ON THE EXISTENCE OF NON-FINITE COVERINGS OF STABLE CURVES OVER COMPLETE DISCRETE VALUATION RINGS

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Abstract

Let R be a complete discrete valuation ring with algebraically residue field of characteristic $p > 0$ and X a stable curve over R. In the present paper, we study the geometry of coverings of X. Under certain assumptions, we prove that, by replacing R by a finite extension of R, there exists a morphism of stable curves $f : Y \to X$ over R such that the morphism $f_\eta : Y_\eta \to X_\eta$ induced by f on generic fibers is finite étale and the morphism $f_s : Y_s \to X_s$ induced by f on special fibers is non-finite.

Keywords: stable curve, stable covering, vertical point, admissible covering.

Mathematics Subject Classification: Primary 14H30; Secondary 11G20.

Introduction

Let R be a complete discrete valuation ring with algebraically closed residue field k, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R. Write $\eta, \overline{\eta}$ and s for the generic point of S, the geometric generic point of S, and the closed point of S corresponding to the natural morphisms $\operatorname{Spec} K \to S$, $\operatorname{Spec} \overline{K} \to S$, and $\operatorname{Spec} k \to S$, respectively. Let X be a stable curve over S of genus g_X. Write X_η, $X_{\overline{\eta}}$, and X_s for the generic fiber of X, the geometric generic fiber of X, and the special fiber of X, respectively. Moreover, we suppose that X_η is smooth over η.

In the present paper, we study the geometry of coverings of X. Let $f_\eta : Y_\eta \longrightarrow X_\eta$ be an arbitrary Galois étale covering over η. By replacing S by a finite extension of S, f_η extends uniquely to a stable covering $f : Y \to X$ over S (cf. Definition 1.3 and Remark 1.3.1). Note that f_s is not a finite morphism in general. A closed point x of X is called a vertical point of X if, by replacing S by a finite extension of S, there exists a stable covering such that the fiber of x is not a finite set (cf. Definition 1.5). Write X^{ver} for the set of the vertical points of X. We may post a question as follows:

Question 0.1. *What is X^{ver}?*

If $\mathrm{char}(K) = \mathrm{char}(k) = 0$, up to composition with an inner automorphism, there is a natural isomorphism between the étale fundamental group of $X_{\overline{\eta}}$ and the admissible fundamental group of X_s (cf. Definition 1 and [V, Théorème 2.2]). Thus, for any finite étale covering of X_η, by replacing S by a finite extension of S, the morphism of special fibers induced by the étale covering of X_η is an admissible covering. Then we obtain X^{ver} is empty.

If $\mathrm{char}(k) = p > 0$, we may consider whether or not the set X^{ver} is empty. The motivation of Question 0.1 partly comes from anabelian geometry. Question 0.1 was first considered by S. Mochizuki (cf. [M2, the proof of Theorem 9.2]). If $\mathrm{char}(K) = 0$, Mochizuki proved that X^{ver} is not empty. Then he reduced the Grothendieck conjecture for proper, hyperbolic curves over number fields to the Grothendieck conjecture for proper, singular, stable curves over finite fields, which is then reduced to the Grothendieck conjecture for affine curves over finite fields which had been proven by A. Tamagawa.

In the present paper, we focus on Question 0.1 in the case where $\mathrm{char}(K) = \mathrm{char}(k) = p > 0$. The main theorem of the present paper is as follows.

Theorem 0.2. *Suppose that* $\mathrm{char}(K) = \mathrm{char}(k) = p > 0$.

(i) Suppose that X is smooth over S, $X_{\overline{\eta}}$ cannot be defined over an algebraic closure of $\mathbb{F}_p$, and X_s can be defined over an algebraic closure of $\mathbb{F}_p$. Then X^{ver} is not empty.

(ii) Suppose that X_s is an irreducible singular stable curve over s. Then X^{ver} is not empty.

Remark 0.2.1. Suppose that $R = \mathbb{F}_q[[t]]$, where $q = p^r$ such that $r \neq 0$. Let X_1 and X_2 be stable curves over S. Moreover, we suppose that the generic fibers of $X_{1,\eta}$ and $X_{2,\eta}$ are smooth over S. The Grothendieck conjecture for curves over local fields of positive characteristic asks whether or not the natural morphism

$$\mathrm{Isom}_{\mathrm{schemes}}(X_{1,\eta}, X_{2,\eta}) \to \mathrm{Isom}(\pi_1(X_{1,\eta}), \pi_1(X_{2,\eta}))/\mathrm{Inn}(\pi_1(X_{2,\eta}))$$

is a bijection. Here, the left-hand side is the set of $\mathbb{F}_p$-isomorphisms of curves and the right-hand side is the set of continuous isomorphisms of profinite groups. The difficult part for proving this conjecture is the surjectivity. Suppose that the fundamental groups of the generic fibers of X_1 and X_2 are isomorphic. By applying Theorem 0.2, similar arguments to the arguments given in the proofs of [M2] imply that the special fibers of X_1 and X_2 are isomorphic. However, we do not know how to recover the X_1 and X_2 from the special fibers and the fundamental groups. At the present, no results are known about the Grothendieck conjecture for curves over local fields of positive characteristic.

On the other hand, Tamagawa approached the Grothendieck conjecture for curves in positive characteristic by changing the point of view. He showed that, the isomorphism class of a hyperbolic curve in positive characteristic may possibly be determined by the isomorphism class of its **geometric** tame fundamental group (i.e., without Galois action). More precisely, Tamagawa proved that (cf. [T1]),

> if U is a hyperbolic curve of type $(0, n)$ over an algebraically closed field of characteristic $p > 0$, then we can detect completely whether U can be defined over $\overline{\mathbb{F}}_p$ or not, group-theoretically from the tame fundamental group of U; moreover, if U is defined over $\overline{\mathbb{F}}_p$, then the isomorphism class of the tame fundamental group of U determines completely the isomorphism class of U as scheme.

Furthermore, for the case of higher genus, the following finiteness theorem proved by M. Raynaud, F. Pop, and M. Saïdi under certain conditions and by Tamagawa in full generality (cf. [R3], [PS], [T2]):

> Let U be a hyperbolic curve over $\overline{\mathbb{F}}_p$, then there are only finitely many isomorphism classes of hyperbolic curves over $\overline{\mathbb{F}}_p$ whose tame fundamental groups are isomorphic to the tame fundamental group of U.

Recently, the author extended the Tamagawa's results above to the case of (possibly singular) pointed stable curves (cf. [Y2]). Furthermore, the author proved that, the set of open morphisms between the tame fundamental groups of two curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$ is not empty if and only if the curves are isomorphic as schemes (cf. [Y3]); this result can be regarded as a weak Hom-version Grothendieck conjecture for curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$. Moreover, the weak Hom-version Grothendieck conjecture for curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$ implies the Hom-version of the Grothendieck conjecture for curves of genus 0 over a field which is finitely generated over a finite field (cf. [Y4]).

Remark 0.2.2. If $\mathrm{char}(K) = 0$ and $\mathrm{char}(k) = p > 0$, Tamagawa proved the following result: suppose that R is a finite extension of the Witt ring $W(\overline{\mathbb{F}}_p)$; then X^{ver} is equal to the set of closed points of X. Moreover, by applying Tamagawa's idea, we extend Tamagawa's result to the case where k is an arbitrary algebraically closed field of characteristic $p > 0$ (cf. Appendix of the present paper).

Moreover, for a given stable covering, we have the following theorem.

Theorem 0.3. *Let G be a finite group, $f : Y \to X$ a G-stable covering over S, and $f_s : Y_s \to X_s$ the morphism induced by f on the special fibers.*

(i) Suppose that X_s is smooth over s. Then f_s is a finite morphism if and only if f_s is an étale covering.

(ii) Suppose that X_s is irreducible, and G is a solvable group. Then f_s is a finite morphism if and only if f_s is an admissible covering.

Finally, let G be a finite p-group, $f : Y \to X$ a non-finite G-stable covering over S, and $x \in X^{\mathrm{ver}}$ such that $f^{-1}(x)$ is not finite. We mention that an explicit formula for p-rank (cf. Definition 1.2) of $f^{-1}(x)$ is given by Raynaud, Saïdi, and the author (cf. [R1], [S], [Y1]).

1 Preliminaries

In this section, we recall some definitions and results which will be used in the present paper.

Definition 1.1. Let C_1 and C_2 be two stable curves over an algebraically closed field l and $\phi : C_2 \to C_1$ a morphism of stable curves over $\mathrm{Spec}\, l$.

We shall call ϕ a **Galois admissible covering** over $\mathrm{Spec}\, l$ (or Galois admissible covering for short) if the following conditions hold: (i) there exists a finite group $G \subseteq \mathrm{Aut}_k(C_2)$ such that $C_2/G = C_1$, and ϕ is equal to the quotient morphism $C_2 \to C_2/G$; (ii) for each $c_2 \in C_2^{\mathrm{sm}}$, ϕ is étale at c_2, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$; (iii) for any $c_2 \in C_2^{\mathrm{sing}}$, the image $\phi(c_2)$ is contained in C_1^{sing}, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$; (iv) for each $c_2 \in C_2^{\mathrm{sing}}$, the local morphism between two nodes (cf. (iii)) induced by ϕ may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{C_1,\phi(c_2)} \cong l[[u,v]]/uv & \to & \hat{\mathcal{O}}_{C_2,c_2} \cong l[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(l)) = 1$ if $\mathrm{char}(l) = p > 0$; moreover, write $D_{c_2} \subseteq G$ for the decomposition group of c_2; then $\tau(s) = \zeta_{\#D_{c_2}} s$ and $\tau(t) = \zeta_{\#D_{c_2}}^{-1} t$ for each $\tau \in D_{c_2}$, where $\zeta_{\#D_{c_2}}$ is a primitive $\#D_{c_2}$-th root of unit.

We shall call ϕ an **admissible covering** if there exists a morphism of stable curves $\phi' : C_2' \to C_2$ over $\mathrm{Spec}\, l$ such that the composite morphism $\phi \circ \phi' : C_2' \to C_1$ is a Galois admissible covering over $\mathrm{Spec}\, l$.

Let Z be the disjoint union of finitely many stable curves over $\mathrm{Spec}\, l$. We shall call a morphism $Z \to C_1$ over $\mathrm{Spec}\, l$ **multi-admissible** if the restriction of $Z \to C_1$ to each connected component of Z is admissible. We define a category $\mathrm{Cov}^{\mathrm{adm}}(C_1)$ as follows: (i) the objects of $\mathrm{Cov}^{\mathrm{adm}}(C_1)$ are the empty object and the multi-admissible coverings of C_1; (ii) for any $A, B \in \mathrm{Cov}^{\mathrm{adm}}(C_1)$, $\mathrm{Hom}(A, B)$ consists of all the morphisms whose restriction to each connected component of B is a multi-admissible covering. It is well-known that $\mathrm{Cov}^{\mathrm{adm}}(C_1)$ is a Galois category. Thus, by choosing a base point $c_1 \in C_1$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(C_1, c_1)$ which is called the **admissible fundamental group** of C_1. For simplicity, we omit the base point and denote the admissible fundamental group by $\pi_1^{\mathrm{adm}}(C_1)$.

For more details on admissible coverings and the fundamental groups for (pointed) stable curves, see [M1], [M2].

Remark 1.1.1. If C_1 is a smooth projective curve over $\mathrm{Spec}\, l$, then every admissible covering of C_1 is étale. Thus, we have $\pi_1^{\mathrm{adm}}(C_1) = \pi_1(C_1)$, where $\pi_1(-)$ denotes the étale fundamental group of $(-)$.

Remark 1.1.2. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack of pointed stable curves of type (g, n) over $\mathrm{Spec}\, \mathbb{Z}$ and $\mathcal{M}_{g,n}$ the open substack of $\overline{\mathcal{M}}_{g,n}$ parametrizing pointed smooth curves. Write $\overline{\mathcal{M}}_{g,n}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ relative to $\mathrm{Spec}\, \mathbb{Z}$.

Let C_1 be a stable curve over an algebraically closed field l of genus g. Then $C_1 \to \mathrm{Spec}\, l$ induces a morphism $\mathrm{Spec}\, l \to \overline{\mathcal{M}}_{g,0}$. Write $s_{C_1}^{\log}$ for the log scheme whose underlying scheme is $\mathrm{Spec}\, l$, and whose log structure is the pulling-back log structure induced by the morphism $\mathrm{Spec}\, l \to \overline{\mathcal{M}}_{g,0}$. We obtain a natural morphism $s_{C_1}^{\log} \to \overline{\mathcal{M}}_{g,0}^{\log}$ induced by the morphism $\mathrm{Spec}\, l \to \overline{\mathcal{M}}_{g,0}$ and a stable log curve $C_1^{\log} := s_{C_1}^{\log} \times_{\overline{\mathcal{M}}_{g,0}^{\log}} \overline{\mathcal{M}}_{g,1}^{\log}$ over $s_{C_1}^{\log}$ whose underlying scheme is C_1. Then the admissible fundamental group $\pi_1^{\mathrm{adm}}(C_1)$ of C_1 is naturally isomorphic to the geometric log étale fundamental group of $C_1^{\log}$ (i.e., $\mathrm{Ker}(\pi_1(C_1^{\log}) \to \pi_1(s_{C_1}^{\log}))$).

Definition 1.2. Let C be a stable curve over an algebraically closed field of characteristic $p > 0$. We define the p**-rank** $\sigma(C)$ of C to be

$$\sigma(C) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(C, \mathbb{F}_p).$$

From now on, we fix some notations. Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R. Write $\eta, \overline{\eta}$ and s for the generic point of S, the geometric generic point of S, and the closed point of S corresponding to the natural morphisms $\mathrm{Spec}\, K \to S$, $\mathrm{Spec}\, \overline{K} \to S$, and $\mathrm{Spec}\, k \to S$, respectively. Let X be a stable curve over S of genus g_X. Write $X_\eta := X \times_S \eta$ for the generic fiber of X, $X_{\overline{\eta}} := X \times_S \overline{\eta}$ for the geometric generic fiber of X, and $X_s := X \times_S s$ for the special fiber of X, respectively. Moreover, we suppose that X_η is smooth over η. Write Γ_{X_s} for the dual graph of X_s and $v(\Gamma_{X_s})$ for the set of vertices of Γ_{X_s}.

Definition 1.3. Let $f : Y \to X$ a morphism of stable curves over S, and G a finite group. We shall call f a G**-stable covering** over S if the morphism $f_\eta : Y_\eta \to X_\eta$ over η induced by f on generic fibers is a Galois étale covering whose Galois group is isomorphic to G.

Remark 1.3.1. Let $W_\eta \to X_\eta$ be any geometrically connected Galois étale covering over η whose Galois group is isomorphic to G. By replacing S by a finite extension of S, W_η admits a stable model W over S. Moreover, the action of G on W_η induces an action of G on W. Then [R1, Appendice Corollaire] implies that W/G is a semi-stable curve over S. Since X is a stable curve over S, we have a natural morphism $W/G \to X$ over S. Thus, the morphism $W_\eta \to X_\eta$ may extend to a G-stable covering over S.

On the other hand, we would like to mention that Definition 1.3 is a special case of [Y1, Definition 3.3]. For more details on semi-stable coverings and the existence of semi-stable coverings, see [Y1, Definition 3.3 and Proposition 3.4].

Remark 1.3.2. Let $f : Y \to X$ be a G-stable covering over S, and y any closed point of Y. Then f induces a morphism $f_y : \mathrm{Spec}\, \widehat{\mathcal{O}}_{Y,y} \to \mathrm{Spec}\, \widehat{\mathcal{O}}_{X,f(y)}$ over S. Suppose that $f_s : Y_s \to X_s$ over s on special fibers induced by f is generically étale.

If y is a smooth point, then $f(y) \in X$ is a smooth point too. Then Zariski-Nagata purity implies that the morphism f_y is étale.

If y is a singular point, then $f(y) \in X$ is a singular point too. Then Zariski-Nagata purity and [T3, Lemma 2.1 (iii)] imply that the morphism of local rings $\widehat{\mathcal{O}}_{X_s,f(y)} \to \widehat{\mathcal{O}}_{Y_s,y}$ induced by f_y satisfies the condition (iv) of Definition 1.1.

Thus, we have that f_s is an admissible covering over s if and only if f_s is generically étale.

The following highly non-trivial result was proved by Tamagawa (cf. [T2, Theorem 8.1], [T3, Corollary 3.11]).

Proposition 1.4. *Let*

$$sp : \pi_1(X_{\overline{\eta}}) \to \pi_1^{\mathrm{adm}}(X_s)$$

be the specialization map.

(i) Suppose that X is smooth over S, $X_{\overline{\eta}}$ cannot be defined over an algebraic closure field of $\mathbb{F}_p$, and X_s can be defined over an algebraically closed field of $\mathbb{F}_p$. Then there exists a finite étale covering $f : Y \to X$ over S such that $Y_{\overline{\eta}} := Y \times_\eta \overline{\eta}$ and Y_s are connected, and $\sigma(Y_{\overline{\eta}}) - \sigma(Y_s) > 0$. In particular, sp is not an isomorphism.

(ii) Suppose that X_s is a singular curve. Then there exists a finite group G, and, by replacing S by a finite extension of S, there exists a G-stable covering $f : Y \to X$ over S

such that $f_s : Y_s \to X_s$ is an admissible covering and $\sigma(Y_{\overline{\eta}}) - \sigma(Y_s) > 0$. In particular, sp is not an isomorphism.

Remark 1.4.1. Let $f : Y \to X$ be a G-stable covering over S. Proposition 1.4 implies that the morphism $f_s : Y_s \to X_s$ on special fibers induced by f is not an admissible covering in general.

Definition 1.5. Let x be a closed point of X. We shall call x a **vertical point** of X if there exists a finite group G, and, after replacing S by a finite extension of S, there exists a G-stable covering $f : Y \to X$ over S such that $\dim(f^{-1}(x)) = 1$. We use the notation X^{ver} to denote the set of vertical points of X.

There is a criterion for the existence of vertical points of G-stable coverings proved by Tamagawa (cf. [T3, Proposition 4.3 (ii) (WRamS2) $\Rightarrow$ $(\mathrm{Vert})_x$]).

Proposition 1.6. *Let $x \in X$ be a closed point and $f : Y \to X$ a G-stable covering over S. Suppose that for each irreducible component $Z := \overline{\{z\}}$ of $\operatorname{Spec} \widehat{\mathcal{O}}_{X_s,x}$, and for each point w of the fiber $Y \times_X z$, the natural morphism from the integral closure W^s of Z in $k(w)^s$ to Z is wildly ramified, where $k(w)^s$ denotes the maximal separable subextension of $k(w)$ in $k(z)$. Then x is a vertical point of X.*

2 Geometry of coverings of curves

We maintain the notations introduced in Section 1.

Lemma 2.1. *Let $f : Y \to X$ be a finite G-stable covering over S and y a closed point of Y. If y is a node (resp. smooth point) of the special fiber Y_s of Y, then $f(y)$ is a node (resp. smooth point) of X_s.*

Proof. Write $I_y \subseteq G$ for the inertia group of y. Consider the quotient morphism $q_{I_y} : Y \to Y/I_y$. Note that [R1, Appendice Corollaire] implies that Y/I_y is a semi-stable curve over S. The morphism f induces a natural morphism $d_{I_y} : Y/I_y \to X$ such that $f = d_{I_y} \circ q_{I_y}$. Then d_{I_y} is étale at $q_{I_y}(y)$. Thus, to verify the lemma, we may assume that $I_y = G$.

Suppose that y is a smooth point of Y_s. Then x is a smooth point (cf. [R1, Proposition 5]).

Suppose that y is a node of Y_s. Let Y_1 and Y_2 be the irreducible components (which may be equal) of Y_s which contain y. Write $D_{Y_1} \subseteq G$ and $D_{Y_2} \subseteq G$ for the decomposition groups of Y_1 and Y_2, respectively. The proof of [R1, Proposition 5] implies that one of the following statements holds: (i) if D_{Y_1} and D_{Y_2} are not equal to G, then $f(y)$ is a smooth point; (ii) if $D_{Y_1} = D_{Y_2} = G$, then $f(y)$ is a node.

Next, we prove that the case (i) does not happen. If D_{Y_1} and D_{Y_2} are not equal to G, then, for each $\tau \in G \setminus D_{Y_1}$ (resp. $\tau \in G \setminus D_{Y_2}$), $\tau(Y_1) = Y_2$ and $\tau(Y_2) = Y_1$. Thus, we obtain that $D := D_1 = D_2$. Moreover, D is a normal subgroup of G. By replacing I_y by I_y/D and Y by Y/D and applying the case (ii), we may assume that D is trivial. Then f_s is étale at the generic points of Y_1 and Y_2. Consider the local morphism $f_y : \operatorname{Spec} \mathcal{O}_{Y,y} \to \operatorname{Spec} \mathcal{O}_{X,f(y)}$ induced by f. Since f_y is étale at all the points of $\operatorname{Spec} \mathcal{O}_{Y,y}$ corresponding to the prime ideals of $\mathcal{O}_{Y,y}$ of height 1, the Zariski-Nagata purity theorem implies that f_y is étale. This means that if $f(y)$ is a smooth point, y is a smooth point too. This is a contradiction. We complete the proof of the lemma. $\square$

Remark 2.1.1. The lemma also has been treated by Raynaud (cf. [R2, Lemme 6.3.5]).

Definition 2.2. Let $f : Y \to X$ be a $\mathbb{Z}/p\mathbb{Z}$-stable covering over S, v an element of $v(\Gamma_{X_s})$, and X_v the irreducible component of X_s corresponding to v. Write β_{X_v} for the generic point of X_v and $k(\beta_{X_v})$ for the residue field of β_{X_v}. We shall call f a **v-wildly ramified covering** if there exists a point $\beta_{Y_v} \in f_s^{-1}(\beta_{X_v})$ such that the extension of residue fields $k(\beta_{Y_v})/k(\beta_{X_v})$ induced by f is purely inseparable. We shall call f a **wildly ramified covering** if there exists an element $v \in v(\Gamma_{X_s})$ such that f is a v-wildly ramified covering.

Next, we prove our main theorem.

Theorem 2.3. *Suppose that* $\mathrm{char}(K) = \mathrm{char}(k) = p > 0$.

(i) Suppose that X is smooth over S, $X_{\overline{\eta}}$ cannot be defined over an algebraic closure of $\mathbb{F}_p$, and X_s can be defined over an algebraic closure of $\mathbb{F}_p$. Then X^{ver} is not empty.

(ii) Suppose that X_s is an irreducible singular stable curve over s. Then X^{ver} is not empty.

Proof. First, let us prove (i). Proposition 1.4 (i) implies that, by replacing X by a finite étale covering of X, we may assume that $\sigma(X_{\overline{\eta}}) - \sigma(X_s) > 0$.

For any geometrically connected Galois étale covering $V_\eta \to X_\eta$ whose Galois group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, by replacing S by a finite extension of S, we may assume that V_η admits a stable model V over S. Suppose that X^{ver} is empty. Then the $\mathbb{Z}/p\mathbb{Z}$-stable covering $V \to X$ over S induced by $V_\eta \to X_\eta$ is a finite morphism, and Lemma 2.1 implies that the special fiber V_s of V is a smooth curve over s. Write β_{V_s} for the generic point of V_s and $I_{V_s} \subseteq \mathbb{Z}/p\mathbb{Z}$ for the inertia group of β_{V_s}. By applying [R1, Proposition 5], we obtain that the quotient V/I_{V_s} is a smooth curve over S. Since I_{V_s} is the inertia group of β_{V_s}, the quotient morphism $V_s \to V_s/I_{V_s}$ is a radical morphism. Moreover, since the natural morphism $V_s/I_{V_s} \to (V/I_{V_s})_s := V/I_{V_s} \times_S s$ is a radical morphism (cf. [KM, Corollary A7.2.2]), the natural morphism of special fibers $V_s \to (V/I_{V_s})_s$ induced by the quotient $V \to V/I_{V_s}$ is a radical morphism. Then we have $g_{V_s} = g_{V_{\overline{\eta}}} > g_{V_{\overline{\eta}}/I_{V_s}} = g_{V_s/I_{V_s}} = g_{V_s}$, where $g_{(-)}$ denotes the genus of $(-)$. We obtain that I_{V_s} is trivial. Thus, V_s is an étale covering over X_s. This means that $\sigma(X_{\overline{\eta}}) - \sigma(X_s) = 0$. This is a contradiction. We complete the proof of (i).

Next, we start to prove (ii). By replacing S by a finite extension of S, there exist a finite group G and a G-stable covering $d : X' \to X$ over S such that (i) the morphism $d_s : X'_s \to X_s$ induced by d on the special fibers is an admissible covering; (ii) each irreducible component of X'_s is smooth over s; (iii) $\sigma(X'_{\overline{\eta}}) - \sigma(X'_s) > 0$ (cf. Proposition 1.4 (ii)), where $X'_{\overline{\eta}} := X' \times_S \overline{\eta}$. In order to prove (ii), from now on, we suppose that $X^{\mathrm{ver}} = \emptyset$.

By replacing S by a finite extension of S again, we may assume that each Galois étale covering of $X'_\eta := X' \times_S \eta$ over η, whose Galois group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, admits a stable model over S.

Claim 1: There exists a $\mathbb{Z}/p\mathbb{Z}$-stable covering of $f : Y \to X'$ such that f is a wildly ramified covering.

Let us prove Claim 1. If Claim 1 is not true, by Definition 6, we have that, for any $\mathbb{Z}/p\mathbb{Z}$-stable covering $Z \to X'$, the morphism of the special fibers $Z_s \to X'_s$ induced by $Z \to X'$ is generically étale. Then Remark 1.3.2 implies that $Z_s \to X'_s$ is an admissible covering. This means that $\sigma(X'_{\overline{\eta}}) = \sigma(X'_s)$. Thus, we obtain a contradiction. This completes the proof of Claim 1.

Since $g_{Y_s} > g_{X'_s}$, we obtain that $f_s : Y_s \to X'_s$ is not a radicial morphism. Write $\Gamma_{X'_s}$ for the dual graph of X'_s, $v(\Gamma_{X'_s})$ for the set of the vertices of $\Gamma_{X'_s}$, and $e(\Gamma_{X'_s})$ for the set of the edges of $\Gamma_{X'_s}$. Thus, there exist two vertices w_1 and w_2 of $v(\Gamma_{X'_s})$ which are linked by an edge $e \in e(\Gamma_{X'_s})$ such that *f is a w_1-wildly ramified covering, and f is not a w_2-wildly ramified covering.* Moreover, we have the following claim.

Claim 2: There exists a $\mathbb{Z}/p\mathbb{Z}$-stable covering $g : Z \to X'$ over S such that *g is a w_2-wildly ramified covering, and g is not a w_1-wildly ramified covering.*

Let us prove Claim 2. Let $v, v' \in v(\Gamma_{X'_s})$ be two vertices which are linked by an edge of $\Gamma_{X'_s}$. Set $v \rightsquigarrow v'$ if each v-wildly ramified coverings is a v'-wildly ramified coverings. We define a subgraph Γ_v of $\Gamma_{X'_s}$ associated to $v \in v(\Gamma_{X'_s})$ to be the maximal subgraph of $\Gamma_{X'_s}$ such that the following conditions hold: (i) A vertex $v'' \in v(\Gamma_{X'_s})$ is a vertex of Γ_v if there is a path $v_0e_{01}v_1e_{1,2}\ldots v_{n-1}e_{n-1,n}v_n$ in $\Gamma_{X'_s}$, where $v_i \in v(\Gamma_{X'_s})$ and $e_{i,i+1} \in e(\Gamma_{X'_s})$ for $i = 0, \ldots, n-1$, such that (a) $v_0 = v$ and $v_n = v''$; (b) v_i and v_{i+1} are linked by $e_{i,i+1}$; (c) $v_0 \rightsquigarrow v_1, \ldots, v_{n-1} \rightsquigarrow v_n$; (ii) for any edge $e \in e(\Gamma_{X'_s})$, write v_1^e and v_2^e for the vertices of $\Gamma_{X'_v}$ which are abutted by e; an edge $e \in e(\Gamma_{X'_s})$ is an edge of Γ_v if v_1^e and v_2^e are vertices of Γ_v. If Claim 2 is not true, by the definition of Γ_{w_2}, we have $w_1 \in \Gamma_{w_2}$. Then we obtain $\Gamma_{w_1} \subseteq \Gamma_{w_2}$. On the other hand, since the $v(\Gamma_{X'_s})$ is transitive under the action of G, we have $\Gamma_{w_1} = \Gamma_{w_2}$. In particular, we have $v_1 \rightsquigarrow v_2$. This contradicts Claim 1, so Claim 2 follows.

Write Y_η (resp. Y_s, Z_η, Z_s) for the generic fiber of Y (resp. the special fiber Y, the generic fiber of Z, the special fiber of Z). By replacing S by a finite extension of S, we may assume that $W_\eta := Y_\eta \times_{X'_\eta} Z_\eta$ admits a stable model W over S. Write W_s for the special fiber W. Then the natural projections $W_\eta \to Y_\eta$ and $W_\eta \to Z_\eta$ over X'_η induce two morphisms of stable curves $W \to Y$ and $W \to Z$ over X', respectively. Write T for the fiber product $Y \times_{X'} Z$, T_η (resp. T_s) for the generic fiber of T (resp. the special fiber of T). Then we obtain a natural morphism $n : W \to T$ over X' induced by the morphisms $W \to Y$ and $W \to Z$. Write h for the morphism $W \to X'$ induced by the natural morphism $W_\eta \to X'_\eta$ and h' for the natural morphism $T \to X'$. Note that we have $h = h' \circ n$. Since we assume that $X^{\mathrm{ver}} = \emptyset$, h is a finite morphism. Thus, n is a finite morphism too. This means that W is equal to the normalization of T.

Write X'_{w_1} (resp. X'_{w_2}) for the irreducible component of X'_s corresponding to w_1 (resp. w_2), Y_1 (resp. Y_2) for the closed subscheme $Y \times_{X'} X'_{w_1} \subset Y_s$ (resp. $Y \times_{X'} X'_{w_2} \subset Y_s$), Z_1 (resp. Z_2) for the closed subscheme $Z \times_{X'} X'_{w_1} \subset Z_s$ (resp. $Z \times_{X'} X'_{w_2} \subset Z_s$), T_1 (resp. T_2) for the closed subscheme $T \times_{X'} X'_{w_1} \subset T_s$ (resp. $T \times_{X'} X'_{w_2} \subset T_s$), and W_1 (resp. W_2)

for the closed subscheme $W \times_{X'} X'_{w_1} \subset W$ (resp. $W \times_{X'} X'_{w_2} \subset W_s$). By the construction of Y and Z, we have $T \to Y$ is étale at the generic point of Y_1, and $T \to Z$ is étale at the generic point of Z_2. Thus, $\mathcal{O}_{T,\beta_{T_1}}$ and $\mathcal{O}_{T,\beta_{T_2}}$ are normal, where β_{T_1} and β_{T_2} denote the respective generic points of T_1 and T_2. Then $n|_{W_1} : W_1 \to T_1$ and $n|_{W_2} : W_2 \to T_2$ are birational. Moreover, since T_1 and T_2 are smooth over s, W_1 and W_2 are smooth over s too. Then $n|_{W_1}$ and $n|_{W_2}$ are isomorphisms.

Write $x_e \in X'_{w_1} \cap X'_{w_2}$ for the node corresponding to e and u_e for the unique closed point of W such that $h(u_e) = x_e$. Write $\widehat{X}'_{w_1} := \overline{\{\widehat{\beta}_{X'_{w_1}}\}}$ and $\widehat{X}'_{w_2} := \overline{\{\widehat{\beta}_{X'_{w_2}}\}}$ for the irreducible components of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{X'_s,x_e}$, respectively, where $\widehat{\beta}_{X'_{w_1}}$ and $\widehat{\beta}_{X'_{w_2}}$ denote the generic points of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{X'_s,x_e}$. Note that since X'_{w_1} and X'_{w_2} are smooth over s, $\mathcal{O}_{\widehat{X}'_{w_1}}$ and $\mathcal{O}_{\widehat{X}'_{w_2}}$ are discrete valuation rings. Write $\widehat{h}_s$ for the morphism $\mathrm{Spec}\,\widehat{\mathcal{O}}_{W_s,u_e} \to \mathrm{Spec}\,\widehat{\mathcal{O}}_{X'_s,x_e}$ induced by h, $\widehat{\beta}_{W_1} := (\widehat{h}_s)^{-1}(\widehat{\beta}_{X'_{w_1}})$ and $\widehat{\beta}_{W_2} := (\widehat{h}_s)^{-1}(\widehat{\beta}_{X'_{w_2}})$ for the generic points of the irreducible components of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{W_s,u_e}$, $k(\widehat{\beta}_{W_1})$ and $k(\widehat{\beta}_{W_2})$ for the residue fields of $\widehat{\beta}_{W_1}$ and $\widehat{\beta}_{W_2}$, respectively. Write $\widehat{W}^s_1$ and $\widehat{W}^s_2$ for the respective integral closures of $\widehat{X}'_{w_1}$ and $\widehat{X}'_{w_2}$ in $k(\widehat{\beta}_{W_1})^s$ and $k(\widehat{\beta}_{W_2})^s$, where $k(\widehat{\beta}_{W_1})^s$ and $k(\widehat{\beta}_{W_2})^s$ denote the respective maximal separable subextension of $k(\widehat{\beta}_{X'_{w_1}})$ and $k(\widehat{\beta}_{X'_{w_2}})$ in $k(\widehat{\beta}_{W_1})$ and $k(\widehat{\beta}_{W_2})$. Note that $\mathcal{O}_{\widehat{W}^s_1}$ and $\mathcal{O}_{\widehat{W}^s_2}$ are discrete valuation rings.

Claim 3: The morphism $\mathcal{O}_{\widehat{X}'_{w_1}} \to \mathcal{O}_{\widehat{W}^s_1}$ (resp. $\mathcal{O}_{\widehat{X}'_{w_2}} \to \mathcal{O}_{\widehat{W}^s_2}$) induced by the natural morphism $\widehat{W}^s_1 \to \widehat{X}'_{w_1}$ (resp. $\widehat{W}^s_2 \to \widehat{X}'_{w_2}$) is a wildly ramified extension.

Let us prove Claim 3. Write $\beta_{X'_{w_1}}$ (resp. $\beta_{X'_{w_2}}$) for the generic point of the irreducible component X'_{w_1} (resp. X'_{w_2}). Write $t_e \in T_1 \cap T_2$ (resp. β_{T_1}, β_{T_2}) for $n(u_e)$ (resp. the generic point of T_1, the generic point of T_2). We have

$$T_1 \to T^s_1 \to X'_{w_1}$$

and

$$T_2 \to T^s_2 \to X'_{w_2},$$

where T^s_1 and T^s_2 denote the smooth projective curves over s whose function fields are the maximal separable subextensions of $k(\beta_{T_1})/k(\beta_{X'_{w_1}})$ and $k(\beta_{T_2})/k(\beta_{X'_{w_2}})$, respectively. Then by the construction of T, we have T^s_1 and T^s_2 are isomorphic to Z_1 and Y_2, respectively. Thus, $T^s_1 \to X_{w_1}$ and $T^s_2 \to X_{w_2}$ are wildly ramified at the closed points $(T_1 \to T^s_1)(t_e)$ and $(T_2 \to T^s_2)(t_e)$, respectively. Since $n|_{W_1}$ and $n|_{W_2}$ are isomorphisms, we have $\widehat{W}^s_1 \cong \mathrm{Spec}\,\widehat{\mathcal{O}}_{T^s_1,t_e}$ and $\widehat{W}^s_2 \cong \mathrm{Spec}\,\widehat{\mathcal{O}}_{T^s_2,t_e}$. Then Claim 3 follows.

Then Proposition 1.6 and Claim 3 imply that x_e is a vertical point. Thus, X^{ver}_s is not empty. This is a contradiction. Then we complete the proof of the theorem. □

Remark 2.3.1. Suppose that $\mathrm{char}(K) = 0$. If k is an algebraic closure of $\mathbb{F}_p$, A. Tamagawa proved that X^{ver} is equal to the set of closed points of X. If k is an arbitrary algebraically closed field of characteristic p, we may prove that X^{ver} is dense in X_s and contains all the nodes of X_s (see Appendix of the present paper).

Theorem 2.4. *Let G be a finite group, $f : Y \to X$ a G-stable covering over S, and $f_s : Y_s \to X_s$ the morphism induced by f on the special fibers.*

(i) Suppose that X_s is smooth over s. Then f_s is a finite morphism if and only if f_s is an étale covering.

(ii) Suppose that X_s is irreducible, and G is a solvable group. Then f_s is a finite morphism if and only if f_s is an admissible covering.

Proof. First, let us prove (i). We only need to prove that, if f_s is a finite morphism, then f_s is an étale covering. By Lemma 2.1, we obtain that Y is smooth over S. Write I_{Y_s} for the inertia group of the special fiber Y_s. Thus, the finite morphism $Y/I_{Y_s} \to X$ is étale. Since $Y = Y_s/I_{Y_s} \to (Y/I_{Y_s})_s := Y/I_{Y_s} \times_S s$ is a radicial morphism, thus the genus of Y_s and $(Y/I_{Y_s})_s$ are equal. On the other hand, the genus of $(Y/I_{Y_s}) \times_S \eta$ is equal to the genus of $(Y/I_{Y_s})_s$, and the genus of Y_η is larger than $(Y/I_{Y_s}) \times_S \eta$. Thus, we have $I_{Y_s} = \{1\}$. Then f_s is étale.

Next, let us prove (ii). We only need to prove that, if f_s is a finite morphism, then f_s is an admissible covering.

If $(\#G, p) = 1$, where $\#(-)$ denotes the cardinality of $(-)$, then the specialization theorem of prime to p admissible fundamental groups (cf. [V, Théorème 2.2]) implies that f_s is an admissible covering. Thus, we may assume that $p | \#G$.

Suppose that $\#G = p$. For each irreducible component Y_v of special fiber Y_s, write $I_{Y_v} \subseteq G$ for the inertia group of Y_v. Since G is abelian, and X_s is irreducible, we obtain that I_{Y_v} does not depend on the choices of Y_v. Then we use the notation I to denote I_{Y_v}. Similar arguments to the arguments given in the proof of (i) imply that $g_{Y_s} = g_{Y_s/I} = g_{Y_{\overline{\eta}}/I} = g_{Y_{\overline{\eta}}}$. We have $I = \{1\}$. Then Remark 1.3.2 implies that f_s is an admissible covering.

Suppose that $\#G > p$. Let

$$\{1\} \subset A := G^{(m)} \subset G^{(m-1)} \subset \cdots \subset G$$

be the derived series of G. Write A_p for the Sylow p-subgroup of A and G' for G/A_p. Then we obtain a A_p-stable covering $f^{A_p} : Y \to Y/A_p$ over S and a G'-stable covering $f^{G'} : Y/A_p \to X$ over S such that $f = f^{G'} \circ f^{A_p}$. By induction, we obtain that $f_s^{G'}$ is an admissible covering. Let us prove that $f_s^{A_p}$ is an admissible covering.

Let $Y_v \subseteq Y_s$ be an irreducible component and $I_{Y_v} \subseteq A_p$ the inertia group of Y_v. For each irreducible component $Y_{v'} \subseteq Y_s$, write $I_{Y_{v'}} \subseteq A_p$ for the inertia group of $Y_{v'}$. Since X_s is irreducible, we have $\#I_{Y_{v'}} = \#I_{Y_v}$. If I_{Y_v} is trivial, then Remark 1.3.2 implies that $f_s^{A_p}$ is an admissible covering. Thus, we may assume that I_{Y_v} is not trivial. Moreover, we have the following Claim.

> **Claim:** For each irreducible component $Y_{v'} \subseteq Y_s$, we have $I_{Y_{v'}} = I_{Y_v}$.
>
> Let us prove the claim. Write A'_p for the quotient group A_p/I_{Y_v}. Consider the quotient of Y by I_{Y_v}, we obtain two morphisms of stable curves $f^{I_{Y_v}} : Y \to Y/I_{Y_v}$ and $f^{A'_p} : Y/I_{Y_v} \to Y/A_p$ such that $f^{A_p} = f^{A'_p} \circ f^{I_{Y_v}}$.
>
> If Y_s is irreducible, then Claim is trivial. Thus, we may assume that Y_s is not irreducible. Let $Y_w (\neq Y_v)$ be an irreducible component such that $Y_v \cap Y_w \neq \emptyset$, $I_{Y_w} \subseteq A_p$ the inertia subgroup of Y_w, and $y \in Y_v \cap Y_w$ a node of Y_s. Write $Y_v^{I_{Y_v}}$

(resp. $Y_w^{I_{Y_v}}$, $y_{I_{Y_v}}$, $Y_v^{A_p}$, $Y_w^{A_p}$, y_{A_p}) for $f^{I_{Y_v}}(Y_v)$ (resp. $f^{I_{Y_v}}(Y_w)$, $f^{I_{Y_v}}(y)$, $f^{A_p}(Y_v)$, $f^{A_p}(Y_w)$, $f^{A_p}(y)$). By the construction, we obtain that $f_s^{I_{Y_v}}|_{Y_v}$ is radicial, and $f_s^{A'_p}|_{Y_v^{I_{Y_v}}}$ is generically étale.

If $I_{Y_v} \neq I_{Y_w}$, then $f_s^{A'_p}|_{Y_w^{I_{Y_v}}}$ is not generically étale. Thus, the local morphism $\mathcal{O}_{Y_v^{A_p}, y_{A_p}} \to \mathcal{O}_{Y_v^{I_{Y_v}}, y_{I_{Y_v}}}$ is a wildly ramified extension. Write $K(Y_{v_1})^{\mathrm{sep}}$ for the separable closure of function field $K(Y_v^{A_p})$ in the function field $K(Y_v)$. Note that $K(Y_v)^{\mathrm{sep}}$ is the function field of $Y_v^{I_{Y_v}}$. Thus, the morphism of $\mathcal{O}_{Y_v^{A_p}, y_{A_p}}$ to the integral closure of $\mathcal{O}_{Y_v^{A_p}, y_{A_p}}$ in $K(Y_v)^{\mathrm{sep}}$ is a wildly ramified extension.

On the other hand, write $K(Y_w)^{\mathrm{sep}}$ for the separable closure of function field $K(Y_w^{A_p})$ in the function field $K(Y_w)$. Similar arguments to the arguments given above imply that the morphism of $\mathcal{O}_{Y_w^{A_p}, y_{A_p}}$ to the integral closure of $\mathcal{O}_{Y_w^{A_p}, y_{A_p}}$ in $K(Y_w)^{\mathrm{sep}}$ is a wildly ramified extension. By Proposition 1.6, we have that y_{A_p} is a vertical point. This contradict to the assumption that f_s is finite. Thus, we have $I_{Y_v} = I_{Y_w}$. Moreover, since Y_s is connected, similar arguments to the arguments given above imply that, for any irreducible component $Y_{v'}$, the inertia subgroup $I_{Y_{v'}}$ is equal to I_{Y_v}.

We use the notation I to denote I_{Y_v}. Similar arguments to the arguments given in the proof of (i) imply that $g_{Y_s} = g_{Y_s/I} = g_{Y_{\overline{\eta}}/I} < g_{Y_{\overline{\eta}}}$. This is a contradiction. Then I is trivial. Then Remark 1.3.2 implies that f_s is an admissible covering.

□

3 Appendix

In this subsection, we assume that $\mathrm{char}(K) = 0$ and $\mathrm{char}(k) = p > 0$.

Let $M_{g,r,k}$ be the coarse moduli space of $\mathcal{M}_{g,r} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, k$. For any point $x \in M_{g,r,k}$, write $\overline{x}$ for a geometric point above x and $(C_{\overline{x}}, D_{\overline{x}})$ for a pointed smooth curve corresponding to the point $\overline{x}$ (well-defined up to isomorphism). Then the isomorphism type of the geometric tame fundamental group $\pi_1^{\mathrm{tame}}((C_{\overline{x}}, D_{\overline{x}}))$ is independent of the choice of $\overline{x}$ and $(C_{\overline{x}}, D_{\overline{x}})$ (and the implicit base point on $(C_{\overline{x}}, D_{\overline{x}})$ used to define $\pi_1^{\mathrm{tame}}((C_{\overline{x}}, D_{\overline{x}}))$. We have the following result proved by Saïdi and Tamagawa.

Proposition 3.1. *Let $U \subseteq M_{g,r,k}$ a subvariety of positive dimension. Then the geometric tame fundamental group π_1^{tame} is not constant on U.*

Proof. See [ST, Theorem 3.12]. □

By using the idea of the proof of Tamagawa's result (cf. [T3, Theorem 0.2 (v)]), we have the following theorem.

Theorem 3.2. *The set of vertical points X^{ver} contains all the nodes of X_s, and the closure of X^{ver} in X_s is equal to X_s.*

Proof. By applying [M2, Lemma 2.9], we may assume that each irreducible component of X_s is smooth over s, and the genus is ≥ 2.

Let $S^{\log}$ be a log regular scheme whose underlying scheme is S, and whose log structure is determined by the closed point of S. Then there is a natural morphism $S^{\log} \to \overline{\mathcal{M}}^{\log}_{g_X,0}$ determined by $X \to S$. Thus, we obtain a stable log curve $X^{\log} := \overline{\mathcal{M}}^{\log}_{g_X,1} \times_{\overline{\mathcal{M}}^{\log}_{g_X,0}} S^{\log}$ whose underlying scheme is X.

Let x be a closed point of X_s. Write X_v for an irreducible component which contains x. We define a pointed smooth curve $U_v^{\bullet}$ of type (g_v, r_v) to be $(X_v, X_v \cap X_s^{\mathrm{sing}})$, where X_s^{sing} denotes the set of nodes of X_s. Then we obtain a morphism $s \to \overline{\mathcal{M}}^{\log}_{g_v,r_v}$ determined by $U_v^{\bullet} \to s$ and a stable log curve $X_v^{\log} := \overline{\mathcal{M}}^{\log}_{g_v,r_v+1} \times_{\overline{\mathcal{M}}^{\log}_{g_v,r_v}} s$ over s whose underlying scheme is X_v.

Moreover, we consider the 2^{nd} log configuration space $Y^{\log} := \overline{\mathcal{M}}^{\log}_{g_X,2} \times_{\overline{\mathcal{M}}^{\log}_{g_X,0}} S^{\log}$ (resp. $W_v^{\log} := \overline{\mathcal{M}}^{\log}_{g_v,r_v+2} \times_{\overline{\mathcal{M}}^{\log}_{g_v,r_v}} s$) associated to $X \to S$ (resp. $U_v^{\bullet} \to s$). Note that there is a natural morphism $Y^{\log} \to X^{\log}$ (resp. $W_v^{\log} \to X_v^{\log}$) induced by the natural morphism of log stacks $\overline{\mathcal{M}}^{\log}_{g_X,2} \to \overline{\mathcal{M}}^{\log}_{g_X,1}$ (resp. $\overline{\mathcal{M}}^{\log}_{g_v,r_v+2} \to \overline{\mathcal{M}}^{\log}_{g_v,r_v+1}$) given by forgetting the last marked point. Write Y (resp. W_v) for the underlying scheme of $Y^{\log}$ (resp. $W_v^{\log}$). Write β_X (resp. β_v, $\widehat{\beta}_X$, $\widehat{\beta}_v$) for the generic point of X (resp. the generic point of X_v, the generic point of $\operatorname{Spec} \widehat{\mathcal{O}}_{X,\beta_v}$, the generic point of $\operatorname{Spec} \widehat{\mathcal{O}}_{X_v,x}$).

Write $\beta_v^{\log}$ for the log scheme whose underlying scheme is β_v, and whose log structure is the pulling-back log structure of $X^{\log}$ via the natural morphism $\beta_v \to X$. Since the restricting log structure of $X_v^{\log}$ on the generic point β_v is trivial, we have a natural morphism $\beta_v \to X_v^{\log}$. Write $(Z'_v)^{\log}$ for $W_v^{\log} \times_{X_v^{\log}} \beta_v$. Write $(Z''_v)^{\log}$ for the stable log curve whose underlying scheme is $W_v \times_{X_v} \beta_v$, and whose log structure is the pulling-back log structure of $Y^{\log} \times_{X^{\log}} \beta_v^{\log}$ via the natural morphism $W_v \times_{X_v} \beta_v \to Y^{\log} \times_{X^{\log}} \widetilde{\beta_v^{\log}}$.

Let $(-)$ be a point (resp. a log point). We use the notation $\overline{(-)}$ (resp. $\widetilde{(-)}$) to denote a geometric point (resp. log geometric point) associated to $(-)$. By the specialization theorem of log étale fundamental groups (cf. [V, Théorème 2.2]), we obtain the following commutative diagram of fundamental groups. Moreover, all seven rows in the commutative diagram are exact.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(Y \times_X \overline{\beta}_X) \arrow[r] & \pi_1(Y \times_X X_\eta) \arrow[r] & \pi_1(X_\eta) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(Y \times_X \overline{\beta}_X) \arrow[r] \arrow[u, "a_1"] & \pi_1(Y \times_X \beta_X) \arrow[r] \arrow[u, "b_1"] & \pi_1(\beta_X) \arrow[r] \arrow[u, "c_1"] & 1 \\
1 \arrow[r] & \pi_1(Y \times_X \overline{\widehat{\beta}_X}) \arrow[r] \arrow[u, "a_2"] \arrow[d, "a_3"'] & \pi_1(Y \times_X \widehat{\beta}_X) \arrow[r] \arrow[u, "b_2"] \arrow[d, "b_3"'] & \pi_1(\widehat{\beta}_X) \arrow[r] \arrow[u, "c_2"] \arrow[d, "c_3"'] & 1 \\
1 \arrow[r] & \pi_1(Y^{\log} \times_{X^{\log}} \widetilde{\beta}_v^{\log}) \arrow[r] & \pi_1(Y^{\log} \times_{X^{\log}} \beta_v^{\log}) \arrow[r] & \pi_1(\beta_v^{\log}) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(W_v^{\log} \times_{(Z_v'')^{\log}} \widetilde{\beta}_v^{\log}) \arrow[r] \arrow[u, "a_4"] \arrow[d, "a_5"'] & \pi_1(W_v^{\log} \times_{(Z_v'')^{\log}} \beta_v^{\log}) \arrow[r] \arrow[u, "b_4"] \arrow[d, "b_5"'] & \pi_1(\beta_v^{\log}) \arrow[r] \arrow[u, equal] \arrow[d, "c_5"'] & 1 \\
1 \arrow[r] & \pi_1((Z_v')^{\log} \times_{\beta_v} \overline{\beta}_v) \arrow[r] & \pi_1((Z_v')^{\log}) \arrow[r] & \pi_1(\beta_v) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1((Z_v')^{\log} \times_{\beta_v} \overline{\widehat{\beta}_v}) \arrow[r] \arrow[u, "a_6"] & \pi_1((Z_v')^{\log} \times_{\beta_v} \widehat{\beta}_v) \arrow[r] \arrow[u, "b_6"] & \pi_1(\widehat{\beta}_v) \arrow[r] \arrow[u, "c_6"] & 1.
\end{tikzcd}$$

Note that the definition of geometric (log) étale fundamental groups implies that a_1, a_2, a_5, and a_6 are isomorphisms (resp. $b_1, b_2, b_5, c_1, c_2, c_3$, and c_5 are surjections, b_2, b_6, c_2, and c_6 are injections). Since there is a natural morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{X,\beta_v} \to X$, we obtain a log scheme $T^{\log}$ whose underlying scheme is $\operatorname{Spec} \widehat{\mathcal{O}}_{X,\beta_v}$, and whose log structure is the pulling-back log structure of $X^{\log}$. Then by considering $Y^{\log} \times_{X^{\log}} T^{\log}$ and the specialization theorem of log étale fundamental groups, we obtain that a_3, b_3 are surjections. Moreover, [T3, Proposition 2.5] and [T3, Example 2.10] imply that a_4 is an injection. Then we obtain b_4 is an injection too.

For each $i = 1, ..., 7$, write $1 \to \Delta_i \to \Pi_i \to G_i \to 1$ for the i^{th} row of the above commutative diagram, $\rho_i : G_i \to \operatorname{Out}(\Delta_i)$ for the outer representation, and Im_i for the image of ρ_i. Then by [T3, Remark 2.3], we obtain

$$\mathrm{Im}_1 = \mathrm{Im}_2 \hookleftarrow \mathrm{Im}_3 \to \mathrm{Im}_4 \twoheadrightarrow \mathrm{Im}_5 = \mathrm{Im}_6 \hookleftarrow \mathrm{Im}_7.$$

Moreover, [T3, Lemma 5.2] implies that $\mathrm{Im}_3 \to \mathrm{Im}_4$ is a surjection. Write $D_{\widehat{\beta}_X}$ for the image of $c_1 \circ c_2$, I_{β_v} for the kernel of $c_5 \circ c_3$, and I_x for $\pi_1(\widehat{\beta}_v)$. Then we obtain a morphism $\pi_1(\widehat{\beta}_X)/I_{\beta_v} \to \mathrm{Im}_6$. On the other hand, since $\mathrm{Im}_1 \hookleftarrow \mathrm{Im}_3$, we have that $\rho_3(\pi_1(\widehat{\beta}_X)) = \rho_1(D_{\widehat{\beta}_X})$. Thus, we obtain a morphism $D_{\widehat{\beta}_X}/c_1 \circ c_2(I_{\beta_v}) \to \mathrm{Im}_6$. Furthermore, we have a commutative diagram as follows:

$$
\begin{tikzcd}
\pi_1(\widehat{\beta}_X) \arrow[r] \arrow[d] & D_{\widehat{\beta}_X}/c_1 \circ c_2(I_{\beta_v}) \arrow[r] & \mathrm{Im}_6 \arrow[d, equal] \\
\pi_1(\beta_v) \arrow[r, equal] & \pi_1(\beta_v) \arrow[r] \arrow[u] & \mathrm{Im}_6 \\
& I_x \arrow[r] \arrow[u] & \mathrm{Im}_7. \arrow[u]
\end{tikzcd}
$$

Write $E_x^{\log}$ for the log scheme whose underlying scheme is $\mathrm{Spec}\,\widehat{\mathcal{O}}_{X_v,x}$, and whose log structure is the pulling-back log structure of $X_v^{\log}$ induced by the natural morphism $\mathrm{Spec}\,\widehat{\mathcal{O}}_{X_v,x} \to X_v$. Write $x^{\log}$ for the log scheme obtained by restricting $E_x^{\log}$ to the closed point $x \in \mathrm{Spec}\,\widehat{\mathcal{O}}_{X_v,x}$, and $\widetilde{x}^{\log}$ for a geometric log point associated to $x^{\log}$. Consider a stable log curve $W_v^{\log} \times_{X_v^{\log}} E_x^{\log} \to E_x^{\log}$ over $E_x^{\log}$. Then we obtain the specialization map

$$sp_x : \pi_1(W_v^{\log} \times_{X_v^{\log}} \widetilde{\beta}_v^{\log}) \to \pi_1(W_v^{\log} \times_{X_v^{\log}} \widetilde{x}^{\log}).$$

Suppose that sp_x is not an isomorphism. Thus, by applying [T3, Proposition 4.1 (ii)], we have the image of wild inertia subgroup I_x^{w} in $D_{\widehat{\beta}_X}/c_1 \circ c_2(I_{\beta_v})$ is infinite. Then by Proposition 1.6, we have $x \in X^{\mathrm{ver}}$.

If x is a node of X_s, then the underlying scheme of $W_v^{\log} \times_{X_v^{\log}} \widetilde{x}^{\log}$ is a singular curve. Then Proposition 1.4 implies that sp_x is not an isomorphism. This means that X^{ver} contains all the nodes of X_s. If x is a smooth closed point of X_s, then the underlying scheme of $W_v^{\log} \times_{X_v^{\log}} \widetilde{x}^{\log}$ is a smooth curve over x. By applying Proposition 3.1, we obtain that the closure of X^{ver} in X_s is equal to X_s. This completes the proof of the theorem.

□

Remark 3.2.1. Note that since the geometric étale fundamental group of X_η is topologically finitely generated, there are only countably many étale coverings that each may contribute at most finitely many points to X^{ver}. Thus, X^{ver} is countable.

References

[M1] S. Mochizuki, The geometry of the compactification of the Hurwitz scheme, *Publ. Res. Inst. Math. Sci.* **31** (1995), 355-441.

[M2] S. Mochizuki, The profinite Grothendieck conjecture for closed hyperbolic curves over number fields. *J. Math. Sci. Univ. Tokyo* **3** (1996), 571-627.

[PS] F. Pop, M. Saïdi, On the specialization homomorphism of fundamental groups of curves in positive characteristic. *Galois groups and fundamental groups,* 107-118, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.

[R1] M. Raynaud, p-groupes et réduction semi-stable des courbes. *The Grothendieck Festschrift, Vol. III*, 179-197, *Progr. Math.*, **88**, *Birkhäuser Boston, Boston, MA,* 1990.

[R2] M. Raynaud, Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar, *Invent. Math.* **116** (1994), 425-462.

[R3] M. Raynaud, Sur le groupe fondamental d'une courbe complète en caractéristique $p > 0$. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 335-351, *Proc. Sympos. Pure Math.,* **70**, *Amer. Math. Soc., Providence, RI,* 2002.

[S] M. Saïdi, p-rank and semi-stable reduction of curves, *C. R. Acad. Sci. Paris,* t. **326**, Série I, 63-68, 1998.

[ST] M. Saïdi, A. Tamagawa, Variation of fundamental groups of curves in positive characteristic. *J. Algebraic Geom.* **26** (2017), 116.

[T1] A. Tamagawa, On the tame fundamental groups of curves over algebraically closed fields of characteristic > 0. *Galois groups and fundamental groups,* 47-105, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.

[T2] A. Tamagawa, Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups. *J. Algebraic Geom.* **13** (2004), 675-724.

[T3] A. Tamagawa, Resolution of nonsingularities of families of curves. *Publ. Res. Inst. Math. Sci.* **40** (2004), 1291-1336.

[V] I. Vidal, Contributions à la cohomologie étale des schémas et des log-schémas, Thèse, U. Paris-Sud (2001).

[Y1] Y. Yang, Formulas for local and global p-ranks of coverings of curves, RIMS Preprint 1863. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/.

[Y2] Y. Yang, On the admissible fundamental groups of curves over algebraically closed fields of characteristic $p > 0$, RIMS Preprint 1869. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/.

[Y3] Y. Yang, On the weak Hom-version of Grothendicek conjecture for curves of type $(0, n)$ over algebraically closed fields of characteristic $p > 0$. See also http://www.kurims.kyoto-u.ac.jp/~yuyang/.

[Y4] Y. Yang, On the Hom-version of the Grothendieck conjecture for curves of genus 0 in positive characteristic, in preparation.

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