{L}_Q$ by the natural morphism $X_t \to X$. Then the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^t_Q \otimes \mathcal{L}_{Q,t}$ exists if and only if the Raynaud-Tamagawa theta divisor associated to $\mathcal{B}^{t_i}_{Q_i} \otimes \mathcal{L}_{Q_i,t_i}$ exists for each $i \in \{1, 2\}$.

3 New-ordinariness of cyclic admissible coverings of generic curves

In this section, we discuss new-ordinariness of prime-to-p cyclic admissible coverings of generic curves. We maintain the notation introduced in Section 2. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$. First, we have the following result.

Lemma 3.1. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$. Suppose that $n = p - 1$, and that $X^\bullet$ is a generic pointed stable curve of type $(0, 3)$. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists. Moreover, we have*

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X, \mathcal{L})),\ ([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}.$$

Proof. This follows immediately from [B, Corollary 6.8]. □

Let R be a discrete valuation ring with algebraically closed residue field k_R, K_R the quotient field of R, and $\overline{K}_R$ an algebraic closure of K_R. Suppose that $k \subseteq K_R$. Let

$$\mathcal{X}^\bullet = (\mathcal{X}, D_{\mathcal{X}} \stackrel{\mathrm{def}}{=} \{e_1, \ldots, e_{n_X}\})$$

be a pointed stable curve of type (g_X, n_X) over R. We shall write $\mathcal{X}^\bullet_\eta = (\mathcal{X}_\eta, D_{\mathcal{X}_\eta} \stackrel{\text{def}}{=} \{e_{\eta,1}, \dots, e_{\eta,n_X}\})$, $\mathcal{X}^\bullet_{\overline{\eta}} = (\mathcal{X}_{\overline{\eta}}, D_{\mathcal{X}_{\overline{\eta}}} \stackrel{\text{def}}{=} \{e_{\overline{\eta},1}, \dots, e_{\overline{\eta},n_X}\})$, $\mathcal{X}^\bullet_s = (\mathcal{X}_s, D_{\mathcal{X}_s} \stackrel{\text{def}}{=} \{e_{s,1}, \dots, e_{s,n_X}\})$ for the generic fiber $\mathcal{X}^\bullet \times_R K_R$ of $\mathcal{X}^\bullet$, the geometric generic fiber $\mathcal{X}^\bullet \times_R \overline{K}_R$ of $\mathcal{X}^\bullet$, and the special fiber $\mathcal{X}^\bullet \times_R k_R$ of $\mathcal{X}^\bullet$, respectively. Write $\Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ and $\Pi_{\mathcal{X}^\bullet_s}$ for the admissible fundamental groups of $\mathcal{X}^\bullet_{\overline{\eta}}$ and $\mathcal{X}^\bullet_s$, respectively. Then we obtain a specialization map

$$sp_R : \Pi_{\mathcal{X}^\bullet_{\overline{\eta}}} \twoheadrightarrow \Pi_{\mathcal{X}^\bullet_s}$$

which is surjective.

We shall say that $X^\bullet$ *admits a (DEG-A)* if the following conditions hold, where "(DEG)" means "degeneration":

(i) The geometric generic fiber $\mathcal{X}^\bullet_{\overline{\eta}}$ of $\mathcal{X}^\bullet$ is $\overline{K}_R$-isomorphic to $X^\bullet \times_k \overline{K}_R$. Then without loss of generality, we may identify $e_{\overline{\eta},r}$, $r \in \{1, \dots, n_X\}$, with $x_r \times_k \overline{K}_R$ via this isomorphism. Note that since the admissible fundamental groups do not depend on the base fields, $\Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ is naturally isomorphic to $\Pi_{X^\bullet}$.

(ii) $\mathcal{X}^\bullet_s$ is a component-generic pointed stable curve over k_R.

(iii) If $n_X \leq 1$, we have $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is isotrivial (i.e., the image of the natural morphism $\operatorname{Spec} R \to \overline{\mathcal{M}}_{g_X,n_X,k_R} \to \overline{M}_{g_X,n_X,k_R}$ determined by $\mathcal{X}^\bullet \to \operatorname{Spec} R$ is a point, where $\overline{\mathcal{M}}_{g_X,n_X,k_R} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{F}_p} k_R$ and $\overline{M}_{g_X,n_X,k_R}$ denotes the coarse moduli space of $\overline{\mathcal{M}}_{g_X,n_X,k_R}$).

(iv) If $n_X = 2$, the underlying curve of $\mathcal{X}^\bullet_s$ is

$$\mathcal{X}_s = C \cup P$$

such that the following conditions hold: (a) C is a projective curve of genus g_X over k_R. (b) P is k_R-isomorphic to $\mathbb{P}^1_{k_R}$. (c) $\#(C \cap P) = 1$. (d) $D_{\mathcal{X}_s} \subseteq P$.

(v) If $n_X \geq 3$, the underlying curve of $\mathcal{X}^\bullet_s$ is

$$\mathcal{X}_s = C \cup (\bigcup_{v=1}^{n_X-2} P_v)$$

such that the following conditions hold: (a) C is an empty set when $g_X = 0$; otherwise, C is a projective curve of genus g_X over k_R. (b) P_v, $v \in \{1, \dots, n_X - 2\}$, is k_R-isomorphic to $\mathbb{P}^1_{k_R}$. (c) If C is not empty, we have $\#(C \cap P_1) = 1$ and $C \cap P_v = \emptyset$ for each $v \in \{2, \dots, n_X - 2\}$. (d) For each $v \in \{1, \dots, n_X - 2\}$, $\#(P_v \cap P_{v+1}) = 1$ and $P_v \cap P_{v'} = \emptyset$ when $v' \notin \{v-1, v, v+1\}$. (e) If $n_X = 3$, we have $D_{\mathcal{X}_s} \cap P_1 = \{e_{s,1}, e_{s,2}, e_{s,3}\}$. (f) If $n_X = 4$, we have $D_{\mathcal{X}_s} \cap P_1 = \{e_{s,1}, e_{s,2}\}$ and $D_{\mathcal{X}_s} \cap P_2 = \{e_{s,3}, e_{s,4}\}$. (g) If $n_X \geq 5$, we have $D_{\mathcal{X}_s} \cap P_1 = \{e_{s,1}, e_{s,2}\}$, $D_{\mathcal{X}_s} \cap P_{n_X-2} = \{e_{s,n_X-1}, e_{s,n_X}\}$, and $D_{\mathcal{X}_s} \cap P_v = \{e_{s,v+1}\}$, $v \in \{2, \dots, n_X - 3\}$.

Moreover, we shall say that $X^\bullet$ *admits a (DEG-B)* if the following conditions hold:

(i) The geometric generic fiber $\mathcal{X}^\bullet_{\overline{\eta}}$ of $\mathcal{X}^\bullet$ is $\overline{K}_R$-isomorphic to $X^\bullet \times_k \overline{K}_R$. Then without loss of generality, we may identify $e_{\overline{\eta},r}$, $r \in \{1, \dots, n_X\}$, with $x_r \times_k \overline{K}_R$ via this isomorphism.

(ii) $\mathcal{X}^\bullet_s$ is a component-generic pointed stable curve over k_R.

(iii) $g_X \geq 1$ and $n_X \geq 2$.

(iv) The underlying curve of $\mathcal{X}^\bullet_s$ is

$$\mathcal{X}_s = C \cup P$$

such that the following conditions hold: (a) C is a projective curve of genus g_X over k_R. (b) P is k_R-isomorphic to $\mathbb{P}^1_{k_R}$. (c) $\#(C \cap P) = 1$. (d) $D_{\mathcal{X}_s} \subseteq P$.

Lemma 3.2. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ and $\alpha \in \mathrm{Rev}^{\mathrm{adm}}_D(X^\bullet)$ such that $\alpha \neq 0$. Suppose that $X^\bullet$ is generic, and that $n = p - 1$. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists. Moreover, we have*

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X, \mathcal{L})),\ ([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}.$$

Proof. Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. We note that, to verify the lemma, we only need to prove the lemma in the case where $Y^\bullet$ is *connected*. Then we may assume that $Y^\bullet$ is connected.

Since $X^\bullet$ is a generic curve, $X^\bullet$ admits a (DEG-A). Furthermore, we write $Q_{\overline{\eta}}$ (resp. Q_s) for the effective divisor on $\mathcal{X}_{\overline{\eta}}$ (resp. $\mathcal{X}_s$) induced by D and $\alpha_{\overline{\eta}} \in \mathrm{Rev}^{\mathrm{adm}}_{Q_{\overline{\eta}}}(\mathcal{X}^\bullet_{\overline{\eta}})$ for the element induced by α. Then we have

$$\gamma_{(\alpha,D)} = \gamma_{(\alpha_{\overline{\eta}},Q_{\overline{\eta}})}.$$

Suppose that $X^\bullet$ satisfies (DEG-A)-(iii). If $n_X = 1$ and $g_X = 1$, then the lemma is trivial. If $n_X \leq 1$ and $g_X \geq 2$, then the lemma follows immediately from [N, Proposition 4].

Suppose that $X^\bullet$ satisfies (DEG-A)-(v). Moreover, we suppose that $C \neq \emptyset$, and that $n_X \geq 5$. For each $v \in \{1, \dots, n_X - 3\}$, we write

$$y_v \text{ and } z_{v+1}$$

for the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_v \hookrightarrow \mathcal{X}_s$ and the inverse image of $P_v \cap P_{v+1}$ of the natural closed immersion $P_{v+1} \hookrightarrow \mathcal{X}_s$, respectively. We define

$$P^\bullet_1 = (P_1, D_{P_1} \overset{\mathrm{def}}{=} \{e_{s,1}, e_{s,2}, y_1\} \cup (C \cap P_1)),$$

$$P^\bullet_{n_X-2} = (P_{n_X-2}, D_{P_{n_X-2}} \overset{\mathrm{def}}{=} \{z_{n_X-2}, e_{s,n_X-1}, e_{s,n_X}\}),$$

and

$$P^\bullet_v = (P_v, D_{P_v} \overset{\mathrm{def}}{=} \{z_v, e_{s,v+1}, y_v\}),\ v \in \{2, \dots, n_X - 3\},$$

to be smooth pointed stable curves of types $(0,4)$, $(0,3)$, and $(0,3)$ over k_R, respectively. Moreover, we define

$$C^\bullet = (C, D_C \stackrel{\text{def}}{=} C \cap P_1)$$

to be a smooth pointed stable curve of type $(g_X, 1)$ over k_R. Note that since $C^\bullet$ is generic, we have $\sigma_C = g_X$. Let

$$f^\bullet_{\overline{\eta}} \stackrel{\text{def}}{=} f^\bullet \times_k \overline{K}_R : \mathcal{Y}^\bullet_{\overline{\eta}} = (\mathcal{Y}_{\overline{\eta}}, D_{\mathcal{Y}_{\overline{\eta}}}) \stackrel{\text{def}}{=} Y^\bullet \times_k \overline{K}_R \to \mathcal{X}^\bullet_{\overline{\eta}}$$

be the Galois admissible covering over $\overline{K}_R$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $f^\bullet$, and $\Pi_{\mathcal{Y}^\bullet_{\overline{\eta}}} \subseteq \Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ the admissible fundamental group of $\mathcal{Y}^\bullet_{\overline{\eta}}$. By the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]), we have

$$sp^{p'}_R : \Pi^{p'}_{\mathcal{X}^\bullet_{\overline{\eta}}} \xrightarrow{\sim} \Pi^{p'}_{\mathcal{X}^\bullet_s},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Then we obtain a normal open subgroup $\Pi^{p'}_{\mathcal{Y}^\bullet_s} \stackrel{\text{def}}{=} sp^{p'}_R(\Pi^{p'}_{\mathcal{Y}^\bullet_{\overline{\eta}}}) \subseteq \Pi^{p'}_{\mathcal{X}^\bullet_s}$. Write $\Pi_{\mathcal{Y}^\bullet_s} \subseteq \Pi_{\mathcal{X}^\bullet_s}$ for the inverse image of $\Pi^{p'}_{\mathcal{Y}^\bullet_s}$ of the natural surjection $\Pi_{\mathcal{X}^\bullet_s} \twoheadrightarrow \Pi^{p'}_{\mathcal{X}^\bullet_s}$. Then $\Pi_{\mathcal{Y}^\bullet_s}$ determines a Galois admissible covering

$$f^\bullet_s : \mathcal{Y}^\bullet_s = (\mathcal{Y}_s, D_{\mathcal{Y}_s}) \to \mathcal{X}^\bullet_s$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_s \in \mathrm{Rev}^{\mathrm{adm}}_{Q_s}(\mathcal{X}^\bullet_s)$ for an element induced by $f^\bullet_s$.

The structure of the maximal prime-to-p quotients of admissible fundamental groups implies that f_s is étale over C. Then we obtain that f_s is étale over $C \cap P_1$. Thus, f_s is étale over D_C. Let $Y_v \stackrel{\text{def}}{=} f_s^{-1}(P_v)$, $v \in \{1, \dots, n_X - 2\}$. We put

$$Y^\bullet_v \stackrel{\text{def}}{=} (Y_v, D_{Y_v} \stackrel{\text{def}}{=} f_s^{-1}(D_{P_v})),\ v \in \{1, \dots, n_X - 2\}.$$

Then $f^\bullet_s$ induces a Galois multi-admissible covering

$$f^\bullet_v : Y^\bullet_v \to P^\bullet_v,\ v \in \{1, \dots, n_X - 2\},$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $([\mathcal{L}_v], Q_v) \in \widetilde{\mathscr{P}}_{P^\bullet_v, n}$ for the element induced by $f^\bullet_v$ for each $v \in \{1, \dots, n_X - 2\}$. Note that since f_s is étale over $C \cap P_1$, we have that $\mathrm{Supp}(Q_1) \subseteq \{e_{s,1}, e_{s,2}, y_1\}$. Moreover, the $k_R[\mu_n]$-module $H^1_{\text{ét}}(Y_v, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Y_v, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{Y_v}(j),$$

where $\zeta \in \mu_n$ acts on $M_{Y_v}(j)$ as the ζ^j-multiplication. Then Lemma 3.1 implies that

$$\gamma_{([\mathcal{L}_v], Q_v)} = \dim_{k_R}(M_{Y_v}(1)) = \dim_{k_R}(H^1(P_v, \mathcal{L}_v)).$$

Let $Z \stackrel{\text{def}}{=} f_s^{-1}(C)$ and $\pi_0(Z)$ the set of connected components of Z. Then $f^\bullet_s$ induces a Galois étale covering (not necessarily connected)

$$f^\bullet_C : Z^\bullet = (Z, D_Z \stackrel{\text{def}}{=} f_s^{-1}(D_C)) \to C^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, $f_C^\bullet$ induces an element $\alpha_C \in \mathrm{Rev}_0^{\mathrm{adm}}(C^\bullet)$. Suppose that $\#\pi_0(Z) \neq n$. Then we have $\alpha_C \neq 0$. The $k_R[\mu_n]$-module $H^1_{\text{ét}}(Z, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Z, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_Z(j),$$

where $\zeta \in \mu_n$ acts on $M_Z(j)$ as the ζ^j-multiplication. [N, Proposition 4] implies that

$$\gamma_{(\alpha_C, 0)} = \dim_{k_R}(M_Z(1)) = g_X - 1.$$

Suppose that $\#\pi_0(Z) = n$. Then we have $\alpha_C = 0$. Since C is ordinary, we obtain immediately that

$$\gamma_{(\alpha_C, 0)} = g_X.$$

Write $\Gamma_{\mathcal{Y}_s^\bullet}$ for the dual semi-graph of $\mathcal{Y}_s^\bullet$. The natural $k[\mu_n]$-submodule

$$H^1(\Gamma_{\mathcal{Y}_s^\bullet}, \mathbb{F}_p) \otimes k \subseteq H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k$$

admits the following canonical decomposition

$$H^1(\Gamma_{\mathcal{Y}_s^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{\mathcal{Y}_s^\bullet}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{\mathcal{Y}_s^\bullet}}(j)$ as the ζ^j-multiplication. Then we see immediately that

$$\gamma_{(\alpha_s, Q_s)} = \gamma_{(\alpha_C, 0)} + \sum_{v=1}^{n_X - 2} \gamma_{([\mathcal{L}_v], Q_v)} + \dim_k(M_{\Gamma_{\mathcal{Y}_s^\bullet}}(1)) = g_X + s(D_{\alpha_s}) - 1.$$

On the other hand, the $k_R[\mu_n]$-modules $H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ and $H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R$ admit the following canonical decompositions

$$H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{\overline{\eta}}}(j)$$

and

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_s}(j),$$

respectively, where $\zeta \in \mu_n$ acts on $M_{\mathcal{Y}_{\overline{\eta}}}(j)$ and $M_{\mathcal{Y}_s}(j)$ as the ζ^j-multiplication. Moreover, we have an injection as $k_R[\mu_n]$-modules

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R \hookrightarrow H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$$

induced by the specialization map $\Pi_{\mathcal{Y}_{\overline{\eta}}^\bullet} \twoheadrightarrow \Pi_{\mathcal{Y}_s^\bullet}$. Thus, we have

$$g_X + s(D_{\alpha_s}) - 1 = \gamma_{(\alpha_s, Q_s)} = \dim_{k_R}(M_{\mathcal{Y}_s}(1))$$
$$\leq \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \dim_{k_R}(M_{\mathcal{Y}_{\overline{\eta}}}(1)) \leq g_X + s(D_{\alpha_{\overline{\eta}}}) - 1.$$

Since $s(D) = s(D_{\alpha_{\overline{\eta}}}) = s(D_{\alpha_s})$, we obtain that

$$\gamma_{([\mathcal{L}],D)} = g_X + s(D) - 1 = \dim_k(H^1(X, \mathcal{L})).$$

This completes the proof of the lemma if $X^\bullet$ satisfies (DEG-A)-(v), $C \neq \emptyset$, and $n_X \geq 5$. By applying similar arguments to the arguments given in the proof above, one can prove the lemma immediately when $X^\bullet$ satisfies (DEG-A)-(v) and either $C = \emptyset$ or $n_X \leq 4$ holds.

Moreover, similar arguments to the arguments given in the proof above imply the lemma holds when $X^\bullet$ satisfies (DEG-A)-(iv). We complete the proof of the lemma. □

Lemma 3.3. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^\sim[D_X]^0$ be an effective divisor on X of degree $s(D)n$ and $x \in D_X$. For each $i \in \{0, \ldots, t-1\}$, we put*

$$d_x^{(i)} \overset{\text{def}}{=} \text{ord}_x(D^{(i)}),$$

and write

$$d_x^{(i)} = \sum_{r=0}^{t-1} d_{x,r}^{(i)} p^r$$

for the p-adic expansion. Then the following statements are equivalent:

(i)

$$s(D)n = \deg(D) = \deg(D^{(i)})$$

holds for each $i \in \{0, 1, \ldots, t-1\}$.

(ii)

$$\sum_{x \in D_X} d_{x,r} = s(D)(p-1)$$

holds for each $r \in \{0, \ldots, t-1\}$.

(iii)

$$\sum_{x \in D_X} d_{x,r}^{(i)} = s(D)(p-1)$$

holds for each $i \in \{0, \ldots, t-1\}$ and each $r \in \{0, \ldots, t-1\}$.

Proof. We see that (ii) ⇒ (iii) and (iii) ⇒ (i) follows immediately from the definition of $D^{(i)}$. Let us prove (i) ⇒ (ii).

Let $r \in \{0, \ldots, t-1\}$. We have

$$d_x^{(r+1)} = d_{x,r}p^{t-1} + \frac{d_x^{(r)} - d_{x,r}}{p} = \frac{1}{p}d_x^{(r)} + \frac{p^t - 1}{p}d_{x,r} = \frac{1}{p}d_x^{(r)} + \frac{n}{p}d_{x,r}.$$

Note that (i) implies that

$$s(D)n = \sum_{x \in D_X} d_x^{(r+1)} = \sum_{x \in D_X} d_x^{(r)}.$$

Then we have

$$s(D)n = \sum_{x \in D_X} d_x^{(r+1)} = \frac{1}{p}\sum_{x \in D_X} d_x^{(r)} + \frac{n}{p}\sum_{x \in D_X} d_{x,r}$$

$$= \frac{1}{p}s(D)n + \frac{n}{p}\sum_{x\in D_X} d_{x,r}.$$

This means that

$$\sum_{x\in D_X} d_{x,r} = s(D)(p-1).$$

We complete the proof of the lemma. □

We introduce the following condition concerning effective divisors on X.

Definition 3.4. Let $t \in \mathbb{N}$ be an arbitrary positive natural number and $n = p^t - 1$. Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$. We shall say that D is *Frobenius-stable* if one of statements mentioned in Lemma 3.3 holds.

Proposition 3.5. *Let $D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ and $\alpha \in \mathrm{Rev}_D^{\mathrm{adm}}(X^\bullet)$ such that $\alpha \neq 0$. Suppose that $X^\bullet$ is generic, and that D is Frobenius stable. Then the Raynaud-Tamagawa theta divisor $\Theta_{\mathcal{E}_D}$ associated to $\mathcal{E}_D$ exists. Moreover, for each $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,n}$, we have*

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X,\mathcal{L})) = \dim_k(H^1(X,\mathcal{L}^{(i)})) = \gamma_{([\mathcal{L}^{(i)}],D^{(i)})},\ i \in \{0,\dots,t-1\}.$$

Proof. Since D is Frobenius-stable, we have that $\dim_k(H^1(X,\mathcal{L})) = \dim_k(H^1(X,\mathcal{L}^{(i)}))$ for each $i \in \{0,\dots,t-1\}$. Then to verify the proposition, it is sufficient to prove that $\gamma_{([\mathcal{L}^{(i)}],D^{(i)})} = \dim_k(H^1(X,\mathcal{L}^{(i)}))$ holds for each $i \in \{0,\dots,t-1\}$. Furthermore, it is easy to see that it is sufficient to prove that

$$\gamma_{([\mathcal{L}],D)} = \dim_k(H^1(X,\mathcal{L})).$$

Suppose that $g_X = 0$. We maintain the notation introduced in Lemma 3.3. We define

$$D_r \stackrel{\mathrm{def}}{=} \sum_{x\in D_X} d_{x,r}x,\ r \in \{0,\dots,t-1\},$$

to be an effective divisor on X. Note that $\deg(D_r) = s(D)(p-1)$ for each $r \in \{0,\dots,t-1\}$, and that

$$D = \sum_{r=0}^{t-1} D_r p^r.$$

Then the proposition follows immediately from Lemma 2.9 and Lemma 3.2.

Suppose that $g_X \geq 1$ and $n_X \leq 1$. Then the proposition follows immediately from [N, Proposition 4].

Suppose that $g_X \geq 1$ and $n_X \geq 2$. Since $X^\bullet$ is generic, we see that $X^\bullet$ admits a (DEG-B). Let $f^\bullet : Y^\bullet = (Y, D_Y) \to X^\bullet$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by α. We note that, to verify the proposition, we only need to prove the proposition in the case where $Y^\bullet$ is *connected.* Then we may assume that $Y^\bullet$ is connected. Furthermore, we write $Q_{\overline{\eta}}$ (resp. Q_s) for the effective divisor on $\mathcal{X}_{\overline{\eta}}$ (resp. $\mathcal{X}_s$) induced by D and $\alpha_{\overline{\eta}} \in \mathrm{Rev}_{Q_{\overline{\eta}}}^{\mathrm{adm}}(\mathcal{X}_{\overline{\eta}}^\bullet)$ for the element induced by α. Then we have

$$\gamma_{(\alpha,D)} = \gamma_{(\alpha_{\overline{\eta}},Q_{\overline{\eta}})}.$$

We define

$$P^\bullet \stackrel{\text{def}}{=} (P, D_P \stackrel{\text{def}}{=} D_{\mathcal{X}_s})$$

and

$$C^\bullet = (C, D_C \stackrel{\text{def}}{=} C \cap P)$$

to be smooth pointed stable curves of types $(0, n_X + 1)$ and $(g_X, 1)$ over k, respectively. Let

$$f^\bullet_{\overline{\eta}} \stackrel{\text{def}}{=} f^\bullet \times_k \overline{K}_R : \mathcal{Y}^\bullet_{\overline{\eta}} = (\mathcal{Y}_{\overline{\eta}}, D_{\mathcal{Y}_{\overline{\eta}}}) \stackrel{\text{def}}{=} Y^\bullet \times_k \overline{K}_R \to \mathcal{X}^\bullet_{\overline{\eta}}$$

be the Galois admissible covering over $\overline{K}_R$ with Galois group $\mathbb{Z}/n\mathbb{Z}$ induced by $f^\bullet$, and $\Pi_{\mathcal{Y}^\bullet_{\overline{\eta}}} \subseteq \Pi_{\mathcal{X}^\bullet_{\overline{\eta}}}$ the admissible fundamental group of $\mathcal{Y}^\bullet_{\overline{\eta}}$. By the specialization theorem of maximal prime-to-p quotients of admissible fundamental groups (cf. [V, Théorème 2.2 (c)]), we have

$$sp^{p'}_R : \Pi^{p'}_{\mathcal{X}^\bullet_{\overline{\eta}}} \xrightarrow{\sim} \Pi^{p'}_{\mathcal{X}^\bullet_s},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Then we obtain a normal open subgroup $\Pi^{p'}_{\mathcal{Y}^\bullet_s} \stackrel{\text{def}}{=} sp^{p'}_R(\Pi^{p'}_{\mathcal{Y}^\bullet_{\overline{\eta}}}) \subseteq \Pi^{p'}_{\mathcal{X}^\bullet_s}$. Write $\Pi_{\mathcal{Y}^\bullet_s} \subseteq \Pi_{\mathcal{X}^\bullet_s}$ for the inverse image of $\Pi^{p'}_{\mathcal{Y}^\bullet_s}$ of the natural surjection $\Pi_{\mathcal{X}^\bullet_s} \twoheadrightarrow \Pi^{p'}_{\mathcal{X}^\bullet_s}$. Then $\Pi_{\mathcal{Y}^\bullet_s}$ determines a Galois admissible covering

$$f^\bullet_s : \mathcal{Y}^\bullet_s = (\mathcal{Y}_s, D_{\mathcal{Y}_s}) \to \mathcal{X}^\bullet_s$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $\alpha_s \in \mathrm{Rev}^{\mathrm{adm}}_{Q_s}(\mathcal{X}^\bullet_s)$ for an element induced by $f^\bullet_s$.

The structure of the maximal prime-to-p quotients of admissible fundamental groups implies that f_s is étale over C. Then we obtain that f_s is étale over $C \cap P$. Thus, f_s is étale over D_C. Let $Y_v \stackrel{\text{def}}{=} f_s^{-1}(P)$. We put

$$Y^\bullet \stackrel{\text{def}}{=} (Y, D_Y \stackrel{\text{def}}{=} f_s^{-1}(D_P)).$$

Then $f^\bullet_s$ induces a Galois multi-admissible covering

$$f^\bullet_P : Y^\bullet \to P^\bullet,$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Write $([\mathcal{L}_P], Q_P) \in \widetilde{\mathscr{P}}_{P^\bullet, n}$ for the element induced by $f^\bullet_P$. Note that since f_s is étale over $C \cap P$, we have that $\mathrm{Supp}(Q_P) \subseteq D_P$. Moreover, the $k_R[\mu_n]$-module $H^1_{\text{ét}}(Y, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Y, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_Y(j),$$

where $\zeta \in \mu_n$ acts on $M_Y(j)$ as the ζ^j-multiplication. Then the case of $g = 0$ implies that

$$\gamma_{([\mathcal{L}_P], Q_P)} = \dim_{k_R}(M_Y(1)) = \dim_{k_R}(H^1(P, \mathcal{L}_P)) = s(D_{\alpha_s}) - 1.$$

Let $Z \stackrel{\text{def}}{=} f_s^{-1}(C)$ and $\pi_0(Z)$ the set of connected components of Z. Then $f^\bullet_s$ induces a Galois étale covering (not necessarily connected)

$$f^\bullet_C : Z^\bullet = (Z, D_Z \stackrel{\text{def}}{=} f_s^{-1}(D_C)) \to C^\bullet$$

over k_R with Galois group $\mathbb{Z}/n\mathbb{Z}$. Moreover, $f_C^\bullet$ induces an element $\alpha_C \in \mathrm{Rev}_0^{\mathrm{adm}}(C^\bullet)$. Suppose that $\#\pi_0(Z) \neq n$. Then we have $\alpha_C \neq 0$. The $k_R[\mu_n]$-module $H^1_{\text{ét}}(Z, \mathbb{F}_p) \otimes k_R$ admits the following canonical decomposition

$$H^1_{\text{ét}}(Z, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_Z(j),$$

where $\zeta \in \mu_n$ acts on $M_Z(j)$ as the ζ^j-multiplication. [N, Proposition 4] implies that

$$\gamma_{(\alpha_C, 0)} = \dim_{k_R}(M_Z(1)) = g_X - 1.$$

Suppose that $\#\pi_0(Z) = n$. Then we have $\alpha_C = 0$. Since C is ordinary, we obtain immediately that

$$\gamma_{(\alpha_C, 0)} = g_X.$$

Write $\Gamma_{\mathcal{Y}_s^\bullet}$ for the dual semi-graph of $\mathcal{Y}_s^\bullet$. The natural $k[\mu_n]$-submodule

$$H^1(\Gamma_{\mathcal{Y}_s^\bullet}, \mathbb{F}_p) \otimes k \subseteq H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k$$

admits the following canonical decomposition

$$H^1(\Gamma_{\mathcal{Y}_s^\bullet}, \mathbb{F}_p) \otimes k = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\Gamma_{\mathcal{Y}_s^\bullet}}(j),$$

where $\zeta \in \mu_n$ acts on $M_{\Gamma_{\mathcal{Y}_s^\bullet}}(j)$ as the ζ^j-multiplication. Then we see immediately that

$$\dim_k(M_{\Gamma_{\mathcal{Y}_s^\bullet}}(1)) = \begin{cases} 0, & \text{if } \#\pi_0(Z) = n, \\ 1, & \text{if } \#\pi_0(Z) = 1. \end{cases}$$

Then we have

$$\gamma_{(\alpha_s, Q_s)} = \gamma_{(\alpha_C, 0)} + \gamma_{([\mathcal{L}_P], Q_P)} + \dim_k(M_{\Gamma_{\mathcal{Y}_s^\bullet}}(1)) = g_X + s(D_{\alpha_s}) - 1.$$

On the other hand, the $k_R[\mu_n]$-modules $H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$ and $H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R$ admit the following canonical decompositions

$$H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_{\overline{\eta}}}(j)$$

and

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M_{\mathcal{Y}_s}(j),$$

respectively, where $\zeta \in \mu_n$ acts on $M_{\mathcal{Y}_{\overline{\eta}}}(j)$ and $M_{\mathcal{Y}_s}(j)$ as the ζ^j-multiplication. Moreover, we have an injection as $k_R[\mu_n]$-modules

$$H^1_{\text{ét}}(\mathcal{Y}_s, \mathbb{F}_p) \otimes k_R \hookrightarrow H^1_{\text{ét}}(\mathcal{Y}_{\overline{\eta}}, \mathbb{F}_p) \otimes k_R$$

induced by the specialization map $\Pi_{\mathcal{Y}_{\overline{\eta}}^\bullet} \twoheadrightarrow \Pi_{\mathcal{Y}_s^\bullet}$. Thus, we have

$$g_X + s(D_{\alpha_s}) - 1 = \gamma_{(\alpha_s, Q_s)} = \dim_{k_R}(M_{\mathcal{Y}_s}(1))$$

$$\leq \gamma_{(\alpha_{\overline{\eta}}, Q_{\overline{\eta}})} = \dim_{k_R}(M_{\mathcal{Y}_{\overline{\eta}}}(1)) \leq g_X + s(D_{\alpha_{\overline{\eta}}}) - 1.$$

Since $s(D) = s(D_{\alpha_{\overline{\eta}}}) = s(D_{\alpha_s})$, we obtain that

$$\gamma_{([\mathcal{L}], D)} = g_X + s(D) - 1 = \dim_k(H^1(X, \mathcal{L})).$$

This completes the proof of the proposition. □

The main result of the present paper is as follows.

Theorem 3.6. *Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p and $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$. Let $t \in \mathbb{N}$ be a positive natural number such that $p^t = 1$ in $(\mathbb{Z}/m\mathbb{Z})^{\times}$, $n \overset{\text{def}}{=} p^t - 1$, and $m' \overset{\text{def}}{=} n/m$. Write D' for the divisor $m'D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order m. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet, m}$ and*

$$f^\bullet : Y^\bullet \to X^\bullet$$

the Galois multi-admissible covering with Galois group $\mathbb{Z}/m\mathbb{Z}$ induced by $([\mathcal{L}], D)$. Suppose that $X^\bullet$ is generic. Then $f^\bullet$ is new-ordinary if and only if $D'(jm')$, $j \in \{1, \ldots, m-1\}$, is Frobenius-stable.

Proof. We see immediately that, to verify the proposition, it is sufficient to prove the theorem when $D = D'$.

Since we have the following canonical isomorphism

$$f_* \mathcal{O}_Y \cong \mathcal{O}_X \oplus \bigoplus_{j=1}^{n} \mathcal{L}(j),$$

we obtain that

$$g_Y = g_X + \sum_{j=1}^{n-1} \dim_k(H^1(X, \mathcal{L}(j))).$$

On the other hand, we have

$$\sigma_Y = \sigma_X + \sum_{j=1}^{n-1} \gamma_{([\mathcal{L}(j)], D(j))}$$

and $\gamma_{([\mathcal{L}(j)], D(j))} \leq \dim_k(H^1(X, \mathcal{L}(j)))$ for each $j \in \{1, \ldots, n\}$.

Suppose that $D(j)$, $j \in \{1, \ldots, n\}$, is Frobenius-stable. Then Proposition 3.5 implies that $Y^\bullet$ is ordinary. Conversely, suppose that $Y^\bullet$ is ordinary. Let $j \in \{1, \ldots, n\}$ and $i \in \{0, \ldots, t-1\}$. Then we have $\gamma_{([\mathcal{L}^{(i)}(j)], D^{(i)}(j))} = \dim_k(H^1(X, \mathcal{L}^{(i)}(j)))$. Moreover, we have

$$\gamma_{([\mathcal{L}^{(i)}(j)], D^{(i)}(j))} \leq \dim_k(H^1(X, \mathcal{L}^{(i')}(j)))$$

holds for each $i' \in \{0, \ldots, t-1\}$. This implies that

$$g_X + s(D^{(i)}(j)) - 1 = \dim_k(H^1(X, \mathcal{L}^{(i)}(j))) = \dim_k(H^1(X, \mathcal{L}^{(i')}(j))) = g_X + s(D^{(i')}(j)) - 1$$

holds for each $i' \in \{0, \ldots, t-1\}$. This means that the statement of Lemma 3.3 (i) holds. Then $D(j)$, $j \in \{1, \ldots, n\}$, is Frobenius-stable. □

Remark 3.6.1. Let $0 \leq \sigma \leq g_X$ be an integer, $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ the moduli stack of pointed stable curves of type (g_X, n_X), $\overline{\mathcal{M}}_{g_X,n_X} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{Z}} k$, and $\overline{\mathcal{M}}^{\sigma}_{g_X,n_X}$ the locally closed reduced substack of $\overline{\mathcal{M}}_{g_X,n_X}$ whose points represent pointed stable curves with p-rank σ. Suppose that $X^\bullet$ be a pointed stable curve corresponding to a geometric point over a generic point of $\overline{\mathcal{M}}^{\sigma}_{g_X,n_X}$. Let ℓ be a prime number prime to p. Then E. Ozman and R. Pries proved the following result (cf. [OP, Theorem 1.1]):

> Let $\ell \neq p$ be a prime number and $n_X = 0$. Suppose that $g_X \geq 2$, and that $0 \leq \sigma \leq g_X$ with $\sigma \neq 0$ if $g_X = 2$. Then every Galois admissible covering of $X^\bullet$ (i.e., Galois étale covering of X) with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ is new-ordinary.

By applying Ozman-Pries' result mentioned above and similar arguments to the arguments given in the proof of Theorem 3.6, we may obtained the following result:

> Let $\ell \neq p$ be a prime number. Suppose that $0 \leq \sigma \leq g_X$ with $\sigma \neq 0$ if $g_X = 2$. Let $D \in (\mathbb{Z}/\ell\mathbb{Z})^{\sim}[D_X]^0$. Let $t \in \mathbb{N}$ be a positive natural number such that $p^t = 1$ in $(\mathbb{Z}/\ell\mathbb{Z})^\times$, $n \stackrel{\text{def}}{=} p^t - 1$, and $m' \stackrel{\text{def}}{=} n/\ell$. Write D' for the divisor $m'D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ when we identify $\mathbb{Z}/\ell\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order ℓ. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,\ell}$ and
>
> $$f^\bullet : Y^\bullet \to X^\bullet$$
>
> the Galois multi-admissible covering with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ induced by $([\mathcal{L}], D)$. Suppose that $X^\bullet$ be a pointed stable curve corresponding to a geometric point over a generic point of $\overline{\mathcal{M}}^{\sigma}_{g_X,n_X}$. Then $f^\bullet$ is new-ordinary if and only if $D'(j)$, $j \in \{1, \dots, n\}$, is Frobenius-stable.

Remark 3.6.2. The author is interested in whether or not Theorem 3.6 can be generalized to the case of arbitrary abelian admissible coverings. More precisely, we have the following question:

> **Question:** Suppose that $X^\bullet$ is generic. Let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois multi-admissible covering whose Galois group is abelian. Can we find a necessary and sufficient condition for the ordinariness of Y?

4 Applications

In this section, we give some applications of Theorem 3.6. We maintain the notation introduced in previous sections.

By applying Theorem 3.6, one may construct new-ordinary ramified coverings easily for generic curves. For instance, we have the following proposition.

Proposition 4.1. *Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$. Suppose that $X^\bullet$ is generic. Then the following statements hold.*

(i) Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. Suppose that $n_X \leq 1$. Then there exists a new-ordinary Galois admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that f is étale.

(ii) Let $m \in \mathbb{N}$ be an arbitrary positive natural number prime to p. Suppose that $n_X \geq 2$ is an even number. We put $D_X \stackrel{\text{def}}{=} \{x_1, x_2, \ldots, x_{2d-1}, x_{2d}\}$. Let

$$D \stackrel{\text{def}}{=} \sum_{r=1}^{d} (a_r x_{2r-1} + b_r x_{2r})$$

such that $1 \leq a_r, b_r \leq m-1$ and $a_r + b_r = m$ for each $r \in \{1, \ldots, d\}$. Note that we have $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet, m}$ and

$$f^\bullet : Y^\bullet \to X^\bullet$$

the Galois multi-admissible covering with Galois group $\mathbb{Z}/m\mathbb{Z}$ induced by $([\mathcal{L}], D)$. Then $f^\bullet$ is new-ordinary. In particular, there exists a new-ordinary Galois admissible covering whose Galois group is $\mathbb{Z}/m\mathbb{Z}$, and whose branch locus of f is equal to D_X.

(iii) Let t be an arbitrary positive natural number, and $n \stackrel{\text{def}}{=} p^t - 1$. Suppose that $p \geq 5$, and that $n_X \geq 3$ is an odd number. Then there exists a new-ordinary Galois multi-admissible covering

$$f^\bullet : Y^\bullet \to X^\bullet$$

with Galois group $\mathbb{Z}/n\mathbb{Z}$ such that the branch locus of f is equal to D_X.

Proof. (i) follows immediately from [N, Proposition 4]. Let us prove (ii). In order to verify (ii), we only need to prove that the restriction of $f^\bullet$ on an arbitrary connected component of $Y^\bullet$ is new-ordinary. Let $t' \in \mathbb{N}$ be a positive natural number such that $p^{t'} = 1$ in $(\mathbb{Z}/m\mathbb{Z})^\times$. We put

$$D' \stackrel{\text{def}}{=} m'D = \sum_{r=1}^{d} (m'a_r x_{2r-1} + m'b_r x_{2r}) \in (\mathbb{Z}/n'\mathbb{Z})^{\sim}[D_X]^0$$

when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n'\mathbb{Z}$ of order m, where $n' \stackrel{\text{def}}{=} p^{t'} - 1$ and $m' \stackrel{\text{def}}{=} n'/m$. Note that we have $n' = m'a_r + m'b_r$ for each $r \in \{1, \ldots, r\}$. Moreover, we see immediately that

$$\deg(D'(j)^{(i)}) = dn', \; i \in \{0, \ldots, t'-1\}, \; j \in \{1, \ldots, n'-1\}.$$

This means that $D'(j)$, $j \in \{1, \ldots, n'\}$, is Frobenius-stable.

Since $\mathcal{L}^{\otimes n'} \cong \mathcal{O}_X(-D)^{\otimes m'} \cong \mathcal{O}_X(-D')$, we have $([\mathcal{L}], D') \in \widetilde{\mathscr{P}}_{X^\bullet, n'}$. Let $g^\bullet : Z^\bullet \to X^\bullet$ be the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n'\mathbb{Z}$ corresponding to $([\mathcal{L}], D')$. Then Theorem 3.6 implies that $g^\bullet$ is new-ordinary. Let $W^\bullet$ be an arbitrary connected component of $Z^\bullet$ and

$$g^\bullet|_{W^\bullet} : W^\bullet \to X^\bullet$$

the Galois admissible covering over k induced by $g^\bullet$. Then we have $g^\bullet|_{W^\bullet}$ is new-ordinary. This completes the proof of (ii).

Next, let us prove (iii). Let $n_X = 2d+1$ and $c \overset{\text{def}}{=} n/(p-1)$. We put $D_X \overset{\text{def}}{=} \{x_1, x_2, \ldots, x_{2d-1}, x_{2d}, x_{2d+1}\}$,

$$Q \overset{\text{def}}{=} cx_{2d-1} + cx_{2d} + c(p-3)x_{2d+1},\ Q' \overset{\text{def}}{=} \sum_{r=1}^{d-1}(x_{2r-1} + (n-1)x_{2r}),$$

and

$$D \overset{\text{def}}{=} Q + Q'.$$

Let $j \in \{1, \ldots, n-1\}$. We put $\alpha_j \overset{\text{def}}{=} j - (p-1)[j/(p-1)]$ and $\beta_j \overset{\text{def}}{=} j(p-3) - [j(p-3)/(p-1)]$. Note that we have that $(p-1)|(2\alpha_j + \beta_j)$, and that $2\alpha_j + \beta_j \in \{p-1, 2(p-1)\}$. Then we have

$$jQ = cjx_{2d-1} + cjx_{2d} + cj(p-3)x_{2d+1}$$

$$= c(\alpha_j x_{2d-1} + \alpha_j x_{2d} + \beta_j x_{2d+1}) + (n[j/(p-1)]x_{2d-1} + n[j/(p-1)]x_{2d} + n[j(p-3)/(p-1)]x_{2d+1}).$$

Thus, we have that

$$Q(j) = c\alpha_j x_{2d-1} + c\alpha_j x_{2d} + c\beta_j x_{2d+1}.$$

Then we obtain that

$$\deg(Q(j)^{(i)}) = (2\alpha_j + \beta_j)n/(p-1),\ i \in \{0, \ldots, t-1\}.$$

Thus, we see immediately that $Q(j)$, $j \in \{1, \ldots, p-1\}$, is Frobenius-stable.

On the other hand, we see immediately that

$$\deg(Q'(j)^{(i)}) = dn,\ i \in \{0, \ldots, t-1\},\ j \in \{1, \ldots, n-1\}.$$

This means that $Q'(j)$, $j \in \{1, \ldots, n-1\}$, is Frobenius-stable. Thus, we have that $D(j)$, $j \in \{1, \ldots, n-1\}$, is Frobenius-stable. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet, n}$ and $f^\bullet : Y^\bullet \to X^\bullet$ the Galois multi-admissible covering over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to $([\mathcal{L}], D)$. Then Theorem 3.6 implies that $Y^\bullet$ is ordinary. This completes the proof of (ii). $\square$

Remark 4.1.1. Suppose that $p \leq 3$, and that $n_X \geq 3$ is an odd number. Let $f^\bullet : Y^\bullet \to X^\bullet$ be a Galois admissible covering with Galois group $\mathbb{Z}/(p-1)\mathbb{Z}$ over k. Then we see immediately that the cardinality of the branch locus of f is $\leq n_X - 1$. Thus, (iii) does not hold when $p \leq 3$.

Next, we consider an inverse Galois problem for $X^\bullet$. Let m be an arbitrary positive natural number prime to p, $n' \overset{\text{def}}{=} p^{t'} - 1$ such that $m|n'$, and $m' \overset{\text{def}}{=} n'/m$. Let G be a finite group which is an extension of $H = \mathbb{Z}/m\mathbb{Z}$ by a p-group P. Then we have

$$G \cong P \rtimes H.$$

Write $\Phi(P) \overset{\text{def}}{=} P^p[P, P]$ for the Frattini subgroup of P and $\overline{P} \overset{\text{def}}{=} P/\Phi(P)$. We put $\overline{G} \overset{\text{def}}{=} G/\Phi(P)$. Then we obtain a $\mathbb{F}_p$-representation

$$\rho : H \to \mathrm{Aut}(\overline{P}).$$

Let $Z(H)$ be the set of irreducible characters of H with values in k and ζ_m a primitive mth root and $\chi_i,\ i \in \mathbb{Z}/m\mathbb{Z}$, the irreducible character such that $\chi_i(1) = \zeta_m^i$. Then we see immediately that $Z(H) = \{\chi_i\}_{i\in\mathbb{Z}/m\mathbb{Z}}$. Let $\rho_{\chi_i} : H \to \mathrm{GL}(\overline{P}_{\chi_i})$ be an irreducible k-representation of H of character χ_i of degree 1. The canonical decomposition of $\overline{P}\otimes_{\mathbb{F}_p} k$ as a $k[H]$-module is given by

$$\overline{P} \otimes_{\mathbb{F}_p} k = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} \overline{P}_{\chi_i}^{m_{\chi_i}}.$$

Then we have the following result.

Proposition 4.2. *We maintain the notation introduced above. Let $X^\bullet$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$. Suppose that $X^\bullet$ is generic. Let $D \in (\mathbb{Z}/m\mathbb{Z})^{\sim}[D_X]^0$. Write D' for the divisor $m'D \in (\mathbb{Z}/n\mathbb{Z})^{\sim}[D_X]^0$ when we identify $\mathbb{Z}/m\mathbb{Z}$ with the unique subgroup of $\mathbb{Z}/n'\mathbb{Z}$ of order m. Let $([\mathcal{L}], D) \in \widetilde{\mathscr{P}}_{X^\bullet,m}$ and $\alpha \in \mathrm{Hom}(\Pi_{X^\bullet}^{\mathrm{ab}}, \mathbb{Z}/m\mathbb{Z})$ the element induced by $([\mathcal{L}], D)$. We put*

$$\phi : \Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{ab}} \xrightarrow{\alpha} H = \mathbb{Z}/m\mathbb{Z}.$$

Suppose that $D'(im')$, $i \in \{1, \ldots, m-1\}$, is Frobenius-stable. Then an embedding problem $(\phi : \Pi_{X^\bullet} \to H, G \twoheadrightarrow H)$ has a solution if and only if

$$m_{\chi_i} \leq \begin{cases} g_X, & \text{if } i = 0, \\ g_X + s(D'(im')) - 1, & \text{if } i \in \{1, \ldots, m-1\}. \end{cases}$$

Proof. The proposition follows immediately from Theorem 3.6 and [B, Proposition 2.4 and Proposition 2.5]. □

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