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The Anabelian Geometry of Curves over Algebraically Closed Fields of Positive Characteristic: A Survey

By

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Abstract

In the present paper, we overview some recent developments in the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

§ 1. Introduction

Let k be a field, and let Z be a hyperbolic curve over k (See §2 for the definition of hyperbolic curves). Roughly speaking, the ultimate goal of the anabelian geometry of curves is the following question:

Question 1.1. *Can we reconstruct the isomorphism class of Z group-theoretically from various versions of its fundamental group?*

The various formulations of Question 1.1 are called **Grothendieck's anabelian conjecture** or the Grothendieck conjecture, for short.

When k is an **arithmetic field** (e.g. a number field, a finite field, a p-adic field), the Grothendieck conjecture has been proven in many cases. Suppose that k is of characteristic 0. For example, if k is a number field, then the Grothendieck conjecture was proved by H. Nakamura in the case of genus 0 ([5], [6]), by A. Tamagawa in the case of affine hyperbolic curves ([12]), and by S. Mochizuki in the case of projective hyperbolic curves ([2]). Moreover, Mochizuki also obtained a very general version (i.e.,

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Hom-version) of the Grothendieck conjecture when k is a sub-p-adic field (i.e., a subfield of a finitely generated extension of a p-adic number field) ([3]).

On the other hand, Tamagawa also considered the Grothendieck conjecture in positive characteristic and proved it for affine hyperbolic curves over finite fields ([12]). Afterwards, Mochizuki generalized this result to the case of projective hyperbolic curves ([4]), and J. Stix generalized this result to the case where the base fields are finitely generated over $\mathbb{F}_p$ ([10], [11]). Note that all the proofs of the Grothendieck conjecture for curves over **arithmetic fields** require the use of **the highly non-trivial outer Galois representation** induced by the fundamental exact sequence of étale (or tame) fundamental groups.

Suppose that the base field is algebraically closed. In this situation, the Galois group of the base field is **trivial**, and the étale (or tame) fundamental group coincides with the geometric fundamental group, thus in a total absence of a Galois action of the base field. In the case of algebraically closed fields of characteristic 0, by applying GAGA, we have that the étale fundamental groups of curves depend only on the genera and the cardinality of the sets of cusps. This means that the hyperbolic curves over algebraically closed fields of characteristic 0 cannot be determined by their étale fundamental groups. Thus the anabelian geometry of curves does not exist in this situation. On the other hand, some developments of M. Raynaud, F. Pop, M. Saïdi, and Tamagawa ([7], [8], [13], [15], [16]) from the 1990's showed evidence for very strong anabelian phenomena for curves over **algebraically closed fields of characteristic** $p > 0$. This kind of anabelian phenomena go beyond Grothendieck's original anabelian geometry, and shows that the étale (or tame) fundamental group of a smooth pointed stable curve over an algebraically closed field must encode "**moduli**" of the curve. This is the reason that we do not have an explicit description of the étale (or tame) fundamental group of any pointed stable curve in positive characteristic.

In the present paper, we give a survey of the currently known results on the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$.

§ 2. Basic definitions and notations

We fix some notations which will be used in the present paper. Let n and g be non-negative integers such that $2g-2+n>0$. A pointed stable curve $X^\bullet := (X, D_X)$ of type (g,n) over a scheme S consists of a flat, proper morphism $\pi : X \to S$, together with a set of n distinct sections $D_X := \{s_i : S \to X\}_{i=1}^n$ such that for each geometric point $\overline{s}$ of S:

(i) The geometric fiber $X_{\overline{s}}$ is a reduced and connected curve of genus g with at most ordinary double points (i.e., nodes).

(ii) $X_{\overline{s}}$ is smooth at the points of $s_i(\overline{s})$ $(1 \leq i \leq n)$.

(iii) $s_i(\overline{s}) \neq s_j(\overline{s})$ for $i \neq j$.

(iv) For every nonsingular rational component E of $X_{\overline{s}}$, the sum of the number of points of E where E meets another component of $X_{\overline{s}}$ and the number of points in $\{s_i(\overline{s})\}_{i=1}^n$ included in E is at least 3.

Let $X^\bullet$ be a pointed stable curve of type (g, n) over S. We shall call D_X the set of marked points of $X^\bullet$ and X the underlying scheme of $X^\bullet$. We shall call that $X^\bullet$ is smooth if the morphism of schemes $\pi : X \to S$ is smooth. Let l be a field, $\bar{l}$ an algebraic closure of l, and Z a smooth curve over l. We shall call Z hyperbolic if there exists a smooth pointed stable curve $\overline{Z}^\bullet$ over $\bar{l}$ such that $Z \times_l \bar{l}$ is $\bar{l}$-isomorphic to $\overline{Z}^\bullet \setminus D_{\overline{Z}}$.

Let k be an algebraically closed field, and

$$X^\bullet := (X, D_X)$$

be a pointed stable curve of type (g, n) over k. Recall that a semi-graph $\mathbb{G}$ consists of the following data:

(i) A set $v(\mathbb{G})$ whose elements we refer to as vertices;

(ii) A set $e(\mathbb{G})$ whose elements we refer to as edges; moreover, any element $e \in e(\mathbb{G})$ is a set of cardinality 2 satisfying the following property: for each $e \neq e' \in e(\mathbb{G})$, we have $e \cap e' = \emptyset$;

(iii) a set of maps $\{\zeta_e^{\mathbb{G}}\}_{e \in e(\mathbb{G})}$ such that $\zeta_e^{\mathbb{G}} : e \to v(\mathbb{G}) \cup \{v(\mathbb{G})\}$ is a map from the set e to the set $v(\mathbb{G}) \cup \{v(\mathbb{G})\}$.

We can define a dual semi-graph $\Gamma_{X^\bullet}$ associated to $X^\bullet$ as follows:

(i) $v(\Gamma_{X^\bullet}) := \{v_E\}_{E \in \mathrm{Irr}(X)}$, where $\mathrm{Irr}(X)$ denotes the set of irreducible components of X;

(ii) write $\mathrm{Node}(X)$ for the set of nodes of X; then we put $e(\Gamma_{X^\bullet}) := e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{op}}(\Gamma_{X^\bullet})$, where $e^{\mathrm{cl}}(\Gamma_{X^\bullet}) := \{e_s\}_{s \in \mathrm{Node}(X)}$ and $e^{\mathrm{op}}(\Gamma_{X^\bullet}) := \{e_m\}_{m \in D_X}$;

(iii) for each $e_s = \{b_s^1, b_s^2\} \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, we define $\zeta_{e_s}^{\Gamma_{X^\bullet}}(b_s^1)$, $\zeta_{e_s}^{\Gamma_{X^\bullet}}(b_s^2)$ in order that $\{\zeta_{e_s}^{\Gamma_{X^\bullet}}(b_s^1),\ \zeta_{e_s}^{\Gamma_{X^\bullet}}(b_s^2)\} := \{v_E \in v(\Gamma_{X^\bullet}) \mid s \in E\}$;

(iv) for each $e_m = \{b_m^1, b_m^2\} \in e^{\mathrm{op}}(\Gamma_{\mathscr{C}})$, we define $\zeta_{e_m}^{\Gamma_{X^\bullet}}(b_m^1)$ to be the unique element v_E in $v(\Gamma_{X^\bullet})$ with $m \in E$, and we set $\zeta_{e_m}^{\Gamma_{X^\bullet}}(b_m^2) := v(\Gamma_{X^\bullet})$.

Let p be a prime number and $\overline{\mathbb{F}}_p$ an algebraic closure of $\mathbb{F}_p$. In the remainder of the present paper, we suppose that $\overline{\mathbb{F}}_p \subseteq k$ is an algebraically closed field of characteristic $p > 0$. Let

$$\overline{\mathcal{M}}_{g,n}$$

be the moduli stack of pointed stable curves of type (g, n) over $\overline{\mathbb{F}}_p$ and $\mathcal{M}_{g,n}$ the open substack of $\overline{\mathcal{M}}_{g,n}$ parameterizing pointed smooth curves. Write

$$\overline{M}_{g,n}$$

for the coarse moduli space of the moduli stack $\overline{\mathcal{M}}_{g,n}$. Moreover, write

$$\overline{\mathcal{M}}_{g,n}^{\log}$$

for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ relative to $\overline{\mathbb{F}}_p$. Let $s := \operatorname{Spec} k$, and let $s^{\log} \to \overline{\mathcal{M}}_{g,n}^{\log}$ be a morphism from an fs log point $s^{\log}$ (i.e., an fs log scheme whose underlying scheme is s) whose underlying morphism $s \to \overline{\mathcal{M}}_{g,n}$ is determined by $X^\bullet \to s$. Thus, we obtain a stable log curve

$$X^{\log} := s^{\log} \times_{\overline{\mathcal{M}}_{g,n}^{\log}} \overline{\mathcal{M}}_{g,n+1}^{\log}$$

whose underlying scheme is X. By choosing suitable base points of $X^{\log}$ and $s^{\log}$, respectively, we obtain a natural surjection of log étale fundamental groups

$$\pi_1(X^{\log}) \twoheadrightarrow \pi_1(s^{\log})$$

of $X^{\log}$ and $s^{\log}$. Moreover, we denote by

$$\Pi_{X^\bullet} := \ker(\pi_1(X^{\log}) \twoheadrightarrow \pi_1(s^{\log}))$$

the geometric log étale fundamental group of $X^{\log}$, and we shall call $\Pi_{X^\bullet}$ the **admissible fundamental group** of $X^\bullet$ which depends only on $X^\bullet$. For each open subgroup $H \subseteq \Pi_{X^\bullet}$, there exists an associated covering

$$X_H^\bullet := (X_H, D_{X_H}) \to X^\bullet$$

over k, called an **admissible covering**. If $X^\bullet$ is smooth over k, then the admissible fundamental group $\Pi_{X^\bullet}$ is naturally (outer) isomorphic to the tame fundamental group of $X \setminus D_X$, and, for each open subgroup $H \subseteq \Pi_{X^\bullet}$, the morphism

$$X_H \setminus D_{X_H} \to X \setminus D_X$$

over k induced by the admissible covering $X_H^\bullet \to X^\bullet$ is a tame covering. Write

$$\pi : \widetilde{X}^{\bullet,\mathrm{adm}} \to X^\bullet$$

for the universal admissible covering space of $X^\bullet$ (which is not a scheme but a pro-scheme) corresponding to the admissible fundamental group $\Pi_{X^\bullet}$. For each $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup$

$e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ and for each $v \in v(\Gamma_{X^\bullet})$, write $\widetilde{e}$ and $\widetilde{v}$ for any elements of the inverse images $\pi^{-1}(e)$ and $\pi^{-1}(v)$, respectively. We denote by

$$I_{\widetilde{e}} \subseteq \Pi_{X^\bullet}$$

and

$$\Pi_{\widetilde{v}} \subseteq \Pi_{X^\bullet}$$

the stabilizer of $\widetilde{e}$ and $\widetilde{v}$, respectively. We shall call $I_{\widetilde{e}}$ an inertia subgroup of e. Note that $I_{\widetilde{e}} \cong \widehat{\mathbb{Z}}(1)^{(p')}$, where $\widehat{\mathbb{Z}}(1)^{(p')}$ denotes the maximal prime-to-p quotient of $\widehat{\mathbb{Z}}(1)$. On the other hand, write X_v for the irreducible component of X corresponding to v and $\mathrm{nl}_v : \widetilde{X}_v \to X_v$ for the normalization morphism of X_v. We write g_v for the genus of $\widetilde{X}_v$, and n_v for the cardinality of $\mathrm{nl}_v^{-1}((\mathrm{Node}(X) \cup D_X) \cap X_v)$. Then we obtain a smooth pointed stable curve

$$X_v^\bullet := (\widetilde{X}_v, \mathrm{nl}_v^{-1}((\mathrm{Node}(X) \cup D_X) \cap X_v))$$

of type (g_v, n_v) over k. We shall call $X_v^\bullet$ a **pointed irreducible component** of $X^\bullet$. By choosing a base point of $X_v^\bullet$, we obtain the admissible fundamental group

$$\Pi_{X_v^\bullet}$$

of $X_v^\bullet$; moreover, we have a natural outer isomorphism of profinite groups

$$\Pi_{X_v^\bullet} \xrightarrow{\sim} \Pi_{\widetilde{v}}.$$

For more details on the theory of admissible coverings and admissible fundamental groups for pointed stable curves, see [1], [2].

§ 3. Group-theoretical reconstructions of various invariants

We maintain the notations introduced in Section 1. Before reconstructing the isomorphism class of $X^\bullet$ as scheme group-theoretically from $\Pi_{X^\bullet}$, we should reconstruct the "topological structure" (e.g. the type (g, n)) of $X^\bullet$ group-theoretically from $\Pi_{X^\bullet}$. When the base field is "an arithmetic field" (e.g. a number field, a p-adic field, a finite field, etc.), the topological structure of a pointed stable curve can be reconstructed group-theoretically from its admissible fundamental group by applying the non-trivial outer Galois action induced by the fundamental exact sequence of fundamental groups (e.g. the theory of weight). When the base field is an algebraically closed field of characteristic $p > 0$, the reconstruction of the topological structure of a pointed stable curve group-theoretically from its admissible fundamental group is highly non-trivial.

In the case of smooth pointed stable curves, Tamagawa proved the following theorem ([15, Theorem 0.1 and Theorem 5.2]).

Theorem 3.1. *Suppose that* $X^\bullet$ *is* **smooth** *over* k*. Then there exists a group-theoretic algorithm whose input datum is* $\Pi_{X^\bullet}$*, and whose output data are*

- *the type* (g, n)*;*
- *the conjugacy class of* $I_{\widetilde{e}}$ *in* $\Pi_{X^\bullet}$ *for each* $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$ *and each* $\widetilde{e} \in \pi^{-1}(e)$*.*

Remark. Tamagawa also obtained an étale fundamental group version of Theorem 3.1 ([13, Theorem 0.1, Theorem 2.5, and Theorem 2.7]). Note that, since the tame fundamental group of a smooth pointed stable curve is a quotient of the étale fundamental group of the smooth pointed stable curve, Theorem 3.1 is stronger than the étale fundamental group version of Theorem3.1.

The author generalized Tamagawa's result to the case of arbitrary pointed stable curves as follows ([21, Theorem 0.5]).

Theorem 3.2. *There exists a group-theoretic algorithm whose input datum is* $\Pi_{X^\bullet}$*, and whose output data are*

- *the type* (g, n) *and the dual semi-graph* $\Gamma_{X^\bullet}$*;*
- *the conjugacy class of* $I_{\widetilde{e}}$ *in* $\Pi_{X^\bullet}$ *for each* $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \cup e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ *and each* $\widetilde{e} \in \pi^{-1}(e)$*;*
- *the type* (g_v, n_v) *and the conjugacy class of* $\Pi_{\widetilde{v}}$ *in* $\Pi_{X^\bullet}$ *for each* $v \in v(\Gamma_{X^\bullet})$ *and each* $\pi^{-1}(v)$*.*

Proof. The key tools of the proof of the theorem are the pointed stable curve version of p-average theorem ([17, Theorem 3.10]) and the general theory of p-Galois admissible coverings (i.e., Galois admissible covering whose Galois group is a p-group). □

Remark. Theorem 3.2 can be regarded as a "mono-anabelian version of the combinatorial Grothendieck conjecture for semi-graphs of anabelioids of PSC-type". Moreover, a "bi-anablian version" of Theorem 3.2 was proved in [19, Theorem 1.2].

§ 4. Various versions of the Grothendieck conjecture for curves over algebraically closed fields of characteristic $p > 0$ and results

We maintain the notations introduced in previous sections. In this section, under certain conditions, we reconstruct the scheme structures of pointed stable curves (i.e., the isomorphism classes of pointed stable curves) group-theoretically from their admissible fundamental groups.

Let us introduce some notations. Let $q \in \overline{M}_{g,n}$ be an arbitrary point, $k(q)$ the residue field of q, k_q an algebraic closure of $k(q)$. Write $X_q^\bullet := (X_q, D_{X_q})$ for the pointed stable curve of type (g,n) over k_q determined by the natural morphism $\mathrm{Spec}\, k_q \to \overline{M}_{g,n}$ and $\Gamma_{X_q^\bullet}$ for the dual semi-graph of $X_q^\bullet$. Since the isomorphism class of the admissible fundamental group of $X_q^\bullet$ depends only on q, we write

$$\Pi_q^{\mathrm{adm}}$$

for the admissible fundamental group of $X_q^\bullet$.

First, we consider the case of smooth pointed stable curves. Let S_n be the n^{th} symmetric group. Note that there exists a natural action of S_n on $\mathcal{M}_{g,n}$. We denote by

$$\mathcal{M}_{g,[n]} := [\mathcal{M}_{g,n}/S_n]$$

the quotient stack, and denote by

$$M_{g,[n]}$$

the coarse moduli space of $\mathcal{M}_{g,[n]}$. Note that we have a morphism

$$[\pi] : M_{g,n} \to M_{g,[n]}$$

induced by the quotient morphism $\mathcal{M}_{g,n} \to \mathcal{M}_{g,[n]}$. For any **closed points** $c_1, c_2 \in M_{g,n}^{\mathrm{cl}}$, where $(-)^{\mathrm{cl}}$ denotes the set of closed points of $(-)$, we define an equivalence relation as follows:

> $c_1 \sim c_2$ if there exists $m \in \mathbb{Z}$ such that $[\pi](c_2) = [\pi](c_1^{(m)})$, where $c_1^{(m)}$ denotes the closed point corresponding to the m^{th} Frobenius twist of the curve corresponding to c_1.

Moreover, let $q_1, q_2 \in M_{g,n}$ be arbitrary two points. We denote by V_{q_1} and V_{q_2} for the topological closure of $\{q_1\}$ and $\{q_2\}$ in $M_{g,n}$. Write

$$V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$$

if, for each closed point $c_2 \in V_{q_2}$, there exists a closed point $c_1 \in V_{q_1}$ such that $c_2 \in \{c_1^{(m)}\}_{m \in \mathbb{Z}}$. Moreover, we write

$$V_{q_1} =_{\mathrm{ec}} V_{q_2}$$

when $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$ and $V_{q_1} \subseteq_{\mathrm{ec}} V_{q_2}$. We shall call that V_{q_1} **essentially contains** V_{q_2} if $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$, and that V_{q_1} **is essentially equal to** V_{q_2} if $V_{q_1} =_{\mathrm{ec}} V_{q_2}$. One can check that $V_{q_1} =_{\mathrm{ec}} V_{q_2}$ if and only if $X_{q_1}^\bullet$ and $X_{q_2}^\bullet$ are isomorphic as schemes (see [20, Proposition 7.2]).

The following conjecture was posted by Tamagawa ([14, Conjecture 1.33]):

Conjecture 4.1. *Let $q_1, q_2 \in M_{g,n}$ be arbitrary two points.*

(weak Isom-version) *The set of continuous isomorphisms of profinite groups*

$$\mathrm{Isom}_{\text{pro-gps}}(\Pi_{q_1}^{\mathrm{adm}}, \Pi_{q_2}^{\mathrm{adm}})$$

is non-empty if and only if $V_{q_1} =_{\mathrm{ec}} V_{q_2}$.

For Conjecture 4.1, Tamagawa proved the following theorem ([15, Theorem 0.2]):

Theorem 4.2. *Let $q_1, q_2 \in M_{0,n}$ be arbitrary* **closed points**. *Then Conjecture 4.1 holds.*

Recently, by following Tamagawa's ideas, A. Sarashina proved the following result ([9, Theorem 1.2]):

Theorem 4.3. *Let $q_1, q_2 \in M_{1,1}$ be arbitrary* **closed points**. *Suppose that $p \neq 2$. Then Conjecture 4.1 holds.*

Remark. In fact, Sarashina only treated the case of étale fundamental groups. By applying Theorem 3.1, we can prove that Sarashina's result also holds for the case of admissible (or tame) fundamental groups ([18, Theorem 6 (ii)]).

For the case of higher genus, we have the following finiteness theorem ([8, Théorème 2.1.2], [11, Theorem B], [16, Theorem 0.1]).

Theorem 4.4. *Let $q \in M_{g,n}$ be a* **closed** *point and S_q the set of closed points $M_{g,n}$ such that $\Pi_q^{\mathrm{adm}} \cong \Pi_{q'}^{\mathrm{adm}}$ for each $q' \in S_q$. Then we have*

$$\#S_q < \infty.$$

Remark. Theorem 4.4 means that over $\overline{\mathbb{F}}_p$, there are only finitely many isomorphism classes of **smooth** pointed stable curves have the same admissible (or tame) fundamental group.

Remark. Theorem 4.4 was proved by Raynaud and Pop-Saidi under certain assumptions on Jacobian, and by Tamagawa in the general case.

The author posed a generalized version of Conjecture 4.1 as follows ([20, Section 7.1]):

Conjecture 4.5. *Let $q_1, q_2 \in M_{g,n}$ be arbitrary two points.*

(weak Hom-version) *The set of open continuous homomorphisms of profinite groups*

$$\mathrm{Hom}_{\text{pro-gps}}(\Pi_{q_1}^{\mathrm{adm}}, \Pi_{q_2}^{\mathrm{adm}})$$

is non-empty if and only if $V_{q_1} \supseteq_{\mathrm{ec}} V_{q_2}$.

Remark. Note that, we have

$$\text{Conjecture 4.5} \Rightarrow \text{Conjecture 4.1}.$$

For Conjecture 4.5, we have the following result ([20, Theorem 6.2 and Remark 6.2.1]):

Theorem 4.6. *Let $q_1, q_2 \in M_{g,n}$ be arbitrary two points, and assume that q_1 is a* **closed point** *(i.e., $\dim(V_{q_1}) = 0$). Suppose that (g, n) is equal to either $(0, n)$ or $(1, 1)$. Moreover, suppose that $p \neq 2$ when $(g, n) = (1, 1)$. Then Conjecture 4.5 holds. In particular, q_2 is also a closed point, and $V_{q_1} =_{\mathrm{ec}} V_{q_2}$ (i.e., $q_1 \sim q_2$).*

Proof. We can prove that the inertia subgroups of marked points of $X^{\bullet}_{q_1}$ and $X^{\bullet}_{q_2}$ can be reconstructed group-theoretically from open continuous surjective homomorphisms between $\Pi^{\mathrm{adm}}_{q_1}$ and $\Pi^{\mathrm{adm}}_{q_2}$. By using this fact, we can prove Theorem 4.6 by similar arguments to those in the proofs of Theorem 4.2 and Theorem 4.3. □

Next, let us consider the case of pointed stable curves. We have the following results, which generalizes Theorem 4.2, Theorem 4.3, and Theorem 4.4 to the case of (possibly singular) pointed stable curves ([21, Corollary 0.6]):

Theorem 4.7. *(i) Let $q_1, q_2 \in \overline{M}_{g,n}$ be arbitrary two points, and assume that q_1 is a* **closed point**. *Suppose that $X^{\bullet}_{q_1}$ is* **irreducible**, *that the genus of the normalization of X_{q_1} is 0, and that $\Pi^{\mathrm{adm}}_{q_1} \cong \Pi^{\mathrm{adm}}_{q_2}$. Then q_2 is also a closed point, and $X^{\bullet}_{q_1} \cong X^{\bullet}_{q_2}$ as schemes.*

(ii) Let $q \in \overline{M}_{g,n}$ be a **closed point** *and $\overline{S}_q$ the set of closed points $q' \in \overline{M}_{g,n}$ such that $\Pi^{\mathrm{adm}}_{q} \cong \Pi^{\mathrm{adm}}_{q'}$. Then*

$$\#\overline{S}_q < \infty.$$

Proof. The theorem follows from Theorem 3.2, Theorem 4.2, Theorem 4.3, and Theorem 4.4. □

Remark. If the curves corresponding to q_1 and q_2 are not irreducible, then Weak Isom-version Conjecture does not hold in general ([21, Corollary 0.6]).

At present, all the results recalled above (Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 4.6, and Theorem 4.7) were only proved in the case of curves over $\overline{\mathbb{F}}_p$. One of the main goals of the anabelian geometry of curves in positive characteristic is to extend the results to the case of curves over **arbitrary** algebraically closed fields of characteristic $p > 0$.

Finally, we will pose a conjecture (Conjecture 4.10) which makes clear the relationship between Conjecture 4.1 or Conjecture 4.5 for **closed points** of moduli spaces and

Conjecture 4.1 or Conjecture 4.5 for **arbitrary points** of moduli spaces. Let $q \in M_{g,n}$ be an arbitrary point. The main difficulty of proving Conjecture 4.1 or Conjecture 4.5 for arbitrary points is that we do not know how to reconstruct the admissible fundamental groups of the closed points of V_q group-theoretically from Π_q^{adm}. Once the admissible fundamental groups of the closed points of V_q are reconstructed group-theoretically from Π_q^{adm}, then, by applying Conjecture 4.1 or Conjecture 4.5 for closed points, the set of closed points of V_q would be reconstructed from Π_q^{adm}. Thus, Conjecture 4.1 or Conjecture 4.5 for arbitrary points would follows from Conjecture 4.1 or Conjecture 4.5 for closed points.

On the other hand, since Π_q^{adm} is topologically finitely generated, the isomorphism class of Π_q^{adm} as profinite groups is determined by the set $\pi_A^{\mathrm{adm}}(q)$, where $\pi_A^{\mathrm{adm}}(-)$ denotes the set of finite quotients of the fundamental group $\Pi_{(-)}^{\mathrm{adm}}$. Then we may consider the following question:

Question 4.8. *(i) For each closed point t of V_q, which collection of finite groups contained in $\pi_A^{\mathrm{adm}}(q)$ is equal to $\pi_A^{\mathrm{adm}}(t)$?*

(ii) For each closed point t of $M_{g,n}$, if $\pi_A^{\mathrm{adm}}(t) \subseteq \pi_A^{\mathrm{adm}}(q)$, then is t a closed point of V_q?

To approach Question 4.8, we introduce a kind of collection of finite groups contained in $\pi_A^{\mathrm{adm}}(q)$ called a **pointed collection** as follows:

Definition 4.9. Let $G \in \pi_A^{\mathrm{adm}}(q)$ be an arbitrary finite group and $U_G \subseteq M_{g,n}$ the subset such that, for each $q' \in U_G$, we have $G \in \pi_A^{\mathrm{adm}}(q')$. Note that U_G is an open subset of $M_{g,n}$ ([20, Proposition 7.3]). We denote by q_{gen} the generic point of $M_{g,n}$, and let

$$\mathcal{C} \subseteq \pi_A^{\mathrm{adm}}(q_{\mathrm{gen}}) = \bigcup_{q \in M_{g,n}^{\mathrm{cl}}} \pi_A^{\mathrm{adm}}(q)$$

be a collection of finite groups contained in $\pi_A^{\mathrm{adm}}(q_{\mathrm{gen}})$.

We shall call that $\mathcal{C}$ is a **pointed collection** if the following conditions are satisfied:

(i) $(\bigcap_{G \in \mathcal{C}} U_G) \cap M_{g,n}^{\mathrm{cl}} \neq \emptyset$;

(ii) $\#((\bigcap_{G \in \mathcal{C}} U_G) \cap M_{g,n}^{\mathrm{cl}}) < \infty$;

(iii) $U_{G'} \cap (\bigcap_{G \in \mathcal{C}} U_G) \cap M_{g,n}^{\mathrm{cl}} = \emptyset$ for each $G' \in \pi_A^{\mathrm{adm}}(q_{\mathrm{gen}})$ such that $G' \notin \mathcal{C}$.

On the other hand, for each closed point $t \in M_{g,n}^{\mathrm{cl}}$, we may define a collection associated to t as follows:

$$\mathcal{C}_t := \{G \in \pi_A^{\mathrm{adm}}(q_{\mathrm{gen}}) \mid t \in U_G\}.$$

Note that, if $t \in V_q^{\mathrm{cl}}$, then $\mathcal{C}_t \subseteq \pi_A^{\mathrm{adm}}(q)$. Moreover, we denote by

$$\mathscr{C}_q := \{\mathcal{C} \text{ pointed collection} \mid \mathcal{C} \subseteq \pi_A^{\mathrm{adm}}(q)\}$$

the set of pointed collections contained in $\pi_A^{\mathrm{adm}}(q)$.

We conjectured the set of closed points V_q^{cl} can be reconstructed from $\pi_A^{\mathrm{adm}}(q)$ as follows ([20, Section 7.2]):

Conjecture 4.10.

(pointed collection conjecture) *For each $t \in M_{g,n}^{\mathrm{cl}}$, the collection $\mathcal{C}_t$ associated to t is a pointed collection. Moreover, the natural map $\theta_q : V_q^{\mathrm{cl}}/\sim \to \mathscr{C}_q$ that $[t] \mapsto \mathcal{C}_t$ is a bijection, where $[t]$ denotes the image of t in $V_q^{\mathrm{cl}}/\sim$.*

Remark. Note that since $\pi_A^{\mathrm{adm}}(q) = \bigcup_{t \in V_q^{\mathrm{cl}}} \pi_A^{\mathrm{adm}}(t)$, we obtain that Conjecture 4.10 is equivalent to Conjecture 4.5.

For Conjecture 4.10, by applying Theorem 4.6, we have the following result ([20, Theorem 7.6]):

Theorem 4.11. *(i) Let q be an arbitrary point of $M_{0,n}^{\mathrm{cl}}$. Then, for each $t \in M_{0,n}^{\mathrm{cl}}$, the collection $\mathcal{C}_t$ is a pointed collection, and for each pointed collection $\mathcal{C} \in \mathscr{C}_q$, there exists $s \in M_{0,n}^{\mathrm{cl}}$ such that $\mathcal{C} = \mathcal{C}_s$. Moreover, the map θ_q is an injection.*

(ii) Let q be an arbitrary point of $M_{0,n}^{\mathrm{cl}}$, and

$$X_q \setminus D_{X_q} \cong \mathbb{P}^1_{\overline{k(q)}} \setminus \{1, 0, \infty, a_1, \dots, a_{n-3}\}.$$

Suppose that $a_i, i \in \{1, \dots, n-4\}$, is an element of $\overline{\mathbb{F}}_p$. Then Conjecture 4.10 holds.

Remark. Suppose that $g = 0$. Theorem 4.11 (i) gave an answer of Question 4.8 (i), and Theorem 4.11 (ii) gave an answer of Question 4.8 (ii) in a special case. In particular, Conjecture 4.10 holds when q is a closed point of $M_{0,n}$.

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