Tame Anabelian Geometry and Topological Structures of Moduli Spaces of Curves over Algebraically Closed Fields of Characteristic $p > 0$

Yu Yang

Abstract

In the present paper, we study the tame anabelian geometry of curves in positive characteristic. Let (X_i, D_{X_i}), $i \in \{1, 2\}$, be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$ and $\pi_1^{\mathrm{t}}(U_{X_i})$ the tame fundamental group of $U_{X_i} \stackrel{\mathrm{def}}{=} X_i \setminus D_{X_i}$. We prove that, if $g_X = 0$ and $k_1 = k_2 = \overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$, then U_{X_1} is isomorphic to U_{X_2} as schemes if and only if there exists an open continuous homomorphism between $\pi_1^{\mathrm{t}}(U_{X_1})$ and $\pi_1^{\mathrm{t}}(U_{X_2})$. This result generalizes the weak Isom-version of the Grothendieck conjecture for curves of type $(0, n_X)$ over $\overline{\mathbb{F}}_p$ which has been proven by A. Tamagawa. Moreover, this result shows a new anabelian phenomenon which says that the *topological structures* of moduli spaces of curves can be reconstructed group-theoretically from tame fundamental groups. We formulate some new conjectures concerning tame fundamental groups of curves over algebraically closed fields of characteristic $p > 0$ from the point of view of moduli spaces. The new conjectures are generalized versions of the weak Isom-version of the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$ which was formulated by Tamagawa.

Keywords: hyperbolic curve, tame fundamental group, moduli space, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14H10, 14G32.

Contents

1 Introduction

In the present paper, we study the tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. Before we explain the main result of the present paper, let us recall some general facts concerning the anabelian geometry of curves.

Let k be a field, and let (X, D_X) be a smooth pointed stable curve of type (g_X, n_X) over k. Here, X denotes the underlying smooth projective curve of (X, D_X), D_X denotes the set of marked points of (X, D_X), g_X denotes the genus of X, and n_X denotes the cardinality of D_X. We put $U_X \stackrel{\text{def}}{=} X \setminus D_X$. Then U_X is a *hyperbolic curve* over k (i.e., $2g_X + n_X - 2 > 0$). Roughly speaking, the ultimate goal of the anabelian geometry of curves is the following question.

> Can we recover the isomorphism class of U_X group-theoretically from various versions of its fundamental group?

The various formulations of this question are called *Grothendieck's anabelian conjecture* or the Grothendieck conjecture, for short. When k is an *arithmetic field* (e.g. number field, p-adic field, finite field, etc.), the Grothendieck conjecture has been proven in many cases. Suppose that k is of characteristic 0. For example, if k is a number field, then the Grothendieck conjecture was proved by H. Nakamura, A. Tamagawa, and S. Mochizuki (cf. [N1], [N2], [T1], [M1]). Moreover, Tamagawa also considered an analogue of the Grothendieck conjecture in positive characteristic and proved the Grothendieck conjecture for affine curves over finite fields (cf. [T1]). Afterwards, J. Stix generalized this result to the case where the base fields are finitely generated over $\mathbb{F}_p$ (cf. [Sti1], [Sti2]). All the proofs of the Grothendieck conjecture for curves over arithmetic fields mentioned above require the use of *the highly non-trivial outer Galois representations* induced by the fundamental exact sequences of étale (or tame) fundamental groups. In the case of algebraically closed fields of characteristic 0, since the étale (or tame) fundamental groups of curves depend only on the types of curves, the anabelian geometry of curves does not exist in this situation.

On the other hand, when the characteristic of k is $p > 0$, the situation is quite different from that in characteristic 0. Some developments of F. Pop, M. Saïdi, M. Raynaud, and Tamagawa (cf. [PS], [R2], [T2], [T3], [T4], [T5]) from the late of 1990s showed evidence for very strong anabelian phenomena for curves over *algebraically closed fields of characteristic* $p > 0$. In this situation, the Galois group of the base field is *trivial*, and the étale (or tame) fundamental group coincides with the geometric étale (or tame) fundamental group, thus there is a total absence of a Galois action of the base field. This kinds of anabelian phenomena go beyond Grothendieck's anabelian geometry (cf. [G1], [G2]). The main problem considered in this situation is as follows:

Can we recover the isomorphism class of a curve group-theoretically from various versions of its **geometric** fundamental group?

The formulation of this problem was given by Tamagawa. Let us explain some details about this arguments. Let

$$(X_i, D_{X_i}),\ i \in \{1, 2\},$$

be a smooth pointed stable curve of type (g_{X_i}, n_{X_i}) over an algebraically closed field k_i of characteristic $p > 0$. Write $\overline{\mathbb{F}}_{p,i}$ for the algebraic closure of $\mathbb{F}_p$ in k_i and k_i^{m} for the minimal algebraically closed subfield of k_i over which (X_i, D_{X_i}) can be defined. Thus, by considering the function field of X_i, one verifies immediately that there exists a natural smooth pointed stable curve (i.e., a minimal model of (X_i, D_{X_i}) (cf. [T3, Definition 1.30 and Lemma 1.31]))

$$(X_i^{\mathrm{m}}, D_{X_i^{\mathrm{m}}})$$

over k_i^{m}, where the function field of X_i^{m} is a subfield of the function field of X_i, such that $U_{X_i} \overset{\mathrm{def}}{=} X_i \setminus D_{X_i}$ may be identified with $U_{X_i^{\mathrm{m}}} \times_{k_i^{\mathrm{m}}} k_i$, where $U_{X_i^{\mathrm{m}}} \overset{\mathrm{def}}{=} X_i^{\mathrm{m}} \setminus D_{X_i^{\mathrm{m}}}$. For suitable choices of base points, we obtain the tame fundamental groups $\pi_1^{\mathrm{t}}(U_{X_i})$ and $\pi_1^{\mathrm{t}}(U_{X_i^{\mathrm{m}}})$ of U_{X_i} and $U_{X_i^{\mathrm{m}}}$, respectively. Note that $\pi_1^{\mathrm{t}}(U_{X_i})$ is naturally isomorphic to $\pi_1^{\mathrm{t}}(U_{X_i^{\mathrm{m}}})$. Write $\mathrm{Isom}_{\text{pro-gps}}(-,-)$ and $\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(-,-)$ for the set of continuous isomorphisms and the set of open continuous homomorphisms of profinite groups between the two profinite groups in parentheses, respectively. In this situation, Tamagawa posed the following conjecture which is called *the weak Isom-version of the Grothendieck conjecture for curves over algebraically closed fields of characteristic* $p > 0$ (=Weak Isom-version Conjecture):

Weak Isom-version Conjecture . *The set of continuous isomorphisms of profinite groups*

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$$

is non-empty if and only if $U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$ *as schemes.*

Moreover, Tamagawa proved the Weak Isom-version Conjecture when $g_{X_1} = g_{X_2} = 0$ and $k_1 = k_2$ is an algebraic closure of $\mathbb{F}_p$. More precisely, Tamagawa proved the following result (cf. [T4, Theorem 0.2]).

Theorem 1.1. *Suppose that* $g_{X_i} = 0$, $i \in \{1, 2\}$. *Then we can detect whether* $k_i^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,i}$ *or not, group-theoretically from* $\pi_1^{\mathrm{t}}(U_{X_i})$. *Moreover, suppose that* $k_1^{\mathrm{m}} = \overline{\mathbb{F}}_{p,1}$. *Then the set of continuous isomorphisms of profinite groups*

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$$

is non-empty if and only if

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes.

Remark 1.1.1. The tame fundamental group of a curve over an algebraically closed field of characteristic $p > 0$ is very mysterious. Note that, if $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$, we always have that the maximal prime-to-p quotients $\pi_1^{\mathrm{t}}(U_{X_1})^{p'}$ and $\pi_1^{\mathrm{t}}(U_{X_2})^{p'}$ of $\pi_1^{\mathrm{t}}(U_{X_1})$

and $\pi_1^{\rm t}(U_{X_2})$, respectively, are isomorphic as profinite groups. Then, roughly speaking, Theorem 1.1 means that the "p-part" of the tame fundamental group of a smooth pointed stable curve over an algebraically closed field of characteristic $p > 0$ contains the entire information of the curve as a scheme, and that the tame fundamental group must encode "moduli" of the curve. This is why we do not have an explicit description of the tame fundamental group (or the set of finite quotients of the tame fundamental group) of any pointed stable curve in positive characteristic.

Remark 1.1.2. Before Tamagawa proved Theorem 1.1, he also obtained an étale fundamental group version of Theorem 1.1 in a completely different way (by using wildly ramified coverings) which is much simpler than the case of tame fundamental groups (cf. [T2, Theorem 3.5]). Note that, for any smooth pointed stable curve over an algebraically closed field of positive characteristic, since the tame fundamental group can be recovered group-theoretically from the étale fundamental group (cf. [T2, Corollary 1.10]), the tame fundamental group version is stronger than the étale fundamental group version.

Remark 1.1.3. We do not know whether or not the Weak Isom-version Conjecture holds in general. On the other hand, we have the following *finiteness theorem* which was proved by Pop, Saïdi, Raynaud, and Tamagawa (cf. [PS], [R2], [T5]):

> over an algebraic closure of $\mathbb{F}_p$, only finitely many isomorphism classes of smooth pointed stable curves have the same tame fundamental groups.

Moreover, the finiteness theorem also holds for arbitrary (possibly singular) pointed stable curves (cf. [Y1]).

In the present paper, we consider the tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ from the point of view of moduli spaces and formulate some generalized conjectures of the Weak Isom-version Conjecture. First, let us reformulate the Weak Isom-version Conjecture from the point of view of moduli spaces. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and let $\mathcal{M}_{g,n,\mathbb{Z}}$ be the moduli stack over $\mathbb{Z}$ parameterizing smooth pointed stable curves of type (g, n). We denote by

$$\mathcal{M}_{g,n} \overset{\rm def}{=} \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$$

the moduli stack over $\overline{\mathbb{F}}_p$, and denote by $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$. Note that the set of marked points of a smooth pointed stable curve admits a natural action of the n-symmetric group S_n. Moreover, we denote by $\mathcal{M}_{g,[n]} \overset{\rm def}{=} [\mathcal{M}_{g,n}/S_n]$ the quotient stack, and denote by $M_{g,[n]}$ the coarse moduli space of $\mathcal{M}_{g,[n]}$. We obtain a morphism $\omega : M_{g,n} \to M_{g,[n]}$ induced by the natural quotient morphism $\mathcal{M}_{g,n} \to \mathcal{M}_{g,[n]}$. Let $q \in M_{g,n}$ be an arbitrary point, $k(q)$ the residue field of q, k_q an algebraically closed field which contains $k(q)$, and $V_q \overset{\rm def}{=} \overline{\{q\}}$ the topological closure of $\{q\}$ in $M_{g,n}$. Write $(X_{k_q}, D_{X_{k_q}})$ for the smooth pointed stable curve of type (g, n) over k_q determined by the natural morphism $\operatorname{Spec} k_q \to M_{g,n}$ and $U_{X_{k_q}} \overset{\rm def}{=} X_{k_q} \setminus D_{X_{k_q}}$. Since the isomorphism class of the tame fundamental group $\pi_1^{\rm t}(U_{X_{k_q}})$ depends only on q, we shall write $\pi_1^{\rm t}(q)$ for the tame fundamental group $\pi_1^{\rm t}(U_{X_{k_q}})$.

For any *closed points* $c_1, c_2 \in M_{g,n}^{\rm cl}$, where $(-)^{\rm cl}$ denotes the set of closed points of $(-)$, we define an equivalence relation as follows:

$c_1 \sim_{\mathrm{ec}} c_2$ if there exists $m \in \mathbb{Z}$ such that $\omega(c_2) = \omega(c_1^{(m)})$, where $c_1^{(m)}$ denotes the closed point corresponding to the curve obtained by mth Frobenius twist of the curve corresponding to c_1.

Moreover, let $q_1, q_2 \in M_{g,n}$ be two arbitrary points. We denote by $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$ if, for each closed point $c_2 \in V_{q_2}$, there exists a closed point $c_1 \in V_{q_1}$ such that $c_2 \in \{c_1^{(m)}\}_{m \in \mathbb{Z}}$. Moreover, we denote by $V_{q_1} =_{\mathrm{ec}} V_{q_2}$ (or $q_1 \sim_{\mathrm{ec}} q_2$) when $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$ and $V_{q_1} \subseteq_{\mathrm{ec}} V_{q_2}$. We shall say that V_{q_1} *essentially contains* V_{q_2} if $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$, and V_{q_1} *is essentially equal to* V_{q_2} if $V_{q_1} =_{\mathrm{ec}} V_{q_2}$. Furthermore, $V_{q_1} =_{\mathrm{ec}} V_{q_2}$ if and only if the minimal models of curves corresponding to q_1 and q_2 are isomorphic as schemes (cf. Proposition 8.2). Then we reformulate the Weak Isom-version Conjecture as follows.

Weak Isom-version Conjecture . *The set of continuous isomorphisms of profinite groups*

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty if and only if $V_{q_1} =_{\mathrm{ec}} V_{q_2}$.

The new formulation of the Weak Isom-version Conjecture means that the quotient of the moduli space $M_{g,n}/\sim_{\mathrm{ec}}$ can be reconstructed as a *set* from the isomorphism classes of the tame fundamental groups of curves corresponding to points of $M_{g,n}$. However, the Weak Isom-version Conjecture cannot tell us any further information of $M_{g,n}/\sim_{\mathrm{ec}}$. The main question of interest in the present paper is as follows.

Main Question . *Can we reconstruct the natural* **topological structure** *of* $M_{g,n}/\sim_{\mathrm{ec}}$ *induced by the topological structure of* $M_{g,n}$ *from the tame fundamental groups of curves corresponding to points of* $M_{g,n}$*?*

This means that, for any irreducible closed sets V_1 and V_2 of $M_{g,n}/\sim_{\mathrm{ec}}$, whether or not we can determine $V_1 \supseteq V_2$ group-theoretically from the tame fundamental groups of the generic points of V_1 and V_2. In order to reconstruct the *topological structure* of $M_{g,n}/\sim_{\mathrm{ec}}$, we pose a *weak Hom-version of the Grothendieck conjecture for curves with fixed types over algebraically closed fields of characteristic* $p > 0$ (=Weak Hom-version Conjecture) as follows.

Weak Hom-version Conjecture . *The set of open continuous homomorphisms of profinite groups*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty if and only if $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$.

First, we note that the Weak Hom-version Conjecture implies the Weak Isom-version Conjecture. Second, the Weak Hom-version Conjecture means that the quotient of the moduli space $M_{g,n}/\sim_{\mathrm{ec}}$ can be reconstructed as a *topological space* from the sets of open continuous homomorphisms of the tame fundamental groups of curves corresponding to points of $M_{g,n}$. The main result of the present paper is the following (cf. Theorem 8.6 (iv)):

Theorem 1.2. *The Weak Hom-version Conjecture holds when* q_1 *is a closed point of* $M_{0,n}$.

Remark 1.2.1. Let us explain the philosophy of the Weak Hom-version Conjecture more precisely. Let $\Pi_{g,n}$ be the set of isomorphism classes of tame fundamental groups of smooth pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$. Then we obtain a natural surjective map (as sets)

$$\pi^{\mathrm{t}}_{g,n} : M_{g,n}/\sim_{\mathrm{ec}} \to \Pi_{g,n},\ [q] \mapsto [\pi^{\mathrm{t}}_1(q)],$$

where $[q]$ denotes the image of $q \in M_{g,n}$ in $M_{g,n}/\sim_{\mathrm{ec}}$, and $[\pi^{\mathrm{t}}_1(q)]$ the isomorphism class of $\pi^{\mathrm{t}}_1(q)$. The Weak Isom-version Conjecture says that $\pi^{\mathrm{t}}_{g,n}$ is a bijection as sets. Moreover, Tamagawa's theorem (i.e., Theorem 1.1) implies the following result:

- The map $\pi^{\mathrm{t}}_{0,n}|_{M^{\mathrm{cl}}_{0,n}/\sim_{\mathrm{ec}}}$ is an injection, where $(-)^{\mathrm{cl}}$ denotes the set of closed points of $(-)$. In particular,
$$\pi^{\mathrm{t}}_{0,4} : M_{0,4}/\sim_{\mathrm{ec}} \to \Pi_{0,4}$$
is a bijection.

Recently, in [Y6], [Y7], the author introduced a topology for the set $\Pi_{g,n}$ which is called *anabelian topology*. Together with the anabelian topology, $\Pi_{g,n}$ can be regarded as a *topology space* which is called *the moduli space of admissible fundamental groups of type* (g, n) *in positive characteristic* p. Moreover, we prove that $\pi^{\mathrm{t}}_{g,n}$ is a continuous morphism and conjectured that $\pi^{\mathrm{t}}_{g,n}$ is a *homeomorphism* (=Homeomorphism Conjecture). In this formulation, Theorem 1.2 and Theorem 8.6 imply the following results:

- Let q be an arbitrary closed point of $M_{0,n}$. Then $[\pi^{\mathrm{t}}_1(q)]$ is a closed point of $\Pi_{0,n}$. In particular,
$$\pi^{\mathrm{t}}_{0,4} : M_{0,4}/\sim_{\mathrm{ec}} \to \Pi_{0,4}$$
is a homeomorphism. Moreover, we have $\dim(M_{0,n}) \leq \dim(\Pi_{0,n})$.

Moreover, we would like to mention that, based on the theory developed in [Y2], [Y3], [Y4], [Y5], and the present paper, in [Y6], [Y7], the author proved Homeomorphism Conjecture when $\dim(\overline{M}_{g,n}) = 1$ (i.e., $(g, n) \in \{(0, 4), (1, 1)\}$), where $\overline{M}_{g,n}$ denotes the Deligne-Mumford compactification of $M_{g,n}$.

Theorem 1.2 follows from the following result (cf. Theorem 7.3) which is a generalization of Tamagawa's result (i.e., Theorem 1.1):

Theorem 1.3. *Suppose that* $g_{X_i} = 0$, $i \in \{1, 2\}$. *We can detect whether* $k^{\mathrm{m}}_i \cong \overline{\mathbb{F}}_{p,i}$, *holds or not, group-theoretically from* $\pi^{\mathrm{t}}_1(U_{X_i})$. *Moreover, suppose that* $k^{\mathrm{m}}_1 \cong \overline{\mathbb{F}}_{p,1}$, *and that* $n_{X_1} = n_{X_2}$. *Then the set of open continuous homomorphisms*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(U_{X_1}), \pi^{\mathrm{t}}_1(U_{X_2}))$$

is non-empty if and only if

$$U_{X^{\mathrm{m}}_1} \cong U_{X^{\mathrm{m}}_2}$$

as schemes. In particular, if this is the case, we have $k^{\mathrm{m}}_2 \cong \overline{\mathbb{F}}_{p,2}$ *and*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(U_{X_1}), \pi^{\mathrm{t}}_1(U_{X_2})) = \mathrm{Isom}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(U_{X_1}), \pi^{\mathrm{t}}_1(U_{X_2})).$$

Remark 1.3.1. Note that Theorem 1.3 is essentially different from Theorem 1.1. The reason is that, in general, there are many *non-isomorphic* open continuous homomorphisms between two tame fundamental groups of smooth pointed stable curves with *a fixed type* over algebraically closed fields of characteristic $p > 0$ (cf. [T5, Theorem 0.3]).

The present paper is organized as follows. In Section 2, we give some definitions and propositions which will be used in the present paper. In Section 3, we give a cohomological explanation for sets of marked points of smooth pointed stable curves. In Section 4, by applying the theory developed in Section 3, we prove that the inertia subgroups of marked points can be mono-anabelian reconstructed from the tame fundamental groups. In Section 5, we prove that the mono-anabelian reconstructions obtained in Section 4 are compatible with open surjective homomorphisms of tame fundamental groups. In Section 6, we prove that the field structures can be mono-anabelian reconstructed from the tame fundamental groups, and that the mono-anabelian reconstructions are compatible with open surjective homomorphisms of tame fundamental groups. In Section 7, we prove Theorem 1.3. In Section 8, we reformulate the Essential Dimension Conjecture and the Weak Isom-version Conjecture from the point of view of moduli spaces, and formulate the Weak Hom-version Conjecture. Moreover, we pose the Pointed Collection Conjecture and prove Theorem 1.2.

Acknowledgements

The author would like to thank Jakob Stix, Akio Tamagawa, and Kazuki Tokimoto for helpful comments. The author would like to thank the referee for carefully reading the manuscript, and for giving me many comments which substantially helped improving the quality of the paper. This paper was written in 2017 summer, and this research was supported by JSPS KAKENHI Grant Number 16J08847.

2 Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, and let

$$(X, D_X)$$

be a smooth pointed stable curve of type (g_X, n_X) over k. Here, X denotes the underlying smooth projective curve of (X, D_X), D_X denotes the set of marked points of (X, D_X), g_X denotes the genus of X, and n_X denotes the cardinality of D_X. Then the curve $U_X \overset{\text{def}}{=} X \setminus D_X$ is a *hyperbolic* curve over k (i.e., $2g_X + n_X - 2 > 0$). By choosing a base point of $x \in U_X$, we obtain the tame fundamental group $\pi_1^{\mathrm{t}}(U_X, x)$ of U_X and the étale fundamental group $\pi_1(X, x)$ of X. For simplicity of notation, we omit the base point and denote by

$$\pi_1^{\mathrm{t}}(U_X) \text{ and } \pi_1(X)$$

the tame fundamental group $\pi_1^{\mathrm{t}}(U_X, x)$ of U_X and the étale fundamental group $\pi_1(X, x)$ of X, respectively. Note that there is a natural continuous surjective homomorphism

$$\pi_1^{\mathrm{t}}(U_X) \twoheadrightarrow \pi_1(X).$$

Definition 2.1. Let (Y, D_Y) and (X, D_X) be smooth pointed stable curves over k, and let $f : (Y, D_Y) \to (X, D_X)$ be a morphism of smooth pointed stable curves over k. We shall say that f is *étale* (resp. *tame*, *Galois étale*, *Galois tame*) if f is étale over X (resp. f is étale over U_X and is at most tamely ramified over D_X, f is a Galois covering and is étale, f is a Galois covering and is tame).

The following highly non-trivial result concerning the limit of p-averages of $\pi_1^{\mathrm{t}}(U_X)$ was proved by Tamagawa (cf. [T4, Theorem 0.5]).

Proposition 2.2. *Let $r \in \mathbb{N}$ be a natural number, and let K_{p^r-1} be the kernel of the natural surjection $\pi_1^{\mathrm{t}}(U_X) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_X)^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}$, where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$. Then we have*

$$\mathrm{Avr}_p(\pi_1^{\mathrm{t}}(U_X)) \overset{\mathrm{def}}{=} \lim_{r\to\infty} \frac{\dim_{\mathbb{F}_p}(K_{p^r-1}^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(\pi_1^{\mathrm{t}}(U_X)^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z})} = \begin{cases} g_X - 1, & \text{if } n_X \leq 1, \\ g_X, & \text{if } n_X > 1. \end{cases}$$

Here, $\#(-)$ denotes the cardinality of $(-)$.

Remark 2.2.1. Tamagawa proved Proposition 2.2 as a main theorem of [T4] by developing a tamely ramified version of Raynaud's theta divisors. Moreover, the author generalized Proposition 2.2 to the case of (possibly singular) pointed stable curves (cf. [Y4]).

Corollary 2.3. *Let (X_i, D_{X_i}), $i \in \{1, 2\}$, be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic p, $U_{X_i} \overset{\mathrm{def}}{=} X_i \setminus D_{X_i}$, and $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ an arbitrary continuous surjective homomorphism of the tame fundamental groups of U_{X_1} and U_{X_2}. Let $H_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ be an open normal subgroup such that $([\pi_1^{\mathrm{t}}(U_{X_2}) : H_2], p) = 1$ and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2)$. Write g_{H_i}, $i \in \{1, 2\}$, for the genus of the smooth pointed stable curve over k_i corresponding to $H_i \subseteq \pi_1^{\mathrm{t}}(U_{X_i})$. Then we have that*

$$g_{H_1} \geq g_{H_2}.$$

Proof. The surjection ϕ induces a surjection

$$\phi^{p'} : \pi_1^{\mathrm{t}}(U_{X_1})^{p'} \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})^{p'},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Moreover, since $\pi_1^{\mathrm{t}}(U_{X_i})^{p'}$, $i \in \{1, 2\}$, is topologically finitely generated, and $\pi_1^{\mathrm{t}}(U_{X_1})^{p'}$ is isomorphic to $\pi_1^{\mathrm{t}}(U_{X_2})^{p'}$ as abstract profinite groups (since the types of (X_1, D_{X_1}) and (X_2, D_{X_2}) are equal to (g_X, n_X)), we obtain that $\phi^{p'}$ is an isomorphism (cf. [FJ, Proposition 16.10.6]).

On the other hand, since $[\pi_1^{\mathrm{t}}(U_{X_1}) : H_1] = [\pi_1^{\mathrm{t}}(U_{X_2}) : H_2]$ and $([\pi_1^{\mathrm{t}}(U_{X_2}) : H_2], p) = 1$, we obtain that the natural homomorphism

$$\phi_H^{p'} : H_1^{p'} \twoheadrightarrow H_2^{p'}$$

induced by $\phi_H \overset{\mathrm{def}}{=} \phi|_{H_1}$ is also an isomorphism. This implies that

$$\#(H_1^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}) = \#(H_2^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z})$$

for all $r \in \mathbb{N}$. Let $K_{H_i,p^{r-1}}$, $i \in \{1,2\}$, be the kernel of the natural surjection

$$H_i \twoheadrightarrow H_i^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}.$$

Then the surjection ϕ_H implies that

$$\mathrm{Avr}_p(H_1) \stackrel{\mathrm{def}}{=} \lim_{r\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_1,p^r-1}\otimes \mathbb{F}_p)}{\#(H_1^{\mathrm{ab}}\otimes \mathbb{Z}/(p^r-1)\mathbb{Z})}$$

$$\geq \mathrm{Avr}_p(H_2) \stackrel{\mathrm{def}}{=} \lim_{r\to\infty} \frac{\dim_{\mathbb{F}_p}(K^{\mathrm{ab}}_{H_2,p^r-1}\otimes \mathbb{F}_p)}{\#(H_2^{\mathrm{ab}}\otimes \mathbb{Z}/(p^r-1)\mathbb{Z})}.$$

Thus, the corollary follows from Proposition 2.2. □

Let K be the function field of X, and let $\widetilde{K}$ be the maximal Galois extension of K in a fixed separable closure of K, unramified over U_X and at most tamely ramified over D_X. Then we may identify $\pi_1^{\mathrm{t}}(U_X)$ with $\mathrm{Gal}(\widetilde{K}/K)$. We define a universal tame covering of (X, D_X) to be

$$(\widetilde{X}, D_{\widetilde{X}}),$$

where $\widetilde{X}$ denotes the normalization of X in $\widetilde{K}$, and $D_{\widetilde{X}}$ denotes the inverse image of D_X in $\widetilde{X}$. For each $\widetilde{e} \in D_{\widetilde{X}}$, we denote by $I_{\widetilde{e}}$ the inertia subgroup of $\pi_1^{\mathrm{t}}(U_X)$ associated to $\widetilde{e}$ (i.e., the stabilizer of $\widetilde{e}$). Note that we have $I_{\widetilde{e}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $\widehat{\mathbb{Z}}(1)^{p'}$ denotes the prime-to-p part of $\widehat{\mathbb{Z}}(1)$.

We recall some well-known results concerning the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. Fist, we define the term "can be reconstructed" from the point of view of mono-anabelian geometry.

Definition 2.4. Let $\mathcal{F}$ be a geometric object and $\Pi_{\mathcal{F}}$ a profinite group associated to the geometric object $\mathcal{F}$. Suppose that we are given an invariant $\mathrm{Inv}_{\mathcal{F}}$ depending on the isomorphism class of $\mathcal{F}$ (in a certain category), and that we are given an additional structure $\mathrm{Add}_{\mathcal{F}}$ (e.g., a family of subgroups, a family of quotient groups) on the profinite group $\Pi_{\mathcal{F}}$ depending functorially on $\mathcal{F}$. We shall say that $\mathrm{Inv}_{\mathcal{F}}$ can be *mono-anabelian reconstructed from* $\Pi_{\mathcal{F}}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}}$, and whose output datum is $\mathrm{Inv}_{\mathcal{F}}$. We shall say that $\mathrm{Add}_{\mathcal{F}}$ *can be mono-anabelian reconstructed from* $\Pi_{\mathcal{F}}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}}$.

Let $\mathcal{F}_i$, $i \in \{1,2\}$, be a geometric object and $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object $\mathcal{F}_i$. Suppose that we are given an additional structure $\mathrm{Add}_{\mathcal{F}_i}$ on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on $\mathcal{F}_i$. We shall say that a map (or a morphism) $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$ *can be mono-anabelian reconstructed* from an open continuous homomorphism $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$.

Remark 2.4.1. Let us explain the philosophy of *mono-anabelian geometry* introduced by Mochizuki (cf. [M2]). The classical point of view of anabelian geometry (i.e., the anabelian geometry considered in [G1], [G2]) focuses on a comparison between two geometric objects via their fundamental groups. Moreover, the term "group-theoretical", in the classical point of view, means that "preserved by an arbitrary isomorphism between the fundamental groups under consideration". The classical point of view is referred to as *bi-anabelian geometry.* On the other hand, mono-anabelian geometry focuses on the establishing a *group-theoretical algorithm* whose *input datum* is an abstract topological group which is isomorphic to the fundamental group of a given geometric object of interest (resp. a continuous homomorphism of abstract topological groups which are isomorphic to a continuous homomorphism of the fundamental groups of given geometric objects of interest), and whose *output datum* is a geometric object which is isomorphic to the given geometric object of interest (resp. a morphism of geometric objects which is isomorphic to a morphism of given geometric objects of interest). In the point of view of mono-anabelian geometry, the term "group-theoretical algorithm" is used to mean that "the algorithm in a discussion is phrased in language that only depends on the topological group structure of the fundamental groups under consideration" (cf. [M2] for more details concerning the philosophy of mono-anabelian geometry). Note that, in general, mono-anabelian-type results imply bi-anabelian-type results.

The following result was proved by Tamagawa which follows from Proposition 2.2, [T4, Lemma 5.1], and [T4, Theorem 5.2].

Proposition 2.5. *(i) The pair* (g_X, n_X) *can be mono-anabelian reconstructed from* $\pi_1^{\mathrm{t}}(U_X)$*.*

(ii) Let $\widetilde{e}$ *and* $\widetilde{e}'$ *be two points of* $D_{\widetilde{X}}$ *distinct from each other. Then the intersection of* $I_{\widetilde{e}}$ *and* $I_{\widetilde{e}'}$ *is trivial in* $\pi_1^{\mathrm{t}}(U_X)$*. Moreover, the map*

$$D_{\widetilde{X}} \to \mathrm{Sub}(\pi_1^{\mathrm{t}}(U_X)),\ \widetilde{e} \mapsto I_{\widetilde{e}},$$

is an injection, where $\mathrm{Sub}(-)$ *denotes the set of closed subgroups of* $(-)$*.*

(iii) Write $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$ *for the set of inertia subgroups in* $\pi_1^{\mathrm{t}}(U_X)$*, namely the image of the map* $D_{\widetilde{X}} \to \mathrm{Sub}(\pi_1^{\mathrm{t}}(U_X))$*. Then* $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$ *can be mono-anabelian reconstructed from* $\pi_1^{\mathrm{t}}(U_X)$*. In particular, the set of marked points* D_X *and* $\pi_1(X)$ *can be mono-anabelian reconstructed from* $\pi_1^{\mathrm{t}}(U_X)$*.*

Remark 2.5.1. The main purposes of the next three sections is as follows.

> We will give a new mono-anabelian reconstruction of $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$, and prove that the mono-anabelian reconstruction (i.e., the group-theoretical algorithm) is compatible with open continuous homomorphisms of tame fundamental groups of smooth pointed stable curves with a fixed type.

Remark 2.5.2. Proposition 2.5 can be extended to the case of (possibly singular) pointed stable curves over algebraically closed fields of characteristic $p > 0$ (i.e., the combinatorial Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$). More precisely, the author proved that, for any pointed stable curve (X, D_X) of type (g_X, n_X) over an algebraically closed field of characteristic $p > 0$, the type (g_X, n_X), the dual semi-graph of (X, D_X), and the tame fundamental groups of the smooth pointed stable curves associated to the normalization of irreducible components of (X, D_X) can be mono-anabelian reconstructed from *the geometric log étale fundamental group* (or equivalently, *the admissible fundamental group*) of (X, D_X) (cf. [Y1, Theorem 1.2], [Y3, Theorem 0.5]). As an application, the author generalized [T4, Theorem 0.2] to the case of (possibly singular) pointed stable curves (cf. [Y1, Theorem 1.3 (a)], [Y3, Corollary 0.6]).

3 The set of marked points

We maintain the notation introduced in Section 2. Moreover, in this section, we suppose that

- $g_X \geq 2$ and $n_X > 0$.

Let $h : (W, D_W) \to (X, D_X)$ be a connected Galois tame covering over k. We put

$$\mathrm{Ram}_h \stackrel{\mathrm{def}}{=} \{e \in D_X \mid h \text{ is ramified over } e\}.$$

Let (Y, D_Y) be a smooth pointed stable curve over k. We shall say that

$$(\ell, d, f : (Y, D_Y) \to (X, D_X))$$

is *a triple associated to* (X, D_X) if the following conditions hold: (i) ℓ and d are prime numbers distinct from each other such that $(\ell, p) = (d, p) = 1$ and $\ell \equiv 1 \pmod{d}$; then all dth roots of unity are contained in $\mathbb{F}_\ell$; (ii) f is a Galois *étale* covering over k whose Galois group is isomorphic to μ_d, where $\mu_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of dth roots of unity. Then we have a natural injection

$$H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \hookrightarrow H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$$

induced by the natural surjection $\pi_1^{\mathrm{t}}(U_Y) \twoheadrightarrow \pi_1(Y)$. Note that every non-zero element of $H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ induces a connected Galois tame covering of (Y, D_Y) of degree ℓ. We obtain an exact sequence

$$0 \to H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell) \to \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell \to 0$$

with a natural action of μ_d.

Let

$$(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell)_{\mu_d} \subseteq \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell$$

be the subset of elements on which μ_d acts via the character $\mu_d \hookrightarrow \mathbb{F}_\ell^\times$ and

$$M_Y^* \subseteq H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$$

the subset of elements whose images are non-zero elements of $(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell)_{\mu_d}$. For each $\alpha \in M_Y^*$, write

$$g_\alpha : (Y_\alpha, D_{Y_\alpha}) \to (Y, D_Y)$$

for the tame covering induced by α. We define a map to be

$$\epsilon : M_Y^* \to \mathbb{Z},\ \epsilon(\alpha) = \# D_{Y_\alpha},$$

and put

$$M_Y \stackrel{\text{def}}{=} \{\alpha \in M_Y^* \mid \#\mathrm{Ram}_{g_\alpha} = d\} = \{\alpha \in M_Y^* \mid \epsilon(\alpha) = \ell(dn_X - d) + d\}.$$

Note that M_Y is not empty. For each $\alpha \in M_Y$, since the image of α is contained in $(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell)_{\mu_d}$, we obtain that the action of μ_d on $\mathrm{Ram}_{g_\alpha} \subseteq D_Y$ is transitive. Thus, there exists a unique marked point $e_\alpha \in D_X$ such that $f(y) = e_\alpha$ for each $y \in \mathrm{Ram}_{g_\alpha}$. Next, we define a pre-equivalence relation $\sim$ on M_Y as follows:

> Let $\alpha, \beta \in M_Y$. Then $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in M_Y^*$, we have $\lambda\alpha + \mu\beta \in M_Y$.

For each $e \in D_X$, we put

$$M_{Y,e} \stackrel{\text{def}}{=} \{\alpha \in M_Y \mid g_\alpha \text{ is ramified over } f^{-1}(e)\}.$$

Then, for any marked points $e, e' \in D_X$ distinct from each other, we have $M_{Y,e} \cap M_{Y,e'} = \emptyset$ and the disjoint union

$$M_Y = \bigsqcup_{e \in D_X} M_{Y,e}.$$

Moreover, we have the following proposition.

Proposition 3.1. *The pre-equivalence relation $\sim$ on M_Y is an equivalence relation. Moreover, the map*

$$\vartheta_X : M_Y/\sim \to D_X, \ [\alpha] \mapsto e_\alpha,$$

is a bijection, where $[\alpha]$ denotes the image of α in $M_Y/\sim$.

Proof. Let $\beta, \gamma \in M_Y$. If $\mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\beta + \mu\gamma \neq 0$, we have

$$\mathrm{Ram}_{g_{\lambda\beta+\mu\gamma}} = \mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}.$$

Thus we obtain that $\beta \sim \gamma$. On the other hand, if $\beta \sim \gamma$, we have $\mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}$. Otherwise, we have $\#\mathrm{Ram}_{g_{\beta+\gamma}} = 2d$. This means that

$$\beta \sim \gamma \text{ if and only if } \mathrm{Ram}_{g_\beta} = \mathrm{Ram}_{g_\gamma}.$$

Then $\sim$ is an equivalence relation on M_Y.

Let us prove that ϑ_X is a bijection. It is easy to see that ϑ_X is an injection. On the other hand, for each $e \in D_X$, the structure of the maximal pro-ℓ tame fundamental groups implies that we may construct a connected tame Galois covering of $h : (Z, D_Z) \to (Y, D_Y)$ such that the element of $H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ induced by h is contained in M_Y. Then ϑ_X is a surjection. This completes the proof of Proposition 3.1. □

Remark 3.1.1. We claim that the set $M_Y/\sim$ does not depend on the choices of ℓ, d, and the étale covering $f : (Y, D_Y) \to (X, D_X)$. Let

$$(\ell^*, d^*, f^* : (Y^*, D_{Y^*}) \to (X, D_X))$$

be an arbitrary triple associated to (X, D_X). Hence we obtain a resulting set $M_{Y^*}/\sim$ and a natural bijection

$$\vartheta_X^* : M_{Y^*}/\sim \to D_X.$$

We will prove that there exists a natural bijection $\delta : M_{Y^*}/\sim \xrightarrow{\sim} M_Y/\sim$ such that $\vartheta_X^* = \vartheta_X \circ \delta$.

First, suppose that $\ell \neq \ell^*$ and $d \neq d^*$. Then we may construct a natural bijection

$$\delta : M_{Y^*}/\sim \xrightarrow{\sim} M_Y/\sim$$

as follows. Let $\alpha \in M_Y$ and $\alpha^* \in M_{Y^*}$. Write $(Y_\alpha, D_{Y_\alpha}) \to (Y, D_Y)$ and $(Y_{\alpha^*}, D_{Y_{\alpha^*}}) \to (Y^*, D_{Y^*})$ for the Galois tame coverings induced by α and α^*, respectively. We consider the following fiber product in the category of smooth pointed stable curves

$$(Y_\alpha, D_{Y_\alpha}) \times_{(X, D_X)} (Y_{\alpha^*}, D_{Y_{\alpha^*}})$$

which is a smooth pointed stable curve over k. Thus, we obtain a connected tame covering $(Y_\alpha, D_{Y_\alpha}) \times_{(X, D_X)} (Y_{\alpha^*}, D_{Y_{\alpha^*}}) \to (X, D_X)$ of degree $dd^*\ell\ell^*$. Then it is easy to check that

$$\vartheta_X([\alpha]) = \vartheta_X^*([\alpha^*])$$

if and only if the cardinality of the set of marked points of $(Y_\alpha, D_{Y_\alpha}) \times_{(X, D_X)} (Y_{\alpha^*}, D_{Y_{\alpha^*}})$ is equal to $dd^*(\ell\ell^*(n_X - 1) + 1)$. We put

$$[\alpha] \overset{\text{def}}{=} \delta([\alpha^*])$$

if $\vartheta_X([\alpha]) = \vartheta_X^*([\alpha^*])$. Moreover, by the construction above, we obtain that $\vartheta_X^* = \vartheta_X \circ \delta$. In general case, we may choose a triple

$$(\ell^{**}, d^{**}, f^{**} : (Y^{**}, D_{Y^{**}}) \to (X, D_X))$$

associated to (X, D_X) such that $\ell^{**} \neq \ell$, $\ell^{**} \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain a resulting set $M_{Y^{**}}/\sim$ and a natural bijection $\vartheta_X^{**} : M_{Y^{**}}/\sim \to D_X$. Then the proof given above implies that there are natural bijections $\delta_1 : M_{Y^{**}}/\sim \xrightarrow{\sim} M_Y/\sim$ and $\delta_2 : M_{Y^{**}}/\sim \xrightarrow{\sim} M_{Y^*}/\sim$. Thus, we may put

$$\delta \overset{\text{def}}{=} \delta_1 \circ \delta_2^{-1} : M_{Y^*}/\sim \xrightarrow{\sim} M_Y/\sim .$$

Remark 3.1.2. Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an arbitrary open normal subgroup and

$$f_H : (X_H, D_{X_H}) \to (X, D_X)$$

the Galois tame covering over k induced by the natural inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$. Let

$$(\ell, d, f : (Y, D_Y) \to (X, D_X))$$

be a triple associated to (X, D_X) such that $(\#(\pi_1^{\mathrm{t}}(U_X)/H), \ell) = (\#(\pi_1^{\mathrm{t}}(U_X)/H), d) = 1$. Then we obtain a triple

$$(\ell, d, g : (Z, D_Z) \overset{\text{def}}{=} (Y, D_Y) \times_{(X, D_X)} (X_H, D_{X_H}) \to (X_H, D_{X_H}))$$

associated to (X_H, D_{X_H}) induced by $(\ell, d, f : (Y, D_Y) \to (X, D_X))$, where $(Y, D_Y) \times_{(X,D_X)} (X_H, D_{X_H})$ denotes the fiber product in the category of smooth pointed stable curves. The triple associated to (X_H, D_{X_H}) induces a set

$$M_Z/\sim$$

which can be identified with the set of marked points D_{X_H} of (X_H, D_{X_H}) by applying Proposition 3.1. Moreover, for each $e_X \in D_X$ and each $\alpha_{Y,e_X} \in M_{Y,e_X}$, α_{Y,e_X} induces an element

$$\alpha_Z = \sum_{e_{X_H} \in f_H^{-1}(e_X)} \alpha_{Z,e_{X_H}}$$

over (Z, D_Z) via the natural morphism $(Z, D_Z) \to (Y, D_Y)$, where $\alpha_{Z,e_{X_H}} \in M_{Z,e_{X_H}}$. On the other hand, for each $e'_{X_H} \in D_{X_H}$ and each $e'_X \in D_X$, we have that

> $f_H(e'_{X_H}) = e'_X$ if and only if there exists an element $\alpha_{Y,e'_X} \in M_{Y,e'_X}$ such that the following conditions hold:
>
> (i) the element α'_Z, induced by α_{Y,e'_X} via the natural morphism $(Z, D_Z) \to (Y, D_Y)$, can be represented by a linear combination
>
> $$\alpha'_Z = \sum_{e_{X_H} \in S_{X_H}} \alpha'_{Z,e_{X_H}},$$
>
> where S_{X_H} is a subset of D_{X_H}, and $\alpha_{Z,e_{X_H}} \in M_{Z,e_{X_H}}$;
>
> (ii) $e'_{X_H} \in S_{X_H}$.

Lemma 3.2. *Let $(\ell, d, f : (Y, D_Y) \to (X, D_X))$ be a triple associated to (X, D_X) and g_Y the genus of Y. Then we have*

$$\#M_{Y,e} = \ell^{2g_Y+1} - \ell^{2g_Y},\ e \in D_X.$$

Moreover, we have

$$\#M_Y = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$

Proof. Let $e \in D_X$. Write $D_e \subseteq D_Y$ for the set $f^{-1}(e)$. Then $M_{Y,e}$ can be naturally regarded as a subset of $H^1_{\text{ét}}(Y \setminus D_e, \mathbb{F}_\ell)$ via the natural open immersion $Y \setminus D_e \hookrightarrow Y$. Write L_e for the $\mathbb{F}_\ell$-vector space generated by $M_{Y,e}$ in $H^1_{\text{ét}}(Y \setminus D_e, \mathbb{F}_\ell)$. Then we have $M_{Y,e} = L_e \setminus H^1_{\text{ét}}(Y, \mathbb{F}_\ell)$. Write H_e for the quotient $L_e/H^1_{\text{ét}}(Y, \mathbb{F}_\ell)$. We have an exact sequence as follows:

$$0 \to H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to L_e \to H_e \to 0.$$

Since the action of μ_d on $f^{-1}(e)$ is transitive, we obtain $\dim_{\mathbb{F}_\ell}(H_e) = 1$. On the other hand, since $\dim_{\mathbb{F}_\ell}(H^1_{\text{ét}}(Y, \mathbb{F}_\ell)) = 2g_Y$, we obtain

$$\#M_{Y,e} = \ell^{2g_Y+1} - \ell^{2g_Y}.$$

Thus, we have

$$\#M_Y = n_X(\ell^{2g_Y+1} - \ell^{2g_Y}).$$

This completes the proof of the lemma. □

4 Mono-anabelian reconstructions of inertia subgroups of marked points

We maintain the notation introduced in previous sections. In this section, we give a new mono-anabelian reconstruction of $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$. First, we have the following lemma.

Lemma 4.1. *(i) The prime number p (i.e., the characteristic of k) can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$.*

(ii) The étale fundamental group $\pi_1(X)$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$.

Proof. (i) Let $\mathfrak{P}$ be the set of prime numbers, and let Q be an arbitrary open subgroup of $\pi_1^{\mathrm{t}}(U_X)$ and r_Q an integer such that

$$\#\{l \in \mathfrak{P} \mid r_Q = \dim_{\mathbb{F}_l}(Q^{\mathrm{ab}} \otimes \mathbb{F}_l)\} = \infty.$$

Then we see immediately that the characteristic of k is the unique prime number p such that there exists an open subgroup $T \subseteq \pi_1^{\mathrm{t}}(U_X)$ and $r_T \neq \dim_{\mathbb{F}_p}(T^{\mathrm{ab}} \otimes \mathbb{F}_p)$.

(ii) Let H be an arbitrary open normal subgroup of $\pi_1^{\mathrm{t}}(U_X)$. We denote by

$$(X_H, D_{X_H})$$

the smooth pointed stable curve of type (g_{X_H}, n_{X_H}) over k induced by H, and denote by

$$f_H : (X_H, D_{X_H}) \to (X, D_X)$$

the morphism of smooth pointed stable curves over k induced by the natural inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$. We note that f_H is étale if and only if

$$g_{X_H} - 1 = \#(\pi_1^{\mathrm{t}}(U_X)/H)(g_X - 1).$$

We put

$$\mathrm{Et}(\pi_1^{\mathrm{t}}(U_X)) \stackrel{\mathrm{def}}{=} \{H \subseteq \pi_1^{\mathrm{t}}(U_X) \text{ is an open normal subgroup} \mid g_{X_H} - 1 = \#(\pi_1^{\mathrm{t}}(U_X)/H)(g_X - 1)\}.$$

Moreover, Proposition 2.5 (i) implies that g_{X_H} and g_X can be mono-anabelian reconstructed from H and $\pi_1^{\mathrm{t}}(U_X)$, respectively. Then the set $\mathrm{Et}(\pi_1^{\mathrm{t}}(U_X))$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$. We obtain that

$$\pi_1(X) = \pi_1^{\mathrm{t}}(U_X) / \bigcap_{H \in \mathrm{Et}(\pi_1^{\mathrm{t}}(U_X))} H.$$

This completes the proof of the lemma. □

Suppose that $g_X \geq 2$. Let us define a group-theoretical object corresponding to a triple which was introduced in Section 3. We shall say that

$$(\ell, d, y)$$

is *a triple associated to* $\pi_1^{\mathrm{t}}(U_X)$ if the following conditions hold: (i) ℓ and d are prime numbers distinct from each other such that $(\ell, p) = (d, p) = 1$ and $\ell \equiv 1 \pmod d$; then all dth roots of unity are contained in $\mathbb{F}_\ell$; (ii) $y \in \mathrm{Hom}(\pi_1(X), \mu_d)$ such that $y \neq 0$, where $\mu_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of dth roots of unity.

Moreover, by applying Lemma 4.1, there is a triple (ℓ, d, y) associated to $\pi_1^{\mathrm{t}}(U_X)$ which can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$. Let $f : (Y, D_Y) \to (X, D_X)$ be a Galois étale covering induced by y. Then we see immediately that $(\ell, d, f : (Y, D_Y) \to (X, D_X))$ is a triple associated to (X, D_X) defined in Section 3. We denote by $\pi_1^{\mathrm{t}}(U_Y)$ the kernel of the composition of the surjections

$$\pi_1^{\mathrm{t}}(U_X) \twoheadrightarrow \pi_1(X) \overset{y}{\twoheadrightarrow} \mu_d.$$

Since $H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \cong \mathrm{Hom}(\pi_1(Y), \mathbb{F}_\ell)$ and $H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell) \cong \mathrm{Hom}(\pi_1^{\mathrm{t}}(U_Y), \mathbb{F}_\ell)$, Lemma 4.1 implies immediately that the following exact sequence

$$0 \to H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell) \to \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell \to 0$$

can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_Y)$. Thus, Proposition 2.5 (i) implies that the set $M_Y/\sim$ defined in Section 3 can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_Y)$. Note that, by Remark 3.1.1, the set $M_Y/\sim$ does not depend on the choices of triples. Then we put

$$D_X^{\mathrm{gp}} \overset{\mathrm{def}}{=} M_Y/\sim,$$

where $(-)^{\mathrm{gp}}$ means "group-theoretical". By Proposition 3.1, we may identify D_X^{gp} with the set of marked points D_X of (X, D_X) via the bijection $\vartheta_X : D_X^{\mathrm{gp}} \overset{\sim}{\to} D_X$ defined in Proposition 3.1.

Proposition 4.2. *Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an arbitrary open normal subgroup and*

$$f_H : (X_H, D_{X_H}) \to (X, D_X)$$

the morphism of smooth pointed stable curves over k induced by the natural inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$. Suppose that $g_X \geq 2$. Then the sets D_X^{gp} and $D_{X_H}^{\mathrm{gp}}$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$ and H, respectively. Moreover, the inclusion $H \hookrightarrow \pi_1^{\mathrm{t}}(U_X)$ induces a map $\gamma_{H, \pi_1^{\mathrm{t}}(U_X)} : D_{X_H}^{\mathrm{gp}} \to D_X^{\mathrm{gp}}$ such that the following commutative diagram holds:

$$\begin{array}{ccc} D_{X_H}^{\mathrm{gp}} & \xrightarrow{\vartheta_{X_H}} & D_{X_H} \\ {\scriptstyle \gamma_{H,\pi_1^{\mathrm{t}}(U_X)}}\downarrow & & \downarrow{\scriptstyle \gamma_{f_H}} \\ D_X^{\mathrm{gp}} & \xrightarrow{\vartheta_X} & D_X, \end{array}$$

where γ_{f_H} denotes the map of the sets of marked points induced by f_H.

Proof. We only need to prove the "Moreover" part of Proposition 4.2. We maintain the notation introduced in Remark 3.1.2. Then, for each $e_X \in D_X$ and each $e_{X_H} \in D_{X_H}$, the sets M_{Y, e_X} and $M_{Z, e_{X_H}}$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$ and H, respectively. Then the "Moreover" part follows from Remark 3.1.2. □

Remark 4.2.1. We maintain the notation introduced in Proposition 4.2. Let $\pi_1(X_H)$ be the étale fundamental group of X_H. Then we have a natural surjection $H \twoheadrightarrow \pi_1(X_H)$. Note that $\pi_1(X_H)$ admits a n action of $\pi_1^{\mathrm{t}}(U_X)/H$ induced by the outer action of $\pi_1^{\mathrm{t}}(U_X)/H$ on H induced by the exact sequence

$$1 \to H \to \pi_1^{\mathrm{t}}(U_X) \to \pi_1^{\mathrm{t}}(U_X)/H \to 1.$$

Moreover, the action of $\pi_1^{\mathrm{t}}(U_X)/H$ on $\pi_1(X_H)$ induces an action of $\pi_1^{\mathrm{t}}(U_X)/H$ on $D_{X_H}^{\mathrm{gp}}$. On the other hand, it is easy to check that the action of $\pi_1^{\mathrm{t}}(U_X)/H$ on $D_{X_H}^{\mathrm{gp}}$ coincides with the natural action of $\pi_1^{\mathrm{t}}(U_X)/H$ on D_{X_H} when we identify D_X^{gp} with D_X.

The main result of the present section is as follows.

Theorem 4.3. *Write* $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$ *for the set of inertia subgroups in* $\pi_1^{\mathrm{t}}(U_X)$. *Then* $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_X))$ *can be mono-anabelian reconstructed from* $\pi_1^{\mathrm{t}}(U_X)$.

Proof. Let $C_X \stackrel{\mathrm{def}}{=} \{H_i\}_{i\in\mathbb{Z}_{>0}}$ be a set of open normal subgroups of $\pi_1^{\mathrm{t}}(U_X)$ such that $\varprojlim_i \pi_1^{\mathrm{t}}(U_X)/H_i \cong \pi_1^{\mathrm{t}}(U_X)$ (i.e., a cofinal system of open normal subgroups).

Let $\widetilde{e} \in D_{\widetilde{X}}$. For each $i \in \mathbb{Z}_{>0}$, we write $(X_{H_i}, D_{X_{H_i}})$ for the smooth pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ induced by H_i and $e_{X_{H_i}} \in D_{X_{H_i}}$ for the image of $\widetilde{e}$. Then we obtain a sequence of marked points

$$\mathcal{I}_{\widetilde{e}}^{C_X} : \cdots \mapsto e_{X_{H_2}} \mapsto e_{X_{H_1}}$$

induced by C_X. Note that the sequence $\mathcal{I}_{\widetilde{e}}^{C_X}$ admits a natural action of $\pi_1^{\mathrm{t}}(U_X)$. We may identify the inertia subgroup $I_{\widetilde{e}}$ associated to $\widetilde{e}$ with the stabilizer of $\mathcal{I}_{\widetilde{e}}^{C_X}$.

Moreover, since Proposition 2.5 (i) implies that $(g_{X_{H_i}}, n_{X_{H_i}})$ can be mono-anabelian reconstructed from H_i, by choosing a suitable set of open normal subgroups C_X, we may assume that $g_{X_{H_1}} \geq 2$. If $n_{X_{H_1}} = 0$, Theorem 4.3 is trivial. Then we may assume that $n_{X_{H_1}} > 0$.

On the other hand, Proposition 4.2 implies that, for each H_i, $i \in \mathbb{Z}_{>0}$, the set $D_{X_{H_i}}^{\mathrm{gp}}$ can be mono-anabelian reconstructed from H_i. For each $e_{X_{H_i}} \in D_{X_{H_i}}$, we denote by

$$e_{X_{H_i}}^{\mathrm{gp}} \stackrel{\mathrm{def}}{=} \vartheta_{X_{H_i}}^{-1}(e_{X_{H_i}}).$$

Then the sequence of marked points $\mathcal{I}_{\widetilde{e}}^{C_X}$ induces a sequence

$$\mathcal{I}_{\widetilde{e}^{\mathrm{gp}}}^{C_X} : \cdots \mapsto e_{X_{H_2}}^{\mathrm{gp}} \mapsto e_{X_{H_1}}^{\mathrm{gp}}.$$

Then Remark 4.2.1 implies that the stabilizer of $\mathcal{I}_{\widetilde{e}^{\mathrm{gp}}}^{C_X}$ is equal to the stabilizer of $\mathcal{I}_{\widetilde{e}}^{C_X}$. This completes the proof of the theorem. □

5 Mono-anabelian reconstructions of inertia subgroups of marked points via surjections

Let (X_i, D_{X_i}), $i \in \{1, 2\}$, be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$, $U_{X_i} \stackrel{\mathrm{def}}{=} X_i \setminus D_{X_i}$, $\pi_1^{\mathrm{t}}(U_{X_i})$ the tame fundamental group of U_{X_i}, and $\pi_1(X_i)$ the étale fundamental group of X_i. Then Lemma 4.1 implies that $\pi_1(X_i)$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})$. Moreover, in this section, we suppose that

- $n_X > 0$;
- $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ is an arbitrary continuous surjective homomorphism of profinite groups.

Note that, since (X_i, D_{X_i}), $i \in \{1, 2\}$, is a smooth pointed stable curve of type (g_X, n_X), ϕ induces a natural surjection $\phi^{p'} : \pi_1^{\mathrm{t}}(U_{X_1})^{p'} \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Moreover, since $\pi_1^{\mathrm{t}}(U_{X_i})^{p'}$, $i \in \{1, 2\}$, is topologically finitely generated, and $\pi_1^{\mathrm{t}}(U_{X_1})^{p'}$ is isomorphic to $\pi_1^{\mathrm{t}}(U_{X_2})^{p'}$ as abstract profinite groups, we obtain that

$$\phi^{p'} : \pi_1^{\mathrm{t}}(U_{X_1})^{p'} \xrightarrow{\sim} \pi_1^{\mathrm{t}}(U_{X_2})^{p'}$$

is an isomorphism (cf. [FJ, Proposition 16.10.6]).

In this section, we will prove that the *mono-anabelian reconstructions* obtained in Section 4 are *compatible with* the open continuous homomorphism $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$. First, we explain the main idea. Let $H_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ be an arbitrary open normal subgroup and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \pi_1^{\mathrm{t}}(U_{X_1})$. We write $(X_{H_i}, D_{X_{H_i}})$, $i \in \{1, 2\}$, for the smooth pointed smooth curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i induced by H_i. In order to prove the compatibility, we need to prove that, for any prime number $\ell \neq p$, the weight-monodromy filtration of $H_2^{\mathrm{ab}} \otimes \mathbb{F}_\ell$ induces the weight-monodromy filtration of $H_1^{\mathrm{ab}} \otimes \mathbb{F}_\ell$ via the natural surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$. Note that the weight 1 part of $H_i^{\mathrm{ab}} \otimes \mathbb{F}_\ell$ corresponds to $\pi_1(X_{H_i})^{\mathrm{ab}} \otimes \mathbb{F}_\ell$, and the weight 2 part of $H_i^{\mathrm{ab}} \otimes \mathbb{F}_\ell$ corresponds to the image of the subgroup of H_i generated by the inertia subgroups of the marked points of $D_{X_{H_i}}$. The key observation is as follows:

> The inequality of the limit of p-averages (cf. Proposition 2.2)
>
> $$\mathrm{Avr}_p(H_1) \geq \mathrm{Avr}_p(H_2)$$
>
> of H_1 and H_2 induced by the surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$ plays a role of the comparability of (outer) Galois actions in the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

Lemma 5.1. *Let ℓ be a prime number distinct from p. Then the isomorphism $(\phi^{p'})^{-1}$: $\pi_1^{\mathrm{t}}(U_{X_2})^{p'} \xrightarrow{\sim} \pi_1^{\mathrm{t}}(U_{X_1})^{p'}$ induces an isomorphism*

$$\psi_X^\ell : H^1_{\mathrm{\acute{e}t}}(X_1, \mathbb{F}_\ell) \cong \mathrm{Hom}(\pi_1(X_1), \mathbb{F}_\ell) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(X_2), \mathbb{F}_\ell) \cong H^1_{\mathrm{\acute{e}t}}(X_2, \mathbb{F}_\ell).$$

Proof. Let

$$f_1 : (Y_1, D_{Y_1}) \to (X_1, D_{X_1})$$

be an étale covering of degree ℓ over k_1. Write

$$f_2 : (Y_2, D_{Y_2}) \to (X_2, D_{X_2})$$

for the connected Galois tame covering of degree ℓ over k_2 induced by $\phi^{p'}$. Then we will prove that f_2 is an étale covering over k_2.

Write g_{Y_1} and g_{Y_2} for the genus of Y_1 and Y_2, respectively. Since f_1 is an étale covering of degree ℓ, the Riemann-Hurwitz formula implies that

$$g_{Y_1} = \ell(g_{X_1} - 1) + 1.$$

On the other hand, the Riemann-Hurwitz formula implies that

$$g_{Y_2} = \ell(g_{X_2} - 1) + 1 + \frac{1}{2}(\ell - 1)\#\mathrm{Ram}_{f_2}.$$

By applying Corollary 2.3, The surjection ϕ implies that

$$g_{Y_1} \geq g_{Y_2}.$$

This means that $\#\mathrm{Ram}_{f_2} = 0$. So f_2 is an étale covering over k_2. Then the morphism $(\phi^{p'})^{-1}$ induces an injection

$$\psi_X^{\ell} : \mathrm{Hom}(\pi_1(X_1), \mathbb{F}_\ell) \hookrightarrow \mathrm{Hom}(\pi_1(X_2), \mathbb{F}_\ell).$$

Furthermore, since $\dim_{\mathbb{F}_\ell}(\mathrm{Hom}(\pi_1(X_1), \mathbb{F}_\ell)) = \dim_{\mathbb{F}_\ell}(\mathrm{Hom}(\pi_1(X_2), \mathbb{F}_\ell)) = 2g_X$, we obtain that ψ_X^{ℓ} is a bijection. This completes the proof of the lemma. $\square$

Lemma 5.2. *Suppose that $g_X \geq 2$. Then the surjection $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ induces a bijection*

$$\rho_\phi : D_{X_1}^{\mathrm{gp}} \xrightarrow{\sim} D_{X_2}^{\mathrm{gp}},$$

and the bijection ρ_ϕ can be mono-anabelian reconstructed from ϕ.

Proof. Let ℓ and d be prime numbers distinct from each other such that $(\ell, p) = (d, p) = 1$. Suppose that $\ell \equiv 1 \pmod{d}$. Then we have that all dth roots of unity are contained in $\mathbb{F}_\ell$. Write $\mu_d \subseteq \mathbb{F}_\ell^\times$ for the subgroup of dth roots of unity. Let $y_2 \in \mathrm{Hom}(\pi_1(X_2), \mu_d)$ such that $y_2 \neq 0$, and let (ℓ, d, y_2) be a triple associated to $\pi_1^{\mathrm{t}}(U_{X_2})$. Then Lemma 5.1 implies that ϕ induces a triple (ℓ, d, y_1) associated to $\pi_1^{\mathrm{t}}(U_{X_1})$, where $y_1 \overset{\mathrm{def}}{=} (\psi_X^d)^{-1}(y_2) \in \mathrm{Hom}(\pi_1(X_1), \mu_d)$.

Let $f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i})$, $i \in \{1, 2\}$, be the étale covering of degree d over k_i induced by y_i. Then the triple (ℓ, d, y_i) associated to $\pi_1^{\mathrm{t}}(U_{X_i})$ induces a triple

$$(\ell, d, f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i}))$$

associated to (X_i, D_{X_i}) over k_i. Note that (Y_1, D_{Y_1}) and (Y_2, D_{Y_2}) are same types.

Write $\pi_1^{\mathrm{t}}(U_{Y_i})$, $i \in \{1, 2\}$, for the kernel of $\pi_1^{\mathrm{t}}(U_{X_i}) \twoheadrightarrow \pi_1(X_i) \overset{y_i}{\twoheadrightarrow} \mu_d$. By replacing (X_i, D_{X_i}) by (Y_i, D_{Y_i}), Lemma 5.1 implies that $(\phi|_{\pi_1^{\mathrm{t}}(U_{Y_1})}^{p'})^{-1}$ induces a commutative diagram as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\text{ét}}(Y_1, \mathbb{F}_\ell) & \longrightarrow & H^1_{\text{ét}}(U_{Y_1}, \mathbb{F}_\ell) & \longrightarrow & \mathrm{Div}^0_{D_{Y_1}}(Y_1) \otimes \mathbb{F}_\ell & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \psi_Y^{\ell}} & & \Big\downarrow{\scriptstyle \psi_Y^{\mathrm{t},\ell}} & & \Big\downarrow & & \\ 0 & \longrightarrow & H^1_{\text{ét}}(Y_2, \mathbb{F}_\ell) & \longrightarrow & H^1_{\text{ét}}(U_{Y_2}, \mathbb{F}_\ell) & \longrightarrow & \mathrm{Div}^0_{D_{Y_2}}(Y_2) \otimes \mathbb{F}_\ell & \longrightarrow & 0, \end{array}$$

where all the vertical arrows are isomorphisms. We note that $H^1_{\text{ét}}(Y_i, \mathbb{F}_\ell)$, $H^1_{\text{ét}}(U_{Y_i}, \mathbb{F}_\ell)$, and $\text{Div}^0_{D_{Y_i}}(Y_i) \otimes \mathbb{F}_\ell$, $i, \in \{1, 2\}$, are naturally isomorphic to $\text{Hom}(\pi_1(Y_i), \mathbb{F}_\ell)$, $\text{Hom}(\pi_1^{\text{t}}(U_{Y_i}), \mathbb{F}_\ell)$, and

$$\text{Hom}(\pi_1^{\text{t}}(U_{Y_i}), \mathbb{F}_\ell)/\text{Hom}(\pi_1(Y_i), \mathbb{F}_\ell)$$

(which is induced by the natural surjection $\pi_1^{\text{t}}(U_{Y_i}) \twoheadrightarrow \pi_1(Y_i)$), respectively. Then Lemma 4.1 implies that the commutative diagram above can be mono-anabelian reconstructed from $\phi|_{\pi_1^{\text{t}}(U_{Y_1})} : \pi_1^{\text{t}}(U_{Y_1}) \twoheadrightarrow \pi_1^{\text{t}}(U_{Y_2})$.

Write $M_{Y_i} \subseteq M^*_{Y_i}$ for the subsets of $H^1_{\text{ét}}(U_{Y_i}, \mathbb{F}_\ell)$ defined in Section 3. Since the actions of μ_d on the exact sequences are compatible with the isomorphisms appearing in the commutative diagram above, we have

$$\psi_Y^{\text{t},\ell}(M^*_{Y_1}) = M^*_{Y_2}.$$

Let $\alpha_1 \in M_{Y_1}$ and

$$g_{\alpha_1} : (Y_{\alpha_1}, D_{Y_{\alpha_1}}) \to (Y_1, D_{Y_1})$$

the Galois tame covering of degree ℓ over k_1 induced by α_1. Write

$$g_{\alpha_2} : (Y_{\alpha_2}, D_{Y_{\alpha_2}}) \to (Y_2, D_{Y_2})$$

for the Galois tame covering of degree ℓ over k_2 induced by $\alpha_2 \stackrel{\text{def}}{=} \psi_Y^{\text{t},\ell}(\alpha_1)$. Write $g_{Y_{\alpha_1}}$ and $g_{Y_{\alpha_2}}$ for the genus of Y_{α_1} and Y_{α_2}, respectively. Then Corollary 2.3 and the Riemann-Hurwitz formula imply that

$$g_{Y_{\alpha_1}} - g_{Y_{\alpha_2}} = \frac{1}{2}(d - \#\text{Ram}_{g_{\alpha_2}})(\ell - 1) \geq 0.$$

This means that

$$d - \#\text{Ram}_{g_{\alpha_2}} \geq 0.$$

Since $\alpha_2 \in M^*_{Y_2}$, we have $d \mid \#\text{Ram}_{g_{\alpha_2}}$. Thus, either $\#\text{Ram}_{g_{\alpha_2}} = 0$ or $\#\text{Ram}_{g_{\alpha_2}} = d$ holds.

If $\#\text{Ram}_{g_{\alpha_2}} = 0$, then g_{α_2} is an étale covering over k_2. Then Lemma 5.1 implies that g_{α_1} is an étale covering over k_1. This provides a contradiction to the fact that $\alpha_1 \in M_{Y_1}$. Then we have $\#\text{Ram}_{g_{\alpha_2}} = d$. This means that $\alpha_2 \in M_{Y_2}$. Thus, we obtain

$$\psi_Y^{\text{t},\ell}(M_{Y_1}) \subseteq M_{Y_2}.$$

On the other hand, Lemma 3.2 implies that $\#M_{Y_1} = \#M_{Y_2}$. We have

$$\psi_Y^{\text{t},\ell} : M_{Y_1} \xrightarrow{\sim} M_{Y_2}.$$

Then Proposition 3.1 implies that $\psi_Y^{\text{t},\ell}$ induces a bijection

$$\rho_\phi : D^{\text{gp}}_{X_1} \xrightarrow{\sim} D^{\text{gp}}_{X_2}.$$

Moreover, since M_{Y_i} and $M^*_{Y_i}$ can be mono-anabelian reconstructed from $\pi_1^{\text{t}}(U_{Y_i})$, the bijection ρ_ϕ can be mono-anabelian reconstructed from ϕ. This completes the proof of the lemma. □

Let $H_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ be an arbitrary open normal subgroup and $H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2)$. We write $(X_{H_i}, D_{X_{H_i}})$, $i \in \{1,2\}$, for the smooth pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i induced by H_i and

$$f_{H_i} : (X_{H_i}, D_{X_{H_i}}) \to (X_i, D_{X_i})$$

for the Galois tame coverings over k_i induced by the inclusion $H_i \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i})$. Moreover, Proposition 4.2 implies that the inclusion $H_i \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i})$ induces a map $\gamma_{H_i,\pi_1^{\mathrm{t}}(U_{X_i})} : D^{\mathrm{gp}}_{X_{H_i}} \to D^{\mathrm{gp}}_{X_i}$ such that the following commutative diagram holds:

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{H_i}} & \xrightarrow{\vartheta_{X_{H_i}}} & D_{X_{H_i}} \\ {\scriptstyle \gamma_{H_i,\pi_1^{\mathrm{t}}(U_{X_i})}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma_{f_{H_i}}} \\ D^{\mathrm{gp}}_{X_i} & \xrightarrow{\vartheta_{X_i}} & D_{X_i}, \end{array}$$

where $\gamma_{f_{H_i}}$ denotes the map of the sets of marked points induced by f_{H_i}. We may identify $\pi_1^{\mathrm{t}}(U_{X_1})/H_1$ with $\pi_1^{\mathrm{t}}(U_{X_2})/H_2$ via the isomorphism $\pi_1^{\mathrm{t}}(U_{X_1})/H_1 \xrightarrow{\sim} \pi_1^{\mathrm{t}}(U_{X_2})/H_2$ induced by ϕ, and denote by $G \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_1})/H_1 \cong \pi_1^{\mathrm{t}}(U_{X_2})/H_2$. Then we have the following lemma.

Lemma 5.3. *Suppose that $g_X \geq 2$, and that $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$. Then the commutative diagram of profinite groups*

(1)
$$\begin{array}{ccc} H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \big\downarrow & & \big\downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}) \end{array}$$

induces a commutative diagram

(2)
$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D^{\mathrm{gp}}_{X_{H_2}} \\ {\scriptstyle \gamma_{H_1,\pi_1^{\mathrm{t}}(U_{X_1})}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma_{H_2,\pi_1^{\mathrm{t}}(U_{X_2})}} \\ D^{\mathrm{gp}}_{X_1} & \xrightarrow{\rho_{\phi}} & D^{\mathrm{gp}}_{X_2}. \end{array}$$

Moreover, the commutative diagram (2) *can be mono-anabelian reconstructed from* (1).

Proof. Proposition 4.2 and Lemma 5.2 implies the diagram

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D^{\mathrm{gp}}_{X_{H_2}} \\ {\scriptstyle \gamma_{H_1,\pi_1^{\mathrm{t}}(U_{X_1})}} \big\downarrow & & \big\downarrow {\scriptstyle \gamma_{H_2,\pi_1^{\mathrm{t}}(U_{X_2})}} \\ D^{\mathrm{gp}}_{X_1} & \xrightarrow{\rho_{\phi}} & D^{\mathrm{gp}}_{X_2} \end{array}$$

can be mono-anabelian reconstructed from the commutative diagram of profinite groups

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \big\downarrow & & \big\downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}). \end{array}$$

Then to verify Lemma 5.3, we only need to check that the diagram is commutative.

Let $e^{\rm gp}_{X_{H_1}} \in D^{\rm gp}_{X_{H_1}}$, $e^{\rm gp}_{X_{H_2}} \stackrel{\rm def}{=} \rho_{\phi|_{H_1}}(e^{\rm gp}_{X_{H_1}}) \in D^{\rm gp}_{X_{H_2}}$, $e^{\rm gp}_{X_1} \stackrel{\rm def}{=} \gamma_{H_1,\pi_1^{\rm t}(U_{X_1})}(e^{\rm gp}_{X_{H_1}}) \in D^{\rm gp}_{X_1}$, $e^{\rm gp}_{X_2} \stackrel{\rm def}{=} (\gamma_{H_2,\pi_1^{\rm t}(U_{X_2})} \circ \rho_{\phi|_{H_1}})(e^{\rm gp}_{X_{H_1}}) \in D^{\rm gp}_{X_2}$, and $e^{\rm gp,*}_{X_1} \stackrel{\rm def}{=} \rho_\phi^{-1}(e^{\rm gp}_{X_2}) \in D^{\rm gp}_{X_1}$. Let us prove that

$$e^{\rm gp}_{X_1} = e^{\rm gp,*}_{X_1}.$$

We put $S^{\rm gp}_{X_{H_1}} \stackrel{\rm def}{=} \gamma^{-1}_{H_1,\pi_1^{\rm t}(U_{X_1})}(e^{\rm gp,*}_{X_1})$ and $S^{\rm gp}_{X_{H_2}} \stackrel{\rm def}{=} \gamma^{-1}_{H_2,\pi_1^{\rm t}(U_{X_2})}(e^{\rm gp}_{X_2})$, respectively. Note that $e^{\rm gp}_{X_{H_2}} \in S^{\rm gp}_{X_{H_2}}$. To verify $e^{\rm gp}_{X_1} = e^{\rm gp,*}_{X_1}$, it is sufficient to prove that $e^{\rm gp}_{X_{H_1}} \in S^{\rm gp}_{X_{H_1}}$. Moreover, for each $i \in \{1, 2\}$, we put

$$e_{X_i} \stackrel{\rm def}{=} \vartheta_{X_i}(e^{\rm gp}_{X_i}),\ e_{X_{H_i}} \stackrel{\rm def}{=} \vartheta_{X_{H_i}}(e^{\rm gp}_{X_i}),\ e^*_{X_1} \stackrel{\rm def}{=} \vartheta_{X_1}(e^{\rm gp,*}_{X_1})$$

and

$$S_{X_i} \stackrel{\rm def}{=} S^{\rm gp}_{X_i},\ S_{X_{H_i}} \stackrel{\rm def}{=} S^{\rm gp}_{X_{H_i}}.$$

Then to verify the lemma, we only need to prove that $e_{X_{H_1}} \in \vartheta_{X_{H_1}}(S_{X_{H_1}})$.

Let ℓ and d be prime numbers distinct from each other such that $(\ell, p) = (d, p) = 1$. Suppose that $\ell \equiv 1 \pmod d$. Then we have that all dth roots of unity are contained in $\mathbb{F}_\ell$. Moreover, we may choose that $(\ell, \#G) = 1$ and $(d, \#G) = 1$. Let $y_2 \in \mathrm{Hom}(\pi_1(X_2), \mu_d)$ such that $y_2 \neq 0$, and let (ℓ, d, y_2) be a triple associated to $\pi_1^{\rm t}(U_{X_2})$. Then Lemma 5.1 implies that ϕ induces a triple (ℓ, d, y_1) associated to $\pi_1^{\rm t}(U_{X_1})$, where $y_1 \stackrel{\rm def}{=} (\psi^d_X)^{-1}(y_2) \in \mathrm{Hom}(\pi_1(X_1), \mu_d)$.

Let $f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i})$, $i \in \{1, 2\}$, be the tame covering of degree d over k_i induced by y_i. Then the triple (ℓ, d, y_i) associated to $\pi_1^{\rm t}(U_{X_i})$ induces a triple

$$(\ell, d, f_i : (Y_i, D_{Y_i}) \to (X_i, D_{X_i}))$$

associated to (X_i, D_{X_i}) over k_i. Note that since f_1 and f_2 are étale, (Y_1, D_{Y_1}) and (Y_2, D_{Y_2}) are same types. On the other hand, we have a triple

$$(\ell, d, g_2 : (Z_2, D_{Z_2}) \stackrel{\rm def}{=} (Y_2, D_{Y_2}) \times_{(X_2, D_{X_2})} (X_{H_2}, D_{X_{H_2}}) \to (X_{H_2}, D_{X_{H_2}}))$$

associated to $(X_{H_2}, D_{X_{H_2}})$ induced by the natural inclusion $H_2 \hookrightarrow \pi_1^{\rm t}(U_{X_2})$ and the triple $(\ell, d, f_2 : (Y_2, D_{Y_2}) \to (X_2, D_{X_2}))$. By Lemma 5.1 again, we obtain a triple

$$(\ell, d, g_1 : (Z_1, D_{Z_1}) \stackrel{\rm def}{=} (Y_1, D_{Y_1}) \times_{(X_1, D_{X_1})} (X_{H_1}, D_{X_{H_1}}) \to (X_{H_1}, D_{X_{H_1}}))$$

associated to $(X_{H_1}, D_{X_{H_1}})$ induced by $\phi|_{H_1}$ and the triple $(\ell, d, g_2 : (Z_2, D_{Z_2}) \to (X_{H_2}, D_{X_{H_2}}))$.

Let $\alpha_2 \in M_{Y_2, e_{X_2}}$. The final paragraph of the proof of Lemma 5.2 implies that we have a bijection $M_{Y_1} = \bigsqcup_{e \in D_{X_1}} M_{Y_1,e} \xrightarrow{\sim} M_{Y_2} = \bigsqcup_{e \in D_{X_2}} M_{Y_2,e}$ induced by ϕ. Then α_2 induces an element

$$\alpha_1 \in M_{Y_1, e^*_{X_1}}.$$

Write $(Y_{\alpha_1}, D_{Y_{\alpha_1}})$ and $(Y_{\alpha_2}, D_{Y_{\alpha_2}})$ for the smooth pointed stable curves over k_1 and k_2 induced by α_1 and α_2, respectively. We consider the connected Galois tame covering

$$(Y_{\alpha_2}, D_{Y_{\alpha_2}}) \times_{(X_2, D_{X_2})} (X_{H_2}, D_{X_{H_2}}) \to (Z_2, D_{Z_2})$$

of degree ℓ over k_2, and write β_2 for an element of $M_{Z_2}^*$ corresponding to this connected Galois tame covering. Then we have

$$\beta_2 = \sum_{c_2 \in S_{X_{H_2}}} t_{c_2}\beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in M_{Z_2,c_2}$. On the other hand, the proof of Lemma 5.2 implies that β_2 induces an element

$$\begin{aligned}\beta_1 &\stackrel{\text{def}}{=} \sum_{c_2 \in S_{X_{H_2}} \setminus \{e_{X_{H_2}}\}} t_{c_2}\beta_{\rho^{-1}_{\phi|_{H_1}}(c_2)} + t_{e_{X_{H_2}}}\beta_{\rho^{-1}_{\phi|_{H_1}}(e_{X_{H_2}})} \\ &= \sum_{c_2 \in S_{X_{H_2}} \setminus \{e_{X_{H_2}}\}} t_{c_2}\beta_{\rho^{-1}_{\phi|_{H_1}}(c_2)} + t_{e_{X_{H_2}}}\beta_{e_{X_{H_1}}} \in M_{Z_1}^*.\end{aligned}$$

Then we have that the coefficient $t_{e_{X_{H_2}}}$ of $\beta_{e_{X_{H_1}}}$ is not equal to 0. Thus, the composition

$$(Y_{\alpha_1}, D_{Y_{\alpha_1}}) \times_{(X_1, D_{X_1})} (X_{H_1}, D_{X_{H_1}}) \to (Z_1, D_{Z_1}) \xrightarrow{g_1} (X_{H_1}, D_{X_{H_1}})$$

is tamely ramified over $e_{X_{H_1}}$. This means that $e_{X_{H_1}}$ is contained in $S_{X_{H_1}}$. We complete the proof of the lemma. □

Remark 5.3.1. Remark 4.2.1 implies that $D^{\text{gp}}_{X_{H_i}}$, $i \in \{1, 2\}$, admits a natural action of G. Moreover, the commutative diagram

$$\begin{array}{ccc} D^{\text{gp}}_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D^{\text{gp}}_{X_{H_2}} \\ \gamma_{H_1, \pi_1^{\text{t}}(U_{X_1})}\downarrow & & \downarrow\gamma_{H_2, \pi_1^{\text{t}}(U_{X_2})} \\ D^{\text{gp}}_{X_1} & \xrightarrow{\rho_\phi} & D^{\text{gp}}_{X_2} \end{array}$$

is compatible with the actions of G.

Lemma 5.4. *We maintain the notation introduced before Lemma 5.3. Suppose that $g_X \geq 2$, and that G is an abelian group. Then we have*

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

Proof. We write m for $\#G$ and put

$$K_2 \stackrel{\text{def}}{=} \ker(\pi_1^{\text{t}}(U_{X_2}) \twoheadrightarrow \pi_1^{\text{t}}(U_{X_2})^{\text{ab}} \otimes \mathbb{Z}/m\mathbb{Z}).$$

Then we see immediately that K_2 is contained in H_2. Let $K_1 \stackrel{\text{def}}{=} \phi^{-1}(K_2) \subseteq H_1$. Write $(X_{K_i}, D_{X_{K_i}})$ for the smooth pointed stable curves of type $(g_{X_{K_i}}, n_{X_{K_i}})$ over k_i induced by K_i and $f_{K_i} : (X_{K_i}, D_{X_{K_i}}) \to (X_i, D_{X_i})$ for the tame covering over k_i induced by the inclusion $K_i \hookrightarrow \pi_1^{\text{t}}(U_{X_i})$. We identify $\pi_1^{\text{t}}(U_{X_1})/K_1$ with $\pi_1^{\text{t}}(U_{X_2})/K_2$ via the isomorphism induced by ϕ, and denote by $A \stackrel{\text{def}}{=} \pi_1^{\text{t}}(U_{X_1})/K_1 \cong \pi_1^{\text{t}}(U_{X_2})/K_2$.

Since each p-Galois tame covering is étale (i.e., Galois tame coverings whose Galois group is a p-group), we see immediately that

$$(g_{X_{K_1}}, n_{X_{K_1}}) = (g_{X_{K_2}}, n_{X_{K_2}}).$$

Then Lemma 5.3 implies that the commutative diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\phi|_{K_1}} & K_2 \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}) \end{array}$$

of profinite groups induces a commutative diagram

$$\begin{array}{ccc} D^{\mathrm{gp}}_{X_{K_1}} & \xrightarrow{\rho_{\phi|_{K_1}}} & D^{\mathrm{gp}}_{X_{K_2}} \\ {\scriptstyle \gamma_{K_1, \pi_1^{\mathrm{t}}(U_{X_1})}} \downarrow & & {\scriptstyle \gamma_{K_2, \pi_1^{\mathrm{t}}(U_{X_2})}} \downarrow \\ D^{\mathrm{gp}}_{X_1} & \xrightarrow{\rho_{\phi}} & D^{\mathrm{gp}}_{X_2}. \end{array}$$

Moreover, Remark 5.3.1 implies that the commutative diagram above admits a natural action of A. Then, for each $e^{\mathrm{gp}}_{X_{K_1}} \in D^{\mathrm{gp}}_{X_{K_1}}$, the inertia subgroup in A associated to $e^{\mathrm{gp}}_{X_{K_1}}$ (i.e., the stabilizer of $e^{\mathrm{gp}}_{X_{K_1}}$ under the action of A) is equal to the inertia subgroup in A associated to $\rho_{\phi|_{K_1}}(e^{\mathrm{gp}}_{X_{K_1}}) \in D^{\mathrm{gp}}_{X_{K_2}}$. On the other hand, write F for the kernel of the natural morphism $A \twoheadrightarrow G$ induced by the inclusion $K_i \hookrightarrow H_i$, $i \in \{1, 2\}$. Since $(X_{H_i}, D_{X_{H_i}}) \cong (X_{K_i}, D_{X_{K_i}})/F$, $i \in \{1, 2\}$, the Riemann-Hurwitz formula implies that

$$(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}}).$$

This completes the proof of the lemma. □

Lemma 5.5. *We maintain the notation introduced before Lemma 5.4. Suppose that $g_X \geq 2$ and $n_X \geq 2$. Then there exists an open normal subgroup $P_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ which is contained in H_2 such that the following holds:*

> *Write $(X_{P_i}, D_{X_{P_i}})$, $i \in \{1, 2\}$, for the smooth pointed stable curve of type $(g_{X_{P_i}}, n_{X_{P_i}})$ over k_i induced by P_i, where $P_1 = \phi^{-1}(P_2)$. We have*
>
> $$(g_{X_{P_1}}, n_{X_{P_1}}) = (g_{X_{P_2}}, n_{X_{P_2}}).$$

Proof. First, suppose that G is a *simple* finite group. By applying Lemma 5.4, we may assume that G is *non-abelian*. Moreover, we claim that we may assume that n_X is a positive *even* number. Let us prove this claim. Suppose that $p \neq 2$. Let $R_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ be an open subgroup such that $\#(\pi_1^{\mathrm{t}}(U_{X_2})/R_2) = 2$, and that $R_2 \supseteq \ker(\pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1(X_2))$ (i.e., the cyclic Galois tame covering corresponding to R_2 is étale). Let $R_1 \overset{\mathrm{def}}{=} \phi^{-1}(R_2) \subseteq \pi_1^{\mathrm{t}}(U_{X_1})$. Then we have that $\#(\pi_1^{\mathrm{t}}(U_{X_1})/R_1) = 2$, and that Lemma 5.1 implies that $R_1 \supseteq \ker(\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1(X_1))$. By replacing H_i and $\pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$, by $H_i \cap R_i$

and R_i, respectively, we may assume that n_X is a positive even number. Suppose that $p = 2$. Let ℓ be a prime number such that $(\ell, 2) = (\ell, \#G) = 1$. By [R1, Théorème 4.3.1], there exists an open subgroup $R_2^* \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ such that $\#(\pi_1^{\mathrm{t}}(U_{X_2})/R_2^*) = \ell$, that $R_2^* \supseteq \ker(\pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1(X_2))$, and that

$$\dim_{\mathbb{F}_p}(R_2^{*,\mathrm{ab}} \otimes \mathbb{F}_p) > 0.$$

Let $R_1^* \stackrel{\mathrm{def}}{=} \phi^{-1}(R_2^*) \subseteq \pi_1^{\mathrm{t}}(U_{X_1})$. Then we have that $\#(\pi_1^{\mathrm{t}}(U_{X_1})/R_1^*) = \ell$, that $\dim_{\mathbb{F}_p}(R_1^{*,\mathrm{ab}} \otimes \mathbb{F}_p) > 0$, and that Lemma 5.1 implies that $R_1^* \supseteq \ker(\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1(X_1))$. Thus, we may take an open subgroup $R_2' \subseteq R_2^*$ such that

$$\pi_1^{\mathrm{t}}(U_{X_2})/R_2' \cong \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/\ell\mathbb{Z},$$

and that $R_2' \supseteq \ker(\pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1(X_2))$. We put $R_1' \stackrel{\mathrm{def}}{=} \phi^{-1}(R_2')$. Then the construction of R_1' implies that $\pi_1^{\mathrm{t}}(U_{X_1})/R_1' \cong \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/\ell\mathbb{Z}$, and that $R_1' \supseteq \ker(\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1(X_1))$. By replacing H_i and $\pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$, by $H_i \cap R_i'$ and R_i', respectively, we may assume that n_X is a positive even number. This completes the proof of the claim.

Write m for $\#G$. Since n_X is a positive *even* number, we may choose a Galois tame covering

$$f_2 : (Y_2, D_{Y_2}) \to (X_2, D_{X_2})$$

over k_2 with Galois group $\mathbb{Z}/m\mathbb{Z}$ such that f_2 is totally ramified over every marked point of D_{X_2}. Write (g_{Y_2}, n_{Y_2}) for the type of (Y_2, D_{Y_2}), $Q_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ for the open normal subgroup induced by f_2, $Q_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(Q_2) \subseteq \pi_1^{\mathrm{t}}(U_{X_1})$,

$$f_1 : (Y_1, D_{Y_1}) \to (X_1, D_{X_1})$$

for the Galois tame covering over k_1 with Galois group $\mathbb{Z}/m\mathbb{Z}$ induced by the natural inclusion $Q_1 \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_1})$, and (g_{Y_1}, n_{Y_1}) for the type of (Y_1, D_{Y_1}). Then Lemma 5.4 implies that

$$(g_{Y_1}, n_{Y_1}) = (g_{Y_2}, n_{Y_2})$$

and f_1 is also totally ramified over every marked point of D_{X_1}.

We consider the Galois tame covering

$$(Z_i, D_{Z_i}) \stackrel{\mathrm{def}}{=} (X_{H_i}, D_{X_{H_i}}) \times_{(X_i, D_{X_i})} (Y_i, D_{Y_i}) \to (X_i, D_{X_i}),\ i \in \{1, 2\},$$

over k_i with Galois group $G \times \mathbb{Z}/m\mathbb{Z}$ which is the composition of $(Z_i, D_{Z_i}) \to (Y_i, D_{Y_i})$ and $(Y_i, D_{Y_i}) \to (X_i, D_{X_i})$. Note that since G is a non-abelian simple finite group, (Z_i, D_{Z_i}) is connected. Moreover, by Abhyankar's lemma, we obtain that $(Z_i, D_{Z_i}) \to (Y_i, D_{Y_i})$ is an étale covering over k_i. Then the Riemann-Hurwitz formula implies that

$$(g_{Z_1}, n_{Z_1}) = (g_{Z_2}, n_{Z_2}).$$

Next, let us prove the lemma in the case where G is an arbitrary finite group. Let $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \stackrel{\mathrm{def}}{=} G$ be a sequence of subgroups of G such that G_i/G_{i-1} is a simple group for all $i \in \{2, \ldots n\}$. In order to verify the lemma, we see that it is sufficient to to prove the lemma when $n = 2$. Let N_2 be the kernel of the natural homomorphism

$$\pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow G \twoheadrightarrow G_1$$

and $N_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(N_2)$. Then by replacing G by G_1 and by applying the lemma for the simple group G_1, we obtain an open normal subgroup $M_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ which is contained in N_2 such that

$$(g_{X_{M_1}}, n_{X_{M_1}}) = (g_{X_{M_2}}, n_{X_{M_2}}),$$

where $M_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(M_2)$, and $(g_{X_{M_i}}, n_{X_{M_i}})$, $i \in \{1, 2\}$, denotes the type of the smooth pointed stable curve corresponding to M_i.

If $M_i \subseteq H_i$, $i \in \{1, 2\}$, then we may put $P_i \stackrel{\mathrm{def}}{=} M_i$. If H_i, $i \in \{1, 2\}$, does not contain M_i, we put $O_i \stackrel{\mathrm{def}}{=} M_i \cap H_i$. Then we have

$$M_i/O_i \cong G/G_1.$$

Note that G/G_1 is a simple group. Then the lemma follows from the lemma when we replace (X_i, D_{X_i}) and G by $(X_{M_i}, D_{X_{M_i}})$ and the simple group G/G_1. This completes the proof of the lemma. □

Next, we prove the main theorem of the present section.

Theorem 5.6. *Let $(\widetilde{X}_i, D_{\widetilde{X}_i})$, $i \in \{1, 2\}$, be the universal tame covering of (X_i, D_{X_i}) defined in Section 2. Then we have the following:*

(i) $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_{X_i}))$ *can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})$.*

(ii) The group-theoretical algorithm of the mono-anabelian reconstruction concerning $\mathrm{Ine}(\pi_1^{\mathrm{t}}(U_{X_i}))$, *$i \in \{1, 2\}$, is compatible with the surjection $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \to \pi_1^{\mathrm{t}}(U_{X_2})$. Moreover, let $\widetilde{e}_2 \in D_{\widetilde{X}_2}$ and $I_{\widetilde{e}_2} \in \mathrm{Ine}(\pi_1^{\mathrm{t}}(U_{X_2}))$ the inertia subgroup associated to $\widetilde{e}_2$. Then there exists an inertia subgroup $I_{\widetilde{e}_1} \in \mathrm{Ine}(\pi_1^{\mathrm{t}}(U_{X_1}))$ associated to a point $\widetilde{e}_1 \in D_{\widetilde{X}_1}$ such that*

$$\phi(I_{\widetilde{e}_1}) = I_{\widetilde{e}_2},$$

and the restriction homomorphism $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \twoheadrightarrow I_{\widetilde{e}_2}$ is an isomorphism.

Proof. (i) follows from Theorem 4.3. Let us prove (ii). Let $m >> 0$ be an integer number such that $(m, p) = 1$. We put $K_i \stackrel{\mathrm{def}}{=} \ker(\pi_1^{\mathrm{t}}(U_{X_i}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{ab}} \otimes \mathbb{Z}/m\mathbb{Z})$, $i \in \{1, 2\}$. Write (X_{K_i}, D_{K_i}) for the smooth pointed stable curve of type $(g_{X_{K_i}}, n_{X_{K_i}})$ over k_i induced by K_i. Moreover, $m >> 0$ implies that

$$g_{X_{K_1}} = g_{X_{K_2}} \geq 2, \; n_{X_{K_1}} = n_{X_{K_2}} \geq 2.$$

By applying Lemma 5.5, we may choose a set of open subgroups $C_{X_2} \stackrel{\mathrm{def}}{=} \{H_{2,j}\}_{j \in \mathbb{Z}_{>0}}$ of $\pi_1^{\mathrm{t}}(U_{X_2})$ such that the following conditions hold:

(a) $H_{2,1} = K_2$;

(b) $\varprojlim_j \pi_1^{\mathrm{t}}(U_{X_2})/H_{2,j} \cong \pi_1^{\mathrm{t}}(U_{X_2})$ (i.e., C_{X_2} is a cofinal system);

(c) write $\{H_{1,j} \overset{\text{def}}{=} \phi^{-1}(H_{2,j})\}_{j\in\mathbb{Z}_{>0}}$ for the set of open subgroups of $\pi_1^{\mathrm{t}}(U_{X_1})$ induced by ϕ, and, for each $j \in \mathbb{Z}_{>0}$, write $(X_{H_{i,j}}, D_{X_{H_{i,j}}})$, $i \in \{1, 2\}$, for the smooth pointed stable curve of type $(g_{X_{H_{i,j}}}, n_{X_{H_{i,j}}})$ over k_i induced by $H_{i,j}$; then we have $(g_{X_{H_{1,j}}}, n_{X_{H_{1,j}}}) = (g_{X_{H_{2,j}}}, n_{X_{H_{2,j}}})$.

For each $j \in \mathbb{Z}_{>0}$, we write $e_{X_{H_{2,j}}} \in D_{X_{H_{2,j}}}$ for the image of $\widetilde{e}_2$. Then we obtain a sequence of marked points

$$\mathcal{I}_{\widetilde{e}_2}^{C_{X_2}} : \cdots \mapsto e_{H_{2,2}} \mapsto e_{H_{2,1}}.$$

Proposition 4.2 implies that, for each $H_{2,j}$, $j \in \mathbb{Z}_{>0}$, the set $D^{\mathrm{gp}}_{X_{H_{2,j}}}$ can be mono-anabelian reconstructed from $H_{2,j}$. For each $e_{X_{H_{2,j}}} \in D_{X_{H_{2,j}}}$, we denote by

$$e^{\mathrm{gp}}_{X_{H_{2,j}}} \overset{\text{def}}{=} \vartheta^{-1}_{X_{H_{2,j}}}(e_{X_{H_{2,j}}}).$$

Then the sequence of marked points $\mathcal{I}_{\widetilde{e}_2}^{C_X}$ induces a sequence

$$\mathcal{I}_{\widetilde{e}_2^{\mathrm{gp}}}^{C_X} : \cdots \mapsto e^{\mathrm{gp}}_{X_{H_{2,2}}} \mapsto e^{\mathrm{gp}}_{X_{H_{2,1}}}.$$

Then Remark 4.2.1 implies that the inertia subgroup associated to $\widetilde{e}_2$ is equal to the stabilizer of $\mathcal{I}_{\widetilde{e}_2^{\mathrm{gp}}}^{C_X}$.

By Lemma 5.3 and Lemma 5.5, $\mathcal{I}_{\widetilde{e}_2^{\mathrm{gp}}}^{C_{X_2}}$ induces a sequence as follows:

$$\cdots \mapsto e^{\mathrm{gp}}_{X_{H_{1,2}}} \overset{\text{def}}{=} \rho^{-1}_{\phi|_{H_{1,2}}}(e^{\mathrm{gp}}_{X_{H_{2,2}}}) \in D^{\mathrm{gp}}_{X_{H_{1,2}}} \mapsto e^{\mathrm{gp}}_{X_{H_{1,1}}} \overset{\text{def}}{=} \rho^{-1}_{\phi|_{H_{1,1}}}(e^{\mathrm{gp}}_{X_{H_{2,1}}}) \in D^{\mathrm{gp}}_{X_{H_{1,1}}}$$

with an action of $I_{\widetilde{e}_2}$. Then Theorem 4.3 implies that we have a sequence

$$\cdots \mapsto e_{X_{H_{1,2}}} \overset{\text{def}}{=} \vartheta_{X_{H_{1,2}}}(e^{\mathrm{gp}}_{X_{H_{1,2}}}) \in D_{X_{H_{1,2}}} \mapsto e_{X_{H_{1,1}}} \overset{\text{def}}{=} \vartheta_{X_{H_{1,1}}}(e^{\mathrm{gp}}_{X_{H_{1,1}}}) \in D_{X_{H_{1,1}}}$$

with an action of $I_{\widetilde{e}_2}$

Let $K_{\ker(\phi)}$ be the subfield of $\widetilde{K}$ induced by the closed subgroup $\ker(\phi)$ of $\pi_1^{\mathrm{t}}(U_{X_1})$. We put

$$(\widetilde{X}_{1,\ker(\phi)}, D_{\widetilde{X}_{1,\ker(\phi)}}),$$

where $\widetilde{X}_{1,\ker(\phi)}$ denotes the normalization of X_1 in $K_{\ker(\phi)}$, and $D_{\widetilde{X}_{1,\ker(\phi)}}$ denotes the inverse image of D_{X_1} in $\widetilde{X}_{1,\ker(\phi)}$. Then the sequence

$$\cdots \mapsto e_{X_{H_{1,2}}} \mapsto e_{X_{H_{1,1}}}.$$

determines a point $\widetilde{e}_{1,\ker(\phi)} \in D_{\widetilde{X}_{1,\ker(\phi)}}$. We choose a point of $\widetilde{e}_1 \in D_{\widetilde{X}_1}$ such that the image of $\widetilde{e}_1$ in $D_{\widetilde{X}_{1,\ker(\phi)}}$ is $\widetilde{e}_{1,\ker(\phi)}$. Then we have $\phi(I_{\widetilde{e}_1}) = I_{\widetilde{e}_2}$. Moreover, since $I_{\widetilde{e}_1}$ and $I_{\widetilde{e}_2}$ are isomorphic to $\widehat{\mathbb{Z}}(1)^{p'}$, the restriction homomorphism $\phi|_{I_{\widetilde{e}_1}}$ is an isomorphism. This completes the proof of (ii). □

6 Mono-anabelian reconstructions of additive structures

Let (X_i, D_{X_i}), $i \in \{1, 2\}$, be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$, $U_{X_i} \stackrel{\mathrm{def}}{=} X_i \setminus D_{X_i}$, $\pi_1^{\mathrm{t}}(U_{X_i})$ the tame fundamental group of U_{X_i}, and $\pi_1(X_i)$ the étale fundamental group of X_i. In this section, we suppose that

- $n_X > 0$;
- $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ is an arbitrary open continuous surjective homomorphism of profinite groups.

By applying Theorem 5.6, we will reconstruct "additive structures" associated to inertia subgroups group-theoretically from continuous surjective homomorphisms of tame fundamental groups. Let $\widetilde{e}_2$ be an arbitrary point of $D_{\widetilde{X}_2}$ and $\widetilde{e}_1$ a point of $D_{\widetilde{X}_1}$ such that $\phi(I_{\widetilde{e}_1}) = I_{\widetilde{e}_2}$. Write $\overline{\mathbb{F}}_{p,i}$, $i \in \{1, 2\}$, for the algebraic closure of $\mathbb{F}_p$ in k_i. We put

$$\mathbb{F}_{\widetilde{e}_i} \stackrel{\mathrm{def}}{=} (I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}) \bigsqcup \{*_{\widetilde{e}_i}\}, \quad i \in \{1, 2\},$$

where $\{*_{\widetilde{e}_i}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_i^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with $\bigcup_{(p,m)=1} \mu_m(k_i)$. Moreover, let $a_{\widetilde{e}_i}$ be a generator of $I_{\widetilde{e}_i}$. Then we have a natural bijection

$$I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \stackrel{\sim}{\to} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}, \ a_{\widetilde{e}_i} \otimes 1 \mapsto 1 \otimes 1.$$

Thus, we obtain the following bijections

$$I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \stackrel{\sim}{\to} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'} \stackrel{\sim}{\to} \bigcup_{(p,m)=1} \mu_m(k_i) \stackrel{\sim}{\to} \overline{\mathbb{F}}_{p,i}^{\times}.$$

This means that $\mathbb{F}_{\widetilde{e}_i}$ can be identified with $\overline{\mathbb{F}}_{p,i}$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widetilde{e}_i} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_i^{p'}$, and whose zero element is $*_{\widetilde{e}_i}$. We have the following proposition.

Proposition 6.1. *The field structure of* $\mathbb{F}_{\widetilde{e}_i}$, $i \in \{1, 2\}$, *can be mono-anabelian reconstructed from* $\pi_1^{\mathrm{t}}(U_{X_i})$. *Moreover,* ϕ *induces an isomorphism*

$$\theta_{\phi, \widetilde{e}_1, \widetilde{e}_2} : \mathbb{F}_{\widetilde{e}_1} \stackrel{\sim}{\to} \mathbb{F}_{\widetilde{e}_2}$$

as fields, and $\theta_{\phi, \widetilde{e}_1, \widetilde{e}_2}$ *can be mono-anabelian reconstructed from* $\phi|_{I_{\widetilde{e}_1}}$.

Proof. The first part of Proposition 6.1 was proved by Tamagawa (cf. [T4, Proposition 5.3]). We only treat the "Moreover" part of Proposition 6.1.

First, we claim that we may assume that

$$n_X = 3.$$

If $g_X = 0$, then $n_X \geq 3$. Suppose that $g_X \geq 1$. Theorem 5.6 (ii) implies that $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ induces an open continuous surjection

$$\phi^{\text{ét}} : \pi_1(X_1) \twoheadrightarrow \pi_1(X_2).$$

Let $H'_2 \subseteq \pi_1(X_2)$ be an open normal subgroup such that $\#(\pi_1(X_2)/H'_2) \geq 3$ and $H'_1 \stackrel{\text{def}}{=} (\phi^{\text{ét}})^{-1}(H'_2)$. Write $H_i \subseteq \pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$, for the inverse image of H'_i of the natural surjection $\pi_1^{\mathrm{t}}(U_{X_i}) \twoheadrightarrow \pi_1(X_i)$, and $(X_{H_i}, D_{X_{H_i}})$ for the smooth pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i induced by H_i. Note that $g_{X_{H_1}} = g_{X_{H_2}} \geq 2$ and $n_{X_{H_1}} = n_{X_{H_2}} \geq 3$. By replacing (X_i, D_{X_i}) by $(X_{H_i}, D_{X_{H_i}})$, we may assume that $g_X \geq 2$ and $n_X \geq 3$. The surjection ϕ induces a bijection

$$D_{X_1} \overset{\vartheta_{X_1}^{-1}}{\overset{\sim}{\to}} D_{X_1}^{\text{gp}} \overset{\rho_\phi}{\overset{\sim}{\to}} D_{X_2}^{\text{gp}} \overset{\vartheta_{X_2}}{\overset{\sim}{\to}} D_{X_2}.$$

Let $D'_{X_1} \stackrel{\text{def}}{=} \{e_{1,1}, e_{1,2}, e_{1,3}\} \subseteq D_{X_1}$ and $D'_{X_2} \stackrel{\text{def}}{=} \{e_{2,1} \stackrel{\text{def}}{=} \vartheta_{X_2} \circ \rho_\phi \circ \vartheta_{X_1}^{-1}(e_{1,1}), e_{2,2} \stackrel{\text{def}}{=} \vartheta_{X_2} \circ \rho_\phi \circ \vartheta_{X_1}^{-1}(e_{1,2}), e_{2,3} \stackrel{\text{def}}{=} \vartheta_{X_2} \circ \rho_\phi \circ \vartheta_{X_1}^{-1}(e_{1,3})\} \subseteq D_{X_2}$. Then (X_i, D'_{X_i}), $i \in \{1, 2\}$, is a smooth pointed stable curve of type $(g_X, 3)$ over k_i. Write I_i, $i \in \{1, 2\}$, for the closed subgroup of $\pi_1^{\mathrm{t}}(U_{X_i})$ generated by the inertia subgroups associated to the elements of $D_{\widetilde{X}_i}$ whose images in D_{X_i} are contained in $D_{X_i} \setminus D'_{X_i}$. Then we have a natural isomorphism

$$\pi_1^{\mathrm{t}}(X_i \setminus D'_{X_i}) \cong \pi_1^{\mathrm{t}}(U_{X_i})/I_i, \ i \in \{1, 2\}.$$

Moreover, Theorem 5.6 (ii) implies that ϕ induces an open continuous surjective homomorphism

$$\phi' : \pi_1^{\mathrm{t}}(X_1 \setminus D'_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(X_2 \setminus D'_{X_2}).$$

Thus, by replacing (X_i, D_{X_i}), $\pi_1^{\mathrm{t}}(U_{X_i})$, and ϕ by (X_i, D'_{X_i}), $\pi_1^{\mathrm{t}}(X_i \setminus D'_{X_i})$, and ϕ', respectively, we may assume that

$$n_X = 3.$$

Let $r \in \mathbb{N}$. We denote by $\mathbb{F}_{p^r, \widetilde{e}_i}$, $i \in \{1, 2\}$, the unique subfield of $\mathbb{F}_{\widetilde{e}_i}$ whose cardinality is equal to p^r. On the other hand, we fix any finite field $\mathbb{F}_{p^r}$ of cardinality p^r and an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. By Theorem 5.6 (i), we have $\mathbb{F}_{p^r, \widetilde{e}_i}^\times = I_{\widetilde{e}_i}/(p^r - 1)$, $i \in \{1, 2\}$, can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})$. Then reconstructing the field structure of $\mathbb{F}_{p^r, \widetilde{e}_i}$ is equivalent to reconstructing

$$\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r, \widetilde{e}_i}, \mathbb{F}_{p^r})$$

as a subset of $\mathrm{Hom}_{\text{group}}(\mathbb{F}_{p^r, \widetilde{e}_i}^\times, \mathbb{F}_{p^r}^\times)$. Note that, in order to reconstruct the field structure of $\mathbb{F}_{\widetilde{e}_i}$, it is sufficient to reconstruct the subset $\mathrm{Hom}_{\text{fields}}(\mathbb{F}_{p^r, \widetilde{e}_i}, \mathbb{F}_{p^r})$ for r in a cofinal subset of $\mathbb{N}$ with respect to division.

Let

$$\chi_i \in \mathrm{Hom}_{\text{groups}}(\pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}, \mathbb{F}_{p^r}^\times), \ i \in \{1, 2\}.$$

Write H_{χ_i}, $i \in \{1, 2\}$, for the kernel of $\pi_1^{\mathrm{t}}(U_{X_i}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z} \overset{\chi_i}{\to} \mathbb{F}_{p^r}^\times$, M_{χ_i} for $H_{\chi_i}^{\text{ab}} \otimes \mathbb{F}_p$, and $(X_{H_{\chi_i}}, D_{X_{H_{\chi_i}}})$ for the smooth pointed stable curve over k_i induced by H_{χ_i}. We define

$$M_{\chi_i}[\chi_i] \stackrel{\text{def}}{=} \{a \in M_{\chi_i} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma(a) = \chi_i(\sigma)a \text{ for all } \sigma \in \pi_1^{\mathrm{t}}(U_{X_i})^{\text{ab}} \otimes \mathbb{Z}/(p^r - 1)\mathbb{Z}\}$$

and $\gamma_{\chi_i}(M_{\chi_i}) \overset{\mathrm{def}}{=} \dim_{\mathbb{F}_p}(M_{\chi_i}[\chi_i])$. Then [T4, Remark 3.7] implies that $\gamma_{\chi_i}(M_{\chi_i}) \leq g_X + 1$. Moreover, we define two maps

$$\mathrm{Res}_{i,r} : \mathrm{Hom}_{\mathrm{groups}}(\pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}_{p^r}^{\times}) \to \mathrm{Hom}_{\mathrm{groups}}(\mathbb{F}_{p^r,\widetilde{e}_i}^{\times}, \mathbb{F}_{p^r}^{\times})$$

and

$$\Gamma_{i,r} : \mathrm{Hom}_{\mathrm{groups}}(\pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}_{p^r}^{\times}) \to \mathbb{Z}_{\geq 0},\ \chi_i \mapsto \gamma_{\chi_i}(M_{\chi_i}),$$

where the map $\mathrm{Res}_{i,r}$ is the restriction with respect to the natural inclusion

$$\mathbb{F}_{p^r,\widetilde{e}_i}^{\times} \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}.$$

Let m_0 be the product of all prime numbers $\leq p-2$ if $p \neq 2,3$ and $m_0 = 1$ if $p = 2,3$. Let r_0 be the order of p in the multiplicative group $(\mathbb{Z}/m_0\mathbb{Z})^{\times}$. Then [T4, Claim 5.4] implies the following result holds:

> there exists a constant $C(g_X)$ which only depends on g_X such that, for each $r > \log_p(C(g_X)+1)$ divisible by r_0, we have
>
> $$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_i}, \mathbb{F}_{p^r}) = \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(\mathbb{F}_{p^r,\widetilde{e}_i}^{\times}, \mathbb{F}_{p^r}^{\times}) \setminus \mathrm{Res}_{i,r}(\Gamma_{i,r}^{-1}(\{g_X+1\})),\ i \in \{1,2\},$$
>
> where $\mathrm{Hom}^{\mathrm{surj}}_{\mathrm{groups}}(-,-)$ denotes the set of surjections whose elements are contained in $\mathrm{Hom}_{\mathrm{groups}}(-,-)$.

Let $\kappa_2 \in \mathrm{Hom}_{\mathrm{groups}}(\pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}_{p^r}^{\times})$. Then ϕ induces a character

$$\kappa_1 \in \mathrm{Hom}_{\mathrm{groups}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}} \otimes \mathbb{Z}/(p^r-1)\mathbb{Z}, \mathbb{F}_{p^r}^{\times}).$$

Moreover, the surjection $\phi|_{H_{\kappa_1}}$ induces a surjection

$$M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2].$$

Suppose that $\kappa_2 \in \Gamma_{2,r}^{-1}(\{g_X+1\})$. The surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$ implies that $\gamma_{\kappa_1}(M_{\kappa_1}) = g_X + 1$. This means that $\kappa_1 \in \Gamma_{1,r}^{-1}(\{g_X+1\})$. On the other hand, by Theorem 5.6 (ii), we have an isomorphism $\phi|_{I_{\widetilde{e}_1}} : I_{\widetilde{e}_1} \xrightarrow{\sim} I_{\widetilde{e}_2}$. Then the isomorphism $\phi|_{I_{\widetilde{e}_1}}$ induces an injection

$$\mathrm{Res}_{2,r}(\Gamma_{2,r}^{-1}(\{g_X+1\})) \hookrightarrow \mathrm{Res}_{1,r}(\Gamma_{1,r}^{-1}(\{g_X+1\})).$$

Since $\#\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r}) = \#\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r})$, we obtain that $\phi|_{I_{\widetilde{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_2}, \mathbb{F}_{p^r}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{p^r,\widetilde{e}_1}, \mathbb{F}_{p^r}).$$

Thus, $\phi|_{I_{\widetilde{e}_1}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{\widetilde{e}_2}, \overline{\mathbb{F}}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fields}}(\mathbb{F}_{\widetilde{e}_1}, \overline{\mathbb{F}}_p).$$

If we choose $\overline{\mathbb{F}}_p = \mathbb{F}_{\widetilde{e}_2}$, then the image of $\mathrm{id}_{\mathbb{F}_{\widetilde{e}_2}}$ via the bijection above induces an isomorphism

$$\theta_{\phi,\widetilde{e}_1,\widetilde{e}_2} : \mathbb{F}_{\widetilde{e}_1} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_2}$$

as fields. This completes the proof of Proposition 6.1. □

7 Reconstructions of the isomorphism classes of curves of type $(0, n)$ via open continuous homomorphisms

In this section, we apply the results obtained in previous sections to tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

We fix some notations. Let (X_i, D_{X_i}), $i \in \{1, 2\}$, be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_i of characteristic $p > 0$, $U_{X_i} \overset{\text{def}}{=} X_i \setminus D_{X_i}$, $\pi_1^{\mathrm{t}}(U_{X_i})$ the tame fundamental group of U_{X_i}, $\pi_1(X_i)$ the étale fundamental group of X_i, and $(\widetilde{X}_i, D_{\widetilde{X}_i})$ the universal tame covering of (X_i, D_{X_i}) associated to $\pi_1^{\mathrm{t}}(U_{X_i})$. Let k_i^{m}, $i \in \{1, 2\}$, be the *minimal* algebraically closed subfield of k_i over which U_{X_i} can be defined. Thus, by considering the function field of X_i, we obtain a smooth pointed stable curve (i.e., *a minimal model of* (X_i, D_{X_i}) (cf. [T3, Definition 1.30 and Lemma 1.31]))

$$(X_i^{\mathrm{m}}, D_{X_i^{\mathrm{m}}})$$

such that $U_{X_i} \cong U_{X_i^{\mathrm{m}}} \times_{k_i^{\mathrm{m}}} k_i$ as k_i-schemes, where $U_{X_i^{\mathrm{m}}} \overset{\text{def}}{=} X_i^{\mathrm{m}} \setminus D_{X_i^{\mathrm{m}}}$. Note that $\pi_1^{\mathrm{t}}(U_{X_i^{\mathrm{m}}})$ is naturally isomorphic to $\pi_1^{\mathrm{t}}(U_{X_i})$. We shall denote by $\overline{\mathbb{F}}_{p,i}$ the algebraic closure of $\mathbb{F}_p$ in k_i. Moreover, we put

$$d_{(X_i, D_{X_i})} \overset{\text{def}}{=} \begin{cases} 0, & \text{if } k_i^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,i}, \\ 1, & \text{if } k_i^{\mathrm{m}} \not\cong \overline{\mathbb{F}}_{p,i}. \end{cases}$$

Lemma 7.1. *Let $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \to \pi_1^{\mathrm{t}}(U_{X_2})$ be an arbitrary open continuous homomorphism. Then ϕ is a surjection.*

Proof. We denote by Π_ϕ the image of ϕ which is an open subgroup of $\pi_1^{\mathrm{t}}(U_{X_2})$. Let (X_ϕ, D_{X_ϕ}) be the smooth pointed stable curve of type (g_{X_ϕ}, n_{X_ϕ}) over k_2 induced by Π_ϕ and

$$(X_\phi, D_{X_\phi}) \to (X_2, D_{X_2})$$

the tame covering of smooth pointed stable curves over k_2 induced by the inclusion $\Pi_\phi \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$. The Riemann-Hurwitz formula implies that $g_{X_\phi} \geq g_X$ and $n_{X_\phi} \geq n_X$. Note that $\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \Pi_\phi \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ implies that

$$2g_X + n_X - 1 \geq 2g_{X_\phi} + n_{X_\phi} - 1 \geq 2g_X + n_X - 1.$$

Then we obtain that $2g_X + n_X - 1 = 2g_{X_\phi} + n_{X_\phi} - 1$. Moreover, Corollary 2.3 and the natural surjection $\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \Pi_\phi$ induced by ϕ imply that $g_X \geq g_{X_\phi}$. Then we obtain that

$$g_X = g_{X_\phi}.$$

Thus, we have

$$(g_X, n_X) = (g_{X_\phi}, n_{X_\phi}).$$

This means that the tame covering $(X_\phi, D_{X_\phi}) \to (X_2, D_{X_2})$ is totally ramified over every marked point of D_{X_2}. Then the Riemann-Hurwitz formula implies that $[\pi_1^{\mathrm{t}}(U_{X_2}) : \Pi_\phi] \neq 1$ if and only if $(g_X, n_X) = (0, 2)$. Thus, we obtain that ϕ is a surjection. $\square$

In the remainder of this section, we suppose that

- $(g_X, n_X) = (0, n)$.

We fix two marked points $e_{1,\infty}, e_{1,0} \in D_{X_1}$ distinct from each other. Moreover, we choose any field $k_1' \cong k_1$, and choose any isomorphism $\varphi_1 : X_1 \xrightarrow{\sim} \mathbb{P}^1_{k_1'}$ as schemes such that $\varphi_1(e_{1,\infty}) = \infty$ and $\varphi_1(e_{1,0}) = 0$. Then the set of k_1-rational points $X_1(k_1) \setminus \{e_{1,\infty}\}$ is equipped with a structure of $\mathbb{F}_p$-module via the bijection φ_1. Note that since any k_1'-isomorphism of $\mathbb{P}^1_{k_1'}$ fixing ∞ and 0 is a scalar multiplication, the $\mathbb{F}_p$-module structure of $X_1(k_1) \setminus \{e_{1,\infty}\}$ does not depend on the choices of k_1' and φ_1 but depends only on the choices of $e_{1,\infty}$ and $e_{1,0}$. Then we shall say that $X_1(k_1) \setminus \{e_{1,\infty}\}$ *is equipped with a structure of* $\mathbb{F}_p$*-module with respect to* $e_{1,\infty}$ *and* $e_{1,0}$.

Lemma 7.2. *Let $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})$ be an open continuous surjective homomorphism. By Lemma 5.2, ϕ induces a bijection $\rho_\phi : D^{\mathrm{gp}}_{X_1} \xrightarrow{\sim} D^{\mathrm{gp}}_{X_2}$. We may identify $D^{\mathrm{gp}}_{X_i}$, $i \in \{1, 2\}$, with D_{X_i} via the bijection $\vartheta_{X_i} : D^{\mathrm{gp}}_{X_i} \xrightarrow{\sim} D_{X_i}$. Write $e_{2,\infty}$ and $e_{2,0}$ for $\rho_\phi(e_{1,\infty})$ and $\rho_\phi(e_{1,0})$, respectively. Let*

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} e_1 = e_{1,0}$$

be a linear condition with respect to $e_{1,\infty}$ and $e_{1,0}$ on (X_1, D_{X_1}), where $b_{e_1} \in \mathbb{F}_p$ for each $e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}$. Then the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} \rho_\phi(e_1) = \rho_\phi(e_{1,0}) = e_{2,0}$$

with respect to $e_{2,\infty}$ and $e_{2,0}$ on (X_2, D_{X_2}) also holds.

Proof. Let $\widetilde{e}_{2,\infty} \in D_{\widetilde{X}_2}$ be a point over $e_{2,\infty}$. The set $\mathbb{F}_{\widetilde{e}_{2,\infty}} \stackrel{\mathrm{def}}{=} (I_{\widetilde{e}_{2,\infty}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}_2) \coprod \{*_{\widetilde{e}_{2,\infty}}\}$ admits a structure of field, and Proposition 6.1 implies that the field structure can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_2})$. Theorem 5.6 implies that there exists a point $\widetilde{e}_{1,\infty} \in D_{\widetilde{X}_1}$ over $e_{1,\infty}$ such that $\phi(I_{\widetilde{e}_{1,\infty}}) = \widetilde{e}_{2,\infty}$. By Proposition 6.1 again, the set $\mathbb{F}_{\widetilde{e}_{1,\infty}} \stackrel{\mathrm{def}}{=} (I_{\widetilde{e}_{1,\infty}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}_1) \coprod \{*_{\widetilde{e}_{1,\infty}}\}$ admits a structure of field which can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_1})$, and ϕ induces an isomorphism

$$\theta_{\phi, \widetilde{e}_{1,\infty}, \widetilde{e}_{2,\infty}} : \mathbb{F}_{\widetilde{e}_{1,\infty}} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_{2,\infty}}$$

as fields.

For each $e_1 \in D_{X_1}$, we choose $b'_{e_1} \in \mathbb{Z}_{\geq 0}$ such that $b'_{e_1} \equiv b_{e_1} \pmod{p}$ and

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1} \geq 2.$$

Let $r \geq 1$ such that $p^r - 2 \geq \sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b'_{e_1}$. For each $\widetilde{e}_1 \in D_{\widetilde{X}_1}$ over e_1, write $I_{\widetilde{e}_1, \mathrm{ab}}$ for the image of the natural morphism

$$I_{\widetilde{e}_1} \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}.$$

Moreover, since the image of $I_{\widetilde{e}_1,\mathrm{ab}}$ does not depend on the choices of $\widetilde{e}_1$, we may write I_{e_1} for $I_{\widetilde{e}_1,\mathrm{ab}}$. The structure of maximal prime-to-p quotient of $\pi_1^{\mathrm{t}}(U_{X_1})$ implies that $\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}$ is generated by $\{I_{e_1}\}_{e_1\in D_{X_1}}$, and that there exists a generator a_{e_1}, $e_1 \in D_{X_1}$, of I_{e_1} such that

$$\prod_{e_1\in D_{X_1}} a_{e_1} = 1.$$

We define

$$I_{e_{1,\infty}} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_{1,\infty}} \mapsto 1,$$

$$I_{e_{1,0}} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_{1,0}} \mapsto (\sum_{e_1\in D_{X_1}\setminus\{e_{1,\infty},e_{1,0}\}} b'_{e_1}) - 1,$$

and

$$I_{e_1} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_1} \mapsto -b'_{e_1},\ e_1 \in D_{X_1}\setminus\{e_{1,\infty},e_{1,0}\}.$$

Then the homomorphisms of inertia subgroups defined above induces a sujection

$$\delta_1 : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z}.$$

Note that $\ker(\delta_1)$ does not depend on the choices of the generators $\{a_{e_1}\}_{e_1\in D_{X_1}}$.

Let $I_{\widetilde{e}_2} \overset{\mathrm{def}}{=} \phi(I_{\widetilde{e}_1})$, $\widetilde{e}_1 \in D_{\widetilde{X}_1}$, and I_{e_2}, $e_2 \in D_{X_2}$, the image of the natural morphism

$$I_{\widetilde{e}_2} \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{ab}}.$$

Since $(p, p^r-1) = 1$, by Theorem 5.6, δ_1 and the isomorphism $\phi^{p'} : \pi_1^{\mathrm{t}}(U_{X_1})^{p'} \overset{\sim}{\to} \pi_1^{\mathrm{t}}(U_{X_2})^{p'}$ imply the following morphisms of inertia subgroups:

$$I_{e_{2,\infty}} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_{2,\infty}} \mapsto 1,$$

$$I_{e_{2,0}} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_{2,0}} \mapsto (\sum_{e_1\in D_{X_1}\setminus\{e_{1,\infty},e_{2,0}\}} b'_{e_1}) - 1,$$

and

$$I_{e_2} \to \mathbb{Z}/(p^r-1)\mathbb{Z},\ a_{e_2} \mapsto -b'_{e_1},\ e_2 \in D_{X_2}\setminus\{e_{2,\infty},e_{2,0}\},$$

where a_{e_2}, $e_2 \in D_{X_2}$, denotes the element induced by a_{e_1}, $e_1 \in D_{X_1}$, via ϕ. Then the homomorphism of inertia subgroups defined above induces a sujection

$$\delta_2 : \pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{ab}} \twoheadrightarrow \mathbb{Z}/(p^r-1)\mathbb{Z}.$$

We put

$$H_{\delta_i} \overset{\mathrm{def}}{=} \ker(\delta_i),\ M_{\delta_i} \overset{\mathrm{def}}{=} H_{\delta_i}^{\mathrm{ab}} \otimes \mathbb{F}_p,\ i \in \{1,2\}.$$

Write $(X_{H_{\delta_i}}, D_{X_{H_{\delta_i}}})$ for the smooth pointed stable curve over k_i induced by H_{δ_i}, where $H_{\delta_1} = \phi^{-1}(H_{\delta_2})$. The $\mathbb{F}_p$-vector space M_{δ_i} admits a natural action of $I_{\widetilde{e}_{i,\infty}}$ via conjugation which coincides with the action via the following character

$$\chi_{I_{\widetilde{e}_{i,\infty}},r} : I_{\widetilde{e}_{i,\infty}} \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_i}) \overset{\delta_i}{\twoheadrightarrow} \mathbb{Z}/(p^r-1)\mathbb{Z} = I_{\widetilde{e}_{i,\infty}}/(p^r-1) \hookrightarrow \mathbb{F}_{\widetilde{e}_{i,\infty}}^{\times},\ i \in \{1,2\}.$$

We put $M_{\delta_i}[\chi_{I_{\widetilde{e}_{i,\infty}},r}] \overset{\text{def}}{=} \{a \in M_{\delta_i} \otimes_{\mathbb{F}_p} \mathbb{F}_{\widetilde{e}_{i,\infty}} \mid \sigma(a) = \chi_{I_{\widetilde{e}_{i,\infty}},r}(\sigma)a \text{ for all } \sigma \in I_{\widetilde{e}_{i,\infty}}\}$ (in fact, $\dim_{\mathbb{F}_{\widetilde{e}_{i,\infty}}}(M_{\delta_i}[\chi_{I_{\widetilde{e}_{i,\infty}},r}])$ is the first generalized Hasse-Witt invariant associated to the tame covering of U_{X_i} corresponding to $H_{\delta_i} \subseteq \pi_1^{\mathrm{t}}(U_{X_i})$ (cf. [Y5, Section 2.2])). Since the action of $I_{\widetilde{e}_{i,\infty}}$ on M_{δ_i} is semi-simple, we obtain a surjection

$$M_{\delta_1}[\chi_{I_{\widetilde{e}_{1,\infty}},r}] \twoheadrightarrow M_{\delta_2}[\chi_{I_{\widetilde{e}_{2,\infty}},r}]$$

induced by $\phi|_{H_{\delta_1}}$ and $\theta_{\phi,\widetilde{e}_{1,\infty},\widetilde{e}_{2,\infty}}$. On the other hand, the third and the final paragraphs of the proof of [T2, Lemma 3.3] implies that the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} e_1 = e_{1,0}$$

with respect to $e_{1,\infty}$ and $e_{1,0}$ on (X_1, D_{X_1}) holds if and only if $M_{\delta_1}[\chi_{I_{\widetilde{e}_{1,\infty}},r}] = 0$. Thus, we obtain $M_{\delta_2}[\chi_{I_{\widetilde{e}_{2,\infty}},r}] = 0$. Then the third and the final paragraphs of the proof of [T2, Lemma 3.3] implies that the linear condition

$$\sum_{e_1 \in D_{X_1} \setminus \{e_{1,\infty}, e_{1,0}\}} b_{e_1} \rho_\phi(e_1) = e_{2,0}$$

with respect to $e_{2,\infty}$ and $e_{2,0}$ on (X_2, D_{X_2}) holds. This completes the proof of the lemma. □

Remark 7.2.1. Note that, if $X_1 = \mathbb{P}^1_k$, then the linear condition is as follows:

$$\sum_{e_1 \in D_{X_1} \setminus \{\infty, 0\}} b_{e_1} e_1 = 0$$

with respect to ∞ and 0.

Now, let us prove the first main theorem of the present paper.

Theorem 7.3. *We maintain the notation introduced above. Then we have the following:*
(i) $d_{(X_i, D_{X_i})}$, $i \in \{1, 2\}$, *can be mono-anabelian reconstructed from* $\pi_1^{\mathrm{t}}(U_{X_i})$.
(ii) Suppose that $k_1^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,1}$. *Then the set of open continuous homomorphisms*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$$

is non-empty if and only if

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes. In particular, if this is the case, we have $k_2^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,2}$ *and*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2})) = \mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2})).$$

Proof. First, let us prove the (ii). The "if" part of (ii) is trivial. We only prove the "only if" part of (ii). Suppose that

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$$

is a non-empty set, and let $\phi \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$. Then Lemma 7.1 implies that ϕ is a surjection.

We identify $D^{\mathrm{gp}}_{X_i}$, $i \in \{1, 2\}$, with D_{X_i} via the bijection $\vartheta_{X_i} : D^{\mathrm{gp}}_{X_i} \xrightarrow{\sim} D_{X_i}$. Since ϕ is a surjection, Lemma 5.2 implies that ϕ induces a bijection

$$\rho_\phi : D_{X_1} \xrightarrow{\sim} D_{X_2}.$$

We put $e_{2,0} \stackrel{\mathrm{def}}{=} \rho_\phi(e_{1,0})$ and $e_{2,\infty} \stackrel{\mathrm{def}}{=} \rho_\phi(e_{1,\infty})$. Let $\widetilde{e}_{2,0} \in D_{\widetilde{X}_2}$ be a point over $e_{2,0}$. Theorem 5.6 implies that there exists a point $\widetilde{e}_{1,0} \in D_{\widetilde{X}_1}$ over $e_{1,0}$ such that $\phi(I_{\widetilde{e}_{1,0}}) = I_{\widetilde{e}_{2,0}}$. Then $\mathbb{F}_{\widetilde{e}_{i,0}} \stackrel{\mathrm{def}}{=} (I_{\widetilde{e}_{i,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})^{p'}_i) \bigsqcup \{*_{\widetilde{e}_{i,0}}\}$, $i \in \{1, 2\}$, admits a structure of field. Moreover, Proposition 6.1 implies that the field structure can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})$, and that ϕ induces a field isomorphism

$$\theta_{\phi, \widetilde{e}_{1,0}, \widetilde{e}_{2,0}} : \mathbb{F}_{\widetilde{e}_{1,0}} \xrightarrow{\sim} \mathbb{F}_{\widetilde{e}_{2,0}}.$$

Proposition 2.5 (i) implies that n can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$. If $n = 3$, (ii) is trivial, so we may assume that

$$n \geq 4.$$

Moreover, since $k_1^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,1}$, without loss of generality, we may assume that

$$k_1 = \overline{\mathbb{F}}_{p,1} = \mathbb{F}_{\widetilde{e}_{1,0}},$$

that $X_1 = \mathbb{P}^1_{\overline{\mathbb{F}}_{p,1}}$, and that

$$D_{X_1} = \{e_{1,\infty} = \infty, e_{1,0} = 0, e_{1,1} = 1, e_{1,2}, \ldots, e_{1,n-2}\}.$$

Here, $e_{1,2}, \ldots, e_{1,n-2} \in \overline{\mathbb{F}}_{p,1} \setminus \{e_{1,0}, e_{1,1}\}$ are distinct from each other.

Step 1: In this step, we will construct a linear relation on a certain tame covering of (X_1, D_{X_1}).

We see that there exists a natural number r prime to p such that $\mathbb{F}_p(\zeta_r)$ contains rth roots of $e_{1,2}, \ldots, e_{1,n-2}$, where ζ_r denotes a fixed primitive rth root of unity in $\overline{\mathbb{F}}_{p,1}$. Let $s \stackrel{\mathrm{def}}{=} [\mathbb{F}_p(\zeta_r) : \mathbb{F}_p]$. For each $e_{1,u} \in \{e_{1,2}, \ldots, e_{1,n-2}\}$, we fix an rth root $e_{1,u}^{1/r}$ in $\overline{\mathbb{F}}_{p,1}$. Then we have

$$e_{1,u}^{1/r} = \sum_{v=0}^{s-1} b_{1,uv} \zeta_r^v, \ u \in \{2, \ldots, n-2\},$$

where $b_{1,uv} \in \mathbb{F}_p$ for each $u \in \{2, \ldots, n-2\}$ and each $v \in \{0, \ldots, s-1\}$.

Let $X_1 \setminus \{e_{1,\infty}\} = \operatorname{Spec} \overline{\mathbb{F}}_{p,1}[x_1]$,

$$f_{H_1} : (X_{H_1}, D_{X_{H_1}}) \to (X_1, D_{X_1})$$

the Galois tame covering over $\overline{\mathbb{F}}_{p,1}$ with Galois group $\mathbb{Z}/r\mathbb{Z}$ determined by the equation $y_1^r = x_1$, and H_1 the open normal subgroup of $\pi_1^{\mathrm{t}}(U_{X_1})$ induced by the tame covering f_{H_1}. Then f_{H_1} is totally ramified over $\{e_{1,\infty} = \infty, e_{1,0} = 0\}$ and is étale over $D_{X_1} \setminus \{\infty, 0\}$. Note that $X_{H_1} = \mathbb{P}^1_{\overline{\mathbb{F}}_{p,1}}$, and that the points of $D_{X_{H_1}}$ over $\{e_{1,\infty}, e_{1,0}\}$ are $\{e_{H_1,\infty} \overset{\text{def}}{=} \infty, e_{H_1,0} \overset{\text{def}}{=} 0\}$. We put

$$e_{H_1,u} \overset{\text{def}}{=} e_{1,u}^{1/r} \in D_{X_{H_1}},\ u \in \{2, \ldots, n-2\},$$

and

$$e_{H_1,1}^v \overset{\text{def}}{=} \zeta_r^v \in D_{X_{H_1}},\ v \in \{0, \ldots, s-1\}.$$

Thus, we obtain a linear condition

$$e_{H_1,u} = \sum_{v=0}^{s-1} b_{1,uv} e_{H_1,1}^v$$

with respect to $e_{H_1,\infty}$ and $e_{H_1,0}$ on $(X_{H_1}, D_{X_{H_1}})$ for each $u \in \{2, \ldots, n-2\}$.

Step 2: In this step, we will prove that the linear relation on a certain tame covering of (X_1, D_{X_1}) constructed in Step 1 induces a linear relation on a certain tame covering of (X_2, D_{X_2}) via the surjection ϕ.

Write H_2 for $\phi(H_1)$. Since $(r, p) = 1$, we have the following commutative diagram of profinite groups:

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi|_{H_1}} & H_2 \\ \downarrow & & \downarrow \\ \pi_1^{\mathrm{t}}(U_{X_1}) & \xrightarrow{\phi} & \pi_1^{\mathrm{t}}(U_{X_2}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/r\mathbb{Z} & = & \mathbb{Z}/r\mathbb{Z}. \end{array}$$

We denote by

$$f_{H_2} : (X_{H_2}, D_{X_{H_2}}) \to (X_2, D_{X_2})$$

the Galois tame covering over $\overline{\mathbb{F}}_{p,2}$ with Galois group $\mathbb{Z}/r\mathbb{Z}$ induced by H_2. Note that Lemma 5.4 implies that $(X_{H_1}, D_{X_{H_1}})$ and $(X_{H_2}, D_{X_{H_2}})$ are equal types. Moreover, Lemma 5.3 implies that the following commutative diagram can be mono-anabelian reconstructed from the commutative diagram of profinite groups above:

$$\begin{array}{ccc} D_{X_{H_1}} & \xrightarrow{\rho_{\phi|_{H_1}}} & D_{X_{H_2}} \\ \downarrow & & \downarrow \\ D_{X_1} & \xrightarrow{\rho_\phi} & D_{X_2}. \end{array}$$

We put

$$e_{2,\infty} \overset{\text{def}}{=} \rho_\phi(e_{1,\infty}),\ e_{2,u} \overset{\text{def}}{=} \rho_\phi(e_{1,u}),\ u \in \{0, \ldots, n-2\},$$

$$e_{H_2,\infty} \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e_{H_1,\infty}),\ e_{H_2,0} \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e_{H_1,0}),\ e_{H_2,u} \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e_{H_1,u}),\ u \in \{2, \ldots, n-2\},$$

and

$$e_{H_2,1}^v \overset{\text{def}}{=} \rho_{\phi|_{H_1}}(e_{H_1,1}^v),\ v \in \{0, \ldots, s-1\}.$$

Remark 5.3.1 implies that f_{H_2} is totally ramified over $\{e_{2,\infty}, e_{2,0}\}$ and is étale over $X_2 \setminus \{e_{2,\infty}, e_{2,0}\}$. Then we may assume that $X_2 = \mathbb{P}^1_{k_2}$, and that $e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1$. We regard $e_{2,u}$, $u \in \{2, \dots, n-2\}$, as an element of $k_2 \setminus \{e_{2,0}, e_{2,1}\}$. Moreover, we have $e_{H_2,\infty} = \infty$ and $e_{H_2,0} = 0$.

We put $\xi_r \stackrel{\text{def}}{=} \theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(\zeta_r)$ is an rth root of unity in $\mathbb{F}_{\widetilde{e}_{2,0}}$. Since $\zeta_r(e^v_{H_1,1}) = e^{v+1}_{H_1,1}$, we obtain

$$\xi_r(e^v_{H_2,1}) = e^{v+1}_{H_2,1},\ v \in \{0, \dots, s-2\}.$$

By applying Lemma 7.2 for $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$, the following linear condition

$$e_{H_2,u} = \sum_{v=0}^{s-1} b_{1,uv} \xi^v_r(e^0_{H_2,1})$$

with respect to $e_{H_2,\infty}$ and $e_{H_2,0}$ on $(X_{H_2}, D_{X_{H_2}})$ holds for each $u \in \{2, \dots, n-2\}$. Since $(e_{H_2,u})^r = e_{2,u}$, $u \in \{2, \dots, n-2\}$, we obtain

$$e_{2,u} = (\sum_{v=0}^{s-1} b_{1,uv} \xi^v_r(e^0_{H_2,1}))^r.$$

Moreover, if we put $e^0_{H_2,1} = 1$, then we obtain that

$$e_{2,u} = (\sum_{v=0}^{s-1} b_{1,uv} \xi^v_r)^r$$

for each $u \in \{2, \dots, n-2\}$. Since $\theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(\zeta_r) = \xi_r$, we have

$$U_{X_1} = U_{X_1^{\mathrm{m}}} = \mathbb{P}^1_{\overline{\mathbb{F}}_{p,1}} \setminus \{e_{1,\infty} = \infty, e_{1,0} = 0, e_{1,1} = 1, e_{1,2}, \dots, e_{1,n-2}\}$$

$$\stackrel{\sim}{\rightarrow} \mathbb{P}^1_{\mathbb{F}_{\widetilde{e}_{2,0}}} \setminus \{e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1, \theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(e_{1,2}), \dots, \theta_{\phi,\widetilde{e}_{1,0},\widetilde{e}_{2,0}}(e_{1,n-2})\}$$

$$\cong \mathbb{P}^1_{\overline{\mathbb{F}}_{p,2}} \setminus \{e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1, e_{2,2}, \dots, e_{2,n-2}\}$$

and

$$\mathbb{P}^1_{\overline{\mathbb{F}}_{p,2}} \setminus \{e_{2,\infty} = \infty, e_{2,0} = 0, e_{2,1} = 1, e_{2,2}, \dots, e_{2,n-2}\} \times_{\overline{\mathbb{F}}_{p,2}} k_2 \cong U_{X_2}.$$

This means that

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes. In particular, we have $k_2^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,2}$.

Finally, we prove that

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(U_{X_1}), \pi^{\mathrm{t}}_1(U_{X_2})) = \mathrm{Isom}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(U_{X_1}), \pi^{\mathrm{t}}_1(U_{X_2})).$$

The "$\supseteq$" part is trivial. We only need to prove the "$\subseteq$" part. We may assume that

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(U_{X_1}), \pi^{\mathrm{t}}_1(U_{X_2})) \neq \emptyset,$$

and let $\phi' \in \mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2}))$. Then $\pi_1^{\mathrm{t}}(U_{X_1})$ is isomorphic to $\pi_1^{\mathrm{t}}(U_{X_2})$ as abstract profinite groups. By Lemma 7.1, ϕ' is a surjection. Then [FJ, Proposition 16.10.6] implies that ϕ' is an isomorphism. Thus, we obtain

$$\phi' \in \mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1}), \pi_1^{\mathrm{t}}(U_{X_2})).$$

This completes the proof of (ii).

Next, let us prove (i). Without loss of generality, we only treat the case where $i = 1$. Moreover, let $(X, D_X) \stackrel{\mathrm{def}}{=} (X_1, D_{X_1})$,

$$D_X = \{e_\infty = \infty, e_0 = 0, e_1 = 1, e_2, \dots, e_{n-2}\},$$

$k \stackrel{\mathrm{def}}{=} k_1$, and $\overline{\mathbb{F}}_p \stackrel{\mathrm{def}}{=} \overline{\mathbb{F}}_{\widetilde{e}_0}$. Let (r, Q) be a pair such that the following conditions hold:

(i) $(r, p) = 1$;

(ii) Q is an open normal subgroup of $\pi_1^{\mathrm{t}}(U_X)$ such that $\pi_1^{\mathrm{t}}(U_X)/Q \cong \mathbb{Z}/r\mathbb{Z}$, and that the Galois tame covering

$$f_Q : (X_Q, D_{X_Q}) \to (X, D_X)$$

over k induced by Q is totally ramified over $\{e_\infty, e_0\}$ and is étale over $D_X \setminus \{e_\infty, e_0\}$.

By applying Theorem 5.6, we see immediately that the set of pairs defined above can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$.

We fix a primitive rth root of unity ζ_r in $\overline{\mathbb{F}}_p$ and put $s_r \stackrel{\mathrm{def}}{=} [\mathbb{F}_p(\zeta_r) : \mathbb{F}_p]$. Moreover, we put

$$e_{Q,\infty} \stackrel{\mathrm{def}}{=} \infty,\ e_{Q,0} \stackrel{\mathrm{def}}{=} 0,\ e_{Q,1}^v \stackrel{\mathrm{def}}{=} \zeta_r^v \in D_{X_Q},\ v \in \{0, \dots s_r - 1\},$$

and let $e_{Q,u} \in D_{X_Q}$, $u \in \{2, \dots, n\}$, such that $f_Q(e_{Q,u}) = e_u$. Moreover, we put

$$L_{Q,u} \stackrel{\mathrm{def}}{=} \{e_{Q,u} - \sum_{v=0}^{s_r-1} b_{uv} e_{Q,1}^v \mid b_{uv} \in \mathbb{F}_p\} \cap \{0\},\ u \in \{2, \dots, n-2\}.$$

By applying similar arguments to the arguments given above, we have that $d_{(X,D_X)} = 0$ if and only if there exists a pair (r, Q) defined above such that $L_{Q,u} \neq \emptyset$ for each $u \in \{2, \dots, n-2\}$. Then the third and the final paragraphs of the proof of [T2, Lemma 3.3] implies that $L_{Q,u}$, $u \in \{2, \dots, n-2\}$, can be mono-anabelian reconstructed from Q. Thus, $d_{(X,D_X)}$ can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_X)$. This completes the proof of the theorem. □

Remark 7.3.1. Note that Theorem 7.3 and [T4, Theorem 0.2] also hold if we replace $\pi_1^{\mathrm{t}}(U_{X_i})$, $i \in \{1, 2\}$, by its maximal pro-solvable quotient $\pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{sol}}$. Then we obtain the following solvable version of Theorem 7.3 which is slightly stronger than the original theorem:

We maintain the notation introduced above. Then $d_{(X_i, D_{X_i})}$, $i \in \{1,2\}$, can be mono-anabelian reconstructed from $\pi_1^{\mathrm{t}}(U_{X_i})^{\mathrm{sol}}$. Moreover, suppose that $k_1^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,1}$. Then the set of open continuous homomorphisms

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{sol}}, \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{sol}})$$

is non-empty if and only if

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes. In particular, if this is the case, we have $k_2^{\mathrm{m}} \cong \overline{\mathbb{F}}_{p,2}$ and

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{sol}}, \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{sol}}) = \mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{sol}}, \pi_1^{\mathrm{t}}(U_{X_2})^{\mathrm{sol}}).$$

Remark 7.3.2. In this remark, we discuss Tamagawa's conjecture concerning essential dimensions of curves over algebraically closed fields of characteristic $p > 0$. Let (C, D_C) be a smooth pointed stable curve over an algebraically closed field l of characteristic $p > 0$ and $\overline{\mathbb{F}}_p$ the algebraic closure of $\mathbb{F}_p$ in l. We define the *essential dimension* of (C, D_C) to be

$$\mathrm{ed}((C, D_C)) \stackrel{\mathrm{def}}{=} \mathrm{td}(l^{\mathrm{m}}),$$

where l^{m} denotes the minimal algebraically closed subfield of l over which (C, D_C) can be defined, and $\mathrm{td}(l^{\mathrm{m}})$ denotes the transcendence degree of l^{m} over $\overline{\mathbb{F}}_p$.

Let (C_i, D_{C_i}), $i \in \{1,2\}$, be a smooth pointed stable curve of type (g_C, n_C) over an algebraically closed field l_i of characteristic $p > 0$ and $U_{C_i} \stackrel{\mathrm{def}}{=} C_i \setminus D_{C_i}$. Tamagawa posed a conjecture concerning the essential dimensions of curves in positive characteristic as follows (cf. [T3, Conjecture 5.3 (ii)]).

Essential Dimension Conjecture . *Suppose that*

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{C_1}), \pi_1^{\mathrm{t}}(U_{C_2}))$$

is non-empty. Then we have $\mathrm{ed}((C_1, D_{C_1})) = \mathrm{ed}((C_2, D_{C_2}))$.

Tamagawa's result concerning the Weak Isom-version Conjecture (cf. [T4, Theorem 0.2]) implies that the Essential Dimension Conjecture holds when $\mathrm{ed}((C_1, D_{C_1})) = 0$ and $g_C = 0$. We may ask the following question:

Suppose that

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(U_{C_1}), \pi_1^{\mathrm{t}}(U_{C_2}))$$

is non-empty. Does $\mathrm{ed}((C_1, D_{C_1})) \geq \mathrm{ed}((C_2, D_{C_2}))$ hold?

Theorem 7.3 implies that the answer is positive when $\mathrm{ed}((C_1, D_{C_1})) = 0$ and $g_C = 0$.

8 Tame anabelian geometry and moduli spaces

In this section, we consider the tame anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ from the point of view of moduli spaces.

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and let $\mathcal{M}_{g,n,\mathbb{Z}}$ be the moduli stack over $\mathbb{Z}$ parameterizing smooth pointed stable curves of type (g, n). We denote by

$$\mathcal{M}_{g,n} \stackrel{\text{def}}{=} \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$$

the moduli stack over $\overline{\mathbb{F}}_p$, and denote by $M_{g,n}$ the coarse moduli space of $\mathcal{M}_{g,n}$. The set of marked points of a smooth pointed stable curve admits a natural action of the n-symmetric group S_n (i.e., order of marked points), then we denote by $\mathcal{M}_{g,[n]} \stackrel{\text{def}}{=} [\mathcal{M}_{g,n}/S_n]$ the quotient stack, and denote by $M_{g,[n]}$ the coarse moduli space of $\mathcal{M}_{g,[n]}$. We obtain a morphism

$$\omega : M_{g,n} \to M_{g,[n]}$$

induced by the quotient morphism $\mathcal{M}_{g,n} \to \mathcal{M}_{g,[n]}$. We define an equivalence relation on the set of closed points of $M^{\text{cl}}_{g,n}$ as follows:

> Let $c_1, c_2 \in M^{\text{cl}}_{g,n}$ be *closed points*, where $(-)^{\text{cl}}$ denotes the set of closed points of $(-)$. Then $c_1 \sim_{\text{ec}} c_2$ if there exists $m \in \mathbb{Z}$ such that $\omega(c_2) = \omega(c_1^{(m)})$, where $c_1^{(m)}$ denotes the closed point corresponding to the curve obtained by mth Frobenius twist of the curve corresponding to c_1.

Let $q \in M_{g,n}$ be an arbitrary point, $k(q)$ the residue field of q, and k_q an algebraically closed field which contains $k(q)$. Write $(X_{k_q}, D_{X_{k_q}})$ for the smooth pointed stable curve of type (g, n) over k_q determined by the natural morphism $\operatorname{Spec} k_q \to \operatorname{Spec} k(q) \to M_{g,n}$ and $U_{X_{k_q}}$ for $X_{k_q} \setminus D_{X_{k_q}}$. In particular, if k_q is an algebraic closure of $k(q)$, we shall write

$$(X_q, D_{X_q})$$

for $(X_{k_q}, D_{X_{k_q}})$. Since the isomorphism class of the tame fundamental group $\pi_1^{\text{t}}(U_{X_{k_q}})$ depends only on q (i.e., does not depend on the choices of k_q), we shall write $\pi_1^{\text{t}}(q)$ and $\pi_A^{\text{t}}(q)$ for $\pi_1^{\text{t}}(U_{X_{k_q}})$ and the set of finite quotients of $\pi_1^{\text{t}}(U_{X_{k_q}})$, respectively. Note that, for any points $q_1, q_2 \in M_{g,n}$, we have that $\pi_1^{\text{t}}(q_1) \cong \pi_1^{\text{t}}(q_2)$ as profinite groups if and only if $\pi_A^{\text{t}}(q_1) = \pi_A^{\text{t}}(q_2)$ as sets (cf. [FJ, Proposition 16.10.7]). We denote by

$$V_q \stackrel{\text{def}}{=} \overline{\{q\}}$$

the topological closure of $\{q\}$ in $M_{g,n}$.

Definition 8.1. Let $q_1, q_2 \in M_{g,n}$. We denote by

$$V_{q_1} \supseteq_{\text{ec}} V_{q_2}$$

if, for each closed point $c_2 \in V^{\text{cl}}_{q_2}$, there exists a closed point $c_1 \in V^{\text{cl}}_{q_1}$ such that $c_1 \sim_{\text{ec}} c_2$. Moreover, we denote by

$$V_{q_1} =_{\text{ec}} V_{q_2}$$

if $V_{q_1} \supseteq_{\text{ec}} V_{q_2}$ and $V_{q_1} \subseteq_{\text{ec}} V_{q_2}$. We shall say that V_{q_1} is *essentially contains* V_{q_1} if $V_{q_1} \supseteq_{\text{ec}} V_{q_2}$, and that V_{q_1} is *essentially equal* to V_{q_2} if $V_{q_1} =_{\text{ec}} V_{q_2}$. Moreover, we also denote by $q_1 \sim_{\text{ec}} q_2$ when $V_{q_1} =_{\text{ec}} V_{q_2}$.

We denote by $\mathcal{M}_{g,n,\mathbb{F}_p} \overset{\text{def}}{=} \mathcal{M}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{F}_p$ the moduli stack over $\mathbb{F}_p$, and denote by $M_{g,n,\mathbb{F}_p}$ the coarse moduli space of $\mathcal{M}_{g,n,\mathbb{F}_p}$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{M}_{g,n} & \xrightarrow{\upsilon} & M_{g,n} & \xrightarrow{\omega^{\mathrm{c}}_{\overline{\mathbb{F}}_p}} & M_{g,n,\mathbb{F}_p} \times_{\mathbb{F}} \overline{\mathbb{F}}_p. \\ \omega \big\downarrow & & \omega^{\mathrm{c}} \big\downarrow & & \big\downarrow \\ \mathcal{M}_{g,n,\mathbb{F}_p} & \xrightarrow{\upsilon_{\mathbb{F}_p}} & M_{g,n,\mathbb{F}_p} & = & M_{g,n,\mathbb{F}_p}. \end{array}$$

Note that the $\overline{\mathbb{F}}_p$-morphism $\omega^{\mathrm{c}}_{\overline{\mathbb{F}}_p}$ is finite and radicial (cf. [KM], Proposition A7.2.1 and Corollary A7.2.2). We have the following proposition.

Proposition 8.2. *We denote by p_i, $i \in \{1, 2\}$, the image $\omega^{\mathrm{c}}(q_i)$, and denote by V_{p_i} the topological closure of p_i in $M_{g,n,\mathbb{F}_p}$. Then $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$ if and only if $V_{p_1} \supseteq V_{p_2}$. In particular, $V_{q_1} =_{\mathrm{ec}} V_{q_2}$ if and only if $V_{p_1} = V_{p_2}$. Moreover, $V_{q_1} =_{\mathrm{ec}} V_{q_2}$ if and only if $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes.*

Proof. First, we suppose that q_1 and q_2 are closed points of $M_{g,n}$. If $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$, then, by the definition of "$\supseteq_{\mathrm{ec}}$", we obtain $q_1 \sim q_2$. Thus, $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes. Then we obtain that $p_1 = p_2$. Conversely, if $V_{p_1} \supseteq V_{p_2}$, then we have $p_1 = p_2$. Thus, we obtain $q_1 \sim q_2$.

Next, we prove the proposition in general. If $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$, then the case of closed points implies that

$$V_{p_1}^{\mathrm{cl}} \supseteq V_{p_2}^{\mathrm{cl}}.$$

Since V_{p_1} and V_{p_2} are irreducible, we obtain that $V_{p_1} \supseteq V_{p_2}$. Conversely, if $V_{p_1} \supseteq V_{p_2}$, we observe that V_{q_1} and V_{q_2} are irreducible components of $(\omega^{\mathrm{c}})^{-1}(V_{p_1})$ and $(\omega^{\mathrm{c}})^{-1}(V_{p_2})$, respectively. Then the result of the case of closed points implies that $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$.

Let us prove the "moreover" part of the proposition. If $V_{q_1} =_{\mathrm{ec}} V_{q_2}$, then we obtain $p_1 = p_2$. Thus, we have

$$\overline{k(q_1)} \cong \overline{k(q_2)} \cong \overline{k(p_1)} = \overline{k(p_2)}$$

as fields. Moreover, the isomorphism induces a natural isomorphism

$$(X_{q_1}, D_{X_{q_1}}) = \mathcal{M}_{g,n+1,\mathbb{F}_p} \times_{\mathcal{M}_{g,n,\mathbb{F}_p}} \operatorname{Spec} \overline{k(q_1)} \cong \mathcal{M}_{g,n+1,\mathbb{F}_p} \times_{\mathcal{M}_{g,n,\mathbb{F}_p}} \operatorname{Spec} \overline{k(q_2)} = (X_{q_2}, D_{X_{q_2}}).$$

Thus, we obtain that $U_{X_{q_1}} \cong U_{X_{q_2}}$ as schemes.

On the other hand, let $s : U_{X_{q_1}} \xrightarrow{\sim} U_{X_{q_2}}$ be an isomorphism. We obtain the following commutative diagram

$$\begin{array}{ccc} U_{X_{q_1}} & \xrightarrow{s} & U_{X_{q_2}} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} \overline{k(q_1)} = \operatorname{Spec} \Gamma(X_{q_1}, \mathcal{O}_{X_{q_1}}) & \xrightarrow{s'} & \operatorname{Spec} \Gamma(X_{q_2}, \mathcal{O}_{X_{q_2}}) = \operatorname{Spec} \overline{k(q_2)}, \end{array}$$

where s' is an $\mathbb{F}_p$-isomorphism induced by s. This commutative diagram implies that the image of the natural composition morphism

$$\operatorname{Spec} \overline{k(q_1)} \to \mathcal{M}_{g,n} \to \mathcal{M}_{g,n,\mathbb{F}_p} \to M_{g,n,\mathbb{F}_p}$$

coincides with the image of the natural composition morphism

$$\operatorname{Spec} \overline{k(q_2)} \to \mathcal{M}_{g,n} \to \mathcal{M}_{g,n,\mathbb{F}_p} \to M_{g,n,\mathbb{F}_p},$$

where $\operatorname{Spec} \overline{k(q_i)} \to \mathcal{M}_{g,n}$, $i \in \{1,2\}$, is the morphism determined by $(X_{q_i}, D_{X_{q_i}})$ over $\overline{k(q_i)}$. Then we have $V_{q_1} =_{\mathrm{ec}} V_{q_2}$. This completes the proof of the proposition. □

Since (g, n) can be mono-anabelian reconstructed from tame fundamental groups, by applying Proposition 8.2, we may formulate the Essential Dimension Conjecture, the Weak Isom-version Conjecture, and the Weak Hom-version Conjecture as follows.

Essential Dimension Conjecture . *Suppose that*

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty. Then $\dim(V_{q_1}) = \dim(V_{q_2})$.

Weak Isom-version Conjecture . *The set of continuous isomorphisms of profinite groups*

$$\mathrm{Isom}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty if and only if $V_{q_1} =_{\mathrm{ec}} V_{q_2}$.

Weak Hom-version Conjecture . *The set of open continuous homomorphisms of profinite groups*

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi_1^{\mathrm{t}}(q_1), \pi_1^{\mathrm{t}}(q_2))$$

is non-empty if and only if $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$.

Let us consider the Weak Hom-version Conjecture. Let q be an arbitrary point of $M_{g,n}$ and $G \in \pi_A^{\mathrm{t}}(q)$ an arbitrary finite group. We put

$$U_G \stackrel{\mathrm{def}}{=} \{q' \in M_{g,n} \mid G \in \pi_A^{\mathrm{t}}(q')\} \subseteq M_{g,n}.$$

Then we have the following result.

Proposition 8.3. *Let q be an arbitrary point of $M_{g,n}$ and $G \in \pi_A^{\mathrm{t}}(q)$ an arbitrary finite group. Then the set U_G contains an open neighborhood of q in $M_{g,n}$.*

Proof. Proposition 8.3 was proved by K. Stevenson when $n = 0$ and q is a closed point of $M_{g,0}$ (cf. [Ste, Proposition 4.2]). Moreover, similar arguments to the arguments given in the proof of [Ste, Proposition 4.2] imply Proposition 8.3 also holds. □

Remark 8.3.1. In fact, U_G is an *open* subset of $M_{g,n}$, and Proposition 8.3 can be extended to $\overline{M}_{g,n}$ (cf. [Y2, Theorem 0.3]), where $\overline{M}_{g,n}$ denotes the coarse moduli space of the Deligne-Mumford compatification of $\mathcal{M}_{g,n}$.

Remark 8.3.2. Proposition 8.3 means that, for any finite group G, either G is not a quotient of the tame fundamental groups of any smooth pointed stable curves of type (g, n) over algebraically closed fields of characteristic $p > 0$, or is a quotient of the tame fundamental group of almost every such curve.

Let $G \in \pi_A^{\mathrm{t}}(q)$ be an arbitrary finite group. Then the subset $U_G \subseteq M_{g,n}$ depends only on G. We denote by q_{gen} the generic point of $M_{g,n}$, and let

$$\mathcal{C} \subseteq \pi_A^{\mathrm{t}}(q_{\mathrm{gen}}) = \bigcup_{q \in M_{g,n}^{\mathrm{cl}}} \pi_A^{\mathrm{t}}(q)$$

be a set of finite groups whose elements are contained in $\pi_A^{\mathrm{t}}(q_{\mathrm{gen}})$.

Definition 8.4. We shall say that $\mathcal{C}$ is a *pointed collection* if the following conditions are satisfied:

(i) $0 < \#((\bigcap_{G \in \mathcal{C}} U_G) \cap M_{g,n}^{\mathrm{cl}}) < \infty$;

(ii) $U_{G'} \cap (\bigcap_{G \in \mathcal{C}} U_G) \cap M_{g,n}^{\mathrm{cl}} = \emptyset$ for each $G' \in \pi_A^{\mathrm{t}}(q_{\mathrm{gen}})$ such that $G' \notin \mathcal{C}$.

On the other hand, for each closed point $t \in M_{g,n}^{\mathrm{cl}}$, we may define a set associated to t as follows:

$$\mathcal{C}_t \overset{\mathrm{def}}{=} \{G \in \pi_A^{\mathrm{t}}(q_{\mathrm{gen}}) \mid t \in U_G\}.$$

Note that, if $t \in V_q^{\mathrm{cl}}$, then $\mathcal{C}_t \subseteq \pi_A^{\mathrm{t}}(q)$. Moreover, we denote by

$$\mathscr{C}_q \overset{\mathrm{def}}{=} \{\mathcal{C} \text{ is a pointed collection} \mid \mathcal{C} \subseteq \pi_A^{\mathrm{t}}(q)\}.$$

At the present, no published results are known concerning the Weak Hom-version Conjecture (or the Weak Isom-version Conjecture) for *non-closed* points.

> The main difficulty of proving the Weak Hom-version Conjecture for non-closed points of $M_{g,n}$ is that, for each $q \in M_{g,n}$, we do not know how to reconstruct the tame fundamental groups of closed points of V_q group-theoretically from $\pi_1^{\mathrm{t}}(q)$.

Once the tame fundamental groups of the closed points of V_q can be reconstructed group-theoretically from $\pi_1^{\mathrm{t}}(q)$, then the Weak Hom-version Conjecture for closed points of $M_{g,n}$ implies that the set of closed points of V_q can be reconstructed group-theoretically from $\pi_1^{\mathrm{t}}(q)$. Thus, the Weak Hom-version Conjecture for non-closed points of $M_{g,n}$ follows from the Weak Hom-version Conjecture for closed points of $M_{g,n}$.

Since the isomorphism class of $\pi_1^{\mathrm{t}}(q)$ as a profinite group can be determined by the set $\pi_A^{\mathrm{t}}(q)$, we conjecture that the set of closed points V_q^{cl} can be reconstructed from $\pi_A^{\mathrm{t}}(q)$ (or equivalently $\pi_1^{\mathrm{t}}(q)$) as follows.

Pointed Collection Conjecture . *For each $t \in M_{g,n}^{\mathrm{cl}}$, the set $\mathcal{C}_t$ associated to t is a pointed collection. Moreover, the natural map*

$$\mathrm{colle}_q : V_q^{\mathrm{cl}}/\sim_{\mathrm{ec}} \to \mathscr{C}_q, \ [t] \mapsto \mathcal{C}_t,$$

is a bijection, where $[t]$ denotes the image of t in $V_q^{\mathrm{cl}}/\sim_{\mathrm{ec}}$.

Write $p_q \in M_{g,n,\mathbb{F}_p}$ for the image $\omega^{\mathrm{c}}(q)$. Then we have $V_q^{\mathrm{cl}}/\sim = V_{p_q}^{\mathrm{cl}}$. This means that the Pointed Collection Conjecture holds if and only if the Weak Hom-version Conjecture holds.

In the remainder of this subsection, we consider the Pointed Collection Conjecture for some special points of $M_{0,n}$.

Definition 8.5. Let $q \in M_{0,n}$ be a point and

$$U_{X_q} \cong \mathbb{P}^1_{\overline{k(q)}} \setminus \{a_1 = 1, a_2 = 0, a_3 = \infty, a_4, \ldots, a_n\}$$

as $\overline{k(q)}$-schemes. We shall say that q is a *coordinated point* if either $q = q_{\text{gen}}$ or the following conditions are satisfied:

(i) $\dim(V_q) = \dim(M_{0,n}) - 1$;

(ii) there exists $i \in \{4, \ldots, n\}$ such that $a_i \in \overline{\mathbb{F}}_p$;

(iii) let $\omega^i_{n,n-1} : M_{0,n} \to M_{0,n-1}$ be the morphism induced by the morphism $\mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ obtained by forgetting the ith marked point; then $\omega^i_{n,n-1}(q)$ is the generic point of $M_{0,n-1}$.

Remark 8.5.1. Let t be a closed point of $M_{0,n}$. Then there exists a set of coordinated points $P_t \overset{\text{def}}{=} \{q_{t,4}, \ldots, q_{t,n}\}$ such that

$$\{t\} = \bigcap_{q_{t,j} \in P_t} V_{q_{t,j}}.$$

Now, we prove the main result of the present paper.

Theorem 8.6. *(i) For each closed point $t \in M^{\text{cl}}_{0,n}$, the set $\mathcal{C}_t$ associated to t is a pointed collection. Moreover, for each pointed collection $\mathcal{C} \in \mathscr{C}_{q_{\text{gen}}}$, there exists a closed point $s \in M^{\text{cl}}_{0,n}$ such that $\mathcal{C} = \mathcal{C}_s$.*

(ii) Let $q \in M_{0,n}$ be an arbitrary point. Then the map colle_q *is an injection.*

(iii) Let $q \in M_{0,n}$ be an arbitrary point. Suppose that there exists a set of coordinated points P_q such that

$$V_q = \bigcap_{u \in P_q} V_u.$$

Then the Pointed Collection Conjecture holds for q. In particular, the Pointed Collection Conjecture holds for each closed point of $M_{0,n}$.

(iv) Let $q_i \in M_{0,n}$, $i \in \{1, 2\}$, be an arbitrary point. Suppose that there exists a set of coordinated points P_{q_1} such that

$$V_{q_1} = \bigcap_{u \in P_{q_1}} V_u.$$

Then the Weak Hom-version Conjecture holds. In particular, the Weak Hom-version Conjecture holds when q_1 is a closed point.

Proof. Let us prove (i). We put $F_t \overset{\text{def}}{=} \{t' \in M^{\text{cl}}_{0,n} \mid t \sim_{\text{ec}} t'\}$. Let t'' be an arbitrary point of $\bigcap_{G \in \pi^{\text{t}}_A(t)} U_G$. Then, for each $G \in \pi^{\text{t}}_A(t)$, we have

$$\mathrm{Hom}^{\text{surj}}_{\text{pro-gps}}(\pi^{\text{t}}_1(t''), G)$$

is non-empty, where $\mathrm{Hom}^{\text{surj}}_{\text{pro-gps}}(-,-)$ denotes the subset of $\mathrm{Hom}^{\text{open}}_{\text{pro-gps}}(-,-)$ whose elements are surjections. Since $\pi^{\text{t}}_1(t'')$ is topologically finitely generated, we obtain that the set $\mathrm{Hom}^{\text{surj}}_{\text{pro-gps}}(\pi^{\text{t}}_1(t''), G)$ is finite. Then the set of open continuous homomorphisms

$$\varprojlim_{G \in \pi^{\text{t}}_A(t)} \mathrm{Hom}^{\text{surj}}_{\text{pro-gps}}(\pi^{\text{t}}_1(t''), G) = \mathrm{Hom}^{\text{surj}}_{\text{pro-gps}}(\pi^{\text{t}}_1(t''), \pi^{\text{t}}_1(t))$$

is non-empty. Thus, Theorem 7.3 implies that $t'' \in F_t$. This means that

$$(\bigcap_{G \in \pi_A^{\mathrm{t}}(t)} U_G) \cap M_{g,n}^{\mathrm{cl}} = F_t.$$

Since U_{X_t} can be defined over a finite field, F_t is a finite set. Then $\mathcal{C}_t$ is a pointed collection.

Let $\mathcal{C} \in \mathscr{C}_{q_{\mathrm{gen}}}$ be a pointed collection and s a closed point of $\bigcap_{G \in \mathcal{C}} U_G$. By replacing t by s, and by applying similar arguments to the arguments given in the proof above, we see that $\mathcal{C} = \mathcal{C}_s$.

(ii) follows immediately from Theorem 7.3. Let us prove (iii). If $n = 4$, then $M_{0,4}$ is a one dimension scheme. For each $q \in M_{0,4}$, the Pointed Collection Conjecture follows immediately from Theorem 7.3. Then we may assume that

$$n \geq 5.$$

(ii) implies that we only need to prove that colle_q is a surjection. If q is a closed point of $M_{0,n}$, then the lemma follows immediately from Theorem 7.3.

Suppose that q is a non-closed point. This means that $\dim(V_q) \geq 1$. If $q = q_{\mathrm{gen}}$, (iii) follows from (i) and (ii). Let us treat the case where $q \neq q_{\mathrm{gen}}$. First, suppose that q is a coordinated point, and that

$$U_{X_q} \cong \mathbb{P}^1_{\overline{k(q)}} \setminus \{1, 0, \infty, a_4, \dots, a_n\}.$$

Without loss of generality, we may assume that $a_n \in \overline{\mathbb{F}}_p$.

For each pointed collection $\mathcal{C} \subseteq \mathscr{C}_q$, by applying (i), there exists a closed point $t_1 \in M_{g,n}^{\mathrm{cl}}$ such that $\mathcal{C}_{t_1} = \mathcal{C}$. We have an open continuous surjective homomorphism

$$\pi_1^{\mathrm{t}}(q) \twoheadrightarrow \pi_1^{\mathrm{t}}(t_1).$$

Let $\omega_{n,4}^{\backslash n} : M_{0,n} \to M_{0,4}$ be the morphism induced by the morphism $\mathcal{M}_{0,n} \to \mathcal{M}_{0,4}$ obtained by forgetting the marked points except the first, the second, the third, and the nth marked points. Write t_1'' and q'' for $\omega_{n,4}^{\backslash n}(t_1)$ and $\omega_{n,4}^{\backslash n}(q)$, respectively. Note that t_1'' and q'' are closed points of $M_{0,4}$. Then Theorem 5.6 implies that the surjection of tame fundamental groups above induces an open continuous surjective homomorphism

$$\pi_1^{\mathrm{t}}(q'') \twoheadrightarrow \pi_1^{\mathrm{t}}(t_1'').$$

Thus, by Theorem 7.3, we obtain that

$$q'' \sim_{\mathrm{ec}} t_1''.$$

Then without loss of generality, we may assume that

$$U_{X_{t_1}} \cong \mathbb{P}^1_{\overline{\mathbb{F}}_p} \setminus \{1, 0, \infty, b_4, \dots, b_{n-1}, a_n\}$$

over $\overline{\mathbb{F}}_p$, where $b_i \in \overline{\mathbb{F}}_p$ for each $i \in \{4, \dots, n-1\}$.

On the other hand, let $\omega_{n,n-1}^{n} : M_{0,n} \to M_{0,n-1}$ be the morphism induced by the morphism $\mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ obtained by forgetting the nth marked point. Write t_1' and q'

for $\omega^n_{n,n-1}(t_1)$ and $\omega^n_{n,n-1}(q)$, respectively. Since q is a coordinated point, q' is the generic point of $M_{0,n-1}$. Then $t'_1 \in V^{\mathrm{cl}}_{q'}$. Moreover, we observe that

$$V_q = \omega^{-1}_{n,n-1}(q').$$

Thus, we obtain $t_1 = \omega^{-1}_{n,n-1}(t'_1)$ is a closed point of V_q. Then the Pointed Collection Conjecture holds for q when q is a coordinated point.

Next, we prove the general case. If $V_q = \bigcap_{u \in P_q} V_u$, then $V^{\mathrm{cl}}_q = \bigcap_{u \in P_q} V^{\mathrm{cl}}_u$ and $\bigcap_{u \in P_q} \mathscr{C}_u = \mathscr{C}_q$. Moreover, since we have a bijection

$$\bigcap_{u \in P_q} V^{\mathrm{cl}}_u \overset{\sim}{\to} \bigcap_{u \in P_q} \mathscr{C}_u,$$

then

$$\mathrm{colle}_q : V^{\mathrm{cl}}_q / \sim_{\mathrm{ec}} = \bigcap_{u \in P_q} V^{\mathrm{cl}}_u / \sim_{\mathrm{ec}} \to \bigcap_{u \in P_q} \mathscr{C}_u = \mathscr{C}_q$$

is a bijection. This completes the proof of (iii).

Let us prove (iv). We only need to prove the "only if" part of the Weak Hom-version Conjecture. Suppose that V_{q_2} is not essentially contained in V_{q_1}. This implies that there exists a closed point $t_2 \in V^{\mathrm{cl}}_{q_2}$ such that $F_{t_2} \cap V_{q_1} = \emptyset$, where $F_{t_2} \overset{\mathrm{def}}{=} \{t'_2 \in M^{\mathrm{cl}}_{0,n} \mid t_2 \sim_{\mathrm{ec}} t'_2\}$. By (iii), we have $\mathcal{C}_{t_2} \notin \mathscr{C}_{q_1}$. Thus, by Lemma 7.1, we obtain that

$$\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(q_1), \pi^{\mathrm{t}}_1(t_2)) = \emptyset.$$

This provides a contradiction to the assumption that $\mathrm{Hom}^{\mathrm{open}}_{\text{pro-gps}}(\pi^{\mathrm{t}}_1(q_1), \pi^{\mathrm{t}}_1(q_2))$ is non-empty. We complete the proof of (iv). □

Remark 8.6.1. Let $q \in M_{g,n}$ be an arbitrary point. Stevenson posed a question as follows (cf. [Ste, Question 4.3] for the case of $n = 0$): Does $\bigcap_{G \in \pi^{\mathrm{t}}_A(q)} U_G$ contain any closed points of $M_{g,n}$? By [T5, Theorem 0.3], $\bigcap_{G \in \pi^{\mathrm{t}}_A(q)} U_G$ contains a closed point of $M_{g,n}$ if and only if q is a closed point of $M_{g,n}$. Furthermore, when $g = 0$ and q is a closed point, the proof of Theorem 8.6 (i) implies that

$$\Big(\bigcap_{G \in \pi^{\mathrm{t}}_A(q)} U_G\Big) \cap M^{\mathrm{cl}}_{0,n} = F_q,$$

where $F_q \overset{\mathrm{def}}{=} \{q' \in M^{\mathrm{cl}}_{0,n} \mid q \sim_{\mathrm{ec}} q'\}$.

Remark 8.6.2. Let t be an arbitrary closed point of $M_{0,n}$ and $F_t \overset{\mathrm{def}}{=} \{t' \in M^{\mathrm{cl}}_{0,n} \mid t \sim_{\mathrm{ec}} t'\}$. We ask the following question:

$$\text{Does } \Big|\bigcap_{G \in \pi^{\mathrm{t}}_A(t)} U_G\Big| = \bigcup_{t' \in F_t} |\mathrm{Spec}\, \mathcal{O}_{M_{0,n},t'}| \text{ hold?}$$

Here $|(-)|$ denotes the underlying topological space of $(-)$. If the answer is "Yes", one may prove the Weak Hom-version Conjecture for any points of $M_{0,n}$.

References

[FJ] M. D. Fried, M. Jarden, Field arithmetic. Third edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete.* 3. Folge. A Series of Modern Surveys in Mathematics **11**. *Springer-Verlag*, Berlin, 2008.

[G1] A. Grothendieck, La longue marche à travers la théorie de Galois, manuscript (1981).

[G2] A. Grothendieck, Letter to G. Faltings (German), manuscript (1984).

[KM] N. Katz, B. Mazur, Arithmetic moduli of elliptic curves, *Ann. of Math. Stud.*, **108**, Princeton University Press, 1985.

[M1] S. Mochizuki, The local pro-p anabelian geometry of curves. *Invent. Math.* **138** (1999), 319–423.

[M2] S. Mochizuki, Topics in absolute anabelian geometry III: global reconstruction algorithms. *J. Math. Sci. Univ. Tokyo* **22** (2015), 939–1156.

[N1] H. Nakamura, Rigidity of the arithmetic fundamental group of a punctured projective line. *J. Reine Angew. Math.* **405** (1990), 117-130.

[N2] H. Nakamura, Galois rigidity of the étale fundamental groups of punctured projective lines. *J. Reine Angew. Math.* **411** (1990), 205-216.

[PS] F. Pop, M. Saïdi, On the specialization homomorphism of fundamental groups of curves in positive characteristic. *Galois groups and fundamental groups,* 107–118, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.

[R1] M. Raynaud, Sections des fibrés vectoriels sur une courbe. *Bull. Soc. math. France* **110** (1982), 103–125.

[R2] M. Raynaud, Sur le groupe fondamental d'une courbe complète en caractéristique $p > 0$. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 335-351, *Proc. Sympos. Pure Math.,* **70**, *Amer. Math. Soc., Providence, RI,* 2002.

[Ste] K. Stevenson, Galois groups of unramified covers of projective curves in characteristic p. *Journal of Algebra* **182** (1996), 770–804.

[Sti1] J. Stix, Affine anabelian curves in positive characteristic. *Compositio Math.* **134** (2002), 75-85.

[Sti2] J. Stix, Projective anabelian curves in positive characteristic and descent theory for log-étale covers. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2002. *Bonner Mathematische Schriften*, **354**. Universität Bonn, Mathematisches Institut, Bonn, 2002.

[T1] A. Tamagawa, The Grothendieck conjecture for affine curves. *Compositio Math.* **109** (1997), 135–194.

[T2] A. Tamagawa, On the fundamental groups of curves over algebraically closed fields of characteristic > 0. *Internat. Math. Res. Notices* (1999), 853-873.

[T3] A. Tamagawa, Fundamental groups and geometry of curves in positive characteristic. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 297–333, *Proc. Sympos. Pure Math.*, **70**, *Amer. Math. Soc., Providence, RI*, 2002.

[T4] A. Tamagawa, On the tame fundamental groups of curves over algebraically closed fields of characteristic > 0. *Galois groups and fundamental groups*, 47–105, *Math. Sci. Res. Inst. Publ.*, **41**, *Cambridge Univ. Press, Cambridge*, 2003.

[T5] A. Tamagawa, Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups. *J. Algebraic Geom.* **13** (2004), 675–724.

[Y1] Y. Yang, On the admissible fundamental groups of curves over algebraically closed fields of characteristic $p > 0$, *Publ. Res. Inst. Math. Sci.* **54** (2018), 649–678.

[Y2] Y. Yang, Finite quotients of fundamental groups and moduli spaces of curves in positive characteristic. See http://www.kurims.kyoto-u.ac.jp/~yuyang/

[Y3] Y. Yang, The combinatorial mono-anabelian geometry of curves over algebraically closed fields of positive characteristic I: combinatorial Grothendieck conjecture, preprint. See http://www.kurims.kyoto-u.ac.jp/~yuyang/

[Y4] Y. Yang, On the averages of generalized Hasse-Witt invariants of pointed stable curves in positive characteristic, *Math. Z.* **295** (2020), 1–45.

[Y5] Y. Yang, Maximums of generalized Hasse-Witt invariants of pointed stable curves in positive characteristic. See http://www.kurims.kyoto-u.ac.jp/~yuyang/

[Y6] Y. Yang, Moduli spaces of fundamental groups of curves in positive characteristic I, preprint. See http://www.kurims.kyoto-u.ac.jp/~yuyang/

[Y7] Y. Yang, Moduli spaces of fundamental groups of curves in positive characteristic II, in preparation.

Yu Yang

Address: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail: yuyang@kurims.kyoto-u.ac.jp