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On the Existence, Geometry and p-Ranks of Vertical Fibers of Coverings of Curves

By

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Abstract

Let R be a complete DVR with algebraically closed residue field of characteristic $p > 0$ and X, Y stable curves over R with smooth generic fibers. Let $f : Y \to X$ be a morphism over R such that the morphism of generic fibers induced by f is a Galois étale covering. A closed point x of X is called a vertical point if $\dim f^{-1}(x) = 1$. In this case, $f^{-1}(x)$ is called the vertical fiber associated to x. We study the existence, the geometry, and the p-ranks of vertical fibers under certain assumptions.

§ 1. Preliminaries

Let R be a complete discrete valuation ring with algebraically closed residue field k, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R. Write $\eta, \overline{\eta}$, and s for the generic point, the geometric generic point, and the closed point of S corresponding to the natural morphisms $\mathrm{Spec}\, K \to S$, $\mathrm{Spec}\, \overline{K} \to S$, and $\mathrm{Spec}\, k \to S$, respectively. Let X be a stable curve of genus g_X over S. Write X_η, $X_{\overline{\eta}}$ and X_s for the generic fiber, the geometric generic fiber and the special fiber, respectively. Moreover, we suppose that X_η is nonsingular.

§ 1.1. Admissible fundamental groups and specialization

Definition 1.1. Let $\phi : Z \to X_s$ be a morphism of stable curves over s. We shall call ϕ a **Galois admissible covering** over s (or Galois admissible covering for short)

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if the following conditions hold: (i) there exists a finite group $G \subseteq \mathrm{Aut}_k(Z)$ such that $Z/G = X_s$, and ϕ is equal to the quotient morphism $Z \to Z/G$; (ii) for each $z \in Z^{\mathrm{sm}}$, ϕ is étale at z, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$; (iii) for any $z \in Z^{\mathrm{sing}}$, the image $\phi(z)$ is contained in X_s^{sing}, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$; (iv) let $z \in Z^{\mathrm{sing}}$ and $D_z \subseteq G$ the decomposition group of z; the local morphism between two nodes (cf. (iii)) induced by ϕ may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X_s,\phi(z)} \cong k[[u,v]]/uv & \to & \hat{\mathcal{O}}_{Z,z} \cong k[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) = p > 0$; moreover, $\tau(s) = \zeta_{\#D_z} s$ and $\tau(t) = \zeta_{\#D_z}^{-1} t$ for each $\tau \in D_z$, where $\#D_z$ denotes the order of D_z, and $\zeta_{\#D_z}$ is a primitive $\#D_z$-th root of unit. We shall call ϕ an **admissible covering** if there exists a morphism of stable curves $\phi' : Z' \to Z$ over s such that the composite morphism $\phi \circ \phi' : Z' \to X_s$ is a Galois admissible covering over s.

Let Y be the disjoint union of finitely many stable curves over s. We shall call a morphism $\psi : Y \to X_s$ over s **multi-admissible** if the restriction of ψ to each connected component of Y is an admissible covering.

We use the notation $\mathrm{Cov}^{\mathrm{adm}}(X_s)$ to denote the category which consists of (empty object and) all the multi-admissible coverings of X_s. It is well-known that $\mathrm{Cov}^{\mathrm{adm}}(X_s)$ is a Galois category. Thus, by choosing a base point $x \in X_s$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X_s, x)$ which is called the **admissible fundamental group** of X_s. For simplicity, we omit the base point and denote the admissible fundamental group by $\pi_1^{\mathrm{adm}}(X_s)$.

Remark. Note that by the definition of admissible coverings, if $\mathrm{char}(k) = p > 0$, the maximal pro-p quotient of the admissible fundamental group $\pi_1^{\mathrm{adm}}(X_s)$ is isomorphic to the maximal pro-p quotient of the étale fundamental group $\pi_1(X_s)$.

Remark. Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of pointed stable curves of type (g, r) over $\mathrm{Spec}\,\mathbb{Z}$ and $\mathcal{M}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ parametrizing pointed smooth curves. Write $\overline{\mathcal{M}}_{g,r}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subset \overline{\mathcal{M}}_{g,r}$ relative to $\mathrm{Spec}\,\mathbb{Z}$. We use the notation $\overline{\mathcal{M}}_g$ (resp. $\overline{\mathcal{M}}_g^{\log}$) to denote the stack $\overline{\mathcal{M}}_{g,0}$ (resp. the log stack $\overline{\mathcal{M}}_{g,0}^{\log}$).

Let $s^{\log} \to \overline{\mathcal{M}}_{g_X}^{\log}$ be a morphism from an fs log point $s^{\log}$ (i.e., an fs log scheme whose underlying scheme is s) whose underlying morphism $s \to \overline{\mathcal{M}}_{g_X}$ is determined by $X_s \to s$. Thus, we obtain a stable log curve $X_s^{\log} := s^{\log} \times_{\overline{\mathcal{M}}_{g_X}^{\log}} \overline{\mathcal{M}}_{g_X,1}^{\log}$ whose underlying scheme is X_s. Then the admissible fundamental group of X_s is isomorphic to the geometric log étale fundamental group of $X_s^{\log}$.

For more details on admissible coverings, log admissible coverings and the fundamental groups for (pointed) stable curves, see [3], [12].

By applying the theory of deformation of stable log curves, we obtain a **specialization morphism** from the geometric étale fundamental group of the generic fiber to the admissible fundamental group of the special fiber:

$$Sp : \pi_1(X_{\overline{\eta}}) \to \pi_1^{\mathrm{adm}}(X_s).$$

Sp is always a surjection, but Sp is not an injection in general. Moreover, we have the following theorem.

Theorem 1.2. *(i) ([1, Exposé X Corollaire 3.9], [11, Théorème 2.2]) If* $\mathrm{char}(K) = \mathrm{char}(k) = 0$*, then* Sp *is an isomorphism.*

(ii) If $\mathrm{char}(K) = 0$ *and* $\mathrm{char}(k) = p > 0$*, then* Sp *is not an isomorphism (cf. the following Remark).*

(iii) If $\mathrm{char}(K) = \mathrm{char}(k) = p > 0$*, then we have the following results:*

(a) ([4, Theorem A and Theorem B], [7, Proposition 2.2.5], [9, Theorem 0.1]) if $k = \overline{\mathbb{F}}_p$*,* X_s *is smooth over* s *and* X *is not a trivial family over* S*, then* Sp *is not an isomorphism;*

(b) ([10, Corollary 3.11]) if X_s *is singular, then* Sp *is not an isomorphism.*

Remark. By the first remark under Definition 1.1, if $\mathrm{char}(k) = p > 0$, we have that the maximal pro-p quotient $\pi_1^p(X_s)$ of $\pi_1(X_s)$ is isomorphic to the maximal pro-p quotient $\pi_1^{p\text{-adm}}(X_s)$ of $\pi_1^{\mathrm{adm}}(X_s)$. Then Theorem 1.2 (ii) follows from the following fact (cf. the third remark of Definition 1.3):

$$\dim_{\mathbb{F}_p}(\mathrm{H}^1_{\text{ét}}(X_{\overline{\eta}}, \mathbb{F}_p)) = 2g_X > g_X \geq \dim_{\mathbb{F}_p}(\mathrm{H}^1_{\text{ét}}(X_s, \mathbb{F}_p)).$$

§ 1.2. Some definitions

In this subsection, we give some definitions. From now on, we assume that $\mathrm{char}(k) = p > 0$.

Definition 1.3. Write $\pi_1^p(X_s)$ for the maximal pro-p quotient of the étale fundamental group $\pi_1(X_s)$ of X_s. It is well-known that $\pi_1^p(X_s)$ is a finitely generated free pro-p group. We define the ***p*-rank** $\sigma(X_s)$ of X_s as follows:

$$\sigma(X_s) := \mathrm{rank}(\pi_1^p(X_s)) = \dim_{\mathbb{F}_p}(\mathrm{H}^1_{\text{ét}}(X_s, \mathbb{F}_p)).$$

Remark. For a semi-stable curve Z over k, we may also define the p-rank $\sigma(Z)$ of Z as follows:

$$\sigma(Z) := \mathrm{rank}(\pi_1^p(Z)) = \dim_{\mathbb{F}_p}(\mathrm{H}^1_{\text{ét}}(Z, \mathbb{F}_p)).$$

Remark. If X_s is smooth, then the p-rank $\sigma(X_s)$ is equal to the dimension of the p-torsion points of the Jacobian J_{X_s} of X_s as an $\mathbb{F}_p$-vector space.

Suppose that X_s is a singular curve. Write Γ_{X_s} for the dual graph of X_s and $v(\Gamma_{X_s})$ for the set of vertices of Γ_{X_s}. For $v \in v(\Gamma_{X_s})$, write X_v for the irreducible component of X_s corresponding to v and $\widetilde{X_v}$ for the normalization of X_v. Then the p-rank $\sigma(X_s)$ of X_s is equal to

$$\sum_{v \in v(\Gamma_{X_s})} \sigma(\widetilde{X_v}) + \mathrm{rank}(\mathrm{H}^1(\Gamma_{X_s}, \mathbb{Z})),$$

where $\mathrm{rank}(\mathrm{H}^1(\Gamma_{X_s}, \mathbb{Z}))$ denotes the rank of $\mathrm{H}^1(\Gamma_{X_s}, \mathbb{Z})$ as a finitely generated free $\mathbb{Z}$-module.

Remark. Note that we have $\sigma(X_s) \leq g_X$.

Definition 1.4. We shall call X_s **ordinary** if $\sigma(X_s) = g_X$.

Definition 1.5. Let $f : Y \to X$ be a morphism of stable curves over S and G a finite group. We shall call f a **stable covering** (resp. ***G*-stable covering**) if the morphism of generic fibers f_η is an étale covering (resp. a Galois étale covering with Galois group G).

Remark. Let $f_\eta : Y_\eta \to X_\eta$ be a morphism of smooth, geometrically connected projective curves over $\mathrm{Spec}\, K$ and G a finite group. Suppose that f_η is a G-étale covering. Then by applying the stable reduction theorem for curves, after possibly replacing S by a finite extension of S, we can extend f_η to a G-stable covering over S (cf. [2, Theorem 0.2]).

Definition 1.6. Let $f : Y \to X$ be a stable covering. Suppose that the morphism of special fibers $f_s : Y_s \to X_s$ is not finite. A closed point $x \in X$ is called a **vertical point** associated to f, or for simplicity, a vertical point when there is no fear of confusion, if $\dim(f^{-1}(x)) = 1$. The inverse image $f^{-1}(x)$ is called the **vertical fiber** associated to x.

§ 2. Questions and results

Let G be a finite group and $f : Y \to X$ a G-stable covering. By Theorem 1.2, *Sp* is not an isomorphism in general. It is natural to pose the following question:

Question 2.1. *Is f_s always a finite morphism? When Y_s an ordinary curve? How to compute the p-ranks of vertical fibers?*

Remark. The motivations of Question 2.1 are as follows:

(1) to understand the reduction of an étale covering of X_η;

(2) to understand the structure of the admissible fundamental groups of stable curves over an algebraically closed field of positive characteristic.

§ 2.1. Existence of vertical fibers

Since Sp is not an isomorphism in general by Theorem 1.2, the morphism of special fibers induced by a stable covering is not an admissible covering in general. In this subsection, we consider whether or not there exists a non-finite stable covering of X (i.e., the existence of vertical fibers). Moreover, we consider a sufficient condition for a given G-stable covering over S to restrict an admissible covering of the special fibers.

First, we define the following set which consists of the vertical points:

$$X^{\mathrm{ver}} := \{x \in X_s \mid x \text{ is a vertical point associated to a stable covering of } X\}.$$

Theorem 2.2. *If $\mathrm{char}(K) = 0$, we have the following results:*

(i) ([10, Theorem 0.2]) if $k = \overline{\mathbb{F}}_p$, then $X^{\mathrm{ver}} = X^{\mathrm{cl}}$, where X^{cl} denotes the set of closed points of X.

(ii) ([13, Theorem 2.5]) the closure of X^{ver} in X_s is equal to X_s and X_s^{sing} is contained in X^{ver}, where X_s^{sing} denotes the singular locus of X_s.

Theorem 2.3. *If $\mathrm{char}(K) = p > 0$ and X_s is irreducible, we have the following results:*

(i) ([13, Theorem 2.7]) if $k = \overline{\mathbb{F}}_p$, X_s is smooth over s and X is not a trivial family over S, then $X^{\mathrm{ver}} \neq \emptyset$.

(ii) ([13, Theorem 2.8]) if X_s is singular, then $X^{\mathrm{ver}} \neq \emptyset$.

(iii) ([14, Theorem 1.3]) for any finite group G, a G-stable covering $f : Y \to X$ is finite if and only if f_s is an admissible covering.

§ 2.2. p-ranks of vertical fibers

In this subsection, we study the p-ranks of vertical fibers of stable coverings. The following theorem was proved by M. Raynaud (cf. [5, Théorème 1]).

Theorem 2.4. *Let G be a p-group, $f : Y \to X$ a G-stable covering, and $x \in X$ a vertical point associated to f. Suppose that x is a smooth point of X_s. Then the p-rank of each connected component of the vertical fiber $f^{-1}(x)$ is equal to 0. In particular, the dual graph of each connected component of the vertical fiber $f^{-1}(x)$ is a tree.*

Raynaud considered the vertical fibers associated to smooth vertical points. In the following, we consider a similar assertion for the vertical fibers associated to singular vertical points.

Let G be a finite p-group, $f : Y \to X$ a G-stable covering and x a vertical point associated to f. Suppose that x is a singular point of X_s. Then there are two irreducible components X_{v_1} and X_{v_2} (which may be equal) of X_s such that $x \in X_{v_1} \bigcap X_{v_2}$. Write Y' for the normalization of X in Y and $\psi : Y' \to X$ for the resulting normalization morphism. Let y' be a closed point of Y' such that $\psi(y') = x$. In order to compute the p-rank of each connected component of the vertical fibers associated to x, by applying the Zariski-Nagata purity and replacing $f : Y \to X$ by the quotient morphism $Y \to Y/I_{y'}$, we may assume that the inertia subgroup $I_{y'} \subseteq G$ of y' is equal to G. Let Y'_{v_1} (resp. Y'_{v_2}) be an irreducible component of Y'_s such that $\psi(Y'_{v_1}) = X_{v_1}$ and $y' \in Y'_{v_1}$ (resp. $\psi(Y'_{v_2}) = X_{v_2}$ and $y' \in Y'_{v_2}$). Write $I_{Y'_{v_1}} \subseteq I_{y'}$ (resp. $I_{Y'_{v_2}} \subseteq I_{y'}$) for the inertia subgroup of Y'_{v_1} (resp. Y'_{v_2}). Write V_x for the vertical fiber $f^{-1}(x)$. Note that since $I_{y'}$ is equal to G, V_x is connected.

The following theorem was proved by M. Saïdi (cf. [8, Theorem]).

Theorem 2.5. *If $I_{y'}$ is isomorphic to a cyclic p-group $\mathbb{Z}/p^r\mathbb{Z}$, then we have $\sigma(V_x) \leq p^r - 1$.*

We generalize Saïdi's result to the case where $I_{y'}$ is a finite abelian p-group as follows:

Theorem 2.6. *(1) ([15, Lemma 2.1]) Write Γ_x for the dual graph of the vertical fiber V_x. If $I_{y'}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, we have the following results: (a) If $I_{Y'_{v_1}} = \mathbb{Z}/p\mathbb{Z}$ and $I_{Y'_{v_2}}$ is trivial, then $\sigma(V_x) = 0$. (b) If $I_{Y'_{v_1}} = I_{Y'_{v_2}} = \mathbb{Z}/p\mathbb{Z}$, then one of the following conditions are satisfied: (i) $\sigma(V_x) = 0$; (ii) $\sigma(V_x) = p - 1$ and $\mathrm{rank}(\mathrm{H}^1(\Gamma_x, \mathbb{Z})) = p - 1$; (iii) $\sigma(V_x) = p - 1$ and Γ_x is a tree.*

(2) ([16, Theorem 1.4]) If $I_{y'}$ is a finite abelian p-group of order p^r, then there exists a bound of $\sigma(V_x)$ which only depends on p^r.

Remark. We can construct some examples for Theorem 2.6 (1-a) and (1-b-ii) (cf. [15, Section 4]).

§ 2.3. Ordinariness

In Subsection 2.2, we studied the p-ranks of vertical fibers of stable coverings. We also have some global results concerning the p-ranks of the special fibers of stable coverings. In order to study an étale covering of X_η with bad reduction, Raynaud (cf. [6, Proposition 3]) proved the following theorem:

Theorem 2.7. *Let G be a finite group and $f : Y \to X$ a G-stable covering. Suppose that X is smooth over S and f_s is not generically étale. Then Y_s is not ordinary.*

By applying Theorem 2.6 (1), we partially generalize Theorem 2.7 to the case where X is not necessarily smooth over S and G is solvable as follows:

Theorem 2.8. *([15, Theorem 3.4]) Let G be a finite solvable group and $f : Y \to X$ a G-stable covering. Suppose that the genus of the normalization of each irreducible component of X_s is > 1, and f_s is not generically étale. Then Y_s is not ordinary.*

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References

[1] Grothendieck, A. and Raynaud, M., Revêtements étales et groupe fondamental (SGA 1). Séminaire de géométrie algébrique du Bois Marie 196061. *Lecture Notes in Math.*, **224**, Springer, Berlin. 1971.
[2] Liu, Q., Stable reduction of finite covers of curves. *Compos. Math.* **142** (2006), 101-118.
[3] Mochizuki, S., The geometry of the compactification of the Hurwitz scheme, *Publ. Res. Inst. Math. Sci.* **31** (1995), 355-441.
[4] Pop, P. and Saïdi, M., On the specialization homomorphism of fundamental groups of curves in positive characteristic. *Galois groups and fundamental groups,* 107-118, *Math. Sci. Res. Inst. Publ.,* **41**, *Cambridge Univ. Press, Cambridge,* 2003.
[5] Raynaud, M., p-groupes et réduction semi-stable des courbes. *The Grothendieck Festschrift, Vol. III*, 179-197, *Progr. Math.*, **88**, *Birkhäuser Boston, Boston, MA,* 1990.
[6] Raynaud, M., Mauvaise réduction des courbes et p-rang. *C. R. Acad. Sci. Paris Sér. I Math.* **319** (1994), no. 12, 1279-1282.
[7] Raynaud, M., Sur le groupe fondamental d'une courbe complète en caractéristique $p > 0$. *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 335-351, *Proc. Sympos. Pure Math.,* **70**, *Amer. Math. Soc., Providence, RI,* 2002.
[8] Saïdi, M., p-rank and semi-stable reduction of curves. *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), no. 1, 63-68.
[9] Tamagawa, A., Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups. *J. Algebraic Geom.* **13** (2004), 675-724.
[10] Tamagawa, A., Resolution of nonsingularities of families of curves. *Publ. Res. Inst. Math. Sci.* **40** (2004), 1291-1336.
[11] Vidal, I., *Contributions à la cohomologie étale des schémas et des log-schémas*, Thèse, U. Paris-Sud (2001).
[12] Yang, Y., *Degeneration of period matrices of stable curves*, RIMS Preprint 1835.

[13] Yang, Y., *On the existence of vertical fibers of coverings of curves over a complete discrete valuation ring*, RIMS Preprint 1843.
[14] Yang, Y., *Finite morphisms of stable curves*, preprint.
[15] Yang, Y., *On the ordinariness of coverings of stable curves*, RIMS Preprint 1845.
[16] Yang, Y., *On the boundedness and graph-theoreticity of p-ranks of coverings of curves*, RIMS Preprint 1864.