On the Existence of Specialization Isomorphisms of Admissible Fundamental Groups in Positive Characteristic

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Abstract

Let p be a prime number, and let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over an algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ parameterizing pointed stable curves of type (g,n), $\mathcal{M}_{g,n}$ the open substack of $\overline{\mathcal{M}}_{g,n}$ parameterizing smooth pointed stable curves, $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$, $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$, $q \in \overline{M}_{g,n}$ an arbitrary point, and Π_q the admissible fundamental group of a pointed stable curve corresponding to a geometric point over q. In the present paper, we prove that, there exists $q_1, q_2 \in \overline{M}_{g,n} \setminus M_{g,n}$ such that q_1 is a specialization of q_2, that $q_1 \neq q_2$, and that a specialization homomorphism $sp : \Pi_{q_2} \twoheadrightarrow \Pi_{q_1}$ is an isomorphism.

Keywords: pointed stable curve, admissible fundamental group, specialization homomorphism, moduli space, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14F35, 14G32.

Contents

1 Introduction

Let p be a prime number, $\overline{\mathbb{F}}_p$ an algebraic closure of the finite field $\mathbb{F}_p$, $\overline{\mathcal{M}}_{g,n}$ the moduli stack over $\overline{\mathbb{F}}_p$ parameterizing pointed stable curves of type (g,n), $\mathcal{M}_{g,n}$ the open substack of $\overline{\mathcal{M}}_{g,n}$ parameterizing smooth pointed stable curves, $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$, and $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$. Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of q and $k(q) \subseteq k$ an algebraically closed field. Then the natural morphism $\operatorname{Spec} k \to \overline{M}_{g,n}$ determines a pointed stable curve $X_q^\bullet$ of type (g,n) over k. We denote by $\Pi_{X_q^\bullet}$ the admissible fundamental group of $X_q^\bullet$ (cf. Definition 2.2). Note that $\Pi_{X_q^\bullet}$ is isomorphic to the tame fundamental group of $X_q^\bullet$ when $q \in M_{g,n}$. Since the isomorphism class of $\Pi_{X_q^\bullet}$ does not depend on the choices of base field k, we may write Π_q for $\Pi_{X_q^\bullet}$.

The admissible fundamental group Π_q is very mysterious. In fact, some developments of F. Pop-M. Saïdi, M. Raynaud, A. Sarashina, A. Tamagawa, and the author (cf. [PS], [R], [Sa], [T1], [T2], [T3], [T4], [Y1], [Y2], [Y3]) showed evidence for very strong anabelian phenomena for curves over algebraically closed fields of characteristic $p > 0$. In this situation, the Galois group of the base field is trivial, and the admissible fundamental group coincides with the geometric fundamental group, thus in a total absence of a Galois action of the base field. This kind of anabelian phenomena go beyond Grothendieck's anabelian geometry, and shows that the admissible fundamental group of a pointed stable curve over an algebraically closed field must encode "moduli" of the curve. This is the reason that we do not have an explicit description of the admissible fundamental group of any pointed stable curve in positive characteristic.

Let $q_1, q_2 \in \overline{M}_{g,n}$ such that q_1 is a specialization of q_2, and that $q_1 \neq q_2$. Then we have a (surjective) specialization homomorphism of admissible fundamental groups

$$sp : \Pi_{q_2} \twoheadrightarrow \Pi_{q_1}.$$

We consider the following question.

Question 1.1. *Whether or not sp is an isomorphism?*

First, we have the following result.

Theorem 1.2. *We maintain the notation introduced above. Suppose that q_1 is a closed point of $\overline{M}_{g,n}$. Let $sp : \Pi_{q_2} \twoheadrightarrow \Pi_{q_1}$ be a specialization homomorphism of admissible fundamental groups. Then sp is not an isomorphism.*

Remark 1.2.1. Suppose that $q_1, q_2 \in M_{g,n}$. Then Theorem 1.2 was proved by Pop-Saïdi, Raynaud in the special case (cf. [PS], [R]), and was proved by Tamagawa in the general case (cf. [T4]). Moreover, the author generalized Tamagawa's result to the case where q_1 and q_2 are arbitrary points of $\overline{M}_{g,n}$ (cf. [Y1]).

Question 1.1 is motivated by the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. Let us explain this from the point of view of moduli spaces. Let $q_i \in \overline{M}_{g,n}$, $i \in \{1, 2\}$, k_{q_i} an algebraic closure of the residue field of q_i, and $X^\bullet_{k_{q_i}}$ the pointed stable curve over k_{q_i} determined by the natural morphism $\operatorname{Spec} k_{q_i} \to \overline{\mathcal{M}}_{g,n}$. We define an equivalence relation $\sim_f$ on $\overline{M}_{g,n}$ as follows: $q_1 \sim_f q_2$ if $X^\bullet_{k_{q_1}} \cong X^\bullet_{k_{q_2}}$ as schemes. Let $\overline{\Pi}_{g,n}$ and $\Pi_{g,n}$ be the set of isomorphism classes of admissible fundamental groups of pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$ and the set of isomorphism classes of admissible (or tame) fundamental groups of smooth pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$, respectively. Write $|\overline{M}_{g,n}|$ and $|M_{g,n}|$ for the underlying topology spaces of $\overline{M}_{g,n}$ and $M_{g,n}$, $|\overline{M}_{g,n}|_f$ and $|M_{g,n}|_f$ for the quotient spaces $|\overline{M}_{g,n}|/\sim_f$ and $|M_{g,n}|/\sim_f$, respectively. Then we have two natural maps

$\pi_1^{\mathrm{adm}}: |\overline{M}_{g,n}|_f \to \overline{\Pi}_{g,n},\ [q] \mapsto [\Pi_q]$, and $\pi_1^{\mathrm{tame}}: |M_{g,n}|_f \to \Pi_{g,n},\ [q] \mapsto [\Pi_q]$, which fit into the following commutative diagram

$$\begin{array}{ccc} |M_{g,n}|_f & \xrightarrow{\pi_1^{\mathrm{tame}}} & \Pi_{g,n} \\ \downarrow & & \downarrow \\ |\overline{M}_{g,n}|_f & \xrightarrow{\pi_1^{\mathrm{adm}}} & \overline{\Pi}_{g,n}, \end{array}$$

where $[q]$ denotes the image of q of the natural morphism of topology spaces $|\overline{M}_{g,n}| \twoheadrightarrow |\overline{M}_{g,n}|_f$, and $[\Pi_q]$ denotes the isomorphism class of Π_q. Note that $\pi_1^{\mathrm{adm}}|_{|M_{g,n}|_f} = \pi_1^{\mathrm{tame}}$. At the end of the 1990s, Tamagawa formulated the weak Isom-version of the Grothendieck conjecture for *smooth* pointed stable curves over algebraically closed fields of characteristic $p > 0$ (=the Weak Isom-version Conjecture for smooth pointed stable curves) as follows:

Conjecture 1.3. *The natural map* $\pi_1^{\mathrm{tame}}: |M_{g,n}|_f \to \Pi_{g,n}$ *is a bijection.*

The bijectivity of π_1^{tame} is very difficult. At the present, Conjecture 1.3 has been proved only in some special cases (cf. [T1], [T2], [Sa]). On the other hand, a natural question is how to formulate the Weak Isom-version Conjecture for *arbitrary* pointed stable curves. In the present paper, we introduce an equivalence relation $\sim_{fe}$ on $\overline{M}_{g,n}$ which is called *Frobenius equivalence*, and which coincides with $\sim_f$ when the points are contained in $M_{g,n}$ (cf. Definition 3.4). Then Proposition 3.7 shows that π_1^{adm} factors through $|\overline{M}_{g,n}|_{fe} \overset{\mathrm{def}}{=} |\overline{M}_{g,n}|/\sim_{fe}$. We put

$$\pi_{1,fe}^{\mathrm{adm}}: |\overline{M}_{g,n}|_{fe} \to \overline{\Pi}_{g,n},\ [q] \mapsto [\Pi_q],$$

where $[q]$ denotes the image of q of the natural morphism $|\overline{M}_{g,n}| \to |\overline{M}_{g,n}|_{fe}$. We obtain the following commutative diagram:

$$\begin{array}{ccc} |M_{g,n}|_f & \xrightarrow{\pi_1^{\mathrm{tame}}} & \Pi_{g,n} \\ \downarrow & & \downarrow \\ |\overline{M}_{g,n}|_{fe} & \xrightarrow{\pi_{1,fe}^{\mathrm{adm}}} & \overline{\Pi}_{g,n}. \end{array}$$

Then the Weak Isom-version Conjecture for arbitrary pointed stable curves can be formulated as follows, which generalizes Conjecture 1.3:

Conjecture 1.4. *The natural map* $\pi_{1,fe}^{\mathrm{adm}}: |\overline{M}_{g,n}|_{fe} \to \overline{\Pi}_{g,n}$ *is a bijection.*

Moreover, we have the following result (cf. Theorem 3.8):

Theorem 1.5. *Conjecture 1.3 is equivalent to Conjecture 1.4.*

On the other hand, as a direct consequence of Theorem 1.2, we have the following finiteness theorem (cf. [T4, Theorem 0.1], [Y1, Theorem 1.3 (b)]):

Let $q \in \overline{M}_{g,n}$ be a *closed* point. Then we have

$$\#(\pi_1^{\mathrm{adm}})^{-1}([\Pi_q]) < \infty.$$

This means that there are only finitely many $\overline{\mathbb{F}}_p$-isomorphism classes of pointed stable curves over $\overline{\mathbb{F}}_p$ whose admissible fundamental groups are isomorphic to Π_q.

If Theorem 1.2 can be generalized to the case of *arbitrary* points of $\overline{M}_{g,n}$, then the finiteness theorem holds for arbitrary points of $\overline{M}_{g,n}$. Question 1.1 for arbitrary points of $\overline{M}_{g,n}$ is one of main open problems in the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. At the time of writing, nothing is known about Question 1.1 if q_1 is not a closed point of $\overline{M}_{g,n}$.

In the present paper, we consider Question 1.1 when $q_1, q_2 \in \overline{M}_{g,n} \setminus M_{g,n}$. The main result of the present paper is as follows (see Theorem 4.3 for precise form):

Theorem 1.6. *Suppose that* $\dim(\overline{M}_{g,n}) \geq 3$. *There exist* $q_1, q_2 \in \overline{M}_{g,n} \setminus M_{g,n}$ *such that* q_1 *is a specialization of* q_2*, that* $q_1 \neq q_2$*, and that a specialization homomorphism* $sp : \Pi_{q_2} \twoheadrightarrow \Pi_{q_1}$ *is an isomorphism.*

Theorem 1.6 implies that Theorem 1.2 does not hold for arbitrary points of $\overline{M}_{g,n}$ in general. Moreover, we see that there exists a point $q \in \overline{M}_{g,n}$ such that

$$\#(\pi_1^{\mathrm{adm}})^{-1}([\Pi_q]) = \infty.$$

This means that the finiteness theorem does not hold for $|\overline{M}_{g,n}|_f$ (cf. Remark 4.4). On the other hand, we may ask whether or not the following generalized version of finiteness theorem hold.

Question 1.7. *Let* $q \in \overline{M}_{g,n}$. *Does* $\#(\pi_{1,fe}^{\mathrm{adm}})^{-1}([\Pi_q]) < \infty$ *hold?*

The present paper is organized as follows. In Section 2, we give some definitions which will be used in the present paper. In Section 3, we introduce an equivalence relation on $\overline{M}_{g,n}$ which is called Frobenius equivalence. Moreover, we prove Theorem 1.5 . In Section 4, by applying the results obtained in Section 3, we prove Theorem 1.6.

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2 Admissible coverings and admissible fundamental groups

In this section, we recall some definitions concerning admissible fundamental groups which will be used in the present paper.

Definition 2.1. Let $\mathbb{G} \stackrel{\text{def}}{=} (v(\mathbb{G}), e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})})$ be a semi-graph (cf. [Y4, Definition 2.1]). Here, $v(\mathbb{G})$, $e^{\text{op}}(\mathbb{G})$, $e^{\text{cl}}(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of open edges of $\mathbb{G}$, the set of closed edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively. Note that, for each $e \in e^{\text{op}}(\mathbb{G}) \cup e^{\text{cl}}(\mathbb{G})$, $e \stackrel{\text{def}}{=} \{b_e^1, b_e^2\}$ is a set of cardinality 2. Then e is a closed edge if $\zeta_e^{\mathbb{G}}(e) \subseteq v(\mathbb{G})$, and e is an open edge if $\zeta_e^{\mathbb{G}}(e) = \{\zeta_e^{\mathbb{G}}(e) \cap v(\mathbb{G}), \{v(\mathbb{G})\}\}$.

We define an *one-point compactification* $\mathbb{G}^{\text{cpt}}$ of $\mathbb{G}$ as follows: if $e^{\text{op}}(\mathbb{G}) = \emptyset$, we set $\mathbb{G}^{\text{cpt}} = \mathbb{G}$; otherwise, the set of vertices of $\mathbb{G}^{\text{cpt}}$ is $v(\mathbb{G}^{\text{cpt}}) \stackrel{\text{def}}{=} v(\mathbb{G}) \coprod \{v_\infty\}$, the set of closed edges of $\mathbb{G}^{\text{cpt}}$ is $e^{\text{cl}}(\mathbb{G}^{\text{cpt}}) \stackrel{\text{def}}{=} e^{\text{cl}}(\mathbb{G}) \cup e^{\text{op}}(\mathbb{G})$, the set of open edges of $\mathbb{G}$ is empty, and each edge $e \in e^{\text{op}}(\mathbb{G}) \subseteq e(\mathbb{G}^{\text{cpt}})$ connects v_∞ with the vertex that is abutted by e.

Let $v \in v(\mathbb{G})$. We shall say that $\mathbb{G}$ is 2-*connected* at v if $\mathbb{G} \setminus \{v\}$ is either empty or connected. Moreover, we shall say that $\mathbb{G}$ is 2-connected if $\mathbb{G}$ is 2-connected at each $v \in v(\mathbb{G})$. Note that, if $\mathbb{G}$ is connected, then $\mathbb{G}^{\text{cpt}}$ is 2-connected at each $v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\text{cpt}})$ if and only if $\mathbb{G}^{\text{cpt}}$ is 2-connected.

Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve over an algebraically closed field k of characteristic $p > 0$, where X denotes the underlying curve and D_X denotes the set of marked points. Write g_X for the genus of X and n_X for the cardinality $\#D_X$ of D_X. We shall say that the pair (g_X, n_X) is the topological type (or type for short) of $X^\bullet$. Write $\Gamma_{X^\bullet}$ for the dual semi-graph of $X^\bullet$ and $r_X \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(H^1(\Gamma_{X^\bullet}, \mathbb{Q}))$ for the Betti number of the semi-graph $\Gamma_{X^\bullet}$. Let $v \in v(\Gamma_{X^\bullet})$ and $e \in e^{\text{op}}(\Gamma_{X^\bullet}) \cup e^{\text{cl}}(\Gamma_{X^\bullet})$. We write X_v for the irreducible component of X corresponding to v, write x_e for the node of X corresponding to e if $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, and write x_e for the marked point of X corresponding to e if $e \in e^{\text{op}}(\Gamma_{X^\bullet})$. Moreover, write $\widetilde{X}_v$ for the *smooth* compactification of $U_{X_v} \stackrel{\text{def}}{=} X_v \setminus X_v^{\text{sing}}$, where $(-)^{\text{sing}}$ denotes the singular locus of $(-)$. We define a smooth pointed stable curve of type (g_v, n_v) over k to be

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \stackrel{\text{def}}{=} (\widetilde{X}_v \setminus U_{X_v}) \cup (D_X \cap X_v)).$$

We shall say $\widetilde{X}_v^\bullet$ the smooth pointed stable curve associated to v.

Definition 2.2. Let $Y^\bullet = (Y, D_Y)$ be a pointed stable curve over k, $f^\bullet : Y^\bullet \to X^\bullet$ a morphism of pointed stable curves over k, and $f : Y \to X$ the morphism of underlying curves induced by $f^\bullet$.

We shall say $f^\bullet$ a *Galois admissible covering* over k (or Galois admissible covering for short) if the following conditions are satisfied: (i) There exists a finite group $G \subseteq \text{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$. (ii) For each $y \in Y^{\text{sm}} \setminus D_Y$, f is étale at y, where $(-)^{\text{sm}}$ denotes the smooth locus of $(-)$. (iii)

For any $y \in Y^{\mathrm{sing}}$, the image $f(y)$ is contained in X^{sing}, where $(-)^{\mathrm{sing}}$ denotes the set of singular points of $(-)$. (iv) For each $y \in Y^{\mathrm{sing}}$, we write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y. Then we have that $(\#D_y, \mathrm{char}(k)) = 1$, and that the local morphism between two nodes induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[u,v]]/uv & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^{\#D_y} \\ v & \mapsto & t^{\#D_y}. \end{array}$$

Moreover, we have that $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta_{\#D_y}^{-1} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit, and $\#(-)$ denotes the cardinality of $(-)$. (v) The local morphism between two marked points induced by f may be described as follows:

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X,f(y)} \cong k[[a]] & \to & \widehat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m, \mathrm{char}(k)) = 1$ (i.e., a tamely ramified extension).

Moreover, we shall say $f^\bullet$ an *admissible covering* if there exists a morphism of pointed stable curves $h^\bullet : W^\bullet \to Y^\bullet$ over k such that the composite morphism $f^\bullet \circ h^\bullet : W^\bullet \to X^\bullet$ is a Galois admissible covering over k. We shall say an admissible covering $f^\bullet$ *étale* if f is an étale morphism.

Let $Z^\bullet$ be a disjoint union of finitely many pointed stable curves over k. We shall say that a morphism $f_Z^\bullet : Z^\bullet \to X^\bullet$ over k is a *multi-admissible covering* if the restriction of $f_Z^\bullet$ to each connected component of $Z^\bullet$ is admissible. For any category $\mathscr{C}$, we write $\mathrm{Ob}(\mathscr{C})$ for the class of objects of $\mathscr{C}$, and write $\mathrm{Hom}(\mathscr{C})$ for the class of morphisms of $\mathscr{C}$. We denote by

$$\mathrm{Cov}^{\mathrm{adm}}(X^\bullet) \overset{\mathrm{def}}{=} (\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)), \mathrm{Hom}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)))$$

the category which consists of the following data: (i) $\mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$ consists of an empty object and all the pairs $(Z^\bullet, f_Z^\bullet : Z^\bullet \to X^\bullet)$, where $Z^\bullet$ is a disjoint union of finitely many pointed stable curves over k, and $f_Z^\bullet$ is a multi-admissible covering over k; (ii) for any $(Z^\bullet, f_Z^\bullet)$, $(Y^\bullet, f_Y^\bullet) \in \mathrm{Ob}(\mathrm{Cov}^{\mathrm{adm}}(X^\bullet))$, we define

$$\mathrm{Hom}((Z^\bullet, f_Z^\bullet), (Y^\bullet, f_Y^\bullet)) \overset{\mathrm{def}}{=} \{g^\bullet \in \mathrm{Hom}_k(Z^\bullet, Y^\bullet) \mid f_Y^\bullet \circ g^\bullet = f_Z^\bullet\},$$

where $\mathrm{Hom}_k(Z^\bullet, Y^\bullet)$ denotes the set of k-morphisms of pointed stable curves. It is well known that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\mathrm{sm}} \backslash D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the *admissible fundamental group* of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by

$$\Pi_{X^\bullet}.$$

Remark 2.2.1. Suppose that $X^\bullet$ is smooth over k. By the definition of admissible fundamental groups, the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X^\bullet$.

Remark 2.2.2. The structure of maximal pro-prime-to-p quotient of $\Pi_{X^\bullet}$ is well-known, which is isomorphic to the pro-prime-to-p completion of the following group (cf. [V, Théorème 2.2 (c)])

$$\langle a_1, \dots, a_{g_X}, b_1, \dots, b_{g_X}, c_1, \dots, c_{n_X} \mid \prod_{i=1}^{g_X} [a_i, b_i] \prod_{j=1}^{n_X} c_j = 1 \rangle.$$

3 Frobenius equivalence and admissible fundamental groups

We maintain the notation introduced in Section 2. Let $X^\bullet$ be a pointed stable curve over an algebraically closed field k of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$.

Definition 3.1. Let Π be an arbitrary profinite group and $m, n \in \mathbb{N}$ positive natural numbers. We define the closed normal subgroup

$$D_n(\Pi)$$

of Π to be the topological closure of $[\Pi, \Pi]\Pi^n$, where $[\Pi, \Pi]$ denotes the commutator subgroup of Π. Moreover, we define the closed normal subgroup

$$D_n^{(m)}(\Pi)$$

of Π inductively by $D_n^{(1)}(\Pi) \overset{\text{def}}{=} D_n(\Pi)$ and $D_n^{(i+1)}(\Pi) \overset{\text{def}}{=} D_n(D_n^{(i)}(\Pi))$, $i \in \{1, \dots, m-1\}$. Note that $\#(\Pi/D_n^{(m)}(\Pi)) \leq \infty$ when Π is topologically finitely generated.

Lemma 3.2. *Let n be a positive natural number prime to p and*

$$f^\bullet : Y^\bullet \to X^\bullet$$

the Galois admissible covering over k induced by the open normal subgroup $D_n^{(3)}(\Pi_{X^\bullet}) \subseteq \Pi_{X^\bullet}$. Write $\Gamma_{Y^\bullet}$ for the dual semi-graph of $Y^\bullet$. Then $\Gamma_{Y^\bullet}^{\text{cpt}}$ is 2-connected.

Proof. The lemma follows immediately from the structures of the maximal pro-prime-to-p quotients of the admissible fundamental groups of smooth pointed stable curves and the definition of Galois admissible coverings. □

Lemma 3.3. *Let n be a positive natural number prime to p and*

$$f^\bullet : Y^\bullet \to X^\bullet$$

the Galois admissible covering over k induced by the open normal subgroup $D_n(\Pi_{X^\bullet}) \subseteq \Pi_{X^\bullet}$. Let $\Gamma_{Y^\bullet}$ be the dual semi-graph of $Y^\bullet$, $w \in v(\Gamma_{Y^\bullet})$, v the image of w of the morphism $\Gamma_{Y^\bullet} \to \Gamma_{X^\bullet}$ of dual semi-graphs induced by $f^\bullet$, and $\widetilde{f}^\bullet_{w,v} : \widetilde{Y}^\bullet_w \to \widetilde{X}^\bullet_v$ the Galois admissible covering of the smooth pointed stable curves associated to w and v over k induced by $f^\bullet$. Suppose that $\Gamma_{X^\bullet}^{\text{cpt}}$ is 2-connected. Then the ramification index of $\widetilde{f}^\bullet_{w,v}$ over every point of $D_{\widetilde{X}_v}$ is divided by n.

Proof. Suppose that $X^\bullet$ is smooth over k. Then the lemma follows immediately from the structures of the maximal pro-prime-to-p quotients of the admissible fundamental groups of smooth pointed stable curves.

Suppose that $X^\bullet$ is singular. Write $\Pi_{\widetilde{X}_v^\bullet}$ for the admissible fundamental group of $\widetilde{X}_v^\bullet$. Since $\Gamma_{X^\bullet}^{\mathrm{cpt}}$ is 2-connected, [Y4, Corollary 3.5] implies that the natural morphism

$$\Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}} \to \Pi_{X^\bullet}^{\mathrm{ab}}$$

is injective. Since $\Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}}$ and $\Pi_{X^\bullet}^{\mathrm{ab}}$ are free abelian groups, we have

$$\Pi_{\widetilde{X}_v^\bullet}/D_n(\Pi_{\widetilde{X}_v^\bullet}) = \Pi_{\widetilde{X}_v^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z} \hookrightarrow \Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z} = \Pi_{X^\bullet}^{\mathrm{ab}}/D_n(\Pi_{X^\bullet}^{\mathrm{ab}}).$$

Note that

$$\widetilde{f}_{w,v}^\bullet : \widetilde{Y}_w^\bullet \to \widetilde{X}_v^\bullet$$

is the Galois admissible covering over k induced by the open normal subgroup $D_n(\Pi_{\widetilde{X}_v^\bullet}) \subseteq \Pi_{\widetilde{X}_v^\bullet}$. Thus, the lemma follows immediately from the lemma when $X^\bullet$ is smooth over k. This completes the proof of the lemma. □

Definition 3.4. Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve over an algebraically closed field k_i of characteristic $p > 0$ and $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$.

(i) Suppose that $X_i^\bullet$, $i \in \{1, 2\}$, is smooth over k_i. Let $U_{X_i} \overset{\mathrm{def}}{=} X_i \setminus D_{X_i}$ and $U_{X_i}^{\min}$ the minimal model of U_{X_i} (cf. [T2, Definition 1.30 and Lemma 1.31]). We shall say that $X_1^\bullet$ is *Frobenius equivalent* to $X_2^\bullet$ if $U_{X_1}^{\min}$ is isomorphic to $U_{X_2}^{\min}$ as schemes.

(ii) We shall say that $X_1^\bullet$ is *Frobenius equivalent* to $X_2^\bullet$ if the following conditions are satisfied: (a) There exists an isomorphism $\rho : \Gamma_{X_1^\bullet} \overset{\sim}{\to} \Gamma_{X_2^\bullet}$ of dual semi-graphs. (b) Let $v_1 \in v(\Gamma_{X_1^\bullet})$, $v_2 \overset{\mathrm{def}}{=} \rho(v_1) \in v(\Gamma_{X_2^\bullet})$, $\widetilde{X}_{1,v_1}^\bullet$ the smooth pointed stable curve associated to v_1, and $\widetilde{X}_{2,v_2}^\bullet$ the smooth pointed stable curve associated to v_2. We have that $\widetilde{X}_{1,v_1}^\bullet$ is Frobenius equivalent to $\widetilde{X}_{2,v_2}^\bullet$. (c) Let $\Gamma_{\widetilde{X}_{i,v_i}^\bullet}$, $i \in \{1, 2\}$, be the dual semi-graph of $\widetilde{X}_{i,v_i}^\bullet$ and $\widetilde{\rho}_{v_1,v_2} : \Gamma_{\widetilde{X}_{1,v_1}^\bullet} \overset{\sim}{\to} \Gamma_{\widetilde{X}_{2,v_2}^\bullet}$ the isomorphism of dual semi-graphs induced by ρ. Let $\Pi_{\widetilde{X}_{i,v_i}^\bullet}$, $i \in \{1, 2\}$, be the admissible fundamental group of $\widetilde{X}_{i,v_i}^\bullet$. There exists an isomorphism $\phi_{v_1,v_2} : \Pi_{\widetilde{X}_{1,v_1}^\bullet} \overset{\sim}{\to} \Pi_{\widetilde{X}_{2,v_2}^\bullet}$ such that the isomorphism of dual semi-graphs $\Gamma_{\widetilde{X}_{1,v_1}^\bullet} \overset{\sim}{\to} \Gamma_{\widetilde{X}_{2,v_2}^\bullet}$ induced by ϕ_{v_1,v_2} (cf. [T3, Theorem 5.2] or [Y1, Theorem 1.2 and Remark 1.2.1]) coincides with $\widetilde{\rho}_{v_1,v_2}$.

Lemma 3.5. *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve over an algebraically closed field k_i of characteristic $p > 0$ and $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ a Galois étale covering over k_1 with Galois group G. Suppose that $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$. Then there exists a Galois étale covering $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ over k_2 with Galois group G such that $Y_1^\bullet$ is Frobenius equivalent to $Y_2^\bullet$.*

Proof. The lemma follows immediately from the definition of Frobenius equivalent and Galois admissible coverings. □

Lemma 3.6. *Let $X_i^\bullet$, $i \in \{1,2\}$, be a pointed stable curve over an algebraically closed field k_i of characteristic $p>0$ and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Let n be a positive natural number prime to p and*

$$f_i^\bullet : Y_i^\bullet \to X_i^\bullet,\ i \in \{1,2\},$$

the Galois admissible covering over k_i induced by the open normal subgroup $D_n(\Pi_{X_i^\bullet})$. Suppose that $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$. Then $Y_1^\bullet$ is Frobenius equivalent to $Y_2^\bullet$.

Proof. Let $i \in \{1,2\}$, and let $\Gamma_{Y_i^\bullet}$ be the dual semi-graph of $Y_i^\bullet$, $w_i \in v(\Gamma_{Y_i^\bullet})$, v_i the image of w_i of the morphism $\Gamma_{Y_i^\bullet} \to \Gamma_{X_i^\bullet}$ induced by $f_i^\bullet$, and $\widetilde{f}^\bullet_{w_i,v_i} : \widetilde{Y}^\bullet_{i,w_i} \to \widetilde{X}^\bullet_{i,v_i}$ the Galois admissible covering of the smooth pointed stable curves associated to w_i and v_i over k_i induced by $f_i^\bullet$. Write $\Pi_{X_i^\bullet}$ and $\Pi_{\widetilde{X}^\bullet_{i,v_i}}$ for the admissible fundamental groups of $X_i^\bullet$ and $\widetilde{X}^\bullet_{i,v_i}$, respectively. We have a natural homomorphism

$$\phi^{\mathrm{ab}}_{i,v_i} : \Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_{i,v_i}} \to \Pi^{\mathrm{ab}}_{X_i^\bullet}.$$

We write K_{i,v_i} for $\ker(\phi^{\mathrm{ab}}_{i,v_i})$. On the other hand, since $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$, there exist an isomorphism $\rho : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$ and an isomorphism $\phi_{v_1,\rho(v_1)} : \Pi_{\widetilde{X}^\bullet_{1,v_1}} \xrightarrow{\sim} \Pi_{\widetilde{X}^\bullet_{2,v_2}}$ satisfying the conditions mentioned in Definition 3.4 (ii). Suppose that $v_2 = \rho(v_1)$. We see that, to verify the lemma, it is sufficient to prove that $\widetilde{Y}^\bullet_{1,w_1}$ is Frobenius equivalent to $\widetilde{Y}^\bullet_{2,w_2}$.

We denote by N_{i,v_i} the kernel of the composition of natural homomorphisms

$$\Pi_{\widetilde{X}^\bullet_{i,v_i}} \twoheadrightarrow \Pi_{\widetilde{X}^\bullet_{i,v_i}}/D_n(\Pi_{\widetilde{X}^\bullet_{i,v_i}}) = \Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_{i,v_i}} \otimes \mathbb{Z}/n\mathbb{Z} \to \Pi^{\mathrm{ab}}_{X_i^\bullet} \otimes \mathbb{Z}/n\mathbb{Z} = \Pi_{X_i^\bullet}/D_n(\Pi_{X_i^\bullet}).$$

Then N_{i,v_i} is equal to the inverse image of $K_{i,v_i} \otimes \mathbb{Z}/n\mathbb{Z}$ of the natural homomorphism $\Pi_{\widetilde{X}^\bullet_{i,v_i}} \twoheadrightarrow \Pi^{\mathrm{ab}}_{\widetilde{X}^\bullet_{i,v_i}} \otimes \mathbb{Z}/n\mathbb{Z}$. Let $\Gamma_{\widetilde{X}^\bullet_{i,v_i}}$ be the semi-graph of $\widetilde{X}^\bullet_{i,v_i}$ and $\widetilde{\gamma}_{i,v_i} : \Gamma_{\widetilde{X}^\bullet_{i,v_i}} \to \Gamma_{X_i^\bullet}$ the natural morphism of the dual semi-graphs of $\widetilde{X}^\bullet_{i,v_i}$ and $X_i^\bullet$. By applying [Y4, Proposition 3.4] and [T3, Theorem 5.2] (or [Y1, Theorem 1.2 and Remark 1.2.1]), we see that K_{i,v_i} depends only on the isomorphism class of $\Pi_{\widetilde{X}^\bullet_{i,v_i}}$ and the morphism of dual semi-graphs $\widetilde{\gamma}_{i,v_i}$. Then $N_{1,v_1} = \phi^{-1}_{v_1,v_2}(N_{2,v_2})$. Note that $\widetilde{f}^\bullet_{w_i,v_i} : \widetilde{Y}^\bullet_{i,w_i} \to \widetilde{X}^\bullet_{i,v_i}$ is the Galois admissible covering over k_i induced by the open normal subgroup $N_{v_i} \subseteq \Pi_{\widetilde{X}^\bullet_{v_i}}$. Since $\widetilde{X}^\bullet_{1,v_1}$ is Frobenius equivalent to $\widetilde{X}^\bullet_{2,v_2}$, we obtain that $\widetilde{Y}^\bullet_{1,w_1}$ is Frobenius equivalent to $\widetilde{Y}^\bullet_{2,w_2}$. □

Next, we prove the main result of this section.

Proposition 3.7. *Let $X_i^\bullet$, $i \in \{1,2\}$, be a pointed stable curve over an algebraically closed field k_i of characteristic $p>0$ and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Suppose that $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$. Then $\Pi_{X_1^\bullet}$ is isomorphic to $\Pi_{X_2^\bullet}$.*

Proof. Let $i \in \{1,2\}$. Write $\pi_A(X_i^\bullet)$ for the set of finite quotients of $\Pi_{X_i^\bullet}$. [FJ, Proposition 16.10.6] implies that $\Pi_{X_1^\bullet}$ is isomorphic to $\Pi_{X_2^\bullet}$ if and only if $\pi_A(X_1^\bullet) = \pi_A(X_2^\bullet)$. Let $G \in \pi_A(X_1^\bullet)$. We will prove $G \in \pi_A(X_2^\bullet)$.

Let G_p be a Sylow-p subgroup of G and $n \stackrel{\text{def}}{=} [G : G_p]$. Let

$$f_1^\bullet : Y_1^\bullet \to X_1^\bullet$$

be a Galois admissible covering over k_1 with Galois group G. We put

$$Q_i \stackrel{\text{def}}{=} D_n^{(4)}(\Pi_{X_i^\bullet}).$$

Moreover, let

$$g_i^\bullet : Z_i^\bullet \to X_i^\bullet$$

be the Galois admissible covering over k_i with Galois group Q_i. By applying Lemma 3.6, we have that $Z_1^\bullet$ is Frobenius equivalent to $Z_2^\bullet$. Write $\Gamma_{Z_i^\bullet}$ for the dual semi-graph of $Z_i^\bullet$. Lemma 3.2 implies that $\Gamma_{Z_i^\bullet}^{\text{cpt}}$ is 2-connected.

Let $W_1^\bullet$ be a connected component of $Y_1^\bullet \times_{X_1^\bullet} Z_1^\bullet$. We write $h_1^\bullet$ and $r_1^\bullet$ for the composition of natural morphisms

$$W_1^\bullet \to Y_1^\bullet \times_{X_1^\bullet} Z_1^\bullet \to Z_1^\bullet \to X_1^\bullet$$

and

$$W_1^\bullet \to Y_1^\bullet \times_{X_1^\bullet} Z_1^\bullet \to Z_1^\bullet,$$

respectively. We see that $h_1^\bullet : W_1^\bullet \to X_1^\bullet$ is a Galois admissible covering over k_1 with Galois group G'. Moreover, G is a quotient of G'. On the other hand, Lemma 3.3 and Abhyankar's lemma imply that $r_1^\bullet : W_1^\bullet \to Z_1^\bullet$ is a Galois étale covering over k_1 with Galois group G''.

By applying Lemma 3.5, there exists a Galois étale covering $r_2^\bullet : W_2^\bullet \to Z_2^\bullet$ over k_2 with Galois group G'' such that $W_1^\bullet$ is Frobenius equivalent to $W_2^\bullet$. Let

$$h_2^\bullet \stackrel{\text{def}}{=} r_2^\bullet \circ g_2^\bullet : W_2^\bullet \to X_2^\bullet$$

be the admissible covering over k_2.

Write $\Gamma_{X_i^\bullet}$ for the dual semi-graph of $X_i^\bullet$. Since $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$, there exists an isomorphism $\rho : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$ of dual semi-graphs. Let $v_1 \in v(X_1^\bullet)$, $v_2 \stackrel{\text{def}}{=} \rho(v_1) \in v(\Gamma_{X_2^\bullet})$, and $\widetilde{X}_{i,v_i}$ the smooth pointed stable curve associated to v_i. Then we obtain a multi-admissible covering

$$\widetilde{h}_{i,v_i}^\bullet : \widetilde{W}_{i,v_i}^\bullet \to \widetilde{X}_{i,v_i}^\bullet$$

over k_i induced by $h_i^\bullet$. Note that $\widetilde{h}_{1,v_1}^\bullet$ is a Galois multi-admissible covering with Galois group G'. Since $W_1^\bullet$ is Frobenius equivalent to $W_2^\bullet$, we have that $\widetilde{h}_{2,v_2}^\bullet$ is a Galois multi-admissible covering over k_2 with Galois group G'. Thus, $h_2^\bullet$ is a Galois admissible covering over k_2 with Galois group G'. This implies $G' \in \pi_A(X_2^\bullet)$. Then we have $G \in \pi_A(X_2^\bullet)$. This completes the proof of the proposition. $\square$

Next, we formulate the weak Isom-version of the Grothendieck conjecture of pointed stable curves over algebraically closed fields of characteristic $p > 0$ (=the Weak Isom-version Conjecture). The Weak Isom-version Conjecture for smooth pointed stable curves was formulated by Tamagawa.

Weak Isom-version Conjecture . *Let $X_i^\bullet$, $i \in \{1,2\}$, be a pointed stable curve over an algebraically closed field k_i of characteristic $p>0$ and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Then $X_1^\bullet$ is Frobenius equivalent to $X_2^\bullet$ if and only if $\Pi_{X_1^\bullet}$ is isomorphic to $\Pi_{X_2^\bullet}$.*

Then we have the following result.

Theorem 3.8. *The Weak Isom-version Conjecture holds for pointed stable curves if and only if the Weak Isom-version Conjecture holds for smooth pointed stable curves.*

Proof. The theorem follows from [Y1, Theorem 1.2 and Remark 1.2.1] and Proposition 3.7. □

4 Specialization isomorphisms of admissible fundamental groups

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$, and let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack over $\overline{\mathbb{F}}_p$ parameterizing pointed stable curves of type (g,n), $\mathcal{M}_{g,n}$ the open substack of $\overline{\mathcal{M}}_{g,n}$ parameterizing smooth pointed stable curves, $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$, and $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$. We shall write $|\overline{\mathcal{M}}_{g,n}|$ for the underlying topology space of the moduli stack $\overline{\mathcal{M}}_{g,n}$. Note that $|\overline{\mathcal{M}}_{g,n}|$ coincides with the underlying topology space of $\overline{M}_{g,n}$.

Let $X^\bullet$ be a pointed stable curve of type (g,n) over an algebraically closed field k of characteristic $p>0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$. Then $X^\bullet \to \operatorname{Spec} k$ determines a morphism $\alpha : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g,n}$. We obtain a point $q_{X^\bullet} \in |\overline{\mathcal{M}}_{g,n}|$ induced by α. Let $v \in v(\Gamma_{X^\bullet})$, $\widetilde{X}_v^\bullet$ the pointed stable curve over k of type (g_v,n_v) associated to v, and $\overline{\mathcal{M}}_{g_v,n_v}$ the moduli stack over $\overline{\mathbb{F}}_p$ parameterizing pointed stable curves of type (g_v,n_v). Then $\widetilde{X}_v^\bullet \to \operatorname{Spec} k$ determines a morphism $\alpha_v : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g_v,n_v}$. We obtain a point $q_{\widetilde{X}_v^\bullet} \in |\overline{\mathcal{M}}_{g_v,n_v}|$ induced by α_v. Moreover, α_v, $v \in v(\Gamma_{X^\bullet})$, induces a morphism

$$\beta : \operatorname{Spec} k \to \prod_{v\in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v}.$$

Let

$$\kappa : \prod_{v\in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v} \to \overline{\mathcal{M}}_{g,n}$$

be the clutching morphism of moduli stacks such that $\alpha = \kappa \circ \beta$.

Let $X_i^\bullet$, $i \in \{1,2\}$, be a pointed stable curve of type (g,n) over k_i and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Since the isomorphism classes of admissible fundamental groups do not depend on the choices of base fields, we have that $\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet}$ if $q_{X_1^\bullet} = q_{X_2^\bullet}$. Let $q \in |\overline{\mathcal{M}}_{g,n}|$. We shall write Π_q for a profinite group which is isomorphic to the admissible fundamental group of a pointed stable curve $X_q^\bullet$ over an algebraically closed field such that $q_{X_q^\bullet} = q$. Next, we define an equivalence relation $\sim_{fe}$ on $|\overline{\mathcal{M}}_{g,n}|$ as follows: Let $q_1, q_2 \in |\overline{\mathcal{M}}_{g,n}|$; then $q_1 \sim_{fe} q_2$ if $X_{q_1}^\bullet$ is Frobenius equivalent to $X_{q_2}^\bullet$. We put

$$|\overline{\mathcal{M}}_{g,n}|_{fe} \overset{\text{def}}{=} |\overline{\mathcal{M}}_{g,n}|/\sim_{fe} .$$

For each $q \in |\overline{\mathcal{M}}_{g,n}|$, we denote by $[q]$ the image of q of the natural morphism $|\overline{\mathcal{M}}_{g,n}| \twoheadrightarrow |\overline{\mathcal{M}}_{g,n}|_{fe}$.

The morphism of moduli stacks κ defined above induces a natural morphism of underlying topology spaces

$$\kappa_{fe} : |\prod_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v}| \to |\overline{\mathcal{M}}_{g,n}|_{fe}.$$

On the other hand, we have a natural surjection

$$\lambda : |\prod_{v \in v(\Gamma_{X^\bullet})} \overline{\mathcal{M}}_{g_v,n_v}| \twoheadrightarrow \prod_{v \in v(\Gamma_{X^\bullet})} |\overline{\mathcal{M}}_{g_v,n_v}|_{fe}.$$

Lemma 4.1. *We maintain the notation introduced above.*

(i) Let $q \in |\overline{\mathcal{M}}_{g,n}|$. Suppose that $\lambda(\kappa_{fe}^{-1}([q_{X^\bullet}])) = \lambda(\kappa_{fe}^{-1}([q]))$. Then we have

$$\Pi_q \cong \Pi_{q_{X^\bullet}}.$$

(ii) Suppose that $q_{\widetilde{X}^\bullet_v}$, $v \in v(\Gamma_{X^\bullet})$, is not a closed point of $|\overline{\mathcal{M}}_{g_v,n_v}|$, and that $\kappa_{fe}^{-1}([q_{X^\bullet}])$ is a generic point of $\lambda^{-1}(\lambda(\kappa_{fe}^{-1}([q_{X^\bullet}])))$. Let $q \in V_{q_{X^\bullet}}$ such that $q \neq q_{X^\bullet}$, and that $\lambda(\kappa_{fe}^{-1}([q_{X^\bullet}])) = \lambda(\kappa_{fe}^{-1}([q]))$, where $V_{q_{X^\bullet}}$ denotes the topological closure of $q_{X^\bullet}$ in $|\overline{\mathcal{M}}_{g,n}|$. Moreover, let

$$sp : \Pi_{q_{X^\bullet}} \twoheadrightarrow \Pi_q$$

be a specialization homomorphism. Then sp is an isomorphism.

Proof. (i) Let $X^\bullet_q$ be the pointed stable curve of type (g, n) over an algebraically closed field such that $q_{X^\bullet_q} = q$. Then the condition $\lambda(\kappa_{fe}^{-1}([q_{X^\bullet}])) = \lambda(\kappa_{fe}^{-1}([q]))$ implies that $X^\bullet$ is Frobenius equivalent to $X^\bullet_q$. Thus, Proposition 3.7 implies that

$$\Pi_q \cong \Pi_{q_{X^\bullet}}.$$

(ii) By (i), we obtain that $\Pi_{q_{X^\bullet}}$ is isomorphic to Π_q as abstract profinite groups. Moreover, [FJ, Proposition 16.10.6] implies that the surjection $sp : \Pi_{q_{X^\bullet}} \twoheadrightarrow \Pi_q$ is an isomorphism. This completes the proof of the lemma. □

Lemma 4.2. *Let X be a variety over an algebraically closed field k such that $\dim(X) \geq 1$, and let $\mathrm{pr}_1 : X \times_k X \to X$ and $\mathrm{pr}_2 : X \times_k X \to X$ be natural projections. Then there exists a subvariety of $Z \subseteq X \times_k X$ such that $\dim(Z) < \dim(X \times_k X)$, and that $\mathrm{pr}_1|_Z : Z \to X$ and $\mathrm{pr}_2|_Z \to X$ are surjective.*

Proof. Trivial. □

The main theorem of the present paper is as follows:

Theorem 4.3. *Suppose that $\dim(\overline{\mathcal{M}}_{g,n}) \geq 3$. There exist $q_1, q_2 \in |\overline{\mathcal{M}}_{g,n}| \setminus |\mathcal{M}_{g,n}|$ such that $q_1 \in V_{q_2}$, that $q_1 \neq q_2$, and that $\Pi_{q_1} \cong \Pi_{q_2}$.*

Proof. **Case 1:** Suppose that $g \in 2\mathbb{Z}_{\geq 0}$ and $n \in 2\mathbb{Z}_{\geq 0}$. Then we have a clutching morphism of moduli stack

$$\kappa : \overline{\mathcal{M}}_{\frac{g}{2},\frac{n}{2}+1} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{\frac{g}{2},\frac{n}{2}+1} \to \overline{\mathcal{M}}_{g,n}.$$

Since $\dim(\overline{\mathcal{M}}_{g,n}) \geq 3$, we have $\dim(\overline{\mathcal{M}}_{\frac{g}{2},\frac{n}{2}+1}) \geq 1$. Then we obtain a morphism of topology spaces

$$\kappa_{fe} : |\overline{\mathcal{M}}_{\frac{g}{2},\frac{n}{2}+1} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{\frac{g}{2},\frac{n}{2}+1}| \to |\overline{\mathcal{M}}_{g,n}|_{fe}.$$

Let $[q_2]$ be the generic point of the image of κ. Then Lemma 4.2 implies that there exists a point $q_1 \in |\overline{\mathcal{M}}_{g,n}|$ such that $q_1 \in V_{q_2}$, and that $q_1 \neq q_2$. Then the theorem follows from Lemma 4.1 (ii).

Case 2: Suppose that $g \in 2\mathbb{Z}_{\geq 0}$, and that $n = 2m+1$ is an odd number. Let k be an algebraic closure of the residue field of $\overline{M}_{\frac{g}{2},m+1}$ and $X_1^\bullet = (X_1, D_{X_1} \stackrel{\text{def}}{=} \{x_0, \ldots, x_m\})$ the pointed stable curve over k corresponding to the natural morphism $\alpha_1 : \operatorname{Spec} k \to \overline{M}_{\frac{g}{2},m+1}$. Write

$$\pi_{m+2,m+1} : \overline{\mathcal{M}}_{\frac{g}{2},m+2} \to \overline{\mathcal{M}}_{\frac{g}{2},m+1}$$

for morphism of moduli stacks determined by forgetting the last marked point. Let α_2' be a k-rational point of $\overline{\mathcal{M}}_{\frac{g}{2},m+2} \times_{\overline{\mathcal{M}}_{\frac{g}{2},m+1},\alpha_1} k$ and α_2 the composition of the natural morphisms

$$\operatorname{Spec} k \stackrel{\alpha_2'}{\to} \overline{\mathcal{M}}_{\frac{g}{2},m+2} \times_{\overline{\mathcal{M}}_{\frac{g}{2},m+1},\alpha_1} k \stackrel{\text{pr}_1}{\to} \overline{\mathcal{M}}_{\frac{g}{2},m+2}.$$

We note that $\pi_{m+2,m+1} \circ \alpha_2 = \alpha_1$. Let $X_2^\bullet = (X_2, D_{X_2})$ be a pointed stable curve over k corresponding $\alpha_2 : \operatorname{Spec} k \to \overline{\mathcal{M}}_{\frac{g}{2},m+2}$. Then we have that $X_1 = X_2$, and that $D_{X_2} = D_{X_1} \cup \{x_{2,m+1}\}$, where $x_{2,m+1}$ is a k-rational point of $X_2 \setminus D_{X_1}$. By gluing $X_1^\bullet$ and $X_2^\bullet$ along x_0, we obtain a pointed stable curve $X^\bullet$ of type (g,n) over k. Then $X^\bullet$ induces a clutching morphism of moduli stack

$$\kappa : \overline{\mathcal{M}}_{\frac{g}{2},m+1} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{\frac{g}{2},m+2} \to \overline{\mathcal{M}}_{g,n}$$

and a morphism of topology spaces

$$\kappa_{fe} : |\overline{\mathcal{M}}_{\frac{g}{2},m+1} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{\frac{g}{2},m+2}| \to |\overline{\mathcal{M}}_{g,n}|_{fe}.$$

Let $[q_1] \in |\overline{\mathcal{M}}_{g,n}|_{fe}$ such that $q_1 = q_{X^\bullet}$. Let

$$\lambda : |\overline{\mathcal{M}}_{\frac{g}{2},m+1} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{\frac{g}{2},m+2}| \twoheadrightarrow |\overline{\mathcal{M}}_{\frac{g}{2},m+1}|_{fe} \times |\overline{\mathcal{M}}_{\frac{g}{2},m+2}|_{fe}$$

be the natural morphism of topology spaces. Moreover, let $[q_2]$ be a generic point of the image of $\kappa_{fe}(\lambda^{-1}(([q_{X_1^\bullet}], [q_{X_2^\bullet}])))$ such that $q_1 \in V_{q_2}$. Let $q_2' \in \overline{M}_{g,n}$ be the point of coarse moduli space induced by q_2 and $\overline{k(q_2')}$ an algebraic closure of the residue field of q_2'. Then the construction implies that

$$\text{td}(\overline{k(q_2')}/\overline{\mathbb{F}}_p) > \text{td}(k/\overline{\mathbb{F}}_p),$$

where td$(-)$ denotes the transcendence degree of fields extension $(-)$. This means that $q_2 \neq q_1$. Then the theorem follows from Lemma 4.1 (ii).

Case 3: Suppose that g and n are odd numbers. Then we have a clutching morphism of moduli stacks

$$\kappa : \overline{\mathcal{M}}_{g-1,n} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{g,n}.$$

Since $g-1$ is an even number, the theorem follows immediately from Case 2. This completes the proof of the theorem. $\square$

Example 4.4. Let k be an algebraic closure of the residue field of the generic point of $\overline{M}_{0,4}$ and $x \in k \setminus \overline{\mathbb{F}}_p$. We put

$$X_m^\bullet = (X_m \stackrel{\text{def}}{=} \mathbb{P}^1_k, D_{X_m} \stackrel{\text{def}}{=} \{0, 1, \infty, x^{p^m}\}),\ m \in \mathbb{Z}_{\geq 0}.$$

Note that $X_m^\bullet$ is a pointed stable curve corresponding to a geometric generic point of $\overline{M}_{0,4}$, and that $X_{m_1}^\bullet$ is not k-isomorphic to $X_{m_2}^\bullet$ if $m_1 \neq m_2$. For each $m \in \mathbb{Z}_{\geq 0}$, by gluing $X_0^\bullet$ and $X_m^\bullet$ along 0, we obtain a pointed stable curve

$$X_{0,m}^\bullet$$

of type $(0,6)$ over k. Let

$$\kappa : \overline{\mathcal{M}}_{0,4} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{0,4} \to \overline{\mathcal{M}}_{0,6}$$

be the clutching morphism of moduli stacks determined by gluing two pointed stable curves of type $(0,4)$ along the first marked points.

Let $q_{X_{0,m}^\bullet} \in |\overline{\mathcal{M}}_{0,6}|$ be the point determined by $X_{0,m}^\bullet$. We see immediately that $\dim(V_{q_{X_{0,m}^\bullet}}) = 1$. Let

$$\kappa_{fe} : |\overline{\mathcal{M}}_{0,4} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{0,4}| \to |\overline{\mathcal{M}}_{0,6}|_{fe}$$

be the morphism of topology spaces induced by κ, $\lambda : |\overline{\mathcal{M}}_{0,4} \times_{\overline{\mathbb{F}}_p} \overline{\mathcal{M}}_{0,4}| \twoheadrightarrow |\overline{\mathcal{M}}_{0,4}|_{fe} \times |\overline{\mathcal{M}}_{0,4}|_{fe}$ the natural surjection, and $q \in |\overline{\mathcal{M}}_{0,6}|$ is the unique point such that $[q]$ is a generic point of the image of κ_{fe}. Note that $q_{X_{0,m}^\bullet} \in V_q$, that $\lambda(\kappa_{fe}^{-1}([q_{X_{0,m}^\bullet}])) = \lambda(\kappa_{fe}^{-1}([q]))$, and that $\dim(V_q) = 2$. Then $q \neq q_{X_{0,m}^\bullet}$ and $\Pi_q \cong \Pi_{q_{X_{0,m}^\bullet}}$. Thus, if

$$sp : \Pi_q \twoheadrightarrow \Pi_{q_{X_{0,m}^\bullet}}$$

is a specialization homomorphism. Then sp is an isomorphism. Since

$$\Pi_{q_{X_{0,i}^\bullet}} \cong \Pi_{q_{X_{0,j}^\bullet}},\ i, j \in \mathbb{Z}_{\geq 0},$$

the finiteness theorem does not hold in general.

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