Proof of the Density Threshold Conjecture for Pinwheel Scheduling

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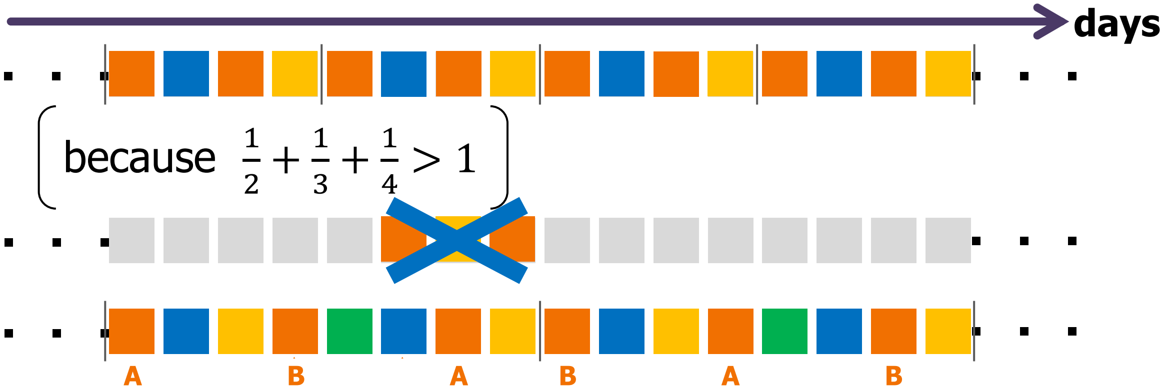
henceforth $a_1 \leq a_2 \leq \cdots \leq a_k$

Pinwheel scheduling [HMRTV89]

Each task $i = 1, \dots, k$ must be done **at least once in any a_i consecutive days**.
Can we achieve this by doing one task every day?

If we can, the k-tuple $A = (a_1, \dots, a_k)$ is said to be **schedulable**.

Examples



$(2, 4, 4)$ YES

$(2, 3, 4)$ NO because $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$

$(2, 3, 100)$ NO

$(3, 4, 5, 8)$ YES

$(3, 4, 6, 8)$ YES

$(4, 5, 6, 6, 8)$ YES

A B

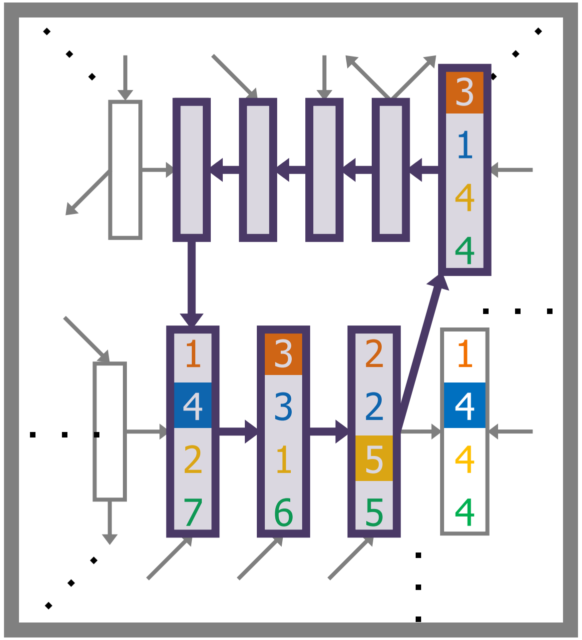
Schedulability is preserved by

RELAX replacing a number with a bigger one

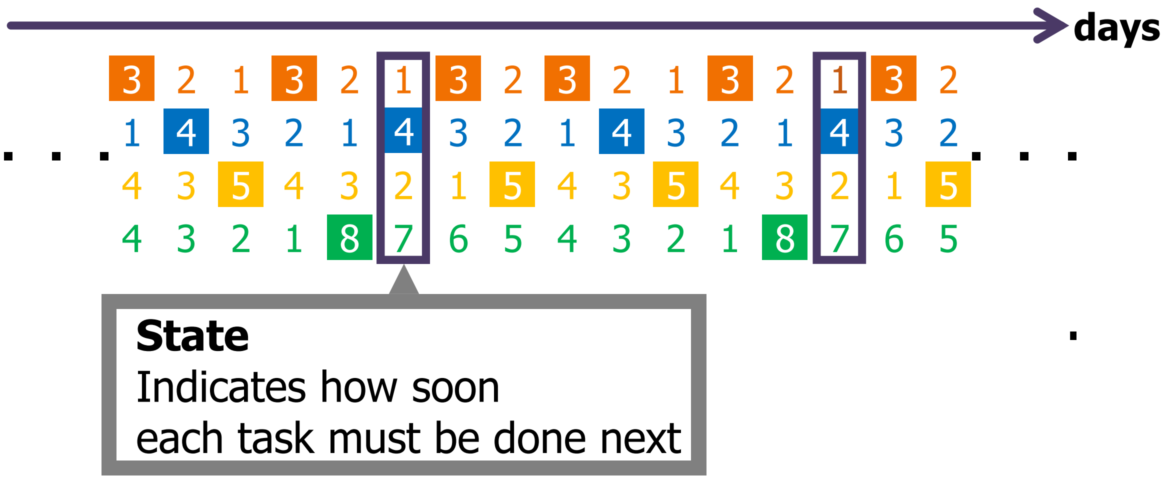
SPLIT replacing a number with two copies of its double
(or N copies of its N-fold)

[HMRTV89] R. Holte, A. Mok, L. Rosier, I. Tulchinsky, D. Varvel. The pinwheel: a real-time scheduling problem. In Proc. 22nd Hawaii International Conference on System Sciences, pp. 693–702, 1989.

Schedulability can be tested in **PSPACE**:
Build the state transition graph and check for a cycle.



Graph for $(3, 4, 5, 8)$
(with $\leq 3 \cdot 4 \cdot 5 \cdot 8$ vertices)



3	2	1	3	2	1	3	2	3	2	1	3	2	1	3	2
1	4	3	2	1	4	3	2	1	4	3	2	1	4	3	2
4	3	5	4	3	2	1	5	4	3	5	4	3	2	1	5
4	3	2	1	8	7	6	5	4	3	2	1	8	7	6	5

State
Indicates how soon
each task must be done next

Definition

The **density** of $A = (a_1, \dots, a_k)$ is $\mathrm{D}(A) = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$.

For A to be schedulable, $\mathrm{D}(A) \le 1$ is necessary.
This is not always sufficient, but:

Theorem [HMRTV89]

A is schedulable if $\mathrm{D}(A) \le 1$ and each number in A divides the next.

Proof: Reduce to a single kind of tasks.

Example $\underset{\text{YES}}{(4,8,8,16,16)} \xleftarrow{\text{SPLIT}} \underset{\text{YES}}{(4,8,8,8)} \xleftarrow{\text{SPLIT}} \underset{\text{YES}}{(4,4,4)}$

Theorem [HRTV92]

A is schedulable if $\mathrm{D}(A) \le 1$ and A has only two distinct numbers.

Corollary [HMRTV89]

A is schedulable if $\mathrm{D}(A) \le \frac{1}{2}$.

Can we make this number bigger?

Proof: Round down to powers of 2.

Example $\underset{\text{YES}}{(6,12,13,24,25)} \xleftarrow{\text{RELAX}} \underset{\text{YES}}{(4,8,8,16,16)}$

Using these theorems more cleverly, we get "thriftier" rounding-down methods that improve the corollary (next page)

[HMRTV89] R. Holte, A. Mok, L. Rosier, I. Tulchinsky, D. Varvel. The pinwheel: a real-time scheduling problem. In Proc. 22nd Hawaii International Conference on System Sciences, pp. 693–702, 1989.

[HRTV92] R. Holte, L. Rosier, I. Tulchinsky, D. Varvel. Pinwheel scheduling with two distinct numbers. Theoretical Computer Science 100, 105–135, 1992.

Theorem (restated) [HMRTV89]

A is schedulable if $\mathrm{D}(A) \leq \frac{1}{2}$.

$(= 0.5)$

Theorem [CC93]

A is schedulable if $\mathrm{D}(A) \leq \frac{2}{3}$.

$(= 0.666\ldots)$

Theorem [CC92]

A is schedulable if $\mathrm{D}(A) \leq \frac{7}{10}$.

$(= 0.7)$

Theorem [FL02]

A is schedulable if $\mathrm{D}(A) \leq \frac{3}{4}$.

$(= 0.75)$

Theorem (this talk)

~~**Conjecture**~~ [CC93]

A is schedulable if $\mathrm{D}(A) \leq \frac{5}{6}$.

$(= 0.833\ldots)$

(Best possible,
because of $(2, 3, \bullet)$)

Partial progress:

Theorem [LL97]

Conjecture is true when
A has only three distinct numbers.

Theorem [FL02]

Conjecture is true
when $a_1 = 2$.

Theorem [GSW22]

Conjecture is true
when $k \leq 12$.

▭ involve computers

[CC92] M.Y. Chan, F. Chin. General schedulers for the pinwheel problem based on double-integer reduction. IEEE Transactions on Computers 41, 755–768, 1992.

[CC93] M.Y. Chan, F. Chin. Schedulers for larger classes of pinwheel instances. Algorithmica 9, 425–462, 1993.

[FL02] P.C. Fishburn, J.C. Lagarias. Pinwheel scheduling: achievable densities. Algorithmica 34, 14–38, 2002.

[GSW22] L. Gąsieniec, B. Smith and S. Wild. Towards the 5/6-density conjecture of pinwheel scheduling. In Proc. SIAM Symposium on Algorithm Engineering and Experiments (ALENEX), pp. 91–103, 2022.

[HMRTV89] R. Holte, A. Mok, L. Rosier, I. Tulchinsky, D. Varvel. The pinwheel: a real-time scheduling problem. In Proc. 22nd Hawaii International Conference on System Sciences, pp. 693–702, 1989.

[LL97] S. Lin, K. Lin. A pinwheel scheduler for three distinct numbers with a tight schedulability bound. Algorithmica 19, 411–426, 1997.

Main Theorem (restated)

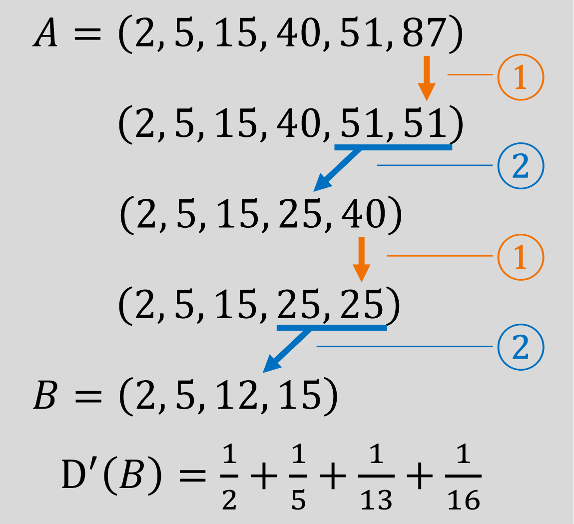
A is schedulable if $\mathrm{D}(A) \le \frac{5}{6}$.

Lemma

Verified by computer
(using transition graphs)

B is schedulable if $\mathrm{D}'(B) < \frac{5}{6} + \frac{1}{22}$ and B consists only of elements < 22.

Modified density D', defined by regarding $11, \dots, 21$ as $12, \dots, 22$



$A = (2, 5, 15, 40, 51, 87)$

$\downarrow$ ①

$(2, 5, 15, 40, \underline{51, 51})$

$\swarrow$ ②

$(2, 5, 15, 25, 40)$

$\downarrow$ ①

$(2, 5, 15, \underline{25, 25})$

$\swarrow$ ②

$B = (2, 5, 12, 15)$

$\mathrm{D}'(B) = \frac{1}{2} + \frac{1}{5} + \frac{1}{13} + \frac{1}{16}$

Proof of Theorem, assuming Lemma

Suppose that A was unschedulable.

Let B be the result of repeatedly applying ①② to A until all elements are ≤ 22.

① If we have a single biggest element, decrease it.

② If we have two copies of the biggest element, say a, a, replace them by one $\left\lfloor \frac{a}{2} \right\rfloor$.

Then B is unschedulable and $\mathrm{D}'(B) < \mathrm{D}(A) + \frac{1}{22} \le \frac{5}{6} + \frac{1}{22}$, contradicting the Lemma.

Because ①② are inverse operations of **RELAX** **SPLIT**

Main

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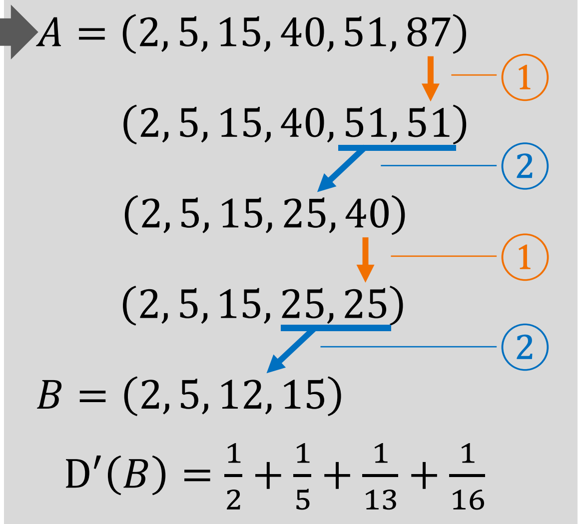

$A = (2, 5, 15, 40, 51, 87)$ ①

$(2, 5, 15, 40, \underline{51, 51})$ ②

$(2, 5, 15, 25, 40)$ ①

$(2, 5, 15, \underline{25, 25})$ ②

$B = (2, 5, 12, 15)$

$\mathrm{D}'(B) = \frac{1}{2} + \frac{1}{5} + \frac{1}{13} + \frac{1}{16}$

Proof

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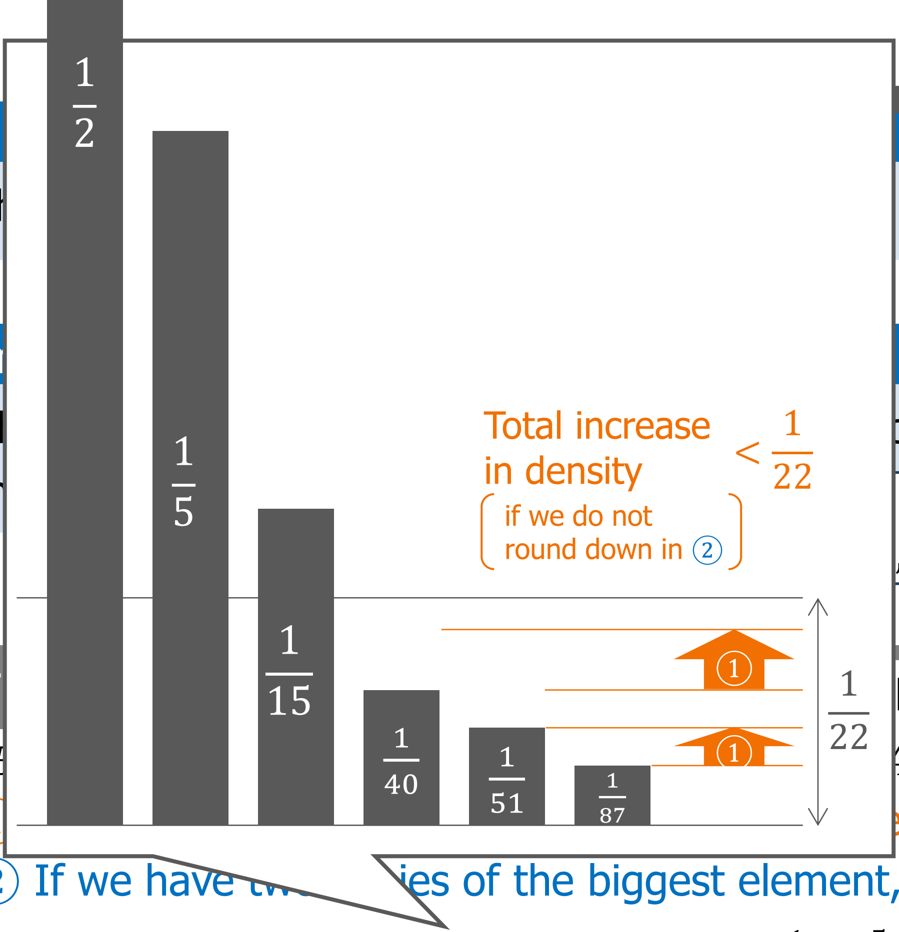

Total increase in density $< \frac{1}{22}$

(if we do not round down in ②)

Because ①② are inverse operations of RELAX SPLIT

Future work

Proof of density bounds without brute-force search

Complexity of deciding schedulability

- Is it in $\mathbf{NP}$? (Note: some instances admit only super-polynomially long solutions)

  We know it is in $\mathbf{PSPACE}$, by the method of transition graphs

- Is it $\mathbf{NP}$-hard?

  We know it is $\mathbf{NP}$-hard if we can write "l copies of a", with l in binary, in the input

The covering version ("point patrolling" [KS20])

What if you can schedule task i **at most** once per a_i days (and must cover all days)?

- Lowest density guarantee lies somewhere between $1.264\ldots$ and 1.4125

Optimization and more applied versions of the problem

- Bamboo garden trimming [GJKLLMR24]:
  Minimize the maximum factor by which we violate the frequency requirements
- Implications for various applied settings

[GJKLLMR24] L. Gąsieniec, T. Jurdziński, R. Klasing, C. Levcopoulos, A. Lingas, J. Min, T. Radzik. Perpetual maintenance of machines with different urgency requirements. Journal of Computer and System Sciences 139, 103476, 2024.

[KS20] A. Kawamura, M. Soejima. Simple strategies versus optimal schedules in multi-agent patrolling. Theoretical Computer Science 839, 195–206, 2020.