Construction of Hodge Theaters

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In this talk, I will explain the first paper of Prof. Mochizuki's theory of IUT. The talk is divided into three parts as follows:

- The motivation of Hodge theaters
- The goal of Hodge theaters
- The construction of Hodge theaters (in particular, the étale picture!)

§1: Motivation of Hodge theaters

- $F/\mathbb{Q}$: a number field
- $\mathcal{O}_F$: the ring of integers of F
- E/F: an elliptic curve s.t. E admits a semi-stable model $\mathcal{E}/\mathcal{O}_F$
- $\mathbb{V}(F)$: the set of places of F
- $\mathbb{V}(F)^{\mathrm{non,bad}}$: the set of non-arch. places v s.t. $E_v \stackrel{\mathrm{def}}{=} E|_{F_v}$ has bad reduction, where F_v denotes the local field at v
- ℓ: a prime number distinct from p_v for all $v \in \mathbb{V}(F)^{\mathrm{non,bad}}$, where p_v denotes the characteristic of the residue field of F_v
- ht_E: Faltings height of E

Goal: We hope that ht_E can be bounded for all elliptic curves satisfying the above conditions.

Note that $E_v \cong \mathbb{G}_m / q_v^{\mathbb{Z}} \twoheadleftarrow \mathbb{G}_m \supseteq \mu_\ell$ for all $v \in \mathbb{V}(F)^{\mathrm{non,bad}}$, and that

$$0 \to \mu_\ell \to E_v[\ell] \to \mathbb{Z}/\ell\mathbb{Z} \to 1.$$

Global multiplicative subspaces (=GMS)

We shall call $H \subseteq E[\ell]$ a "GMS" if $H|_{F_v}$ coincides with μ_ℓ for all $v \in \mathbb{V}(F)^{\mathrm{non,bad}}$. This means that there exists a Galois étale covering $Y \to E$ corresponding to H such that $Y_v \to E_v$ is a topological covering of dual semi-graphs for all $v \in \mathbb{V}(F)^{\mathrm{non,bad}}$.

If "GMS" exists, then by some standard discussions of Diophantine geometry, we may obtain that ht_E can be bounded. However, we have

$$\#\{E/F \text{ s.t. GMS exists}\} < \infty.$$

Goal of IUT: We want to do similar discussions for arbitrary elliptic curves over number fields.

The first step: We need an analogue of "GMS" for arbitrary elliptic curves.

IUT's answer: It's "Hodge theaters"

§2: Goal of Hodge theaters

Reference

Section 1 of "S. Mochizuki, The étale theta function and its Frobenioid-theoretic manifestations. *Publ. Res. Inst. Math. Sci.* **45** (2009), 227-349."

Local theory (over non-archimedean bad places)

- $\ell >> 0$: a prime number
- $p \geq 3$: a prime number s.t. $p \neq \ell$
- k: a p-adic field
- $\mathcal{O}_k$: the ring of integers of k
- $X \cong \mathbb{G}_{m,k}/q^{\mathbb{Z}}$: an elliptic curve over k with bad reduction (i.e., Tate curve)
- $X^{\log}$: the log stable curve over k determined by the zero point of X

Moreover, we assume that

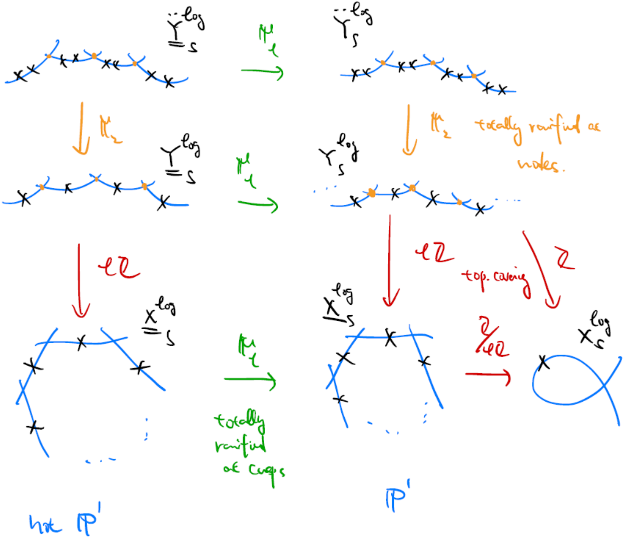
- $\sqrt{-1} \in k$
- $X[2\ell](\overline{k}) = X[2\ell](k)$, where $\overline{k}$ denotes an algebraic closure of k

Then we have the following commutative diagram of cartesian squares of tempered coverings:

$$
\begin{tikzcd}
\underline{\underline{\ddot{Y}}}^{\log} \arrow[r, "\mu_\ell"] \arrow[d, "\mu_2"'] & \ddot{Y}^{\log} \arrow[d, "\mu_2"'] & \\
\underline{\underline{Y}}^{\log} \arrow[r, "\mu_\ell"] \arrow[d, "\ell\mathbb{Z}"'] & Y^{\log} \arrow[d, "\ell\mathbb{Z}"'] & \\
\underline{\underline{X}}^{\log} \arrow[r, "\mu_\ell"] & \underline{X}^{\log} \arrow[r, "\mathbb{Z}/\ell\mathbb{Z}"] & X^{\log}
\end{tikzcd}
$$

The above coverings are defined in the next page via a picture of special fibers.

The special fibers of the above commutative digram is as follows:



On the other hand, let us fix a cusp $O_{\underline{X}}$ (i.e., zero cusp) of $\underline{X}$. Then the image of $O_{\underline{X}}$ of $\underline{X} \to X$ is the zero cusp (or the zero point) O_X. The curve $(\underline{X}, O_{\underline{X}})$ can be regarded as an elliptic curve over k. Thus, we obtain

$$\begin{array}{ccc} \underline{X}^{\log} & \xrightarrow{\mathbb{Z}/\ell\mathbb{Z}} & X^{\log} \\ {\scriptstyle \pm 1}\downarrow & & {\scriptstyle \pm 1}\downarrow \\ \underline{C}^{\log} & \xrightarrow{\text{degree } \ell} & C^{\log}, \end{array}$$

where $\underline{C}^{\log} \stackrel{\text{def}}{=} [\underline{X}^{\log}/\{\pm 1\}]$ and $C^{\log} \stackrel{\text{def}}{=} [X^{\log}/\{\pm 1\}]$ denote the quotient stacks.

Moreover, there exists a unique irreducible component $0_{\underline{X}_s} \in \mathrm{Irr}(\underline{X}_s)$ such that the reduction of $O_{\underline{X}}$ is contained in $0_{\underline{X}_s}$, where $\mathrm{Irr}(\underline{X}_s)$ denotes the set of irreducible components of $\underline{X}$.

Let $0_{Y_s} \in \mathsf{Irr}(Y_s)$ be a lifting of $0_{\underline{X}_s}$. Then we obtain a labeling

$$\mathbb{Z} \xrightarrow{\sim} \mathsf{Irr}(Y_s) \xrightarrow{\sim} \mathsf{Irr}(\ddot{Y}_s) \xrightarrow{\sim} \mathsf{Irr}(\underline{\underline{\ddot{Y}}}_s)$$

such that $0 \mapsto 0_{Y_s}$. Moreover, we put

- $\mu_- \in \underline{X}(k)$: 2-torsion point whose reduction is contained in $0_{\underline{X}_s}$
- $\mu_-^Y \in Y(k)$: the unique lifting of μ_- s.t. the reduction is contained in 0_{Y_s}
- $\xi_j^Y \in Y(k)$: $j \cdot \mu_-^Y$ (with the action of $j \in \mathbb{Z} \cong \mathsf{Aut}(Y^{\log}/X^{\log})$)

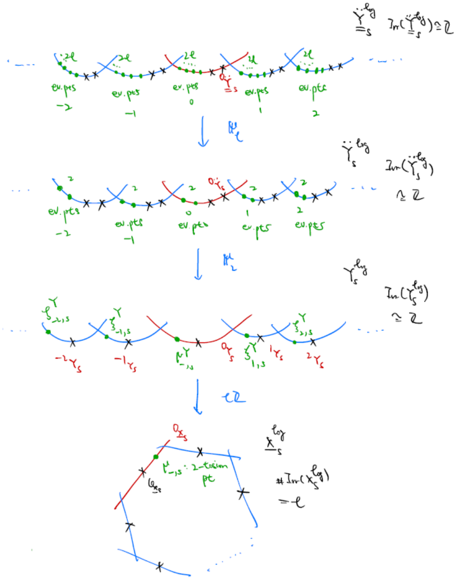
We have the following definition.

Definition 1

We shall call a lifting of $\xi_j^Y \in Y(k)$ in $\ddot{Y}(k)$ an evaluation point of $\ddot{Y}^{\log}$ labeled by $j \in \mathbb{Z}$ $(\overset{\sim}{\to} \mathrm{Irr}(Y_s))$. Moreover, we shall call a lifting of an evaluation point of $\ddot{Y}(k)$ labeled j in $\underline{\underline{\ddot{Y}}}(k)$ an evaluation point of $\underline{\underline{\ddot{Y}}}^{\log}$ labeled by $j \in \mathbb{Z}$ $(\overset{\sim}{\to} \mathrm{Irr}(\ddot{Y}_s) \overset{\sim}{\to} \mathrm{Irr}(Y_s))$.

Moreover, we have the following picture:

We have the following diagram of special fibers:



Non-archimedean Θ-functions

We put

$$\underline{\underline{\ddot{\mathcal{Y}}}},\ \underline{\underline{\mathcal{Y}}},\ \ddot{\mathcal{Y}},\ \mathcal{Y}$$

the p-adic formal schemes whose Raynaud generic fibers are $\underline{\underline{\ddot{Y}}}$, $\underline{\underline{Y}}$, $\ddot{Y}$, Y, and whose special fibers are $\underline{\underline{\ddot{Y}}}_s$, $\underline{\underline{Y}}_s$, $\ddot{Y}_s$, Y_s, respectively.

Write $0_{\ddot{Y}_s} \in \mathrm{Irr}(\ddot{Y}_s)$ for the irreducible component over $0_{Y_s} \in \mathrm{Irr}(Y_s)$ and $\ddot{\mathcal{U}} \subseteq \ddot{\mathcal{Y}}$, $\mathcal{U} \subseteq \mathcal{Y}$ for the open formal subschemes such that

$$\ddot{\mathcal{U}}_s \cong 0_{\ddot{Y}_s} \setminus \ddot{Y}_s^{\mathrm{sing}}\ (\cong \mathbb{G}_m),\ \mathcal{U}_s \cong 0_{Y_s} \setminus Y_s^{\mathrm{sing}}\ (\cong \mathbb{G}_m).$$

Then $\mathcal{U}$ is isomorphic to the p-adic formal completion of $\mathbb{G}_{m,\mathcal{O}_k}$ with multiplicative coordinate $U \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$. Moreover, we put

$$\ddot{U} \overset{\mathrm{def}}{=} \sqrt{U} \in \Gamma(\ddot{\mathcal{U}}, \mathcal{O}_{\ddot{\mathcal{U}}}).$$

We have the following function on $\ddot{Y}$:

$$\ddot{\Theta}(\ddot{U}) = q^{-\frac{1}{8}} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot q^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{U}^{2n+1}.$$

Moreover, we define a function

$$\underline{\underline{\Theta}} \stackrel{\mathrm{def}}{=} \ddot{\Theta}(\text{an evaluation pt labeled by } 0) \cdot \ddot{\Theta}^{-1}$$

on $\underline{\underline{\ddot{Y}}}$ which can be regarded as an "ℓ-th root" of $\ddot{\Theta}$ (in the sense of cohomological classes). Note that there are exactly two evaluation points labeled by 0 in $\ddot{Y}_s$, and that we have

$$\ddot{\Theta}(\text{an ev. pt labeled by } 0) = -\ddot{\Theta}(\text{another ev. pt labeled by } 0).$$

Values of $\underline{\underline{\Theta}}$ at evaluation points

We put $\underline{\underline{q}} \stackrel{\mathrm{def}}{=} q^{\frac{1}{2\ell}}$. Then $\underline{\underline{\Theta}}(\text{an evaluation pt labeled by } j) \in \mu_{2\ell} \cdot \underline{\underline{q}}^{j^2}$.

Let $\mathbb{V}(F)^{\mathrm{non,bad}}$ be the notation introduced in §1. Moreover, we denote by $\underline{\underline{\Theta}}_v$ the function defined above at the place $v \in \mathbb{V}(F)^{\mathrm{non,bad}}$. Then we have

$$\begin{array}{cccccccc} & & \pm 1 & \pm 2 & \ldots & \pm j & \ldots \\ \underline{\underline{\Theta}}_v & \mapsto & \underline{\underline{q}}_v & \underline{\underline{q}}_v^4 & \ldots & \underline{\underline{q}}_v^{j^2} & \ldots \end{array}$$

<u>**The Goal of Hodge theaters:**</u> Roughly speaking, Hodge theater (at least, the étale part) is a virtual "GMS" for an arbitrary elliptic curve over a number field which manages

> *$\underline{\underline{\Theta}}_v$-values for all non-archimedean bad places (with their labels) via anabelian geometry.*

§3: Initial Θ-data

Reference

Section 3 of "S. Mochizuki, Inter-universal Teichmüller theory I: Construction of Hodge theaters. *Publ. Res. Inst. Math. Sci.* **57** (2021), 3–207."

Firstly, we have the following notation:

- F : a number field s.t. $\sqrt{-1} \in F$
- E : an elliptic curve over F s.t. E has stable reduction at all $v \in \mathbb{V}(F)^{\mathrm{non}}$
- $\ell > 5$: a prime number s.t. $\ell \neq p_v$ for all $v \in \mathbb{V}(F)^{\mathrm{non,bad}}$
- $K \stackrel{\mathrm{def}}{=} F(E[\ell])$
- F_{mod} : the field of moduli of E
- $X \stackrel{\mathrm{def}}{=} E \setminus \{O_E\}$
- $C \stackrel{\mathrm{def}}{=} [X/\{\pm 1\}]$

Furthermore, we assume that

- $E[6](\overline{F}) = E[6](F)$, where $\overline{F}$ denotes an algebraic closure of F
- $C_K \stackrel{\mathrm{def}}{=} C \times_F K$ is a "K-core" (i.e., a terminal object in the category of étale coverings and quotients of X_K over K)
- F/F_{mod} is Galois
- $SL_2(\mathbb{F}_\ell) \subseteq \mathrm{Im}(G_F \to \mathrm{Aut}(E[\ell])$ $(\cong GL_2(\mathbb{F}_\ell))$

Let $\underline{X}_K \to X_K \overset{\text{def}}{=} X \times_F K$ be an étale covering with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Note that $\#(\underline{X}_K^{\text{cpt}} \setminus \underline{X}_K) = \ell$. Just like the local theory recalled above, we fix a cusp $O_{\underline{X}_K} \in \underline{X}_K^{\text{cpt}} \setminus \underline{X}_K$ and call it zero cusp. Then $(\underline{X}_K^{\text{cpt}}, O_{\underline{X}_K})$ is an elliptic curve over K. In particular, there exists a $\{\pm 1\}$-action on $\underline{X}_K$. Thus, we have

$$\begin{array}{ccc} \underline{X}_K & \xrightarrow{\mathbb{Z}/\ell\mathbb{Z}} & X_K \\ {\scriptstyle \pm 1}\big\downarrow & & {\scriptstyle \pm 1}\big\downarrow \\ \underline{C}_K & \xrightarrow{\text{degree } \ell} & C_K, \end{array}$$

where $\underline{C}_K \overset{\text{def}}{=} [\underline{X}_K/\{\pm 1\}]$ denotes the quotient stack. Moreover, we fix a non-zero cusp

$$\underline{\epsilon}$$

of $\underline{C}_K$ (i.e., the image of a cusp $\in \underline{X}_K^{\text{cpt}} \setminus (\underline{X}_K \cup \{O_{\underline{X}_K}\})$).

We put the following

- $\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$: a subset s.t. the natural map $\underline{\mathbb{V}} \hookrightarrow \mathbb{V}(K) \twoheadrightarrow \mathbb{V}(F_{\mathrm{mod}})$ is a bijection
- $\underline{\mathbb{V}}^{\mathrm{bad}} \subseteq \underline{\mathbb{V}}$: a non-empty subset s.t. for every $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the following are satisfied: (i) E has bad reduction at $\underline{v}$; (ii) $\underline{X}_{\underline{v}} \to X_{\underline{v}}$ induces a topological covering of their dual semi-graphs; (iii) the reduction $\underline{\epsilon}_{\underline{v}}$ of $\underline{\epsilon}$ is the cusp of $\underline{C}_{\underline{v}}^{\log}$ labeled by $\overline{1} \in \mathbb{F}_\ell/\{\pm 1\}$ ($\cong$ Cusp($\underline{C}_{\underline{v}}^{\log}$)); (iv) the image of the natural map $\underline{\mathbb{V}}^{\mathrm{bad}} \hookrightarrow \mathbb{V}(K) \twoheadrightarrow \mathbb{V}(F_{\mathrm{mod}}) \twoheadrightarrow \mathbb{V}(\mathbb{Q})$ does not contain 2. Then for each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, we have the local theory explained in §2.
- $\underline{\mathbb{V}}^{\mathrm{good}} \overset{\mathrm{def}}{=} \underline{\mathbb{V}} \setminus \underline{\mathbb{V}}^{\mathrm{bad}}$. Note that $E_{\underline{v}}$, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, has bad reduction in general.

We shall call

$$(\overline{F}/F, E, \ell, \underline{C}_K, \underline{\mathbb{V}}, \underline{\mathbb{V}}^{\mathrm{bad}}, \underline{\epsilon})$$

an initial Θ-data. From now on, we fix an initial Θ-data, and in the reminder of my talk, I will explain the following diagram which is constructed from the given initial Θ-data (I only explain the constructions at non-archimedean places which are the most important cases in the original form of IUT):

$\mathfrak{D}$-$\Theta^{\pm ell}NF$-$\mathcal{H}T$

$\mathfrak{D}$-$\Theta^{\pm ell}$-$\mathcal{H}T$ $\qquad\qquad$ $\mathfrak{D}$-ΘNF-$\mathcal{H}T$

$$
\begin{tikzcd}
{}^\dagger\mathfrak{D}_{\succ} \arrow[rr, squiggly, "\text{gluing by ``}{}^\dagger\phi^{\Theta^\pm}_{\pm}|_J = {}^\dagger\phi^{\Theta}_{*}\text{''}"] & & {}^\dagger\mathfrak{D}_{>} \\
{}^\dagger\mathfrak{D}_T \arrow[u, "\begin{array}{c}\mathfrak{D}\text{-}\Theta^{\pm}\text{-bridge} \\ {}^\dagger\phi^{\Theta^\pm}_{\pm}\end{array}"'] \arrow[d, "\begin{array}{c}\mathfrak{D}\text{-}\Theta^{ell}\text{-bridge} \\ {}^\dagger\phi_{\pm}\end{array}"'] \arrow[rr, squiggly, "\text{gluing by identify } J = (T\setminus\{0\})/\{\pm1\}"] & & {}^\dagger\mathfrak{D}_J \arrow[u, "\begin{array}{c}\mathfrak{D}\text{-}\Theta\text{-bridge} \\ {}^\dagger\phi^{\Theta}_{*}\end{array}"'] \arrow[d, "\begin{array}{c}\mathfrak{D}\text{-}NF\text{-bridge} \\ {}^\dagger\phi^{NF}_{*}\end{array}"'] \\
{}^\dagger\mathfrak{D}^{\circledcirc\pm} & & {}^\dagger\mathfrak{D}^{\circledcirc}
\end{tikzcd}
$$

$T \cong \mathbb{F}_l$ $\qquad\qquad$ $J \cong \mathbb{F}_l^{*} \overset{def}{=} \mathbb{F}_l^{\times}/\{\pm1\}$

with $\mathbb{F}_l^{\rtimes\pm} \overset{def}{=} \mathbb{F}_l \rtimes \{\pm1\}$-symm.
arising from geometry
(i.e., ⊞-symm)

with $\mathbb{F}_l^{*}$-symm.
arising from arithmetic
(i.e., ⊠-symm.)

Many constructions appeared in the above picture are not difficult to understand from geometry of coverings of curves. On the other hand, in IUT, we need to share information via various links between different Hodge theaters (or different universes) by using fundamental groups (via anabelian geometry), then we need some group-theoretical descriptions.

§4: Construction of $\mathcal{D}$-ΘNF-$\mathcal{HT}$

Reference

Section 4 of "S. Mochizuki, Inter-universal Teichmüller theory I: Construction of Hodge theaters. *Publ. Res. Inst. Math. Sci.* **57** (2021), 3–207."

In the section, I explain the right-hand side:

$\mathcal{D}$ - ΘNF - HT

$$J = \mathbb{F}_l^{*} \overset{\text{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}$$

$$\begin{tikzcd}
\mathcal{D}_{>} \\
\mathcal{D}_{J} \arrow[u, "\phi_*^{\Theta}"'] \arrow[d, "\phi_*^{NF}"] \\
\mathcal{D}^{\circledcirc}
\end{tikzcd}$$

ϕ_*^{Θ}: $\mathcal{D}$-Θ-bridge

ϕ_*^{NF}: $\mathcal{D}$-NF-bridge

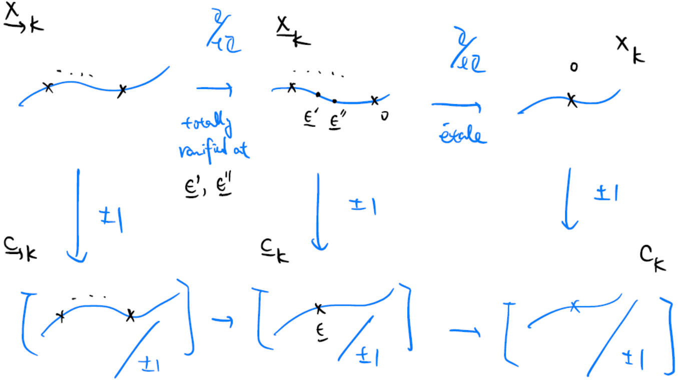
with $\mathbb{F}_l^{*}$-symm.
arising from
arithmetic
(i.e., $\boxtimes$-symm.)

$\mathcal{D}$-prime-strips

We put

$$\mathcal{D}_{\underline{v}} \stackrel{\mathrm{def}}{=} \begin{cases} \pi_1^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}}^{\log}) \text{ (or } \mathcal{B}^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}}^{\log})^0\text{)}, & \text{if } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \\ \pi_1^{\text{ét}}(\underset{\rightarrow}{X}_{\underline{v}}) \text{ (or } \mathcal{B}^{\text{ét}}(\underset{\rightarrow}{X}_{\underline{v}})^0\text{)}, & \text{if } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \end{cases}$$

where $\mathcal{B}(-)^0$ denotes the subcategory of the Galois category $\mathcal{B}(-)$ consisting of connected objects, and $\underset{\rightarrow}{X}_{\underline{v}} \stackrel{\mathrm{def}}{=} \underset{\rightarrow}{X}_K \times_K K_{\underline{v}}$ is determined by $(\underline{C}_K, \underline{\epsilon})$ via the picture in the next page:


X
totally
ramified at
ε', ε''
étale
±1

Let $\mathcal{D} \overset{\text{def}}{=} \{\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$. Then we put

$$\mathcal{D}_{>} \overset{\text{def}}{=} \{\mathcal{D}_{>,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \ (\cong {}^{\dagger}\mathcal{D}),$$

where $D_{>,\underline{v}} \cong \mathcal{D}_{\underline{v}}$ for all $\underline{v} \in \underline{\mathbb{V}}$.

Let $\mathbb{F}_{\ell}^{\divideontimes} \overset{\text{def}}{=} \mathbb{F}_{\ell}/\{\pm 1\}$, $\underline{\mathbb{V}}_j$, $j \in \mathbb{F}_{\ell}^{\divideontimes}$, a copy of $\underline{\mathbb{V}}$, and $\mathcal{D}_j \overset{\text{def}}{=} \{\mathcal{D}_{\underline{v}_j}\}_{\underline{v}_j\in\underline{\mathbb{V}}_j}$. Then we put

$$\mathcal{D}_J \overset{\text{def}}{=} \{\mathcal{D}_j\}_{j\in J},$$

where $J \overset{\text{def}}{=} \mathbb{F}_{\ell}^{\divideontimes}$. Moreover, we put

$$\mathcal{D}^{\circledcirc} \overset{\text{def}}{=} \pi_1^{\text{ét}}(\underline{C}_K) \ (\text{or } \mathcal{B}^{\text{ét}}(\underline{C}_K)^0).$$

$\boxtimes$-symmetry (on cusps)

For every $\underline{v} \in \underline{\mathbb{V}}$, recall the following commutative diagram:

$$\begin{tikzcd} \underline{X}_{\underline{v}} \overset{\mathrm{def}}{=} \underline{X}_K \times_K K_v \arrow[r, "\mathbb{Z}/\ell\mathbb{Z}"] \arrow[d, "\pm 1"'] & X_{\underline{v}} \overset{\mathrm{def}}{=} X_K \times_K K_v \arrow[d, "\pm 1"'] \\ \underline{C}_{\underline{v}} \overset{\mathrm{def}}{=} \underline{C}_K \times_K K_v \arrow[r, "\text{degree } \ell"] & C_{\underline{v}} \overset{\mathrm{def}}{=} C_K \times_K K_v. \end{tikzcd}$$

We put

$$\mathsf{LabCusp}_{\underline{v}} \overset{\mathrm{def}}{=} \text{the set of non-zero cusps of } \underline{C}_{\underline{v}}.$$

Then we have

$$\mathsf{LabCusp}_{\underline{v}} \overset{\sim}{\to} \mathbb{F}_\ell^{⋇} (\overset{\mathrm{def}}{=} \mathbb{F}_\ell^{\times}/\{\pm 1\}) = \{\ell^{⋇}, \ell^{⋇} - 1, \ldots, 2, 1, -2, \ldots, -\ell^{⋇}\},$$

where $\underline{\epsilon}_{\underline{v}} \mapsto$ the image of 1 in $\mathbb{F}_\ell^{⋇}$ and $\ell^{⋇} \overset{\mathrm{def}}{=} (\ell - 1)/2$.

Moreover, we put

$$\text{LabCusp}_K \stackrel{\text{def}}{=} \text{the set of non-zero cusps of } \underline{C}_K.$$

Then there exists a natural bijection

$$\text{LabCusp}_{\underline{v}} \xrightarrow{\sim} \text{LabCusp}_K, \ \underline{v} \in \underline{\mathbb{V}},$$

via the natural homomorphism $\underline{C}_{\underline{v}} \to \underline{C}_K$. This means that the sets

$$\{\text{LabCusp}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

can be managed by LabCusp_K via the above bijections.

On the other hand, since $\underline{C}_K$ is a "K-core", we have

$$\mathsf{Aut}(\underline{C}_K) \hookrightarrow \mathsf{Gal}(K/F_{\mathrm{mod}}).$$

Moreover, we define a subgroup

$$\mathsf{Aut}_{\underline{\epsilon}}(\underline{C}_K) \stackrel{\mathrm{def}}{=} \{\sigma \in \mathsf{Aut}(\underline{C}_K) \text{ s.t. } \sigma(\underline{\epsilon}) = \underline{\epsilon}\}.$$

Let $E[\ell](\overline{F}) \twoheadrightarrow Q(\cong \mathbb{F}_\ell)$ be the quotient determined by the Galois étale covering $\underline{X}_K \to X_K$. Then we have the following

$$\mathsf{LabCusp}_K \xrightarrow{\sim} \big(\{Q \curvearrowright O_{\underline{X}_K}\} \setminus \{O_{\underline{X}_K}\}\big)/\{\pm 1\}$$

$$\xrightarrow{\sim} \big(Q \setminus \{0\}\big)/\{\pm 1\}.$$

Thus, we obtain an exact sequence

$$1 \to \mathsf{Aut}_{\underline{\epsilon}}(\underline{C}_K) \to \mathsf{Aut}(\underline{C}_K) \to \mathsf{Aut}(Q)/\{\pm 1\}(\cong \mathbb{F}_\ell^{\circledast}) \to 1.$$

This means

$$\mathsf{Aut}(\underline{C}_K)/\mathsf{Aut}_{\underline{\epsilon}}(\underline{C}_K) \cong \mathbb{F}_\ell^{\circledast}.$$

By using anabelian geometry, we have a group-theoretical version of the above isomorphism:

$$\mathsf{Aut}(\mathcal{D}^{\circledcirc})/\mathsf{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathbb{F}_\ell^{\circledast}$$

and $\mathsf{Aut}(\mathcal{D}^{\circledcirc})/\mathsf{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})$ is a sub-quotient of the Galois group of the extension of number fields $\mathsf{Gal}(K/F_{\mathrm{mod}})$. Moreover, we obtain the following action (=⊠-symmetry arising from arithmetic):

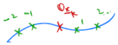
$$\mathsf{Aut}(\mathcal{D}^{\circledcirc})/\mathsf{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \curvearrowright \mathsf{LabCusp}_K \ (= \mathbb{F}_\ell^{\circledast} \curvearrowright \mathbb{F}_\ell^{\circledast}).$$

$\mathbb{F}[\ell](\bar{F})$

$\downarrow$

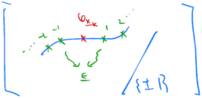
$\circlearrowleft \quad Q \cong \mathbb{F}_\ell$

$\underline{X}_K$:



$\downarrow \pm 1$

$\underline{C}_K$:



$Q \cdot \mathcal{O}_{\underline{x}_K} \setminus \{\mathcal{O}_{\underline{x}_K}\}$

$\cong$

$\mathrm{Cusp}(\underline{X}_K) \setminus \{\mathcal{O}_{\underline{x}_K}\} \cong \mathbb{F}_\ell^\times$

$\downarrow \pm 1$

$\mathrm{LabCusp}_K = \left(\mathrm{Cusp}(\underline{X}_K) \setminus \{\mathcal{O}_{\underline{x}_K}\}\right) / \{\pm 1\}$

$\cong \mathbb{F}_\ell^*$

$\cong \left(Q \cdot \mathcal{O}_{\underline{x}_K} \setminus \{\mathcal{O}_{\underline{x}_K}\}\right) / \{\pm 1\}$

$\mathrm{Aut}(Q) \twoheadrightarrow \mathrm{Aut}(Q)/\{\pm 1\}$ action

$\Rightarrow$ "$\boxtimes$-Symmetry" arising from arithmetic : $\mathbb{F}_\ell^* \cong \mathrm{Aut}(Q)/\{\pm 1\} \curvearrowright \mathrm{LabCusp}_K \cong \mathbb{F}_\ell^*$

$\cong$ by using anabelian geometry.

$\mathrm{Aut}(\mathcal{D}^{\circledcirc}) / \mathrm{Aut}_\epsilon(\mathcal{D}^{\circledcirc})$

a subquotient of $\mathrm{Gal}(K/F_{\mathrm{mod}})$

(Model) $\mathcal{D}$-NF-bridge

Let $\underline{v} \in \underline{\mathbb{V}}$. We put

$$\phi^{\mathrm{NF}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}$$

induced by $\underline{\underline{X}}_{\underline{v}} \to \underline{C}_{v} \to \underline{C}_{K}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and $\underrightarrow{X}_{\underline{v}} \to \underline{C}_{v} \to \underline{C}_{K}$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$. We put (as a poly-morphism (i.e., a set of morphisms))

$$\phi^{\mathrm{NF}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \mathrm{Aut}(\mathcal{D}_{\underline{v}}) : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}.$$

Recall that $\underline{\mathbb{V}}_j$, $j \in \mathbb{F}_\ell^{\circledast}$, a copy of $\underline{\mathbb{V}}$. We write

$$\mathcal{D}_j \stackrel{\mathrm{def}}{=} \{\mathcal{D}_{\underline{v}_j}\}_{\underline{v}_j \in \underline{\mathbb{V}}_j},$$

where $\mathcal{D}_{\underline{v}_j} \cong \mathcal{D}_{\underline{v}}$. Let

$$\phi^{\mathrm{NF}}_1 \stackrel{\mathrm{def}}{=} \{\phi^{\mathrm{NF}}_{\underline{v}_1}\}_{\underline{v}_1 \in \underline{\mathbb{V}}_1} : \mathcal{D}_1 \to \mathcal{D}^{\circledcirc}$$

be the poly-morphism determined by $\phi^{\mathrm{NF}}_{\underline{v}_1}$, $\underline{v}_1 \in \underline{\mathbb{V}}_1$.

Since $\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \cdot \phi_1^{\mathrm{NF}} = \phi_1^{\mathrm{NF}}$ (i.e., stable under the action of $\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})$ by definition), we obtain an action of $\mathbb{F}_\ell^{\circledast} \xrightarrow{\sim} \mathrm{Aut}(\mathcal{D}^{\circledcirc})/\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})$ on ϕ_1^{NF}. Moreover, we put

$$\phi_j^{\mathrm{NF}} \stackrel{\mathrm{def}}{=} j \cdot \phi_1^{\mathrm{NF}} : \mathcal{D}_j \to \mathcal{D}^{\circledcirc}, \ j \in \mathbb{F}_\ell^{\circledast}.$$

We shall call the poly-morphism

$$\phi_{\circledast}^{\mathrm{NF}} \stackrel{\mathrm{def}}{=} \{\phi_j^{\mathrm{NF}}\}_{j \in \mathbb{F}_\ell^{\circledast}} : \mathcal{D}_J \text{ (or } \mathcal{D}_{\circledast}) \stackrel{\mathrm{def}}{=} \{D_j\}_{j \in \mathbb{F}_\ell^{\circledast}} \to \mathcal{D}^{\circledcirc}$$

the (model) $\mathcal{D}$-NF-bridge (recall $J = \mathbb{F}_\ell^{\circledast}$).

(Model) $\mathcal{D}$-Θ-bridge

We put

$$\mathcal{D}_{>} \stackrel{\text{def}}{=} \{\mathcal{D}_{>,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},$$

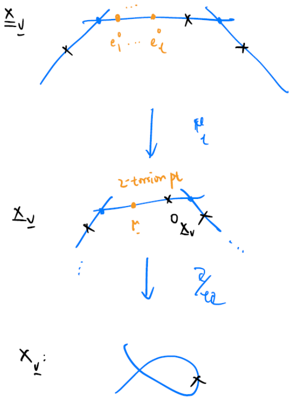
where $\mathcal{D}_{>,\underline{v}} \cong \mathcal{D}_{\underline{v}}$.

Let $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$. Then we have the following morphism

$$\widetilde{\phi}^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j}\ (\cong \mathcal{D}_{\underline{v}}) \stackrel{①}{\twoheadrightarrow} \mathsf{Gal}(\overline{K}_{\underline{v}}/K_{\underline{v}}) \stackrel{②}{\to} \mathcal{D}_{\underline{v}},\ j \in \mathbb{F}_{\ell}^{*},$$

where ① is the natural surjection $\pi_1^{\text{tp}}(\underline{\underline{X}}_{\underline{v}}) \twoheadrightarrow \mathsf{Gal}(\overline{K}_{\underline{v}}/K_{\underline{v}})$. Let us explain ②.

Recall the tempered coverings whose special fibers are as following:



Note that $\{e_1^j, \ldots, e_\ell^j\}_{j \in \mathbb{F}_\ell^*}$ are $K_{\underline{v}}$ -rational points of $\underline{\underline{X}}_{\underline{v}}$. Then we define ② to be "the Galois section determined by a point of $\{e_1^j, \ldots, e_\ell^j\}_{j \in \mathbb{F}_\ell^*}$" (roughly speaking, $\{e_1^j, \ldots, e_\ell^j\}_{j \in \mathbb{F}_\ell^*}$ is a finite approximation of evaluation points explained in §2 and the Galois sections contains the informations of values of $\underline{\underline{\Theta}}_{\underline{v}}$ explained in §2). Then we have information about values of theta functions.

We put (as a poly-morphism)

$$\phi^{\Theta}_{\underline{v}_j} \stackrel{\text{def}}{=} \mathsf{Aut}(\mathcal{D}_{>,\underline{v}}) \circ \widetilde{\phi}^{\Theta}_{\underline{v}_j} \circ \mathsf{Aut}(\mathcal{D}_{\underline{v}_j}) : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}},\ j \in \mathbb{F}_\ell^*.$$

On the other hand, let $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$. We put (as a full poly-isomorphism)

$$\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} (\cong \mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathcal{D}_{>,\underline{v}} (\cong \mathcal{D}_{\underline{v}}),\ j \in \mathbb{F}_\ell^*.$$

Moreover, for global case, we put

$$\phi_j^{\Theta} \overset{\mathrm{def}}{=} \{\phi_{\underline{v}_j}^{\Theta}\}_{\underline{v}_j \in \underline{\mathbb{V}}_j} : \mathcal{D}_j \overset{\mathrm{def}}{=} \{\mathcal{D}_{\underline{v}_j}\}_{\underline{v}_j \in \underline{\mathbb{V}}_j} \to \mathcal{D}_{>} \overset{\mathrm{def}}{=} \{\mathcal{D}_{>,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}.$$

Then we have

$$\phi_{\divideontimes}^{\Theta} \overset{\mathrm{def}}{=} \{\phi_j^{\Theta}\}_{j \in J \overset{\mathrm{def}}{=} \mathbb{F}_\ell^{\divideontimes}} : \mathcal{D}_J \text{ (or } \mathcal{D}_{\divideontimes}) \overset{\mathrm{def}}{=} \{\mathcal{D}_j\}_{j \in J} \to \mathcal{D}_{>}$$

and shall call $\phi_{\divideontimes}^{\Theta}$ the (model) $\mathcal{D}$-Θ-bridge.

Summary

We have $\mathcal{D}$-ΘNF-$\mathcal{HT}$ as following

$$\mathcal{D}_{>} \overset{\mathrm{def}}{=} \{\mathcal{D}_{>,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \overset{\phi^{\Theta}_{*}}{\leftarrow} \mathcal{D}_{J} \overset{\mathrm{def}}{=} \{D_j\}_{j\in J\overset{\mathrm{def}}{=}\mathbb{F}^{*}_{\ell}} \overset{\mathrm{def}}{=} \{\{\mathcal{D}_{\underline{v}_j}\}_{\underline{v}_j\in\underline{\mathbb{V}}_j}\}_{j\in J} \overset{\phi^{\mathrm{NF}}_{*}}{\rightarrow} \mathcal{D}^{\circledcirc}$$

and, for each $j \in J$, the maps of sets of cusps (as $\mathbb{F}^{*}_{\ell}$-torsors)

$$\phi^{\mathrm{LC}}_{j} : \mathsf{LabCusp}(\mathcal{D}^{\circledcirc}) \overset{\sim}{\to} \mathsf{LabCusp}(\mathcal{D}_j) \overset{\sim}{\to} \mathsf{LabCusp}(\mathcal{D}_{>}),\ \underline{\epsilon} \mapsto j.$$

§5: Construction of $\mathcal{D}$-Θ^{ell}-$\mathcal{HT}$

Reference

Section 5 and Section 6 of "S. Mochizuki, Inter-universal Teichmüller theory I: Construction of Hodge theaters. *Publ. Res. Inst. Math. Sci.* **57** (2021), 3–207."

In this section, we explain the left-hand side:

$\mathcal{D}$-$\Theta^{\pm ell}$-HT

$$\begin{array}{ccc}
 & & \mathcal{D}_{\succ} \\
\mathcal{D}\text{-}\Theta^{\pm}\text{-bridge} & \phi^{\Theta^{\pm}}_{\pm} & \uparrow \\
 & & \mathcal{D}_{T} \\
\mathcal{D}\text{-}\Theta^{ell}\text{-bridge} & \phi^{\Theta^{ell}}_{\pm} & \downarrow \\
 & & \mathcal{D}^{\odot\pm}
\end{array}$$

$T \cong \mathbb{F}_l$

with $\mathbb{F}_l^{\rtimes\pm} \overset{\text{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}$

- symmetry arising from geometry (i.e., $\boxplus$-symm.)

$\boxplus$-symmetry (on cusps)

For each $\underline{v} \in \underline{\mathbb{V}}$, we put

$$\mathsf{LabCusp}^{\pm}_{\underline{v}} \stackrel{\mathrm{def}}{=} \text{the set of cusps of } \underline{X}_{\underline{v}}.$$

Then we have the natural action of $\mathrm{Gal}(\underline{X}_{\underline{v}}/\underline{C}_{\underline{v}}) \cong \{\pm 1\}$ on

$$\mathsf{LabCusp}^{\pm}_{\underline{v}} \xrightarrow{\sim} \mathbb{F}_{\ell} = \{\ell^{*}, \ldots, 1, 0, -1, \ldots, -\ell^{*}\}.$$

On the other hand, we put

$$\mathsf{LabCusp}^{\pm}_{K} \stackrel{\mathrm{def}}{=} \text{the set of cusps of } \underline{X}_{K}.$$

Then we may manage the sets of cusps $\{\mathsf{LabCusp}^{\pm}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ via the natural bijection induced by $\underline{X}_{\underline{v}} \to \underline{X}_{K}$:

$$\mathsf{LabCusp}^{\pm}_{\underline{v}} \xrightarrow{\sim} \mathsf{LabCusp}^{\pm}_{K}, \ \underline{v} \in \underline{\mathbb{V}}.$$

We put

$$\mathcal{D}_{\succ} \stackrel{\mathrm{def}}{=} \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},$$

where $\mathcal{D}_{\succ,\underline{v}} \cong \mathcal{D}_{\underline{v}}$. Note that $\mathsf{LabCusp}^{\pm}_{\underline{v}}$ can be mono-anabelian reconstructed from $\mathcal{D}_{\succ,\underline{v}}$. On the other hand, we put

$$\mathcal{D}^{\circledcirc\pm} \stackrel{\mathrm{def}}{=} \pi_1^{\text{ét}}(\underline{X}_K) \text{ (or } \mathcal{B}^{\text{ét}}(\underline{X}_K)^0).$$

Note that $\mathsf{LabCusp}^{\pm}_{K}$ can be mono-anabelian reconstructed from $\mathcal{D}^{\circledcirc\pm}$. Then we obtain a group-theoretical version of the above bijection of cusps:

$$\mathsf{LabCusp}^{\pm}(\mathcal{D}_{\succ}) \xrightarrow{\sim} \mathsf{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm}).$$

We may identify $\mathsf{LabCusp}^{\pm}(\mathcal{D}_{\succ})$ with $\mathsf{LabCusp}^{\pm}_{K}(\mathcal{D}^{\circledcirc\pm})$ via the above bijection. Moreover, there is a natural action

$$\mathsf{Aut}_k(\underline{X}_K)(\cong \mathbb{F}_\ell^{\rtimes\pm} \stackrel{\mathrm{def}}{=} \mathbb{F}_\ell \rtimes \{\pm 1\}) \curvearrowright \mathsf{LabCusp}^{\pm}_{K}(\cong \mathbb{F}_\ell).$$

In fact, the above action can be expressed group-theoretically. We put

$$\mathsf{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \overset{\mathrm{def}}{=} \ker(\mathsf{Aut}(\mathcal{D}^{\circledcirc\pm}) \to \mathbb{F}_\ell^{\circledast}),$$

where the homomorphism is determined by the quotient $E[\ell](\overline{\mathbb{F}}) \to Q$ and $Q \cong \mathbb{F}_\ell$ is introduced in §4, and put

$$\mathsf{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathsf{Aut}(\mathcal{D}^{\circledcirc\pm})$$

the subgroup of automorphisms which fix the cusps of $\underline{X}_K$. Then we have the following action (=⊞-symmetry arising from geometry):

$$(\mathsf{Aut}_K(\underline{X}_K) \cong)\ \mathsf{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathsf{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \curvearrowright \mathsf{LabCusp}_K^{\pm}(\mathcal{D}^{\circledcirc\pm})$$

which can be regarded as

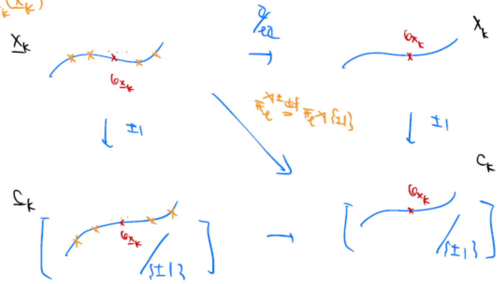
$$\mathbb{F}_\ell^{\rtimes\pm} \curvearrowright \mathbb{F}_\ell$$

by using $\underline{\epsilon}$.

$\mathrm{LabCusp}^{\pm}_{\underline{k}} \overset{\mathrm{def}}{=}$ the set of $\mathrm{Cusp}(\underline{X}_k)$

$\curvearrowleft$ action $\cong \{\ell^* = \frac{\ell-1}{2}, \cdots, 1, 0, -1, \cdots, -\ell^*\}$.

$\mathrm{Aut}_k(\underline{X}_k)$



$\Longrightarrow$ "$\boxplus$-symmetry" : $\mathrm{Aut}_k(\underline{X}_k) \curvearrowright \mathrm{LabCusp}^{\pm}_{\underline{k}}$

$\mathbb{F}_\ell^{\rtimes\pm} \cong \mathrm{Aut}_k(\underline{X}_k)$, $\quad \mathbb{F}_\ell \cong \mathrm{LabCusp}^{\pm}_{\underline{k}}$

arising from geometry

$\mathrm{Aut}_k(\underline{X}_k) \cong \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) / \mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm})$ $\leftarrow$ by using anabelian geometry.

(Model) $\mathcal{D}$-$\Theta^{\pm}$-bridge

Let $T \cong \mathsf{LabCusp}^{\pm}_{K}(\mathcal{D}^{\circledcirc\pm}) \cong \{\ell^{*}, \ldots, 1, 0, -1, \ldots, -\ell^{*}\}$ with action of $\mathbb{F}_{\ell}^{\rtimes\pm}$ and $\underline{\mathbb{V}}_t$, $t \in T$, a copy of $\underline{\mathbb{V}}$. We put a poly-isomorphism

$$\phi^{\Theta^{\pm}}_{\underline{v}_t} : \mathcal{D}_{\underline{v}_t}(\cong \mathcal{D}_{\underline{v}}) \overset{\mathsf{Aut}_{+}(\mathcal{D}^{\circledcirc\pm})\text{-orbit}}{\xrightarrow{\sim}} \mathcal{D}_{\succ,\underline{v}},$$

where, roughly speaking, $\mathsf{Aut}_{+}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathsf{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$ is the subgroup such that $\sigma(\text{"positive labels"}) = \text{"positive labels"}$ for all $\sigma \in \mathsf{Aut}_{+}(\mathcal{D}^{\circledcirc\pm})$. Moreover, we put

$$\phi^{\Theta^{\pm}}_{t} \overset{\text{def}}{=} \{\phi^{\Theta^{\pm}}_{\underline{v}_t}\}_{\underline{v}_t \in \underline{\mathbb{V}}_t} : \mathcal{D}_t \overset{\text{def}}{=} \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v}_t \in \underline{\mathbb{V}}_t} \to \mathcal{D}_{\succ} \overset{\text{def}}{=} \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}.$$

Then we shall put

$$\phi^{\Theta^{\pm}}_{\pm} \overset{\text{def}}{=} \{\phi^{\Theta^{\pm}}_{t}\}_{t \in T} : \mathcal{D}_T \ (\text{or } \mathcal{D}_{\pm}) \overset{\text{def}}{=} \{\mathcal{D}_t\}_{t \in T} \to \mathcal{D}_{\succ}$$

and call $\phi^{\Theta^{\pm}}_{\pm}$ the (model) $\mathcal{D}$-$\Theta^{\pm}$-bridge.

(Model) $\mathcal{D}$-Θ^{ell}-bridge

For $\underline{v} \in \underline{\mathbb{V}}$, we put

$$\phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc\pm}$$

the morphism determined by the natural morphism $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and $\underline{\underrightarrow{X}}_{\underline{v}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$. Write

$$\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \circ \phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} \circ \mathrm{Aut}_{+}(\mathcal{D}_{\underline{v}_0}) : \mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm},$$

$$\phi^{\Theta^{\mathrm{ell}}}_{0} \stackrel{\mathrm{def}}{=} \{\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0}\}_{\underline{v}_0 \in \underline{\mathbb{V}}_0} : \mathcal{D}_0 \stackrel{\mathrm{def}}{=} \{\mathcal{D}_{\underline{v}_0}\}_{\underline{v}_0 \in \underline{\mathbb{V}}_0} \to \mathcal{D}^{\circledcirc\pm}.$$

Note that since

$$t \in T \ (\cong \mathbb{F}_\ell) \subseteq \mathbb{F}_\ell^{\rtimes\pm} \cong \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{cusp}}(\mathcal{D}^{\circledcirc\pm}) \curvearrowright \phi^{\Theta^{\mathrm{ell}}}_{0},$$

we put

$$\phi^{\Theta^{\mathrm{ell}}}_{t} \stackrel{\mathrm{def}}{=} t \cdot \phi^{\Theta^{\mathrm{ell}}}_{0} : \mathcal{D}_t \stackrel{\mathrm{def}}{=} \{D_{\underline{v}_t}\}_{\underline{v}_t \in \underline{\mathbb{V}}_t} \to \mathcal{D}^{\circledcirc\pm}.$$

We shall put

$$\phi_{\pm}^{\Theta^{\mathrm{ell}}} \overset{\mathrm{def}}{=} \{\phi_t^{\Theta^{\mathrm{ell}}}\}_{t\in T} : \mathcal{D}_T \overset{\mathrm{def}}{=} \{\mathcal{D}_t\}_{t\in T} \to \mathcal{D}^{\circledcirc\pm}$$

and call $\phi_{\pm}^{\Theta^{\mathrm{ell}}}$ the (model) $\mathcal{D}$-Θ^{ell}-bridge.

Summary

We have $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-$\mathcal{HT}$ as following:

$$\mathcal{D}_{\succ} \overset{\mathrm{def}}{=} \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \overset{\phi_{\pm}^{\Theta^{\pm}}}{\longleftarrow} \mathcal{D}_T \overset{\mathrm{def}}{=} \{D_t\}_{t\in T(\cong\mathbb{F}_\ell)} \overset{\mathrm{def}}{=} \{\{\mathcal{D}_{\underline{v}_t}\}_{\underline{v}_t\in\underline{\mathbb{V}}_t}\}_{t\in T} \overset{\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\longrightarrow} \mathcal{D}^{\circledcirc\pm}.$$

Note that we do not have any information about theta functions by the definition of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-$\mathcal{HT}$. To obtain that, we need to "glue" $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-$\mathcal{HT}$ with $\mathcal{D}$-ΘNF-$\mathcal{HT}$.

§6: $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters

Reference

Section 6 of "S. Mochizuki, Inter-universal Teichmüller theory I: Construction of Hodge theaters. *Publ. Res. Inst. Math. Sci.* **57** (2021), 3–207."

$\mathcal{D}$-ΘNF-Hodge theater

We shall call

$$^{\dagger}\mathcal{D}_{>} \overset{\phi_{*}^{\Theta}}{\leftarrow} {}^{\dagger}\mathcal{D}_{J} \overset{\phi_{*}^{\mathrm{NF}}}{\rightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc}$$

a $\mathcal{D}$-ΘNF-Hodge theater if it is "isomorphic" to (i.e., poly-isomorphisms $^{\dagger}\mathcal{D}_{>} \xrightarrow{\sim} \mathcal{D}_{>}$, $^{\dagger}\mathcal{D}_{J} \xrightarrow{\sim} \mathcal{D}_{J}$, $^{\dagger}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} \mathcal{D}^{\circledcirc}$ satisfy certain compatible conditions)

$$\mathcal{D}_{>} \overset{\phi_{*}^{\Theta}}{\leftarrow} \mathcal{D}_{J} \overset{\phi_{*}^{\mathrm{NF}}}{\rightarrow} \mathcal{D}^{\circledcirc}.$$

$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater

We shall call

$$^{\dagger}\mathcal{D}_{\succ} \overset{\phi_{\pm}^{\Theta^{\pm}}}{\leftarrow} {}^{\dagger}\mathcal{D}_{T} \overset{\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\rightarrow} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater if it is "isomorphic" to (i.e., poly-isomorphisms $^{\dagger}\mathcal{D}_{\succ} \overset{\sim}{\to} \mathcal{D}_{\succ}$, $^{\dagger}\mathcal{D}_{T} \overset{\sim}{\to} \mathcal{D}_{T}$, $^{\dagger}\mathcal{D}^{\circledcirc\pm} \overset{\sim}{\to} \mathcal{D}^{\circledcirc\pm}$ satisfy certain compatible conditions)

$$\mathcal{D}_{\succ} \overset{\phi_{\pm}^{\Theta^{\pm}}}{\leftarrow} \mathcal{D}_{T} \overset{\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\rightarrow} \mathcal{D}^{\circledcirc\pm}.$$

$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater

We put the following identifications (i.e., "a gluing")

$$(T \setminus \{0\})/\{\pm 1\} = J.$$

Then we can construct a $\mathcal{D}$-Θ-bridge

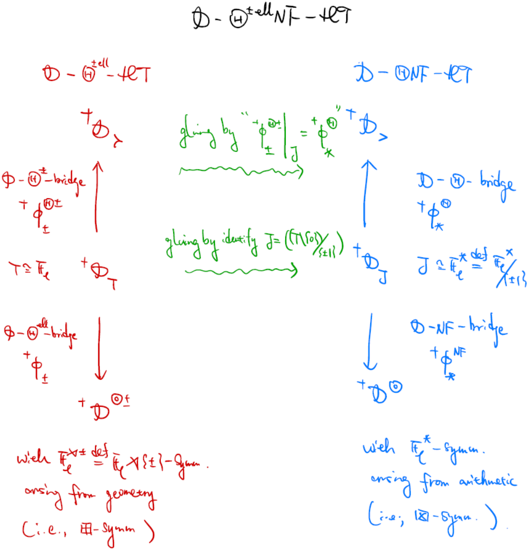
$$^{\dagger}\phi^{\Theta}_{\divideontimes}[^{\dagger}\phi^{\Theta^{\pm}}_{\pm}] : {}^{\dagger}\mathcal{D}_J \stackrel{\mathrm{def}}{=} \{\mathcal{D}_t\}_{t\in J} \to {}^{\dagger}\mathcal{D}_{>}$$

from any $\mathcal{D}$-$\Theta^{\pm}$-bridge $^{\dagger}\phi^{\Theta^{\pm}}_{\pm} : {}^{\dagger}\mathcal{D}_T \to {}^{\dagger}\mathcal{D}_{\succ}$. Then we shall call a triple

$$(\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-}\mathcal{HT}, \mathcal{D}\text{-}\Theta\mathrm{NF}\text{-}\mathcal{HT}, {}^{\dagger}\phi^{\Theta}_{\divideontimes}[^{\dagger}\phi^{\Theta^{\pm}}_{\pm}] \cong {}^{\dagger}\phi^{\Theta}_{\divideontimes})$$

$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater.

Then we obtain the following diagram mentioned above.



Frobenioids

Let $\mathcal{D}$ be a $\mathcal{D}$-prime-strip (i.e., a fundamental group or (the subcategory of connected objects of) a Galois category). We put

$$\mathfrak{F}$$

the Frobenioid whose base is $\mathcal{D}$. Roughly speaking, $\mathfrak{F}$ is a category over $\mathcal{D}$ whose objects are "rational functions" on objects (i.e., coverings) of $\mathcal{D}$.

$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater

We obtain the following which is called $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater, and whose base is $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater:

$\Theta^{\pm ell}NF$-HT

$\Theta^{\pm ell}$-HT

${}^{\dagger}\mathfrak{F}_{\succ}$

$\Theta^{\pm}$-bridge $\quad {}^{\dagger}\psi^{\Theta^{\pm}}_{\pm} \uparrow$

${}^{\dagger}\mathfrak{F}_{T}$

Θ^{ell}-bridge $\quad {}^{\dagger}\psi^{\Theta^{ell}}_{\pm} \downarrow$

${}^{\dagger}\mathfrak{D}^{\odot\pm}$

with $\boxplus$-symm.

gluing by "${}^{\dagger}\psi^{\Theta^{\pm}}_{\pm}\big|_{J} = {}^{\dagger}\psi^{\Theta}_{*}$"
$\rightsquigarrow$

gluing by $J = \mathbb{F}_l^{\times}/\{\pm 1\}$
$\rightsquigarrow$

ΘNF-HT

${}^{\dagger}\mathfrak{F}_{>}$

$\uparrow {}^{\dagger}\psi^{\Theta}_{*} \quad$ Θ-bridge

${}^{\dagger}\mathfrak{F}_{J}$

$\downarrow {}^{\dagger}\psi^{NF}_{*} \quad$ NF-bridge

${}^{\dagger}\mathfrak{F}^{\circledcirc}$

with $\boxtimes$-symm.

$\vdots$

$\mathfrak{F}^{\circledast}$

Thank you for the attention !