TOPOLOGICAL AND GROUP-THEORETICAL SPECIALIZATIONS OF FUNDAMENTAL GROUPS OF CURVES IN POSITIVE CHARACTERISTIC

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Abstract. In the present paper, we study some new anabelian phenomena of curves over algebraically closed fields of characteristic $p > 0$, and formulate two new conjectures concerning open continuous homomorphisms of admissible fundamental groups that are motivated by the theory of moduli spaces of fundamental groups. Moreover, we prove the conjectures hold for genus 0 under certain assumptions.

Keywords: pointed stable curve, admissible fundamental group, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14G32; Secondary 14H30.

Contents

1. Introduction

1.1. **Anabelian geometry.** In the 1980s, A. Grothendieck suggested a theory of arithmetic geometry called anabelian geometry ([G]), roughly speaking, which focuses on the following question: Can we reconstruct the geometric information of a variety group-theoretically from various versions of its algebraic fundamental group? The varieties which can be completely determined by their fundamental groups are called "anabelian varieties" by Grothendieck, and to classify the anabelian varieties in all dimensions over all fields is called "anabelian dream" of him. In the particular case of dimension 1, he conjectured that all smooth pointed stable curves, or hyperbolic curves (defined over certain fields) are anabelian varieties.

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1.1.1. Let p be a prime number and $\#(-)$ the cardinality of $(-)$. Let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve of type (g_X, n_X) over a field k of characteristic $\mathrm{char}(k)$, where X denotes the underlying curve which is a semi-stable curve over k, D_X denotes the (finite) set of marked points satisfying [K, Definition 1.1 (iv)], g_X denotes the genus of X, and $n_X \overset{\mathrm{def}}{=} \#(D_X)$. In the present introduction, "curves" means pointed stable curves unless indicated otherwise.

1.1.2. *Grothendieck's anabelian philosophy.* Suppose that $X^\bullet$ is smooth over k. When k is an "arithmetic" field (e.g. a number field, a p-adic field, a finite field, etc.), Grothendieck's anabelian conjectures for curves (or the Grothendieck conjectures for short), roughly speaking, are based on the following *anabelian philosophy* (see [G, p289 (6)] for a precise statement):

> **Hom-version:** The sets of dominant morphisms of smooth pointed stable curves can be determined group-theoretically from the sets of open continuous homomorphisms of their algebraic fundamental groups.

In particular, we have the following two versions:

> **Isom-version:** The sets of isomorphisms of smooth pointed stable curves can be determined group-theoretically from the sets of isomorphisms of their algebraic fundamental groups.
>
> **Weak Isom-version:** The isomorphism class of $X^\bullet$ can be determined group-theoretically from the isomorphism class of its algebraic fundamental group.

Grothendieck's anabelian philosophy tells us, *over arithmetic fields, what geometric behavior of curves should be anabelian.*

1.1.3. *Anabelian geometry of curves over arithemetic fields.* Grothendieck's anabelian conjectures over arithmetic fields have been proven in many cases (e.g. see [P], [MNT], [T1] for surveys). All the proofs of the Grothendieck conjectures for curves over arithmetic fields mentioned above require the use of *the non-trivial outer Galois representations* induced by the fundamental exact sequences of fundamental groups.

1.1.4. *Beyond the arithmetical action.* Next, we consider the case where $X^\bullet$ is an arbitrary pointed stable curve, and suppose that k is an algebraically closed field. Let $\pi_1^{\mathrm{adm}}(X^\bullet)$ be the admissible fundamental group of $X^\bullet$ (see 2.2.2). Note that if $X^\bullet$ is smooth over k, then $\pi_1^{\mathrm{adm}}(X^\bullet)$ is naturally isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(X^\bullet)$. When $\mathrm{char}(k) = 0$, since the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet)$ depends only on the type (g_X, n_X), the anabelian geometry of curves does not exist in this situation. On the other hand, if $\mathrm{char}(k) = p$, the situation is quite different from that in characteristic 0. The admissible fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet)$ is very mysterious and its structure is no longer known. In the remainder of the introduction, we assume that k *is an algebraically closed field of characteristic* p.

After M. Raynaud and D. Harbater proved Abhyankar's conjecture, Harbater asked whether or not the geometric information of a curve over k can be carried out from its geometric fundamental groups ([Ha1], [Ha2]). Since the late 1990s, based on the philosophy concerning "Weak Isom-version" explained in 1.1.2, some results of Raynaud ([R]), F. Pop-M. Saïdi ([PS]), A. Tamagawa ([T2], [T4], [T5]), and the author of the present paper ([Y1], [Y2]) showed evidences for very strong *anabelian phenomena for curves over algebraically closed fields of characteristic* p (see [T3] for more about this conjectural world based on Grothendieck's anabelian philosophy mentioned in 1.1.2). In this situation, the arithmetic fundamental group coincides with the geometric fundamental group, thus there is a total absence of a Galois action of the base field. This kind of anabelian phenomenon is the reason why we do not have an explicit description of the geometric fundamental group of any pointed stable curve in characteristic p.

The anabelian geometry of curves over algebraically closed fields of characteristic p is very difficult. At present, in this situation, the Grothendieck conjecture was proved only in the case of weak Isom-version under the assumption that curves are defined over $\overline{\mathbb{F}}_p$ of type $(0, n_X)$, and that curves are defined over $\overline{\mathbb{F}}_p$ of type $(1,1)$ if $p \neq 2$ ([S, Theorem 1.1], [T4, Theorem 0.2], [T6], [Y1, Theorem 1.2 and Theorem 1.3], [Y4, Theorem 3.8]).

Since Tamagawa discovered that there also exists the anabelian geometry for certain smooth pointed stable curves over algebraically closed fields of characteristic p, 28 years have passed. However, the "Weak Isom-version" is still the only anabelian phenomenon that we know in this situation, and we *cannot* even imagine what phenomena arose from curves and their fundamental groups should be anabelian until the author of the present paper observed a new kind anabelian phenomenon explained below.

1.2. **Motivation.**

1.2.1. *A new kind of anabelian phenomenon.* When we try to formulate a "Hom-version" conjecture for curves over algebraically closed fields of characteristic p based on Grothendieck's anabelian philosophy mentioned in 1.1.2 (i.e. an analogue of the conjecture posed in [G, p289 (6)]), we see that the set of dominate morphisms between two pointed stable curves are *possibly empty*, and that the set of open continuous homomorphisms of their admissible fundamental groups are *not empty* in general (e.g. specialization homomorphisms of a non-isotrivial family of pointed stable curves). Then the relation of two pointed stable curves *cannot* be determined by the set of open continuous homomorphisms of their admissible fundamental groups if we only consider anabelian geometry in the sense of "Hom-version" mentioned in 1.1.2. In fact, the existence of specialization homomorphisms is the reason that Tamagawa cannot formulate a "Hom-version" conjecture for tame fundamental groups of smooth pointed stable curves in general ([T3, Remark 1.34]).

On the other hand, the author observed a new phenomenon that has never been seen before: It is possible that *the sets of deformations* of a smooth pointed stable curve can be reconstructed group-theoretically from open continuous homomorphisms of their admissible fundamental groups. This observation implies *a new kind of anabelian phenomenon that cannot be explained by using Grothendieck's original anabelian philosophy mentioned in 1.1.2*: The *topological structures* of moduli spaces of curves in positive characteristic are encoded in the *sets of open continuous homomorphisms* of geometric fundamental groups of curves in positive characteristic.

This new kind of anabelian phenomenon can be precisely captured by using the so-called *moduli spaces of admissible fundamental groups* and the *homeomorphism conjecture* introduced in [Y6], [Y7]. Let us briefly explain them in 1.2.2.

1.2.2. *Moduli spaces of admissible fundamental groups and the homeomorphism conjecture.* Let $\overline{\mathcal{M}}_{g,n,\mathbb{Z}}$ be the moduli stack over $\mathbb{Z}$ parameterizing pointed stable curves of type (g,n) and $\overline{M}_{g,n}$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$. In [Y6, Section 3.2], the author introduced a topological space $\overline{\Pi}_{g,n}$ in a group-theoretical way, whose underlying set consists of the isomorphism classes (as profinite groups) of admissible fundamental groups of curves of type (g,n), and whose topology is determined by the sets of finite quotients of admissible fundamental groups of curves of type (g,n). We shall call $\overline{\Pi}_{g,n}$ the *moduli space of admissible fundamental groups of type* (g,n).

There exists a natural map (as sets) $\overline{M}_{g,n} \to \overline{\Pi}_{g,n}$ defined by $q \mapsto [\Pi_q]$, where Π_q denotes the admissible fundamental group of the curve corresponding to a geometric point over q, and $[\Pi_q]$ denotes the isomorphism class of Π_q. By introducing the so-called *Frobenius equivalence* $\sim_{fe}$ on $\overline{M}_{g,n}$ (see [Y4, Definition 3.4]), we have a *continuous* surjective map ([Y6, Theorem 3.6])

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \stackrel{\mathrm{def}}{=} \overline{M}_{g,n}/\sim_{fe} \twoheadrightarrow \overline{\Pi}_{g,n}, \ [q] \mapsto [\Pi_q],$$

where $[q]$ denotes the equivalence class of q, and $\overline{\mathfrak{M}}_{g,n}$ is the quotient topological space whose topology is induced by the Zariski topology of $\overline{M}_{g,n}$. Moreover, we posed the so-called *homeomorphism conjecture* ([Y6, Section 3.3]) which says that $\pi^{\mathrm{adm}}_{g,n}$ is a homeomorphism.

The homeomorphism conjecture generalizes all the conjectures appeared in the theory of admissible (or tame) anabelian geometry of curves over algebraically closed fields of characteristic p, and means that the moduli spaces of curves in positive characteristic can be reconstructed group-theoretically *as topological spaces* from sets of open continuous homomorphisms of admissible fundamental groups of pointed stable curves in positive characteristic. Moreover, it sheds some new light on the theory of the anabelian geometry of curves over algebraically closed fields of characteristic p based on the following new anabelian philosophy:

> The *anabelian properties* of pointed stable curves over algebraically closed fields of characteristic p are equivalent to the *topological properties* of the topological space $\overline{\Pi}_{g,n}$.

The above philosophy supplies a point of view to see *what anabelian phenomena that we can reasonably expect for pointed stable curves over algebraically closed fields of characteristic p.*

1.2.3. *Towards the homeomorphism conjecture for higher dimensional moduli spaces.* The homeomorphism conjecture has been proved by the author in the case where $\dim(\overline{M}_{g,n}) \leq 1$ (e.g. see [Y6, Theorem 6.7] for the case of $g = 0$). The main goal of the anabelian geometry of curves over algebraically closed fields of characteristic p is to prove the homeomorphism conjecture for higher dimensional moduli spaces. The author believes that it can be proved by the following steps:

- Step 1 (closed points): prove the homeomorphism conjecture for closed points of $\overline{\mathfrak{M}}_{g,n}$.
- Step 2 (non-closed points corresponding to smooth curves): prove the homeomorphism conjecture for non-closed points of $\mathfrak{M}_{g,n} \overset{\text{def}}{=} M_{g,n}/\sim_{fe} \subseteq \overline{\mathfrak{M}}_{g,n}$ by using Step 1.
- Step 3 (from smooth to singular): prove the homeomorphism conjecture by using Step 2.

When $g = 0$, Step 1 has been completed by the author ([Y6, Theorem 6.7]). The Step 2 is equivalent to the *weak Hom-version conjecture* and the *pointed collection conjecture* formulated in [HYZ, Section 2].

In the present paper, we treat Step 3 and give a precise formulation via the *group-theoretical specialization conjecture* explained below. On the other hand, in the remainder of the introduction, we also treat the case of *maximal pro-solvable quotients* of admissible fundamental groups (or pro-solvable admissible fundamental groups for short). Note that the pro-solvable version is stronger than the original version in general since the pro-solvable admissible fundamental groups can be reconstructed group-theoretically from admissible fundamental groups.

1.3. Various specializations via fundamental groups.

1.3.1. Let $X_i^\bullet$, $i \in \{1, 2\}$, be an arbitrary pointed stable curve of type (g_X, n_X) over k_i of characteristic p, $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$ (2.2.1), and $\Pi_{X_i^\bullet}$ either the admissible fundamental group of $X_i^\bullet$ or the maximal pro-solvable quotient of admissible fundamental group of $X_i^\bullet$. We put $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ the set of open continuous homomorphisms between $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, and let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ be an arbitrary open continuous homomorphism. Note that ϕ is a surjection since the types of $X_1^\bullet$ and $X_2^\bullet$ are equal (see [Y6, Lemma 4.3]).

To complete Step 3 mentioned in 1.2.3 (i.e. to prove the homeomorphism conjecture for arbitrary pointed stable curves by using the homeomorphism conjecture for smooth curves), we need to establish a precise group-theoretical correspondence via ϕ between various "*pointed stable sub-curves*" (2.2.3, 2.2.4, 2.2.5) of $X_1^\bullet$ and $X_2^\bullet$ (e.g. pointed stable curves associated to irreducible components of $X_1^\bullet$ and $X_2^\bullet$). Namely, we need the following:

> (i) Give a group-theoretical description of various pointed stable sub-curves of $X_i^\bullet$ via the closed subgroups of $\Pi_{X_i^\bullet}$.
> (ii) Establish a correspondence between the closed subgroups of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ appeared in (i) via ϕ.

1.3.2. *Combinatorial data, topological data, and geometric data.* For (i) mentioned above, we introduce the following sets

$$\mathrm{Com}(\Gamma_{X_i^\bullet}),\ \mathrm{Typ}(X_i^\bullet),\ \mathrm{Geo}(\Pi_{X_i^\bullet})$$

which we call the combinatorial data associated to $\Gamma_{X_i^\bullet}$, the topological data associated to $X_i^\bullet$, and the geometric data associated to $\Pi_{X_i^\bullet}$, respectively (see Section 2.3 and Definition 2.5 for precise definitions). Roughly speaking, $\mathrm{Com}(\Gamma_{X_i^\bullet})$ consists of various sub-semi-graphs of $\Gamma_{X_i^\bullet}$ (see 2.1.2) which encodes the *gluing data* of various pointed stable sub-curves of $X_i^\bullet$, $\mathrm{Typ}(X_i^\bullet)$ consists of the *topological types* of various pointed stable sub-curves of $X_i^\bullet$, and $\mathrm{Geo}(\Pi_{X_i^\bullet})$ consists of the closed subgroups of $\Pi_{X_i^\bullet}$ which are isomorphic to the admissible fundamental groups (or pro-solvable admissible fundamental groups) of various pointed stable sub-curves of $X_i^\bullet$.

Some special cases of the above data have been studied by Tamagawa when $X_i^\bullet$ is smooth over k_i ([T3], [T4]) and by the author when $X_i^\bullet$ is an arbitrary pointed stable curve ([Y1], [Y2]). Moreover, in [Y2], the author proved that the dual semi-graph of a pointed stable curve in positive characteristic can be reconstructed group-theoretically from its pro-solvable admissible fundamental groups. As a corollary, we have that $\mathrm{Com}(\Gamma_{X_i^\bullet})$, $\mathrm{Typ}(X_i^\bullet)$ can be determined by $\mathrm{Geo}(\Pi_{X_i^\bullet})$, and that $\mathrm{Geo}(\Pi_{X_i^\bullet})$ can be reconstructed group-theoretically from $\Pi_{X_i^\bullet}$ (see [Y2, Theorem 0.3] or Theorem 2.6 and Remark 2.6.1 of the present paper for explanations).

1.3.3. *Specializations via fundamental groups.* For (ii) mentioned in 1.3.1 (this is the main topic of the present paper), we have the following conjectures (see 3.1.3 and Proposition 3.9 for more precise formulations and some other equivalent formulations):

Topological Specialization Conjecture . *Suppose that* $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ *is not empty. Then* $X_2^\bullet$ *is a degeneration (or reduction) of* $X_1^\bullet$ *as "topological spaces".*

Group-theoretical Specialization Conjecture . *Let* $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ *be an arbitrary open continuous homomorphism. Then we have*

$$\phi(\mathrm{Geo}(\Pi_{X_1^\bullet})) \subseteq \mathrm{Geo}(\Pi_{X_2^\bullet}).$$

Proposition 3.6 of the present paper says that the topological specialization conjecture can be deduced from the group-theoretical specialization conjecture.

New anabelian phenomena. Let us explain the anabelian phenomena concerning the above conjectures. Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary points such that q_2 is contained in $V(q_1)$, where $V(q_1)$ denotes the topological closure of q_1 in $\overline{M}_{g,n}$. Then there exist a complete discrete valuation ring R and a morphism $\mathrm{Spec}\, R \to \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ such that the image of the morphism is $\{q_1, q_2\}$. Let $\overline{\eta}$ and $\overline{s}$ be a geometric generic point and a geometric closed point over the generic point and the closed point of $\mathrm{Spec}\, R$, respectively. Write $\mathcal{X}^\bullet$ for the pointed stable curve over R determined by the morphism $\mathrm{Spec}\, R \to \overline{\mathcal{M}}_{g,n}$, $\mathcal{X}_\eta^\bullet$ for the generic fiber, $\mathcal{X}_s^\bullet$ for the special fiber, $X_{q_1}^\bullet$ for $X_{\overline{\eta}}^\bullet \overset{\mathrm{def}}{=} \mathcal{X}_\eta^\bullet \times_\eta \overline{\eta}$, and $X_{q_2}^\bullet$ for $X_{\overline{s}}^\bullet \overset{\mathrm{def}}{=} \mathcal{X}_s^\bullet \times_s \overline{s}$.

By the general theories of log geometry and admissible fundamental groups, we obtain a specialization surjective homomorphism of admissible fundamental groups (=an open continuous homomorphism of admissible fundamental groups arising from scheme theory, see [SGA1], [V])

$$sp_R^{\mathrm{adm}} : \pi_1^{\mathrm{adm}}(X_{q_1}^\bullet) \twoheadrightarrow \pi_1^{\mathrm{adm}}(X_{q_2}^\bullet).$$

Since $\mathcal{X}_s^\bullet$ is a reduction of $\mathcal{X}_\eta^\bullet$, the deformation theory of admissible coverings of $\mathcal{X}^\bullet$ implies that

$$sp_R^{\mathrm{adm}}(\mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_1}^\bullet))) \subseteq \mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_2}^\bullet)),$$

where $\mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_i}^\bullet))$, $i \in \{1, 2\}$, denotes the geometric data associated to $\pi_1^{\mathrm{adm}}(X_{q_i}^\bullet)$. For instance, let $\Pi_1 \in \mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_1}^\bullet))$ be a closed subgroup of $\pi_1^{\mathrm{adm}}(X_{q_1}^\bullet)$ associated to the pointed stable sub-curve $\widetilde{X}_{v_1}^\bullet$ determined by an irreducible component X_{v_1} of $X_{\overline{\eta}} = X_{q_1}$ (see 2.2.4 for $\widetilde{X}_{v_1}^\bullet$). Then $sp_R^{\mathrm{adm}}(\Pi_1)$ is a closed subgroup of $\pi_1^{\mathrm{adm}}(X_{q_2}^\bullet)$ associated to the pointed stable sub-curve determined by the *degeneration* (or *reduction*) of X_{v_1} in $X_{\overline{s}} = X_{q_2}$. This means that we have the following *geometric phenomena*:

- The combinatorial data $\mathrm{Com}(\Gamma_{X_{q_2}^\bullet})$ and the topological data $\mathrm{Typ}(X_{q_2}^\bullet)$ can be controlled by the combinatorial data $\mathrm{Com}(\Gamma_{X_{q_1}^\bullet})$ and the topological data $\mathrm{Typ}(X_{q_1}^\bullet)$ via the "*deformation*" $\mathcal{X}^\bullet$ of $X_{q_2}^\bullet$ over R *arising from scheme theory.*

- The geometric data $\mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_2}^\bullet))$ of $X_{q_2}^\bullet$ can be controlled by the geometric data $\mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_1}^\bullet))$ of $X_{q_1}^\bullet$ via an open continuous homomorphism sp_R^{adm} of admissible fundamental groups *arising from scheme theory*.

On the other hand, the topological specialization conjecture and group-theoretical specialization conjecture mean that there should exist the following *anabelian phenomena*:

- The combinatorial data $\mathrm{Com}(\Gamma_{X_{q_2}^\bullet})$ and the topological data $\mathrm{Typ}(X_{q_2}^\bullet)$ can be controlled by the combinatorial data $\mathrm{Com}(\Gamma_{X_{q_1}^\bullet})$ and the topological data $\mathrm{Typ}(X_{q_1}^\bullet)$ via the "*deformation*" $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{op}}(\Pi_{X_{q_1}^\bullet}, \Pi_{X_{q_2}^\bullet})$ explained in 1.2.1 which is *arose from group theory*.
- The geometry data $\mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_2}^\bullet))$ of $X_{q_2}^\bullet$ can be controlled by the geometry data $\mathrm{Geo}(\pi_1^{\mathrm{adm}}(X_{q_1}^\bullet))$ of $X_{q_1}^\bullet$ via an arbitrary open continuous homomorphism ϕ of admissible fundamental groups which is *arose from group theory*.

1.3.4. The topological specialization conjecture and group-theoretical specialization conjecture are very difficult. They are highly non-trivial even in the simplest case where $X_i^\bullet$, $i \in \{1, 2\}$, is *smooth* over k_i, $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{op}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) = \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ (this condition is equivalent to $\mathrm{Isom}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset$), and $\phi \in \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ is an *isomorphism*, where $\mathrm{Isom}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ denotes the set of isomorphisms of admissible fundamental groups. In this special case, the above conjectures are proved by Tamagawa which are the main results of [T4] (see [T4, Theorem 0.1 and Theorem 5.2]).

If we assume that $\mathrm{Hom}_{\mathrm{pg}}^{\mathrm{op}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) = \mathrm{Isom}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$, and that $\phi \in \mathrm{Isom}_{\mathrm{gp}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ is an isomorphism, then the group-theoretical specialization conjecture is equivalent to the so-called "*combinatorial Grothendieck conjecture*" which is the main conjecture in the theory of combinatorial anabelian geometry developed by Y. Hoshi and S. Mochizuki (e.g. [HM1], [HM2], [M2]) in characteristic 0, and by the author in characteristic p ([Y1], [Y2]). Thus, the group-theoretical specialization conjecture can be regarded as *the ultimate generalization of the combinatorial Grothendieck conjecture in characteristic p.*

On the other hand, the combinatorial Grothendieck conjecture is an "Isom-version" problem, and the group-theoretical specialization conjecture is a "Hom-version" problem. Similar to other theory in anabelian geometry, Hom-version problems are so much harder than the Isom-version problems.

1.3.5. *Main results.* Now, we give the main results of the present paper. For the topological specialization conjecture, we have the following result (see Theorem 4.9 for a more precise statement):

Theorem 1.1. *The topological specialization conjecture holds when $g_X = 0$.*

The group-theoretical specialization conjecture are so much harder than the topological specialization conjecture since we need to treat all open subgroups of admissible fundamental groups. On the other hand, we may ask the following question:

Problem 1.2. *Does the topological specialization conjecture imply the group-theoretical specialization conjecture?*

By applying Theorem 1.1, we have the following result (see Theorem 5.7 for a more precise statement):

Theorem 1.3. *Suppose that the topological specialization conjecture holds for arbitrary types. Then the group-theoretical specialization conjecture holds when $g_X = 0$.*

1.4. **Structure of the present paper.** The present paper is organized as follows.

In Section 2, we recall some notation concerning semi-graphs, pointed stable curves, and admissible fundamental groups. Moreover, we introduce combinatorial data, topological data, and geometric data.

In Section 3, we introduce the topological and group-theoretical specialization homomorphisms of admissible fundamental groups, and formulate the topological specialization conjecture and the group-theoretical specialization conjecture. Moreover, we prove some properties concerning topological and group-theoretical specialization homomorphisms.

In Section 4, we prove Theorem 1.1.

In Section 5, we prove Theorem 1.3.

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2. Geometric data associated to pointed stable curves

In this section, we recall some notation concerning pointed stable curves and their admissible fundamental groups. Moreover, we introduce the so-called *geometric data* associated to admissible fundamental groups of pointed stable curves.

2.1. **Semi-graphs.** In this subsection, we recall some notation concerning semi-graphs ([M1, Section 1]).

2.1.1. (a) Let

$$\mathbf{G} \overset{\mathrm{def}}{=} (v(\mathbf{G}), e(\mathbf{G}), \zeta^{\mathbf{G}} : e(\mathbf{G}) \to v(\mathbf{G}) \cup \{v(\mathbf{G})\})$$

be a semi-graph. Here, $v(\mathbf{G})$, $e(\mathbf{G})$, and $\zeta^{\mathbf{G}}$ denote the set of vertices of $\mathbf{G}$, the set of edges of $\mathbf{G}$, and the set of coincidence maps of $\mathbf{G}$, respectively. Note that $\{v(\mathbf{G})\}$ is a set with exactly one element.

Let $e \in e(\mathbf{G})$ be an edge. Then $e \overset{\mathrm{def}}{=} \{b_e^1, b_e^2\}$ is a set of cardinality 2 for each $e \in e(\mathbf{G})$. The set $e(\mathbf{G})$ consists of closed edges and open edges. If e is a closed edge, then the coincidence map $\zeta^{\mathbf{G}}$ is a map from e to the set of vertices to which e abuts. If e is an open edge, then the coincidence map $\zeta^{\mathbf{G}}$ is a map from e to the set which consists of the unique vertex to which e abuts and the set $\{v(\mathbf{G})\}$ (i.e. either $\zeta^{\mathbf{G}}(b_e^1)$ or $\zeta^{\mathbf{G}}(b_e^2)$ is not contained in $v(\mathbf{G})$).

(b) We shall write $e^{\mathrm{op}}(\mathbf{G}) \subseteq e(\mathbf{G})$ for the set of open edges of $\mathbf{G}$ and $e^{\mathrm{cl}}(\mathbf{G}) \subseteq e(\mathbf{G})$ for the set of closed edges of $\mathbf{G}$. Note that we have

$$e(\mathbf{G}) = e^{\mathrm{op}}(\mathbf{G}) \cup e^{\mathrm{cl}}(\mathbf{G}).$$

Moreover, we denote by $e^{\mathrm{lp}}(\mathbf{G}) \subseteq e^{\mathrm{cl}}(\mathbf{G})$ the subset of closed edges such that $\#(\zeta^{\mathbf{G}}(e)) = 1$ (i.e. a closed edge which abuts to a unique vertex of $\mathbf{G}$), where "lp" means "loop". For each $e \in e(\mathbf{G})$, we denote by $v^{\mathbf{G}}(e) \subseteq v(\mathbf{G})$ the set of vertices of $\mathbf{G}$ to which e abuts. For each $v \in v(\mathbf{G})$, we denote by $e^{\mathbf{G}}(v) \subseteq e(\mathbf{G})$ the set of edges of $\mathbf{G}$ to which v is abutted.

(c) We shall say $\mathbf{G}$ *connected* if $\mathbf{G}$ is connected as a topological space whose topology is induced by the topology of $\mathbb{R}^2$, where $\mathbb{R}$ denotes the real number field. We denote by $r_{\mathbf{G}} \overset{\mathrm{def}}{=} \dim_{\mathbb{Q}}(H^1(\mathbf{G}, \mathbb{Q}))$ the *Betti number* of $\mathbf{G}$, where $\mathbb{Q}$ denotes the rational number field. Moreover, we shall call $\mathbf{G}$ a *tree* if $r_{\mathbf{G}} = 0$.

Let $v \in v(\mathbf{G})$. We shall say that $\mathbf{G}$ is 2-connected at v if $\mathbf{G} \setminus \{v\}$ is either empty or connected. Moreover, we shall say that $\mathbf{G}$ is 2-*connected* if $\mathbf{G}$ is 2-connected at each $v \in v(\mathbf{G})$.

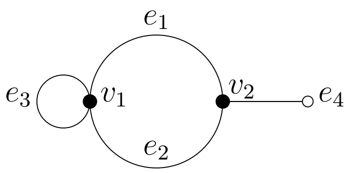
(d) We define an *one-point compactification* $\mathbf{G}^{\mathrm{cpt}}$ of $\mathbf{G}$ as follows: if $e^{\mathrm{op}}(\mathbf{G}) = \emptyset$, we put $\mathbf{G}^{\mathrm{cpt}} = \mathbf{G}$; otherwise, the set of vertices of $\mathbf{G}^{\mathrm{cpt}}$ is the disjoint union $v(\mathbf{G}^{\mathrm{cpt}}) \overset{\mathrm{def}}{=} v(\mathbf{G}) \sqcup \{v_\infty\}$, the set of closed edges of $\mathbf{G}^{\mathrm{cpt}}$ is $e^{\mathrm{cl}}(\mathbf{G}^{\mathrm{cpt}}) \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\mathbf{G}) \cup e^{\mathrm{cl}}(\mathbf{G})$, the set of open edges of $\mathbf{G}^{\mathrm{cpt}}$ is empty, and every edge $e \in e^{\mathrm{op}}(\mathbf{G}) \subseteq e^{\mathrm{cl}}(\mathbf{G}^{\mathrm{cpt}})$ connects v_∞ with the vertex that is abutted by e.

Remark. The motivations of the above notation concerning semi-graphs are the dual semi-graphs of pointed stable curves (see 2.2.1 below).

Example 2.1. Let us give an example of semi-graph to explain the above notation. We use the notation "$\bullet$" and "$\circ$ with a line segment" to denote a vertex and an open edge, respectively.

Let $\mathbf{G}$ be a semi-graph as follows:

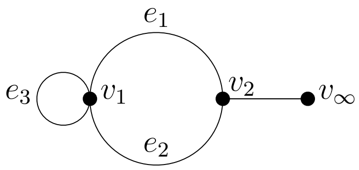
G:



Then we see $v(\mathbf{G}) = \{v_1, v_2\}$, $e^{\rm cl}(\mathbf{G}) = \{e_1, e_2, e_3\}$, $e^{\rm op}(\mathbf{G}) = \{e_4\}$, $\zeta^{\mathbf{G}}(e_1) = \{v_1, v_2\}$, $\zeta^{\mathbf{G}}(e_2) = \{v_1, v_2\}$, $\zeta^{\mathbf{G}}(e_3) = \{v_1\}$, and $\zeta^{\mathbf{G}}(e_4) = \{v_2, \{v(\mathbf{G})\}\}$. Moreover, we have $e^{\rm lp}(\mathbf{G}) = \{e_3\}$, $v^{\mathbf{G}}(e_1) = \{v_1, v_2\}$, $v^{\mathbf{G}}(e_2) = \{v_1, v_2\}$, $v^{\mathbf{G}}(e_3) = \{v_1\}$, $v^{\mathbf{G}}(e_4) = \{v_2\}$, $e^{\mathbf{G}}(v_1) = \{e_1, e_2, e_3\}$, and $e^{\mathbf{G}}(v_2) = \{e_1, e_2, e_4\}$.

Moreover, $\mathbf{G}^{\rm cpt}$ is the following:

$\mathbf{G}^{\rm cpt}$:



2.1.2. (a) Let $\mathbf{G}'$ be a connected semi-graph. We shall say $\mathbf{G}'$ a *sub-semi-graph* of $\mathbf{G}$ if either $\mathbf{G}' = \{e\}$ for some $e \in e(\mathbf{G})$ or the following conditions hold:

(i) $v(\mathbf{G}') \neq \emptyset$ and $v(\mathbf{G}') \subseteq v(\mathbf{G})$.
(ii) $e^{\rm cl}(\mathbf{G}') \subseteq e^{\rm cl}(\mathbf{G})$ is the subset of closed edges such that $v^{\mathbf{G}}(e) \subseteq v(\mathbf{G}')$.
(iii) $e^{\rm op}(\mathbf{G}') \subseteq (e^{\rm cl}(\mathbf{G}) \cup e^{\rm op}(\mathbf{G})) \setminus e^{\rm cl}(\mathbf{G}')$ is the subset of edges such that $\#(v^{\mathbf{G}}(e) \cap v(\mathbf{G}')) = 1$.

Note that the definition of $\mathbf{G}'$ implies that $\mathbf{G}'$ can be completely determined by $v(\mathbf{G}')$ if $v(\mathbf{G}') \neq \emptyset$.

The condition (ii) implies that, if $e \in e^{\rm lp}(\mathbf{G})$ is a loop and $v^{\mathbf{G}}(e) \subseteq v(\mathbf{G}')$, then $e \in e^{\rm cl}(\mathbf{G}')$. If $\mathbf{G}' = \{e\}$ for some $e \in e(\mathbf{G})$, we will use e to denote $\mathbf{G}'$. Moreover, there exists a natural injection $\mathbf{G}' \hookrightarrow \mathbf{G}$, and $\mathbf{G}'$ can be regarded as *a topological subspace* of $\mathbf{G}$ via this injection.

(b) Suppose that $\mathbf{G}'$ is a sub-semi-graph of $\mathbf{G}$ such that $v(\mathbf{G}') \neq \emptyset$. Let $L \subseteq e^{\rm cl}(\mathbf{G}')$ be a *subset of closed edges* of $\mathbf{G}'$ such that $\mathbf{G}' \setminus L$ (i.e. removing L from $\mathbf{G}'$) is *connected*. For any $e \overset{\rm def}{=} \{b_e^1, b_e^2\} \in L$, we put $e^i \overset{\rm def}{=} \{b_{e^i}^1, b_{e^i}^2\}$, $i \in \{1, 2\}$, and shall call e^i the i-edge associated to e. We shall say that $\mathbf{G}'_L$ is *the semi-graph associated to* $\mathbf{G}'$ *and* L if the following conditions hold:

(i) $v(\mathbf{G}'_L) \overset{\rm def}{=} v(\mathbf{G}')$.
(ii) $e^{\rm op}(\mathbf{G}'_L) \overset{\rm def}{=} e^{\rm op}(\mathbf{G}') \cup \{e^1, e^2\}_{e \in L}$, $\zeta^{\mathbf{G}'_L}(e) = \{\zeta^{\mathbf{G}'}(b_e^1), \{v(\mathbf{G}'_L)\}\}$ if $e = \{b_e^1, b_{e_2}^2\} \in e^{\rm op}(\mathbf{G}')$ and $\zeta^{\mathbf{G}'}(b_e^1) \in v(\mathbf{G}')$, $\zeta^{\mathbf{G}'_L}(e^1) \overset{\rm def}{=} \{\zeta^{\mathbf{G}'}(b_e^1), \{v(\mathbf{G}'_L)\}\}$ if e^1 is the 1-edge associated to $e \in L$, and $\zeta^{\mathbf{G}'_L}(e^2) \overset{\rm def}{=} \{\zeta^{\mathbf{G}'}(b_e^2), \{v(\mathbf{G}'_L)\}\}$ if e^2 is the 2-edge associated to $e \in L$.
(iii) $e^{\rm cl}(\mathbf{G}'_L) \overset{\rm def}{=} e^{\rm cl}(\mathbf{G}') \setminus L$, and $\zeta^{\mathbf{G}'_L}(e) \overset{\rm def}{=} \zeta^{\mathbf{G}'}(e)$ if $e \in e^{\rm cl}(\mathbf{G}') \setminus L$.

Then we have a natural map of semi-graphs

$$\delta_{(\mathbf{G}', L)} : \mathbf{G}'_L \to \mathbf{G}'$$

which is defined as follows:

- $\delta_{(\mathbf{G}', L)}(v) = v$ for $v \in v(\mathbf{G}'_L)$.
- $\delta_{(\mathbf{G}', L)}(e) = e$ for $e \in e(\mathbf{G}'_L) \setminus \{e^1, e^2\}_{e \in L}$.
- $\delta_{(\mathbf{G}', L)}(e^i) = e$, $i \in \{1, 2\}$, for i-edge associated to $e \in L$.

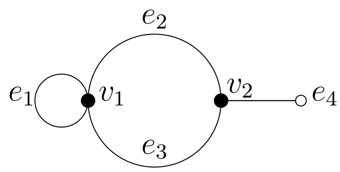
Moreover, we put $\delta_{\mathbf{G}'_L} : \mathbf{G}'_L \overset{\delta_{(\mathbf{G}', L)}}{\to} \mathbf{G}' \hookrightarrow \mathbf{G}$ the composition of maps of semi-graphs. Note that $\delta_{\mathbf{G}'_L}|_{\mathbf{G}'_L \setminus \{e^1, e^2\}_{e \in L}}$ is an injection.

Remark. The motivations of the above notation concerning semi-graphs are the dual semi-graphs of pointed stable sub-curves (see 2.2.3, 2.2.4, and 2.2.5 below).

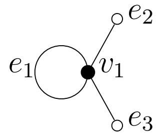
Example 2.2. We give some examples of semi-graphs to explain the above notation. We use the notation "$\bullet$" and "$\circ$" to denote a vertex and an open edge, respectively.

Let $\mathbf{G}$ be a semi-graph, $\mathbf{G}'$ the sub-semi-graph of $\mathbf{G}$ such that $v(\mathbf{G}') = \{v_1\}$, and $L \overset{\text{def}}{=} \{e_1\} \subseteq e^{\text{cl}}(\mathbf{G}')$ a subset of edges of $\mathbf{G}'$ and $\{e_1^1, e_1^2\}$ the set of 1-edge and 2-edge associated to e_1. Then we have the following:

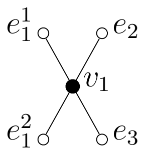
$\mathbf{G}$:



$\mathbf{G}'$:



$\mathbf{G}'_L$:



2.2. **Pointed stable curves and admissible fundamental groups.** In this subsection, we recall some notation concerning pointed stable curves and their admissible fundamental groups.

2.2.1. Let p be a prime number, and let

$$X^\bullet = (X, D_X)$$

be a pointed stable curve over an algebraically closed field k of characteristic p, where X denotes the underlying curve and D_X denotes a finite set of marked points satisfying [K, Definition 1.1 (iv)]. Write g_X for the genus of X and n_X for the cardinality $\#(D_X)$ of D_X. We shall call (g_X, n_X) the *topological type* (or type for short) of $X^\bullet$.

Recall that the *dual semi-graph*

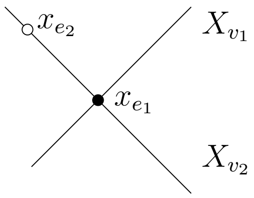
$$\Gamma_{X^\bullet} \overset{\text{def}}{=} (v(\Gamma_{X^\bullet}), e(\Gamma_{X^\bullet}), \zeta^{\Gamma_{X^\bullet}})$$

of $X^\bullet$ is a semi-graph associated to $X^\bullet$ defined as follows:

(i) $v(\Gamma_{X^\bullet})$ is the set of irreducible components of X.
(ii) $e^{\text{op}}(\Gamma_{X^\bullet})$ is the set of marked points D_X.
(iii) $e^{\text{cl}}(\Gamma_{X^\bullet})$ is the set of singular points (or nodes) X^{sing} of X.
(iv) $\zeta^{\Gamma_{X^\bullet}}(e)$, $e \in e^{\text{op}}(\Gamma_{X^\bullet})$, consists of the set $\{v(\Gamma_{X^\bullet})\}$ and the unique irreducible component containing e.
(v) $\zeta^{\Gamma_{X^\bullet}}(e)$, $e \in e^{\text{cl}}(\Gamma_{X^\bullet})$, consists of the irreducible components containing e.

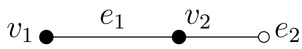
Example 2.3. We give an example to explain dual semi-graphs of pointed stable curves. Let $X^\bullet$ be a pointed stable curve over k whose irreducible components are X_{v_1} and X_{v_2}, whose node is x_{e_1}, and whose marked point is $x_{e_2} \in X_{v_2}$. We use the notation "$\bullet$" and "$\circ$" to denote a node and a marked point, respectively. Then $X^\bullet$ is as follows:

$X^\bullet$:



We write v_1 and v_2 for the vertices of $\Gamma_{X^\bullet}$ corresponding to X_{v_1} and X_{v_2}, respectively, e_1 for the closed edge corresponding to x_{e_1}, and e_2 for the open edge corresponding to x_{e_2}. Moreover, we use the notation "$\bullet$" and "$\circ$ with a line segment" to denote a vertex and an open edge, respectively. Then the dual semi-graph $\Gamma_{X^\bullet}$ of $X^\bullet$ is as follows:

$\Gamma_{X^\bullet}$:



2.2.2. By choosing a base point $x \in X \setminus X^{\mathrm{sm}}$ of $X^\bullet$, where X^{sm} denotes the smooth locus of X, we have the *admissible fundamental group* (see [Y4, Section 2] or [Y5, Section 1.1 and Section 1.2] for the definitions of admissible coverings and admissible fundamental groups)

$$\pi_1^{\mathrm{adm}}(X^\bullet, x)$$

of $X^\bullet$. In the present paper, since we only focus on the isomorphism class of $\pi_1^{\mathrm{adm}}(X^\bullet, x)$, we omit the base point x and write $\pi_1^{\mathrm{adm}}(X^\bullet)$ for $\pi_1^{\mathrm{adm}}(X^\bullet, x)$. Moreover, we put $\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$ the *maximal pro-solvable quotient* of $\pi_1^{\mathrm{adm}}(X^\bullet)$. We shall write $\pi_1^{\text{ét}}(X)$, $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})$, $\pi_1^{\text{ét}}(X)^{\mathrm{sol}}$, and $\pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}$ for the étale fundamental group of X, the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, the maximal pro-solvable quotient of $\pi_1^{\text{ét}}(X^\bullet)$, and the maximal pro-solvable quotient of $\pi_1^{\mathrm{top}}(X^\bullet)$, respectively.

From now on, we denote by

$$\Pi_{X^\bullet}$$

either $\pi_1^{\mathrm{adm}}(X^\bullet)$ *or* $\pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$ *unless indicated otherwise.* If $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)$, we denote by

$$\Pi_{X^\bullet}^{\text{ét}} \overset{\mathrm{def}}{=} \pi_1^{\text{ét}}(X),\ \Pi_{X^\bullet}^{\mathrm{top}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet}).$$

If $\Pi_{X^\bullet} = \pi_1^{\mathrm{adm}}(X^\bullet)^{\mathrm{sol}}$, we denote by

$$\Pi_{X^\bullet}^{\text{ét}} \overset{\mathrm{def}}{=} \pi_1^{\text{ét}}(X)^{\mathrm{sol}},\ \Pi_{X^\bullet}^{\mathrm{top}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(\Gamma_{X^\bullet})^{\mathrm{sol}}.$$

Then we have the following natural surjections

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{top}}.$$

Let $H \subseteq \Pi_{X^\bullet}$ be an arbitrary open subgroup. We write $X_H^\bullet$ for the pointed stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H and $\Gamma_{X_H^\bullet}$ for the dual semi-graph of $X_H^\bullet$. Then we obtain an admissible covering

$$f_H^\bullet : X_H^\bullet \to X^\bullet$$

over k induced by the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, and obtain a natural morphism of dual semi-graphs

$$f_H^{\mathrm{sg}} : \Gamma_{X_H^\bullet} \to \Gamma_{X^\bullet}$$

induced by $f_H^\bullet$, where "sg" means "semi-graph". We shall say that $f_H^\bullet$ is *étale* if the underlying morphism $f_H : X_H \to X$ induced by $f_H^\bullet$ is étale.

Moreover, if H is an open *normal* subgroup, then $\Gamma_{X_H^\bullet}$ admits an action of $\Pi_{X^\bullet}/H$ induced by the natural action of $\Pi_{X^\bullet}/H$ on $X_H^\bullet$. Note that the quotient of $\Gamma_{X_H^\bullet}$ by $\Pi_{X^\bullet}/H$ coincides with $\Gamma_{X^\bullet}$, and that H is isomorphic to the admissible fundamental group $\Pi_{X_H^\bullet}$ of $X_H^\bullet$. We also use the notation $H^{\text{ét}}$ and H^{top} to denote $\Pi^{\text{ét}}_{X_H^\bullet}$ and $\Pi^{\text{top}}_{X_H^\bullet}$, respectively.

2.2.3. We define pointed stable curves associated to various semi-graphs introduced in 2.1.2. Let $\Gamma \subseteq \Gamma_{X^\bullet}$ be a sub-semi-graph (2.1.2 (a)). We write X_Γ for the semi-stable sub-curve of X (i.e. a closed subscheme of X which is a semi-stable curve) whose irreducible components are the irreducible components corresponding to the vertices of $v(\Gamma)$, and whose nodes are the nodes corresponding to the edges of $e^{\text{cl}}(\Gamma)$. Moreover, write D_{X_Γ} for the set of closed points $X_\Gamma \cap \{x_e\}_{e\in e^{\text{op}}(\Gamma)\subseteq e(\Gamma_{X^\bullet})}$, where $x_e \in X$ denotes the closed point corresponding to $e \in e(\Gamma_{X^\bullet})$. We define a pointed stable curve of type (g_Γ, n_Γ) over k to be

$$X_\Gamma^\bullet = (X_\Gamma, D_{X_\Gamma}).$$

Note that the dual semi-graph of $X_\Gamma^\bullet$ is naturally isomorphic to Γ. We shall call $X_\Gamma^\bullet$ *the pointed stable curve of type* (g_Γ, n_Γ) *associated to* Γ. We denote by $\Pi_{X_\Gamma^\bullet}$ the admissible fundamental group of $X_\Gamma^\bullet$.

2.2.4. Let $\Gamma \subseteq \Gamma_{X^\bullet}$ be a sub-semi-graph and $L \subseteq e^{\text{cl}}(\Gamma)$ such that $\Gamma \setminus L$ is connected. Let Γ_L be the semi-graph associated to Γ and L (2.1.2 (b)), and $\text{Node}_L(X_\Gamma) \subseteq X_\Gamma^{\text{sing}}$ the set of nodes of X_Γ corresponding to L. We write $\text{nor}_L : X_{\Gamma_L} \to X_\Gamma$ for the normalization of X_Γ at $\text{Node}_L(X_\Gamma)$. Moreover, we put $D_{X_{\Gamma_L}} \overset{\text{def}}{=} \text{nor}_L^{-1}(D_{X_\Gamma} \cup \text{Node}_L(X_\Gamma))$. We define a pointed stable curve of type $(g_{\Gamma_L}, n_{\Gamma_L})$ to be

$$X_{\Gamma_L}^\bullet = (X_{\Gamma_L}, D_{\Gamma_L}).$$

Note that the dual semi-graph of $X_{\Gamma_L}^\bullet$ is naturally isomorphic to Γ_L. We shall call $X_{\Gamma_L}^\bullet$ *the pointed stable curve of type* $(g_{\Gamma_L}, n_{\Gamma_L})$ *associated to* Γ_L. By the construction of $X_{\Gamma_L}^\bullet$, we see

$$r_{\Gamma_L} = r_\Gamma - \#(L),\ g_{\Gamma_L} = g_\Gamma - \#(L),\ n_{\Gamma_L} = n_\Gamma + 2\#(L).$$

We denote by $\Pi_{X_{\Gamma_L}^\bullet}$ the admissible fundamental group of $X_{\Gamma_L}^\bullet$. Moreover, we have the following natural outer injections (i.e. up to inner automorphism of $\Pi_{X^\bullet}$)

$$\Pi_{X_{\Gamma_L}^\bullet} \hookrightarrow \Pi_{X_\Gamma^\bullet} \hookrightarrow \Pi_{X^\bullet}.$$

2.2.5. Let $v \in v(\Gamma_{X^\bullet})$ and $\Gamma_v \subseteq \Gamma_{X^\bullet}$ the sub-semi-graph such that $v(\Gamma_v) = \{v\}$. Let $e^{\text{lp}}(\Gamma_v)$ be the set of loops of Γ_v (2.1.1 (b)). Note that in this situation, we have $e^{\text{lp}}(\Gamma_v) = e^{\text{cl}}(\Gamma_v)$. Write X_v for the irreducible component corresponding to v and $\text{nor}_v : \widetilde{X}_v \to X_v$ for the normalization of X_v. We put $D_{\widetilde{X}_v} \overset{\text{def}}{=} \text{nor}_v^{-1}((D_X \cap X_v) \cup (X_v \cap X^{\text{sing}}))$. Then we have $\widetilde{X}_v = X_{(\Gamma_v)_{e^{\text{lp}}(\Gamma_v)}}$ and $D_{\widetilde{X}_v} = D_{X_{(\Gamma_v)_{e^{\text{lp}}(\Gamma_v)}}}$. Moreover, we shall call

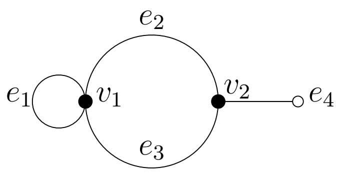
$$\widetilde{X}_v^\bullet \overset{\text{def}}{=} (\widetilde{X}_v, D_{\widetilde{X}_v}) = X^\bullet_{(\Gamma_v)_{e^{\text{lp}}(\Gamma_v)}}$$

the smooth pointed stable curve of type $(g_v, n_v) \overset{\text{def}}{=} (g_{(\Gamma_v)_{e^{\text{lp}}(\Gamma_v)}}, n_{(\Gamma_v)_{e^{\text{lp}}(\Gamma_v)}})$ *associated to* v. If X_v is smooth over k, for simplicity, we use the notation $X_v^\bullet$ to denote $\widetilde{X}_v^\bullet = X_{\Gamma_v}^\bullet$. We denote by $\Pi_{\widetilde{X}_v^\bullet}$ the admissible fundamental group of $\widetilde{X}_v^\bullet$. Suppose that Γ_v is contained in a sub-semi-graph $\Gamma \subseteq \Gamma_{X^\bullet}$. Then we have the following natural outer injections

$$\Pi_{\widetilde{X}_v^\bullet} \hookrightarrow \Pi_{X_{\Gamma_v}^\bullet} \hookrightarrow \Pi_{X_\Gamma^\bullet} \hookrightarrow \Pi_{X^\bullet}.$$

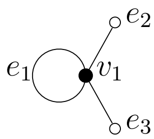
Example 2.4. Suppose that the dual semi-graph $\Gamma_{X^\bullet}$ is as follows:

$\Gamma_{X^\bullet}$:

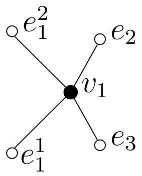


Then we have

$\Gamma_{v_1} = \Gamma_{X^\bullet_{\Gamma_{v_1}}}$:



$\Gamma_{\widetilde{X}^\bullet_{v_1}} = \Gamma_{X^\bullet_{(\Gamma_{v_1})_{e^{\mathrm{lp}}(\Gamma_{v_1})}}}$:



2.3. **Geometric data.** In this subsection, we introduce various subgroups of $\Pi_{X^\bullet}$ which can be regarded as group-theoretical descriptions of pointed stable curves defined in 2.2.3, 2.2.4, and 2.2.5.

2.3.1. **Settings.** Let $X^\bullet = (X, D_X)$ be a pointed stable curve of type (g_X, n_X) over an algebraically closed field k of characteristic $p > 0$, $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$, and $\Pi_{X^\bullet}$ either the admissible fundamental group of $X^\bullet$ or the maximal pro-solvable quotient of the admissible fundamental group of $X^\bullet$.

Write $\mathfrak{P}$ for the set of prime numbers. Let Π be a profinite group, and let $\Sigma \subseteq \mathfrak{P}$ be either the set $\mathfrak{P}$ or a subset such that $p \notin \Sigma$. We denote by Π^Σ the maximal pro-Σ quotient of Π^Σ. In particular, if $\Sigma = \mathfrak{P}$ (resp. $\mathfrak{P} \setminus \{p\}$), we use the notation Π (resp. $\Pi^{p'}$) to denote $\Pi^{\mathfrak{P}}$ (resp. $\Pi^{\mathfrak{P}\setminus\{p\}}$).

2.3.2. We put

$$\widehat{X} \overset{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi^\Sigma_{X^\bullet} \text{ open}} X_H, \; D_{\widehat{X}} \overset{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi^\Sigma_{X^\bullet} \text{ open}} D_{X_H}, \; \widehat{\Gamma}_{X^\bullet} \overset{\mathrm{def}}{=} \varprojlim_{H \subseteq \Pi^\Sigma_{X^\bullet} \text{ open}} \Gamma_{X^\bullet_H}.$$

We shall call

$$\widehat{X}^\bullet = (\widehat{X}, D_{\widehat{X}})$$

the universal admissible covering associated to $\Pi^\Sigma_{X^\bullet}$, and $\widehat{\Gamma}_{X^\bullet}$ the dual semi-graph of $\widehat{X}^\bullet$ which is a simply connected topological space. Note that we have that $\mathrm{Aut}(\widehat{X}^\bullet/X^\bullet) = \Pi^\Sigma_{X^\bullet}$, and that $\widehat{\Gamma}_{X^\bullet}$ admits a natural action of $\Pi^\Sigma_{X^\bullet}$. We denote by

$$\pi_X : \widehat{\Gamma}_{X^\bullet} \twoheadrightarrow \Gamma_{X^\bullet}$$

the natural surjection.

2.3.3. Let $\Gamma \subseteq \Gamma_{X^\bullet}$ be a sub-semi-graph (2.1.2 (a)) and $L \subseteq e^{\mathrm{cl}}(\Gamma)$ a subset of closed edges of Γ such that $\Gamma \setminus L$ is connected. Then we have the semi-graph Γ_L associated to Γ and L (2.1.2 (b)). Let $\widehat{\Gamma} \subseteq \widehat{\Gamma}_{X^\bullet}$ be a connected component of $\pi_X^{-1}(\Gamma)$ and $\widehat{\Gamma \setminus L}$ a connected component of $\pi_X^{-1}(\Gamma \setminus L)$. We denote by

$$\Pi_{\widehat{\Gamma}} \overset{\mathrm{def}}{=} \{\sigma \in \Pi^\Sigma_{X^\bullet} \mid \sigma(\widehat{\Gamma}) = \widehat{\Gamma}\} \subseteq \Pi^\Sigma_{X^\bullet},$$

$$\Pi_{\widehat{\Gamma}_L} \overset{\text{def}}{=} \{\sigma \in \Pi^{\Sigma}_{X^\bullet} \mid \sigma(\widehat{\Gamma \setminus L}) = \widehat{\Gamma \setminus L}\} \subseteq \Pi^{\Sigma}_{X^\bullet}$$

the stabilizer subgroups (or the decomposition subgroups) of $\widehat{\Gamma}$ and $\widehat{\Gamma \setminus L}$ under the action of $\Pi^{\Sigma}_{X^\bullet}$ on $\widehat{\Gamma}_{X^\bullet}$, respectively. Note that the conjugacy class of $\Pi_{\widehat{\Gamma}}$ (resp. $\Pi_{\widehat{\Gamma}_L}$) does not depend on the choices of $\widehat{\Gamma}$ (resp. $\widehat{\Gamma \setminus L}$).

Let $v \in v(\Gamma_{X^\bullet})$ and $\widehat{v} \in \pi_X^{-1}(v)$. We denote by $\Pi_{\widehat{v}} \subseteq \Pi^{\Sigma}_{X^\bullet}$ the stabilizer subgroup of $\widehat{v}$ under the action of $\Pi^{\Sigma}_{X^\bullet}$ on $\widehat{\Gamma}_{X^\bullet}$. We see

$$\Pi_{\widehat{v}} = \Pi_{\widehat{\Gamma}_L}$$

if $\Gamma = \Gamma_v$, $L = e^{\mathrm{lp}}(\Gamma_v)$, and $\widehat{v} \in \widehat{\Gamma \setminus L}$.

2.3.4. By the theory of admissible fundamental groups, the following facts are well-known: $\Pi_{\widehat{\Gamma}}$ is isomorphic to $\Pi^{\Sigma}_{X^\bullet_\Gamma}$, and $\Pi_{\widehat{\Gamma}_L}$ is isomorphic to $\Pi^{\Sigma}_{X^\bullet_{\Gamma_L}}$ (this is the reason that we *do not* use the notation $\Pi_{\widehat{\Gamma \setminus L}}$ to denote the stabilizer subgroup of $\widehat{\Gamma \setminus L}$). In particular, $\Pi_{\widehat{v}}$ is outer isomorphic to $\Pi^{\Sigma}_{\widetilde{X}^\bullet_v}$ for all $v(\Gamma_{X^\bullet})$. Note that we have the following natural injections

$$\Pi_{\widehat{\Gamma}_L} \hookrightarrow \Pi_{\widehat{\Gamma}} \hookrightarrow \Pi^{\Sigma}_{X^\bullet}$$

if $\widehat{\Gamma \setminus L} \subseteq \widehat{\Gamma}$. Let $e \in e(\Gamma_{X^\bullet})$ and $\widehat{e} \in \pi_X^{-1}(e)$. Then $I_{\widehat{e}} \overset{\text{def}}{=} \Pi_{\widehat{e}} \overset{\sim}{\to} \widehat{\mathbb{Z}}(1)^{\Sigma \setminus \{p\}}$ is isomorphic to an inertia subgroup associated to the closed point of X corresponding to e.

Moreover, let $v \in v(\Gamma)$ and $e \in e(\Gamma_v)$ such that $\widehat{e}$ abuts to $\widehat{v}$, and that $\widehat{\Gamma}_v \subseteq \widehat{\Gamma}$. Then we have the following natural injections

$$I_{\widehat{e}} \hookrightarrow \Pi_{\widehat{v}} \hookrightarrow \Pi_{\widehat{\Gamma}_v} \hookrightarrow \Pi_{\widehat{\Gamma}_L} \hookrightarrow \Pi_{\widehat{\Gamma}} \hookrightarrow \Pi^{\Sigma}_{X^\bullet}.$$

Note that $\Pi_{\widehat{v}} \overset{\sim}{\to} \Pi_{\widehat{\Gamma}_v}$ if X_v is non-singular.

2.3.5. We denote by $\mathrm{Ssg}(\Gamma_{X^\bullet})$ the set of sub-semi-graphs of $\Gamma_{X^\bullet}$ and put

$$\mathrm{Com}(\Gamma_{X^\bullet}) \overset{\text{def}}{=} \{(\Gamma, L) \mid \Gamma \setminus L \text{ is connected}\}_{\Gamma \in \mathrm{Ssg}(\Gamma_{X^\bullet}), L \subseteq e^{\mathrm{cl}}(\Gamma)},$$

where "Ssg" means "sub-semi-graph", and "Com" means "combinatorial", and L is possibly an empty set. Furthermore, we put

$$\mathrm{Ssg}(\Pi^{\Sigma}_{X^\bullet}) \overset{\text{def}}{=} \{\Pi_{\widehat{\Gamma}}\}_{\Gamma \in \mathrm{Ssg}(\Gamma_{X^\bullet})} \subseteq \mathrm{Geo}(\Pi^{\Sigma}_{X^\bullet}) \overset{\text{def}}{=} \{\Pi_{\widehat{\Gamma}_L}\}_{(\Gamma, L) \in \mathrm{Com}(\Gamma_{X^\bullet})},$$

where "Geo" means "geometry". In particular, we denote by

$$\mathrm{Ver}(\Pi^{\Sigma}_{X^\bullet}) \overset{\text{def}}{=} \{\Pi_{\widehat{v}}\}_{\widehat{v} \in v(\Gamma_{\widehat{X}^\bullet})} \subseteq \mathrm{Geo}(\Pi^{\Sigma}_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi^{\Sigma}_{X^\bullet}) \overset{\text{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\mathrm{op}}(\Gamma_{\widehat{X}^\bullet})} \subseteq \mathrm{Ssg}(\Pi^{\Sigma}_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi^{\Sigma}_{X^\bullet}) \overset{\text{def}}{=} \{I_{\widehat{e}}\}_{\widehat{e} \in e^{\mathrm{cl}}(\Gamma_{\widehat{X}^\bullet})} \subseteq \mathrm{Ssg}(\Pi^{\Sigma}_{X^\bullet}).$$

Note that $\mathrm{Ssg}(\Pi^{\Sigma}_{X^\bullet})$, $\mathrm{Geo}(\Pi^{\Sigma}_{X^\bullet})$, $\mathrm{Ver}(\Pi^{\Sigma}_{X^\bullet})$, $\mathrm{Edg}^{\mathrm{op}}(\Pi^{\Sigma}_{X^\bullet})$, and $\mathrm{Edg}^{\mathrm{cl}}(\Pi^{\Sigma}_{X^\bullet})$ admit natural actions of $\Pi^{\Sigma}_{X^\bullet}$ (i.e. the conjugacy actions). Moreover, we have the following natural bijections

$$\mathrm{Geo}(\Pi^{\Sigma}_{X^\bullet})/\Pi^{\Sigma}_{X^\bullet} \overset{\sim}{\to} \mathrm{Com}(\Gamma_{X^\bullet}),$$

$$\mathrm{Ssg}(\Pi^{\Sigma}_{X^\bullet})/\Pi^{\Sigma}_{X^\bullet} \overset{\sim}{\to} \mathrm{Ssg}(\Gamma_{X^\bullet}),$$

$$\mathrm{Ver}(\Pi^{\Sigma}_{X^\bullet})/\Pi^{\Sigma}_{X^\bullet} \overset{\sim}{\to} v(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{op}}(\Pi^{\Sigma}_{X^\bullet})/\Pi^{\Sigma}_{X^\bullet} \overset{\sim}{\to} e^{\mathrm{op}}(\Gamma_{X^\bullet}),$$

$$\mathrm{Edg}^{\mathrm{cl}}(\Pi^{\Sigma}_{X^\bullet})/\Pi^{\Sigma}_{X^\bullet} \overset{\sim}{\to} e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

2.3.6. We define combinatorial data, topological data, and geometric data associated to pointed stable curves and their admissible fundamental groups, respectively, as follows:

Definition 2.5. (a) We shall call $\mathrm{Com}(\Gamma_{X^\bullet})$ the *combinatorial data* associated to $X^\bullet$,

$$\mathrm{Typ}(X^\bullet) \overset{\mathrm{def}}{=} \{(g_{\Gamma_L}, n_{\Gamma_L})\}_{(\Gamma,L)\in \mathrm{Com}(\Gamma_{X^\bullet})}$$

the *topological data* associated to $X^\bullet$, and $\mathrm{Geo}(\Pi^\Sigma_{X^\bullet})$ the *geometric data* associated to $\Pi^\Sigma_{X^\bullet}$.

(b) Let $(\Gamma, L) \in \mathrm{Com}(\Gamma_{X^\bullet})$ be a combinatorial datum, Γ_L the semi-graph associated to Γ and L, $\widehat{\Gamma \setminus L} \subseteq \pi_X^{-1}(\Gamma \setminus L)$ a connected component, and $\Pi_{\widehat{\Gamma}_L} (\subseteq \Pi^\Sigma_{X^\bullet}) \in \mathrm{Geo}(\Pi^\Sigma_{X^\bullet})$ the stabilizer subgroup of $\widehat{\Gamma \setminus L}$.

We shall call $\Pi_{\widehat{\Gamma}_L}$ a *geometry-like subgroup* of $\Pi^\Sigma_{X^\bullet}$ associated to Γ_L (or the geometry-like subgroup of $\Pi^\Sigma_{X^\bullet}$ associated to $\widehat{\Gamma \setminus L}$). In particular, we have the following: If $\Gamma = \Gamma_v$ and $L = e^{\mathrm{lp}}(\Gamma_v)$ for some $v \in v(\Gamma_{X^\bullet})$, we shall call $\Pi_{\widehat{v}} \in \mathrm{Ver}(\Pi^\Sigma_{X^\bullet})$ a *vertex-like subgroup* of $\Pi^\Sigma_{X^\bullet}$ associated to v (or the vertex-like subgroup of $\Pi^\Sigma_{X^\bullet}$ associated to $\widehat{v}$). If $\Gamma = \{e\}$ for some $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ and $L = \emptyset$, we shall call $I_{\widehat{e}} \in \mathrm{Edg}^{\mathrm{op}}(\Pi^\Sigma_{X^\bullet})$ an *open-edge-like subgroup* of $\Pi^\Sigma_{X^\bullet}$ associated to e (or the open-edge-like subgroup of $\Pi^\Sigma_{X^\bullet}$ associated to $\widehat{e}$). If $\Gamma = \{e\}$ for some $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ and $L = \emptyset$, we shall call $I_{\widehat{e}} \in \mathrm{Edg}^{\mathrm{cl}}(\Pi^\Sigma_{X^\bullet})$ a *closed-edge-like subgroup* of $\Pi^\Sigma_{X^\bullet}$ associated to e (or the closed-edge-like subgroup of $\Pi^\Sigma_{X^\bullet}$ associated to $\widehat{e}$).

Remark 2.5.1. Let us explain the geometric motivation of Definition 2.5. One of main goals of the theory of anabelian geometry is to prove that algebraic varieties can be completely determined group-theoretically from various versions of their algebraic fundamental groups. Then for a given algebraic variety, before we start to study the anabelian properties of the algebraic variety, we need to find the corresponding group-theoretical descriptions of its geometry informations (i.e. descriptions of its geometric informations by using closed subgroups of its algebraic fundamental group).

In the case of pointed stable curves, Definition 2.5 means that the conjugacy class

$$\{\sigma^{-1}\Pi_{\widehat{\Gamma}_L}\sigma\}_{\sigma\in\Pi^\Sigma_{X^\bullet}}$$

corresponds to the pointed stable curve of type $(g_{\Gamma_L}, n_{\Gamma_L})$ associated to Γ_L defined in 2.2.4.

For the geometric data, we have the following result.

Theorem 2.6. *We maintain the notation introduced in Definition 2.5. Suppose $\Sigma = \mathfrak{P}$. Then there exists a group-theoretical algorithm whose input datum is $\Pi_{X^\bullet}$, and whose output data are* $\mathrm{Geo}(\Pi_{X^\bullet})$, $\mathrm{Com}(\Gamma_{X^\bullet})$, *and* $\mathrm{Typ}(X^\bullet)$. *In particular,* $\Pi \in \mathrm{Geo}(\Pi_{X^\bullet})$ *determines group-theoretically a unique element* $(\Gamma_\Pi, L_\Pi) \in \mathrm{Com}(\Gamma_{X^\bullet})$ *and a unique element* $(g_\Pi, n_\Pi) \overset{\mathrm{def}}{=} (g_{(\Gamma_\Pi)_{L_\Pi}}, n_{(\Gamma_\Pi)_{L_\Pi}}) \in \mathrm{Typ}(X^\bullet)$.

Remark 2.6.1. Suppose that $X^\bullet$ is *smooth* (in this situation, $\mathrm{Geo}(\Pi_{X^\bullet}) = \{\Pi_{X^\bullet}\} \cup \mathrm{Edg}^{\mathrm{op}}(\Pi_{X^\bullet})$). Then Theorem 2.6 was proved by Tamagawa ([T4, Theorem 0.1 and Theorem 5.2]). Moreover, this result is the most important (and the most difficult) step in his proof of *the weak Isom-version of the Grothendieck conjecture for (tame fundamental groups!) of smooth curves of genus* 0 *over an algebraic closure of* $\mathbb{F}_p$ ([T4, Theorem 0.2]).

Suppose that $X^\bullet$ is an *arbitrary* pointed stable curve. Theorem 2.6 was proved by the author of the present paper ([Y1, Theorem 1.2], [Y2, Theorem 0.3]).

2.3.7. We maintain the notation introduced above. Let $(\Gamma_a, L_a), (\Gamma_b, L_b) \in \mathrm{Com}(\Gamma_{X^\bullet})$. Then Γ_a, Γ_b can be regarded as topological subspaces of $\Gamma_{X^\bullet}$ (2.1.2 (a)). Suppose that $\Gamma_a \cap \Gamma_b$ is non-empty, and that

$$\Gamma_a \cap \Gamma_b \subseteq e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

Moreover, we write $\Pi_a \overset{\mathrm{def}}{=} \Pi_{\widehat{(\Gamma_a)}_{L_a}} \subseteq \Pi^\Sigma_{X^\bullet}$, $\Pi_b \overset{\mathrm{def}}{=} \Pi_{\widehat{(\Gamma_b)}_{L_b}} \subseteq \Pi^\Sigma_{X^\bullet}$ for the geometry-like subgroups associated to some $\widehat{\Gamma_a \setminus L_a}$, $\widehat{\Gamma_b \setminus L_b} \subseteq \widehat{\Gamma}_{X^\bullet}$, respectively. We have the following lemma.

Lemma 2.7. *Suppose that $\Pi_a \cap \Pi_b \subseteq \Pi^\Sigma_{X^\bullet}$ is not trivial. Then $\Pi_a \cap \Pi_b$ is a closed-edge-like subgroup of $\Pi^\Sigma_{X^\bullet}$.*

Proof. If either Γ_a or Γ_b is an edge of $\Gamma_{X^\bullet}$, then the lemma is trivial. Thus, we may assume that $v(\Gamma_a)$ and $v(\Gamma_b)$ are not empty.

Let $H \subseteq \Pi^{\Sigma}_{X^\bullet}$ be an arbitrary open subgroup, $H_a \stackrel{\text{def}}{=} H \cap \Pi_a$, and $H_b \stackrel{\text{def}}{=} H \cap \Pi_b$. Then we have the natural injections (see 2.2.2 for $(-)^{\text{ét}}$)

$$H_a^{\text{ét,ab}} \hookrightarrow H^{\text{ét,ab}},\ H_b^{\text{ét,ab}} \hookrightarrow H^{\text{ét,ab}}.$$

Moreover, since $\Gamma_a \cap \Gamma_b \subseteq e^{\text{cl}}(\Gamma_{X^\bullet})$, $H_a^{\text{ét,ab}} \cap H_b^{\text{ét,ab}}$ is trivial.

Let $J \subseteq \Pi_a \cap \Pi_b$ be a non-trivial pro-cyclic subgroup (i.e. a subgroup generalized by one element) and $J_H \stackrel{\text{def}}{=} J \cap H$. Then the image of the natural homomorphism

$$J_H \to H^{\text{ét,ab}}$$

is trivial. By applying [HM1, Lemma 1.6], J is contained in a unique closed-edge subgroup $I_{\widehat{e}_J}$ of $\Pi^{\Sigma}_{X^\bullet}$ for some $\widehat{e}_J \in e^{\text{cl}}(\widehat{\Gamma}_{X^\bullet})$. Write e_J for the image of $\widehat{e}_J$ of the natural map $\pi_X : \widehat{\Gamma}_{X^\bullet} \twoheadrightarrow \Gamma_{X^\bullet}$. We see immediately that $I_{\widehat{e}_J} \subseteq \Pi_a \cap \Pi_b$, that $\widehat{e}_J$ connects $\widehat{\Gamma_a \setminus L_a}$ with $\widehat{\Gamma_b \setminus L_b}$, and that $e_J \in \Gamma_a \cap \Gamma_b \subseteq e^{\text{cl}}(\Gamma_{X^\bullet})$. Write $\widehat{E}$ for the set of edges connecting $\widehat{\Gamma_a \setminus L_a}$ with $\widehat{\Gamma_b \setminus L_b}$. Then [M2, Proposition 1.2 (i)] implies that $\Pi_a \cap \Pi_b$ coincides with the subgroup generated by $\{I_{\widehat{e}}\}_{\widehat{e} \in \widehat{E}}$. Moreover, by applying similar arguments to the arguments given in the proof of [HM1, Lemma 1.8], we obtain

$$\Pi_a \cap \Pi_b = I_{\widehat{e}_J}.$$

This completes the proof of the lemma. $\square$

3. Topological and group-theoretical specializations

3.1. Specializations and conjectures.

3.1.1. **Settings.** Let $\overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}}$ be the moduli stack parameterizing pointed stable curves of type (g_X, n_X) over $\operatorname{Spec}\mathbb{Z}$, $\overline{\mathbb{F}}_p$ an algebraic closure of the finite field $\mathbb{F}_p$ of characteristic $p > 0$, $\overline{\mathcal{M}}_{g_X,n_X} \stackrel{\text{def}}{=} \overline{\mathcal{M}}_{g_X,n_X,\mathbb{Z}} \times_{\mathbb{Z}} \overline{\mathbb{F}}_p$, and $\overline{M}_{g_X,n_X}$ the coarse moduli space of $\overline{\mathcal{M}}_{g_X,n_X}$. For $q \in \overline{M}_{g_X,n_X}$, we shall write $V(q)$ for the topological closure of q in $\overline{M}_{g_X,n_X}$.

Let $i \in \{1, 2\}$, and let $q_i \in \overline{M}_{g_X,n_X}$ be an arbitrary point of $\overline{M}_{g_X,n_X}$ and k_i an algebraically closed field containing the residue field $k(q_i)$ of q_i. Then the natural morphism $\operatorname{Spec} k_i \to \overline{M}_{g_X,n_X}$ determines a pointed stable curve

$$X_i^\bullet = (X_i, D_{X_i})$$

of type (g_X, n_X) over k_i. We denote by $\Gamma_{X_i^\bullet}$ the dual semi-graph of $X_i^\bullet$, $r_{\Gamma_{X_i^\bullet}}$ the Betti number of $\Gamma_{X_i^\bullet}$, and $\Pi_{X_i^\bullet}$ either the admissible fundamental group of $X_i^\bullet$ or the maximal pro-solvable quotient of the admissible fundamental group of $X_i^\bullet$. Let $\operatorname{Com}(\Gamma_{X_i^\bullet})$, $\operatorname{Typ}(X_i^\bullet)$, and $\operatorname{Geo}(\Pi_{X_i^\bullet})$ be the combinatorial data associated to $X_i^\bullet$ (Definition 2.5 (a)), the topological data associated to $X_i^\bullet$ (Definition 2.5 (a)), and the geometric data associated to $\Pi_{X_i^\bullet}$ (Definition 2.5 (a)), respectively.

We denote by

$$\operatorname{Hom}^{\text{op}}_{\text{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$$

the set of open continuous homomorphisms of profinite groups $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$. Let $\phi \in \operatorname{Hom}^{\text{op}}_{\text{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ be an arbitrary element. Then [Y6, Lemma 4.3] implies that ϕ is a *surjection*.

Let Σ be an arbitrary set of prime numbers such that $p \notin \Sigma$. We write $pr^{\Sigma}_{\Pi_{X_i^\bullet}} : \Pi_{X_i^\bullet} \twoheadrightarrow \Pi^{\Sigma}_{X_i^\bullet}$, $i \in \{1, 2\}$, for the natural surjection. Note that the structures of maximal pro-prime-to-p quotients of admissible fundamental groups of pointed stable curves (e.g. see [Y6, 1.2.4]) imply that ϕ induces an *isomorphism* $\phi^{\Sigma} : \Pi^{\Sigma}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{\Sigma}_{X_2^\bullet}$.

3.1.2. We maintain the notation introduced in 3.1.1.

Definition 3.1. (a) We shall call that q_2 is a *topological specialization* of q_1 if there exists a point $q'_2 \in V(q_1)$ such that the following conditions are satisfied:

(i) There exists an isomorphism of dual semi-graphs $\psi^{\mathrm{sg}} : \Gamma_{q'_2} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$, where $\Gamma_{q'_2}$ denotes the dual semi-graph of a pointed stable curve corresponding to a geometric point over $\mathrm{Spec}\, k(q'_2) \to \overline{M}_{g_X,n_X}$ (note that the isomorphism class of $\Gamma_{q'_2}$ does not depend on the choices of geometric points over $\mathrm{Spec}\, k(q'_2) \to \overline{M}_{g_X,n_X}$). In particular, ψ^{sg} induces a bijection $\psi^{\mathrm{com}} : \mathrm{Com}(\Gamma_{q'_2}) \xrightarrow{\sim} \mathrm{Com}(\Gamma_{X_2^\bullet})$.

(ii) Let $(\Gamma'_2, L'_2) \in \mathrm{Com}(\Gamma_{q'_2})$ be an arbitrary element and $(\Gamma_2, L_2) \overset{\mathrm{def}}{=} \psi^{\mathrm{com}}((\Gamma'_2, L'_2)) \in \mathrm{Com}(\Gamma_{q'_2})$. Then we have $(g_{(\Gamma'_2)_{L'_2}}, n_{(\Gamma'_2)_{L'_2}}) = (g_{(\Gamma_2)_{L_2}}, n_{(\Gamma_2)_{L_2}})$ (2.2.4).

We shall call an open continuous homomorphism $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ a *topological specialization homomorphism* if q_2 is a topological specialization of q_1.

On the other hand, since q'_2 is contained in $V(q_1)$, the corresponding degeneration implies that there exists a natural map $sp^{\mathrm{com}}_{q_1,q'_2} : \mathrm{Com}(\Gamma_{X_1^\bullet}) \to \mathrm{Com}(\Gamma_{q'_2})$. We put

$$sp^{\mathrm{com}}_{X_1^\bullet,X_2^\bullet} \overset{\mathrm{def}}{=} \psi^{\mathrm{com}} \circ sp^{\mathrm{com}}_{q_1,q'_2} : \mathrm{Com}(\Gamma_{X_1^\bullet}) \to \mathrm{Com}(\Gamma_{q'_2}) \xrightarrow{\sim} \mathrm{Com}(\Gamma_{X_2^\bullet}).$$

Note that the restriction map $sp^{\mathrm{com}}_{X_1^\bullet,X_2^\bullet}|_{e^{\mathrm{op}}(\Gamma_{X_1^\bullet})} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \to e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ is a bijection. The map $sp^{\mathrm{com}}_{X_1^\bullet,X_2^\bullet}$ will be used to define "strong topological specialization homomorphism" (see Definiton 4.1 below).

(b) Let $\Pi_1 \in \mathrm{Ver}(\Pi_{X_1^\bullet})$ be an arbitrary vertex-like subgroup of $\Pi_{X_1^\bullet}$ and $\Pi'_2 \in \mathrm{Ver}(\Pi_{X_2^\bullet})$ an arbitrary vertex-like subgroup of $\Pi_{X_2^\bullet}$. We shall call an open continuous homomorphism $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ a *group-theoretical specialization homomorphism* if the following conditions are satisfied:

(i) $\Pi_2 \overset{\mathrm{def}}{=} \phi(\Pi_1) \in \mathrm{Geo}(\Pi_{X_2^\bullet})$ (note that $\Pi_2 \notin \mathrm{Ver}(\Pi_{X_1^\bullet})$ in general).
(ii) There exists $\Pi'_1 \in \mathrm{Ver}(\Pi_{X_1^\bullet})$ such that $\Pi'_2 \subseteq \phi(\Pi'_1)$.
(iii) Let $(g_{\Pi_i}, n_{\Pi_i}) \in \mathrm{Typ}(X_i^\bullet)$, $i \in \{1, 2\}$, be the topological datum associated to $X_i^\bullet$ determined group-theoretically by Π_i (Theorem 2.6). Then we have $(g_{\Pi_1}, n_{\Pi_1}) = (g_{\Pi_2}, n_{\Pi_2})$.

Remark 3.1.1. In the next subsection, we will prove that if ϕ is a group-theoretical specialization homomorphism, then ϕ is a topological specialization homomorphism (see Proposition 3.6 below).

3.1.3. Motivated by the homeomorphism conjecture formulated in [Y6, Section 3.3], we formulate the following conjectures concerning topological and group-theoretical specialization homomorphisms:

Topological Specialization Conjecture . *Let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ be an arbitrary open continuous homomorphism. Then ϕ is a topological specialization homomorphism (Definition 3.1 (a)). In particular, q_2 is a topological specialization of q_1 if and only if $\mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset$.*

Group-theoretical Specialization Conjecture . *Let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ be an arbitrary open continuous homomorphism. Then ϕ is a group-theoretical specialization homomorphism (Definition 3.1 (b)).*

Remark. We may formulate a more general version of the group-theoretical specialization conjecture as follows:

(**A general version of the group-theoretical specialization conjecture**) *We maintain the notation introduced in 3.1.1. Let $\Pi_1 \in \mathrm{Geo}(\Pi_{X_1^\bullet})$ and $\Pi'_2 \in \mathrm{Geo}(\Pi_{X_2^\bullet})$ be arbitrary geometry-like subgroups. Then the following statements hold:*

(i) $\Pi_2 \overset{\mathrm{def}}{=} \phi(\Pi_1) \in \mathrm{Geo}(\Pi_{X_2^\bullet})$.
(ii) There exists $\Pi'_1 \in \mathrm{Geo}(\Pi_{X_1^\bullet})$ such that $\Pi'_2 \subseteq \phi(\Pi'_1)$.
(iii) Let $(g_{\Pi_i}, n_{\Pi_i}) \in \mathrm{Typ}(X_i^\bullet)$, $i \in \{1, 2\}$, be the topological datum associated to $X_i^\bullet$ determined group-theoretically by Π_i (Theorem 2.6). Then we have $(g_{\Pi_1}, n_{\Pi_1}) = (g_{\Pi_2}, n_{\Pi_2})$.

In fact, we can prove that *the group-theoretical specialization conjecture implies the general version of the group-theoretical specialization conjecture*. In the present paper, we do not discuss the general version.

Remark. Theorem 2.6 says that the topological specialization conjecture and the group-theoretical specialization conjecture hold for $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ if ϕ is an *isomorphism*.

3.1.4. For an arbitrary open continuous homomorphism ϕ, by using two group-theoretical formulas concerning generalized Hasse-Witt invariants (see [Y3, Theorem 1.3], [Y5, Theorem 1.2]), we have the following result (see [Y6, Theorem 4.11] for (a) and [Y6, Theorem 5.30] for (b)):

Theorem 3.2. *Let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ be an arbitrary open continuous homomorphism. Then the following statements hold:*

(a) The open continuous homomorphism ϕ induces group-theoretically a surjection

$$\phi^{\mathrm{edg,op}} : \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})$$

between the sets of open-edge-like subgroups of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$. Moreover, we obtain a bijection

$$\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) = \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) = \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet}$$

induced by $\phi^{\mathrm{edg,op}}$.

(b) Suppose $g_X = 0$, $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_2^\bullet}))$, and $\#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) = \#(e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}))$. Then ϕ is a topological specialization homomorphism and a group-theoretical specialization homomorphism. In particular, for any open subgroup $H_2 \subseteq \Pi_{X_2^\bullet}$, $\phi|_{H_1} : H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2) \to H_2$ induces group-theoretically an isomorphism of dual semi-graphs

$$\phi|^{\mathrm{sg}}_{H_1} : \Gamma_{X^\bullet_{H_1}} \xrightarrow{\sim} \Gamma_{X^\bullet_{H_2}},$$

where $\Gamma_{X^\bullet_{H_i}}$, $i \in \{1,2\}$, denotes the dual semi-graph of the pointed stable curve $X^\bullet_{H_i}$ corresponding to H_i.

Remark 3.2.1. Theorem 3.2 (b) also holds for pointed stable curves of an arbitrary type under certain conditions, see [Y6, Theorem 5.26].

By applying Theorem 3.2, we have the following corollary.

Corollary 3.3. *(a) Suppose that $X_i^\bullet$, $i \in \{1,2\}$, is smooth over k_i. Then the topological specialization conjecture and the group-theoretical specialization conjecture hold.*

(b) Suppose $(g_X, n_X) = (0,4)$. Then the topological specialization conjecture and the group-theoretical specialization conjecture hold.

Proof. (a) follows immediately from Theorem 3.2 (a) and the definitions of topological and group-theoretical specialization homomorphisms. Let us prove (b).

Suppose that $X_1^\bullet$ is smooth over k_1. Then Theorem 3.2 (a) implies that ϕ is a topological specialization homomorphism and a group-theoretical specialization homomorphism.

Suppose that $X_1^\bullet$ is singular. Then [Y6, Lemma 6.3] implies that $X_2^\bullet$ is also singular. Moreover, the assumption $(g_X, n_X) = (0,4)$ implies $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_2^\bullet})) = 2$ and $\#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) = \#(e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})) = 1$. Then (b) follows immediately from Theorem 3.2 (b). $\square$

3.2. **Topological and group-theoretical specialization homomorphisms.** In this subsection, we will prove that the group-theoretical specialization conjecture implies the topological specialization conjecture (see Proposition 3.6). Moreover, we prove that the definition of group-theoretical specialization homomorphisms (i.e. Definition 3.1 (b)) can be simplified (see Proposition 3.9).

3.2.1. **Settings.** We maintain the notation introduced in 3.1.1.

3.2.2. Let $\Gamma_{i,a}$, $\Gamma_{i,b}$, $i \in \{1,2\}$, be sub-semi-graphs (2.1.2) of $\Gamma_{X_i^\bullet}$. Then $\Gamma_{i,a}$, $\Gamma_{i,b}$ can be regarded as topological subspaces of $\Gamma_{X_i^\bullet}$ (2.1.2). Moreover, let $\Pi_{i,a} \stackrel{\text{def}}{=} \Pi_{\widehat{\Gamma}_{i,a}} \in \mathrm{Geo}(\Pi_{X_i^\bullet})$, $\Pi_{i,b} \stackrel{\text{def}}{=} \Pi_{\widehat{\Gamma}_{i,b}} \in \mathrm{Geo}(\Pi_{X_i^\bullet})$ be the geometry-like subgroups associated to some $\widehat{\Gamma}_{i,a}$, $\widehat{\Gamma}_{i,b} \subseteq \widehat{\Gamma}_{X_i^\bullet}$, respectively. We have the following lemma.

Lemma 3.4. *Suppose $\Gamma_{1,a} \cap \Gamma_{1,b} \subseteq e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$, $\phi(\Pi_{1,a}) = \Pi_{2,a}$, and $\phi(\Pi_{1,b}) = \Pi_{2,b}$. Moreover, suppose that $\Pi_{1,a} \cap \Pi_{1,b} \subseteq \Pi_{X_1^\bullet}$ is not trivial. Then $\Pi_{2,a} \cap \Pi_{2,b} \subseteq \Pi_{X_2^\bullet}$ is a closed-edge-like subgroup of $\Pi_{X_2^\bullet}$.*

Proof. Since $\Pi_{1,a} \cap \Pi_{1,b} \subseteq \Pi_{X_1^\bullet}$ is not trivial, we have that $\Pi_{2,a} \cap \Pi_{2,b} \subseteq \Pi_{X_2^\bullet}$ is non-trivial, and that Lemma 2.7 implies that $\Pi_{1,a} \cap \Pi_{1,b} \subseteq \Pi_{X_1^\bullet}$ is a closed-edge-like subgroup of $\Pi_{X_1^\bullet}$. Moreover, $pr^{p'}_{\Pi_{X_1^\bullet}}(\Pi_{1,a} \cap \Pi_{1,b}) = \Pi^{p'}_{1,a} \cap \Pi^{p'}_{1,b} \subseteq \Pi^{p'}_{X_1^\bullet}$ is a closed-edge-like subgroup of $\Pi^{p'}_{X_1^\bullet}$, where $(-)^{p'}$ denotes the maximal pro-prime-to-p quotient of $(-)$ (see 2.3.1). Write $\phi^{p'} : \Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet}$ for the isomorphism induced by ϕ.

Suppose that either $\Pi_{1,a}$ or $\Pi_{1,b}$ is a closed-edge-like subgroup of $\Pi_{X_1^\bullet}$. Without loss of generality, we may assume that $\Pi_{1,a}$ is a closed-edge-like subgroup of $\Pi_{X_1^\bullet}$. Then we have $\Pi_{1,a} \cong \widehat{\mathbb{Z}}(1)^{p'}$. Since $\Pi_{2,a} = \phi(\Pi_{1,a}) \in \mathrm{Geo}(\Pi_{X_2^\bullet})$, the structures of maximal pro-prime-to-p quotients of admissible fundamental groups of pointed stable curves (e.g. see [Y6, 1.2.4]) imply that $\Pi_{2,a}$ is either a closed-edge-like subgroup or an open-edge-like subgroup of $\Pi_{X_2^\bullet}$.

By applying Theorem 3.2, we obtain that $\phi^{p'}$ induces a bijection
$$\mathrm{Edg}^{\mathrm{op}}(\Pi^{p'}_{X_1^\bullet}) \xrightarrow{\sim} \mathrm{Edg}^{\mathrm{op}}(\Pi^{p'}_{X_2^\bullet}).$$
If $\Pi_{2,a}$ is an open-edge-like subgroup of $\Pi_{X_2^\bullet}$, then we have $pr^{p'}_{\Pi_{X_2^\bullet}}(\Pi_{2,a}) = \Pi^{p'}_{2,a} \in \mathrm{Edg}^{\mathrm{op}}(\Pi^{p'}_{X_2^\bullet})$. Moreover, we obtain $pr^{p'}_{\Pi_{X_1^\bullet}}(\Pi_{1,a}) = \Pi^{p'}_{1,a} \in \mathrm{Edg}^{\mathrm{op}}(\Pi^{p'}_{X_1^\bullet})$. This contradicts the fact that $\Pi^{p'}_{1,a}$ is a closed-edge-like subgroup of $\Pi^{p'}_{X_1^\bullet}$. Thus, $\Pi_{2,a} = \Pi_{2,a} \cap \Pi_{2,b} \subseteq \Pi_{X_2^\bullet}$ is a closed-edge-like subgroup of $\Pi_{X_2^\bullet}$.

Suppose that $\Pi_{1,a}$, $\Pi_{1,b}$ are not closed-edge-like subgroups of $\Pi_{X_1^\bullet}$. To verify the lemma, by applying Lemma 2.7, it's sufficient to prove that $\Gamma_{2,a} \cap \Gamma_{2,b} \subseteq e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$. If $\Gamma_{2,a} \cap \Gamma_{2,b}$ is empty, then $\Pi_{2,a} \cap \Pi_{2,b}$ is trivial. Then we may assume that $\Gamma_{2,a} \cap \Gamma_{2,b}$ is not empty. By using similar arguments to the arguments given in the third paragraph, we see that $\Gamma_{2,a} \cap \Gamma_{2,b} \cap e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ is empty. On the other hand, since
$$(\phi^{p'} \circ pr^{p'}_{\Pi_{X_1^\bullet}})(\Pi_{1,a} \cap \Pi_{1,b}) = \phi^{p'}(\Pi^{p'}_{1,a} \cap \Pi^{p'}_{1,b}) = \Pi^{p'}_{2,a} \cap \Pi^{p'}_{2,b} \cong \widehat{\mathbb{Z}}(1)^{p'},$$
the structures of maximal pro-prime-to-p quotients of admissible fundamental groups of pointed stable curves imply that $\Gamma_{2,a} \cap \Gamma_{2,b} \cap v(\Gamma_{X_2^\bullet})$ is empty. Thus, we have $\Gamma_{2,a} \cap \Gamma_{2,b} \subseteq e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$. We complete the proof of the lemma. □

3.2.3. We have the following lemma.

Lemma 3.5. *Suppose that the condition given in Definition 3.1 (b)-(i) holds. Then $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ induces group-theoretically a map (neither an injection nor a surjection in general)*
$$\phi^{\mathrm{edg,cl}} : \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet}) \to \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})$$
between the sets of closed-edge-like subgroups of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$. Moreover, we obtain an injection
$$\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) = \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \hookrightarrow e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) = \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet}$$
induced by $\phi^{\mathrm{edg,cl}}$.

Proof. Let $\widehat{e}_1 \in e^{\mathrm{cl}}(\widehat{\Gamma}_{X_1^\bullet})$ be a closed edge, $e_1 \in e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$ the image of $\widehat{e}_1$ of the natural map $\pi_{X_1} : \widehat{\Gamma}_{X_1^\bullet} \twoheadrightarrow \Gamma_{X_1^\bullet}$, and $I_{\widehat{e}_1} \in \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet})$ the closed-edge-like subgroup of $\Pi_{X_1^\bullet}$ associated to $\widehat{e}_1$.

Suppose $e_1 \notin e^{\mathrm{lp}}(\Gamma_{X_1^\bullet})$ (see 2.1.1 (b) for $e^{\mathrm{lp}}(\Gamma_{X_1^\bullet})$). Then the singular point of X_1 corresponding to e_1 is contained in two different irreducible components of X_1. Since the condition given in Definition 3.1 (b)-(i) holds, Lemma 3.4 implies $\phi(I_{\widehat{e}_1}) \in \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})$.

Suppose $e_1 \in e^{\mathrm{lp}}(\Gamma_{X_1^\bullet})$. Let ℓ be a prime number distinct from p,

$$H_2 \stackrel{\mathrm{def}}{=} \ker(\Pi_{X_2^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell),$$

$$H_1 \stackrel{\mathrm{def}}{=} \phi^{-1}(H_2) = \ker(\Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_1^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_\ell),$$

$\phi_{H_1} \stackrel{\mathrm{def}}{=} \phi|_{H_1} : H_1 \to H_2$ the open continuous homomorphism induced by ϕ, $X_{H_i}^\bullet$, $i \in \{1,2\}$, the pointed stable curve corresponding to H_i, and $\Gamma_{X_{H_i}^\bullet}$ the dual semi-graph of $X_{H_i}^\bullet$. We see immediately that $e^{\mathrm{lp}}(\Gamma_{X_{H_i}^\bullet})$ is empty. We put

$$I_{\widehat{e}_{H_1}} \stackrel{\mathrm{def}}{=} I_{\widehat{e}_1} \cap H_1 \in \mathrm{Edg}^{\mathrm{cl}}(H_1) = \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet}) \cap H_1.$$

Note that $I_{\widehat{e}_1}$ is the normalizer of $I_{\widehat{e}_{H_1}}$ in $\Pi_{X_1^\bullet}$, and that the index $[I_{\widehat{e}_1} : I_{\widehat{e}_{H_1}}]$ is ℓ. The lemma of the case of $e_1 \notin e^{\mathrm{lp}}(\Gamma_{X_1^\bullet})$ proved above implies $I_{\widehat{e}_{H_2}} \stackrel{\mathrm{def}}{=} \phi_{H_1}(I_{\widehat{e}_{H_1}}) \in \mathrm{Edg}^{\mathrm{cl}}(H_2)$. We put

$$I_{\widehat{e}_2}$$

the normalizer of $I_{\widehat{e}_{H_2}}$ in $\Pi_{X_2^\bullet}$. Then we have $I_{\widehat{e}_2} \in \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})$ and $[I_{\widehat{e}_2} : I_{\widehat{e}_{H_2}}] \leq \ell$. On the other hand, since $I_{\widehat{e}_i}$, $i \in \{1,2\}$, is the normalizer of $I_{\widehat{e}_{H_i}}$ in $\Pi_{X_i^\bullet}$, we obtain $\phi(I_{\widehat{e}_1}) \subseteq I_{\widehat{e}_2}$. Moreover, since $\phi^\ell : \Pi_{X_1^\bullet}^\ell \xrightarrow{\sim} \Pi_{X_2^\bullet}^\ell$ is an isomorphism, we see $[I_{\widehat{e}_1} : I_{\widehat{e}_{H_1}}] = [I_{\widehat{e}_2} : I_{\widehat{e}_{H_2}}] = \ell$. This means $\phi(I_{\widehat{e}_1}) = I_{\widehat{e}_2}$. Thus, ϕ induces group-theoretically a map

$$\phi^{\mathrm{edg,cl}} : \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet}) \to \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})$$

between the sets of closed-edge-like subgroups of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$.

Next, we prove the "moreover" part of the lemma. Let

$$\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \stackrel{\mathrm{def}}{=} \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_1^\bullet})/\Pi_{X_1^\bullet} \to e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) \stackrel{\mathrm{def}}{=} \mathrm{Edg}^{\mathrm{cl}}(\Pi_{X_2^\bullet})/\Pi_{X_2^\bullet}$$

be the map induced by $\phi^{\mathrm{edg,cl}}$ and $e_{1,j} \in e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$, $j \in \{a,b\}$, a closed edge such that $\phi^{\mathrm{sg,cl}}(e_{1,a}) = \phi^{\mathrm{sg,cl}}(e_{1,b})$. Let $\widehat{e}_{1,j} \in e^{\mathrm{cl}}(\widehat{\Gamma}_{X_1^\bullet})$, $j \in \{a,b\}$, be a closed edge over $e_{1,j}$ and $I_{\widehat{e}_{1,j}}$ the closed-edge-like subgroup of $\Pi_{X_1^\bullet}$ associated to $\widehat{e}_{1,j}$. Then $pr^{p'}_{\Pi_{X_1^\bullet}}(I_{\widehat{e}_{1,j}}) \in \mathrm{Edg}^{\mathrm{cl}}(\Pi^{p'}_{X_1^\bullet})$ and $pr^{p'}_{\Pi_{X_2^\bullet}}(\phi(I_{\widehat{e}_{1,j}})) \in \mathrm{Edg}^{\mathrm{cl}}(\Pi^{p'}_{X_2^\bullet})$, $j \in \{a,b\}$, are closed-edge-like subgroups of $\Pi^{p'}_{X_1^\bullet}$ and $\Pi^{p'}_{X_2^\bullet}$, respectively. Since $\phi^{\mathrm{sg,cl}}(e_{1,a}) = \phi^{\mathrm{sg,cl}}(e_{1,b})$, the conjugacy classes $pr^{p'}_{\Pi_{X_2^\bullet}}(\phi(I_{\widehat{e}_{1,a}}))$ and $pr^{p'}_{\Pi_{X_2^\bullet}}(\phi(I_{\widehat{e}_{1,b}}))$ in $\Pi^{p'}_{X_2^\bullet}$ are equal. On the other hand, since $\phi^{p'} : \Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet}$ is an isomorphism, we obtain that the conjugacy classes $pr^{p'}_{\Pi_{X_1^\bullet}}(I_{\widehat{e}_{1,a}})$ and $pr^{p'}_{\Pi_{X_1^\bullet}}(I_{\widehat{e}_{1,b}})$ in $\Pi^{p'}_{X_1^\bullet}$ are equal. This means $e_{1,a} = e_{1,b}$. We complete the proof of the lemma. $\square$

3.2.4. Suppose that the condition given in Definition 3.1 (b)-(i) holds. Let $v_1 \in v(\Gamma_{X_1^\bullet})$ be an arbitrary vertex of $\Gamma_{X_1^\bullet}$, $\widehat{v}_1 \in v(\widehat{\Gamma}_{X_1^\bullet})$ a vertex of $\widehat{\Gamma}_{X_1^\bullet}$ over v_1, and $\Pi_{\widehat{v}_1}$ the vertex-like subgroup of $\Pi_{X_1^\bullet}$ associated to $\widehat{v}_1$. Then there exists a unique pair $(\Gamma[v_1], L[v_1]) \in \mathrm{Com}(\Gamma_{X_2^\bullet})$ (see 2.3.5 for $\mathrm{Com}(\Gamma_{X_2^\bullet})$) such that $\phi(\Pi_{\widehat{v}_1}) = \Pi_{\widehat{\Gamma(v_1)}} \stackrel{\mathrm{def}}{=} \Pi_{\widehat{\Gamma[v_1]_{L[v_1]}}}$ (see 2.3.3 for $\Pi_{\widehat{\Gamma[v_1]_{L[v_1]}}}$), where $\Gamma(v_1) \stackrel{\mathrm{def}}{=} \Gamma[v_1]_{L[v_1]}$ denotes the semi-graph associated to $\Gamma[v_1]$ and $L[v_1]$ (2.1.2 (b)). Note that $(\Gamma[v_1], L[v_1])$ depends only on the choice of v_1 (or the conjugacy class of $\Pi_{\widehat{v}_1}$). We have the following proposition.

Proposition 3.6. *Let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ be an arbitrary open continuous homomorphism. Suppose that ϕ is a group-theoretical specialization homomorphism (Definition 3.1 (b)). Then ϕ is a topological specialization homomorphism (Definition 3.1 (a)). In particular, the group-theoretical specialization conjecture implies the topological specialization conjecture.*

Proof. Let $v, w \in v(\Gamma_{X_1^\bullet})$ be arbitrary vertices of $\Gamma_{X_1^\bullet}$ distinct from each other when $\#(v(\Gamma_{X_1^\bullet})) \geq 2$ and $\Gamma_v, \Gamma_w \subseteq \Gamma_{X_1^\bullet}$ the sub-semi-graphs associated to v, w (see 2.2.5), respectively. We put $L_v \stackrel{\mathrm{def}}{=} e^{\mathrm{lp}}(\Gamma_v)$ and $L_w \stackrel{\mathrm{def}}{=} e^{\mathrm{lp}}(\Gamma_w)$. Moreover, we put

$$\Gamma^v \stackrel{\mathrm{def}}{=} (\Gamma_v)_{L_v}, \ \Gamma^w \stackrel{\mathrm{def}}{=} (\Gamma_w)_{L_w}$$

the semi-graphs associated to Γ_v and L_v, Γ_w and L_w, respectively.

Firstly, to verify that ϕ is a topological specialization homomorphism, we need to prove that the dual semi-graph $\Gamma_{X_2^\bullet}$ of $X_2^\bullet$ is isomorphic to the dual semi-graph of a reduction of $X_1^\bullet$ (i.e. we prove that the condition given in Definition 3.1 (a)-(i) holds). This means that we need to check the following conditions (see Theorem 3.2 (a) for $\phi^{\mathrm{sg,op}}$, Lemma 3.5 for $\phi^{\mathrm{sg,cl}}$, and 2.1.2 (b) for δ_{Γ^v}, δ_{Γ^w}, $\delta_{\Gamma(v)}$, $\delta_{\Gamma(w)}$):

(i) $\phi^{\mathrm{sg,op}}(e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \cap \delta_{\Gamma^v}(e^{\mathrm{op}}(\Gamma^v))) = e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \cap \delta_{\Gamma(v)}(e^{\mathrm{op}}(\Gamma(v))$.
(ii) $\phi^{\mathrm{sg,cl}}(L_v) \subseteq e^{\mathrm{cl}}(\Gamma[v])$.
(iii) $\phi^{\mathrm{sg,cl}}(\delta_{\Gamma^v}(e^{\mathrm{op}}(\Gamma^v)) \cap \delta_{\Gamma^w}(e^{\mathrm{op}}(\Gamma^w))) = \delta_{\Gamma(v)}(e^{\mathrm{op}}(\Gamma(v))) \cap \delta_{\Gamma(w)}(e^{\mathrm{op}}(\Gamma(w)))$.
(iv) $\#(e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \cap \delta_{\Gamma^v}(e^{\mathrm{op}}(\Gamma^v))) = \#(e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \cap \delta_{\Gamma(v)}(e^{\mathrm{op}}(\Gamma(v))))$.
(v) $\#(L_v) = \#(\phi^{\mathrm{sg,cl}}(L_v))$.
(vi) $\#(\delta_{\Gamma^v}(e^{\mathrm{op}}(\Gamma^v)) \cap \delta_{\Gamma^w}(e^{\mathrm{op}}(\Gamma^w))) = \#(\delta_{\Gamma(v)}(e^{\mathrm{op}}(\Gamma(v))) \cap \delta_{\Gamma(w)}(e^{\mathrm{op}}(\Gamma(w))))$.

The conditions (i), (iv) say that the degeneration (as a topological space) of the marked points of $X_1^\bullet$ contained in X_v (2.2.5) are the marked points of $X_2^\bullet$ contained in $X_{\Gamma[v]}$ (2.2.3). The conditions (ii), (v) say that the degeneration (as a topological space) of the singular points of $X_1^\bullet$ corresponding to L_v are singular points of $X_2^\bullet$ contained in $X_{\Gamma[v]}$. The conditions (iii), (vi) says that the degeneration (as a topological space) of the gluing of $\{\widetilde{X}_v\}_{v\in v(\Gamma_{X_1^\bullet})}$ (2.2.5) along the singular points of $X_1^\bullet$ that gives rise to $X_1^\bullet$ is the gluing of $\{X_{\Gamma(v)}^\bullet\}_{v\in v(\Gamma_{X_1^\bullet})}$ (2.2.4) along the singular points corresponding to $\{\phi^{\mathrm{sg,cl}}(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}))\}_{v\in v(\Gamma_{X_1^\bullet})}$ of $X_2^\bullet$ that gives rise to $X_2^\bullet$.

We maintain the notation introduced at the beginning of 3.2.4. Let $e \in e^{\mathrm{op}}(\Gamma^v)$ and $I_{\widehat{e}} \subseteq \Pi_{X_1^\bullet}$ the open edge-like subgroup associated to an edge $\widehat{e} \in \pi_{X_1}^{-1}(e)$ such that $I_{\widehat{e}} \in \Pi_{\widehat{v}}$ (or $\widehat{e}$ abuts to $\widehat{v}$). Then by applying Theorem 3.2 (a) for $\phi|_{\Pi_{\widehat{v}}} : \Pi_{\widehat{v}} \to \Pi_{\widehat{\Gamma(v)}}$, we see that $\phi(I_{\widehat{e}})$ is an open edge-like subgroup of $\Pi_{\widehat{\Gamma(v)}}$.

Suppose $\delta_{\Gamma^v}(e) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \cap \delta_{\Gamma^v}(e^{\mathrm{op}}(\Gamma^v))$. Then the condition (i) follows immediately from the "moreover" part of Theorem 3.2 (a) and the commutative diagram

$$\begin{array}{ccc} I_{\widehat{e}} & \longrightarrow & \phi(I_{\widehat{e}}) \\ \downarrow & & \downarrow \\ \Pi_{\widehat{v}} & \xrightarrow{\phi|_{\Pi_{\widehat{v}}}} & \Pi_{\widehat{\Gamma(v)}} \\ \downarrow & & \downarrow \\ \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet}, \end{array}$$

where the vertical arrows are natural injections.

Suppose $\delta_{\Gamma^v}(e) \in L_v$. We see that there exists an element $\sigma \in \Pi_{X_1^\bullet}$ such that $\Pi_{\widehat{v}} \neq \sigma^{-1}\Pi_{\widehat{v}}\sigma$ and $I_{\widehat{e}} \in \Pi_{\widehat{v}} \cap \sigma^{-1}\Pi_{\widehat{v}}\sigma$. Then the condition (ii) follows immediately from the "moreover" part of Theorem 3.2 (a) and the commutative diagram

$$\begin{array}{ccc} I_{\widehat{e}} & \longrightarrow & \phi(I_{\widehat{e}}) \\ \downarrow & & \downarrow \\ \Pi_{\widehat{v}} \cap \sigma^{-1}\Pi_{\widehat{v}}\sigma & \longrightarrow & \Pi_{\widehat{\Gamma(v)}} \cap \phi(\sigma)^{-1}\Pi_{\widehat{\Gamma(v)}}\phi(\sigma) \\ \downarrow & & \downarrow \\ \Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet}, \end{array}$$

where the vertical arrows are natural injections.

Suppose $\delta_{\Gamma^v}(e) \in \delta_{\Gamma^v}(e^{\mathrm{op}}(\Gamma^v)) \cap \delta_{\Gamma^w}(e^{\mathrm{op}}(\Gamma^w)) \subseteq e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$. We have $I_{\widehat{e}} \in \Pi_{\widehat{v}} \cap \Pi_{\widehat{w}}$ for some $\widehat{w} \in \pi_{X_1}^{-1}(w)$. Then the condition (iii) follows immediately from Lemma 3.5 and the commutative diagram

$$\begin{tikzcd} I_{\widehat{e}} \arrow[r] \arrow[d] & \phi(I_{\widehat{e}}) \arrow[d] \\ \Pi_{\widehat{v}} \cap \Pi_{\widehat{w}} \arrow[r] \arrow[d] & \Pi_{\widehat{\Gamma(v)}} \cap \Pi_{\widehat{\Gamma(w)}} \arrow[d] \\ \Pi_{X_1^\bullet} \arrow[r, "\phi"] & \Pi_{X_2^\bullet}, \end{tikzcd}$$

where the vertical arrows are natural injections.

On the other hand, the conditions (iv), (v), (vi) follow immediately from the "moreover" parts of Theorem 3.2 (a) and Lemma 3.5 (i.e. the injectivity of $\phi^{\mathrm{sg,op}}$ and $\phi^{\mathrm{sg,cl}}$).

Next, to verify ϕ is a topological specialization homomorphism, we need to prove that the condition given in Definition 3.1 (a)-(ii) holds. Since ϕ is a group-theoretical specialization homomorphism, Definition 3.1 (a)-(ii) follows immediately from Definition 3.1 (b)-(iii). This completes the proof of the proposition. $\square$

3.2.5. In the remainder of this subsection, we prove that the condition given in Definition 3.1 (b)-(i) implies the conditions given in Definition 3.1 (b)-(ii), (b)-(iii).

Lemma 3.7. *The condition given in Definition 3.1 (b)-(i) implies the condition given in Definition 3.1 (b)-(ii).*

Proof. Let $i \in \{1, 2\}$. Suppose that every irreducible component of $X_i^\bullet$ is smooth over k_i, that $\Gamma^{\mathrm{cpt}}_{X_i^\bullet}$ is 2-connected (see 2.1.1 (c) (d)), and that $g_{v_i} \geq 1$ for all $v_i \in v(\Gamma_{X_i^\bullet})$ (see 2.2.5 for g_{v_i}). We put $M_{X_i^\bullet} \overset{\mathrm{def}}{=} \Pi^{p',\mathrm{ab}}_{X_i^\bullet}$, $M^{\mathrm{top}}_{X_i^\bullet} \overset{\mathrm{def}}{=} \Pi^{\mathrm{top},p',\mathrm{ab}}_{X_i^\bullet}$. Since $\mathrm{Im}(\Pi_{\widehat{v}_i} \to M_{X_i^\bullet})$ does not depend on the choice $\widehat{v}_i \in v(\widehat{\Gamma}_{X_i^\bullet})$ over $v_i \in v(\Gamma_{X_i^\bullet})$, we put $M_{v_i} \overset{\mathrm{def}}{=} \Pi^{p',\mathrm{ab}}_{\widehat{v}_i}$, $v_i \in v(\Gamma_{X_i^\bullet})$. Then we have a surjection

$$M_{X_i^\bullet} \twoheadrightarrow M^{\mathrm{top}}_{X_i^\bullet}$$

induced by the natural surjection $\Pi_{X_i^\bullet} \twoheadrightarrow \Pi^{\mathrm{top}}_{X_i^\bullet}$ (see 2.2.2) whose kernel is equal to

$$M^{\mathrm{ver}}_{X_i^\bullet} \overset{\mathrm{def}}{=} \mathrm{Im}(\bigoplus_{v_i \in v(\Gamma_{X_i^\bullet})} M_{v_i} \to M_{X_i^\bullet}).$$

Moreover, [Y3, Corollary 3.5] implies that the natural homomorphism

$$M_{v_i} \to M^{\mathrm{ver}}_{X_i^\bullet},\ v_i \in v(\Gamma_{X_i^\bullet})$$

is an injection.

On the other hand, we put $M_{\Gamma(v_1)} \overset{\mathrm{def}}{=} \Pi^{p',\mathrm{ab}}_{\widehat{\Gamma(v_1)}}$. Note that $M_{\Gamma(v_1)}$ depends only on $\Gamma(v_1)$. Moreover, we put

$$M^{\mathrm{cur}}_{X_2^\bullet} \overset{\mathrm{def}}{=} \mathrm{Im}(\bigoplus_{v_1 \in v(\Gamma_{X_1^\bullet})} M_{\Gamma(v_1)} \to M_{X_2^\bullet}),\ M^{\text{cur-top}}_{X_2^\bullet} \overset{\mathrm{def}}{=} M_{X_2^\bullet}/M^{\mathrm{cur}}_{X_2^\bullet}.$$

By applying similar arguments to the arguments given in the proof of [Y3, Proposition 3.4], we obtain that the natural homomorphism

$$M_{\Gamma(v_1)} \to M^{\mathrm{cur}}_{X_2^\bullet},\ v_1 \in v(\Gamma_{X_1^\bullet}),$$

is an injection. Since Definition 3.1 (b)-(i) holds, the isomorphism $\phi^{p'}: \Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet}$ induces the following commutative diagram

$$\begin{tikzcd}
M_{v_1} \arrow[r] \arrow[d] & M^{\mathrm{ver}}_{X_1^\bullet} \arrow[r] \arrow[d] & M_{X_1^\bullet} \arrow[r] \arrow[d, "\phi^{p',\mathrm{ab}}"'] & M^{\mathrm{top}}_{X_1^\bullet} \arrow[d] \\
M_{\Gamma(v_1)} \arrow[r] & M^{\mathrm{cur}}_{X_2^\bullet} \arrow[r] & M_{X_2^\bullet} \arrow[r] & M^{\text{cur-top}}_{X_2^\bullet},
\end{tikzcd}$$

where all of the vertical homomorphisms are isomorphisms. Note that since we assume $g_{v_2} \geq 1$ for all $v_2 \in v(\Gamma_{X_2^\bullet})$, either $M_{v_2} \subseteq M_{\Gamma(v_1)}$ (in $M_{X_2^\bullet}$) for some $v_1 \in v(\Gamma_{X_1^\bullet})$ holds or M_{v_2} is not contained in $M^{\mathrm{cur}}_{X_2^\bullet}$ (in $M_{X_2^\bullet}$). Then to verify the lemma, it's sufficient to prove that the image $M_{v_2} \hookrightarrow M_{X_2^\bullet} \twoheadrightarrow M^{\text{cur-top}}_{X_2^\bullet}$ is trivial for all $v_2 \in v(\Gamma_{X_2^\bullet})$. Moreover, it is equivalent to prove that for all $v_2 \in v(\Gamma_{X_2^\bullet})$, the image $M_{v_2} \otimes \mathbb{F}_\ell \hookrightarrow M_{X_2^\bullet} \otimes \mathbb{F}_\ell \twoheadrightarrow M^{\text{cur-top}}_{X_2^\bullet} \otimes \mathbb{F}_\ell$ is trivial for a prime number $\ell \in \mathfrak{P} \setminus \{p\}$.

We put

$$N_{X_2^\bullet} \overset{\mathrm{def}}{=} \{\alpha \in \mathrm{Hom}(M_{X_2^\bullet}, \mathbb{Z}/\ell\mathbb{Z}) \mid \alpha(M^{\mathrm{cur}}_{X_2^\bullet}) = 0,\ \alpha(M_{v_2}) = 0 \text{ for any } v_2 \in v(\Gamma_{X_2^\bullet})\}.$$

Note that $\alpha(M_{v_2}) = 0$ for any $v_2 \in v(\Gamma_{X_2^\bullet})$ does not imply $\alpha(M^{\mathrm{cur}}_{X_2^\bullet}) = 0$ since $\Gamma(v_1)$ is not a tree (2.1.1 (c)) in general. Moreover, the definition of $N_{X_2^\bullet}$ implies that $\alpha \in N_{X_2^\bullet}$ factors through not only $M^{\text{cur-top}}_{X_2} \otimes \mathbb{F}_\ell$ but also $M^{\mathrm{top}}_{X_2} \otimes \mathbb{F}_\ell$.

We calculate $\dim_{\mathbb{F}_\ell}(N_{X_2^\bullet})$. Let $v_1, v_1' \in v(\Gamma_{X_1^\bullet})$. By applying the left-hand side of the above commutative diagram, we have $v_1 = v_1'$ if and only if $\Gamma(v_1) = \Gamma(v_1')$. In particular, we obtain $\#(v(\Gamma_{X_1^\bullet})) = \#(\{\Gamma(v_1)\}_{v_1 \in v(\Gamma_{X_1^\bullet})})$. Moreover, by applying Lemma 3.4 and Lemma 3.5, we have that $\Gamma(v_1)$ and $\Gamma(v_1')$ are connected with a closed edge e_2 of $\Gamma_{X_2^\bullet}$ if and only if v_1 and v_1' are connected with a closed edge e_1 of $\Gamma_{X_1^\bullet}$ such that $\phi^{\mathrm{sg,cl}}(e_1) = e_2$. We put

$$V_2 \overset{\mathrm{def}}{=} v(\Gamma_{X_2^\bullet}) \setminus \bigcup_{v_1 \in v(\Gamma_{X_1^\bullet})} v(\Gamma(v_1)),$$

$$E_2 \overset{\mathrm{def}}{=} e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) \setminus (\phi^{\mathrm{sg,cl}}(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) \cup \bigcup_{v_1 \in v(\Gamma_{X_1^\bullet})} e^{\mathrm{cl}}(\Gamma(v_1))).$$

Then by the Euler-Poincaré formula for semi-graphs, we obtain

$$\begin{aligned}
&\dim_{\mathbb{F}_\ell}(M^{\text{cur-top}}_{X_2^\bullet} \otimes \mathbb{F}_\ell) \geq \dim_{\mathbb{F}_\ell}(N_{X_2^\bullet}) \\
&= \#(\phi^{\mathrm{sg,cl}}(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}))) + \#(E_2) - \#(\{\Gamma(v_1)\}_{v_1 \in v(\Gamma_{X_1^\bullet})}) - \#(V_2) + 1 \\
&\geq \#(\phi^{\mathrm{sg,cl}}(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}))) - \#(\{\Gamma(v_1)\}_{v_1 \in v(\Gamma_{X_1^\bullet})}) + 1 \\
&= \#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) - \#(v(\Gamma_{X_1^\bullet})) + 1 = \dim_{\mathbb{F}_\ell}(M^{\mathrm{top}}_{X_1^\bullet} \otimes \mathbb{F}_\ell).
\end{aligned}$$

On the other hand, the right-hand side of the above commutative diagram implies

$$\dim_{\mathbb{F}_\ell}(M^{\mathrm{top}}_{X_1^\bullet} \otimes \mathbb{F}_\ell) = \dim_{\mathbb{F}_\ell}(M^{\text{cur-top}}_{X_2^\bullet} \otimes \mathbb{F}_\ell).$$

This means $M^{\text{cur-top}}_{X_2^\bullet} \otimes \mathbb{F}_\ell \cong \mathrm{Hom}_{\mathbb{F}_\ell}(N_{X_2^\bullet}, \mathbb{F}_\ell)$. Thus, $M_{v_2} \otimes \mathbb{F}_\ell \hookrightarrow M_{X_2^\bullet} \otimes \mathbb{F}_\ell \twoheadrightarrow M^{\text{cur-top}}_{X_2^\bullet} \otimes \mathbb{F}_\ell$ is trivial for all $v_2 \in v(\Gamma_{X_2^\bullet})$. We complete the proof of the lemma if $\Gamma^{\mathrm{cpt}}_{X_i^\bullet}$, $i \in \{1, 2\}$, is 2-connected, $g_{v_i} \geq 1$ for any $v_i \in v(\Gamma_{X_i^\bullet})$, and every irreducible component of $X_i^\bullet$ is non-singular.

Next, we prove the lemma in the general case. By [Y6, Lemma 5.4], there exist a prime number $\ell' >> 0$ distinct from p and a characteristic subgroup $H_2 \subseteq \Pi_{X_2^\bullet}$ such that the following conditions hold:

- The irreducible components of $X^\bullet_{H_i}$ are smooth over k_i.
- $\Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a finite ℓ'-group, where $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2)$.
- Write $\Gamma_{X^\bullet_{H_i}}$, $i \in \{1, 2\}$, for the dual semi-graph of the pointed stable curve corresponding to H_i. Then $\Gamma^{\mathrm{cpt}}_{X^\bullet_{H_i}}$ is 2-connected.

- $g_{v_{H_i}} \geq 1$, $i \in \{1,2\}$, for all $v_{H_i} \in v(\Gamma_{X^\bullet_{H_i}})$.

Let $\Pi_{\widehat{v}_2}$ be an arbitrary vertex-like subgroup of $\Pi_{X_2^\bullet}$ and $H_{\widehat{v}_2} \stackrel{\text{def}}{=} \Pi_{\widehat{v}_2} \cap H_2$. Then $H_{\widehat{v}_2}$ is a vertex-like subgroup of H_2. By applying the lemma for H_1, H_2, and $\phi|_{H_1} : H_1 \to H_2$ proved above, we obtain that $H_{\widehat{v}_2}$ is contained in $\Pi_{\widehat{\Gamma(v_1)}} \cap H_2$ for some $v_1 \in v(\Gamma_{X_1^\bullet})$. Moreover, we have that $\widehat{v}_2$ is a vertex of $\widehat{\Gamma(v_1)}$. Note that $\Pi_{\widehat{v}_2}$ is the stabilizer of $\widehat{v}_2$. Then $\Pi_{\widehat{v}_2}$ is contained in the stabilizer of $\widehat{\Gamma(v_1)}$ (since $(\sigma^{-1}\Pi_{\widehat{\Gamma(v_1)}}\sigma) \cap \Pi_{\widehat{\Gamma(v_1)}}$, $\sigma \in \Pi_{X_2^\bullet} \setminus \Pi_{\widehat{\Gamma(v_1)}}$, is either trivial or a closed-edge-like subgroup of $\Pi_{X_2^\bullet}$). Thus, we obtain $\Pi_{\widehat{v}_2} \subseteq \Pi_{\widehat{\Gamma(v_1)}}$. This completes the proof of the lemma. □

Lemma 3.8. *The condition given in Definition 3.1 (b)-(i) implies the condition given in Definition 3.1 (b)-(iii).*

Proof. Let $v_1 \in v(\Gamma_{X_1^\bullet})$ and $(\Gamma[v_1], L[v_1]) \in \mathrm{Com}(\Gamma_{X_2^\bullet})$ the pair determined by v_1 introduced at the beginning of 3.2.4. Let $\phi^{\mathrm{sg,cl}} : e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \hookrightarrow e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$ be the map obtained in Lemma 3.5. We have the following claim:

Claim: $\phi^{\mathrm{sg,cl}}(e^{\mathrm{lp}}(v_1)) = L[v_1]$ (see 2.1.1 (b) for $e^{\mathrm{lp}}(v_1)$).
We prove the claim. Let $e_1 \in e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$ (resp. $e_2 \in e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$) and $\widehat{e}_1 \in \pi_{X_1}^{-1}(e_1) \subseteq e^{\mathrm{cl}}(\widehat{\Gamma}_{X_1^\bullet})$ (resp. $\widehat{e}_2 \in \pi_{X_2}^{-1}(e_2) \subseteq e^{\mathrm{cl}}(\widehat{\Gamma}_{X_2^\bullet})$) a closed edge over e_1 (resp. e_2). Then the claim follows immediately from the following: $e_1 \in e^{\mathrm{lp}}(v_1)$ (resp. $e_2 \in L[v_1]$) if and only if $I_{\widehat{e}_1} = \Pi_{\widehat{v}_1'} \cap \Pi_{\widehat{v}_1''}$ for some $\Pi_{\widehat{v}_1'}, \Pi_{\widehat{v}_1''} \subseteq \Pi_{X_1^\bullet}$ (resp. $I_{\widehat{e}_2} = \Pi_{\widehat{\Gamma(v_1)}'} \cap \Pi_{\widehat{\Gamma(v_1)}''}$ for some $\Pi_{\widehat{\Gamma(v_1)}'}, \Pi_{\widehat{\Gamma(v_1)}''} \subseteq \Pi_{X_2^\bullet}$) such that the conjugacy classes of $\Pi_{\widehat{v}_1'}, \Pi_{\widehat{v}_1''}$ in $\Pi_{X_1^\bullet}$ are equal (resp. the conjugacy classes of $\Pi_{\widehat{\Gamma(v_1)}'}, \Pi_{\widehat{\Gamma(v_1)}''}$ in $\Pi_{X_2^\bullet}$ are equal), where $\widehat{v}_1', \widehat{v}_1'' \in \pi_{X_1}^{-1}(v_1) \subseteq v(\widehat{\Gamma}_{X_1^\bullet})$ (resp. $\widehat{\Gamma[v_1] \setminus L[v_1]}', \widehat{\Gamma[v_1] \setminus L[v_1]}''$ are connected components of $\pi_{X_2}^{-1}(\Gamma[v_1] \setminus L[v_1]) \subseteq \widehat{\Gamma}_{X_2^\bullet}$).

We put

$$E_2 \stackrel{\text{def}}{=} \{e_2 \in e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) \mid e_2 \in \Gamma(v_{1,a}) \cap \Gamma(v_{1,b})$$
$$\text{for some } v_{1,a}, v_{1,b} \in v(\Gamma_{X_1^\bullet}) \text{ such that } v_{1,a} \neq v_{1,b}\} \subseteq e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}),$$

where $\Gamma(v_{1,a}) \cap \Gamma(v_{1,b})$ denotes the intersection as topological subspaces of $\Gamma_{X_2^\bullet}$ (2.1.2 (a)). Note that the above claim implies

$$\sum_{v_1 \in v(\Gamma_{X_1^\bullet})} \#(L[v_1]) + \#(E_2) = \#(\phi^{\mathrm{sg,cl}}(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}))) = \#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})).$$

Let $\Pi_{\widehat{v}_1}$, $v_1 \in v(\Gamma_{X_1^\bullet})$, be an arbitrary vertex-like subgroup of $\Pi_{X_1^\bullet}$ and $I_{\widehat{e}_1}$ an open-edge-like subgroup (resp. a closed-edge-like subgroup) of $\Pi_{X_1^\bullet}$ such that $I_{\widehat{e}_1} \subseteq \Pi_{\widehat{v}_1}$. Then Theorem 3.2 (a) (resp. Lemma 3.5) implies that $\phi(I_{\widehat{e}_1})$ is an open-edge-like subgroup (resp. a closed-edge-like subgroup) of $\Pi_{X_2^\bullet}$ contained in a geometry-like subgroup $\Pi_{\widehat{\Gamma(v_1)}}$ of $\Pi_{X_2^\bullet}$. Moreover, we have $n_{v_1} \leq n_{\Gamma(v_1)}$ for all $v_1 \in v(\Gamma_{X_1^\bullet})$. On the other hand, since $n_X = n_{X_1} = n_{X_2}$, we have

$$\sum_{v_1 \in v(\Gamma_{X_1^\bullet})} n_{v_1} = n_{X_1} + 2\#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) = n_{X_2} + \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} 2\#(L[v_1]) + 2\#(E_2) = \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} n_{\Gamma(v_1)}.$$

This implies $n_{v_1} = n_{\Gamma(v_1)}$ for all $v_1 \in v(\Gamma_{X_1^\bullet})$. Then to verify the lemma, it's sufficient to prove $g_{v_1} = g_{\Gamma(v_1)}$ for all $v_1 \in v(\Gamma_{X_1^\bullet})$.

We put $I_{\widehat{v}_1} \subseteq \Pi_{\widehat{v}_1}$ the normal closed subgroup generated by (see 2.1.1 (b) for $e^{\Gamma_{X_1^\bullet}}(v_1)$)

$$\{I_{\widehat{e}_1} \mid \widehat{e}_1 \in \pi_{X_1}^{-1}(e_1),\ e_1 \in e^{\Gamma_{X_1^\bullet}}(v_1)\}.$$

Then since $n_{v_1} = n_{\Gamma(v_1)}$, the surjection $\Pi_{\widehat{v}_1} \twoheadrightarrow \Pi_{\widehat{\Gamma(v_1)}}$ induced by ϕ implies

$$g_{v_1} = \frac{1}{2} \cdot \mathrm{rank}_{\widehat{\mathbb{Z}}^{p'}}((\Pi_{\widehat{v}_1}/I_{\widehat{v}_1})^{p'}) \geq \frac{1}{2} \cdot \mathrm{rank}_{\widehat{\mathbb{Z}}^{p'}}((\phi(\Pi_{\widehat{v}_1})/\phi(I_{\widehat{v}_1}))^{p'}) = g_{\Gamma(v_1)}.$$

By the Euler-Poincaré formula for semi-graphs, we obtain

$$g_{X_1} = \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} g_{v_1} + r_{\Gamma_{X_1^\bullet}} = \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} g_{v_1} + \#(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})) - \#(v(\Gamma_{X_1^\bullet})) + 1.$$

On the other hand, by Lemma 3.7 and the Euler-Poincaré formula for semi-graphs, we have

$$\begin{aligned} g_{X_2} &= \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} g_{\Gamma(v_1)} + \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} \#(L[v_1]) + \#(E_2) - \#(\{\Gamma(v_1)\}_{v_1 \in v(\Gamma_{X_1^\bullet})}) + 1 \\ &= \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} g_{\Gamma(v_1)} + \#(\phi^{\mathrm{sg,cl}}(e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}))) - \#(v(\Gamma_{X_1^\bullet})) + 1 \\ &= \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} g_{\Gamma(v_1)} + r_{\Gamma_{X_1^\bullet}}. \end{aligned}$$

Since $g_X = g_{X_1} = g_{X_2}$, we obtain

$$\sum_{v_1 \in v(\Gamma_{X_1^\bullet})} g_{v_1} = \sum_{v_1 \in v(\Gamma_{X_1^\bullet})} g_{\Gamma(v_1)}.$$

This implies $g_{v_1} = g_{\Gamma(v_1)}$ for all $v_1 \in v(\Gamma_{X_1^\bullet})$. We complete the proof of the lemma. $\square$

Thus, Lemma 3.7 and Lemma 3.8 implies the following:

Proposition 3.9. *Let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$ be an arbitrary open continuous homomorphism. Suppose $\phi(\Pi_1) \in \mathrm{Geo}(\Pi_{X_2^\bullet})$ for all $\Pi_1 \in \mathrm{Ver}(\Pi_{X_1^\bullet})$. Then ϕ is a group-theoretical specialization homomorphism.*

4. Topological specialization conjecture for curves of $g_X = 0$

In this section, we will prove the topological specialization conjecture for pointed stable curves of genus 0 (see Theorem 4.9 for a precise statement).

4.0.1. **Settings.** We maintain the notation introduced in 3.1.1. Suppose that $g_X = 0$, and that $\Pi_{X_i^\bullet}$, $i \in \{1, 2\}$, is the maximal *pro-solvable* quotient of the admissible fundamental group of $X_i^\bullet$. Moreover, we fix the following notation.

Let $E_i \subseteq e^{\mathrm{op}}(\Gamma_{X_i^\bullet})$ be a subset of open edges of $\Gamma_{X_i^\bullet}$ such that $\#(E_i) \leq n_X - 3$ and $\phi^{\mathrm{sg,op}}(E_1) = E_2$, where $\phi^{\mathrm{sg,op}}$ is the bijection of the sets of open edges induced by ϕ (see Theorem 3.2 (a)). We put $\widehat{E}_i \overset{\mathrm{def}}{=} \pi_{X_i}^{-1}(E_i) \subseteq e^{\mathrm{op}}(\widehat{\Gamma}_{X_i^\bullet})$ (see 2.3.2 for π_{X_i}) and

$$I_{E_i} \subseteq \Pi_{X_i^\bullet}$$

the closed normal subgroup generated by $\{I_{\widehat{e}_i}\}_{\widehat{e}_i \in \widehat{E}_i}$. Moreover, Theorem 3.2 (a) implies $\phi(I_{E_1}) = I_{E_2}$.

On the other hand, we write $D_{E_i} \subseteq D_{X_i}$ for the subset of marked points corresponding to E_i. Since $g_X = 0$ and $\#(D_{E_i}) \leq n_X - 3$, by contracting certain (-1)-curves and (-2)-curves, the pointed semi-stable curve $(X_i, D_{X_i} \setminus D_{E_i})$ over k_i determines a pointed stable curve

$$X^\bullet_{E_i} = (X_{E_i}, D_{X_{E_i}})$$

of type $(0, \#(D_{X_i} \setminus D_{E_i}))$ over k_i. Note that we have a natural (contracting) morphism

$$f^\bullet_{E_i} : X^\bullet \to X^\bullet_{E_i}.$$

We shall denote by $f_{E_i} : X \to X_{E_i}$ the morphism of underlying curves induced by $f^\bullet_{E_i}$. Write $\Gamma_{X^\bullet_{E_i}}$ for the dual semi-graph of $X^\bullet_{E_i}$. Then $f^\bullet_{E_i}$ induces a map $f^{\mathrm{sg}}_{E_i} : \Gamma_{X_i^\bullet} \to \Gamma_{X^\bullet_{E_i}}$ of dual semi-graphs.

We denote by

$$\Pi_{X^\bullet_{E_i}}$$

the maximal pro-solvable quotient of the admissible fundamental group of $X^\bullet_{E_i}$. Then we have a natural isomorphism $\Pi_{X^\bullet_i}/I_{E_i} \xrightarrow{\sim} \Pi_{X^\bullet_{E_i}}$. Moreover, ϕ induces an open continuous homomorphism

$$\phi_E : \Pi_{X^\bullet_{E_1}} \to \Pi_{X^\bullet_{E_2}}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} \Pi_{X^\bullet_1} & \xrightarrow{\phi} & \Pi_{X^\bullet_2} \\ \downarrow & & \downarrow \\ \Pi_{X^\bullet_1}/I_{E_1} \cong \Pi_{X^\bullet_{E_1}} & \xrightarrow{\phi_E} & \Pi_{X^\bullet_{E_2}} \cong \Pi_{X^\bullet_2}/I_{E_2}. \end{array}$$

If $E_i = \{e_i\}$ for some $e_i \in e^{\mathrm{op}}(\Gamma_{X^\bullet_i})$, we also use the notation $X^\bullet_{e_i}$, $\Pi_{X^\bullet_{e_i}}$, $\Gamma_{X^\bullet_{e_i}}$, $f^\bullet_{e_i}$, $f^{\mathrm{sg}}_{e_i}$, and ϕ_e to denote $X^\bullet_{E_i}$, $\Pi_{X^\bullet_{E_i}}$, $\Gamma_{X^\bullet_{E_i}}$, $f^\bullet_{E_i}$, $f^{\mathrm{sg}}_{E_i}$, and ϕ_E, respectively.

4.0.2. We introduce a strong version of topological specialization homomorphisms as follows:

Definition 4.1. Let $\phi^{\mathrm{sg,op}} : e^{\mathrm{op}}(\Gamma_{X^\bullet_1}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X^\bullet_2})$ be the bijection induced by ϕ (Theorem 3.2 (a)). We shall call ϕ a *strong topological specialization homomorphism* if the following conditions are satisfied:

- ϕ is a topological specialization homomorphism (Definition 3.1 (a)).
- $\phi^{\mathrm{sg,op}} = sp^{\mathrm{com}}_{X^\bullet_1,X^\bullet_2}|_{e^{\mathrm{op}}(\Gamma_{X^\bullet_1})}$ for some $sp^{\mathrm{com}}_{X^\bullet_1,X^\bullet_2} : \mathrm{Com}(\Gamma_{X^\bullet_1}) \to \mathrm{Com}(\Gamma_{X^\bullet_2})$ (see Definition 3.1 (a) for $sp^{\mathrm{com}}_{X^\bullet_1,X^\bullet_2}$).

The following corollary follows immediately from Corollary 3.3 (b):

Corollary 4.2. *Suppose $(g_X, n_X) = (0, 4)$. Then ϕ is a strong topological specialization homomorphism.*

4.0.3. *Further settings.* We maintain the notation introduced in 4.0.1. Suppose $n_X \geq 5$. Let $e_i \in e^{\mathrm{op}}(\Gamma_{X^\bullet_i})$, $i \in \{1, 2\}$, be an open edge such that $\phi^{\mathrm{sg,op}}(e_1) = e_2$. Write $x_i \stackrel{\mathrm{def}}{=} x_{e_i} \in D_{X_i}$ for the marked point of $X^\bullet_i$ corresponding to e_i. The assumption $n_X \geq 5$ implies that $X^\bullet_{e_i}$ is a pointed stable curve of type $(0, n_X - 1)$ over k_i. Note that one of the following conditions hold:

- $\#(v(\Gamma_{X^\bullet_i})) = \#(v(\Gamma_{X^\bullet_{e_i}}))$.
- $\#(v(\Gamma_{X^\bullet_i})) = \#(v(\Gamma_{X^\bullet_{e_i}})) + 1$.

On the other hand, let $W^\bullet_i$, $i \in \{1, 2\}$, be an arbitrary pointed stable curve over k_i of type $(0, n_W)$, $\Pi_{W^\bullet_i}$ the maximal pro-solvable quotient of the admissible fundamental group of $W^\bullet_i$, and $\phi_W : \Pi_{W^\bullet_1} \to \Pi_{W^\bullet_2}$ an arbitrary open continuous homomorphism. Moreover, we assume the following conditions hold:

- ϕ_W is a strong topological specialization homomorphism if $n_W \leq n_X - 1$.

4.0.4. Firstly, we have the following lemma:

Lemma 4.3. *We maintain the settings introduced in 4.0.3. Moreover, we suppose that $\#(v(\Gamma_{X^\bullet_1})) = \#(v(\Gamma_{X^\bullet_{e_1}})) + 1$ holds. Then we have $\#(v(\Gamma_{X^\bullet_2})) = \#(v(\Gamma_{X^\bullet_{e_2}})) + 1$.*

Proof. Suppose $\#(v(\Gamma_{X^\bullet_2})) = \#(v(\Gamma_{X^\bullet_{e_2}}))$. We will construct a contradiction. Since we suppose $\#(v(\Gamma_{X^\bullet_1})) = \#(v(\Gamma_{X^\bullet_{e_1}})) + 1$, there exists an irreducible component $X_{1,1}$ of X_1 such that one of the following situations holds:

(i) $X_{1,1} \cap D_{X_1} = \{x_1\}$ and $\#(X_{1,1} \cap X^{\mathrm{sing}}_1) = 2$.
(ii) $X_{1,1} \cap D_{X_1} = \{x_1, a_1\}$ and $\#(X_{1,1} \cap X^{\mathrm{sing}}_1) = 1$, where $a_1 \neq x_1$.

On the other hand, let $X_{2,1}$ be the irreducible component of X_2 such that $x_2 \in X_{2,1} \cap D_{X_2}$. Since we assume $\#(v(\Gamma_{X^\bullet_2})) = \#(v(\Gamma_{X^\bullet_{e_2}}))$, we have

$$\#(X_{2,1} \cap X^{\mathrm{sing}}_2) + \#(X_{2,1} \cap D_{X_2}) \geq 4.$$

Case (i). We assume that (i) holds. Then we see immediately that there exist marked points $s_2, b_2, c_2 \in D_{X_2} \setminus \{x_2\}$ of $X_2^\bullet$ distinct from each other satisfying the following condition:

- For $m \in \{s_2, b_2, c_2\}$, put C_m as follows:
  $(*)$ If m is contained in $X_{2,1}$, then $C_m = X_{2,1}$. Otherwise, let C_m be the connected component of $X_2 \setminus X_{2,1}$ containing m.
  $(**)$ Let $m_1, m_2 \in \{s_2, b_2, c_2\}$ be elements distinct from each other. Then $C_{m_1} \neq C_{m_2}$ if $C_{m_1} \neq X_{2,1}$ and $C_{m_2} \neq X_{2,1}$.

Let $e_{s_2}, e_{b_2}, e_{c_2} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ be the open edges corresponding to s_2, b_2, c_2, respectively. We put $e_{s_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{s_2}) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$, $e_{b_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{b_2}) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$, and $e_{c_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{c_2}) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$.

We put $E_i \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{s_i}, e_{b_i}, e_{c_i}\}$. By the above constructions, we see immediately that X_{E_1} is *singular* with two irreducible components, and that X_{E_2} is *non-singular*. On the other hand, by applying [Y6, Lemma 6.3] for $\phi_E : \Pi_{X_{E_1}^\bullet} \to \Pi_{X_{E_2}^\bullet}$, we obtain that X_{E_2} is singular. This contradicts our construction of $X_{E_2}^\bullet$. Then we obtain the lemma under the assumption of (i).

Case (ii). We assume that (ii) holds. Let $e_{a_1} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ be the open edge corresponding to a_1, $e_{a_2} \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{a_1}) \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$, and $a_2 \in D_{X_2}$ the marked point corresponding to e_{a_2}. Moreover, we see immediately that there exist marked points $b_2, c_2 \in D_{X_2} \setminus \{x_2, a_2\}$ of $X_2^\bullet$ distinct from each other satisfying the following condition:

- For $m \in \{a_2, b_2, c_2\}$, put C_m as follows:
  $(*)$ If m is contained in $X_{2,1}$, then $C_m = X_{2,1}$. Otherwise, let C_m be the connected component of $X_2 \setminus X_{2,1}$ containing m.
  $(**)$ Let $m_1, m_2 \in \{a_2, b_2, c_2\}$ be elements distinct from each other. Then $C_{m_1} \neq C_{m_2}$ if $C_{m_1} \neq X_{2,1}$ and $C_{m_2} \neq X_{2,1}$.

We put $e_{b_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{b_2}) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$, $e_{c_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{c_2}) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$. Note that since $\#(X_{1,1} \cap X^{\mathrm{sing}}) = 1$, the marked points b_1, c_1 corresponding to e_{b_1}, e_{c_1}, respectively, are contained in $X_1 \setminus X_{1,1}$.

We put $E_i \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. By the above constructions, we see immediately that X_{E_1} is *singular* with two irreducible components, and that X_{E_2} is *non-singular*. On the other hand, by applying [Y6, Lemma 6.3] for $\phi_E : \Pi_{X_{E_1}^\bullet} \to \Pi_{X_{E_2}^\bullet}$, we obtain that X_{E_2} is singular. This contradicts our construction of $X_{E_2}^\bullet$. Then we obtain the lemma under the assumption of (ii). This completes the proof of the lemma. $\square$

4.0.5. We maintain the settings introduced in 4.0.3. Let $v_i \in v(\Gamma_{X_i^\bullet})$, $i \in \{1, 2\}$, be the vertex of $\Gamma_{X_i^\bullet}$ such that the corresponding irreducible component X_{v_i} contains x_i (see 4.0.3 for x_i). Note that $f_{e_i}(X_{v_i})$ is either a marked point or a node of $X_{e_i}^\bullet$ if $\#(v(\Gamma_{X_i^\bullet})) = \#(v(\Gamma_{X_{e_i}^\bullet})) + 1$ (see 4.0.1 for f_{e_i} and $X_{e_i}^\bullet$), and that $f_{e_i}(X_{v_i})$ is an irreducible component of $X_{e_i}^\bullet$ if $\#(v(\Gamma_{X_i^\bullet})) = \#(v(\Gamma_{X_{e_i}^\bullet}))$.

We define $X_{v_{e_i}}$ to be an irreducible component of X_{e_i} as follows:

- The irreducible component of X_{e_i} containing $f_{e_i}(X_{v_i})$ if $\#(v(\Gamma_{X_i^\bullet})) = \#(v(\Gamma_{X_{e_i}^\bullet})) + 1$ and $f_{e_i}(X_{v_i}) \in D_{X_{e_i}}$.
- The irreducible component $f_{e_i}(X_{v_i})$ of X_{e_i} if $\#(v(\Gamma_{X_i^\bullet})) = \#(v(\Gamma_{X_{e_i}^\bullet}))$.

Moreover, if $\#(v(\Gamma_{X_i^\bullet})) = \#(v(\Gamma_{X_{e_i}^\bullet})) + 1$ and $f_{e_i}(X_{v_i}) \in X_{e_i}^{\mathrm{sing}}$, we define $X_{v_{e_i}^1}$ and $X_{v_{e_i}^2}$ to be the irreducible components of X_{e_i} as follows:

- The irreducible components of X_{e_i} such that $f_{e_i}(X_{v_i}) \in X_{v_{e_i}^1} \cap X_{v_{e_i}^2}$.

We shall write $v_{e_i}, v_{e_i}^1, v_{e_i}^2 \in v(\Gamma_{X_{e_i}^\bullet})$ for the vertices of $\Gamma_{X_{e_i}^\bullet}$ corresponding to $X_{v_{e_i}}, X_{v_{e_i}^1}, X_{v_{e_i}^2}$, respectively, and $X_{v_{e_i}}^\bullet, X_{v_{e_i}^1}^\bullet, X_{v_{e_i}^2}^\bullet$ for the smooth pointed stable curves over k_i associated to $v_{e_i}, v_{e_i}^1, v_{e_i}^2$, respectively (2.2.5).

Note that the type of $X^\bullet_{e_1}$ is $(0, n_X - 1)$. The assumption of 4.0.3 concerning $W^\bullet_i$ says that $\phi_e : \Pi_{X_{e^\bullet_1}} \to \Pi_{X_{e^\bullet_2}}$ is a strong topological specialization homomorphism. Then there exists a map

$$sp^{\mathrm{com}}_{X^\bullet_{e_1}, X^\bullet_{e_2}} : \mathrm{Com}(\Gamma_{X^\bullet_{e_1}}) \to \mathrm{Com}(\Gamma_{X^\bullet_{e_2}})$$

such that $sp^{\mathrm{com}}_{X^\bullet_{e_1}, X^\bullet_{e_2}}|_{e^{\mathrm{op}}(\Gamma_{X^\bullet_{e_1}})} = \phi^{\mathrm{sg,op}}_e$.

Let $(\Gamma_{v_{e_1}}, e^{\mathrm{lp}}(\Gamma_{v_{e_1}}) = \emptyset), (\Gamma_{v^1_{e_1}}, e^{\mathrm{lp}}(\Gamma_{v^1_{e_1}}) = \emptyset), (\Gamma_{v^2_{e_1}}, e^{\mathrm{lp}}(\Gamma_{v^2_{e_1}}) = \emptyset) \in \mathrm{Com}(\Gamma_{X^\bullet_{e_1}})$ be combinatorial data associated to $X^\bullet_{e_1}$ (see 2.2.5 for $\Gamma_{v_{e_1}}, \Gamma_{v^1_{e_1}}, \Gamma_{v^2_{e_1}}$). Then we put $(\Gamma_2, \emptyset) \overset{\mathrm{def}}{=} sp^{\mathrm{com}}_{X^\bullet_{e_1}, X^\bullet_{e_2}}((\Gamma_{v_{e_1}}, \emptyset)), (\Gamma^1_2, \emptyset) \overset{\mathrm{def}}{=} sp^{\mathrm{com}}_{X^\bullet_{e_1}, X^\bullet_{e_2}}((\Gamma_{v^1_{e_1}}, \emptyset)), (\Gamma^2_2, \emptyset) \overset{\mathrm{def}}{=} sp^{\mathrm{com}}_{X^\bullet_{e_1}, X^\bullet_{e_2}}((\Gamma_{v^2_{e_1}}, \emptyset)) \in \mathrm{Com}(\Gamma_{X^\bullet_{e_2}})$. Moreover, we shall denote by

$$X^\bullet_{\Gamma_2} = (X_{\Gamma_2}, D_{X_{\Gamma_2}}), X^\bullet_{\Gamma^1_2} = (X_{\Gamma^1_2}, D_{X_{\Gamma^1_2}}),\ X^\bullet_{\Gamma^2_2} = (X_{\Gamma^2_2}, D_{X_{\Gamma^2_2}})$$

the pointed stable curves over k_2 associated to Γ_2, Γ^1_2, Γ^2_2, respectively (2.2.3). Then we have the following lemmas.

Lemma 4.4. *We maintain the settings introduced in 4.0.3 and the notation introduced at the beginning of 4.0.5. Moreover, we suppose that $\#(v(\Gamma_{X^\bullet_1})) = \#(v(\Gamma_{X^\bullet_{e_1}}))$ and $\#(v(\Gamma_{X^\bullet_2})) = \#(v(\Gamma_{X^\bullet_{e_2}}))$ hold. Then $X_{v_{e_2}}$ is an irreducible component of X_{Γ_2}.*

Proof. Suppose $\#(v(\Gamma_{X^\bullet_1})) = 1$ (i.e. $X^\bullet_1$ is non-singular). Then the lemma is trivial. To verify the lemma, we suppose $\#(v(\Gamma_{X^\bullet_1})) \geq 2$ (i.e. $X^\bullet_1$ is singular). Moreover, suppose that $X_{v_{e_2}}$ is *not* an irreducible component of X_{Γ_2}. We will construct a contradiction.

Since $\#(v(\Gamma_{X^\bullet_2})) = \#(v(\Gamma_{X^\bullet_{e_2}}))$, one of the following holds:

(1) $\#(X_{v_2} \cap X^{\mathrm{sing}}_2) = 1$ (i.e. $\#(\pi_0(X_2 \setminus X_{v_2})) = 1$). Then in this situation, we have $\#(X_{v_2} \cap D_{X_2}) \geq 3$.

(2) $\#(X_{v_2} \cap X^{\mathrm{sing}}_2) \geq 2$ (i.e. $\#(\pi_0(X_2 \setminus X_{v_2})) \geq 2$). Then in this situation, we have

$$\#(\pi_0(X_2 \setminus X_{v_2})) + \#(D_{X_2} \cap X_{v_2}) \geq 4.$$

Thus, there exists a connected component $C_2 \in \pi_0(X_2 \setminus X_{v_2})$ such that X_{Γ_2} is contained in the topological closure $\overline{f_{e_2}(C_2)}$ of $f_{e_2}(C_2)$ in X_{e_2}. Let $a_2 \in D_{X_2} \cap C_2$ be a marked point of $X^\bullet_2$.

On the other hand, let $b_2, c_2 \in D_{X_2} \setminus ((D_{X_2} \cap C_2) \cup \{x_2\})$ be marked points distinct from each other such that the following conditions are satisfied:

- If $\#(\pi_0(X_2 \setminus X_{v_2})) = 1$, then b_2, c_2 are contained in $X_{v_2} \cap D_{X_2}$.
- If $\#(\pi_0(X_2 \setminus X_{v_2})) = 2$ (this implies $\#(D_{X_2} \cap X_{v_2}) \geq 2$), then we have that $b_2 \in D_{X_2} \cap X_{v_2}$, and that c_2 is a marked point contained in the connected component of $\pi_0(X_2 \setminus X_{v_2})$ distinct from C_2.
- If $\#(\pi_0(X_2 \setminus X_{v_2})) \geq 3$, then b_2, c_2 are contained in two different connected components of $\pi_0(X_2 \setminus X_{v_2})$ distinct from C_2.

We denote by $e_{a_2}, e_{b_2}, e_{c_2} \in e^{\mathrm{op}}(\Gamma_{X^\bullet_2})$ the open edges of $\Gamma_{X^\bullet_2}$ corresponding to a_2, b_2, c_2, respectively. Moreover, we put

$$e_{a_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{a_2}),\ e_{b_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{b_2}),\ e_{c_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{c_2}) \in e^{\mathrm{op}}(\Gamma_{X^\bullet_1}).$$

We write $a_1, b_1, c_1 \in D_{X_1}$ for the marked points of $X^\bullet_1$ corresponding to $e_{a_1}, e_{b_1}, e_{c_1}$, respectively.

We put $E_i \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma_{X^\bullet_i}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. Note that $X^\bullet_{E_i}$ is a pointed stable curve of type $(0, 4)$ over k_i. Then we obtain an open continuous homomorphism $\phi_E : \Pi_{X^\bullet_{E_1}} \to \Pi_{X^\bullet_{E_2}}$. Moreover, the above constructions implies that $X^\bullet_{E_2}$ is *smooth* over k_2. On the other hand, since we assume that $\phi_e : \Pi_{X^\bullet_{e_1}} \to \Pi_{X^\bullet_{e_2}}$ is a strong topological specialization homomorphism (4.0.3), this implies that $X^\bullet_{E_1}$ is *singular*, that the irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(a_1)$, respectively, are equal, and that the irreducible components containing $f_{E_1}(b_1)$ and $f_{E_1}(c_1)$, respectively, are equal. This contradicts [Y6, Lemma 6.3]. Then we complete the proof of the lemma. □

Lemma 4.5. *We maintain the settings introduced in 4.0.3 and the notation introduced at the beginning of 4.0.5. Moreover, we suppose that* $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet}))$, *that* $\#(v(\Gamma_{X_2^\bullet})) = \#(v(\Gamma_{X_{e_2}^\bullet}))+1$, *and that* $f_{e_2}(X_{v_2})$ *is a marked point of* $X_{e_2}^\bullet$. *Then* $X_{v_{e_2}}$ *is an irreducible component of* X_{Γ_2}.

Proof. Suppose $\#(v(\Gamma_{X_1^\bullet})) = 1$ (i.e. $X_1^\bullet$ is non-singular). Then the lemma is trivial. To verify the lemma, we suppose $\#(v(\Gamma_{X_1^\bullet})) \geq 2$ (i.e. $X_1^\bullet$ is singular). Moreover, suppose that $X_{v_{e_2}}$ is *not* an irreducible component of X_{Γ_2}. We will construct a contradiction.

Since $f_{e_2}(X_{v_2})$ is a marked point of $X_{e_2}^\bullet$, we have $\#(D_{X_2} \cap X_{v_2}) = 2$. Then we have $D_{X_2} \cap X_{v_2} \overset{\text{def}}{=} \{x_2, a_2\}$. Moreover, we see that there exists a connected component $C_{e_2} \in \pi_0(X_{e_2} \setminus X_{v_{e_2}})$ such that X_{Γ_2} is contained in $\overline{C}_{e_2}$. Let $b_2 \in D_{X_2} \setminus \{x_2, a_2\}$ be a marked point of $X_2^\bullet$ such that $f_{e_2}(b_2)$ is contained in C_{e_2}.

On the other hand, let $c_2 \in D_{X_2} \setminus \{x_2, a_2, b_2\}$ be a marked point of $X_2^\bullet$ such that the following conditions are satisfied:

- If $\#(\pi_0(X_{e_2} \setminus X_{v_{e_2}})) = 1$ (this implies $\#(D_{X_{e_2}} \cap X_{v_{e_2}}) \geq 2$), then $f_{e_2}(c_2)$ is contained in $D_{X_{e_2}} \cap X_{v_{e_2}}$. Note that we have $\{f_{e_2}(a_2), f_{e_2}(c_2)\} \subseteq D_{X_{e_2}} \cap X_{v_{e_2}}$.
- If $\#(\pi_0(X_{e_2} \setminus X_{v_{e_2}})) \geq 2$, then $f_{e_2}(c_2)$ is contained in a connected component of $X_{e_2} \setminus X_{v_{e_2}}$ which is distinct from C_{e_2}.

We denote by $e_{a_2}, e_{b_2}, e_{c_2} \in e^{\text{op}}(\Gamma_{X_2^\bullet})$ the open edges of $\Gamma_{X_2^\bullet}$ corresponding to a_2, b_2, c_2, respectively. Moreover, we put

$$e_{a_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{a_2}),\ e_{b_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{b_2}),\ e_{c_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{c_2}) \in e^{\text{op}}(\Gamma_{X_1^\bullet}).$$

Write $a_1, b_1, c_1 \in D_{X_1}$ for the marked points of $X_1^\bullet$ corresponding to $e_{a_1}, e_{b_1}, e_{c_1}$, respectively.

We put $E_i \overset{\text{def}}{=} e^{\text{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. Then we obtain an open continuous homomorphism $\phi_E : \Pi_{X_{E_1}^\bullet} \to \Pi_{X_{E_2}^\bullet}$. The above constructions implies that $X_{E_2}^\bullet$ is a *singular* pointed stable curve of type $(0, 4)$ over k_2. Moreover, we see that X_{E_2} has two irreducible components, that the irreducible components containing $f_{E_2}(x_2)$ and $f_{E_2}(a_2)$, respectively, are equal, and that the irreducible components containing $f_{E_2}(b_2)$ and $f_{E_2}(c_2)$, respectively, are equal.

On the other hand, since $\phi_e : \Pi_{X_{e_1}^\bullet} \to \Pi_{X_{e_2}^\bullet}$ is a strong topological specialization homomorphism (4.0.3), we see that $X_{E_1}^\bullet$ is *singular*. Moreover, the above constructions imply that the irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(b_1)$, respectively, are equal, and that the irreducible components containing $f_{E_1}(a_1)$ and $f_{E_1}(c_1)$, respectively, are equal.

By applying Corollary 3.3 (b), we obtain that the irreducible components of X_{E_2} containing $f_{E_2}(x_2)$ and $f_{E_2}(b_2)$, respectively, are equal. This contradicts our construction of $X_{E_2}^\bullet$. We complete the proof of the lemma. $\square$

Lemma 4.6. *We maintain the settings introduced in 4.0.3 and the notation introduced at the beginning of 4.0.5. Moreover, we suppose that* $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet}))$, *that* $\#(v(\Gamma_{X_2^\bullet})) = \#(v(\Gamma_{X_{e_2}^\bullet}))+1$, *and that* $f_{e_2}(X_{v_2})$ *is a node of* X_{e_2}. *Then* $f_{e_2}(X_{v_2})$ *is contained in* X_{Γ_2}.

Proof. Suppose $\#(v(\Gamma_{X_1^\bullet})) = 1$ (i.e. $X_1^\bullet$ is non-singular). Then the lemma is trivial. To verify the lemma, we suppose $\#(v(\Gamma_{X_1^\bullet})) \geq 2$ (i.e. $X_1^\bullet$ is singular). Moreover, suppose that $f_{e_2}(X_{v_2})$ is a node which is *not* contained in X_{Γ_2}. We will construct a contradiction.

Since $f_{e_2}(X_{v_2})$ is a node of X_{e_2}, we have $\#(\pi_0(X_{e_2} \setminus f_{e_2}(X_{v_2}))) = 2$. Then there exists a connected component $C_{e_2} \in \pi_0(X_{e_2} \setminus f_{e_2}(X_{v_2}))$ such that X_{Γ_2} is contained in $\overline{C}_{e_2}$. Moreover, since $f_{e_2}(X_{v_2})$ is not contained in X_{Γ_2}, there exists a unique connected component $Z \in \pi_0(X_2 \setminus (X_{v_2} \cup f_{e_2}^{-1}(X_{\Gamma_2})))$ such that $f_{e_2}(Z) \subseteq C_{e_2}$, and that $\overline{Z} \cap X_{v_2} \neq \emptyset$.

Let $a_2 \in (D_{X_2} \setminus \{x_2\}) \cap Z$, $b_2 \in (D_{X_2} \setminus \{x_2\}) \cap (f^{-1}(C_{e_2}) \setminus Z)$, and $c_2 \in (D_{X_2} \setminus \{x_2\})$ a marked point which is not contained in $f_{e_2}^{-1}(C_{e_2})$. We denote by $e_{a_2}, e_{b_2}, e_{c_2} \in e^{\text{op}}(\Gamma_{X_2^\bullet})$ the open edges of $\Gamma_{X_2^\bullet}$ corresponding to a_2, b_2, c_2, respectively. Moreover, we put

$$e_{a_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{a_2}),\ e_{b_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{b_2}),\ e_{c_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{c_2}) \in e^{\text{op}}(\Gamma_{X_1^\bullet}).$$

Write $a_1, b_1, c_1 \in D_{X_1}$ for the marked points corresponding to $e_{a_1}, e_{b_1}, e_{c_1}$, respectively.

We put $E_i \overset{\text{def}}{=} e^{\text{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. Then we obtain an open continuous homomorphism $\phi_E : \Pi_{X^\bullet_{E_1}} \to \Pi_{X^\bullet_{E_2}}$. The above constructions implies that $X^\bullet_{E_2}$ is a *singular* pointed stable curve of type $(0, 4)$ over k_2. Moreover, we see that X_{E_2} has two irreducible components, that the irreducible components containing $f_{E_2}(x_2)$ and $f_{E_2}(c_2)$, respectively, are equal, and that the irreducible components containing $f_{E_2}(a_2)$ and $f_{E_2}(b_2)$, respectively, are equal.

On the other hand, since $\phi_e : \Pi_{X^\bullet_{e_1}} \to \Pi_{X^\bullet_{e_2}}$ is a strong topological specialization homomorphism (4.0.3), we see that $X^\bullet_{E_1}$ is *singular*. Moreover, the above constructions implies that the irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(b_1)$, respectively, are equal, and that the irreducible components containing $f_{E_1}(a_1)$ and $f_{E_1}(c_1)$, respectively, are equal.

By applying Corollary 3.3 (b), we obtain that the irreducible components of X_{E_2} containing $f_{E_2}(x_2)$ and $f_{E_2}(b_2)$, respectively, are equal. This contradicts our construction of $X^\bullet_{E_2}$. We complete the proof of the lemma. $\square$

Lemma 4.7. *We maintain the settings introduced in 4.0.3 and the notation introduced at the beginning of 4.0.5. Moreover, we suppose that $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X^\bullet_{e_1}})) + 1$ holds, and that $f_{e_1}(x_1) \in D_{X_{e_1}}$ is a marked point of $X^\bullet_{e_1}$. Then the following statements hold: (i) $f_{e_2}(x_2) \in D_{X_{e_2}}$ is a marked point of $X^\bullet_{e_2}$. (ii) $X_{v_{e_2}}$ is an irreducible component of X_{Γ_2}.*

Proof. (i) Suppose that $f_{e_2}(x_2)$ is *not* a marked point of $X^\bullet_{e_2}$. We will construct a contradiction. Note that Lemma 4.3 implies that $f_{e_2}(x_2)$ is a node of X_{e_2}.

Since $f_{e_1}(x_1) \in D_{X_{e_1}}$ is a marked point of $X^\bullet_{e_1}$, there exists a unique marked point $a_1 \in D_{X_1} \setminus \{x_1\}$ such that a_1 is contained in X_{v_1}. We write $e_{a_1} \in e^{\text{op}}(\Gamma_{X_1^\bullet})$ for the open edge corresponding to $a_1, e_{a_2} \overset{\text{def}}{=} \phi^{\text{sg,op}}(e_{a_1})$, and $a_2 \in D_{X_2}$ for the marked point of $X_2^\bullet$ corresponding to e_{a_2}. Then we see immediately that a_2 is contained in a connected component $C_2 \in \pi_0(X_2 \setminus X_{v_2})$. Moreover, we note that $\#(D_{X_2} \cap C_2) \geq 2$. Then we take $b_2 \in (D_{X_2} \cap C_2) \setminus \{a_2\}$. On the other hand, let $c_2 \in D_{X_2} \setminus \{x_2, a_2, b_2\}$ be a marked point of $X_2^\bullet$ such that c_2 is contained in a connected component of $X_2 \setminus X_{v_2}$ distinct from C_2. We denote by $e_{b_2}, e_{c_2} \in e^{\text{op}}(\Gamma_{X_2^\bullet})$ the open edges of $\Gamma_{X_2^\bullet}$ corresponding to b_2, c_2, respectively. Moreover, we put

$$e_{b_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{b_2}),\ e_{c_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{c_2}) \in e^{\text{op}}(\Gamma_{X_1^\bullet}).$$

Write $b_1, c_1 \in D_{X_1}$ for the marked points corresponding to e_{b_1} and e_{c_1}, respectively.

We put $E_i \overset{\text{def}}{=} e^{\text{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. Then we obtain an open continuous homomorphism $\phi_E : \Pi_{X^\bullet_{E_1}} \to \Pi_{X^\bullet_{E_2}}$. The above constructions imply that $X^\bullet_{E_2}$ is a *singular* pointed stable curve of type $(0, 4)$ over k_2. Moreover, we see that X_{E_2} has two irreducible components, that the irreducible components containing $f_{E_2}(x_2)$ and $f_{E_2}(c_2)$, respectively, are equal, and that the irreducible components containing $f_{E_2}(a_2)$ and $f_{E_2}(b_2)$, respectively, are equal.

On the other hand, since $\phi_e : \Pi_{X^\bullet_{e_1}} \to \Pi_{X^\bullet_{e_2}}$ is a strong topological specialization homomorphism (4.0.3), we see that $X^\bullet_{E_1}$ is *singular*, that the irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(a_1)$, respectively, are equal, and that the irreducible components containing $f_{E_1}(b_1)$ and $f_{E_1}(c_1)$, respectively, are equal. By applying Corollary 3.3 (b), we obtain that the irreducible components of X_{E_2} containing $f_{E_2}(x_2)$ and $f_{E_2}(a_2)$, respectively, are equal. This contradicts our construction of $X^\bullet_{E_2}$. We complete the proof of (i).

(ii) Suppose that $X_{v_{e_2}}$ is *not* an irreducible component of X_{Γ_2}. We will construct a contradiction. Since $f_{e_1}(x_1)$ (resp. $f_{e_2}(x_2)$) is a marked point, there exist a unique marked point $a_1 \in D_{X_1} \setminus \{x_1\}$ (resp. $b_2 \in D_{X_2} \setminus \{x_2\}$) such that a_1 is contained in X_{v_1} (resp. b_2 is contained in X_{v_2}).

We denote by $e_{a_1} \in e^{\text{op}}(\Gamma_{X_1^\bullet})$, $e_{b_2} \in e^{\text{op}}(\Gamma_{X_2^\bullet})$ the open edges of $\Gamma_{X_1^\bullet}$ and $\Gamma_{X_2^\bullet}$ corresponding to a_1, b_2, respectively. Moreover, we put

$$e_{a_2} \overset{\text{def}}{=} \phi^{\text{sg,op}}(e_{a_1}) \in e^{\text{op}}(\Gamma_{X_2^\bullet}),\ e_{b_1} \overset{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{b_2}) \in e^{\text{op}}(\Gamma_{X_1^\bullet}).$$

Write $a_2 \in D_{X_2}$ and $b_1 \in D_{X_1}$ for the marked points corresponding to e_{a_2} and e_{b_1}, respectively. Note that since we assume that $X_{v_{e_2}}$ is not an irreducible component of X_{Γ_2}, we have $a_1 \neq b_1$ and $a_2 \neq b_2$. Furthermore, we have that $b_1 \notin X_{v_1}$, and that there exists a connected component C_1 of $X_1 \setminus X_{v_1}$

such that b_1 is contained in C_1. We take $c_1 \in (D_{X_1} \cap C_1) \setminus \{b_1\}$ a marked point of $X_1^\bullet$ and write $e_{c_1} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ for the open edge of $\Gamma_{X_1^\bullet}$ corresponding to c_1. We put $e_{c_2} \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{c_1})$ and write $c_2 \in D_{X_2}$ for the marked point corresponding to e_{c_2}.

We put $E_i \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. Then we obtain an open continuous homomorphism $\phi_E : \Pi_{X_{E_1}^\bullet} \to \Pi_{X_{E_2}^\bullet}$. Moreover, the above constructions imply that $X_{E_2}^\bullet$ is a *singular* pointed stable curve of type $(0,4)$ over k_2 such that X_{E_2} has two irreducible components, that the irreducible components containing $f_{E_2}(x_2)$ and $f_{E_1}(b_2)$, respectively, are equal, and that the irreducible components containing $f_{E_2}(a_2)$ and $f_{E_2}(c_2)$, respectively, are equal.

On the other hand, the above constructions imply that $X_{E_1}^\bullet$ is a *singular* pointed stable curve of type $(0,4)$ over k_1 such that the irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(a_1)$, respectively are equal, and that the irreducible components containing $f_{E_1}(b_1)$ and $f_{E_1}(c_1)$, respectively, are equal.

By applying Corollary 3.3 (b), we obtain that the irreducible components of X_{E_2} containing $f_{E_2}(x_2)$ and $f_{E_2}(a_2)$ are equal. This contradicts our construction of $X_{E_2}^\bullet$. We complete the proof of (ii). $\square$

Lemma 4.8. *We maintain the settings introduced in 4.0.3 and the notation introduced at the beginning of 4.0.5. Moreover, we suppose that $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet})) + 1$ holds, and that $f_{e_1}(x_1)$ is a node of X_{e_1}. Then the following statements hold: (i) $f_{e_2}(x_2)$ is a node of X_{e_2}. (ii) $f_{e_2}(x_2)$ is contained in $X_{\Gamma_2^1} \cap X_{\Gamma_2^2}$.*

Proof. (i) Suppose that $f_{e_2}(x_2)$ is *not* a node of X_{e_2}. We will construct a contradiction. Note that Lemma 4.3 implies that $f_{e_2}(x_2)$ is a marked point of $X_{e_2}^\bullet$. Then there exists a unique marked point $a_2 \in D_{X_2} \setminus \{x_2\}$ such that a_2 is contained in X_{v_2}. We write $e_{a_2} \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ for the open edge corresponding to a_2, $e_{a_1} \overset{\mathrm{def}}{=} (\phi^{\mathrm{sg,op}})^{-1}(e_{a_2})$, and $a_1 \in D_{X_1}$ for the marked point of $X_1^\bullet$ corresponding to e_{a_1}. Then a_1 is contained in a connected component $C_1 \in \pi_0(X_1 \setminus X_{v_1})$. Moreover, we note that $\#(D_{X_1} \cap C_1) \geq 2$. We take $b_1 \in (D_{X_1} \cap C_1) \setminus \{a_1\}$. On the other hand, since $\#(\pi_0(X_1 \setminus X_{v_1})) = 2$, there exists a marked point $c_1 \in D_{X_1} \setminus \{x_1, a_1, b_1\}$ such that c_1 is contained in the unique connected component of $X_1 \setminus X_{v_1}$ distinct from C_1. We denote by $e_{b_1}, e_{c_1} \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ the open edges of $\Gamma_{X_1^\bullet}$ corresponding to b_1, c_1, respectively. Moreover, we put

$$e_{b_2} \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{b_1}),\ e_{c_2} \overset{\mathrm{def}}{=} \phi^{\mathrm{sg,op}}(e_{c_1}) \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}).$$

Write $b_2, c_2 \in D_{X_2}$ for the marked points corresponding to e_{b_2} and e_{c_2}, respectively.

We put $E_i \overset{\mathrm{def}}{=} e^{\mathrm{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. Then we obtain an open continuous homomorphism $\phi_E : \Pi_{X_{E_1}^\bullet} \to \Pi_{X_{E_2}^\bullet}$. The above constructions imply that $X_{E_2}^\bullet$ is a *singular* pointed stable curve of type $(0,4)$ over k_2 such that the irreducible components containing $f_{E_2}(x_2)$ and $f_{E_2}(a_2)$, respectively, are equal, and that the irreducible components containing $f_{E_2}(b_2)$ and $f_{E_2}(c_2)$, respectively, are equal.

On the other hand, the above constructions imply that $X_{E_1}^\bullet$ is a *singular* pointed stable curve over k_1 such that the irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(c_1)$, respectively, are equal, and that the irreducible components containing $f_{E_1}(a_1)$ and $f_{E_1}(b_1)$, respectively, are equal. By applying Corollary 3.3 (b), we obtain that the irreducible components of X_{E_2} containing $f_{E_2}(x_2)$ and $f_{E_2}(c_2)$, respectively, are equal. This contradicts our construction of $X_{E_2}^\bullet$. We complete the proof of (i).

(ii) Suppose that the node $f_{e_2}(x_2)$ is *not* contained in $X_{\Gamma_2^1} \cap X_{\Gamma_2^2}$. We will construct a contradiction. Since $f_{e_2}(x_2)$ is a node and is *not* contained in $X_{\Gamma_2^1} \cap X_{\Gamma_2^2}$, either $X_{v_2} \cap f_{e_2}^{-1}(X_{\Gamma_2^1}) = \emptyset$ or $X_{v_2} \cap f_{e_2}^{-1}(X_{\Gamma_2^2}) = \emptyset$ holds. Without loss of generality, we may assume $X_{v_2} \cap f_{e_2}^{-1}(X_{\Gamma_2^2}) = \emptyset$. Then we have $\#(\pi_0(X_2 \setminus (X_{v_2} \cup f_{e_2}^{-1}(X_{\Gamma_2^2})))) \geq 2$. Moreover, let

$$C_2^1, C_2^2 \in \pi_0(X_2 \setminus (X_{v_2} \cup f_{e_2}^{-1}(X_{\Gamma_2^2})))$$

be connected components such that $\overline{C_2^1} \cap X_{v_2} \neq \emptyset$, $\overline{C_2^1} \cap f_{e_2}^{-1}(X_{\Gamma_2^1}) = \emptyset$, and $\overline{C_2^1} \cap f_{e_2}^{-1}(X_{\Gamma_2^2}) = \emptyset$, and that $f_{e_2}^{-1}(X_{\Gamma_2^1}) \subseteq \overline{C_2^2}$.

Let $a_2 \in D_{X_2} \cap C_2^1$, $b_2 \in D_{X_2} \cap C_2^2$, and $c_2 \in D_{X_2} \setminus (C_2^1 \cup C_2^2 \cup \{x_2\})$ be marked points of $X_2^\bullet$. Note that a_2, b_2, c_2 are distinct from x_2. We denote by $e_{a_2}, e_{b_2}, e_{c_2} \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ the open edges of $\Gamma_{X_2^\bullet}$

corresponding to a_2, b_2, c_2, respectively. Moreover, we put

$$e_{a_1} \stackrel{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{a_2}),\ e_{b_1} \stackrel{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{b_2}),\ e_{c_1} \stackrel{\text{def}}{=} (\phi^{\text{sg,op}})^{-1}(e_{c_2}) \in e^{\text{op}}(\Gamma_{X_1^\bullet}).$$

Write $a_1, b_1, c_1 \in D_{X_1}$ for the marked points corresponding to $e_{a_1}, e_{b_1}, e_{c_1}$, respectively.

We put $E_i \stackrel{\text{def}}{=} e^{\text{op}}(\Gamma_{X_i^\bullet}) \setminus \{e_i, e_{a_i}, e_{b_i}, e_{c_i}\}$. Then we obtain an open continuous homomorphism $\phi_E : \Pi_{X_{E_1}^\bullet} \to \Pi_{X_{E_2}^\bullet}$. The above constructions imply that $X_{E_2}^\bullet$ is a *singular* pointed stable curve of type $(0,4)$ over k_2 such that the irreducible components containing $f_{E_2}(x_2)$ and $f_{E_2}(a_2)$, respectively, are equal, and that the irreducible components containing $f_{E_2}(b_2)$ and $f_{E_2}(c_2)$, respectively, are equal.

On the other hand, since $\phi_e : \Pi_{X_{e_1}^\bullet} \to \Pi_{X_{e_2}^\bullet}$ is a strong topological specialization homomorphism (4.0.3), $X_{E_1}^\bullet$ is a *singular* pointed stable curve over k_1 of type $(0,4)$ such that one of the following cases holds:

- The irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(c_1)$, respectively, are equal, and that the irreducible components containing $f_{E_1}(a_1)$ and $f_{E_1}(b_1)$, respectively, are equal.
- The irreducible components containing $f_{E_1}(x_1)$ and $f_{E_1}(b_1)$, respectively, are equal, and that the irreducible components containing $f_{E_1}(a_1)$ and $f_{E_1}(c_1)$, respectively, are equal.

By applying Corollary 3.3 (b), we obtain that one of the following cases holds:

- The irreducible components of X_{E_2} containing $f_{E_2}(x_2)$ and $f_{E_2}(c_2)$, respectively, are equal.
- The irreducible components of X_{E_2} containing $f_{E_2}(x_2)$ and $f_{E_2}(b_2)$, respectively, are equal.

This contradicts our construction of $X_{E_2}^\bullet$. We complete the proof of (ii). □

4.0.6. The main result of the present section is the following:

Theorem 4.9. *We maintain the notation introduced in 3.1.1. Suppose that $g_X = 0$, and that $\Pi_{X_i^\bullet}$, $i \in \{1,2\}$, is either the admissible fundamental group of $X_i^\bullet$ or the maximal pro-solvable quotient of the admissible fundamental group of $X_i^\bullet$. Let $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ be an arbitrary open continuous homomorphism. Then ϕ is a strong topological specialization homomorphism. In particular, the topological specialization conjecture holds.*

Proof. Since the maximal pro-solvable quotient of the admissible fundamental groups can be reconstructed group-theoretically from the admissible fundamental groups, to verify the theorem, we may assume that $\Pi_{X_i^\bullet}$ is the maximal pro-solvable quotient of the admissible fundamental group of $X_i^\bullet$.

Suppose that $n_X = 3$. Then $X_i^\bullet$, $i \in \{1,2\}$, is a smooth pointed stable curve over k_i. The theorem follows immediately from Corollary 3.3 (a). Suppose that $n_X = 4$. Then the theorem follows immediately from Corollary 3.3 (b).

Next, suppose that the theorem holds for $3 \leq n_X \leq n-1$. We will prove the theorem holds for $n_X = n$. We maintain the settings introduced in 4.0.3 and the notation introduced at the beginning of 4.0.5. Since the theorem holds for $n_X \leq n-1$, to verify the theorem holds for $n_X = n$ (i.e. the underlying topological space of X_2 is a degeneration of the underlying topological space of X_1), it's sufficient to prove that the following statements hold:

(i) If $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet}))$ and $\#(v(\Gamma_{X_2^\bullet})) = \#(v(\Gamma_{X_{e_2}^\bullet}))$, then $X_{v_{e_2}}$ is an irreducible component of X_{Γ_2}.

(ii) If $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet}))$, $\#(v(\Gamma_{X_2^\bullet})) = \#(v(\Gamma_{X_{e_2}^\bullet})) + 1$, and $f_{e_2}(X_{v_2})$ is a marked point of $X_{e_2}^\bullet$, then $X_{v_{e_2}}$ is an irreducible component of X_{Γ_2}.

(iii) If $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet}))$, $\#(v(\Gamma_{X_2^\bullet})) = \#(v(\Gamma_{X_{e_2}^\bullet})) + 1$, and $f_{e_2}(X_{v_2})$ is a node of X_{e_2}, then $f_{e_2}(X_{v_2})$ is contained in X_{Γ_2}.

(iv) If $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet})) + 1$ and $f_{e_1}(x_1) \in D_{X_{e_1}}$ is a marked point of $X_{e_1}^\bullet$, then $f_{e_2}(x_2) \in D_{X_{e_2}}$ is a marked point of $X_{e_2}^\bullet$, and $X_{v_{e_2}}$ is an irreducible component of X_{Γ_2}.

(v) If $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet})) + 1$ and $f_{e_1}(x_1)$ is a node of X_{e_1}, then $f_{e_2}(x_2)$ is a node of X_{e_2}, and $f_{e_2}(x_2)$ is contained in $X_{\Gamma_2^1} \cap X_{\Gamma_2^2}$.

Suppose $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet}))$. Since X_{e_2} is a degeneration of X_{e_1} (as topological spaces), X_2 is a degeneration of X_1 if $f_{e_2}(x_2)$ is contained in X_{Γ_2} which is equivalent to (i), (ii), (iii) listed above.

Suppose $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet})) + 1$. If $f_{e_1}(x_1) \in D_{X_{e_1}}$ is a marked point of $X_{e_1}^\bullet$ (i.e. $f_{e_1} : X_1 \to X_{e_1}$ is a blow-up along a smooth closed point of X_{e_1}), since X_{e_2} is a degeneration of X_{e_1} (as topological spaces), X_2 is a degeneration of X_1 if $f_{e_2} : X_2 \to X_{e_2}$ is a blow-up along a smooth closed point of X_{e_2} contained in X_{Γ_2} which is equivalent to (iv).

Suppose $\#(v(\Gamma_{X_1^\bullet})) = \#(v(\Gamma_{X_{e_1}^\bullet})) + 1$. If $f_{e_1}(x_1)$ is a singular point of $X_{e_1}^\bullet$ (i.e. $f_{e_1} : X_1 \to X_{e_1}$ is a blow-up along a singular point of X_{e_1}), since X_{e_2} is a degeneration of X_{e_1} (as topological spaces), X_2 is a degeneration of X_1 if $f_{e_2} : X_2 \to X_{e_2}$ is a blow-up along a singular point of X_{e_2} contained in $X_{\Gamma_2^1} \cap X_{\Gamma_2^2}$ which is equivalent to (v).

The statements (i), (ii), (iii), (iv), (v) follow from Lemma 4.4, Lemma 4.5, Lemma 4.6, Lemma 4.7, and Lemma 4.8, respectively. We complete the proof of the theorem. $\square$

5. Group-theoretical specialization conjecture for $g_X = 0$ under assumptions

In this section, we will prove that the group-theoretical specialization conjecture holds for $g_X = 0$ if we assume that the topological specialization conjecture holds for *arbitrary* types (see Theorem 5.7).

5.1. **Boundary data.**

5.1.1. Let $W^\bullet$ be a pointed stable curve of type $(0, n)$ over an algebraically closed field k of characteristic $p > 0$ and $\Gamma_{W^\bullet}$ the dual semi-graph of $W^\bullet$. Note that since $\Gamma_{W^\bullet}$ is a tree, we have $\mathrm{Ssg}(\Gamma_{W^\bullet}) = \mathrm{Com}(\Gamma_{W^\bullet})$ (2.3.5).

Definition 5.1. Let $\mathbf{B} \in \mathrm{Com}(\Gamma_{W^\bullet})$ be a combinatorial datum associated to $W^\bullet$. We shall call $\mathbf{B}$ a *boundary* combinatorial datum (or, a boundary sub-semi-graph (2.1.2)) of $\Gamma_{W^\bullet}$ if the following conditions are satisfied:

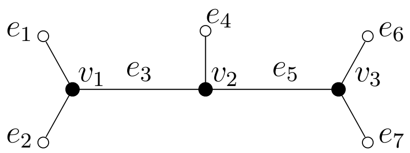
- $v(\mathbf{B}) \neq \emptyset$.
- $\Gamma_{W^\bullet} \setminus \mathbf{B}$ is connected or empty (note that we have $\mathrm{Ssg}(\Gamma_{W^\bullet}) = \mathrm{Com}(\Gamma_{W^\bullet})$).

Let $W_{\mathbf{B}}^\bullet$ be the pointed stable curve associated to $\mathbf{B}$ (2.2.4). We see that $\mathbf{B}$ is a boundary combinatorial datum if and only if $W \setminus W_{\mathbf{B}}$ is connected or empty.

If $v(\Gamma_{W^\bullet}) \setminus v(\mathbf{B}) \neq \emptyset$, there exists a unique *boundary* combinatorial datum $\mathbf{B}^c \in \mathrm{Com}(\Gamma_{W^\bullet})$ such that $v(\mathbf{B}^c) = v(\Gamma_{W^\bullet}) \setminus v(\mathbf{B})$ (i.e. $\mathbf{B}^c$ is the sub-semi-graph determined by $v(\Gamma_{W^\bullet}) \setminus v(\mathbf{B})$), where "c" means "complement".

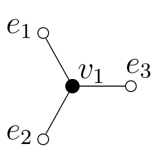
Example 5.2. Let us give an example to explain the above notation. We use the notation "•" and "∘ with a line segment" to denote a vertex and an open edge, respectively. Let $\Gamma_{W^\bullet}$ be a semi-graph as follows:

$\Gamma_{W^\bullet}$:



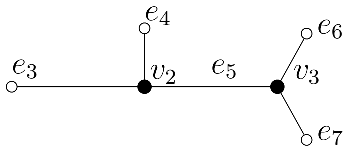
Let $\mathbf{B}$ be a sub-semi-graph as follows:

$\mathbf{B}$:



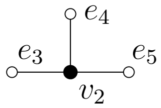
Then $\mathbf{B}$ is boundary, and $\mathbf{B}^c$ is the following:

$\mathbf{B}^{\mathrm{c}}$:



On the other hand, we also give an example of sub-semi-graphs which is not a boundary sub-semi-graph.

$\mathbf{B}'$:



The following lemma will be used in the next subsection.

Lemma 5.3. *Let $\mathbf{B} \in \mathrm{Ssg}(\Gamma_{W^\bullet})$ be a boundary sub-semi-graph and $w \in v(\mathbf{B}^{\mathrm{c}})$ the vertex such that w and $\mathbf{B}$ are connected with a closed edge of $\Gamma_{W^\bullet}$. Write $W^\bullet_{\mathbf{B}}$, $W^\bullet_w$ for the pointed stable curves over k associated to $\mathbf{B}$, w, respectively. Then there exist a pointed stable curve $Z^\bullet$ of type (g_Z, n_Z) over k and an abelian Galois admissible covering $f^\bullet : Z^\bullet \to W^\bullet$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ for some prime number $\ell \neq p$ such that the following conditions are satisfied:*

- $g_Z = 0$.
- *Write $f^{\mathrm{sg}} : \Gamma_{Z^\bullet} \to \Gamma_{W^\bullet}$ for the natural map of dual semi-graphs induced by $f^\bullet$. Let $u \in (f^{\mathrm{sg}})^{-1}(w)$ and $\Gamma \in \mathrm{Ssg}(\Gamma_{Z^\bullet})$ a connected component of $(f^{\mathrm{sg}})^{-1}(\mathbf{B})$. We denote by*
$$Z^\bullet_u,\ Z^\bullet_\Gamma$$
the pointed stable curves of types $(0, n_u)$, $(0, n_\Gamma)$ over k associated to u, Γ, respectively. Then we have
$$n_\Gamma << n_u.$$
This means that for any positive natural number m, $n_u - n_\Gamma > m$ for a suitable choice of ℓ.

Proof. Since $W^\bullet$ is a pointed stable curve of genus 0, we have
$$\#(W_w \cap D_W) + \#(W_w \cap W^{\mathrm{sing}}) \geq 3.$$
Note that $W_w \cap W_{\mathbf{B}} \neq \emptyset$ implies that $\#(W_w \cap W^{\mathrm{sing}}) \geq 1$. Now, we construct two marked points $x_1, x_2 \in D_W$ of $W^\bullet$ as follows.

Suppose $\#(W_w \cap D_W) \geq 2$. We take $x_1, x_2 \in W_w \cap D_W$ marked points of $W^\bullet$ distinct from each other.

Suppose $\#(W_w \cap D_W) = 1$. Then we have $\#(\pi_0(W \setminus W_w)) \geq 2$. Moreover, there exists a connected component $C \in \pi_0(W \setminus W_w)$ such that $C \cap W_{\mathbf{B}} = \emptyset$. We take x_1 the marked point contained in $W_w \cap D_W$ and take x_2 a marked point contained in C.

Suppose $\#(W_w \cap D_W) = 0$. Then we have $\#(\pi_0(W \setminus W_w)) \geq 3$. Moreover, there exist two connected components $C_1, C_2 \in \pi_0(W \setminus W_w)$ distinct from each other such that $C_1 \cap W_{\mathbf{B}} = \emptyset$ and $C_2 \cap W_{\mathbf{B}} = \emptyset$. We take x_1, x_2 marked points contained in C_1, C_2, respectively.

Let $\ell >> 0$ be a prime number prime to p, and let $f^\bullet : Z^\bullet \to W^\bullet$ be a Galois admissible covering with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that $f^\bullet$ is totally ramified over x_1, x_2 and is étale over $D_W \setminus \{x_1, x_2\}$. Then we see immediately that $f^\bullet$ is the desired Galois admissible covering. This completes the proof of the lemma. □

5.2. **Main result.**

5.2.1. **Settings.** We maintain the notation introduced in 3.1.1. Moreover, we assume that the following holds:

- $g_X = 0$.
- $\Pi_{X_i^\bullet}$, $i \in \{1, 2\}$, is the maximal *pro-solvable* quotient of the admissible fundamental group of $X_i^\bullet$.
- $\phi \in \mathrm{Hom}_{\mathrm{pg}}^{\mathrm{op}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$.
- Let $\Pi_1 \in \mathrm{Geo}(\Pi_{X_1^\bullet})$ and $\Pi_2 \overset{\mathrm{def}}{=} \phi(\Pi_1) \subseteq \Pi_{X_2^\bullet}$. If $\Pi_2 \in \mathrm{Geo}(\Pi_{X_2^\bullet})$, then $\phi|_{H_1} : H_1 \to H_2$ is a topological specialization homomorphism for all open subgroups $H_2 \subseteq \Pi_2$, where $H_1 \overset{\mathrm{def}}{=} (\phi|_{H_1})^{-1}(H_2) \subseteq \Pi_1$.

5.2.2. We maintain the notation introduced in 5.2.1. Let $\Gamma_1 \in \mathrm{Ssg}(\Gamma_{X_1^\bullet})$ be a *boundary* sub-semi-graph. Note that, in this situation, this means that $X_1 \setminus X_{\Gamma_1}$ is connected or empty, where $X_{\Gamma_1} \subseteq X_1$ denotes the semi-stable curve corresponding Γ_1 (2.2.3).

By applying Theorem 4.9, we obtain that $\phi : \Pi_{X_1^\bullet} \to \Pi_{X_2^\bullet}$ is a strong topological specialization homomorphism. Namely, $\phi^{\mathrm{sg,op}} = sp^{\mathrm{com}}_{X_1^\bullet, X_2^\bullet}|_{e^{\mathrm{op}}(\Gamma_{X_1^\bullet})}$ holds for some $sp^{\mathrm{com}}_{X_1^\bullet, X_2^\bullet} : \mathrm{Com}(\Gamma_{X_1^\bullet}) \to \mathrm{Com}(\Gamma_{X_2^\bullet})$ (Definition 4.1). We put

$$(\Gamma_2, \emptyset) \overset{\mathrm{def}}{=} sp^{\mathrm{com}}_{X_1^\bullet, X_2^\bullet}((\Gamma_1, \emptyset)).$$

Then Theorem 4.9 implies that $\Gamma_2 \subseteq \Gamma_{X_2^\bullet}$ is a boundary sub-semi-graph. On the other hand, write $E_{\Gamma_1} \subseteq e^{\mathrm{op}}(\Gamma_{X_1^\bullet})$ for the set of open edges of $\Gamma_{X_1^\bullet}$ on which Γ_1 is abutted, $E_{\Gamma_2} \subseteq e^{\mathrm{op}}(\Gamma_{X_2^\bullet})$ for the set of open edges of $\Gamma_{X_2^\bullet}$ on which Γ_2 is abutted. Note that we have

$$\phi^{\mathrm{sg,op}}(E_{\Gamma_1}) = E_{\Gamma_2}.$$

Let $\widehat{\Gamma}_i \subseteq \widehat{\Gamma}_{X_i^\bullet}$, $i \in \{1, 2\}$, be a connected component of $\pi_{X_i}^{-1}(\Gamma_i)$, and $\Pi_{\widehat{\Gamma}_i} \subseteq \Pi_{X_i^\bullet}$ the geometry-like subgroup associated to $\widehat{\Gamma}_i$ (2.3.3). Moreover, we put

$$\mathrm{Edg}^{\mathrm{op}}_{E_{\Gamma_i}}(\Pi_{\widehat{\Gamma}_i}) \overset{\mathrm{def}}{=} \{I_{\widehat{e}_i} \in \Pi_{\widehat{\Gamma}_i} \mid \widehat{e}_i \in \pi_{X_i}^{-1}(e_i),\ e_i \in E_{\Gamma_i}\} \subseteq \mathrm{Edg}^{\mathrm{op}}(\Pi_{X_i^\bullet}).$$

Then we have the following lemma:

Lemma 5.4. *We maintain the notation introduced above. Then* $\Pi_{\widehat{\Gamma}_i}$ *is generated by* $\{I_{\widehat{e}_i} \mid I_{\widehat{e}_i} \in \mathrm{Edg}^{\mathrm{op}}_{E_{\Gamma_i}}(\Pi_{\widehat{\Gamma}_i})\}$.

Proof. This lemma follows immediately from the facts that g_X is equal to 0, and that Γ_i is a boundary sub-semi-graph. □

5.2.3. We maintain the setting and notation introduced in 5.2.1 and 5.2.2. Before we start to prove the group-theoretical specialization conjecture under the settings 5.2.1, we will prove firstly the following:

> $(*)$: There exists a connected component $\widehat{\Gamma}_2 \subseteq \widehat{\Gamma}_{X_2^\bullet}$ of $\pi_{X_2}^{-1}(\Gamma_2)$ such that $\phi(\Pi_{\widehat{\Gamma}_1}) = \Pi_{\widehat{\Gamma}_2}$.

By Lemma 5.4, $(*)$ is equivalent to the following statement:

> $(**)$: Let $I_{\widehat{e}_{1,j}} \in \mathrm{Edg}^{\mathrm{op}}_{E_{\Gamma_1}}(\Pi_{\widehat{\Gamma}_1})$, $j \in \{a, b\}$. Theorem 3.2 (a) implies $\phi(I_{\widehat{e}_{1,j}}) = I_{\widehat{e}_{2,j}}$ for some $\widehat{e}_{2,j} \in e^{\mathrm{op}}(\widehat{\Gamma}_{X_2^\bullet})$. Suppose that $I_{\widehat{e}_{2,j}} \in \Pi_{\widehat{\Gamma}_{2,j}}$ (or equivalently, $\widehat{e}_{2,j} \in \widehat{\Gamma}_{2,j}$) for some connected component $\widehat{\Gamma}_{2,j}$ of $\pi_{X_2}^{-1}(\Gamma_2)$. Then we have $\Pi_{\widehat{\Gamma}_{2,a}} = \Pi_{\widehat{\Gamma}_{2,b}}$.

Let $H_2 \subseteq \Pi_{X_2^\bullet}$ be an arbitrary open subgroup and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_1^\bullet}$. Write $X_{H_i}^\bullet$, $i \in \{1, 2\}$, for the pointed stable curve of type (g_H, n_H) over k_i corresponding to H_i (note that Theorem 3.2 (a) implies that the types of $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ are equal), $f_{H_i}^\bullet : X_{H_i}^\bullet \to X_i^\bullet$ the admissible covering determined by the natural injection $H_i \hookrightarrow \Pi_{X_i^\bullet}$, $\Gamma_{X_{H_i}^\bullet}$ for the dual semi-graph of $X_{H_i}^\bullet$, and $f_{H_i}^{\mathrm{sg}} : \Gamma_{X_{H_i}^\bullet} \to \Gamma_{X_i^\bullet}$ the natural map of dual semi-graphs induced by $f_{H_i}^\bullet$. We maintain the notation introduced in $(**)$. Moreover, for $j \in \{a, b\}$, we denote by

$$e_{H_1,j},\ \Gamma_{H_1},\ e_{H_2,j},\ \Gamma_{H_2,j}$$

the images of $\widehat{e}_{1,j}$, $\widehat{\Gamma}_1$, $\widehat{e}_{2,j}$, $\widehat{\Gamma}_{2,j}$ under the natural maps of semi-graphs $\widehat{\Gamma}_{X_1^\bullet} \to \Gamma_{X_{H_1}^\bullet}$ and $\widehat{\Gamma}_{X_2^\bullet} \to \Gamma_{X_{H_2}^\bullet}$, respectively. Note that we have

$$e_{H_1,a}, e_{H_1,b} \in e^{\mathrm{op}}(\Gamma_{H_1}),\ e_{H_2,a} \in e^{\mathrm{op}}(\Gamma_{H_2,a}),\ e_{H_2,b} \in e^{\mathrm{op}}(\Gamma_{H_2,b}).$$

Moreover, we shall write $X^\bullet_{\Gamma_{H_1}}, X^\bullet_{\Gamma_{H_2,a}}, X^\bullet_{\Gamma_{H_2,b}}$ for the pointed stable curves of types $(g_{\Gamma_{H_1}}, n_{\Gamma_{H_1}})$, $(g_{\Gamma_{H_2,a}}, n_{\Gamma_{H_2,a}})$, $(g_{\Gamma_{H_2,b}}, n_{\Gamma_{H_2,b}})$, corresponding to Γ_{H_1}, $\Gamma_{H_2,a}$, $\Gamma_{H_2,b}$, respectively. In particular, if H_2 is an open normal subgroup of $\Pi_{X_2^\bullet}$, we have $(g_{\Gamma_{H_2,a}}, n_{\Gamma_{H_2,a}}) = (g_{\Gamma_{H_2,b}}, n_{\Gamma_{H_2,b}})$. Then we put $(g_{\Gamma_{H_2}}, n_{\Gamma_{H_2}}) \overset{\mathrm{def}}{=} (g_{\Gamma_{H_2,a}}, n_{\Gamma_{H_2,a}}) = (g_{\Gamma_{H_2,b}}, n_{\Gamma_{H_2,b}})$ when H_2 is an open *normal* subgroup of $\Pi_{X_2^\bullet}$. We see that, to verify $(**)$, it's sufficient to prove the following statement:

$(\star)$: $\Gamma_{H_2,a} = \Gamma_{H_2,b}$ for arbitrary open subgroup $H_2 \subseteq \Pi_{X_2^\bullet}$.

In 5.2.4 below, we will prove the statement $(\star)$ under the settings 5.2.1 (see Proposition 5.6 below).

If $v(\Gamma_{X_i^\bullet}) \setminus v(\Gamma_i) \neq \emptyset$, write Γ_i^{c} for the unique boundary sub-semi-graph of $\Gamma_{X_i^\bullet}$ such that $v(\Gamma_i^{\mathrm{c}}) = v(\Gamma_{X_i^\bullet}) \setminus v(\Gamma_i)$. Moreover, we denote by $X^\bullet_{\Gamma_i^{\mathrm{c}}}$ the pointed stable curve of type $(0, n_{\Gamma_i^{\mathrm{c}}})$ over k_i. On the other hand, since $\Gamma_{X_2^\bullet}$ is a tree, there exists a unique vertex

$$w_2 \in v(\Gamma_2^{\mathrm{c}}) \subseteq v(\Gamma_{X_2^\bullet})$$

such that w_2 and Γ_2 are connected with a closed edge of $\Gamma_{X_2^\bullet}$. We denote by $X^\bullet_{w_2}$ the smooth pointed stable curve of type $(0, n_{w_2})$ over k_2 corresponding to w_2.

By applying Lemma 5.3, there exists an open normal subgroup $P_2 \subseteq \Pi_{X_2^\bullet}$ such that $\Pi_{X_2^\bullet}/P_2 \cong \mathbb{Z}/\ell'\mathbb{Z}$ for some $\ell' \neq p$, and that the Galois admissible covering $f^\bullet_{P_2} : X^\bullet_{P_2} \to X_2^\bullet$ corresponding to the natural injection $P_2 \hookrightarrow \Pi_{X_2^\bullet}$ satisfies the conditions listed in the conclusion of Lemma 5.3. Let $P_1 \overset{\mathrm{def}}{=} \phi^{-1}(P_2)$, and let $X^\bullet_{P_1}$ be the pointed stable curve of type (g_{P_1}, n_{P_1}) over k_1. Theorem 3.2 (a) implies that $(g_{P_1}, n_{P_1}) = (g_{P_2}, n_{P_2})$ and $g_{P_1} = g_{P_2} = 0$, where (g_{P_2}, n_{P_2}) denotes the type of $X^\bullet_{P_2}$. Moreover, we note that $\phi|_{P_1} : P_1 \to P_2$ is a strong topological specialization homomorphism (Definition 4.1 and Theorem 4.9). To verify $(\star)$, it's sufficient to prove the following:

$\Gamma_{H_2,a} = \Gamma_{H_2,b}$ for arbitrary open subgroup $H_2 \subseteq P_2$.

Then by replacing $X_i^\bullet$, $\Pi_{X_i^\bullet}$, $i \in \{1, 2\}$, and ϕ by $X^\bullet_{P_i}$, P_i, and $\phi|_{P_1}$, *in the remainder of this subsection, we may assume*

$$n_{w_2} >> n_{\Gamma_2} (= n_{\Gamma_1}).$$

5.2.4. We maintain the settings and the notation introduced in 5.2.1 and 5.2.3.

Lemma 5.5. *Let $H_2 \subseteq \Pi_{X_2^\bullet}$ be an open normal subgroup of $\Pi_{X_2^\bullet}$ and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Pi_{X_2^\bullet}$. Let $w_{H_2} \in (f^{\mathrm{sg}}_{H_2})^{-1}(w_2) \subseteq v(\Gamma_{X^\bullet_{H_2}})$ be a vertex over w_2 and $X^\bullet_{w_{H_2}}$ the smooth (since $g_X = 0$) pointed stable curve of type $(g_{w_{H_2}}, n_{w_{H_2}})$ associated to w_{H_2}. Suppose $g_{w_{H_2}} >> g_{\Gamma_{H_1}}$ for all w_{H_2}. Then we have $\Gamma_{H_2,a} = \Gamma_{H_2,b}$.*

Proof. To verify the lemma, we suppose $\Gamma_{H_2,a} \neq \Gamma_{H_2,b}$. We will construct a contradiction.

Let ℓ be the minimal odd prime number distinct from p (i.e. ℓ is equal to either 3 or 5). Let $Q_1 \subseteq H_1$ be an open normal subgroup of H_1 such that the following conditions are satisfied (the existence of Q_1 follows immediately from the structure of maximal prime-to-p quotients of admissible fundamental groups):

- $H_1/Q_1 \cong \mathbb{Z}/\ell\mathbb{Z}$.
- Write $X^\bullet_{Q_1}$ for the pointed stable curve over k_1 corresponding to Q_1 and

$$h_1^\bullet : X^\bullet_{Q_1} \to X^\bullet_{H_1}$$

  for the Galois admissible covering determined by $Q_1 \hookrightarrow H_1$ satisfying the following conditions:
  (i) $\#\pi_0((h_1^{-1}(X_{H_1} \setminus X_{\Gamma_{H_1}})) = \ell\#\pi_0(X_{H_1} \setminus X_{\Gamma_{H_1}})$ (i.e. h_1 is a trivial covering over $X_{H_1} \setminus X_{\Gamma_{H_1}}$).

(ii) h_1 is étale over $D_{X_{H_1}} \setminus \{x_{e_{H_1,a}}, x_{e_{H_1,b}}\}$ and is totally ramified over $x_{e_{H_1,j}}$, $j \in \{a, b\}$, where $x_{e_{H_1,j}}$ denotes the marked point corresponding to $e_{H_1,j}$. Note that in this situation, $h_1^{-1}(X_{\Gamma_{H_1}})$ is connected.

We put $Q_2 \overset{\text{def}}{=} \phi(Q_1) \subseteq \Pi_{X_2^\bullet}$. Moreover, we denote by $X_{Q_2}^\bullet$ the pointed stable curve over k_2 corresponding to Q_2 and

$$h_2^\bullet : X_{Q_2}^\bullet \to X_{H_2}^\bullet$$

the Galois admissible covering determined by $Q_2 \hookrightarrow H_2$. We see $H_2/Q_2 \cong \mathbb{Z}/\ell\mathbb{Z}$.

Write $\Gamma_{X_{Q_i}^\bullet}$, $i \in \{1, 2\}$, for the dual semi-graph of $X_{Q_i}^\bullet$. Write $\Gamma_{Q_1} \subseteq \Gamma_{X_{Q_1}^\bullet}$ for the sub-semi-graph such that the underlying curve of the corresponding pointed stable curve $X_{\Gamma_{Q_1}}^\bullet$ is equal to $h_1^{-1}(X_{\Gamma_{H_1}})$. By the construction of $h_1^\bullet$ and the choice of ℓ, we have

$$g_{\Gamma_{Q_1}} = \ell(g_{\Gamma_{H_1}} - 1) + \frac{1}{2}(\ell - 1) + 1 << g_{w_{H_2}},$$

where $g_{\Gamma_{Q_1}}$ denotes the genus of $X_{\Gamma_{Q_1}}^\bullet$.

On the other hand, since we assume that $\phi|_{H_1} : H_1 \to H_2$ is a topological specialization homomorphism (i.e. the settings 5.2.1), there is a map of combinatorial data (i.e. the map of dual semi-graphs induced by a degeneration or reduction)

$$sp_{Q_1,Q_2}^{\text{com}} : \text{Com}(\Gamma_{X_{Q_1}^\bullet}) \to \text{Com}(\Gamma_{X_{Q_2}^\bullet}).$$

We put $(\Gamma_{Q_2}, \emptyset) \overset{\text{def}}{=} sp_{Q_1,Q_2}^{\text{com}}((\Gamma_{Q_1}, \emptyset))$. Then we have the following claim.

Claim: $h_2^{\text{sg}}(\Gamma_{Q_2}) \subseteq \Gamma_{X_{H_2}^\bullet}$ *contains* $\Gamma_{H_2,a}$ *and* $\Gamma_{H_2,b}$, *where* $h_2^{\text{sg}} : \Gamma_{X_{Q_2}^\bullet} \to \Gamma_{X_{H_2}^\bullet}$ *denotes the map of dual semi-graphs induced by* $h_2^\bullet$. *In particular,* $h_2^{\text{sg}}(\Gamma_{Q_2})$ *contains* w_{H_2} *for some* $w_{H_2} \in (f_{H_2}^{\text{sg}})^{-1}(w_2) \subseteq v(\Gamma_{X_{H_2}^\bullet})$.

Let us prove the claim. Since $\Gamma_{X_{Q_2}^\bullet}$ can be regarded as the dual semi-graph of a reduction of $X_{Q_1}^\bullet$ and $sp_{Q_1,Q_2}^{\text{com}}$ is induced by the reduction map, the action of $\mathbb{Z}/\ell\mathbb{Z}$ on $\Gamma_{X_{Q_1}^\bullet}$ (determined by the action of $\mathbb{Z}/\ell\mathbb{Z}$ on $X_{Q_1}^\bullet$ induced by the Galois admissible covering $h_1^\bullet$) induces uniquely an action of $\mathbb{Z}/\ell\mathbb{Z}$ on $\Gamma_{X_{Q_2}^\bullet}$.

Note that we do *not* know whether or not the action of $\mathbb{Z}/\ell\mathbb{Z}$ on $\Gamma_{X_{Q_2}^\bullet}$ defined above coincides with the action $\mathbb{Z}/\ell\mathbb{Z}$ on $\Gamma_{X_{Q_2}^\bullet}$ induced by the Galois admissible covering $h_2^\bullet$. In the remainder of the proof of the claim, we only consider the action of $\mathbb{Z}/\ell\mathbb{Z}$ on $\Gamma_{X_{Q_2}^\bullet}$ induced by the action of $\mathbb{Z}/\ell\mathbb{Z}$ on $\Gamma_{X_{Q_1}^\bullet}$ defined above.

Let $\mathbb{G} \subseteq \Gamma_{X_{Q_1}^\bullet}$ be an arbitrary sub-semi-graph such that $v(\mathbb{G}) \cap v(\Gamma_{Q_1}) = \emptyset$. By the construction of $X_{Q_1}^\bullet$, we see that the decomposition subgroup of $\mathbb{G}$ under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is trivial, and the the decomposition subgroup of Γ_{Q_1} under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is $\mathbb{Z}/\ell\mathbb{Z}$. Then we have the decomposition subgroup of $sp_{Q_1,Q_2}^{\text{com}}(\mathbb{G})$ under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is trivial, and the decomposition subgroup of $\Gamma_{Q_2} = sp_{Q_1,Q_2}^{\text{com}}(\Gamma_{Q_1})$ under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is $\mathbb{Z}/\ell\mathbb{Z}$. Since the decomposition group of $e_{Q_2,j}$, $j \in \{a, b\}$, under the action of $\mathbb{Z}/\ell\mathbb{Z}$ is $\mathbb{Z}/\ell\mathbb{Z}$, where $e_{Q_2,j}$ is defined in 5.2.3 by replacing H_2 by Q_2, we see that $e_{Q_2,j}$, $j \in \{a, b\}$, is contained in $e^{\text{op}}(\Gamma_{Q_2})$. Thus, $h_2^{\text{sg}}(\Gamma_{Q_2}) \subseteq \Gamma_{X_{H_2}^\bullet}$ contains $\Gamma_{H_2,j}$, $j \in \{a, b\}$, since $h_2^{\text{sg}}(e_{Q_2,j}) = e_{H_2,j} \in \Gamma_{H_2,j}$.

On the other hand, since $h_2^{\text{sg}}(\Gamma_{Q_2})$ is connected and $\Gamma_{H_2,a}$ is distinct from $\Gamma_{H_2,b}$, then $h_2^{\text{sg}}(\Gamma_{Q_2})$ contains w_{H_2} for some $w_{H_2} \in (f_{H_2}^{\text{sg}})^{-1}(w_2) \subseteq v(\Gamma_{X_{H_2}^\bullet})$. We complete the proof of the claim.

We return to prove the lemma. By the claim, we obtain

$$g_{\Gamma_{Q_1}} = g_{\Gamma_{Q_2}} \geq g_{w_{H_2}},$$

where $g_{\Gamma_{Q_2}}$ denotes the genus of the pointed stable curve $X_{\Gamma_{Q_2}}^\bullet$ corresponding to Γ_{Q_2}. We obtain a contradiction. This completes the proof of the lemma. $\square$

Proposition 5.6. *The statement* $(\star)$ *mentioned in 5.2.3 holds. In particular, the statement* $(*)$ *mentioned in 5.2.3 holds.*

Proof. Suppose that $(\star)$ does not hold. Then there exists an open subgroup $Q_2 \subseteq \Pi_{X_2^\bullet}$ such that

$$\Gamma_{Q_2,a} \neq \Gamma_{Q_2,b}.$$

Thus, for *any* open subgroup $P_2 \subseteq Q_2$, we have $\Gamma_{P_2,a} \neq \Gamma_{P_2,b}$. Let us construct a contradiction.

Let $\ell >> \#(\Pi_{X_1^\bullet}/Q_1) = \#(\Pi_{X_2^\bullet}/Q_2)$ be a prime number prime to p and $K_2 \subseteq \Pi_{X_2^\bullet}$ an open normal subgroup such that the following conditions are satisfied (the existence of K_2 follows immediately from the structure of maximal prime-to-p quotients of admissible fundamental groups):

- $\Pi_{X_2^\bullet}/K_2 \cong \mathbb{Z}/\ell\mathbb{Z}$.
- Write $f^\bullet_{K_2} : X^\bullet_{K_2} \to X_2^\bullet$ for the Galois admissible covering over k_2 corresponding to $K_2 \hookrightarrow \Pi_{X_2^\bullet}$. Then $f^\bullet_{K_2}$ is totally ramified over

$$(X_{w_2} \cap D_{X_2}) \cup ((X_{w_2} \cap X_2^{\mathrm{sing}}) \setminus (X_{\Gamma_2} \cap X_2^{\mathrm{sing}}))$$

  and is étale over $(X_{\Gamma_2} \cap X_2^{\mathrm{sing}}) \cup (X_{\Gamma_2} \cap D_{X_2})$. Note that in this situation, $f^\bullet_{K_2}$ induces a trivial covering over $X^\bullet_{\Gamma_2}$.

We put $K_1 = \phi^{-1}(K_2)$. Write $f^\bullet_{K_1} : X^\bullet_{K_1} \to X_1^\bullet$ for the Galois admissible covering over k_1 corresponding to $K_1 \hookrightarrow \Pi_{X_1^\bullet}$. Since ϕ is a strong topological specialization homomorphism (Theorem 4.9), we see that

$$(g_{\Gamma_{K_1}}, n_{\Gamma_{K_1}}) = (g_{\Gamma_{K_2,a}}, n_{\Gamma_{K_2,a}}) = (g_{\Gamma_{K_2,b}}, n_{\Gamma_{K_2,b}}).$$

On the other hand, since we assume $n_{w_2} >> n_{\Gamma_2}$, we obtain that

$$g_{w_{K_2}} >> g_{\Gamma_{K_2,a}} = g_{\Gamma_{K_2,b}} = g_{\Gamma_{K_1}},\ w_{K_2} \in (f^{\mathrm{sg}}_{K_2})^{-1}(w_2) \subseteq v(\Gamma_{X^\bullet_{K_2}}).$$

We put $H_2 \overset{\mathrm{def}}{=} K_2 \cap Q_2$ and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2)$. Write $f^\bullet_{H_i} : X^\bullet_{H_i} \to X_i^\bullet$, $i \in \{1,2\}$, for the Galois admissible covering over k_i corresponding to $H_i \hookrightarrow \Pi_{X_i^\bullet}$. The choice of $\ell >> \#(\Pi_{X_1^\bullet}/Q_1) = \#(\Pi_{X_2^\bullet}/Q_2)$ and the fact $g_{w_{K_2}} >> g_{\Gamma_{K_2,a}} = g_{\Gamma_{K_2,b}} = g_{\Gamma_{K_1}}$ imply

$$g_{w_{H_2}} >> g_{\Gamma_{H_2,a}} = g_{\Gamma_{H_2,b}},\ g_{w_{H_2}} >> g_{\Gamma_{H_1}}.$$

Note that we do not know whether or not $g_{\Gamma_{H_2,a}} = g_{\Gamma_{H_2,b}} = g_{\Gamma_{H_1}}$ holds in general. Thus, by Lemma 5.5, we obtain

$$\Gamma_{H_2,a} = \Gamma_{H_2,b}.$$

This contradicts the fact $\Gamma_{H_2,a} \neq \Gamma_{H_2,b}$ since H_2 is contained in Q_2. We complete the proof of the proposition. □

5.2.5. The main theorem of the present section is the following:

Theorem 5.7. *We maintain the notation introduced in 3.1.1. Moreover, we assume that the following holds:*

- $g_X = 0$.
- $\Pi_{X_i^\bullet}$, $i \in \{1,2\}$, *is the maximal pro-solvable quotient of the admissible fundamental group of* $X_i^\bullet$.
- $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{pg}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet})$.
- *Let* $\Pi_1 \in \mathrm{Geo}(\Pi_{X_1^\bullet})$ *and* $\Pi_2 \overset{\mathrm{def}}{=} \phi(\Pi_1) \subseteq \Pi_{X_2^\bullet}$. *If* $\Pi_2 \in \mathrm{Geo}(\Pi_{X_2^\bullet})$, *then* $\phi|_{H_1} : H_1 \to H_2$ *is a topological specialization homomorphism (Definition 3.1 (a)) for all open subgroups* $H_2 \subseteq \Pi_2$, *where* $H_1 \overset{\mathrm{def}}{=} (\phi|_{H_1})^{-1}(H_2) \subseteq \Pi_1$.

Proof. Suppose $\#(v(\Gamma_{X_1^\bullet})) = 1$. Then the theorem is trivial. To verify the theorem, we assume $\#(v(\Gamma_{X_1^\bullet})) \geq 2$.

Let $v_1 \in v(\Gamma_{X_1^\bullet})$ be an arbitrary vertex of $\Gamma_{X_1^\bullet}$, $\widehat{v}_1 \in \pi^{-1}_{X_1}(v_1) \subseteq \widehat{\Gamma}_{X_1^\bullet}$, and $\Pi_{\widehat{v}_1} \in \mathrm{Ver}(\Pi_{X_1^\bullet})$ the vertex-like subgroup associated to $\widehat{v}_1$. To verify the theorem, by Proposition 3.9, it's sufficient to prove $\phi(\Pi_{\widehat{v}_1}) \in \mathrm{Geo}(\Pi_{X_2^\bullet})$.

Since $\Gamma_{X_1^\bullet}$ is a tree, there exists a *boundary* (proper) sub-semi-graph $\Gamma_1 \in \mathrm{Ssg}(\Gamma_{X_1^\bullet})$ (Definition 5.1) such that the following conditions are satisfied:

- $v_1 \in v(\Gamma_1)$ and $\Gamma_{v_1} \in \mathrm{Ssg}(\Gamma_1) \subseteq \mathrm{Ssg}(\Gamma_{X_1^\bullet})$ is a boundary sub-semi-graph of Γ_1 (see 2.3.3 for Γ_{v_1}).
- Write $\Gamma_1^{\mathrm{c}} \in \mathrm{Ssg}(\Gamma_{X_1^\bullet})$ for the unique boundary sub-semi-graph such that $v(\Gamma_1^{\mathrm{c}}) = v(\Gamma_{X_1^\bullet}) \setminus v(\Gamma_1)$. Then v_1 and Γ_1^{c} are connected with a closed edge of $\Gamma_{X_1^\bullet}$.

Let $\widehat{\Gamma}_1$ be the connected component of $\pi_{X_1}^{-1}(\Gamma_1) \subseteq \widehat{\Gamma}_{X_1^\bullet}$ containing $\widehat{v}_1$ and $\Pi_{\widehat{\Gamma}_1} \in \mathrm{Geo}(\Pi_{X_1^\bullet})$ the geometry-like subgroup associated to $\widehat{\Gamma}_1$.

Since Γ_1 is a boundary sub-semi-graph, Proposition 5.6 implies that there exist a sub-semi-graph $\Gamma_2 \in \mathrm{Ssg}(\Gamma_{X_2^\bullet})$ and a connected component $\widehat{\Gamma}_2 \in \pi_{X_2}^{-1}(\Gamma_2)$ such that $\Pi_{\widehat{\Gamma}_2} = \phi(\Pi_{\widehat{\Gamma}_1}) \in \mathrm{Geo}(\Pi_{X_2^\bullet})$, where $\Pi_{\widehat{\Gamma}_2}$ is the geometry-like subgroup associated to $\widehat{\Gamma}_2$.

Moreover, since $\Gamma_{v_1} \in \mathrm{Ssg}(\Gamma_1)$ is a boundary sub-semi-graph of Γ_1, by applying Proposition 5.6 for $\phi|_{\Pi_{\widehat{\Gamma}_1}} : \Pi_{\widehat{\Gamma}_1} \to \Pi_{\widehat{\Gamma}_2}$ (our assumptions say that $\phi|_{H_1} : H_1 \to H_2$ is a topological specialization homomorphism for all open subgroups $H_2 \subseteq \Pi_2$, where $H_1 \overset{\mathrm{def}}{=} (\phi|_{H_1})^{-1}(H_2) \subseteq \Pi_1$), we obtain $\phi(\Pi_{\widehat{v}_1}) \in \mathrm{Geo}(\Pi_{\widehat{\Gamma}_2}) \subseteq \mathrm{Geo}(\Pi_{X_2^\bullet})$. This completes the proof of the theorem. □

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