Raynaud-Tamagawa theta divisors and fundamental groups of curves in positive charactersitic

Yu Yang

RIMS, Kyoto University

February 7, 2020

Notations

- k: a field
- $\overline{k}$: a separable closure of k
- G_k: the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$
- $X_i, i \in \{1, 2\}$: a hyperbolic curve of type (g_{X_i}, n_{X_i}) over k (i.e., $2g_{X_i} + n_{X_i} - 2 > 0$), where g_{X_i} denotes the genus, n_{X_i} denotes the cardinality of the set $\overline{X_i \times \overline{k}} \setminus X_i \times \overline{k}$.
- Π_{X_i}: tame fundamental group of X_i
- Δ_{X_i}: tame fundamental group of $X_i \times_k \overline{k}$ (i.e., geometric tame fundamental group of X_i)

Homotopy exact sequence of fundamental groups

We have the following fundamental exact sequence of fundamental groups for suitable choices of base points:

$$1 \to \Delta_{X_i} \to \Pi_{X_i} \stackrel{\mathsf{pr}_{X_i}}{\to} G_k \to 1.$$

The exact sequence above implies a natural outer Galois representation

$$G_k \to \mathsf{Out}(\Delta_{X_i}) \stackrel{\text{def}}{=} \mathsf{Aut}(\Delta_{X_i})/\mathsf{Inn}(\Delta_{X_i}).$$

- $\mathrm{Isom}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2})$: the set of continuous isomorphisms of profinite groups
- $\mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2}) \stackrel{\text{def}}{=} \{\Phi \in \mathrm{Isom}_{\text{pro-gps}}(\Pi_{X_1}, \Pi_{X_2}) \mid \mathrm{pr}_{X_1} = \mathrm{pr}_{X_2} \circ \Phi\}$

Fundamental problem of anabelian geometry I

Roughly speaking, the main problem of the anabelian geometry (a theory of arithmetic geometry introduced by Alexander Grothendieck in *Esquisse d'un Programme*) of curves is as follows:

Problem 1

How much geometric information (e.g. g_{X_i}, n_{X_i}, etc.) about the isomorphism class of a curve is contained in various versions of its fundamental group?

More precisely, the ultimate goal of anabelian geometry is the following question:

- Can we reconstructed the isomorphism class of X_i group-theoretically from various versions of its fundamental group?

Grothendieck's anabelian conjecture over arithmetic fields

The formulation of the question above is Grothendieck's anabelian conjecture (or the Grothendieck conjecture, for short).

Conjecture 1 (Isom-version Conjecture of characteristic 0)

Let k be an "arithmetic field" of characteristic 0 (e.g. number field, p-adic field). The natural map

$$\mathrm{Isom}_{k\text{-curves}}(X_1, X_2) \to \mathrm{Isom}_{G_k}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

is a bijection.

- Grothendieck only posed his conjecture in the case of number fields.

Previous results concerning the Grothendieck conjecture for curves over arithmetic fields I

The Grothendieck conjecture for curves has been proven in many cases. For example, we have the following results:

- If k is a number field, Conjecture 1 was proved by Hiroaki Nakamura when $g_{X_i} = 0$, by Akio Tamagawa when X_i is affine, and by Shinichi Mochizuki in general.

Previous results concerning the Grothendieck conjecture for curves over arithmetic fields II

- Tamagawa also considered a positive characteristic version of Conjecture 1 and proved the conjecture when k is a finite field and X_i is affine. Mochizuki extended Tamagawa's result to projective case.
- Recently, Mohamed Saïdi and Tamagawa proved that the positive characteristic version of Conjecture 1 also holds if one replaces Δ_{X_i}, $i \in \{1, 2\}$, by the maximal prime-to-p quotient of Δ_{X_i}.

Previous results concerning the Grothendieck conjecture for curves over arithmetic fields III

When $\mathrm{char}(k) = 0$ (resp. $\mathrm{char}(k) > 0$), Δ_{X_i} (resp. the maximal prime-to-p quotient of Δ_{X_i}) is a profinite group which is isomorphic to the profinite completion (resp. the prime-to-p profinite completion) of the topological fundamental group of a Riemann surface of type (g_{X_i}, n_{X_i}). Then the structure of Δ_{X_i} is known. The outer Galois representation

$$G_k \to \mathsf{Out}(\Delta_{X_i})$$

reviewed above contains all the geometric information of the isomorphism class of X_i. All the results mentioned above concerning the Grothendieck conjecture for curves over "arithmetic fields" require the use of the this outer Galois representation (e.g. good and stable reduction criterion, p-adic Hodge theory, Weil conjecture, Galois cohomology, etc.).

Abhyankar's Conjecture I

- l: an algebraically closed field of characteristic $p > 0$
- Z: affine curves of type (g_Z, n_Z) over l
- $\Pi_Z^{\text{ét}}$: etale fundamental group of Z

We have the following famous conjecture concerning the Galois groups of algebraic function fields of characteristic $p > 0$ posed by Shreeram Abhyankar.

Abhyankar's Conjecture II

Conjecture 2 (Abhyankar's conjecture)

For a finite group G, let $p(G)$ be the subgroup generated by all the Sylow p-subgroups of G. Then G is a finite quotient of $\Pi_Z^{\text{ét}}$ if and only if the minimum number of generators of $G/p(G)$ is less than $2g_Z + n_Z - 1$.

- The solvable case of the conjecture was solved by Jean-Pierre Serre (1990) and the full conjecture was proved by Michel Raynaud (1994) and David Harbater (1994).

- Abhyankar's conjecture says that the set of finite quotients of $\Pi_Z^{\text{ét}}$ can be determined completely by the topological structure of Z (i.e., (g_Z, n_Z)).

However, Abhyankar's conjecture cannot tell us any information concerning the global structure of $\Pi_Z^{\text{ét}}$. The structure of $\Pi_Z^{\text{ét}}$ is unknown even in the case where $Z = \mathbb{A}^1_{\overline{\mathbb{F}}_p}$.

On the other hand, Abhyankar's conjecture shows that the (geometric) étale fundamental groups of curves in positive characteristic are highly nontrivial. In his ICM(1994) talk, Harbater asked the following question:

- Can we carry out the geometric information of an affine curve in positive characteristic by using its geometric étale fundamental group?

Around 2000, Raynaud, Florian Pop, Saïdi, and Tamagawa showed evidence for very strong anabelian phenomena for curves over

algebraically closed fields of positive characteristic.

In this setting, G_k is trivial, and $\Pi_{X_i} = \Delta_{X_i}$. Thus in a total absence of a Galois action of the base field. of the étale (or tame) fundamental group of any hyperbolic curve in positive characteristic.

Note that Δ_{X_i} depends only on (g_{X_i}, n_{X_i}) when k is an algebraically closed field of characteristic 0, moreover, (g_{X_i}, n_{X_i}) cannot be reconstructed by Δ_{X_i} when X_i is affine. Then no anabelian geometry exists in this situation.

In the remainder of this talk, I will explain the

tame/admissible anabelian geometry

of curves over algebraically closed fields of characteristic $p > 0$ with the main focus on the topological and combinatorial structures of pointed stable curves.

Fundamental groups of curves in positive characteristic I

- $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$: moduli stacks of smooth pointed stable curves and pointed stable curves of type (g, n) over $\overline{\mathbb{F}}_p$
- $M_{g,n}$ and $\overline{M}_{g,n}$: coarse moduli spaces of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, respectively
- $q \in \overline{M}_{g,n}$: an arbitrary point
- k: an arbitrary algebraically closed field which contains the residue field $k(q)$ of q

Fundamental groups of curves in positive characteristic II

- $X^\bullet = (X, D_X)$: pointed stable curve determined by the natural morphism $\operatorname{Spec} k \to \operatorname{Spec} k(q) \to \overline{M}_{g,n}$
- $X^{\log}$: the log stable curve whose log structure is induced by the log stack $\overline{\mathcal{M}}_{g,n}^{\log}$ (whose log structure is induced by $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$)
- $\Gamma_{X^\bullet}$: the dual semi-graph of $X^\bullet$

Fundamental groups of curves in positive characteristic III

Denote by

$$\Delta^{\mathrm{adm}}$$

the geometric log étale fundamental group of $X^{\log}$ (or admissible fundamental group of $X^{\bullet}$) which depends only on q (i.e., Δ^{adm} does not depend on the choices of k).

Note that $\Delta^{\mathrm{adm}} \cong \pi_1^{\mathrm{tame}}(X \setminus D_X)$ when X is nonsingular.

Main goal

The main goal of this talk is to explain the following results obtained by Tamagawa (smooth case) and the speaker (general case):

- There exists a group-theoretical formula for the topological type (g, n). In particular, (g, n) is a group-theoretical invariant.
- There exists a group-theoretical algorithm whose input datum is Δ^{adm}, and whose output datum is the dual semi-graph $\Gamma_{X^\bullet}$.

Remarks I

- When k is an "arithmetic field", (g, n) can be reconstructed by applying outer Galois actions (e.g. weight-monodromy filtration). However, in the case of tame/admissible fundamental groups of curves over algebraically closed fields of positive characteristic, the reconstruction of (g, n) is highly nontrivial.
- Suppose that $X^\bullet$ is smooth over k. Tamagawa also obtained a group-theoretical formula for (g, n) by using the étale fundamental group of $X \setminus D_X$, whose proof is much simpler (only 1 page!) than the case of tame fundamental groups. Moreover, a result of Tamagawa says that the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group, then the tame fundamental group version is stronger than the étale fundamental group version.

Remarks II

- The most important reason for using tame/admissible fundamental groups is that tame/admissible fundamental groups are "good" invariants if one considers the theory of anabelian geometry of curves in positive characteristic from the point of view of moduli spaces (e.g. lifting and degeneration).

Generalized Hasse-Witt invariants of cyclic coverings I

- H: an open normal subgroup such that $G \stackrel{\text{def}}{=} \Delta^{\text{adm}}/H$ is a cyclic group whose order is prime to p
- $Y^\bullet = (Y, D_Y)$: pointed stable curve over k corresponding to H

Then we obtain a natural representation

$$\rho_H : G \to \mathrm{Aut}_k(H^1_{et}(Y, \mathbb{F}_p) \otimes k)$$

and a decomposition

$$H^1_{et}(Y, \mathbb{F}_p) \otimes k \cong \bigoplus_{\chi : G \to k^\times} H_\chi.$$

Generalized Hasse-Witt invariants of cyclic coverings II

We shall say that

$$\{\dim_k(H_\chi)\}_\chi$$

is the set of generalized Hasse-Witt invariants of cyclic covering $Y^\bullet \to X^\bullet$. Note that since $H^1_{et}(Y, \mathbb{F}_p) \cong H^{\mathrm{ab}} \otimes \mathbb{F}_p$, we have that generalized Hasse-Witt invariants are group-theoretical invariants associated to Δ^{adm}, where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$.

- $\{\dim_k(H_\chi)\}_{\chi,H}$ (or $\{\rho_H\}_H$) plays a role of "outer Galois representations" in the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (i.e., a lot of geometric information concerning $X^\bullet$ can be carried out from $\{\dim_k(H_\chi)\}_{\chi,H}$).

Raynaud-Tamagawa theta divisors I

The theory of Raynaud-Tamagawa theta divisors is a powerful tool to study generalized Hasse-Witt invariants of cyclic coverings. Let me explain this theory roughly in just few slices. For simplicity, we suppose that $X^\bullet$ is smooth over k.

Let $N \stackrel{\text{def}}{=} p^f - 1$, $f \in \mathbb{N}_{>0}$, D an effective divisor on X such that $\mathrm{Supp}(D) \subseteq D_X$ and $\mathrm{ord}_Q(D) < p^f$ for each $Q \in \mathrm{Supp}(D)$, and $\mathcal{I}$ a line bundle on X such that $\mathcal{I}^{\otimes N} \cong \mathcal{O}_X(-D)$. Let F_k^f be the fth absolute Frobenius morphism of k,

$$X_f \stackrel{\text{def}}{=} X \times_{k, F_k^f} k$$

the fth Frobenius twist of X, $F_{X/k}^f : X \to X_1 \to \ldots \to X_f$ the fth relative Frobenius morphism of X, and $\mathcal{I}_f$ the pulling back of the line bundle $\mathcal{I}$ under the natural morphism $X_f \to X$.

Raynaud-Tamagawa theta divisors II

We obtain a vector bundle $\mathcal{B}_D^f \stackrel{\text{def}}{=} (F_{X/k}^f)_*(\mathcal{O}_X(D))/\mathcal{O}_{X_f}$, and put

$$\mathcal{E}_D^f \stackrel{\text{def}}{=} \mathcal{B}_D^f \otimes \mathcal{I}_f$$

on X_f. Consider the following condition $(\star)$:

$$0 = \min\{H^0(X_f, \mathcal{E}_D^f \otimes \mathcal{L}), H^1(X_f, \mathcal{E}_D^f \otimes \mathcal{L})\},\ [\mathcal{L}] \in J_{X_f},$$

where J_{X_f} denotes the Jacobian of X_f. We put

$$\Theta_{\mathcal{E}_D^f} \stackrel{\text{def}}{=} \{[\mathcal{L}] \in J_{X_f} \mid \mathcal{L} \text{ does not satisfy } (\star)\}.$$

In fact, $\Theta_{\mathcal{E}_D^f}$ is a closed subscheme of J_{X_f} with codimension ≤ 1. We shall say $\Theta_{\mathcal{E}_D^f}$ the Raynaud-Tamagawa theta divisor associated to D if $\Theta_{\mathcal{E}_D^f} \neq J_{X_f}$.

Raynaud-Tamagawa theta divisors III

- The theory of $\Theta_{\mathcal{E}_D^f}$ was developed by Raynaud (1982) when $D = 0$, and the ramified version (i.e., $D \neq 0$) was developed by Tamagawa (2003).

- If $D = 0$ (resp. $\deg(D) = N$), the existence of $\Theta_{\mathcal{E}_D^f}$ was proved by Raynaud (resp. Tamagawa). The existence of $\Theta_{\mathcal{E}_D^f}$ is a very difficult problem, and it does not exist in general.

- There exists a $\mathbb{Z}/N\mathbb{Z}$-tame covering of $X^\bullet$ whose ramification divisor is equal to D, and whose generalized Hasse-Witt invariant attains the maximum if and only if there exists $[\mathcal{L}] \in J_{X_f}[N]$ such that $[\mathcal{L}] \notin \Theta_{\mathcal{E}_D^f}$, that $N|\deg(D)$, and that $\mathrm{ord}_Q(D) < p^f - 1$ for each $Q \in \mathrm{Supp}(D)$.

Raynaud-Tamagawa theta divisors IV

If $\Theta_{\mathcal{E}_D^f}$ exists, we may use intersection theory to estimate the cardinality of $\mathbb{Z}/N\mathbb{Z}$-tame covering of $X^\bullet$ whose ramification divisor is equal to D, and whose generalized Hasse-Witt invariant attains the maximum. This is the main idea and purpose of Raynaud and Tamagawa's theory on theta divisors.

By using $\Theta_{\mathcal{E}_D^f}$, Raynaud obtained the following deep theorem, which is the first result concerning the global structure of tame fundamental group of curves over algebraically closed fields of characteristic $p > 0$:

- Let $X^\bullet$ be a projective curve (i.e., $D_X = \emptyset$) over k. Then Δ^{adm} (i.e., the étale fundamental group of X) is not a prime-to-p profinite group. This means that, for each open subgroup $H_1 \subseteq \Delta^{\mathrm{adm}}$, there exists an open subgroup $H_2 \subseteq H_1$ such that $H_2^{\mathrm{ab}} \otimes \mathbb{F}_p \neq 0$.

p-average of admissible fundamental groups I

- K_N: the kernel of the natural surjection $\Delta^{\mathrm{adm}} \twoheadrightarrow \Delta^{\mathrm{adm,ab}} \otimes \mathbb{Z}/N\mathbb{Z}$, where $N \overset{\mathrm{def}}{=} p^f - 1$

Tamagawa introduced an important group-theoretical invariant as following, which is called the limit of p-average of Δ^{adm}:

$$\mathsf{Avr}_p(\Delta^{\mathrm{adm}}) \overset{\mathrm{def}}{=} \lim_{f \to \infty} \frac{\dim_{\mathbb{F}_p}(K_N^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(\Delta^{\mathrm{adm,ab}} \otimes \mathbb{Z}/N\mathbb{Z})}.$$

- Roughly speaking, when $N >> 0$, almost all of the generalized Hasse-Witt invariants of $\mathbb{Z}/N\mathbb{Z}$-admissible coverings are equal to $\mathsf{Avr}_p(\Delta^{\mathrm{adm}})$.

p-average of admissible fundamental groups II

We have the following highly nontrivial theorem which was proved by Tamagawa ($\Gamma_{X^\bullet}$ is 2-connected (e.g. $X^\bullet$ is smooth over k)), and was generalized by the speaker (general case) by using Raynaud-Tamagawa theta divisors. For simplicity, in this talk, I only give the result when $X^\bullet$ is smooth over k.

Theorem 1 (p-average theorem)

Suppose that $X^\bullet$ is smooth over k. Then we have

$$\mathrm{Avr}_p(\Delta^{\mathrm{adm}}) = \begin{cases} g-1, & \textit{if } n \leq 1, \\ g, & \textit{if } n \geq 2. \end{cases}$$

- The smooth version of p-average theorem means that $\mathrm{Avr}_p(\Delta^{\mathrm{adm}})$ contains the information concerning (g, n) when $X^\bullet$ is smooth over k.

Main theorems I: notations

- $b^i \overset{\text{def}}{=} \dim_{\mathbb{Q}_\ell}(H^i_{et}(X \setminus D_X, \mathbb{Q}_\ell))$ (i.e., the Betti number of the ith ℓ-adic étale cohomology group), $i \in \{0, 1, 2\}$, where ℓ is a prime number distinct from p. Moreover, we may prove that b^i, $i \in \{0, 1, 2\}$, is a group-theoretical invariant.

- Let $\ell' \in \mathfrak{Primes} \setminus \{p\}$ be an arbitrary prime number distinct from p. Write $\mathrm{Nom}_{\ell'}(\Delta^{\mathrm{adm}})$ for the set of normal subgroups of Δ^{adm} such that $\#(\Delta^{\mathrm{adm}}/\Delta^{\mathrm{adm}}(\ell')) = \ell'$ for each $\Delta^{\mathrm{adm}}(\ell') \in \mathrm{Nom}_{\ell'}(\Delta^{\mathrm{adm}})$. We put

$$c \overset{\text{def}}{=} \begin{cases} 1, & \text{if } b^2 = 1, \\ 1, & \text{if } b^2 = 0,\ \mathrm{Avr}_p(\Delta^{\mathrm{adm}}(\ell)) - 1 = \ell(\mathrm{Avr}_p(\Delta^{\mathrm{adm}})), \\ & \ell \in \mathfrak{Primes} \setminus \{p\},\ \Delta^{\mathrm{adm}}(\ell) \in \mathrm{Nom}_\ell(\Delta^{\mathrm{adm}}), \\ 0, & \text{otherwise.} \end{cases}$$

Main theorems II: smooth case

By applying the p-average theorem, Tamagawa proved the following result:

Theorem 2 (An anabelian formula for (g, n) (smooth case))

Suppose that $X^\bullet$ is smooth over k. Then we have

$$g = \mathrm{Avr}_p(\Delta^{\mathrm{adm}}) + c, \ \ n = b^1 - 2\mathrm{Avr}_p(\Delta^{\mathrm{adm}}) - 2c - b^2 + 1.$$

- This result is a key step in Tamagawa's proof of the weak Isom-version of Grothendieck conjecture for smooth curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$, which says that the isomorphism classes of smooth curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$ can be determined group-theoretically from the isomorphism classes of their tame fundamental groups.

Main theorems III: singular case

The approach to finding an anabelian formula for (g, n) by applying the limit of p-averages associated to Δ^{adm} explained above cannot be generalized to the case where $X^\bullet$ is an arbitrary (possibly singular) pointed stable curve. The reason is that the singular version of p-average theorem is very complicated in general, and $\mathrm{Avr}_p(\Delta^{\mathrm{adm}})$ depends not only on (g, n) but also on the graphic structure of $\Gamma_{X^\bullet}$.

Main theorems IV: singular case

By proving the existence of Raynaud-Tamagawa theta divisor for certain effective divisor D on X, the speaker obtained the following result:

Theorem 3 (Maximum generalized Hasse-Witt invariant theorem)

There exists a prime-to-p cyclic admissible covering of $X^\bullet$ such that a generalized Hasse-Witt invariant of the cyclic admissible covering attains the maximum

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g-1, & \text{if } n=0, \\ g+n-2, & \text{if } n \neq 0. \end{cases}$$

Moreover, $\gamma_{X^\bullet}^{\max}$ is a group-theoretical invariant.

Main theorems V: singular case

The maximum generalized Hasse-Witt invariant theorem implies the following formula immediately:

Theorem 4 (An anabelian formula for (g, n) (general case))

Let $X^\bullet$ be an arbitrary pointed stable curve of type (g, n) over k. Then we have

$$g = b^1 - \gamma^{\max}_{X^\bullet} - 1, \ n = 2\gamma^{\max}_{X^\bullet} - b^1 - b^2 + 3.$$

Main theorems VI: singular case

On the other hand, $\mathrm{Avr}_p(\Delta^{\mathrm{adm}})$ contains the information concerning the Betti number of $\Gamma_{X^\bullet}$ if $\Gamma_{X^\bullet}$ is "good" enough. This means that the weight-monodromy filtration associated to the first ℓ-adic étale cohomology group of every admissible covering of $X^\bullet$ can be reconstructed group-theoretically from the corresponding open subgroup of Δ^{adm}. Note that, if k is an "arithmetic field", the weight-monodromy filtration can be reconstructed group-theoretically by using the theory of "weights".

This observation is a key in the speaker's proof of combinatorial Grothendieck conjecture in positive characteristic (i.e., all of the topological and combinatorial data concerning $X^\bullet$ can be reconstructed group-theoretically from Δ^{adm}).

Main theorems VI: singular case

Let $v(\Gamma_{X^\bullet})$ be the set of vertices of $\Gamma_{X^\bullet}$. Write

$$\mathrm{nom}_v : \widetilde{X}_v \to X_v,\ v \in v(\Gamma_{X^\bullet}),$$

for the normalization morphism of the irreducible component of X corresponding to v. Then we define a smooth pointed stable curve

$$\widetilde{X}_v^\bullet = (\widetilde{X}_v, D_{\widetilde{X}_v} \overset{\mathrm{def}}{=} \mathrm{nom}_v^{-1}((X^{\mathrm{sing}} \cap X_v) \cup (D_X \cap X_v)),\ v \in v(\Gamma_{X^\bullet}),$$

of type (g_v, n_v) over k, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$. Write Δ_v^{adm} for the admissible (=tame) fundamental group of $\widetilde{X}_v^\bullet$. Then we have a natural outer injection $\Delta_v^{\mathrm{adm}} \hookrightarrow \Delta^{\mathrm{adm}}$.

Main theorems VII: singular case

Let us show the second main result of this talk.

Theorem 5 (Combinatorial Grothendieck conjecture in positive characteristic)

Let $X^\bullet$ be an arbitrary pointed stable curve of type (g, n) over k. Then there exists a group-theoretical algorithm whose input datum is Δ^{adm}, and whose output data are the following:

- g, n, *and* $\Gamma_{X^\bullet}$;
- *the conjugacy class of the inertia subgroup of every marked point of* $X^\bullet$ *in* Δ^{adm};
- *the conjugacy class of the inertia subgroup of every node of* $X^\bullet$ *in* Δ^{adm};
- g_v, n_v, *and the conjugacy class of* Δ_v^{adm} *in* Δ^{adm} *for each* $v \in v(\Gamma_{X^\bullet})$.

Main theorems VII: remarks

- By applying combinatorial Grothendieck conjecture, all the results concerning the tame anabelian geometry of smooth curves over algebraically closed fields of characteristic $p > 0$ can be extended to the case of pointed stable curves.

- Since the group-theoretical algorithm appeared in combinatorial Grothendieck conjecture is not an explicit algorithm, the formula for (g, n) cannot be deduced by combinatorial Grothendieck conjecture.

Main theorems VIII: remarks

- p-average theorem and maximum generalized Hasse-Witt invariant theorem play fundamental roles in the theory of moduli spaces of fundamental groups of curves in positive characteristic developing by the speaker. The aim of this theory is to reconstruct topological structures of moduli spaces of curves in positive characteristic from fundamental groups of curves.

Thank you for the attention !