TOPOLOGY OF MODULI SPACES OF CURVES AND ANABELIAN GEOMETRY IN POSITIVE CHARACTERISTIC

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Abstract. In the present paper, we study a new kind of anabelian phenomenon concerning the smooth pointed stable curves in positive characteristic. It shows that the topology of moduli spaces of curves can be understood from the viewpoint of anabelian geometry. We formulate some new anabelian-geometric conjectures concerning tame fundamental groups of curves over algebraically closed fields of characteristic $p > 0$ from the point of view of moduli spaces. The conjectures are generalized versions of the weak Isom-version of the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$ which was formulated by Tamagawa. Moreover, we prove that the conjectures hold for certain points lying in the moduli space of curves of genus 0.

Keywords: smooth pointed stable curve, tame fundamental group, moduli space, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14H10, 14G32.

Contents

1. Introduction

1.1. The mystery of fundamental groups in positive characteristic.

1.1.1. Let k be an algebraically closed field of characteristic $p \geq 0$, and let (X, D_X) be a *smooth* pointed stable curve of type (g_X, n_X) over k (i.e. $2g_X + n_X - 2 > 0$, see [K, Definition 1.1 (iv)]), where X denotes the underlying curve, D_X denotes the (ordered) finite set of marked points, g_X

denotes the genus of X, and n_X denotes the cardinality $\#(D_X)$ of D_X. We put $U_X \stackrel{\text{def}}{=} X \setminus D_X$. By choosing a base point of U_X, we have the tame fundamental group $\pi_1^{\mathrm{t}}(U_X)$ of U_X.

If $p = 0$, it is well-known that the structure of $\pi_1^{\mathrm{t}}(U_X)$ is isomorphic to the profinite completion of the topological fundamental group of a Riemann surface of type (g_X, n_X). Hence, almost no geometric information about U_X can be carried out from $\pi_1^{\mathrm{t}}(U_X)$. By contrast, if $p > 0$, the situation is quite different from that in characteristic 0. The tame fundamental group $\pi_1^{\mathrm{t}}(U_X)$ contains rich geometric information of (X, D_X), moreover, there exist *anabelian phenomena for curves over algebraically closed fields of characteristic* $p > 0$.

1.1.2. Firstly, let us explain some general background about anabelian geometry. In the 1980s, A. Grothendieck suggested a theory of arithmetic geometry called anabelian geometry ([G]). The central question of the theory is as follows: Can we reconstruct the geometric information of a variety group-theoretically from various versions of its algebraic fundamental group? The original anabelian geometry suggested by Grothendieck focused on varieties over *arithmetic* fields, in particular, the fields finitely generated over $\mathbb{Q}$. In the case of curves in characteristic 0, anabelian geometry has been deeply studied (e.g. [N], [T1]) and, in particular, the most important case (i.e. the fields finitely generated over $\mathbb{Q}$, or more general, sub-p-adic fields) has been completely established ([M]). Note that the actions of the Galois groups of the base fields on the geometric fundamental groups play a crucial role for recovering geometric information of curves over arithmetic fields.

Next, we return to the case where k is an algebraically closed field of characteristic $p > 0$. In [T2], A. Tamagawa discovered that there also exist *anabelian phenomena* for curves over *algebraically closed fields of characteristic* p. This came rather surprisingly since it means that, in positive characteristic, the geometry of curves can be only determined by their geometric fundamental groups *without Galois actions*. Since the late 1990s, this kind of anabelian phenomenon has been studied further by M. Raynaud ([R2]), F. Pop-M. Saïdi ([PS]), Tamagawa ([T2], [T4], [T5]), and the second author of the present paper ([Y1], [Y2], [Y4]). More precisely, they focused on the so-called *weak Isom-version of Grothendieck's anabelian conjecture for curves over algebraically closed fields of characteristic* $p > 0$ (or the "weak Isom-version conjecture" for short) formulated by Tamagawa ([T3, Conjecture 2.2]) which says that curves are isomorphic if and only if their tame (or étale) fundamental groups are isomorphic. At present, this conjecture still wide-open.

1.2. **Reconstructions of moduli spaces of curves via anabelian geometry.** In the present paper, we study a new kind of anabelian phenomenon concerning curves over algebraically closed fields of characteristic $p > 0$ which shows that the topological structures of moduli spaces of curves can be understood by their fundamental groups.

1.2.1. Let $\mathbb{F}_p$ be the prime field of characteristic $p > 0$, and let $\mathcal{M}^{\mathrm{ord}}_{g,n,\mathbb{Z}}$ be the moduli stack over $\mathbb{Z}$ parameterizing smooth n-pointed stable curves of type (g, n) (in the sense of [K]). We put $\mathcal{M}^{\mathrm{ord}}_{g,n,\mathbb{F}_p} \stackrel{\text{def}}{=} \mathcal{M}^{\mathrm{ord}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{F}_p$. Note that the set of marked points of an n-smooth pointed stable curve admits a natural action of the n-symmetric group S_n. Moreover, we denote by $\mathcal{M}_{g,n,\mathbb{F}_p} \stackrel{\text{def}}{=} [\mathcal{M}^{\mathrm{ord}}_{g,n,\mathbb{F}_p}/S_n]$ the quotient stack, and denote by $M_{g,n,\mathbb{F}_p}$ the coarse moduli space of $\mathcal{M}_{g,n,\mathbb{F}_p}$.

Let $q \in M_{g,n,\mathbb{F}_p}$ be an arbitrary point, $k(q)$ the residue field of q, k_q an algebraically closed field containing $k(q)$, and $V_q \stackrel{\text{def}}{=} \overline{\{q\}}$ the topological closure of $\{q\}$ in $M_{g,n,\mathbb{F}_p}$. Write $(X_{k_q}, D_{X_{k_q}})$ for the smooth pointed stable curve of type (g, n) over k_q determined by the natural morphism $\operatorname{Spec} k_q \to M_{g,n,\mathbb{F}_p}$ and put $U_{X_{k_q}} \stackrel{\text{def}}{=} X_{k_q} \setminus D_{X_{k_q}}$. In particular, we put $(X_{k_q}, D_{X_{k_q}}) \stackrel{\text{def}}{=} (X_q, D_{X_q})$ and $U_{X_q} \stackrel{\text{def}}{=} X_q \setminus D_{X_q}$ if k_q is an algebraic closure of $k(q)$. Since the isomorphism class of the tame fundamental group $\pi_1^{\mathrm{t}}(U_{X_{k_q}})$ depends only on q, we shall write $\pi_1^{\mathrm{t}}(q)$ for the tame fundamental group $\pi_1^{\mathrm{t}}(U_{X_{k_q}})$.

1.2.2. We maintain the notation introduced above. The weak Isom-version conjecture of Tamagawa can be reformulated as follows:

Weak Isom-version Conjecture . *Let $q_i \in M_{g,n,\mathbb{F}_p}$, $i \in \{1,2\}$, be an arbitrary point of $M_{g,n,\mathbb{F}_p}$. The set of continuous isomorphisms of profinite groups*

$$\mathrm{Isom}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty if and only if $V_{q_1} = V_{q_2}$ (namely, $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes).

The weak Isom-version conjecture means that *moduli spaces of curves can be reconstructed "as sets" from the isomorphism classes of the tame fundamental groups of curves*. This conjecture has been *only* confirmed by Tamagawa ([T4, Theorem 0.2]) in the case of genus 0, namely, the following:

> Suppose that q_1 is a *closed* point of $M_{0,n,\mathbb{F}_p}$. Then the weak Isom-version conjecture holds.

Next, we pose a new conjecture as follows, which we call *the weak Hom-version of the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$* (=weak Hom-version conjecture), and which generalizes the weak Isom-version conjecture.

Weak Hom-version Conjecture . *Let $q_i \in M_{g,n,\mathbb{F}_p}$, $i \in \{1,2\}$, be an arbitrary point of $M_{g,n,\mathbb{F}_p}$. The set of open continuous homomorphisms of profinite groups*

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty if and only if $V_{q_1} \supseteq V_{q_2}$.

Roughly speaking, this means that a smooth pointed stable curve corresponding to a geometric point over q_2 can be deformed to a smooth pointed stable curve corresponding to a geometric point over q_1 if and only if the set of open continuous homomorphisms of tame fundamental groups $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$ is not empty.

The weak Hom-version conjecture means that *the sets of deformations* of a smooth pointed stable curve can be reconstructed group-theoretically from the sets of open continuous homomorphisms of their tame fundamental groups. Therefore, it provides *a new kind of anabelian phenomenon*:

> The moduli spaces of curves in positive characteristic can be understood not only as sets but also "*as topological spaces*" from the *sets of open continuous homomorphisms* of tame fundamental groups of curves in positive characteristic.

1.3. Main result.

1.3.1. The main result of the present paper confirms the weak Hom-version conjecture for curves of genus 0 (see Theorem 4.4 (iv) for a more general statement):

Theorem 1.1. *The Weak Hom-version Conjecture holds when q_1 is a closed point of $M_{0,n,\mathbb{F}_p}$.*

Theorem 1.1 follows from the following "Hom-type" anabelian result (see Theorem 4.3 for a more precise statement) which is a generalization of Tamagawa's result (i.e. [T4, Theorem 0.2]):

Theorem 1.2. *Let $q_1 \in M_{g,n,\mathbb{F}_p}$ be a closed point and $q_2 \in M_{g,n,\mathbb{F}_p}$ an arbitrary point. Then the set of open continuous homomorphisms*

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty if and only if $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes.

Remark. Note that Theorem 1.2 is *essentially* different from [T4, Theorem 0.2]. The reason is the following: We *do not know* whether or not

$$\mathrm{Isom}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty when $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$ is *non-empty.*

On the other hand, to verify Theorem 1.2, we need to establish various anabelian reconstructions from *open continuous homomorphisms* of tame fundamental groups which are much harder than the case of *isomorphisms* in general. We explain in more detail about this point in the reminder of the introduction.

1.3.2. Let us explain the main differences between the proofs of Tamagawa's result (i.e. [T4, Theorem 0.2]) and our result (i.e. Theorem 1.2), and new ingredients of our proof. First, we recall the key points of the proof of Tamagawa's result. Roughly speaking, Tamagawa's proof consists of two parts:

(1) He proved that *the sets of inertia subgroups* of marked points and *the field structures* associated to inertia subgroups of marked points of smooth pointed stable curves can be reconstructed group-theoretically from tame fundamental groups. This is the most difficult part of Tamagawa's proof.
(2) By using the inertia subgroups and their associated field structures, if $g = 0$, he proved that the coordinates of marked points can be calculated group-theoretically.

The group-theoretical reconstructions in Tamagawa's proofs (1) and (2) are *isomorphic version reconstructions.* This means that the reconstructions should fix an isomorphism class of a tame fundamental group. To explain this, let us show an example. Let U_{X_i}, $i \in \{1, 2\}$, be a curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$ introduced above, $\pi_1^{\mathrm{t}}(U_{X_i})$ the tame fundamental group of U_{X_i}, $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \to \pi_1^{\mathrm{t}}(U_{X_2})$ an open continuous homomorphism, $H_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ an open subgroup, and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2)$. In Tamagawa's proof, since ϕ is an *isomorphism*, we have $H_1 \cong H_2$. Then the group-theoretical reconstruction for types implies that the type $(g_{X_{H_1}}, n_{X_{H_1}})$ and the type $(g_{X_{H_2}}, n_{X_{H_2}})$ of the curves corresponding to H_1 and H_2, respectively, are equal. This is a key point in the proof of Tamagawa's group-theoretical reconstruction of the inertia subgroups of marked points. On the other hand, his method cannot be applied to the present paper. The reason is that we need to treat the case where ϕ is an *arbitrary open continuous homomorphism.* Since H_1 is not isomorphic to H_2 in general (e.g. specialization homomorphism), we *do not* know whether or not $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. This is one of main difficulties of "Hom-type" problems appeared in anabelian geometry. Similar difficulties for generalized Hasse-Witt invariants will appear if we try to reconstruct the field structure associated to inertia subgroups of marked points.

To overcome the difficulties mentioned above, we have the following key observation:

> The inequalities of $\mathrm{Avr}_p(H_i)$ (i.e. the p-averages of generalized Hasse-Witt invariants (see 3.4.3)) induced by ϕ play roles of the comparability of (outer) Galois representations in the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

In the present paper, our method for reconstructing inertia subgroups of marked points is completely different from Tamagawa's reconstruction. We develop a new *group-theoretical algorithm* for reconstructing the inertia subgroups of marked points whose input datum is a profinite group which is isomorphic to $\pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$, and whose output data are inertia subgroups of marked points (Theorem 3.14). Moreover, we prove that the group-theoretical algorithm and the reconstructions for field structures are *compatible* witharbitrary surjection ϕ (Proposition 3.15). By using Theorem 3.14 and Proposition 3.15, we may prove that Tamagawa's calculation of coordinates is compatible with our reconstructions. This implies Theorem 1.2.

1.4. **Some further developments.**

1.4.1. *Moduli spaces of fundamental groups.* Let us explain some further developments for the anabelian phenomenon concerning the weak Hom-verson conjecture. In [Y6], the second author of the present paper introduced a topological space $\Pi_{g,n}$ (or more general, $\overline{\Pi}_{g,n}$) determined group-theoretically by the tame fundamental groups of smooth pointed stable curves (or more general, the geometric log étale fundamental groups of arbitrary pointed stable curves) of type (g, n) which he call *moduli spaces of fundamental groups of curves*, whose underlying set is the sets of isomorphism classes of fundamental groups, and whose topology is determined by the sets of finite quotients of fundamental groups. Moreover, he posed the so-called *homeomorphism conjecture*, roughly speaking, which says that (by quotiening a certain equivalence relation induced by Frobenius actions) the moduli spaces of curves are homeomorphic to the moduli spaces of fundamental groups.

In the present literature, the term "anabelian" is understood to mean that a geometric object can be determined by its fundamental group. On the other hand, the homeomorphism conjecture concerning moduli spaces of fundamental groups supplies a new point of view to understand anabelian phenomena as follows:

> The term "anabelian" means that not only a geometric object can be determined by its fundamental groups, but also a certain *moduli space of geometric objects* can be determined by the fundamental groups of geometric objects.

Under this point of view, *the homeomorphism conjecture is reminiscent of a famous theorem in the theory of classic Teichmüller spaces* which state that the Teichmüller spaces of complex hyperbolic curves are homeomorphic to the spaces of discrete and faithful representations of topological fundamental groups of underlying surfaces into the group $PSL_2(\mathbb{R})$.

In fact, Theorem 1.1 implies that $M_{0,4,\mathbb{F}_p}$ is *homeomorphic* to $\Pi_{0,4}$ *as topological spaces* (note that Tamagawa's result (i.e. [T4, Theorem 0.2]) only says that the natural map $M_{0,4,\mathbb{F}_p} \to \Pi_{0,4}$ is a bijection *as sets*). Based on [Y1], [Y3], [Y4], [Y5], and the main results of the present paper, the main results of [Y6] and [Y7] says that the homeomorphism conjecture holds for 1-*dimensional moduli spaces of pointed stable curves.* Moreover, the weak Hom-version conjecture and the pointed collection conjecture (see Section 2.2 of the present paper) are main steps toward the homeomorphism conjecture for higher dimensional moduli spaces of curves (see [Y8, Section 1.2.3]).

1.4.2. *The sets of finite quotients of tame fundamental groups.* We maintain the notation introduced in 1.1.1. The techniques developed in §3 of the present paper have important applications for understanding the set of finite quotients $\pi_A^{\mathrm{t}}(U_X)$ of the *tame fundamental groups* $\pi_1^{\mathrm{t}}(U_X)$ of U_X.

Note that, if U_X is *affine*, the set $\pi_A^{\text{ét}}(U_X)$ of finite quotients of the *étale fundamental groups* $\pi_1^{\text{ét}}(U_X)$ of U_X can be completely determined by its type (g_X, n_X) (i.e. Abhyankar's conjecture proved by Raynaud for affine lines and D. Harbater in general). However, the structure of $\pi_1^{\text{ét}}(U_X)$ cannot be carried out from $\pi_A^{\text{ét}}(U_X)$ since $\pi_1^{\text{ét}}(U_X)$ is not topologically finitely generated when U_X is affine.

By contrast, the isomorphism class of $\pi_1^{\mathrm{t}}(U_X)$ can be completely determined by $\pi_A^{\mathrm{t}}(U_X)$ since $\pi_1^{\mathrm{t}}(U_X)$ is topologically finitely generated, and one *cannot* to expect that there exists an explicit description for the entire set $\pi_A^{\mathrm{t}}(U_X)$ since there exists *anabelian phenomenon* mentioned above (i.e. $\pi_A^{\mathrm{t}}(U_X)$ depends on the isomorphism class of U_X). On the other hand, for understanding more precisely the relationship between the structures of tame fundamental groups and the anabelian phenomena in positive characteristic world, it is naturally to ask the following interesting problem:

> How does the scheme structure of U_X affect explicitly the set of finite quotients $\pi_A^{\mathrm{t}}(U_X)$?

In [Y9], by applying the techniques developed in §3 of the present paper and [Y5, Theorem 1.2], we obtain the following interesting generalization of [T4, Theorem 0.2] (i.e. a "*finite version*" of the weak Isom-version conjecture):

> Let $q_1 \in M_{g_1,n_1,\mathbb{F}_p}$ and $q_2 \in M_{0,n_2,\mathbb{F}_p}$ be arbitrary points and $\pi_A^{\mathrm{t}}(q_i)$ the set of finite quotients of the tame fundamental group $\pi_1^{\mathrm{t}}(q_i)$. Suppose that q_2 is a closed point of $M_{0,n_2,\mathbb{F}_p}$. Then we can *construct explicitly a finite group* G depending on q_1 and q_2

such that $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes if and only if $G \in \pi_A^{\mathrm{t}}(q_1) \cap \pi_A^{\mathrm{t}}(q_2)$. In particular, if $\pi_1^{\mathrm{t}}(q_1) \not\cong \pi_1^{\mathrm{t}}(q_2)$, then we can construct explicitly a finite group G depending on q_1 and q_2 such that $G \in \pi_A^{\mathrm{t}}(q_1)$ and $G \notin \pi_A^{\mathrm{t}}(q_2)$.

1.5. **Structure of the present paper.** The present paper is organized as follows. In Section 2, we formulate the the weak Hom-version conjecture and the pointed collection conjecture. In Section 3, we give a group-theoretical algorithm for reconstructions of inertia subgroups associated marked points, and prove that the group-theoretical algorithm is compatible with arbitrary open surjective homomorphisms of tame fundamental groups. In Section 4, we prove our main results.

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2. Conjectures

In this section, we formulate two new conjectures concerning anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

2.1. **The weak Hom-version conjecture.** In this subsection, we formulate the first conjecture of the present paper which we call "the weak Hom-version conjecture".

2.1.1. Let k be an algebraically closed field of characteristic $p > 0$, and let

$$(X, D_X)$$

be a smooth pointed stable curve of type (g_X, n_X) over k, where X denotes the (smooth) underlying curve of genus g_X and D_X denotes the (ordered) finite set of marked points with cardinality $n_X \overset{\mathrm{def}}{=} \#(D_X)$ satisfying [K, Definition 1.1 (iv)] (i.e. $2g_X + n_X - 2 > 0$). Note that $U_X \overset{\mathrm{def}}{=} X \setminus D_X$ is a hyperbolic curve over k.

Let (Y, D_Y) and (X, D_X) be smooth pointed stable curves over k, and let $f : (Y, D_Y) \to (X, D_X)$ be a morphism of smooth pointed stable curves over k. We shall say that f is *étale* (resp. *tame*, *Galois étale*, *Galois tame*) if f is étale over X (resp. f is étale over U_X and is at most tamely ramified over D_X, f is a Galois covering and is étale, f is a Galois covering and is tame).

By choosing a base point of $x \in U_X$, we have the tame fundamental group $\pi_1^{\mathrm{t}}(U_X, x)$ of U_X and the étale fundamental group $\pi_1(X, x)$ of X. Since we only focus on the isomorphism classes of fundamental groups in the present paper, for simplicity of notation, we omit the base point and denote by $\pi_1^{\mathrm{t}}(U_X)$ and $\pi_1(X)$ the tame fundamental group $\pi_1^{\mathrm{t}}(U_X, x)$ of U_X and the étale fundamental group $\pi_1(X, x)$ of X, respectively. Note that there is a natural continuous surjective homomorphism $\pi_1^{\mathrm{t}}(U_X) \twoheadrightarrow \pi_1(X)$.

2.1.2. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and let $\mathcal{M}^{\mathrm{ord}}_{g,n,\mathbb{F}_p}$ be the moduli stack over $\mathbb{Z}$ parameterizing smooth pointed stable curves of type (g, n) in the sense of [K, Definition 1.1]. The set of marked points of a smooth pointed stable curve admits a natural action of the n-symmetric group S_n, we put $\mathcal{M}_{g,n,\mathbb{Z}} \overset{\mathrm{def}}{=} [\mathcal{M}^{\mathrm{ord}}_{g,n,\mathbb{Z}}/S_n]$ the quotient stack. Moreover, we denote by $\mathcal{M}^{\mathrm{ord}}_{g,n} \overset{\mathrm{def}}{=} \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, $\mathcal{M}_{g,n,\mathbb{F}_p} \overset{\mathrm{def}}{=} \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{F}_p$, and $\mathcal{M}_{g,n} \overset{\mathrm{def}}{=} \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, and denote by $M^{\mathrm{ord}}_{g,n}$, $M_{g,n,\mathbb{F}_p}$, and $M_{g,n}$ the coarse moduli spaces of $\mathcal{M}^{\mathrm{ord}}_{g,n}$, $\mathcal{M}_{g,n,\mathbb{F}_p}$, and $\mathcal{M}_{g,n}$, respectively.

Let $q \in M_{g,n}^{\rm ord}$ be an arbitrary point and $k(q)$ the residue field of q, and k_q an algebraically closed field containing $k(q)$. Write $(X_{k_q}, D_{X_{k_q}})$ for the smooth pointed stable curve of type (g,n) over k_q determined by the natural morphism $\mathrm{Spec}\, k_q \to \mathrm{Spec}\, k(q) \to M_{g,n}^{\rm ord}$ and $U_{X_{k_q}}$ for $X_{k_q} \setminus D_{X_{k_q}}$. In particular, if k_q is an algebraic closure of $k(q)$, we shall write (X_q, D_{X_q}) for $(X_{k_q}, D_{X_{k_q}})$.

Since the isomorphism class of the tame fundamental group $\pi_1^{\rm t}(U_{X_{k_q}})$ depends only on q (i.e. the isomorphism class does not depend on the choices of k_q), we shall write $\pi_1^{\rm t}(q)$ and $\pi_A^{\rm t}(q)$ for $\pi_1^{\rm t}(U_{X_{k_q}})$ and the set of finite quotients of $\pi_1^{\rm t}(U_{X_{k_q}})$, respectively. [FJ, Proposition 16.10.7] implies that for any points $q_1, q_2 \in M_{g,n}^{\rm ord}$, $\pi_1^{\rm t}(q_1) \cong \pi_1^{\rm t}(q_2)$ as profinite groups if and only if $\pi_A^{\rm t}(q_1) = \pi_A^{\rm t}(q_2)$ as sets.

On the other hand, Let $q \in M_{g,n}^{\rm ord}$ and $q' \in M_{g,n,\mathbb{F}_p}$ be arbitrary points. We denote by $V_q \subseteq M_{g,n}^{\rm ord}$ and $V_{q'} \subseteq M_{g,n,\mathbb{F}_p}$ the topological closures of q and q' in $M_{g,n}^{\rm ord}$ and $M_{g,n,\mathbb{F}_p}$, respectively.

2.1.3. We have the following definition.

Definition 2.1. (i) Let $c_1, c_2 \in M_{g,n}^{\rm ord, cl}$ be *closed points*, where $(-)^{\rm cl}$ denotes the set of closed points of $(-)$. Then $c_1 \sim_{fe} c_2$ if there exists $m \in \mathbb{Z}$ such that $\nu(c_2) = \nu(c_1^{(m)})$, where $c_1^{(m)}$ denotes the closed point corresponding to the curve obtained by mth Frobenius twist of the curve corresponding to c_1. Here "fe" means "Frobenius equivalence".

(ii) Let $q_1, q_2 \in M_{g,n}^{\rm ord}$ be arbitrary points. We denote by $V_{q_1} \supseteq_{fe} V_{q_2}$ if, for each closed point $c_2 \in V_{q_2}^{\rm cl}$, there exists a closed point $c_1 \in V_{q_1}^{\rm cl}$ such that $c_1 \sim_{fe} c_2$. Moreover, we denote by $V_{q_1} =_{fe} V_{q_2}$ if $V_{q_1} \supseteq_{fe} V_{q_2}$ and $V_{q_1} \subseteq_{fe} V_{q_2}$. Moreover, we also denote by $q_1 \sim_{fe} q_2$ if $V_{q_1} =_{fe} V_{q_2}$.

We have the following proposition.

Proposition 2.2. *Let $\omega : M_{g,n}^{\rm ord} \to M_{g,n,\mathbb{F}_p}$ be the morphism induced by the natural morphism $\mathcal{M}_{g,n}^{\rm ord} \to \mathcal{M}_{g,n,\mathbb{F}_p}$. Let $i \in \{1,2\}$, and let $q_i \in M_{g,n}^{\rm ord}$ and $q_i' \overset{\rm def}{=} \omega(q_i) \in M_{g,n,\mathbb{F}_p}$. Then we have $V_{q_1} \supseteq_{fe} V_{q_2}$ if and only if $V_{q_1'} \supseteq V_{q_2'}$. In particular, we have $V_{q_1} =_{fe} V_{q_2}$ if and only if $V_{q_1'} = V_{q_2'}$. Namely, we have $V_{q_1} =_{fe} V_{q_2}$ if and only if $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes.*

Proof. Suppose that q_i, $i \in \{1,2\}$, is a closed point of $M_{g,n}^{\rm ord}$. If $V_{q_1} \supseteq_{fe} V_{q_2}$, we see immeidately $q_1 \sim q_2$. Thus, we obtain $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes. This means $q_1' = q_2'$. Conversely, if $V_{q_1'} \supseteq V_{q_2'}$, then we have $q_1' = q_2'$. Thus, we obtain $q_1 \sim q_2$.

Suppose that q_i, $i \in \{1,2\}$, is an aribtrary point of $M_{g,n}^{\rm ord}$. If $V_{q_1} \supseteq_{fe} V_{q_2}$, then the case of closed points implies $V_{q_1'}^{\rm cl} \supseteq V_{q_2'}^{\rm cl}$. Since $V_{q_1'}$ and $V_{q_2'}$ are irreducible, we obtain $V_{q_1'} \supseteq V_{q_2'}$. Conversely, if $V_{q_1'} \supseteq V_{q_2'}$, we note that V_{q_i} is an irreducible component of $(\omega)^{-1}(V_{q_i'})$. Then the case of closed points implies $V_{q_1} \supseteq_{fe} V_{q_2}$. □

2.1.4. Denote by $\mathrm{Hom}_{\rm pg}^{\rm op}(-,-)$ the set of open continuous homomorphisms of profinite groups, and by $\mathrm{Isom}_{\rm pg}(-,-)$ the set of isomorphisms of profinite groups. We have the following conjecture.

Weak Hom-version Conjecture . *Let $q_i \in M_{g,n}$ (resp. $q_i \in M_{g,n,\mathbb{F}_p}$), $i \in \{1,2\}$, be an arbitrary point. Then we have*

$$\mathrm{Hom}_{\rm pg}^{\rm op}(\pi_1^{\rm t}(q_1), \pi_1^{\rm t}(q_2))$$

is non-empty if and only if $V_{q_1} \supseteq_{fe} V_{q_2}$ (resp. $V_{q_1} \supseteq V_{q_2}$).

The weak Hom-version conjecture means that the *topological structures* of the moduli spaces of smooth pointed stable curves can be understood by the tame fundamental groups of curves. In particular, the weak Hom-version conjecture implies the following conjecture which was essentially formulated by Tamagawa ([T3]).

Weak Isom-version Conjecture . *Let $q_i \in M_{g,n}$ (resp. $q_i \in M_{g,n,\mathbb{F}_p}$), $i \in \{1,2\}$, be an arbitrary point. Then we have*

$$\mathrm{Isom}_{\rm pg}(\pi_1^{\rm t}(q_1), \pi_1^{\rm t}(q_2))$$

is non-empty if and only if $V_{q_1} =_{fe} V_{q_2}$ (resp. $V_{q_1} = V_{q_2}$).

The weak Isom-version conjecture means that the *set structures* of the moduli spaces of smooth pointed stable curves can be understood by the tame fundamental groups of curves.

2.2. **The pointed collection conjecture.** In this subsection, we formulate the second conjecture of the present paper which we call "the pointed collection conjecture".

2.2.1. We maintain the notation introduced in 2.1.2.

2.2.2. Let q be an arbitrary point of $M_{g,n}^{\mathrm{ord}}$ and $G \in \pi_A^{\mathrm{t}}(q)$ an arbitrary finite group. We put

$$U_G \stackrel{\mathrm{def}}{=} \{q' \in M_{g,n}^{\mathrm{ord}} \mid G \in \pi_A^{\mathrm{t}}(q')\} \subseteq M_{g,n}^{\mathrm{ord}}.$$

Then we have the following result.

Proposition 2.3. *Let q be an arbitrary point of $M_{g,n}^{\mathrm{ord}}$ and $G \in \pi_A^{\mathrm{t}}(q)$ an arbitrary finite group. Then the set U_G contains an open neighborhood of q in $M_{g,n}^{\mathrm{ord}}$.*

Proof. Proposition 2.3 was proved by K. Stevenson when $n = 0$ and q is a closed point of $M_{g,0}$ (cf. [Ste, Proposition 4.2]). Moreover, by similar arguments to the arguments given in the proof of [Ste, Proposition 4.2], Proposition 2.3 also holds for $n \geq 0$. □

Definition 2.4. We denote by q_{gen} the generic point of $M_{g,n}^{\mathrm{ord}}$, and let

$$\mathcal{C} \subseteq \pi_A^{\mathrm{t}}(q_{\mathrm{gen}}) = \bigcup_{q \in M_{g,n}^{\mathrm{ord,cl}}} \pi_A^{\mathrm{t}}(q)$$

be a subset of $\pi_A^{\mathrm{t}}(q_{\mathrm{gen}})$. We shall say that $\mathcal{C}$ is a *pointed collection* if the following conditions are satisfied:

(i) $0 < \#((\bigcap_{G \in \mathcal{C}} U_G) \cap M_{g,n}^{\mathrm{ord,cl}}) < \infty$;
(ii) $U_{G'} \cap (\bigcap_{G \in \mathcal{C}} U_G) \cap M_{g,n}^{\mathrm{ord,cl}} = \emptyset$ for each $G' \in \pi_A^{\mathrm{t}}(q_{\mathrm{gen}})$ such that $G' \notin \mathcal{C}$.

On the other hand, for each closed point $t \in M_{g,n}^{\mathrm{ord,cl}}$, we may define a set associated to t as follows:

$$\mathcal{C}_t \stackrel{\mathrm{def}}{=} \{G \in \pi_A^{\mathrm{t}}(q_{\mathrm{gen}}) \mid t \in U_G\}.$$

Note that, if $t \in V_q^{\mathrm{cl}}$ and q is not a closed point, then a result of Tamagawa ([T5, Theorem 0.3]) implies that $\mathcal{C}_t \subseteq \pi_A^{\mathrm{t}}(q)$ and $\mathcal{C}_t \neq \pi_A^{\mathrm{t}}(q)$. Moreover, we denote by

$$\mathscr{C}_q \stackrel{\mathrm{def}}{=} \{\mathcal{C} \text{ is a pointed collection} \mid \mathcal{C} \subseteq \pi_A^{\mathrm{t}}(q)\}.$$

2.2.3. At present, no published results are known concerning the weak Hom-version conjecture (or the weak Isom-version conjecture) for *non-closed* points. The main difficulty of proving the weak Hom-version conjecture (or the weak Isom-version conjecture) for non-closed points of $M_{g,n}^{\mathrm{ord}}$ is the following: For each $q \in M_{g,n}^{\mathrm{ord}}$, we *do not* know how to reconstruct the tame fundamental groups of closed points of V_q group-theoretically from $\pi_1^{\mathrm{t}}(q)$.

Once the tame fundamental groups of the closed points of V_q can be reconstructed group-theoretically from $\pi_1^{\mathrm{t}}(q)$, then the weak Hom-version conjecture for closed points of $M_{g,n}^{\mathrm{ord}}$ implies that the set of closed points of V_q can be reconstructed group-theoretically from $\pi_1^{\mathrm{t}}(q)$. Thus, the weak Hom-version conjecture for non-closed points of $M_{g,n}^{\mathrm{ord}}$ can be deduced from the weak Hom-version conjecture for closed points of $M_{g,n}^{\mathrm{ord}}$.

Let $q \in M_{g,n}^{\mathrm{ord}}$. Since the isomorphism class of $\pi_1^{\mathrm{t}}(q)$ as a profinite group can be determined by the set $\pi_A^{\mathrm{t}}(q)$, the following conjecture tell us how to reconstruct group-theoretically the set of finite quotients of a closed point of V_q from $\pi_A^{\mathrm{t}}(q)$ (or $\pi_1^{\mathrm{t}}(q)$).

Pointed Collection Conjecture . *For each $t \in M_{g,n}^{\mathrm{ord,cl}}$, the set $\mathcal{C}_t$ associated to t is a pointed collection. Moreover, let $q \in M_{g,n}^{\mathrm{ord}}$. Then the natural map*

$$\mathrm{colle}_q : \mathscr{V}_q^{\mathrm{cl}} \to \mathscr{C}_q,\ [t] \mapsto \mathcal{C}_t,$$

is a bijection, where $[t]$ denotes the image of t in $\mathscr{V}_q^{\mathrm{cl}} \stackrel{\mathrm{def}}{=} V_q^{\mathrm{cl}}/\sim_{fe}$.

Write $q' \in M_{g,n,\mathbb{F}_p}$ for the image $\omega(q)$. Then we have $\mathscr{V}_q^{\rm cl} = V_{q'}^{\rm cl}$. This means that the pointed collection conjecture holds if and only if the weak Hom-version conjecture holds.

3. Reconstructions of marked points

3.1. **Anabelian reconstructions.**

3.1.1. **Settings.** We maintain the notation introduced in 2.1.1.

3.1.2. Let us recall the definitions concerning "anabelian reconstructions".

Definition 3.1. Let $\mathcal{F}$ be a geometric object and $\Pi_{\mathcal{F}}$ a profinite group associated to the object $\mathcal{F}$. Suppose that we are given an invariant $\mathrm{Inv}_{\mathcal{F}}$ depending on the isomorphism class of $\mathcal{F}$ (in a certain category), and that we are given an additional structure $\mathrm{Add}_{\mathcal{F}}$ (e.g., a family of subgroups, a family of quotient groups) on the profinite group $\Pi_{\mathcal{F}}$ depending functorially on $\mathcal{F}$.

We shall say that $\mathrm{Inv}_{\mathcal{F}}$ can be *mono-anabelian reconstructed from* $\Pi_{\mathcal{F}}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}}$, and whose output datum is $\mathrm{Inv}_{\mathcal{F}}$. We shall say that $\mathrm{Add}_{\mathcal{F}}$ *can be mono-anabelian reconstructed from* $\Pi_{\mathcal{F}}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}}$.

Let $\mathcal{F}_i$, $i \in \{1, 2\}$, be a geometric object and $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object $\mathcal{F}_i$. Suppose that we are given an additional structure $\mathrm{Add}_{\mathcal{F}_i}$ on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on $\mathcal{F}_i$. We shall say that a map (or a morphism) $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$ *can be mono-anabelian reconstructed* from an open continuous homomorphism $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$.

3.1.3. Let K be the function field of X, and let $\widetilde{K}$ be the maximal Galois extension of K in a fixed separable closure of K, unramified over U_X and at most tamely ramified over D_X. Then we may identify $\pi_1^{\rm t}(U_X)$ with $\mathrm{Gal}(\widetilde{K}/K)$. We define the universal tame covering of (X, D_X) associated to $\pi_1^{\rm t}(U_X)$ to be

$$(\widetilde{X}, D_{\widetilde{X}}),$$

where $\widetilde{X}$ denotes the normalization of X in $\widetilde{K}$, and $D_{\widetilde{X}}$ denotes the inverse image of D_X in $\widetilde{X}$. Then there is a natural action of $\pi_1^{\rm t}(U_X)$ on $(\widetilde{X}, D_{\widetilde{X}})$. For each $\widetilde{e} \in D_{\widetilde{X}}$, we denote by $I_{\widetilde{e}}$ the inertia subgroup of $\pi_1^{\rm t}(U_X)$ associated to $\widetilde{e}$ (i.e. the stabilizer of $\widetilde{e}$ in $\pi_1^{\rm t}(U_X)$). Then we have $I_{\widetilde{e}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $\widehat{\mathbb{Z}}(1)^{p'}$ denotes the prime-to-p part of $\widehat{\mathbb{Z}}(1)$. The following result was proved by Tamagawa ([T4, Lemma 5.1 and Theorem 5.2]).

Proposition 3.2. *(i) The type (g_X, n_X) can be mono-anabelian reconstructed from $\pi_1^{\rm t}(U_X)$.*

(ii) Let $\widetilde{e}$ and $\widetilde{e}'$ be two points of $D_{\widetilde{X}}$ distinct from each other. Then the intersection of $I_{\widetilde{e}}$ and $I_{\widetilde{e}'}$ is trivial in $\pi_1^{\rm t}(U_X)$. Moreover, the map

$$D_{\widetilde{X}} \to \mathrm{Sub}(\pi_1^{\rm t}(U_X)),\ \widetilde{e} \mapsto I_{\widetilde{e}},$$

is an injection, where $\mathrm{Sub}(-)$ *denotes the set of closed subgroups of* $(-)$.

(iii) Write $\mathrm{Ine}(\pi_1^{\rm t}(U_X))$ *for the set of inertia subgroups in $\pi_1^{\rm t}(U_X)$, namely the image of the map $D_{\widetilde{X}} \to \mathrm{Sub}(\pi_1^{\rm t}(U_X))$. Then* $\mathrm{Ine}(\pi_1^{\rm t}(U_X))$ *can be mono-anabelian reconstructed from $\pi_1^{\rm t}(U_X)$. In particular, the set of marked points D_X and $\pi_1(X)$ can be mono-anabelian reconstructed from $\pi_1^{\rm t}(U_X)$.*

The main purposes of the remainder of present section are as follows: We will give a new mono-anabelian reconstruction of $\mathrm{Ine}(\pi_1^{\rm t}(U_X))$, and prove that the mono-anabelian reconstruction (i.e. the group-theoretical algorithm) *is compatible with any open continuous homomorphisms* of tame fundamental groups of smooth pointed stable curves with a fixed type.

3.2. **The set of marked points.**

3.2.1. **Settings.** We maintain the notation introduced in 2.1.1. Moreover, we suppose that $g_X \geq 2$ and $n_X > 0$.

3.2.2. In this subsection, we will prove that the set of marked points can be regarded as a quotient set of a set of cohomological classes of a suitable covering of curves (i.e. Proposition 3.3). The main idea is the following: By taking a suitable étale covering with a prime degree $f : (Y, D_Y) \to (X, D_X)$, for every marked point $x \in D_X$, there exists a set of tame coverings with a prime degree which is totally ramified over the inverse image $f^{-1}(x)$. Then x can be regarded as the set cohomological classes corresponding to such coverings.

3.2.3. Let $h : (W, D_W) \to (X, D_X)$ be a connected Galois tame covering over k. We put

$$\mathrm{Ram}_h \stackrel{\text{def}}{=} \{e \in D_X \mid h \text{ is ramified over } e\}.$$

Let (Y, D_Y) be a smooth pointed stable curve over k. We shall say that

$$(\ell, d, f : (Y, D_Y) \to (X, D_X))$$

is *an mp-triple associated to* (X, D_X) if the following conditions hold: (i) ℓ and d are prime numbers distinct from each other such that $(\ell, p) = (d, p) = 1$ and $\ell \equiv 1 \pmod d$; then all dth roots of unity are contained in $\mathbb{F}_\ell$; (ii) f is a Galois *étale* covering over k whose Galois group is isomorphic to μ_d, where $\mu_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of dth roots of unity. Here, "mp" means "marked points".

Then we have a natural injection $H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \hookrightarrow H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ induced by the natural surjection $\pi_1^{\mathrm{t}}(U_Y) \twoheadrightarrow \pi_1(Y)$. Note that every non-zero element of $H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ induces a connected Galois tame covering of (Y, D_Y) of degree ℓ. We obtain an exact sequence

$$0 \to H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell) \to \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell \to 0$$

with a natural action of μ_d.

3.2.4. Let $(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell)_{\mu_d} \subseteq \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell$ be the subset of elements on which μ_d acts via the character $\mu_d \hookrightarrow \mathbb{F}_\ell^\times$ and $M_Y^* \subseteq H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ the subset of elements whose images are non-zero elements of $(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell)_{\mu_d}$. For each $\alpha \in M_Y^*$, write $g_\alpha : (Y_\alpha, D_{Y_\alpha}) \to (Y, D_Y)$ for the tame covering induced by α. We define $\epsilon : M_Y^* \to \mathbb{Z}$, where $\epsilon(\alpha) \stackrel{\text{def}}{=} \# D_{Y_\alpha}$. Denote by

$$M_Y \stackrel{\text{def}}{=} \{\alpha \in M_Y^* \mid \#\mathrm{Ram}_{g_\alpha} = d\} = \{\alpha \in M_Y^* \mid \epsilon(\alpha) = \ell(dn_X - d) + d\}.$$

Note that since $(\ell, p) = (d, p) = 1$ and $\#(f^{-1}(x)) = \ell$ for all $x \in D_X$, the structure of maximal pro-prime-to-p quotient of $\pi_1^{\mathrm{t}}(U_Y)$ (i.e. it's isomorphic to the pro-prime-to-p completion of the topological fundamental group of a Riemann surface of type (g_Y, n_Y)) implies that M_Y is not empty.

For each $\alpha \in M_Y$, since the image of α is contained in $(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell)_{\mu_d}$, we obtain that the action of μ_d on $\mathrm{Ram}_{g_\alpha} \subseteq D_Y$ is transitive. Thus, there exists a unique marked point $e_\alpha \in D_X$ such that $f(y) = e_\alpha$ for each $y \in \mathrm{Ram}_{g_\alpha}$.

For each $e \in D_X$, we put

$$M_{Y,e} \stackrel{\text{def}}{=} \{\alpha \in M_Y \mid g_\alpha \text{ is ramified over } f^{-1}(e)\}.$$

Then, for any marked points $e, e' \in D_X$ distinct from each other, we have $M_{Y,e} \cap M_{Y,e'} = \emptyset$ and the disjoint union

$$M_Y = \bigsqcup_{e \in D_X} M_{Y,e}.$$

3.2.5. Next, we define a pre-equivalence relation $\sim$ on M_Y as follows: Let $\alpha, \beta \in M_Y$. Then $\alpha \sim \beta$ if $\lambda\alpha + \mu\beta \in M_Y$ for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in M_Y^*$. Then we have the following proposition.

Proposition 3.3. *The pre-equivalence relation $\sim$ on M_Y is an equivalence relation. Moreover, the map*

$$\vartheta_X : M_Y/\sim \to D_X,\ [\alpha] \mapsto e_\alpha,$$

is a bijection, where $[\alpha]$ denotes the image of α in $M_Y/\sim$.

Proof. Let $\beta, \gamma \in M_Y$. If $\mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\beta + \mu\gamma \neq 0$, we have $\mathrm{Ram}_{g_{\lambda\beta+\mu\gamma}} = \mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}$. Thus we obtain that $\beta \sim \gamma$. On the other hand, if $\beta \sim \gamma$, we have $\mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}$. Otherwise, we have $\#\mathrm{Ram}_{g_{\beta+\gamma}} = 2d$. This means that $\beta \sim \gamma$ if and only if $\mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}$. Then $\sim$ is an equivalence relation on M_Y.

Let us prove that ϑ_X is a bijection. It is easy to see that ϑ_X is an injection. On the other hand, for each $e \in D_X$, the structure of the maximal pro-ℓ tame fundamental groups implies that we may construct a connected tame Galois covering of $h : (Z, D_Z) \to (Y, D_Y)$ such that h is totally tamely ramified over $f^{-1}(e)$ (i.e. the element of $H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ induced by h is contained in M_Y). Then ϑ_X is a surjection. This completes the proof of Proposition 3.3. □

Remark 3.3.1. We claim that the set $M_Y/\sim$ does not depend on the choices of mp-triples associated to (X, D_X). Let

$$(\ell^*, d^*, f^* : (Y^*, D_{Y^*}) \to (X, D_X))$$

be an arbitrary mp-triple associated to (X, D_X). Hence we obtain a resulting set $M_{Y^*}/\sim$ and a natural bijection $\vartheta_X^* : M_{Y^*}/\sim \to D_X$. We will prove that there exists a natural bijection $\delta : M_{Y^*}/\sim \xrightarrow{\sim} M_Y/\sim$ such that $\vartheta_X^* = \vartheta_X \circ \delta$.

First, suppose that $\ell \neq \ell^*$ and $d \neq d^*$. Then we may construct a natural bijection $\delta : M_{Y^*}/\sim \xrightarrow{\sim} M_Y/\sim$ as follows. Let $\alpha \in M_Y$ and $\alpha^* \in M_{Y^*}$. Write $(Y_\alpha, D_{Y_\alpha}) \to (Y, D_Y)$ and $(Y_{\alpha^*}, D_{Y_{\alpha^*}}) \to (Y^*, D_{Y^*})$ for the Galois tame coverings induced by α and α^*, respectively. We consider the following fiber product in the category of smooth pointed stable curves

$$(Y_\alpha, D_{Y_\alpha}) \times_{(X,D_X)} (Y_{\alpha^*}, D_{Y_{\alpha^*}})$$

which is a smooth pointed stable curve over k. Thus, we obtain a connected tame covering $(Y_\alpha, D_{Y_\alpha}) \times_{(X,D_X)} (Y_{\alpha^*}, D_{Y_{\alpha^*}}) \to (X, D_X)$ of degree $dd^*\ell\ell^*$. Then it is easy to check that $\vartheta_X([\alpha]) = \vartheta_X^*([\alpha^*])$ if and only if the cardinality of the set of marked points of $(Y_\alpha, D_{Y_\alpha}) \times_{(X,D_X)} (Y_{\alpha^*}, D_{Y_{\alpha^*}})$ is equal to $dd^*(\ell\ell^*(n_X - 1) + 1)$. We put $[\alpha] \overset{\text{def}}{=} \delta([\alpha^*])$ if $\vartheta_X([\alpha]) = \vartheta_X^*([\alpha^*])$. Moreover, by the construction above, we obtain that $\vartheta_X^* = \vartheta_X \circ \delta$. In general case, we may choose an mp-triple

$$(\ell^{**}, d^{**}, f^{**} : (Y^{**}, D_{Y^{**}}) \to (X, D_X))$$

associated to (X, D_X) such that $\ell^{**} \neq \ell$, $\ell^{**} \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain a resulting set $M_{Y^{**}}/\sim$ and a natural bijection $\vartheta_X^{**} : M_{Y^{**}}/\sim \to D_X$. Then the proof given above implies that there are natural bijections $\delta_1 : M_{Y^{**}}/\sim \xrightarrow{\sim} M_Y/\sim$ and $\delta_2 : M_{Y^{**}}/\sim \xrightarrow{\sim} M_{Y^*}/\sim$. Thus, we may put

$$\delta \overset{\text{def}}{=} \delta_1 \circ \delta_2^{-1} : M_{Y^*}/\sim \xrightarrow{\sim} M_Y/\sim .$$

Remark 3.3.2. Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an arbitrary open normal subgroup and $f_H : (X_H, D_{X_H}) \to (X, D_X)$ the Galois tame covering over k induced by the natural inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$. Let

$$(\ell, d, f : (Y, D_Y) \to (X, D_X))$$

be an mp-triple associated to (X, D_X) such that $(\#(\pi_1^{\mathrm{t}}(U_X)/H), \ell) = (\#(\pi_1^{\mathrm{t}}(U_X)/H), d) = 1$. Then we obtain an mp-triple

$$(\ell, d, g : (Z, D_Z) \overset{\text{def}}{=} (Y, D_Y) \times_{(X,D_X)} (X_H, D_{X_H}) \to (X_H, D_{X_H}))$$

associated to (X_H, D_{X_H}) induced by $(\ell, d, f : (Y, D_Y) \to (X, D_X))$, where $(Y, D_Y)\times_{(X,D_X)}(X_H, D_{X_H})$ denotes the fiber product in the category of smooth pointed stable curves. The mp-triple associated to (X_H, D_{X_H}) induces a set $M_Z/\sim$ which can be identified with the set of marked points D_{X_H} of (X_H, D_{X_H}) by applying Proposition 3.3. Moreover, for each $e_X \in D_X$ and each $\alpha_{Y,e_X} \in M_{Y,e_X}$, α_{Y,e_X} induces an element

$$\alpha_Z = \sum_{e_{X_H} \in f_H^{-1}(e_X)} \alpha_{Z,e_{X_H}}$$

over (Z, D_Z) via the natural morphism $(Z, D_Z) \to (Y, D_Y)$, where $\alpha_{Z,e_{X_H}} \in M_{Z,e_{X_H}}$. On the other hand, for each $e'_{X_H} \in D_{X_H}$ and each $e'_X \in D_X$, we have that $f_H(e'_{X_H}) = e'_X$ if and only if there exists an element $\alpha_{Y,e'_X} \in M_{Y,e'_X}$ such that the following conditions hold: (i) the element α'_Z, induced by α_{Y,e'_X} via the natural morphism $(Z, D_Z) \to (Y, D_Y)$, can be represented by a linear combination

$$\alpha'_Z = \sum_{e_{X_H} \in S_{X_H}} \alpha'_{Z,e_{X_H}},$$

where S_{X_H} is a subset of D_{X_H}, and $\alpha_{Z,e_{X_H}} \in M_{Z,e_{X_H}}$; (ii) $e'_{X_H} \in S_{X_H}$.

Lemma 3.4. *Let $(\ell, d, f : (Y, D_Y) \to (X, D_X))$ be an mp-triple associated to (X, D_X) and g_Y the genus of Y. Then we have $\#(M_{Y,e}) = \ell^{2g_Y+1} - \ell^{2g_Y}$, $e \in D_X$. Moreover, we have $\#(M_Y) = n_X(\ell^{2g_Y+1} - \ell^{2g_Y})$.*

Proof. Let $e \in D_X$. Write $D_e \subseteq D_Y$ for the set $f^{-1}(e)$. Then $M_{Y,e}$ can be naturally regarded as a subset of $H^1_{\text{ét}}(Y \setminus D_e, \mathbb{F}_\ell)$ via the natural open immersion $Y \setminus D_e \hookrightarrow Y$. Write L_e for the $\mathbb{F}_\ell$-vector space generated by $M_{Y,e}$ in $H^1_{\text{ét}}(Y \setminus D_e, \mathbb{F}_\ell)$. Then we have $M_{Y,e} = L_e \setminus H^1_{\text{ét}}(Y, \mathbb{F}_\ell)$. Write H_e for the quotient $L_e/H^1_{\text{ét}}(Y, \mathbb{F}_\ell)$. We have an exact sequence as follows:

$$0 \to H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to L_e \to H_e \to 0.$$

Since the action of μ_d on $f^{-1}(e)$ is transitive, we obtain $\dim_{\mathbb{F}_\ell}(H_e) = 1$. On the other hand, since $\dim_{\mathbb{F}_\ell}(H^1_{\text{ét}}(Y, \mathbb{F}_\ell)) = 2g_Y$, we obtain $\#(M_{Y,e}) = \ell^{2g_Y+1} - \ell^{2g_Y}$. Thus, we have $\#(M_Y) = n_X(\ell^{2g_Y+1} - \ell^{2g_Y})$. This completes the proof of the lemma. □

3.3. Reconstructions of inertia subgroups.

3.3.1. **Settings.** We maintain the notation introduced in 2.1.1.

3.3.2. In this subsection, we will prove that the inertia subgroups of marked points can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_X)$ (i.e. Proposition 3.7). The main idea is as follows: Let $H \subseteq \pi_1^{\text{t}}(U_X)$ be an arbitrary normal open subgroup and $(X_H, D_{X_H}) \to (X, D_X)$ the tame covering corresponding to H. Firstly, by using some numerical conditions induced by the Riemann-Hurwitz formula, the étale fundamental group $\pi_1(X)$ can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_X)$. Then the results obtained in Section 3.2 implies that D_X can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_X)$. Moreover, D_{X_H} can be also mono-anabelian reconstructed from H. Secondly, since the natural injection $H \hookrightarrow \pi_1^{\text{t}}(U_X)$ induces a map of sets of cohomological classes obtained in Section 3.2, we obtain that the natural map $D_{X_H} \to D_X$ can be mono-anabelian reconstructed from $H \hookrightarrow \pi_1^{\text{t}}(U_X)$. Thus, by taking a cofinal system of open normal subgroups of $\pi_1^{\text{t}}(U_X)$, we obtain a new mono-anabelian reconstruction of $\text{Ine}(\pi_1^{\text{t}}(U_X))$.

3.3.3. First, we have the following lemma.

Lemma 3.5. *(i) The prime number p (i.e. the characteristic of k) can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_X)$.*

(ii) The étale fundamental group $\pi_1(X)$ can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_X)$.

Proof. (i) Let $\mathfrak{P}$ be the set of prime numbers, and let Q be an arbitrary open subgroup of $\pi_1^{\mathrm{t}}(U_X)$ and r_Q an integer such that

$$\#\{l \in \mathfrak{P} \mid r_Q = \dim_{\mathbb{F}_l}(Q^{\mathrm{ab}} \otimes \mathbb{F}_l)\} = \infty.$$

Then we see immediately that the characteristic of k is the unique prime number p such that there exists an open subgroup $T \subseteq \pi_1^{\mathrm{t}}(U_X)$ and $r_T \neq \dim_{\mathbb{F}_p}(T^{\mathrm{ab}} \otimes \mathbb{F}_p)$.

(ii) Let H be an arbitrary open normal subgroup of $\pi_1^{\mathrm{t}}(U_X)$. We denote by (X_H, D_{X_H}) the smooth pointed stable curve of type (g_{X_H}, n_{X_H}) over k induced by H, and denote by $f_H : (X_H, D_{X_H}) \to (X, D_X)$ the morphism of smooth pointed stable curves over k induced by the natural inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$. We note that f_H is étale if and only if $g_{X_H} - 1 = \#(\pi_1^{\mathrm{t}}(U_X)/H)(g_X - 1)$. We put

$$\mathrm{Et}(\pi_1^{\mathrm{t}}(U_X)) \overset{\mathrm{def}}{=} \{H \subseteq \pi_1^{\mathrm{t}}(U_X) \text{ is an open normal subgroup} \mid$$

$$g_{X_H} - 1 = \#(\pi_1^{\mathrm{t}}(U_X)/H)(g_X - 1)\}.$$

Moreover, Proposition 3.2 (i) implies that g_{X_H} and g_X can be mono-anabelian reconstructed from H and $\pi_1^{\mathrm{t}}(U_X)$, respectively. Then the set $\mathrm{Et}(\pi_1^{\mathrm{t}}(U_X))$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$. We obtain that

$$\pi_1(X) = \pi_1^{\mathrm{t}}(U_X) / \bigcap_{H \in \mathrm{Et}(\pi_1^{\mathrm{t}}(U_X))} H.$$

This completes the proof of the lemma. □

3.3.4. Suppose $g_X \geq 2$. Let us define a group-theoretical object corresponding to an mp-triple which was introduced in 3.2.3. We shall say that

$$(\ell, d, y)$$

is *an mp-triple associated to* $\pi_1^{\mathrm{t}}(U_X)$ if the following conditions hold: (i) ℓ and d are prime numbers distinct from each other such that $(\ell, p) = (d, p) = 1$ and $\ell \equiv 1 \pmod{d}$; then all dth roots of unity are contained in $\mathbb{F}_\ell$; (ii) $y \in \mathrm{Hom}(\pi_1(X), \mu_d)$ such that $y \neq 0$, where $\mu_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of dth roots of unity.

3.3.5. Moreover, by applying Lemma 3.5, there is a triple (ℓ, d, y) associated to $\pi_1^{\mathrm{t}}(U_X)$ which can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$. Let $f : (Y, D_Y) \to (X, D_X)$ be a Galois étale covering induced by y. Then we see immediately that $(\ell, d, f : (Y, D_Y) \to (X, D_X))$ is an mp-triple associated to (X, D_X) defined in 3.2.3. We denote by $\pi_1^{\mathrm{t}}(U_Y)$ the kernel of the composition of the surjections $\pi_1^{\mathrm{t}}(U_X) \twoheadrightarrow \pi_1(X) \overset{y}{\twoheadrightarrow} \mu_d$. Since $H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \cong \mathrm{Hom}(\pi_1(Y), \mathbb{F}_\ell)$ and $H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell) \cong \mathrm{Hom}(\pi_1^{\mathrm{t}}(U_Y), \mathbb{F}_\ell)$, Lemma 3.5 implies immediately that the following exact sequence

$$0 \to H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell) \to \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell \to 0$$

can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_Y)$. Thus, Proposition 3.2 (i) implies that the set $M_Y/\sim$ defined in 3.2.5 can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_Y)$. Note that, by Remark 3.3.1, the set $M_Y/\sim$ does not depend on the choices of mp-triples. Then we put

$$D_X^{\mathrm{gp}} \overset{\mathrm{def}}{=} M_Y/\sim,$$

where $(-)^{\mathrm{gp}}$ means "group-theoretical". By Proposition 3.3, we may identify D_X^{gp} with the set of marked points D_X of (X, D_X) via the bijection $\vartheta_X : D_X^{\mathrm{gp}} \overset{\sim}{\to} D_X$ defined in Proposition 3.3.

Proposition 3.6. *Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an arbitrary open normal subgroup and*

$$f_H : (X_H, D_{X_H}) \to (X, D_X)$$

the morphism of smooth pointed stable curves over k induced by the natural inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$. Suppose $g_X \geq 2$. Then the sets D_X^{gp} and $D_{X_H}^{\mathrm{gp}}$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$

and H, respectively. Moreover, the inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$ induces a map $\gamma_{H,\pi_1^{\mathrm{t}}(U_X)} : D^{\mathrm{gp}}_{X_H} \to D^{\mathrm{gp}}_X$ such that the following commutative diagram holds:

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_H} & \xrightarrow{\vartheta_{X_H}} & D_{X_H} \\ {\scriptstyle \gamma_{H,\pi_1^{\mathrm{t}}(U_X)}}\downarrow & & \downarrow{\scriptstyle \gamma_{f_H}} \\ D^{\mathrm{gp}}_X & \xrightarrow{\vartheta_X} & D_X, \end{array}$$

where γ_{f_H} denotes the map of the sets of marked points induced by f_H.

Proof. We only need to prove the "moreover" part of Proposition 3.6. We maintain the notation introduced in Remark 3.3.2. Note that, for each $e_X \in D_X$ and each $e_{X_H} \in D_{X_H}$, the sets M_{Y,e_X} and $M_{Z,e_{X_H}}$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$ and H, respectively. Then the "moreover" part follows from Remark 3.3.2. □

Remark 3.6.1. We maintain the notation introduced in Proposition 3.6. Let $\pi_1(X_H)$ be the étale fundamental group of X_H. Then we have a natural surjection $H \twoheadrightarrow \pi_1(X_H)$. Note that $\pi_1(X_H)$ admits an action of $\pi_1^{\mathrm{t}}(U_X)/H$ induced by the outer action of $\pi_1^{\mathrm{t}}(U_X)/H$ on H induced by the exact sequence

$$1 \to H \to \pi_1^{\mathrm{t}}(U_X) \to \pi_1^{\mathrm{t}}(U_X)/H \to 1.$$

Moreover, the action of $\pi_1^{\mathrm{t}}(U_X)/H$ on $\pi_1(X_H)$ induces an action of $\pi_1^{\mathrm{t}}(U_X)/H$ on $D^{\mathrm{gp}}_{X_H}$. On the other hand, it is easy to check that the action of $\pi_1^{\mathrm{t}}(U_X)/H$ on $D^{\mathrm{gp}}_{X_H}$ coincides with the natural action of $\pi_1^{\mathrm{t}}(U_X)/H$ on D_{X_H} when we identify D^{gp}_X with D_X.

3.3.6. We have the following result.

Proposition 3.7. *Write* $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$ *for the set of inertia subgroups in* $\pi_1^{\mathrm{t}}(U_X)$*. Then* $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$ *can be mono-anabelian reconstructed from* $\pi_1^{\mathrm{t}}(U_X)$*.*

Proof. Let $C_X \stackrel{\mathrm{def}}{=} \{H_i\}_{i\in\mathbb{Z}_{>0}}$ be a set of open normal subgroups of $\pi_1^{\mathrm{t}}(U_X)$ such that $\varprojlim_i \pi_1^{\mathrm{t}}(U_X)/H_i \cong \pi_1^{\mathrm{t}}(U_X)$ (i.e. a cofinal system of open normal subgroups).

Let $\widetilde{e} \in D_{\widetilde{X}}$. For each $i \in \mathbb{Z}_{>0}$, we write $(X_{H_i}, D_{X_{H_i}})$ for the smooth pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ induced by H_i and $e_{X_{H_i}} \in D_{X_{H_i}}$ for the image of $\widetilde{e}$. Then we obtain a sequence of marked points

$$\mathcal{I}^{C_X}_{\widetilde{e}} : \cdots \mapsto e_{X_{H_2}} \mapsto e_{X_{H_1}}$$

induced by C_X. Note that the sequence $\mathcal{I}^{C_X}_{\widetilde{e}}$ admits a natural action of $\pi_1^{\mathrm{t}}(U_X)$. We may identify the inertia subgroup $I_{\widetilde{e}}$ associated to $\widetilde{e}$ with the stabilizer of $\mathcal{I}^{C_X}_{\widetilde{e}}$.

Moreover, since Proposition 3.2 (i) implies that $(g_{X_{H_i}}, n_{X_{H_i}})$ can be mono-anabelian reconstructed from H_i, by choosing a suitable set of open normal subgroups C_X, we may assume that $g_{X_{H_1}} \geq 2$. If $n_{X_{H_1}} = 0$, Proposition 3.7 is trivial. Then we may assume that $n_{X_{H_1}} > 0$.

On the other hand, Proposition 3.6 implies that, for each H_i, $i \in \mathbb{Z}_{>0}$, the set $D^{\mathrm{gp}}_{X_{H_i}}$ can be mono-anabelian reconstructed from H_i. For each $e_{X_{H_i}} \in D_{X_{H_i}}$, we denote by

$$e^{\mathrm{gp}}_{X_{H_i}} \stackrel{\mathrm{def}}{=} \vartheta^{-1}_{X_{H_i}}(e_{X_{H_i}}).$$

Then the sequence of marked points $\mathcal{I}^{C_X}_{\widetilde{e}}$ induces a sequence

$$\mathcal{I}^{C_X}_{\widetilde{e}^{\mathrm{gp}}} : \cdots \mapsto e^{\mathrm{gp}}_{X_{H_2}} \mapsto e^{\mathrm{gp}}_{X_{H_1}}.$$

By applying the "moreover" part of Proposition 3.6, we see that $\mathcal{I}^{C_X}_{\widetilde{e}^{\mathrm{gp}}}$ can be mono-anabelian reconstructed from C_X. Then Remark 3.6.1 implies that the stabilizer of $\mathcal{I}^{C_X}_{\widetilde{e}^{\mathrm{gp}}}$ is equal to the stabilizer of $\mathcal{I}^{C_X}_{\widetilde{e}}$. This completes the proof of the proposition. □

3.4. Reconstructions of inertia subgroups via surjections.

3.4.1. **Settings.** Let (X_i, D_{X_i}), $i \in \{1,2\}$, be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$, $U_{X_i} \overset{\text{def}}{=} X_i \setminus D_{X_i}$, $\pi_1^{\text{t}}(U_{X_i})$ the tame fundamental group of U_{X_i}, and $\pi_1(X_i)$ the étale fundamental group of X_i. Then Lemma 3.5 implies that $\pi_1(X_i)$ can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_{X_i})$. Moreover, in this subsection, we suppose that $n_X > 0$, and that $\phi : \pi_1^{\text{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\text{t}}(U_{X_2})$ is an arbitrary open continuous surjective homomorphism of profinite groups.

Note that, since (X_i, D_{X_i}), $i \in \{1,2\}$, is a smooth pointed stable curve of type (g_X, n_X), ϕ induces a natural surjection $\phi^{p'} : \pi_1^{\text{t}}(U_{X_1})^{p'} \twoheadrightarrow \pi_1^{\text{t}}(U_{X_2})^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Since $\pi_1^{\text{t}}(U_{X_i})^{p'}$, $i \in \{1,2\}$, is topologically finitely generated, and $\pi_1^{\text{t}}(U_{X_1})^{p'}$ is isomorphic to $\pi_1^{\text{t}}(U_{X_2})^{p'}$ as abstract profinite groups, we obtain that $\phi^{p'} : \pi_1^{\text{t}}(U_{X_1})^{p'} \overset{\sim}{\to} \pi_1^{\text{t}}(U_{X_2})^{p'}$ is an isomorphism ([FJ, Proposition 16.10.6]).

3.4.2. In this subsection, we will prove that the *mono-anabelian reconstructions* obtained in Proposition 3.7 are *compatible* with any open continuous homomorphisms (i.e. Theorem 3.14). We explain the main idea. Let $H_2 \subseteq \pi_1^{\text{t}}(U_{X_2})$ be an arbitrary open normal subgroup and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2) \subseteq \pi_1^{\text{t}}(U_{X_1})$. We write $(X_{H_i}, D_{X_{H_i}})$, $i \in \{1,2\}$, for the smooth pointed smooth curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i induced by H_i. To prove the compatibility, we need to prove that, for any prime number $\ell \neq p$, the weight-monodromy filtration of $H_2^{\text{ab}} \otimes \mathbb{F}_\ell$ induces the weight-monodromy filtration of $H_1^{\text{ab}} \otimes \mathbb{F}_\ell$ via the natural surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$. Note that the weight 1 part of $H_i^{\text{ab}} \otimes \mathbb{F}_\ell$ corresponds to $\pi_1(X_{H_i})^{\text{ab}} \otimes \mathbb{F}_\ell$, and the weight 2 part of $H_i^{\text{ab}} \otimes \mathbb{F}_\ell$ corresponds to the image of the subgroup of H_i generated by the inertia subgroups of the marked points of $D_{X_{H_i}}$. The key observation is as follows:

> The inequality of the limit of p-averages (see Proposition 3.8 (i) below)
> $$\text{Avr}_p(H_1) \geq \text{Avr}_p(H_2)$$
> of H_1 and H_2 induced by the surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$ plays a role of the comparability of "Galois actions" in the theory of the anabelian geometry of curves over *algebraically closed fields of characteristic* $p > 0$.

3.4.3. Firstly, we have the following proposition.

Proposition 3.8. *(i) Let (X, D_X) be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $U_X \overset{\text{def}}{=} X \setminus D_X$, and $\pi_1^{\text{t}}(U_X)$ the tame fundamental group of U_X. Let $r \in \mathbb{N}$ be a natural number, and let K_{p^r-1} be the kernel of the natural surjection $\pi_1^{\text{t}}(U_X) \twoheadrightarrow \pi_1^{\text{t}}(U_X)^{\text{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}$, where $(-)^{\text{ab}}$ denotes the abelianization of $(-)$. Then we have*

$$\text{Avr}_p(\pi_1^{\text{t}}(U_X)) \overset{\text{def}}{=} \lim_{r\to\infty} \frac{\dim_{\mathbb{F}_p}(K_{p^r-1}^{\text{ab}} \otimes \mathbb{F}_p)}{\#(\pi_1^{\text{t}}(U_X)^{\text{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z})} = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X > 1. \end{cases}$$

(ii) We maintain the setting introduced in 3.4.1. Let $H_2 \subseteq \pi_1^{\text{t}}(U_{X_2})$ be an open normal subgroup such that $([\pi_1^{\text{t}}(U_{X_2}) : H_2], p) = 1$ and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2)$. Write g_{H_i}, $i \in \{1,2\}$, for the genus of the smooth pointed stable curve over k_i corresponding to $H_i \subseteq \pi_1^{\text{t}}(U_{X_i})$. Then we have $g_{H_1} \geq g_{H_2}$.

Proof. (i) is the Tamagawa's result concerning the limit of p-averages of $\pi_1^{\text{t}}(U_X)$ ([T4, Theorem 0.5]). Let us prove (ii). The surjection ϕ induces a surjection $\phi^{p'} : \pi_1^{\text{t}}(U_{X_1})^{p'} \twoheadrightarrow \pi_1^{\text{t}}(U_{X_2})^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Moreover, since $\pi_1^{\text{t}}(U_{X_i})^{p'}$, $i \in \{1,2\}$, is topologically finitely generated, and $\pi_1^{\text{t}}(U_{X_1})^{p'}$ is isomorphic to $\pi_1^{\text{t}}(U_{X_2})^{p'}$ as abstract profinite groups (since the types of (X_1, D_{X_1}) and (X_2, D_{X_2}) are equal to (g_X, n_X)), we obtain that $\phi^{p'}$ is an isomorphism (cf. [FJ, Proposition 16.10.6]).

On the other hand, since $[\pi_1^{\text{t}}(U_{X_1}) : H_1] = [\pi_1^{\text{t}}(U_{X_2}) : H_2]$ and $([\pi_1^{\text{t}}(U_{X_2}) : H_2], p) = 1$, we obtain that the natural homomorphism $\phi_H^{p'} : H_1^{p'} \twoheadrightarrow H_2^{p'}$ induced by $\phi_H \overset{\text{def}}{=} \phi|_{H_1} : H_1 \twoheadrightarrow H_2$ is also an isomorphism. This implies

$$\#(H_1^{\text{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}) = \#(H_2^{\text{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z})$$

for all $r \in \mathbb{N}$. Let $K_{H_i,p^{r-1}}$, $i \in \{1,2\}$, be the kernel of the natural surjection $H_i \twoheadrightarrow H_i^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}$. Then the surjection ϕ_H implies

$$\mathrm{Avr}_p(H_1) \overset{\mathrm{def}}{=} \lim_{r\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_1,p^r-1} \otimes \mathbb{F}_p)}{\#(H_1^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z})} \geq \mathrm{Avr}_p(H_2) \overset{\mathrm{def}}{=} \lim_{r\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_2,p^r-1} \otimes \mathbb{F}_p)}{\#(H_2^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z})}.$$

Thus, (ii) follows from (i). □

3.4.4. We have the following lemmas.

Lemma 3.9. *Let ℓ be a prime number distinct from p. Then the isomorphism $(\phi^{p'})^{-1} : \pi_1^{\mathrm{t}}(U_{X_2})^{p'} \xrightarrow{\sim} \pi_1^{\mathrm{t}}(U_{X_1})^{p'}$ induces an isomorphism*

$$\psi_X^\ell : H^1_{\text{ét}}(X_1, \mathbb{F}_\ell) \cong \mathrm{Hom}(\pi_1(X_1), \mathbb{F}_\ell) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(X_2), \mathbb{F}_\ell) \cong H^1_{\text{ét}}(X_2, \mathbb{F}_\ell).$$

Proof. Let $f_1 : (Y_1, D_{Y_1}) \to (X_1, D_{X_1})$ be an étale covering of degree ℓ over k_1. Write $f_2 : (Y_2, D_{Y_2}) \to (X_2, D_{X_2})$ for the connected Galois tame covering of degree ℓ over k_2 induced by $\phi^{p'}$. Then we will prove that f_2 is also an étale covering over k_2.

Write g_{Y_1} and g_{Y_2} for the genus of Y_1 and Y_2, respectively. Since f_1 is an étale covering of degree ℓ, the Riemann-Hurwitz formula implies $g_{Y_1} = \ell(g_{X_1} - 1) + 1$. On the other hand, the Riemann-Hurwitz formula implies $g_{Y_2} = \ell(g_{X_2} - 1) + 1 + \frac{1}{2}(\ell - 1)\#(\mathrm{Ram}_{f_2})$. By applying Proposition 3.8 (ii), the surjection ϕ implies $g_{Y_1} \geq g_{Y_2}$. This means $\#(\mathrm{Ram}_{f_2}) = 0$. So f_2 is an étale covering over k_2. Then the morphism $(\phi^{p'})^{-1}$ induces an injection

$$\psi_X^\ell : \mathrm{Hom}(\pi_1(X_1), \mathbb{F}_\ell) \hookrightarrow \mathrm{Hom}(\pi_1(X_2), \mathbb{F}_\ell).$$

Furthermore, since $\dim_{\mathbb{F}_\ell}(\mathrm{Hom}(\pi_1(X_1), \mathbb{F}_\ell)) = \dim_{\mathbb{F}_\ell}(\mathrm{Hom}(\pi_1(X_2), \mathbb{F}_\ell)) = 2g_X$, we obtain that ψ_X^ℓ is a bijection. This completes the proof of the lemma. □

Lemma 3.10. *Suppose $g_X \geq 2$. Then the surjection $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ induces a bijection*

$$\rho_\phi : D^{\mathrm{gp}}_{X_1} \xrightarrow{\sim} D^{\mathrm{gp}}_{X_2},$$

and the bijection ρ_ϕ can be mono-anabelian reconstructed from ϕ.

Proof. Let (ℓ, d, y_2) be an mp-triple associated to $\pi_1^{\mathrm{t}}(U_{X_2})$ (see 3.3.4). Then Lemma 3.9 implies that ϕ induces an mp-triple (ℓ, d, y_1) associated to $\pi_1^{\mathrm{t}}(U_{X_1})$, where $y_1 \overset{\mathrm{def}}{=} (\psi_X^d)^{-1}(y_2) \in \mathrm{Hom}(\pi_1(X_1), \mu_d)$.

Let $f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i})$, $i \in \{1,2\}$, be the étale covering of degree d over k_i induced by y_i. Then the mp-triple (ℓ, d, y_i) associated to $\pi_1^{\mathrm{t}}(U_{X_i})$ determines an mp-triple

$$(\ell, d, f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i}))$$

associated to (X_i, D_{X_i}) over k_i. Note that the types of (Y_1, D_{Y_1}) and (Y_2, D_{Y_2}) are equal.

Write $\pi_1^{\mathrm{t}}(U_{Y_i})$, $i \in \{1,2\}$, for the kernel of $\pi_1^{\mathrm{t}}(U_{X_i}) \twoheadrightarrow \pi_1(X_i) \overset{y_i}{\twoheadrightarrow} \mu_d$. By replacing (X_i, D_{X_i}) by (Y_i, D_{Y_i}), Lemma 3.9 implies that $(\phi|^{p'}_{\pi_1^{\mathrm{t}}(U_{Y_1})})^{-1}$ induces a commutative diagram as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_{\text{ét}}(Y_1, \mathbb{F}_\ell) & \longrightarrow & H^1_{\text{ét}}(U_{Y_1}, \mathbb{F}_\ell) & \longrightarrow & \mathrm{Div}^0_{D_{Y_1}}(Y_1) \otimes \mathbb{F}_\ell & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_Y^\ell} & & \downarrow{\scriptstyle \psi_Y^{\mathrm{t},\ell}} & & \downarrow & & \\
0 & \longrightarrow & H^1_{\text{ét}}(Y_2, \mathbb{F}_\ell) & \longrightarrow & H^1_{\text{ét}}(U_{Y_2}, \mathbb{F}_\ell) & \longrightarrow & \mathrm{Div}^0_{D_{Y_2}}(Y_2) \otimes \mathbb{F}_\ell & \longrightarrow & 0,
\end{array}$$

where all the vertical arrows are isomorphisms. We note that $H^1_{\text{ét}}(Y_i, \mathbb{F}_\ell)$, $H^1_{\text{ét}}(U_{Y_i}, \mathbb{F}_\ell)$, and $\mathrm{Div}^0_{D_{Y_i}}(Y_i) \otimes \mathbb{F}_\ell$, $i, \in \{1,2\}$, are naturally isomorphic to $\mathrm{Hom}(\pi_1(Y_i), \mathbb{F}_\ell)$, $\mathrm{Hom}(\pi_1^{\mathrm{t}}(U_{Y_i}), \mathbb{F}_\ell)$, and $\mathrm{Hom}(\pi_1^{\mathrm{t}}(U_{Y_i}), \mathbb{F}_\ell)/\mathrm{Hom}(\pi_1(Y_i), \mathbb{F}_\ell)$, respectively. Then Lemma 3.5 implies that the commutative diagram above can be mono-anabelian reconstructed from $\phi|_{\pi_1^{\mathrm{t}}(U_{Y_1})} : \pi_1^{\mathrm{t}}(U_{Y_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{Y_2})$.

Write $M_{Y_i} \subseteq M^*_{Y_i}$ for the subsets of $H^1_{\text{ét}}(U_{Y_i}, \mathbb{F}_\ell)$ defined in 3.2.4. Since the actions of μ_d on the exact sequences are compatible with the isomorphisms appearing in the commutative diagram above, we have $\psi_Y^{\mathrm{t},\ell}(M^*_{Y_1}) = M^*_{Y_2}$. Next, we prove $\psi_Y^{\mathrm{t},\ell}(M_{Y_1}) = M_{Y_2}$.

Let $\alpha_1 \in M_{Y_1}$ and $g_{\alpha_1} : (Y_{\alpha_1}, D_{Y_{\alpha_1}}) \to (Y_1, D_{Y_1})$ the Galois tame covering of degree ℓ over k_1 induced by α_1. Write $g_{\alpha_2} : (Y_{\alpha_2}, D_{Y_{\alpha_2}}) \to (Y_2, D_{Y_2})$ for the Galois tame covering of degree ℓ over

k_2 induced by $\alpha_2 \overset{\text{def}}{=} \psi_Y^{\mathrm{t},\ell}(\alpha_1)$. Write $g_{Y_{\alpha_1}}$ and $g_{Y_{\alpha_2}}$ for the genus of Y_{α_1} and Y_{α_2}, respectively. Then Proposition 3.8 (ii) and the Riemann-Hurwitz formula imply that $g_{Y_{\alpha_1}} - g_{Y_{\alpha_2}} = \frac{1}{2}(d - \#(\mathrm{Ram}_{g_{\alpha_2}}))(\ell-1) \geq 0$. This means $d - \#(\mathrm{Ram}_{g_{\alpha_2}}) \geq 0$. Since $\alpha_2 \in M^*_{Y_2}$, we have $d \mid \#(\mathrm{Ram}_{g_{\alpha_2}})$. Thus, either $\#(\mathrm{Ram}_{g_{\alpha_2}}) = 0$ or $\#(\mathrm{Ram}_{g_{\alpha_2}}) = d$ holds.

If $\#(\mathrm{Ram}_{g_{\alpha_2}}) = 0$, then g_{α_2} is an étale covering over k_2. Then Lemma 3.9 implies that g_{α_1} is an étale covering over k_1. This provides a contradiction to the fact that $\alpha_1 \in M_{Y_1}$. Then we have $\#(\mathrm{Ram}_{g_{\alpha_2}}) = d$. This means $\alpha_2 \in M_{Y_2}$. Thus, we obtain $\psi_Y^{\mathrm{t},\ell}(M_{Y_1}) \subseteq M_{Y_2}$. On the other hand, Lemma 3.4 implies $\#(M_{Y_1}) = \#(M_{Y_2})$. We have $\psi_Y^{\mathrm{t},\ell} : M_{Y_1} \xrightarrow{\sim} M_{Y_2}$. Then Proposition 3.3 implies that $\psi_Y^{\mathrm{t},\ell}$ induces a bijection

$$\rho_\phi : D^{\mathrm{gp}}_{X_1} \xrightarrow{\sim} D^{\mathrm{gp}}_{X_2}.$$

Moreover, since M_{Y_i} and $M^*_{Y_i}$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{Y_i})$, the bijection ρ_ϕ can be mono-anabelian reconstructed from ϕ. This completes the proof of the lemma. □

3.4.5. Let $H_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ be an arbitrary open normal subgroup and $H_1 \overset{\text{def}}{=} \phi^{-1}(H_2)$. We write $(X_{H_i}, D_{X_{H_i}})$, $i \in \{1,2\}$, for the smooth pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i induced by H_i and $f_{H_i} : (X_{H_i}, D_{X_{H_i}}) \to (X_i, D_{X_i})$ for the Galois tame coverings over k_i induced by the inclusion $H_i \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i})$. Moreover, Proposition 3.6 implies that the inclusion $H_i \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i})$ induces a map $\gamma_{H_i,\pi_1^{\mathrm{t}}(U_{X_i})} : D^{\mathrm{gp}}_{X_{H_i}} \to D^{\mathrm{gp}}_{X_i}$ which fits into the following commutative diagram:

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{H_i}} & \xrightarrow{\vartheta_{X_{H_i}}} & D_{X_{H_i}} \\ {\scriptstyle \gamma_{H_i,\pi_1^{\mathrm{t}}(U_{X_i})}}\downarrow & & \downarrow{\scriptstyle \gamma_{f_{H_i}}} \\ D^{\mathrm{gp}}_{X_i} & \xrightarrow{\vartheta_{X_i}} & D_{X_i}, \end{array}$$

where $\gamma_{f_{H_i}}$ denotes the map of the sets of marked points induced by f_{H_i}. We may identify $\pi_1^{\mathrm{t}}(U_{X_1})/H_1$ with $\pi_1^{\mathrm{t}}(U_{X_2})/H_2$ via the isomorphism $\pi_1^{\mathrm{t}}(U_{X_1})/H_1 \xrightarrow{\sim} \pi_1^{\mathrm{t}}(U_{X_2})/H_2$ induced by ϕ, and denote by $G \overset{\text{def}}{=} \pi_1^{\mathrm{t}}(U_{X_1})/H_1 \cong \pi_1^{\mathrm{t}}(U_{X_2})/H_2$. Then we have the following lemma.

Lemma 3.11. *Suppose that $g_X \geq 2$, and that $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then the commutative diagram of profinite groups*

(1)
$$\begin{array}{ccc} H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}) \end{array}$$

induces a commutative diagram

(2)
$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D^{\mathrm{gp}}_{X_{H_2}} \\ {\scriptstyle \gamma_{H_1,\pi_1^{\mathrm{t}}(U_{X_1})}}\downarrow & & \downarrow{\scriptstyle \gamma_{H_2,\pi_1^{\mathrm{t}}(U_{X_2})}} \\ D^{\mathrm{gp}}_{X_1} & \xrightarrow{\rho_\phi} & D^{\mathrm{gp}}_{X_2}. \end{array}$$

Moreover, the commutative diagram (2) *can be mono-anabelian reconstructed from* (1).

Proof. Proposition 3.6 and Lemma 3.10 imply the diagram

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D^{\mathrm{gp}}_{X_{H_2}} \\ {\scriptstyle \gamma_{H_1,\pi_1^{\mathrm{t}}(U_{X_1})}}\downarrow & & \downarrow{\scriptstyle \gamma_{H_2,\pi_1^{\mathrm{t}}(U_{X_2})}} \\ D^{\mathrm{gp}}_{X_1} & \xrightarrow{\rho_\phi} & D^{\mathrm{gp}}_{X_2} \end{array}$$

can be mono-anabelian reconstructed from the commutative diagram of profinite groups

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}). \end{array}$$

To verify Lemma 3.11, it is sufficient to check that the diagram is commutative.

Let $e^{\mathrm{gp}}_{X_{H_1}} \in D^{\mathrm{gp}}_{X_{H_1}}$, $e^{\mathrm{gp}}_{X_{H_2}} \overset{\mathrm{def}}{=} \rho_{\phi|_{H_1}}(e^{\mathrm{gp}}_{X_{H_1}}) \in D^{\mathrm{gp}}_{X_{H_2}}$, $e^{\mathrm{gp}}_{X_1} \overset{\mathrm{def}}{=} \gamma_{H_1,\pi_1^{\mathrm{t}}(U_{X_1})}(e^{\mathrm{gp}}_{X_{H_1}}) \in D^{\mathrm{gp}}_{X_1}$, $e^{\mathrm{gp}}_{X_2} \overset{\mathrm{def}}{=} (\gamma_{H_2,\pi_1^{\mathrm{t}}(U_{X_2})} \circ \rho_{\phi|_{H_1}})(e^{\mathrm{gp}}_{X_{H_1}}) \in D^{\mathrm{gp}}_{X_2}$, and $e^{\mathrm{gp},*}_{X_1} \overset{\mathrm{def}}{=} \rho_{\phi}^{-1}(e^{\mathrm{gp}}_{X_2}) \in D^{\mathrm{gp}}_{X_1}$. Let us prove

$$e^{\mathrm{gp}}_{X_1} = e^{\mathrm{gp},*}_{X_1}.$$

We put $S^{\mathrm{gp}}_{X_{H_1}} \overset{\mathrm{def}}{=} \gamma^{-1}_{H_1,\pi_1^{\mathrm{t}}(U_{X_1})}(e^{\mathrm{gp},*}_{X_1})$ and $S^{\mathrm{gp}}_{X_{H_2}} \overset{\mathrm{def}}{=} \gamma^{-1}_{H_2,\pi_1^{\mathrm{t}}(U_{X_2})}(e^{\mathrm{gp}}_{X_2})$, respectively. Note that $e^{\mathrm{gp}}_{X_{H_2}} \in S^{\mathrm{gp}}_{X_{H_2}}$. To verify $e^{\mathrm{gp}}_{X_1} = e^{\mathrm{gp},*}_{X_1}$, it is sufficient to prove that $e^{\mathrm{gp}}_{X_{H_1}} \in S^{\mathrm{gp}}_{X_{H_1}}$. Moreover, for each $i \in \{1, 2\}$, we put

$$e_{X_i} \overset{\mathrm{def}}{=} \vartheta_{X_i}(e^{\mathrm{gp}}_{X_i}),\ e_{X_{H_i}} \overset{\mathrm{def}}{=} \vartheta_{X_{H_i}}(e^{\mathrm{gp}}_{X_i}),\ e^*_{X_1} \overset{\mathrm{def}}{=} \vartheta_{X_1}(e^{\mathrm{gp},*}_{X_1}),\ S_{X_i} \overset{\mathrm{def}}{=} S^{\mathrm{gp}}_{X_i},\ S_{X_{H_i}} \overset{\mathrm{def}}{=} S^{\mathrm{gp}}_{X_{H_i}}.$$

Then to verify the lemma, we only need to prove that $e_{X_{H_1}} \in \vartheta_{X_{H_1}}(S_{X_{H_1}})$.

Let (ℓ, d, y_2) be an mp-triple associated to $\pi_1^{\mathrm{t}}(U_{X_2})$. Then Lemma 3.9 implies that ϕ induces an mp-triple (ℓ, d, y_1) associated to $\pi_1^{\mathrm{t}}(U_{X_1})$, where $y_1 \overset{\mathrm{def}}{=} (\psi^d_X)^{-1}(y_2) \in \mathrm{Hom}(\pi_1(X_1), \mu_d)$. Let $f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i})$, $i \in \{1, 2\}$, be the tame covering of degree d over k_i induced by y_i. Then the mp-triple (ℓ, d, y_i) associated to $\pi_1^{\mathrm{t}}(U_{X_i})$ induces an mp-triple

$$(\ell, d, f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i}))$$

associated to (X_i, D_{X_i}) over k_i. Note that since f_1 and f_2 are étale, the types of (Y_1, D_{Y_1}) and (Y_2, D_{Y_2}) are equal. On the other hand, we have an mp-triple

$$(\ell, d, g_2 : (Z_2, D_{Z_2}) \overset{\mathrm{def}}{=} (Y_2, D_{Y_2}) \times_{(X_2, D_{X_2})} (X_{H_2}, D_{X_{H_2}}) \to (X_{H_2}, D_{X_{H_2}}))$$

associated to $(X_{H_2}, D_{X_{H_2}})$ induced by the natural inclusion $H_2 \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ and the mp-triple $(\ell, d, f_2 : (Y_2, D_{Y_2}) \to (X_2, D_{X_2}))$. By Lemma 3.9 again, we obtain an mp-triple

$$(\ell, d, g_1 : (Z_1, D_{Z_1}) \overset{\mathrm{def}}{=} (Y_1, D_{Y_1}) \times_{(X_1, D_{X_1})} (X_{H_1}, D_{X_{H_1}}) \to (X_{H_1}, D_{X_{H_1}}))$$

associated to $(X_{H_1}, D_{X_{H_1}})$ induced by $\phi|_{H_1}$ and the triple $(\ell, d, g_2 : (Z_2, D_{Z_2}) \to (X_{H_2}, D_{X_{H_2}}))$.

Let $\alpha_2 \in M_{Y_2, e_{X_2}}$. The final paragraph of the proof of Lemma 3.10 implies that we have a bijection $M_{Y_1} = \bigsqcup_{e \in D_{X_1}} M_{Y_1, e} \xrightarrow{\sim} M_{Y_2} = \bigsqcup_{e \in D_{X_2}} M_{Y_2, e}$ induced by ϕ. Then α_2 induces an element $\alpha_1 \in M_{Y_1, e^*_{X_1}}$. Write $(Y_{\alpha_1}, D_{Y_{\alpha_1}})$ and $(Y_{\alpha_2}, D_{Y_{\alpha_2}})$ for the smooth pointed stable curves over k_1 and k_2 induced by α_1 and α_2, respectively. Consider the connected Galois tame covering

$$(Y_{\alpha_2}, D_{Y_{\alpha_2}}) \times_{(X_2, D_{X_2})} (X_{H_2}, D_{X_{H_2}}) \to (Z_2, D_{Z_2})$$

of degree ℓ over k_2, and write β_2 for an element of $M^*_{Z_2}$ corresponding to this connected Galois tame covering. Then we have

$$\beta_2 = \sum_{c_2 \in S_{X_{H_2}}} t_{c_2} \beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in M_{Z_2, c_2}$. On the other hand, by the proof concerning $\psi^{\mathrm{t},\ell}_Y(M^*_{Y_1}) = M^*_{Y_2}$ in the fourth paragraph of the proof of Lemma 3.10, β_2 induces an element

$$\beta_1 \overset{\mathrm{def}}{=} \sum_{c_2 \in S_{X_{H_2}} \setminus \{e_{X_{H_2}}\}} t_{c_2} \beta_{\rho^{-1}_{\phi|_{H_1}}(c_2)} + t_{e_{X_{H_2}}} \beta_{\rho^{-1}_{\phi|_{H_1}}(e_{X_{H_2}})}$$

$$= \sum_{c_2 \in S_{X_{H_2}} \setminus \{e_{X_{H_2}}\}} t_{c_2} \beta_{\rho^{-1}_{\phi|_{H_1}}(c_2)} + t_{e_{X_{H_2}}} \beta_{e_{X_{H_1}}} \in M^*_{Z_1}.$$

Then we have that the coefficient $t_{e_{X_{H_2}}}$ of $\beta_{e_{X_{H_1}}}$ is not equal to 0. Thus, the composition

$$(Y_{\alpha_1}, D_{Y_{\alpha_1}}) \times_{(X_1, D_{X_1})} (X_{H_1}, D_{X_{H_1}}) \to (Z_1, D_{Z_1}) \xrightarrow{g_1} (X_{H_1}, D_{X_{H_1}})$$

is tamely ramified over $e_{X_{H_1}}$. This means that $e_{X_{H_1}}$ is contained in $S_{X_{H_1}}$. We complete the proof of the lemma. □

Remark 3.11.1. Remark 3.6.1 implies that $D^{\mathrm{gp}}_{X_{H_i}}$, $i \in \{1, 2\}$, admits a natural action of G. Moreover, the commutative diagram

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D^{\mathrm{gp}}_{X_{H_2}} \\ {\scriptstyle \gamma_{H_1, \pi_1^{\mathrm{t}}(U_{X_1})}}\downarrow & & {\scriptstyle \gamma_{H_2, \pi_1^{\mathrm{t}}(U_{X_2})}}\downarrow \\ D^{\mathrm{gp}}_{X_1} & \xrightarrow{\rho_\phi} & D^{\mathrm{gp}}_{X_2} \end{array}$$

is compatible with the actions of G.

3.4.6. Next, we prove that the condition $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$ mentioned in Lemma 3.11 can be omitted. Firstly, we treat the case of abelian groups.

Lemma 3.12. *We maintain the notation introduced in 3.4.5. Suppose that $g_X \geq 2$, and that G is an abelian group. Then we have $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$.*

Proof. We write m for $\#G$ and put $K_2 \overset{\mathrm{def}}{=} \ker(\pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{ab}} \otimes \mathbb{Z}/m\mathbb{Z})$. Then we see immediately that K_2 is contained in H_2. Let $K_1 \overset{\mathrm{def}}{=} \phi^{-1}(K_2) \subseteq H_1$. Write $(X_{K_i}, D_{X_{K_i}})$ for the smooth pointed stable curves of type $(g_{X_{K_i}}, n_{X_{K_i}})$ over k_i induced by K_i and $f_{K_i} : (X_{K_i}, D_{X_{K_i}}) \to (X_i, D_{X_i})$ for the tame covering over k_i induced by the inclusion $K_i \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i})$. We identify $\pi_1^{\mathrm{t}}(U_{X_1})/K_1$ with $\pi_1^{\mathrm{t}}(U_{X_2})/K_2$ via the isomorphism induced by ϕ, and denote by $A \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_1})/K_1 \cong \pi_1^{\mathrm{t}}(U_{X_2})/K_2$.

Since each p-Galois tame covering is étale (i.e., Galois tame coverings whose Galois group is a p-group), we have that $g_{X_{K_1}} = g_{X_{K_2}}$ follows from the Riemann-Hurwitz formula, and that $n_{X_{K_1}} = \#(A)n_X = n_{X_{K_2}}$. Then we obtain $(g_{X_{K_1}}, n_{X_{K_1}}) = (g_{X_{K_2}}, n_{X_{K_2}})$. Thus, Lemma 3.11 implies that the commutative diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\phi|_{K_1}} & K_2 \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}) \end{array}$$

of profinite groups induces a commutative diagram

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{K_1}} & \xrightarrow{\rho_{\phi|_{K_1}}} & D^{\mathrm{gp}}_{X_{K_2}} \\ {\scriptstyle \gamma_{K_1, \pi_1^{\mathrm{t}}(U_{X_1})}}\downarrow & & {\scriptstyle \gamma_{K_2, \pi_1^{\mathrm{t}}(U_{X_2})}}\downarrow \\ D^{\mathrm{gp}}_{X_1} & \xrightarrow{\rho_\phi} & D^{\mathrm{gp}}_{X_2}. \end{array}$$

Moreover, Remark 3.11.1 implies that the commutative diagram above admits a natural action of A. Then, for each $e^{\mathrm{gp}}_{X_{K_1}} \in D^{\mathrm{gp}}_{X_{K_1}}$, the inertia subgroup $I_{e^{\mathrm{gp}}_{X_{K_1}}}$ in A associated to $e^{\mathrm{gp}}_{X_{K_1}}$ (i.e. the stabilizer of $e^{\mathrm{gp}}_{X_{K_1}}$ under the action of A) is equal to the inertia subgroup $I_{e^{\mathrm{gp}}_{X_{K_2}}}$ in A associated to $e^{\mathrm{gp}}_{X_{K_2}} \overset{\mathrm{def}}{=} \rho_{\phi|_{K_1}}(e^{\mathrm{gp}}_{X_{K_1}}) \in D^{\mathrm{gp}}_{X_{K_2}}$. On the other hand, write F for the kernel of the natural morphism $A \twoheadrightarrow G$ induced by the inclusion $K_i \hookrightarrow H_i$, $i \in \{1, 2\}$. Since $(X_{H_i}, D_{X_{H_i}}) \cong (X_{K_i}, D_{X_{K_i}})/F$, the set of ramification indices of the Galois tame covering $(X_{K_i}, D_{X_{K_i}}) \to (X_{H_i}, D_{X_{H_i}})$ with Galois group F are equal to $\{\#(F \cap I_{e^{\rm gp}_{X_{K_i}}})\}_{e^{\rm gp}_{X_{K_i}} \in D^{\rm gp}_{X_{K_i}}}$. Then by the Riemann-Hurwitz formula, we have $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. This completes the proof of the lemma. □

Next, we treat the general case.

Lemma 3.13. *We maintain the notation introduced in 3.4.5. Suppose that $g_X \geq 2$ and $n_X \geq 2$. Then there exists an open normal subgroup $P_2 \subseteq \pi_1^{\rm t}(U_{X_2})$ which is contained in H_2 such that the following holds:*

> *Write $(X_{P_i}, D_{X_{P_i}})$, $i \in \{1, 2\}$, for the smooth pointed stable curve of type $(g_{X_{P_i}}, n_{X_{P_i}})$ over k_i induced by P_i, where $P_1 = \phi^{-1}(P_2)$. We have $(g_{X_{P_1}}, n_{X_{P_1}}) = (g_{X_{P_2}}, n_{X_{P_2}})$.*

Proof. First, suppose that G is a *simple* finite group. By applying Lemma 3.12, we may assume that G is *non-abelian*. Moreover, we claim that we may assume that n_X is a positive *even* number. Let us prove this claim. Suppose $p \neq 2$. Let $R_2 \subseteq \pi_1^{\rm t}(U_{X_2})$ be an open subgroup such that $\#(\pi_1^{\rm t}(U_{X_2})/R_2) = 2$, and that $R_2 \supseteq \ker(\pi_1^{\rm t}(U_{X_2}) \twoheadrightarrow \pi_1(X_2))$ (i.e. the cyclic Galois tame covering corresponding to R_2 is étale). Let $R_1 \stackrel{\rm def}{=} \phi^{-1}(R_2) \subseteq \pi_1^{\rm t}(U_{X_1})$. Then we have that $\#(\pi_1^{\rm t}(U_{X_1})/R_1) = 2$, and that Lemma 3.9 implies $R_1 \supseteq \ker(\pi_1^{\rm t}(U_{X_1}) \twoheadrightarrow \pi_1(X_1))$. By replacing H_i and $\pi_1^{\rm t}(U_{X_i})$, $i \in \{1, 2\}$, by $H_i \cap R_i$ and R_i, respectively, we may assume that n_X is a positive even number. Suppose that $p = 2$. Let ℓ be a prime number such that $(\ell, 2) = (\ell, \#G) = 1$. By [R1, Théorème 4.3.1], there exists an open subgroup $R_2^* \subseteq \pi_1^{\rm t}(U_{X_2})$ such that $\#(\pi_1^{\rm t}(U_{X_2})/R_2^*) = \ell$, that $R_2^* \supseteq \ker(\pi_1^{\rm t}(U_{X_2}) \twoheadrightarrow \pi_1(X_2))$, and that

$$\dim_{\mathbb{F}_p}(R_2^{*,\rm ab} \otimes \mathbb{F}_p) > 0.$$

Let $R_1^* \stackrel{\rm def}{=} \phi^{-1}(R_2^*) \subseteq \pi_1^{\rm t}(U_{X_1})$. Then we have that $\#(\pi_1^{\rm t}(U_{X_1})/R_1^*) = \ell$, that $\dim_{\mathbb{F}_p}(R_1^{*,\rm ab} \otimes \mathbb{F}_p) > 0$, and that Lemma 3.9 implies $R_1^* \supseteq \ker(\pi_1^{\rm t}(U_{X_1}) \twoheadrightarrow \pi_1(X_1))$. Thus, we may take an open subgroup $R_2' \subseteq R_2^*$ such that

$$\pi_1^{\rm t}(U_{X_2})/R_2' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z},$$

and that $R_2' \supseteq \ker(\pi_1^{\rm t}(U_{X_2}) \twoheadrightarrow \pi_1(X_2))$. We put $R_1' \stackrel{\rm def}{=} \phi^{-1}(R_2')$. Then the construction of R_1' implies $\pi_1^{\rm t}(U_{X_1})/R_1' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ and $R_1' \supseteq \ker(\pi_1^{\rm t}(U_{X_1}) \twoheadrightarrow \pi_1(X_1))$. By replacing H_i and $\pi_1^{\rm t}(U_{X_i})$, $i \in \{1, 2\}$, by $H_i \cap R_i'$ and R_i', respectively, we may assume that n_X is a positive even number. This completes the proof of the claim.

Let $\#G \stackrel{\rm def}{=} p^t m'$ such that $(m', p) = 1$. Since n_X is a positive *even* number, we may choose a Galois tame covering

$$f_2 : (Y_2, D_{Y_2}) \to (X_2, D_{X_2})$$

over k_2 with Galois group $\mathbb{Z}/m'\mathbb{Z}$ such that f_2 is totally ramified over every marked point of D_{X_2}. Write (g_{Y_2}, n_{Y_2}) for the type of (Y_2, D_{Y_2}), $Q_2 \subseteq \pi_1^{\rm t}(U_{X_2})$ for the open normal subgroup induced by f_2, $Q_1 \stackrel{\rm def}{=} \phi^{-1}(Q_2) \subseteq \pi_1^{\rm t}(U_{X_1})$,

$$f_1 : (Y_1, D_{Y_1}) \to (X_1, D_{X_1})$$

for the Galois tame covering over k_1 with Galois group $\mathbb{Z}/m'\mathbb{Z}$ induced by the natural inclusion $Q_1 \hookrightarrow \pi_1^{\rm t}(U_{X_1})$, and (g_{Y_1}, n_{Y_1}) for the type of (Y_1, D_{Y_1}). Then Lemma 3.12 implies that $(g_{Y_1}, n_{Y_1}) = (g_{Y_2}, n_{Y_2})$ and f_1 is also totally ramified over every marked point of D_{X_1}.

We consider the Galois tame covering

$$(Z_i, D_{Z_i}) \stackrel{\rm def}{=} (X_{H_i}, D_{X_{H_i}}) \times_{(X_i, D_{X_i})} (Y_i, D_{Y_i}) \to (X_i, D_{X_i}),\ i \in \{1, 2\},$$

over k_i with Galois group $G \times \mathbb{Z}/m'\mathbb{Z}$ which is the composition of $(Z_i, D_{Z_i}) \to (Y_i, D_{Y_i})$ and $(Y_i, D_{Y_i}) \to (X_i, D_{X_i})$. Note that since G is a non-abelian simple finite group, (Z_i, D_{Z_i}) is connected. Moreover, by Abhyankar's lemma, we obtain that $(Z_i, D_{Z_i}) \to (Y_i, D_{Y_i})$ is an étale covering over k_i. Since $(g_{Y_1}, n_{Y_1}) = (g_{Y_2}, n_{Y_2})$ and $(Z_i, D_{Z_i}) \to (Y_i, D_{Y_i})$ is unramified, the Riemann-Hurwitz formula implies $(g_{Z_1}, n_{Z_1}) = (g_{Z_2}, n_{Z_2})$.

Next, let us prove the lemma in the case where G is an arbitrary finite group. Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \stackrel{\rm def}{=} G$ be a sequence of subgroups of G such that G_i/G_{i-1} is a simple group for all

$i \in \{2, \dots n\}$. In order to verify the lemma, we see that it is sufficient to prove the lemma when $n = 2$. Let N_2 be the kernel of the natural homomorphism $\pi_1^{\rm t}(U_{X_2}) \twoheadrightarrow G \twoheadrightarrow G_1$ and $N_1 \overset{\rm def}{=} \phi^{-1}(N_2)$. Then by replacing G by G_1 and by applying the lemma for the simple group G_1, we obtain an open normal subgroup $M_2 \subseteq \pi_1^{\rm t}(U_{X_2})$ which is contained in N_2 such that $(g_{X_{M_1}}, n_{X_{M_1}}) = (g_{X_{M_2}}, n_{X_{M_2}})$, where $M_1 \overset{\rm def}{=} \phi^{-1}(M_2)$, and $(g_{X_{M_i}}, n_{X_{M_i}})$, $i \in \{1, 2\}$, denotes the type of the smooth pointed stable curve corresponding to M_i.

If $M_i \subseteq H_i$, $i \in \{1, 2\}$, then we may put $P_i \overset{\rm def}{=} M_i$. If H_i, $i \in \{1, 2\}$, does not contain M_i, we put $O_i \overset{\rm def}{=} M_i \cap H_i$. Then we have $M_i/O_i \cong G/G_1$. Note that G/G_1 is a simple group. Then the lemma follows from the lemma when we replace (X_i, D_{X_i}) and G by $(X_{M_i}, D_{X_{M_i}})$ and the simple group G/G_1, respectively. This completes the proof of the lemma. □

3.4.7. Now, we prove the main result of the present section.

Theorem 3.14. *Let $(\widetilde{X}_i, D_{\widetilde{X}_i})$, $i \in \{1, 2\}$, be the universal tame covering of (X_i, D_{X_i}) defined in 3.1.3. Let $\phi : \pi_1^{\rm t}(U_{X_1}) \twoheadrightarrow \pi_1^{\rm t}(U_{X_2})$ be arbitrary open continuous surjective homomorphism. Then the group-theoretical algorithm of the mono-anabelian reconstruction concerning $\mathrm{Ine}(\pi_1^{\rm t}(U_{X_i}))$ obtained in Proposition 3.7 is compatible with the surjection $\phi : \pi_1^{\rm t}(U_{X_1}) \twoheadrightarrow \pi_1^{\rm t}(U_{X_2})$. Namely, the following holds: Let $\widetilde{e}_2 \in D_{\widetilde{X}_2}$ and $I_{\widetilde{e}_2} \in \mathrm{Ine}(\pi_1^{\rm t}(U_{X_2}))$ the inertia subgroup associated to $\widetilde{e}_2$. Then there exists an inertia subgroup $I_{\widetilde{e}_1} \in \mathrm{Ine}(\pi_1^{\rm t}(U_{X_1}))$ associated to a point $\widetilde{e}_1 \in D_{\widetilde{X}_1}$ such that*

$$\phi(I_{\widetilde{e}_1}) = I_{\widetilde{e}_2},$$

and that the restriction homomorphism $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \twoheadrightarrow I_{\widetilde{e}_2}$ is an isomorphism.

Proof. If $n_X = 0$, then the theorem is trivial. We suppose $n_X > 0$. Let $m >> 0$ be an integer number such that $(m, p) = 1$. We put $K_i \overset{\rm def}{=} \ker(\pi_1^{\rm t}(U_{X_i}) \twoheadrightarrow \pi_1^{\rm t}(U_{X_i})^{\rm ab} \otimes \mathbb{Z}/m\mathbb{Z})$, $i \in \{1, 2\}$. Write (X_{K_i}, D_{K_i}) for the smooth pointed stable curve of type $(g_{X_{K_i}}, n_{X_{K_i}})$ over k_i induced by K_i. Moreover, the condition $m >> 0$ implies $g_{X_{K_1}} = g_{X_{K_2}} \geq 2$, $n_{X_{K_1}} = n_{X_{K_2}} \geq 2$.

By applying Lemma 3.13, we may choose a set of open subgroups $C_{X_2} \overset{\rm def}{=} \{H_{2,j}\}_{j \in \mathbb{Z}_{>0}}$ of $\pi_1^{\rm t}(U_{X_2})$ such that the following conditions hold: (a) $H_{2,1} = K_2$; (b) $\varprojlim_j \pi_1^{\rm t}(U_{X_2})/H_{2,j} \cong \pi_1^{\rm t}(U_{X_2})$ (i.e. C_{X_2} is a cofinal system); (c) write $\{H_{1,j} \overset{\rm def}{=} \phi^{-1}(H_{2,j})\}_{j \in \mathbb{Z}_{>0}}$ for the set of open subgroups of $\pi_1^{\rm t}(U_{X_1})$ induced by ϕ, and, for each $j \in \mathbb{Z}_{>0}$, write $(X_{H_{i,j}}, D_{X_{H_{i,j}}})$, $i \in \{1, 2\}$, for the smooth pointed stable curve of type $(g_{X_{H_{i,j}}}, n_{X_{H_{i,j}}})$ over k_i induced by $H_{i,j}$; then we have $(g_{X_{H_{1,j}}}, n_{X_{H_{1,j}}}) = (g_{X_{H_{2,j}}}, n_{X_{H_{2,j}}})$.

For each $j \in \mathbb{Z}_{>0}$, we write $e_{X_{H_{2,j}}} \in D_{X_{H_{2,j}}}$ for the image of $\widetilde{e}_2$. Then we obtain a sequence of marked points

$$\mathcal{I}_{\widetilde{e}_2}^{C_{X_2}} : \cdots \mapsto e_{H_{2,2}} \mapsto e_{H_{2,1}}.$$

Proposition 3.6 implies that, for each $H_{2,j}$, $j \in \mathbb{Z}_{>0}$, the set $D^{\rm gp}_{X_{H_{2,j}}}$ can be mono-anabelian reconstructed from $H_{2,j}$. For each $e_{X_{H_{2,j}}} \in D_{X_{H_{2,j}}}$, we denote by

$$e^{\rm gp}_{X_{H_{2,j}}} \overset{\rm def}{=} \vartheta^{-1}_{X_{H_{2,j}}}(e_{X_{H_{2,j}}}).$$

Then the sequence of marked points $\mathcal{I}_{\widetilde{e}_2}^{C_X}$ induces a sequence

$$\mathcal{I}_{\widetilde{e}_2^{\rm gp}}^{C_X} : \cdots \mapsto e^{\rm gp}_{X_{H_{2,2}}} \mapsto e^{\rm gp}_{X_{H_{2,1}}}.$$

Then Remark 3.6.1 implies that the inertia subgroup associated to $\widetilde{e}_2$ is equal to the stabilizer of $\mathcal{I}_{\widetilde{e}_2^{\rm gp}}^{C_X}$.

By Lemma 3.11 and Lemma 3.13, $\mathcal{I}_{\widetilde{e}_2^{\rm gp}}^{C_{X_2}}$ induces a sequence as follows:

$$\cdots \mapsto e^{\rm gp}_{X_{H_{1,2}}} \overset{\rm def}{=} \rho^{-1}_{\phi|_{H_{1,2}}}(e^{\rm gp}_{X_{H_{2,2}}}) \in D^{\rm gp}_{X_{H_{1,2}}} \mapsto e^{\rm gp}_{X_{H_{1,1}}} \overset{\rm def}{=} \rho^{-1}_{\phi|_{H_{1,1}}}(e^{\rm gp}_{X_{H_{2,1}}}) \in D^{\rm gp}_{X_{H_{1,1}}}$$

with an action of $I_{\widetilde{e}_2}$. Then Proposition 3.7 implies that we have a sequence

$$\cdots \mapsto e_{X_{H_{1,2}}} \stackrel{\text{def}}{=} \vartheta_{X_{H_{1,2}}}(e^{\text{gp}}_{X_{H_{1,2}}}) \in D_{X_{H_{1,2}}} \mapsto e_{X_{H_{1,1}}} \stackrel{\text{def}}{=} \vartheta_{X_{H_{1,1}}}(e^{\text{gp}}_{X_{H_{1,1}}}) \in D_{X_{H_{1,1}}}$$

with an action of $I_{\widetilde{e}_2}$

Let $K_{\ker(\phi)}$ be the subfield of $\widetilde{K}$ induced by the closed subgroup $\ker(\phi)$ of $\pi_1^{\text{t}}(U_{X_1})$, $\widetilde{X}_{1,\ker(\phi)}$ the normalization of X_1 in $K_{\ker(\phi)}$, and $D_{\widetilde{X}_{1,\ker(\phi)}}$ the inverse image of D_{X_1} in $\widetilde{X}_{1,\ker(\phi)}$. Then the sequence

$$\cdots \mapsto e_{X_{H_{1,2}}} \mapsto e_{X_{H_{1,1}}}.$$

determines a point $\widetilde{e}_{1,\ker(\phi)} \in D_{\widetilde{X}_{1,\ker(\phi)}}$. We choose a point of $\widetilde{e}_1 \in D_{\widetilde{X}_1}$ such that the image of $\widetilde{e}_1$ in $D_{\widetilde{X}_{1,\ker(\phi)}}$ is $\widetilde{e}_{1,\ker(\phi)}$. Then we have $\phi(I_{\widetilde{e}_1}) = I_{\widetilde{e}_2}$. Moreover, since $I_{\widetilde{e}_1}$ and $I_{\widetilde{e}_2}$ are isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, the restriction homomorphism $\phi|_{I_{\widetilde{e}_1}}$ is an isomorphism. This completes the proof of the theorem. $\square$

3.5. **Reconstructions of additive structures via surjections.**

3.5.1. **Settings.** We maintain the settings introduced in 3.4.1.

3.5.2. Let $\widetilde{e}_2$ be an arbitrary point of $D_{\widetilde{X}_2}$. By applying Theorem 3.14, there exists a point $\widetilde{e}_1 \in D_{\widetilde{X}_1}$ such that $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \xrightarrow{\sim} I_{\widetilde{e}_2}$ is an isomorphism. Write $\overline{\mathbb{F}}_{p,i}$, $i \in \{1,2\}$, for the algebraic closure of $\mathbb{F}_p$ in k_i. We put

$$\mathbb{F}_{\widetilde{e}_i} \stackrel{\text{def}}{=} (I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widetilde{e}_i}\}, \quad i \in \{1,2\},$$

where $\{*_{\widetilde{e}_i}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_i^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with $\bigcup_{(p,m)=1} \mu_m(k_i)$. Moreover, let $a_{\widetilde{e}_i}$ be a generator of $I_{\widetilde{e}_i}$. Then we have a natural bijection

$$I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}, \ a_{\widetilde{e}_i} \otimes 1 \mapsto 1 \otimes 1.$$

Thus, we obtain the following bijections

$$I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \xrightarrow{\sim} \bigcup_{(p,m)=1} \mu_m(k_i) \xrightarrow{\sim} \overline{\mathbb{F}}_{p,i}^{\times}.$$

This means that $\mathbb{F}_{\widetilde{e}_i}$ can be identified with $\overline{\mathbb{F}}_{p,i}$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}$, and whose zero element is $*_{\widetilde{e}_i}$.

3.5.3. The main goal of the present subsection is to prove that $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \xrightarrow{\sim} I_{\widetilde{e}_2}$ induces an isomorphism $\mathbb{F}_{\widetilde{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_2}$ *as fields*. The main idea is as following: First, we reduce the problem to the case where $n_X = 3$ by applying Theorem 3.14. Second, the field structure of $\mathbb{F}_{\widetilde{e}_i}$ (i.e. the set of isomorphisms of $\mathbb{F}_{\widetilde{e}_i}$ and $\overline{\mathbb{F}}_{p,i}$ as fields) can be translated to a certain problem concerning generalized Hasse-Witt invariants (e.g. $\gamma_{\chi_i}(M_{\chi_i})$ in the proof of Proposition 3.15). Then by applying Theorem 3.14 again, we obtained the result by comparing $\gamma_{\chi_1}(M_{\chi_1})$ with $\gamma_{\chi_2}(M_{\chi_2})$.

3.5.4. We have the following proposition.

Proposition 3.15. *The field structure of $\mathbb{F}_{\widetilde{e}_i}$, $i \in \{1,2\}$, can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_{X_i})$. Moreover, the isomorphism $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \xrightarrow{\sim} I_{\widetilde{e}_2}$ induces an isomorphism*

$$\theta_{\phi,\widetilde{e}_1,\widetilde{e}_2} : \mathbb{F}_{\widetilde{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_2}$$

as fields.

Proof. First, we claim that we may assume $n_X = 3$. If $g_X = 0$, then $n_X \geq 3$. Suppose that $g_X \geq 1$. Theorem 3.14 implies that $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ induces an open continuous surjection $\phi^{\text{ét}} : \pi_1(X_1) \twoheadrightarrow \pi_1(X_2)$. Let $H_2' \subseteq \pi_1(X_2)$ be an open normal subgroup such that $\#(\pi_1(X_2)/H_2') \geq 3$ and $H_1' \overset{\text{def}}{=} (\phi^{\text{ét}})^{-1}(H_2')$. Write $H_i \subseteq \pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$, for the inverse image of H_i' of the natural surjection $\pi_1^{\mathrm{t}}(U_{X_i}) \twoheadrightarrow \pi_1(X_i)$, and $(X_{H_i}, D_{X_{H_i}})$ for the smooth pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i induced by H_i. Note that $g_{X_{H_1}} = g_{X_{H_2}} \geq 2$ and $n_{X_{H_1}} = n_{X_{H_2}} \geq 3$. By replacing (X_i, D_{X_i}) by $(X_{H_i}, D_{X_{H_i}})$, we may assume $g_X \geq 2$ and $n_X \geq 3$. The surjection ϕ induces a bijection

$$D_{X_1} \overset{\vartheta_{X_1}^{-1}}{\overset{\sim}{\to}} D_{X_1}^{\text{gp}} \overset{\rho_\phi}{\overset{\sim}{\to}} D_{X_2}^{\text{gp}} \overset{\vartheta_{X_2}}{\overset{\sim}{\to}} D_{X_2}.$$

Let $D_{X_1}' \overset{\text{def}}{=} \{e_{1,1}, e_{1,2}, e_{1,3}\} \subseteq D_{X_1}$ and $D_{X_2}' \overset{\text{def}}{=} \{e_{2,1} \overset{\text{def}}{=} \vartheta_{X_2} \circ \rho_\phi \circ \vartheta_{X_1}^{-1}(e_{1,1}), e_{2,2} \overset{\text{def}}{=} \vartheta_{X_2} \circ \rho_\phi \circ \vartheta_{X_1}^{-1}(e_{1,2}), e_{2,3} \overset{\text{def}}{=} \vartheta_{X_2} \circ \rho_\phi \circ \vartheta_{X_1}^{-1}(e_{1,3})\} \subseteq D_{X_2}$. Then (X_i, D_{X_i}'), $i \in \{1, 2\}$, is a smooth pointed stable curve of type $(g_X, 3)$ over k_i. Write I_i, $i \in \{1, 2\}$, for the closed subgroup of $\pi_1^{\mathrm{t}}(U_{X_i})$ generated by the inertia subgroups associated to the elements of $D_{\widetilde{X}_i}$ whose images in D_{X_i} are contained in $D_{X_i} \setminus D_{X_i}'$. Then we have an isomorphism

$$\pi_1^{\mathrm{t}}(X_i \setminus D_{X_i}') \cong \pi_1^{\mathrm{t}}(U_{X_i})/I_i, \ i \in \{1, 2\}.$$

Moreover, Theorem 3.14 implies that ϕ induces an open continuous surjective homomorphism

$$\phi' : \pi_1^{\mathrm{t}}(X_1 \setminus D_{X_1}') \twoheadrightarrow \pi_1^{\mathrm{t}}(X_2 \setminus D_{X_2}').$$

Thus, by replacing (X_i, D_{X_i}), $\pi_1^{\mathrm{t}}(U_{X_i})$, and ϕ by (X_i, D_{X_i}'), $\pi_1^{\mathrm{t}}(X_i \setminus D_{X_i}')$, and ϕ', respectively, we may assume $n_X = 3$.

Let $r \in \mathbb{N}$. We denote by $\mathbb{F}_{p^r, \widetilde{e}_i}$, $i \in \{1, 2\}$, the unique subfield of $\mathbb{F}_{\widetilde{e}_i}$ whose cardinality is equal to p^r. On the other hand, we fix any finite field $\mathbb{F}_{p^r}$ of cardinality p^r and an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. By Proposition 3.7, we have $\mathbb{F}_{p^r, \widetilde{e}_i}^\times = I_{\widetilde{e}_i}/(p^r - 1)$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})$. Then reconstructing the field structure of $\mathbb{F}_{p^r, \widetilde{e}_i}$ is equivalent to reconstructing $\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r, \widetilde{e}_i}, \mathbb{F}_{p^r})$ as a subset of $\mathrm{Hom}_{\text{group}}(\mathbb{F}_{p^r, \widetilde{e}_i}^\times, \mathbb{F}_{p^r}^\times)$. Note that, in order to reconstruct the field structure of $\mathbb{F}_{\widetilde{e}_i}$, it is sufficient to reconstruct the subset $\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r, \widetilde{e}_i}, \mathbb{F}_{p^r})$ for r in a cofinal subset of $\mathbb{N}$ with respect to division.

Let $\chi_i \in \mathrm{Hom}_{\text{groups}}(\pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_{p^r}^\times)$. Write H_{χ_i} for the kernel of $\pi_1^{\mathrm{t}}(U_{X_i}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z} \overset{\chi_i}{\to} \mathbb{F}_{p^r}^\times$, M_{χ_i} for $H_{\chi_i}^{\text{ab}} \otimes \mathbb{F}_p$, and $(X_{H_{\chi_i}}, D_{X_{H_{\chi_i}}})$ for the smooth pointed stable curve over k_i induced by H_{χ_i}. We define

$$M_{\chi_i}[\chi_i] \overset{\text{def}}{=} \{a \in M_{\chi_i} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma(a) = \chi_i(\sigma)a \text{ for all } \sigma \in \pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}\}$$

and $\gamma_{\chi_i}(M_{\chi_i}) \overset{\text{def}}{=} \dim_{\overline{\mathbb{F}}_p}(M_{\chi_i}[\chi_i])$ (i.e. a generalized Hasse-Witt invariant (see [Y5, Section 2.2])). Then [T4, Remark 3.7] implies $\gamma_{\chi_i}(M_{\chi_i}) \leq g_X + 1$. Moreover, we define two maps

$$\mathrm{Res}_{i,r} : \mathrm{Hom}_{\text{groups}}(\pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_{p^r}^\times) \to \mathrm{Hom}_{\text{groups}}(\mathbb{F}_{p^r, \widetilde{e}_i}^\times, \mathbb{F}_{p^r}^\times),$$

$$\Gamma_{i,r} : \mathrm{Hom}_{\text{groups}}(\pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_{p^r}^\times) \to \mathbb{Z}_{\geq 0}, \ \chi_i \mapsto \gamma_{\chi_i}(M_{\chi_i}),$$

where the map $\mathrm{Res}_{i,r}$ is the restriction with respect to the natural inclusion $\mathbb{F}_{p^r, \widetilde{e}_i}^\times \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}$.

Let m_0 be the product of all prime numbers $\leq p - 2$ if $p \neq 2, 3$ and $m_0 = 1$ if $p = 2, 3$. Let r_0 be the order of p in the multiplicative group $(\mathbb{Z}/m_0\mathbb{Z})^\times$. Then [T4, Claim 5.4] implies the following result holds:

> there exists a constant $C(g_X)$ which only depends on g_X such that, for each $r > \log_p(C(g_X) + 1)$ divisible by r_0, we have
>
> $$\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r, \widetilde{e}_i}, \mathbb{F}_{p^r}) = \mathrm{Hom}_{\text{groups}}^{\text{surj}}(\mathbb{F}_{p^r, \widetilde{e}_i}^\times, \mathbb{F}_{p^r}^\times) \setminus \mathrm{Res}_{i,r}(\Gamma_{i,r}^{-1}(\{g_X + 1\})), \ i \in \{1, 2\},$$
>
> where $\mathrm{Hom}_{\text{groups}}^{\text{surj}}(-, -)$ denotes the set of surjections whose elements are contained in $\mathrm{Hom}_{\text{groups}}(-, -)$.

Let $\kappa_2 \in \mathrm{Hom}_{\mathrm{groups}}(\pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}_{p^r}^\times)$. Then ϕ induces a character

$$\kappa_1 \in \mathrm{Hom}_{\mathrm{groups}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}_{p^r}^\times).$$

Moreover, the surjection $\phi|_{H_{\kappa_1}}$ induces a surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$. Suppose that $\kappa_2 \in \Gamma_{2,r}^{-1}(\{g_X+1\})$. The surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$ implies $\gamma_{\kappa_1}(M_{\kappa_1}) = g_X + 1$. This means $\kappa_1 \in \Gamma_{1,r}^{-1}(\{g_X+1\})$. On the other hand, the isomorphism $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \xrightarrow{\sim} I_{\widetilde{e}_2}$ induces an injection

$$\mathrm{Res}_{2,r}(\Gamma_{2,r}^{-1}(\{g_X+1\})) \hookrightarrow \mathrm{Res}_{1,r}(\Gamma_{1,r}^{-1}(\{g_X+1\})).$$

Since $\#(\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r})) = \#(\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r}))$, we obtain that $\phi|_{I_{\widetilde{e}_1}}$ induces a bijection $\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r})$. Thus, $\phi|_{I_{\widetilde{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{\widetilde{e}_2}, \overline{\mathbb{F}}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{\widetilde{e}_1}, \overline{\mathbb{F}}_p).$$

If we choose $\overline{\mathbb{F}}_p = \mathbb{F}_{\widetilde{e}_2}$, then the image of $\mathrm{id}_{\mathbb{F}_{\widetilde{e}_2}}$ via the bijection above induces an isomorphism $\theta_{\phi,\widetilde{e}_1,\widetilde{e}_2} : \mathbb{F}_{\widetilde{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_2}$ as fields. This completes the proof of the proposition. □

4. Main theorems

4.1. **The first main theorem.** In this subsection, we apply the results obtained in previous sections to prove that the scheme-theoretical structures of curves of type $(0,n)$ over $\overline{\mathbb{F}}_p$ can be controlled group-theoretically via *open continuous homomorphism* (Theorem 4.3).

4.1.1. **Settings.** We fix some notation. Let (X_i, D_{X_i}), $i \in \{1,2\}$, be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p>0$, $U_{X_i} \overset{\mathrm{def}}{=} X_i \setminus D_{X_i}$, $\pi_1^{\mathrm{t}}(U_{X_i})$ the tame fundamental group of U_{X_i}, $\pi_1(X_i)$ the étale fundamental group of X_i, and $(\widetilde{X}_i, D_{\widetilde{X}_i})$ the universal tame covering of (X_i, D_{X_i}) associated to $\pi_1^{\mathrm{t}}(U_{X_i})$ (3.1.3). Let k_i^{m}, $i \in \{1,2\}$, be the *minimal* algebraically closed subfield of k_i over which U_{X_i} can be defined. Thus, by considering the function field of X_i, we obtain a smooth pointed stable curve $(X_i^{\mathrm{m}}, D_{X_i^{\mathrm{m}}})$ (i.e. *a minimal model of* (X_i, D_{X_i}) (cf. [T3, Definition 1.30 and Lemma 1.31])) such that $U_{X_i} \cong U_{X_i^{\mathrm{m}}} \times_{k_i^{\mathrm{m}}} k_i$ as k_i-schemes, where $U_{X_i^{\mathrm{m}}} \overset{\mathrm{def}}{=} X_i^{\mathrm{m}} \setminus D_{X_i^{\mathrm{m}}}$. Note that $\pi_1^{\mathrm{t}}(U_{X_i^{\mathrm{m}}})$ is naturally isomorphic to $\pi_1^{\mathrm{t}}(U_{X_i})$. We shall denote by $\overline{\mathbb{F}}_{p,i}$ the algebraic closure of $\mathbb{F}_p$ in k_i. Moreover, we put

$$d_{(X_i,D_{X_i})} \overset{\mathrm{def}}{=} \begin{cases} 0, & \text{if } k_i^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,i}, \\ 1, & \text{if } k_i^{\mathrm{m}} \not\cong \overline{\mathbb{F}}_{p,i}. \end{cases}$$

4.1.2. Firstly, we have the following lemma.

Lemma 4.1. *Let $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \to \pi_1^{\mathrm{t}}(U_{X_2})$ be an arbitrary open continuous homomorphism. Then ϕ is a surjection.*

Proof. We denote by Π_ϕ the image of ϕ which is an open subgroup of $\pi_1^{\mathrm{t}}(U_{X_2})$. Let (X_ϕ, D_{X_ϕ}) be the smooth pointed stable curve of type (g_{X_ϕ}, n_{X_ϕ}) over k_2 induced by Π_ϕ and $f_\phi : (X_\phi, D_{X_\phi}) \to (X_2, D_{X_2})$ the tame covering of smooth pointed stable curves over k_2 induced by the inclusion $\Pi_\phi \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$. Since f_ϕ is a tame covering, we have that $n_{X_\phi} \geq n_X$. On the other hand, if $g_X = 0$, we have $g_\phi \geq 0$. If $g_X > 0$, the Riemann-Hurwitz formula implies $g_{X_\phi} \geq [\pi_1^{\mathrm{t}}(U_{X_2}) : \Pi_\phi](g_X - 1) + 1 \geq g_X$. Then we have $g_\phi \geq g_X$ and $n_{X_\phi} \geq n_X$. Note that $\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \Pi_\phi \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ implies

$$2g_X + n_X - 1 \geq 2g_{X_\phi} + n_{X_\phi} - 1 \geq 2g_X + n_X - 1.$$

Then we obtain that $2g_X + n_X - 1 = 2g_{X_\phi} + n_{X_\phi} - 1$. Moreover, Proposition 3.8 (ii) and the natural surjection $\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \Pi_\phi$ induced by ϕ imply that $g_X \geq g_{X_\phi}$. Then we obtain that $g_X = g_{X_\phi}$. Thus, we have $(g_X, n_X) = (g_{X_\phi}, n_{X_\phi})$. This means that the tame covering $f_\phi : (X_\phi, D_{X_\phi}) \to (X_2, D_{X_2})$ is totally ramified over every marked point of D_{X_2}.

Let us prove $[\pi_1^{\mathrm{t}}(U_{X_2}) : \Pi_\phi] = 1$. Suppose $[\pi_1^{\mathrm{t}}(U_{X_2}) : \Pi_\phi] \neq 1$. Since f_ϕ is totally ramified, the Riemann-Hurwitz formula implies $g_{X_\phi} > g_X$ if $n_X \neq 0$ and $g_X \neq 0$. This is a contradiction. If

$n_X = 0$, the Riemann-Hurwitz formula implies $g_X = 1$ if $g_X \neq 0$. This contradicts the assumption where (X_i, D_{X_i}) is a pointed stable curve. Then we obtain $g_X = g_{X_\phi} = 0$. Moreover, by applying the Riemann-Hurwitz formula again, since $n_X = n_{X_\phi}$, we obtain $n_X = n_{X_\phi} = 2$. This contradicts the assumption where (X_i, D_{X_i}) is pointed stable curve. Then we have $[\pi_1^{\mathrm{t}}(U_{X_2}) : \Pi_\phi] = 1$. This means that ϕ is a surjection. □

4.1.3. **Further settings.** In the remainder of this subsection, we suppose $(g_X, n_X) = (0, n)$. We fix two marked points $e_{1,\infty}, e_{1,0} \in D_{X_1}$ distinct from each other. Moreover, we choose any field $k_1' \cong k_1$, and choose any isomorphism $\varphi_1 : X_1 \overset{\sim}{\to} \mathbb{P}^1_{k_1'}$ as schemes such that $\varphi_1(e_{1,\infty}) = \infty$ and $\varphi_1(e_{1,0}) = 0$. Then the set of k_1-rational points $X_1(k_1) \setminus \{e_{1,\infty}\}$ is equipped with a structure of $\mathbb{F}_p$-module via the bijection φ_1. Note that since any k_1'-isomorphism of $\mathbb{P}^1_{k_1'}$ fixing ∞ and 0 is a scalar multiplication, the $\mathbb{F}_p$-module structure of $X_1(k_1) \setminus \{e_{1,\infty}\}$ does not depend on the choices of k_1' and φ_1 but depends only on the choices of $e_{1,\infty}$ and $e_{1,0}$. Then we shall say that $X_1(k_1) \setminus \{e_{1,\infty}\}$ *is equipped with a structure of* $\mathbb{F}_p$*-module with respect to* $e_{1,\infty}$ *and* $e_{1,0}$.

By applying Theorem 3.14, in the next lemma, we will prove that Tamagawa's group-theoretical criterion (i.e. [T2, Lemma 3.3]) for linear conditions is compatible with arbitrary open continuous surjective homomorphism.

Lemma 4.2. *Let* $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ *be an open continuous surjective homomorphism. By Lemma 3.10,* ϕ *induces a bijection* $\rho_\phi : D^{\mathrm{gp}}_{X_1} \overset{\sim}{\to} D^{\mathrm{gp}}_{X_2}$. *We may identify* $D^{\mathrm{gp}}_{X_i}$, $i \in \{1, 2\}$, *with* D_{X_i} *via the bijection* $\vartheta_{X_i} : D^{\mathrm{gp}}_{X_i} \overset{\sim}{\to} D_{X_i}$. *Write* $e_{2,\infty}$ *and* $e_{2,0}$ *for* $\rho_\phi(e_{1,\infty})$ *and* $\rho_\phi(e_{1,0})$, *respectively. Let*

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} e_1 = e_{1,0}$$

be a linear condition with respect to $e_{1,\infty}$ *and* $e_{1,0}$ *on* (X_1, D_{X_1}), *where* $b_{e_1} \in \mathbb{F}_p$ *for each* $e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}$. *Then the linear condition*

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} \rho_\phi(e_1) = \rho_\phi(e_{1,0}) = e_{2,0}$$

with respect to $e_{2,\infty}$ *and* $e_{2,0}$ *on* (X_2, D_{X_2}) *also holds.*

Proof. Let $\widetilde{e}_{2,\infty} \in D_{\widetilde{X}_2}$ be a point over $e_{2,\infty}$. The set $\mathbb{F}_{\widetilde{e}_{2,\infty}} \overset{\mathrm{def}}{=} (I_{\widetilde{e}_{2,\infty}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_2^{p'}) \sqcup \{*_{\widetilde{e}_{2,\infty}}\}$ admits a structure of field, and Proposition 3.15 implies that the field structure can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_2})$. Theorem 3.14 implies that there exists a point $\widetilde{e}_{1,\infty} \in D_{\widetilde{X}_1}$ over $e_{1,\infty}$ such that $\phi(I_{\widetilde{e}_{1,\infty}}) = \widetilde{e}_{2,\infty}$. By Proposition 3.15 again, the set $\mathbb{F}_{\widetilde{e}_{1,\infty}} \overset{\mathrm{def}}{=} (I_{\widetilde{e}_{1,\infty}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_1^{p'}) \sqcup \{*_{\widetilde{e}_{1,\infty}}\}$ admits a structure of field which can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_1})$, and ϕ induces an isomorphism $\theta_{\phi, \widetilde{e}_{1,\infty}, \widetilde{e}_{2,\infty}} : \mathbb{F}_{\widetilde{e}_{1,\infty}} \overset{\sim}{\to} \mathbb{F}_{\widetilde{e}_{2,\infty}}$ as fields.

For each $e_1 \in D_{X_1}$, we take $b'_{e_1} \in \mathbb{Z}_{\geq 0}$ such that $b'_{e_1} \equiv b_{e_1} \pmod{p}$ and

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1} \geq 2.$$

Let $r \geq 1$ such that $p^r - 2 \geq \sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1}$. For each $\widetilde{e}_1 \in D_{\widetilde{X}_1}$ over e_1, write $I_{\widetilde{e}_1, \mathrm{ab}}$ for the image of the natural morphism $I_{\widetilde{e}_1} \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}$. Moreover, since the image of $I_{\widetilde{e}_1, \mathrm{ab}}$ does not depend on the choices of $\widetilde{e}_1$, we may write I_{e_1} for $I_{\widetilde{e}_1, \mathrm{ab}}$. The structure of maximal prime-to-p quotient of $\pi_1^{\mathrm{t}}(U_{X_1})$ implies that $\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}$ is generated by $\{I_{e_1}\}_{e_1 \in D_{X_1}}$, and that there exists a generator a_{e_1}, $e_1 \in D_{X_1}$, of I_{e_1} such that $\prod_{e_1 \in D_{X_1}} a_{e_1} = 1$. We define

$$I_{e_{1,\infty}} \to \mathbb{Z}/(p^r - 1)\mathbb{Z}, \ a_{e_{1,\infty}} \mapsto 1,$$

$$I_{e_{1,0}} \to \mathbb{Z}/(p^r - 1)\mathbb{Z}, \ a_{e_{1,0}} \mapsto (\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1}) - 1,$$

and

$$I_{e_1} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_1} \mapsto -b'_{e_1},\ e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}.$$

Then the homomorphisms of inertia subgroups defined above induces a surjection $\delta_1 : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z}$. Note that $\ker(\delta_1)$ does not depend on the choices of the generators $\{a_{e_1}\}_{e_1 \in D_{X_1}}$.

Let $I_{\widetilde{e}_2} \overset{\mathrm{def}}{=} \phi(I_{\widetilde{e}_1})$, $\widetilde{e}_1 \in D_{\widetilde{X}_1}$, and I_{e_2}, $e_2 \in D_{X_2}$, the image of the natural homomorphism $I_{\widetilde{e}_2} \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{ab}}$. Since $(p, p^r-1) = 1$, by Theorem 3.14, δ_1 and the isomorphism $\phi^{p'} : \pi_1^{\mathrm{t}}(U_{X_1})^{p'} \overset{\sim}{\to} \pi_1^{\mathrm{t}}(U_{X_2})^{p'}$ imply the following homomorphisms of inertia subgroups:

$$I_{e_{2,\infty}} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_{2,\infty}} \mapsto 1,$$

$$I_{e_{2,0}} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_{2,0}} \mapsto (\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1}) - 1,$$

and

$$I_{e_2} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_2} \mapsto -b'_{e_1},\ e_2 \in D_{X_2} \setminus \{e_{2,\infty}, e_{2,0}\},$$

where a_{e_2}, $e_2 \in D_{X_2}$, denotes the element induced by a_{e_1}, $e_1 \in D_{X_1}$, via ϕ. Then the homomorphisms of inertia subgroups defined above induces a sujection $\delta_2 : \pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z}$.

We put $H_{\delta_i} \overset{\mathrm{def}}{=} \ker(\delta_i)$, $M_{\delta_i} \overset{\mathrm{def}}{=} H_{\delta_i}^{\mathrm{ab}} \otimes \mathbb{F}_p$, $i \in \{1,2\}$. Note that we have $H_{\delta_1} = \phi^{-1}(H_{\delta_2})$. Write $(X_{H_{\delta_i}}, D_{X_{H_{\delta_i}}})$ for the smooth pointed stable curve over k_i induced by H_{δ_i}. The $\mathbb{F}_p$-vector space M_{δ_i} admits a natural action of $I_{\widetilde{e}_{i,\infty}}$ via conjugation which coincides with the action via the following character

$$\chi_{I_{\widetilde{e}_{i,\infty}}, r} : I_{\widetilde{e}_{i,\infty}} \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i}) \overset{\delta_i}{\twoheadrightarrow} \mathbb{Z}/(p^r-1)\mathbb{Z} = I_{\widetilde{e}_{i,\infty}}/(p^r-1) \hookrightarrow \mathbb{F}_{\widetilde{e}_{i,\infty}}^{\times},\ i \in \{1,2\}.$$

We put $M_{\delta_i}[\chi_{I_{\widetilde{e}_{i,\infty}}, r}] \overset{\mathrm{def}}{=} \{a \in M_{\delta_i} \otimes_{\mathbb{F}_p} \mathbb{F}_{\widetilde{e}_{i,\infty}} \mid \sigma(a) = \chi_{I_{\widetilde{e}_{i,\infty}}, r}(\sigma) a \text{ for all } \sigma \in I_{\widetilde{e}_{i,\infty}}\}$, where $\sigma(a)$, $\sigma \in I_{\widetilde{e}_{i,\infty}}$, is the induced action of the conjugacy action of $I_{\widetilde{e}_{i,\infty}}$ on H_{δ_i}. In fact, $\dim_{\mathbb{F}_{\widetilde{e}_{i,\infty}}}(M_{\delta_i}[\chi_{I_{\widetilde{e}_{i,\infty}}, r}])$ is the first generalized Hasse-Witt invariant associated to the tame covering of U_{X_i} corresponding to $H_{\delta_i} \subseteq \pi_1^{\mathrm{t}}(U_{X_i})$ (see [Y5, Section 2.2]). Since the action of $I_{\widetilde{e}_{i,\infty}}$ on M_{δ_i} is semi-simple, we obtain a surjection $M_{\delta_1}[\chi_{I_{\widetilde{e}_{1,\infty}}, r}] \twoheadrightarrow M_{\delta_2}[\chi_{I_{\widetilde{e}_{2,\infty}}, r}]$ induced by $\phi|_{H_{\delta_1}}$ and $\theta_{\phi, \widetilde{e}_{1,\infty}, \widetilde{e}_{2,\infty}}$. On the other hand, the third and the final paragraphs of the proof of [T2, Lemma 3.3] imply that the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} e_1 = e_{1,0}$$

with respect to $e_{1,\infty}$ and $e_{1,0}$ on (X_1, D_{X_1}) holds if and only if $M_{\delta_1}[\chi_{I_{\widetilde{e}_{1,\infty}}, r}] = 0$. Thus, we obtain $M_{\delta_2}[\chi_{I_{\widetilde{e}_{2,\infty}}, r}] = 0$. Then the third and the final paragraphs of the proof of [T2, Lemma 3.3] imply that the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} \rho_\phi(e_1) = e_{2,0}$$

with respect to $e_{2,\infty}$ and $e_{2,0}$ on (X_2, D_{X_2}) holds. This completes the proof of the lemma. □

Remark 4.2.1. Note that, if $X_1 = \mathbb{P}_k^1$, then the linear condition is as follows:

$$\sum_{e_1 \in D_{X_1} \setminus \{\infty, 0\}} b_{e_1} e_1 = 0$$

with respect to ∞ and 0.

4.1.4. Now, we prove the first main theorem of the present paper.

Theorem 4.3. *We maintain the notation and settings introduced above. Then we have the following:*

(i) $d_{(X_i, D_{X_i})}$, $i \in \{1, 2\}$, *can be mono-anabelian reconstructed from* $\pi_1^{\rm t}(U_{X_i})$.

(ii) Suppose $k_1^{\rm m} \cong \overline{\mathbb{F}}_{p,1}$. *Then the set of open continuous homomorphisms*

$$\mathrm{Hom}^{\rm op}_{\rm pg}(\pi_1^{\rm t}(U_{X_1}), \pi_1^{\rm t}(U_{X_2}))$$

is non-empty if and only if $U_{X_1^{\rm m}} \cong U_{X_2^{\rm m}}$ *as schemes. In particular, if this is the case, we have* $k_2^{\rm m} \cong \overline{\mathbb{F}}_{p,2}$ *and*

$$\mathrm{Hom}^{\rm op}_{\rm pg}(\pi_1^{\rm t}(U_{X_1}), \pi_1^{\rm t}(U_{X_2})) = \mathrm{Isom}_{\rm pg}(\pi_1^{\rm t}(U_{X_1}), \pi_1^{\rm t}(U_{X_2})).$$

Proof. Firstly, let us prove the (ii). The "if" part of (ii) is trivial. We treat the "only if" part of (ii). Suppose that $\mathrm{Hom}^{\rm op}_{\rm pg}(\pi_1^{\rm t}(U_{X_1}), \pi_1^{\rm t}(U_{X_2}))$ is a non-empty set, and let $\phi \in \mathrm{Hom}^{\rm op}_{\rm pg}(\pi_1^{\rm t}(U_{X_1}), \pi_1^{\rm t}(U_{X_2}))$. Then Lemma 4.1 implies that ϕ is a surjection.

We identify $D^{\rm gp}_{X_i}$, $i \in \{1, 2\}$, with D_{X_i} via the bijection $\vartheta_{X_i} : D^{\rm gp}_{X_i} \xrightarrow{\sim} D_{X_i}$. Since ϕ is a surjection, Lemma 3.10 implies that ϕ induces a bijection $\rho_\phi : D_{X_1} \xrightarrow{\sim} D_{X_2}$. We put $e_{2,0} \overset{\rm def}{=} \rho_\phi(e_{1,0})$ and $e_{2,\infty} \overset{\rm def}{=} \rho_\phi(e_{1,\infty})$. Let $\widetilde{e}_{2,0} \in D_{\widetilde{X}_2}$ be a point over $e_{2,0}$. Theorem 3.14 implies that there exists a point $\widetilde{e}_{1,0} \in D_{\widetilde{X}_1}$ over $e_{1,0}$ such that $\phi(I_{\widetilde{e}_{1,0}}) = I_{\widetilde{e}_{2,0}}$. Then $\mathbb{F}_{\widetilde{e}_{i,0}} \overset{\rm def}{=} (I_{\widetilde{e}_{i,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \sqcup \{*_{\widetilde{e}_{i,0}}\}$, $i \in \{1, 2\}$, admits a structure of field. Moreover, Proposition 3.15 implies that the field structure can be mono-anabelian reconstructed from $\pi_1^{\rm t}(U_{X_i})$, and that ϕ induces a field isomorphism $\theta_{\phi, \widetilde{e}_{1,0}, \widetilde{e}_{2,0}} : \mathbb{F}_{\widetilde{e}_{1,0}} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_{2,0}}$.

Proposition 3.2 (i) implies that n can be mono-anabelian reconstructed from $\pi_1^{\rm t}(U_{X_i})$, $i \in \{1, 2\}$. If $n = 3$, (ii) is trivial, so we may assume $n \geq 4$. Moreover, since $k_1^{\rm m} \cong \overline{\mathbb{F}}_{p,1}$, without loss of generality, we may assume that $k_1 = \overline{\mathbb{F}}_{p,1} = \mathbb{F}_{\widetilde{e}_{1,0}}$, that $X_1 = \mathbb{P}^1_{\overline{\mathbb{F}}_{p,1}}$, and that

$$D_{X_1} = \{e_{1,\infty} = \infty, e_{1,0} = 0, e_{1,1} = 1, e_{1,2}, \dots, e_{1,n-2}\}.$$

Here, $e_{1,2}, \dots, e_{1,n-2} \in \overline{\mathbb{F}}_{p,1} \setminus \{e_{1,0}, e_{1,1}\}$ are distinct from each other.

Step 1: In this step, we will construct a linear condition on a certain tame covering of (X_1, D_{X_1}).

We see that there exists a natural number r prime to p such that $\mathbb{F}_p(\zeta_r)$ contains rth roots of $e_{1,2}, \dots, e_{1,n-2}$, where ζ_r denotes a fixed primitive rth root of unity in $\overline{\mathbb{F}}_{p,1}$. Let $s \overset{\rm def}{=} [\mathbb{F}_p(\zeta_r) : \mathbb{F}_p]$. For each $e_{1,u} \in \{e_{1,2}, \dots, e_{1,n-2}\}$, we fix an rth root $e_{1,u}^{1/r}$ in $\overline{\mathbb{F}}_{p,1}$. Then we have

$$e_{1,u}^{1/r} = \sum_{v=0}^{s-1} b_{1,uv} \zeta_r^v, \ \ u \in \{2, \dots, n-2\},$$

where $b_{1,uv} \in \mathbb{F}_p$ for each $u \in \{2, \dots, n-2\}$ and each $v \in \{0, \dots, s-1\}$.

Let $X_1 \setminus \{e_{1,\infty}\} = \mathrm{Spec}\, \overline{\mathbb{F}}_{p,1}[x_1]$, $f_{H_1} : (X_{H_1}, D_{X_{H_1}}) \to (X_1, D_{X_1})$ the Galois tame covering over $\overline{\mathbb{F}}_{p,1}$ with Galois group $\mathbb{Z}/r\mathbb{Z}$ determined by the equation $y_1^r = x_1$, and H_1 the open normal subgroup of $\pi_1^{\rm t}(U_{X_1})$ induced by the tame covering f_{H_1}. Then f_{H_1} is totally ramified over $\{e_{1,\infty} = \infty, e_{1,0} = 0\}$ and is étale over $D_{X_1} \setminus \{\infty, 0\}$. Note that $X_{H_1} = \mathbb{P}^1_{\overline{\mathbb{F}}_{p,1}}$, and that the points of $D_{X_{H_1}}$ over $\{e_{1,\infty}, e_{1,0}\}$ are $\{e_{H_1,\infty} \overset{\rm def}{=} \infty, e_{H_1,0} \overset{\rm def}{=} 0\}$. We put

$$e_{H_1,u} \overset{\rm def}{=} e_{1,u}^{1/r} \in D_{X_{H_1}}, \ \ u \in \{2, \dots, n-2\}, \ \ e_{H_1,1}^v \overset{\rm def}{=} \zeta_r^v \in D_{X_{H_1}}, \ \ v \in \{0, \dots, s-1\}.$$

Thus, we obtain a linear condition

$$e_{H_1,u} = \sum_{v=0}^{s-1} b_{1,uv} e_{H_1,1}^v$$

with respect to $e_{H_1,\infty}$ and $e_{H_1,0}$ on $(X_{H_1}, D_{X_{H_1}})$ for each $u \in \{2, \dots, n-2\}$.

Step 2: In this step, we will prove that the linear condition on a certain tame covering of (X_1, D_{X_1}) constructed in Step 1 induces a linear condition on a certain tame covering of (X_2, D_{X_2}) via the surjection ϕ.

Write H_2 for $\phi(H_1)$. Since $(r, p) = 1$, we have the following commutative diagram of profinite groups:

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/r\mathbb{Z} & = & \mathbb{Z}/r\mathbb{Z}. \end{array}$$

We denote by $f_{H_2} : (X_{H_2}, D_{X_{H_2}}) \to (X_2, D_{X_2})$ the Galois tame covering over $\overline{\mathbb{F}}_{p,2}$ with Galois group $\mathbb{Z}/r\mathbb{Z}$ induced by H_2. Note that Lemma 3.12 implies that $(X_{H_1}, D_{X_{H_1}})$ and $(X_{H_2}, D_{X_{H_2}})$ are equal types. Moreover, Lemma 3.11 implies that the following commutative diagram can be mono-anabelian reconstructed from the commutative diagram of profinite groups above:

$$\begin{array}{ccc} D_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D_{X_{H_2}} \\ \downarrow & & \downarrow \\ D_{X_1} & \xrightarrow{\rho_\phi} & D_{X_2}. \end{array}$$

We put

$$e_{2,\infty} \overset{\text{def}}{=} \rho_\phi(e_{1,\infty}),\ e_{2,u} \overset{\text{def}}{=} \rho_\phi(e_{1,u}),\ u \in \{0, \dots, n-2\},$$
$$e_{H_2,\infty} \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e_{H_1,\infty}),\ e_{H_2,0} \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e_{H_1,0}),\ e_{H_2,u} \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e_{H_1,u}),\ u \in \{2, \dots, n-2\},$$

and

$$e^v_{H_2,1} \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e^v_{H_1,1}),\ v \in \{0, \dots, s-1\}.$$

Remark 3.11.1 implies that f_{H_2} is totally ramified over $\{e_{2,\infty}, e_{2,0}\}$ and is étale over $X_2 \setminus \{e_{2,\infty}, e_{2,0}\}$. Then we may assume that $X_2 = \mathbb{P}^1_{k_2}$, and that $e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1$. We regard $e_{2,u}$, $u \in \{2, \dots, n-2\}$, as an element of $k_2 \setminus \{e_{2,0}, e_{2,1}\}$. Moreover, we have $e_{H_2,\infty} = \infty$ and $e_{H_2,0} = 0$.

We put $\xi_r \overset{\text{def}}{=} \theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(\zeta_r)$ which is an rth root of unity in $\mathbb{F}_{\widetilde{e}_{2,0}}$. Since $\zeta_r(e^v_{H_1,1}) = e^{v+1}_{H_1,1}$, we obtain $\xi_r(e^v_{H_2,1}) = e^{v+1}_{H_2,1}$, $v \in \{0, \dots, s-2\}$. By applying Lemma 4.2 for $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$, the following linear condition

$$e_{H_2,u} = \sum_{v=0}^{s-1} b_{1,uv}\xi^v_r(e^0_{H_2,1})$$

with respect to $e_{H_2,\infty}$ and $e_{H_2,0}$ on $(X_{H_2}, D_{X_{H_2}})$ holds for each $u \in \{2, \dots, n-2\}$. Since $(e_{H_2,u})^r = e_{2,u}$, $u \in \{2, \dots, n-2\}$, we obtain

$$e_{2,u} = (\sum_{v=0}^{s-1} b_{1,uv}\xi^v_r(e^0_{H_2,1}))^r.$$

Moreover, if we put $e^0_{H_2,1} = 1$, then we obtain that

$$e_{2,u} = (\sum_{v=0}^{s-1} b_{1,uv}\xi^v_r)^r$$

for each $u \in \{2, \dots, n-2\}$. Since $\theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(\zeta_r) = \xi_r$, we have

$$U_{X_1} = U_{X_1^{\mathrm{m}}} = \mathbb{P}^1_{\overline{\mathbb{F}}_{p,1}} \setminus \{e_{1,\infty} = \infty, e_{1,0} = 0, e_{1,1} = 1, e_{1,2}, \dots, e_{1,n-2}\}$$

$$\xrightarrow{\sim} \mathbb{P}^1_{\mathbb{F}_{\widetilde{e}_{2,0}}} \setminus \{e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1, \theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(e_{1,2}), \ldots, \theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(e_{1,n-2})\}$$
$$\cong \mathbb{P}^1_{\overline{\mathbb{F}}_{p,2}} \setminus \{e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1, e_{2,2}, \ldots, e_{2,n-2}\}$$

and

$$\mathbb{P}^1_{\overline{\mathbb{F}}_{p,2}} \setminus \{e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1, e_{2,2}, \ldots, e_{2,n-2}\} \times_{\overline{\mathbb{F}}_{p,2}} k_2 \cong U_{X_2}.$$

This means $U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$ as schemes. In particular, we have $k_2^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,2}$.

Finally, we prove that $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2})) = \mathrm{Isom}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$. The "$\supseteq$" part is trivial. We only need to prove the "$\subseteq$" part. We may assume $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2})) \neq \emptyset$. Let $\phi' \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$. Then $\pi_1^{\mathrm{t}}(U_{X_1})$ is isomorphic to $\pi_1^{\mathrm{t}}(U_{X_2})$ as abstract profinite groups. By Lemma 4.1, ϕ' is a surjection. Then [FJ, Proposition 16.10.6] implies that ϕ' is an isomorphism. Thus, we obtain $\phi' \in \mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$. This completes the proof of (ii).

Next, let us prove (i). Without loss of generality, we only treat the case where $i = 1$. Moreover, let $(X, D_X) \overset{\mathrm{def}}{=} (X_1, D_{X_1})$,

$$D_X = \{e_\infty = \infty, e_0 = 0, e_1 = 1, e_2, \ldots, e_{n-2}\},$$

$k \overset{\mathrm{def}}{=} k_1$, and $\overline{\mathbb{F}}_p \overset{\mathrm{def}}{=} \overline{\mathbb{F}}_{\widetilde{e}_0}$. Let (r, Q) be a pair such that the following conditions hold: (i) $(r, p) = 1$; (ii) Q is an open normal subgroup of $\pi_1^{\mathrm{t}}(U_X)$ such that $\pi_1^{\mathrm{t}}(U_X)/Q \cong \mathbb{Z}/r\mathbb{Z}$, and that the Galois tame covering $f_Q : (X_Q, D_{X_Q}) \to (X, D_X)$ over k induced by Q is totally ramified over $\{e_\infty, e_0\}$ and is étale over $D_X \setminus \{e_\infty, e_0\}$.

By applying Theorem 3.14, we see immediately that the set of pairs defined above can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$.

We fix a primitive rth root of unity ζ_r in $\overline{\mathbb{F}}_p$ and put $s_r \overset{\mathrm{def}}{=} [\mathbb{F}_p(\zeta_r) : \mathbb{F}_p]$. Moreover, we put

$$e_{Q,\infty} \overset{\mathrm{def}}{=} \infty,\ e_{Q,0} \overset{\mathrm{def}}{=} 0,\ e_{Q,1}^v \overset{\mathrm{def}}{=} \zeta_r^v \in D_{X_Q},\ v \in \{0, \ldots s_r - 1\},$$

and let $e_{Q,u} \in D_{X_Q}$, $u \in \{2, \ldots, n\}$, such that $f_Q(e_{Q,u}) = e_u$. Denote by

$$L_{Q,u} \overset{\mathrm{def}}{=} \{e_{Q,u} - \sum_{v=0}^{s_r-1} b_{uv} e_{Q,1}^v \mid b_{uv} \in \mathbb{F}_p\} \cap \{0\},\ u \in \{2, \ldots, n-2\}.$$

By applying similar arguments to the arguments given in the proof of (ii) above, we have that $d_{(X,D_X)} = 0$ if and only if there exists a pair (r, Q) defined above such that $L_{Q,u} \neq \emptyset$ for each $u \in \{2, \ldots, n-2\}$. Then the third and the final paragraphs of the proof of [T2, Lemma 3.3] implies that $L_{Q,u}$, $u \in \{2, \ldots, n-2\}$, can be mono-anabelian reconstructed from Q. Thus, $d_{(X,D_X)}$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$. This completes the proof of the theorem. □

Remark 4.3.1. Note that Theorem 4.3 also holds if we replace $\pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$, by its maximal pro-solvable quotient $\pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{sol}}$. Then we obtain the following solvable version of Theorem 4.3 which is slightly stronger than the original theorem:

> *We maintain the notation introduced above. Then $d_{(X_i,D_{X_i})}$, $i \in \{1, 2\}$, can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{sol}}$. Moreover, suppose that $k_1^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,1}$. Then the set of open continuous homomorphisms*
>
> $$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{sol}}, \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{sol}})$$
>
> *is non-empty if and only if $U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$ as schemes. In particular, if this is the case, we have $k_2^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,2}$ and*
>
> $$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{sol}}, \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{sol}}) = \mathrm{Isom}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{sol}}, \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{sol}}).$$

4.2. **The second main theorem.** In this subsection, by using Theorem 4.3, we prove a result concerning pointed collection conjecture and the weak Hom-version conjecture (Theorem 4.4).

4.2.1. **Settings.** We maintain the notation introduced in 2.1.2.

4.2.2. Let $q \in M_{0,n}^{\mathrm{ord}}$ be an arbitrary point, $\overline{k(q)}$ an algebraic closure of $k(q)$, and

$$U_{X_q} \cong \mathbb{P}^1_{\overline{k(q)}} \setminus \{a_1 = 1, a_2 = 0, a_3 = \infty, a_4, \dots, a_n\}$$

as $\overline{k(q)}$-schemes. We shall say that q is a *coordinated point* if either $q = q_{\mathrm{gen}}$ or the following conditions are satisfied:

(i) $\dim(V_q) = \dim(M_{0,n}^{\mathrm{ord}}) - 1$.

(ii) There exists $i \in \{4, \dots, n\}$ such that $a_i \in \overline{\mathbb{F}}_p$.

Let $\omega_{n,4}^{\setminus i} : M_{0,n}^{\mathrm{ord}} \to M_{0,4}^{\mathrm{ord}}$ be the morphism induced by the morphism $\mathcal{M}_{0,n}^{\mathrm{ord}} \to \mathcal{M}_{0,4}^{\mathrm{ord}}$ obtained by forgetting the marked points except the first, the second, the third, and the ith marked points. If q is a coordinated point and $q \neq q_{\mathrm{gen}}$, then we have that $q'' \overset{\mathrm{def}}{=} \omega_{n,4}^{\setminus i}(q)$ is a closed point of $M_{0,4}^{\mathrm{ord}}$, and that $(\omega_{n,4}^{\setminus i})^{-1}(q'') = V_q$ since $(\omega_{n,4}^{\setminus i})^{-1}(q'')$ is an irreducible closed subset of dimension $\dim(M_{0,n}^{\mathrm{ord}}) - 1$ containing V_q.

Let t be a closed point of $M_{0,n}^{\mathrm{ord}}$. Then there exists a set of coordinated points $P_t \overset{\mathrm{def}}{=} \{q_{t,4}, \dots, q_{t,n}\}$ such that

$$\{t\} = \bigcap_{q_{t,j} \in P_t} V_{q_{t,j}}.$$

4.2.3. Now, we prove the second main result of the present paper.

Theorem 4.4. *(i) For each closed point $t \in M_{0,n}^{\mathrm{ord,cl}}$, the set $\mathcal{C}_t$ associated to t is a pointed collection (Definition 2.4). Moreover, for each pointed collection $\mathcal{C} \in \mathscr{C}_{q_{\mathrm{gen}}}$, there exists a closed point $s \in M_{0,n}^{\mathrm{ord,cl}}$ such that $\mathcal{C} = \mathcal{C}_s$.*

(ii) Let $q \in M_{0,n}^{\mathrm{ord}}$ be an arbitrary point. Then the the natural map $\mathrm{colle}_q : \mathscr{V}_q^{\mathrm{cl}} \to \mathscr{C}_q$, $[t] \mapsto \mathcal{C}_t$, is an injection.

(iii) Let $q \in M_{0,n}^{\mathrm{ord}}$ be an arbitrary point. Suppose that there exists a set of coordinated points P_q such that

$$V_q = \bigcap_{u \in P_q} V_u.$$

Then the pointed collection conjecture holds for q. In particular, the pointed collection conjecture holds for each closed point of $M_{0,n}^{\mathrm{ord}}$.

(iv) Let $q_i \in M_{0,n}^{\mathrm{ord}}$, $i \in \{1, 2\}$, be an arbitrary point. Suppose that there exists a set of coordinated points P_{q_1} such that

$$V_{q_1} = \bigcap_{u \in P_{q_1}} V_u.$$

Then the weak Hom-version conjecture holds. In particular, the weak Hom-version conjecture holds when q_1 is a closed point.

Proof. Let us prove (i). We put $F_t \overset{\mathrm{def}}{=} \{t' \in M_{0,n}^{\mathrm{ord,cl}} \mid t \sim_{fe} t'\}$. Let t'' be an arbitrary point of $\bigcap_{G \in \pi_A^{\mathrm{t}}(t)} U_G$. Then, for each $G \in \pi_A^{\mathrm{t}}(t)$, $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{surj}}(\pi_1^{\mathrm{t}}(t''), G)$ is non-empty, where $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{surj}}(-,-)$ denotes the subset of $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{open}}(-,-)$ whose elements are surjections. Since $\pi_1^{\mathrm{t}}(t'')$ is topologically finitely generated, we obtain that the set $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{surj}}(\pi_1^{\mathrm{t}}(t''), G)$ is finite. Then the set of open continuous homomorphisms

$$\varprojlim_{G \in \pi_A^{\mathrm{t}}(t)} \mathrm{Hom}_{\mathrm{pg}}^{\mathrm{surj}}(\pi_1^{\mathrm{t}}(t''), G) = \mathrm{Hom}_{\mathrm{pg}}^{\mathrm{surj}}(\pi_1^{\mathrm{t}}(t''), \pi_1^{\mathrm{t}}(t))$$

is non-empty. Thus, Theorem 4.3 implies $t'' \in F_t$. This means

$$\Big(\bigcap_{G \in \pi_A^{\mathrm{t}}(t)} U_G\Big) \cap M_{g,n}^{\mathrm{ord,cl}} = F_t.$$

Since U_{X_t} can be defined over a finite field, F_t is a finite set. Then $\mathcal{C}_t$ is a pointed collection.

Let $\mathcal{C} \in \mathscr{C}_{q_{\mathrm{gen}}}$ be a pointed collection and s a closed point of $\bigcap_{G\in\mathcal{C}} U_G$. By replacing t by s, and by applying similar arguments to the arguments given in the proof above, we obtain $\mathcal{C} = \mathcal{C}_s$.

(ii) follows immediately from Theorem 4.3. Let us prove (iii). If $n = 4$, then $M^{\mathrm{ord}}_{0,4}$ is a one dimension scheme. For each $q \in M^{\mathrm{ord}}_{0,4}$, the pointed collection conjecture follows immediately from Theorem 4.3. Then we may assume $n \geq 5$. To verify (iii), (ii) implies that we only need to prove that colle_q is a surjection.

Suppose that q is a closed point of $M^{\mathrm{ord}}_{0,n}$. Let $\mathcal{C} \in \mathscr{C}_q$ be an arbitrary pointed collection contained in $\mathscr{C}_q$. By applying (i), there exists a closed point $s \in M^{\mathrm{ord,cl}}_{0,n}$ such that the pointed collection $\mathcal{C}_s$ associated to s is equal to $\mathcal{C}$. Since $\mathcal{C} \in \mathscr{C}_q$, there exists a surjection $\pi_1^{\mathrm{t}}(q) \twoheadrightarrow \pi_1^{\mathrm{t}}(s)$. Then Theorem 4.3 implies $\pi_1^{\mathrm{t}}(q) \overset{\sim}{\to} \pi_1^{\mathrm{t}}(s)$. Thus, we have $\pi_A(q) = \mathcal{C}_s = \mathcal{C}$ (or equivalently, $\mathscr{C}_q = \{\pi_A^{\mathrm{t}}(q)\}$). In particular, colle_q is a surjection if q is a closed point of $M^{\mathrm{ord}}_{0,n}$.

Suppose that q is a non-closed point. This means $\dim(V_q) \geq 1$. If $q = q_{\mathrm{gen}}$, (iii) follows from (i) and (ii). Let us treat the case where $q \neq q_{\mathrm{gen}}$. First, suppose that q is a coordinated point, and that

$$U_{X_q} \cong \mathbb{P}^1_{k(q)} \setminus \{1, 0, \infty, a_4, \dots, a_n\}.$$

Without loss of generality, we may assume $a_n \in \overline{\mathbb{F}}_p$.

For each pointed collection $\mathcal{C} \subseteq \mathscr{C}_q$, by applying (i), there exists a closed point $t_1 \in M^{\mathrm{ord,cl}}_{g,n}$ such that $\mathcal{C}_{t_1} = \mathcal{C}$. Then we have an open continuous surjective homomorphism $\pi_1^{\mathrm{t}}(q) \twoheadrightarrow \pi_1^{\mathrm{t}}(t_1)$. Let $\omega^{\backslash n}_{n,4} : M^{\mathrm{ord}}_{0,n} \to M^{\mathrm{ord}}_{0,4}$ be the morphism induced by the morphism $\mathcal{M}^{\mathrm{ord}}_{0,n} \to \mathcal{M}^{\mathrm{ord}}_{0,4}$ obtained by forgetting the marked points except the first, the second, the third, and the nth marked points. We put $t_1'' \overset{\mathrm{def}}{=} \omega^{\backslash n}_{n,4}(t_1)$ and $q'' \overset{\mathrm{def}}{=} \omega^{\backslash n}_{n,4}(q)$. Note that t_1'' and q'' are closed points of $M_{0,4}$. Write (X_q, D_{X_q}), $(X_{t_1}, D_{X_{t_1}})$, $(X_{q''}, D_{X_{q''}})$, and $(X_{t_1''}, D_{X_{t_1''}})$ for the smooth pointed stable curves corresponding to q, t_1, q'' and t_1'', respectively.

We denote by $I_q \subseteq \pi_1^{\mathrm{t}}(U_{X_q}) = \pi_1^{\mathrm{t}}(q)$ the normal closed subgroup generated by $I_{\widetilde{e}} \in \mathrm{Ine}(\pi_1^{\mathrm{t}}(U_{X_q}))$, $e \in D_{X_q} \setminus D_{X_{q''}}$, where $\widetilde{e} \in D_{\widetilde{X}_q}$ is an element over e (see 3.1.3 for $D_{\widetilde{X}_q}$), and $I_{t_1} \subseteq \pi_1^{\mathrm{t}}(U_{X_{t_1}}) = \pi_1^{\mathrm{t}}(t_1)$ the normal closed subgroup generated by $I_{\widetilde{e}} \in \mathrm{Ine}(\pi_1^{\mathrm{t}}(U_{X_{t_1}}))$, $e \in D_{X_{t_1}} \setminus D_{X_{t_1''}}$. Note that we have $\pi_1^{\mathrm{t}}(q)/I_q \overset{\sim}{\to} \pi_1^{\mathrm{t}}(q'')$ and $\pi_1^{\mathrm{t}}(t_1)/I_{t_1} \overset{\sim}{\to} \pi_1^{\mathrm{t}}(t_1'')$. Moreover, Theorem 3.14 implies that the image of I_q under the surjection $\pi_1^{\mathrm{t}}(q) \twoheadrightarrow \pi_1^{\mathrm{t}}(t_1)$ is I_{t_1}. Then the surjection $\pi_1^{\mathrm{t}}(q) \twoheadrightarrow \pi_1^{\mathrm{t}}(t_1)$ induces an open continuous surjective homomorphism $\pi_1^{\mathrm{t}}(q'') \twoheadrightarrow \pi_1^{\mathrm{t}}(t_1'')$. Thus, by Theorem 4.3, we obtain that $q'' \sim_{fe} t_1''$. Then without loss of generality, we may assume $q'' = t_1''$ and

$$U_{X_{t_1}} \cong \mathbb{P}^1_{\overline{\mathbb{F}}_p} \setminus \{1, 0, \infty, b_4, \dots, b_{n-1}, a_n\}$$

over $\overline{\mathbb{F}}_p$, where $b_i \in \overline{\mathbb{F}}_p$ for each $i \in \{4, \dots, n-1\}$. Furthermore, we see $t_1 \in (\omega^{\backslash n}_{n,4})^{-1}(t_1'') = (\omega^{\backslash n}_{n,4})^{-1}(q'') = V_q$. Thus, t_1 is a closed point of V_q. Then the pointed collection conjecture holds for q when q is a coordinated point.

Next, we prove the general case. If $V_q = \bigcap_{u\in P_q} V_u$, then $V_q^{\mathrm{cl}} = \bigcap_{u\in P_q} V_u^{\mathrm{cl}}$ and $\bigcap_{u\in P_q} \mathscr{C}_u = \mathscr{C}_q$. Moreover, since we have a bijection $\mathrm{colle}_u : \mathscr{V}_u^{\mathrm{cl}} \overset{\sim}{\to} \mathscr{C}_u$ for each $u \in P_q$, we have that

$$\mathrm{colle}_q : \mathscr{V}_q^{\mathrm{cl}} = \bigcap_{u\in P_q} \mathscr{V}_u^{\mathrm{cl}} \to \bigcap_{u\in P_q} \mathscr{C}_u = \mathscr{C}_q$$

is a bijection. This completes the proof of (iii).

Let us prove (iv). We only need to prove the "only if" part of the weak Hom-version conjecture. Suppose that V_{q_2} is not essentially contained in V_{q_1}. This implies that there exists a closed point $t_2 \in V_{q_2}^{\mathrm{cl}}$ such that $F_{t_2} \cap V_{q_1} = \emptyset$, where $F_{t_2} \overset{\mathrm{def}}{=} \{t_2' \in M^{\mathrm{ord,cl}}_{0,n} \mid t_2 \sim_{fe} t_2'\}$. By (iii), we have $\mathcal{C}_{t_2} \notin \mathscr{C}_{q_1}$. Thus, by Lemma 4.1, we obtain that

$$\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(t_2)) = \emptyset.$$

This provides a contradiction to the assumption that $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$ is non-empty. We complete the proof of (iv). $\square$

Remark 4.4.1. Let $q \in M_{g,n}$ be an arbitrary point. Stevenson posed a question as follows (see [Ste, Question 4.3] for the case of $n = 0$): Does $\bigcap_{G\in\pi_A^{\mathrm{t}}(q)} U_G$ contain any closed points of $M_{g,n}$? By [T5, Theorem 0.3], $\bigcap_{G\in\pi_A^{\mathrm{t}}(q)} U_G$ contains a closed point of $M_{g,n}$ if and only if q is a closed point of $M_{g,n}$. Furthermore, when $g = 0$ and q is a closed point, the proof of Theorem 4.4 (i) implies that

$$(\bigcap_{G\in\pi_A^{\mathrm{t}}(q)} U_G) \cap M_{0,n}^{\mathrm{cl}} = F_q,$$

where $F_q \overset{\mathrm{def}}{=} \{q' \in M_{0,n}^{\mathrm{cl}} \mid q \sim_{fe} q'\}$.

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