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On non-linear ordinary and evolution equations

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Recently F. Browder [1] and T. Kato [5] proved the existence and uniqueness of (mild) solutions of the ordinary differential equation

(1) $$\frac{du}{dt} = f(t,u)$$

as well as of the evolution equation

(2) $$\frac{du}{dt} = A(t)u + f(t,u),$$

where the unknown function $u(t)$ takes values in a Hilbert space H, and $\{A(t)\}$ are densely defined closed linear operators of hyperbolic type. Their mothods are based on the monotonicity argument; i.e. they assume that $f(t,u)$ satisfies the monotonicity condition

(3) $$\mathrm{Re}\ (f(t,u) - f(t,v),\ u-v) \leqq M\,\|u-v\|^2 ,$$

and use this property rather extensively. For proving the existence of a solution of the ordinary differential equation (1) they use Peano's existence theorem for the finite dimensional spaces and the so-called Galerkin's method of approximating the space H by a sequence of increasing finite dimensional spaces.

Instead of assuming the monotonicity of $f(t,u)$ we impose a more general condition as follows. Let I_T be an interval $t_0 \leqq t \leqq t_0 + T$ and X be a Banach space. Suppose that there exists a real valued continuous function $\Phi(t,u,v)$ on a set D

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in $I_T \times X \times X$ which satisfies some of the following properties:[1]

(P$_1$) $\Phi(t,u,v) > 0$ if $u \neq v$; $= 0$ if $u = v$.

(P$_2$) $\Phi(t,u_n,v_n) \to 0$ imples $u_n - v_n \to 0$ for each t.

(P$_2'$) $\Phi(t,u_n,v_n) \to 0$ imples $u_n - v_n \to 0$ uniformly in t.

(P$_3$) $\Phi(t,u,v)$ is continuously (Fréchet) differentiable

and

$$D_f\Phi = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial u}f(t,u) + \frac{\partial\Phi}{\partial v}f(t,v) \leqq 0.$$

(P$_4$) For any positive number M, $\frac{\partial\Phi}{\partial t}$, $\frac{\partial\Phi}{\partial u}x$, $\frac{\partial\Phi}{\partial v}x$ are continuous in (u,v) uniformly for $(t,u,v) \in D$ and $|x| \leqq M$.

We see, in particular, that if f satisfies the monotonicity condition (3) then $\Phi(t,u,v)$ defined by $\Phi(t,u,v) = e^{-2Mt}|u-v|^2$ satisfies all these conditions on $I_T \times X \times X$. On the other hand, the example 3 of Section 1 shows that our condition is more general than the monotonicity. For proving the existence of solutions of the ordinary differential equation (1), we don't need to use Galerkin's method. Instead, we approximate the solution by a family of suitable piecewise smooth curves, and our proofs are short and elementary.

For the evolution equation (2), however, our results are by no means satisfactory. To apply our method to the evolution equation we have to assume one more condition, namely,

(P$_5$) $\frac{\partial\Phi}{\partial u} A(t)u + \frac{\partial\Phi}{\partial v} A(t)v \leqq 0.$

Then we can prove the existence of a mild solution of (2) under some conditions on $f(t,u)$.

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1) Originally introduced by Okamura for classical ordinary differential equations about thirty years ago [9], [10].

1. Uniquness and stability of the solution

Let X be a normed space, I_T an interval $[t_o, t_o + T]$, $D(t)$, $t \in I_T$, a subset of X, and $D = \{(t,u);\ t \in I_T,\ u \in D(t)\}$.

Consider the equation

$$(1) \qquad \frac{du}{dt} = f(t,u),$$

where $f(t,u)$ is defined on D. Note that $f(t,u)$ need not be continuous.

We say that $u(t)$ is a solution of (1) on I_T if $u(t)$ is defined on I_T, taking values in X, strongly differentiable in t for $t_o < t \leqq t_o + T$, and satisfying the equation on $t_o < t \leqq t_o + T$. If, moreover, $u(t)$ is continuous on I_T and satisfies $u(t_o) = u_o$ then $u(t)$ is said to be a solution of (1) on I_T taking the initial value u_o.

Theorem 1. Let u_o be in $\overline{D(0)}$, $f(t,u)$ be defined on the set $\{(t,u);\ t_o < t \leqq t_o + T,\ u \in D(t)\}$. Suppose that there exists a real valued continuous function $\Phi(t,u,v)$ on some neighbourhood of $\Delta = \{(t,u,v);\ (t,v) \in D,\ (t,v) \in D\}$ with the properties (P_1) and (P_3). Then there is at most one solution of (1) taking the initial value u_o. Furthermore, if Φ satisfies (P_2) then $u(t)$ depends continuously on u_o for each t.

Proof. Suppose that there exist two solutions $u_1(t)$, $u_2(t)$ satisfying (1) and $u_1(t_o) = u_2(t_o) = u_o$. Then we have

$$\frac{d}{dt}\Phi(t,u_1(t),u_2(t)) = \left[\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial u} f(t,u_1(t)) + \frac{\partial\Phi}{\partial v} f(t,u_2(t))\right]_{\substack{u=u_1(t)\\ v=u_2(t)}}$$

$$\leqq 0 \quad \text{for } t > t_o.$$

Hence

$$(4) \qquad \Phi(t,u_1(t),u_2(t)) - \Phi(t_o,u_1(t_o),u_2(t_o)) \leqq 0.$$

Since $\Phi(t_o,u_1(t_o),u_2(t_o)) = \Phi(t_o,u_o,u_o) = 0$, we see that $\Phi(t,u_1(t),u_2(t)) = 0$. Therefore $u_1(t) = u_2(t)$.

Since $\Phi(t,u,v)$ is continuous, $|u_1(t_o) - u_2(t_o)| \to 0$ implies $\Phi(t_o,u_1(t_o),u_2(t_o)) \to 0$. This implies $\Phi(t,u_1(t),u_2(t)) \to 0$ by (4). If Φ satisfies (P_2), then we have $|u_1(t) - u_2(t)| \to 0$.

<u>Remark.</u> We could weaken the condition (P_3) to the following:

(P_3') $\Phi(t,u(t),v(t))$ is monotone-increasing in t for any two solutions $u(t)$ and $v(t)$ of (1).

Then, it turns out that the existence of such a continuous function satisfying (P_1), (P_2) and (P_3') is also a necessary condition for uniqueness of the solution if $f(t,u)$ is assumed to be bounded and continuous, but we shall not go into the details here.

<u>Examples</u>: 1. If X is a Hilbert space and $f(t,u)$ satisfies the monotonicity condition (3) then, as mentioned in the introduction, $\Phi(t,u,v) = e^{-2Mt}|u-v|^2$ satisfies our conditions.

2. If $A(t)$ is of hyperbolic type then we have the dissipative property

$$\mathrm{Re}(A(t)u,\ u) \leqq 0,$$

so that if we consider

$$\frac{du}{dt} = f(t,u) = A(t)u + g(t,u)$$

where $g(t,u)$ satisfies the monotonicity condition (3), then $f(t,u)$ also satisfies (3) and hence we have the uniqueness of the solution.

3. Let $X = R^1$. Consider the differential equation

$$\frac{du}{dt} = f(t,u) = \begin{cases} 1 - \sqrt{u} & u \geqq 0 \\ 1 & u < 0. \end{cases}$$

Then the function $f(t,u)$ <u>does not</u> satisfy Lipschitz' condition but <u>does</u> satisfy the monotonicity condition. On the other hand, when we consider

$$\frac{du}{dt} = f(t,u) = \begin{cases} 1 + \sqrt{u} & u \geqq 0 \\ 1 & u < 0, \end{cases}$$

we see that the second member of this differential equation <u>does not</u> satisfy the monotonicity condition. But there <u>does</u> exist a function $\Phi(t,u,v)$ which satisfies our conditions (P_1) - (P_4) for this equation, namely,

$$\Phi(t,u,v) = \begin{cases} (\sqrt{u} - \sqrt{v} - \log(1+\sqrt{u}) + \log(1+\sqrt{v}))^2 & u \geqq 0, \quad v \geqq 0 \\ (\sqrt{u} - \log(1+\sqrt{u}) - \frac{1}{2}v)^2 & u \geqq 0, \quad v < 0 \\ (\frac{1}{2}u - \sqrt{v} + \log(1+\sqrt{v}))^2 & u < 0, \quad v \geqq 0 \\ \frac{1}{4}(u - v)^2 & u < 0, \quad v < 0, \end{cases}$$

say.

<u>4</u>. Consider the following simple parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(t,x,u)$$

on a region bounded by $t=t_o$, $t=t_o+T$, $x=\lambda_1(t)$ and $x=\lambda_2(t)$, where $\lambda_1(t)<\lambda_2(t)$ for $t_o<t\leqq t_o+T$. The initial and boundary conditions are: $u=g(x)$ on $t=t_o$, $u=h_1(t)$ and $u=h_2(t)$ on $x=\lambda_1(t)$ and $x=\lambda_2(t)$ respectively, where g, h_1 and h_2 are continuous and $g(\lambda_1(t_o)) = h_1(t_o)$, $g(\lambda_2(t_o)) = h_2(t_o)$. It is known[1)] that if $F(t,x,u_1) - F(t,x,u_2) \leqq K(u_1 - u_2)$ holds for $u_1 > u_2$ then the solution is unique. By the argument of Theorem 1 it can be proved as follows. Let $D(t)$ be a set of all functions in x which are continuous on $\lambda_1(t)<x<\lambda_2(t)$, belong to C^2 on $\lambda_1(t)<x<\lambda_2(t)$, and take values $h_1(t)$ and $h_2(t)$ at $x=\lambda_1(t)$ and $x=\lambda_2(t)$ respectively.

Define $\Phi(t,u,v) = e^{-2Kt}\int_{\lambda_1(t)}^{\lambda_2(t)} |u(x) - v(x)|^2 dx$, and

1) [6], [7].

$f(t,u) = \frac{d^2u(x)}{dx^2} + F(t,x,u(x))$ for $u \in D(t)$. Then we have

$$D_f \Phi = \int_{\lambda_1(t)}^{\lambda_2(t)} [\frac{d^2u(x)}{dx^2} + F(t,x,u(x)) - \frac{d^2v(x)}{dx^2} - F(t,x,v(x))](u(x)-v(x))dx$$

$$- K\int_{\lambda_1(t)}^{\lambda_2(t)} |u(x) - v(x)|^2 dx \leq - \int_{\lambda_1(t)}^{\lambda_2(t)} (\frac{d(u(x) - v(x))}{dx})^2 dx \leq 0,$$

so that the solution is unique.

Let X be a Banach space and $f(t,u)$ be a mapping from $I_T \times X$ to X. We say that $f(t,u)$ is demi-continuous[1] if it is continuous from $I_T \times X$ with the strong topology to X with the weak topology.

Theorem 2. Let X be a Banach space and $f(t,u)$ be defined and demi-continuous on a set $D \subset \{(t,u);\ t_0 < t \leq t_0+T,\ u \in X\}$ for which we assume that $D(t) = \{u\,;(t,u) \in D\}$ is closed in X for each t in $t_0 < t \leq t_0+T$. Furthermore we assume that $f(t,u)$ is dominated by a summable function $M(t)$ in such a way that $|f(t,u)| \leq M(t)$ for $(t,u) \in D$. Suppose that there exists a real valued function $\Phi(t,u,v)$ on a neighbourhood of the set $\Delta = \{(t,u,v);\ (t,u) \in D,\ (t,v) \in D\}$ and satisfying (P_1), (P_2) and (P_3). Assume that there is a continuous curve $u = \psi(t)$ in D for which $\sup_{u \in D(t)} \Phi(t, \psi(t), u) \to 0$ as $t \to t_0$. Assume furthermore that for t_1 in $t_0 < t_1 < t_0+T$ the equation (1) has solutions starting to the right from $(t_1, \psi(t_1))$ reaching to the plane $t = t_0+T$ in D. Then there is a unique solution $u(t)$ of (1) with $u(t_0) = u_0 = \psi(t_0)$.

Proof. Denote by $u = \varphi(t;\ t_1)$ a solution starting from $(t_1, \psi(t_1))$. Then if we take $t_0 < t_1 < t_2$ we have

$$\frac{d}{dt} \Phi(t, \varphi(t;\ t_1), \varphi(t;\ t_2)) \leq 0 \quad \text{for} \quad t_2 \leq t \leq t_0+T,$$

so that

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1) [1].

$$\Phi(t,\varphi(t;\,t_2),\varphi(t;\,t_1)) \leqq \Phi(t_2,\varphi(t_2;\,t_2),\varphi(t_2;\,t_1))$$
$$= \Phi(t_2;\psi(t_2),\ \varphi(t_2;\,t_1)) \to 0 \quad \text{as} \quad t_2 \to t_0.$$

Thus $|\varphi(t;\,t_1) - \varphi(t;\,t_2)| \to 0$ if $t_1,\ t_2 \to t_0$. Let $\varphi(t) = \lim_{t_1 \to t_0} \varphi(t;\,t_1)$. Then $(t,\varphi(t)) \in D$ and $\varphi(t)$ is a solution of (1), since for arbitrary $\hat{t}\ (> t_0)$ we have

$$\varphi(t;\,t_1) - \varphi(\hat{t};\,t_1) = \int_{\hat{t}}^{t} f(\tau,\varphi(\tau;\,t_1))d\tau \quad t_0 < t_1 < t,$$

and hence by letting $t_1 \to t_0$ we get

$\varphi(t) - \varphi(\hat{t}) = \int_{\hat{t}}^{t} f(\tau,\varphi(\tau))d\tau$. Thus φ is a solution of (1) in D. Uniqueness of the solution is obvious, since if $\chi(t)$ is such a solution then

$$\Phi(t,\varphi(t;\,t_1),\chi(t)) \leqq \Phi(t_1,\varphi(t_1;\,t_1),\chi(t_1))$$
$$= \Phi(t_1,\psi(t_1),\chi(t_1)) \to 0$$

shows that $\varphi(t;\,t_1) \to \chi(t_0)$.

Example.[1] Let H be a Hilbert space and $f(t,u)$ be a continuous function on $E = \{(t,u);\ t_0 < t \leqq t_0 + T,\ |u - u_0| \leqq c\}$ to H. Suppose that

i) $f(t,u) \to 0$ as $(t,u) \to 0$ in E,

ii) $\mathrm{Re}(f(t,u) - f(t,v),\ u - v) \leqq \frac{1}{t - t_0}|u - v|^2$ on E.

Then there is a unique solution $\varphi(t)$ of (1) such that $\varphi(t) \to u_0$ as $t - t_0$.

In fact, take $\Phi(t,u,v) = \frac{1}{(t - t_0)^2}|u - v|^2$. Then $D_f\Phi \leqq 0$ on $\{(t,u,v);\ (t,u) \in E,\ (t,v) \in E\}$. Let $\psi(t) \equiv u_0$. We may assume by the condition i) that $|f(t,u)| \leqq m$ if $t_0 < t \leqq t_0 + T_1\ (\leqq t_0 + T)$, $|u - u_0| \leqq c_1\ (\leqq c)$ and also $mT_1 \leqq c_1$.

Now put

$$\lambda(t) = \max\{|f(t,u)|\ ;\ |u - u_0| \leqq m|t - t_0|\}.$$

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1) A generalization and a proof along [9] of Nagumo's uniqueness condition [8].

Then $\lambda(t) \to 0$ as $t \to t_o$. Set

$$\mu(t) = \int_{t_0}^{t} \lambda(\tau)d\tau .$$

Then $D = \{(t,u) ;\ t_o < t \leqq t_o + T_1,\ |u - u_o| \leqq \mu(t)\}$ is contained in E and

$$\sup_{u \in D(t)} \Phi(t, \psi(t), u) = \frac{1}{(t - t_o)^2} |\mu(t)|^2 \to 0 \text{ as } t \to t_o.$$

Since there exists a solution starting to the right from $(t_1, \psi(t_1))$, for each t_1 in $t_o < t_1 < t_o + T_1$, reaching to the plane $t = t_o + T_1$ by the Theorem 3 below, we obtain the result by applying the previous theorem.

2. Existence theorems for ordinary differential equations.

Let X be a Banach space and $f(t,u)$ be a mapping from $I_T \times X$ to X. We assume that $f(t,u)$ sends bounded sets into bounded sets. Then for any $b > 0$ there exists a positive number M such that $|f(t,u)| \leqq M$ for $(t,u) \in I_T \times V_b$, where $V_b = \{u \in X;\ |u - u_o| \leqq b\}$. Let $a > 0$ be such that $aM \leqq b$.

Theorem 2. Suppose that $f(t,u)$ is continuous on $I_a \times V_b$ and that there exists a real valued continuous function $\Phi(t,u,v)$ on $I_a \times V_b \times V_b$ satisfying conditions (P_1), (P_2), (P_3) and (P_4). Then (1) has a solution on I_a taking initial value u_o at t_o.

Proof. Let Δ be a subdivision of I_a: $t_o < t_1 < \ldots < t_n = t_o + a$. For $t_{k-1} \leqq t \leqq t_k$ we define

$$\varphi_\Delta(t) = \varphi_\Delta(t_{k-1}) + \int_{t_{k-1}}^{t} f(\tau, \varphi_\Delta(t_{k-1}))d\tau, \quad \varphi_\Delta(t_o) = u_o.$$

Then

$$\varphi_\Delta(t) = \varphi_\Delta(t_o) + \int_{t_0}^{t_1} f(\tau, \varphi_\Delta(t_o))d\tau + \ldots + \int_{t_{k-2}}^{t_{k-1}} f(\tau, \varphi_\Delta(t_{k-2}))d\tau$$
$$+ \int_{t_{k-1}}^{t} f(\tau, \varphi_\Delta(t_{k-1}))d\tau$$
$$= u_o + \int_{t_0}^{t} f_\Delta(\tau)d\ ,$$

and

$$\varphi_\Delta'(t) = f(t, \varphi_\Delta(t_{k-1})) \quad \text{for } t_{k-1} < t < t_k.$$

Let Δ and Δ' be two subdivisions of I_a, and consider $\varphi_\Delta(t)$ and $\varphi_{\Delta'}(t)$ as above. If t is not a subdivision point of either Δ or Δ', $t_{k-1} < t < t_k$ and $t'_{j-1} < t < t'_j$ say, then

$$\frac{d}{dt}\Phi(t, \varphi_\Delta(t), \varphi_{\Delta'}(t)) = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial u}\varphi'_\Delta(t) + \frac{\partial \Phi}{\partial v}\varphi'_{\Delta'}(t)$$

$$= \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial u} f(t, \varphi_\Delta(t_{k-1})) + \frac{\partial \Phi}{\partial v} f(t, \varphi_{\Delta'}(t'_{j-1}))$$

$$= \left(\frac{\partial \Phi}{\partial t} - \widetilde{\frac{\partial \Phi}{\partial t}}\right) + \widetilde{\frac{\partial \Phi}{\partial t}} + \left[\left(\frac{\partial \Phi}{\partial u} - \widetilde{\frac{\partial \Phi}{\partial u}}\right) + \widetilde{\frac{\partial \Phi}{\partial u}}\right] f(t, \varphi_\Delta(t_{k-1}))$$

$$+ \left[\left(\frac{\partial \Phi}{\partial v} - \widetilde{\frac{\partial \Phi}{\partial v}}\right) + \widetilde{\frac{\partial \Phi}{\partial v}}\right] f(t, \varphi_{\Delta'}(t'_{j-1}))$$

where

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial t}(t, \varphi_\Delta(t), \varphi_{\Delta'}(t)), \quad \widetilde{\frac{\partial \Phi}{\partial t}} = \frac{\partial \Phi}{\partial t}(t, \varphi_\Delta(t_{k-1}), \varphi_{\Delta'}(t'_{j-1})),$$

and similarly for $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$.

By the assumption (P_4), for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if we take $|\Delta| = \max(t_k - t_{k-1}) < \delta$ and $|\Delta'| = \max(t'_j - t'_{j-1}) < \delta$ then we have

$$\left|\frac{\partial \Phi}{\partial t} - \widetilde{\frac{\partial \Phi}{\partial t}}\right| < \frac{\varepsilon}{3}, \quad \left|\left(\frac{\partial \Phi}{\partial u} - \widetilde{\frac{\partial \Phi}{\partial u}}\right) f(t, \varphi_\Delta(t_{k-1}))\right| < \frac{\varepsilon}{3},$$

$$\left|\left(\frac{\partial \Phi}{\partial v} - \widetilde{\frac{\partial \Phi}{\partial v}}\right) f(t, \varphi_{\Delta'}(t'_{j-1}))\right| < \frac{\varepsilon}{3}.$$

Thus

$$\frac{d}{dt}\Phi(t, \varphi_\Delta(t), \varphi_{\Delta'}(t)) < \varepsilon.$$

Therefore

$$\Phi(t, \varphi_\Delta(t), \varphi_{\Delta'}(t)) = \Phi(t, \varphi_\Delta(t), \varphi_{\Delta'}(t)) - \Phi(t_o, \varphi_\Delta(t_o), \varphi_{\Delta'}(t_o)) \leqq \varepsilon a,$$

and this shows that there exists a $\varphi(t)$ such that

$$\varphi_\Delta(t) \to \varphi(t) \quad \text{for each } t.$$

Now fix t in I_a. Then for each subdivision Δ there exists a k such that $t_{k-1} < t \leqq t_k$. Then $|\varphi_\Delta(t_{k-1}) - \varphi_\Delta(t)| \leqq M(t - t_{k-1}) \leqq M|\Delta|$ shows that $\varphi_\Delta(t_{k-1}) \to \varphi(t)$. Therefore $f(t, \varphi_\Delta(t_{k-1})) = f_\Delta(t) \to f(t, \varphi(t))$, and

$$\varphi(t) = u_o + \int_{t_o}^{t} f(\tau, \varphi(\tau))d\tau.$$

QED

So far we have been assuming that $f(t,u)$ is continuous. But the conclusion of Theorem 3 is still true under weaker hypotheses on f.

Theorem 4. *Suppose that X is a real reflexive Banach space. Then the conclusion of Theorem 3 is true if we replace the continuity of f by demi-continuity. In this case, we take the differentiation in the sense of weak topology.*

Proof. Note that $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$ are bounded linear functionals. Therefore by an argument similar to that used in the proof of Theorem 3 we can prove that

$$f_\Delta(t) \rightharpoonup f(t, \varphi(t)) \quad (\rightharpoonup \text{ means weak convergence}).$$

So that we have

$$\varphi'(t) = f(t, \varphi(t)).$$

Remark 1. If f satisfies the monotonicity condition on $I_T \times X$ then we may take $a = T$, for in this case we can prove that $\{\varphi_\Delta(t)\}$ is uniformly bounded on I_T.

Remark 2. The assumptions on $f(t,u)$ can be weakend even further; if $f(t,u)$ is measurable in t, demi-continuous in u and dominated by a summable function $M(t)$ for u's remaining in a bounded set, then we can prove the existence of a local solution in the same fashion. But, for such a Carathéodory-type equation, we don't enter into details.

3. Evolution equations.

First we shall summarize some of the results[1)] that have been obtained so far for the linear equation

$$(5) \qquad \frac{du}{dt} = A(t)u + g(t) \qquad t \in I_T,$$

where $\{A(t)\}$ is a family of densely defined closed linear operators on a Banach space X and $g(t)$ is a function on I_T

1) [2], [3], [4] and [5].

taking values in X. Usually $A(t)$ is unbounded.

We make the standing assumption that there exists an evolution operator $U(t,s)$ associated with $A(t)$. This means that $\{U(t,s)\}$ is a family of bounded linear operators from X to X defined for $t_0 \leqq s \leqq t \leqq t_0 + T$, strongly continuous in the two variables jointly and satisfying the conditions

$$U(t,s)\, U(s,r) = U(t,r), \qquad U(s,s) = I,$$

$$\left.\begin{aligned} \frac{\partial U(t,s)u}{\partial t} &= A(t)\, U(t,s)u \\ \frac{\partial U(t,s)u}{\partial s} &= U(t,s)\, A(s)u \end{aligned}\right\} \quad \text{for some } u \in X \text{ specified in each case.}$$

We understand that $u(t)$ is a <u>strict solution</u> of (5) on I_T with the initial value u_0 if $u(t)$ is strongly continuous on $I_T = [t_0, t_0 + T]$, $u(t_0) = u_0$, strongly continuously differentiable and satisfying (5) on $(t_0, t_0 + T)$. Then, if $g(t)$ is continuous, any strict solution is of the form

$$(6) \qquad u(t) = U(t,t_0)u_0 + \int_{t_0}^{t} U(t,s)g(s)ds.$$

Following F. Browder [1] $u(t)$ is said to be a <u>mild solution</u> of (5) with the initial value u_0 if $u(t)$ is continuous on I_T and satisfies (6).

Roughly speaking there are two cases of (5) which are called "parabolic" and "hyperbolic". The family $A(t)$ is said to be <u>uniformly parabolic</u> if:

i) The spectrum of $A(t)$ is in a sector $S_\omega = \{z;\ |\arg(z-\tau)| < \omega < \frac{\pi}{2}\}$, $|(\lambda - A(t))^{-1}| \leqq M/\lambda$ for $\lambda \notin S_\omega$, and $|A(t)^{-1}| \leqq M$, where ω and M are independent of t.

ii) For some $h = 1/m$, where m is a positive integer, $\mathcal{D}(A(t)^h)$ is independent of t, and

$$\left.\begin{aligned} &|A(t)^h A(s)^{-h}| \leqq M \\ &|A(t)^h A(s)^{-h} - I| \leqq M|t-s|^k \end{aligned}\right\} \quad t,s \in I_T, \quad 1-h < k \leqq 1,$$

where M does not depend on t.

Then it is known that there exists a unique evolution operator with the following properties: $U(t,s)X \subseteq \mathcal{D}(A(t))$ for $s<t$, $U(t,s)u$ is Hölder continuous in t and s for $u \in \mathcal{D} = \mathcal{D}(A(t_0)^h)$, and (6) is a strict solution of (5) if $g(t)$ is Hölder continuous on I_T, where u_0 is an arbitrary element of X.

The family $\{A(t)\}$ is said to be <u>hyperbolic</u> if $A(t)$, for each t, is the infinitesimal generator of a contraction semigroup, $\mathcal{D}(A(t))$ is independent of t[1], and $A(t)A(t_0)^{-1}$, which is a bounded operator, is strongly continuously differentiable. Then, it is known that there exists a unique evolution operator such that $U(t,s)\mathcal{D}(A(s)) \subseteq \mathcal{D}(A(t))$, and (6) is a strict solution of (5) if $g(t) \in \mathcal{D}(A(t))$ and $u_0 \in \mathcal{D}(A(t_0))$.

Now we shall go into the non-linear evolution equation

$$(2) \qquad \frac{du}{dt} = A(t)u + f(t,u).$$

We assume that the continuous function $f(t,u)$ sends bounded sets in $I_T \times X$ into bounded sets in X. Then there exists a positive number M such that $|f(t,u)| \leqq M$ if $t \in I_T$ and $u \in V_b = \{u \in X;\ |u - u_0| \leqq b\}$. Take a positive number a so small that $|U(t,t_0)u_0| + M|U|(t - t_0) \leqq b$ if $|t - t_0| \leqq a$, where $|U| = \sup|U(t,s)|$.

Suppose that there exists a real valued continuous function $\Phi(t,u,v)$ on $I_a \times V_b \times V_b$ having the properties (P_1), (P_2'), (P_3), (P_4) and

$$(P_5) \qquad \frac{\partial \Phi}{\partial u} A(t)u + \frac{\partial \Phi}{\partial v} A(t)v \leqq 0.$$

Let Δ be a subdivision of I_a: $t_0 < t_1 < \dots < t_n = t_0 + a$. For $t_{k-1} \leqq t \leqq t_k$ we define

$$\varphi_\Delta(t) = U(t,t_{k-1})\varphi_\Delta(t_{k-1}) + \int_{t_{k-1}}^{t} U(t,s)f(s,U(s,t_{k-1})\varphi_\Delta(t_{k-1}))ds,$$

$$\varphi_\Delta(t_0) = u_0.$$

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1) This condition can be weakend further, see [3].

Then we can write

$$\varphi_\Delta(t) = U(t,t_0)u_0 + \int_{t_0}^{t} f_\Delta(t,s)ds,$$

where $f_\Delta(t,s) = U(t,s)f(s,U(s,t_{j-1})\varphi_\Delta(t_{j-1}))$, $s \in [t_{j-1}, t_j]$.
Hence $\varphi_\Delta(t) \in V_b$.

When $\{A(t)\}$ is parabolic we assume that $u_0 \in \mathcal{D}(A(t_0))$, and that $f(t,u)$ is Hölder continuous in t and u. Then we have $\varphi_\Delta(t_{k-1}) \in \mathcal{D}(A(t_{k-1}))$, and $U(t,t_{k-1})\varphi_\Delta(t_{k-1})$ is Hölder continuous in t for $t_{k-1} \leqq t \leqq t_k$. Hence $\varphi_\Delta(t)$ is differentiable in (t_{k-1},t_k), and we have

(7) $\varphi_\Delta'(t) = A(t)U(t,t_{k-1})\varphi_\Delta(t_{k-1}) + f(t,U(t_{k-1})\varphi_\Delta(t_{k-1}))$.

When $\{A(t)\}$ is hyperbolic we assume that $u_0 \in \mathcal{D}(A(t_0))$, that $f(t,u) \in \mathcal{D}(A(t))$ whenever $u \in \mathcal{D}(A(t))$, and that $A(t)f(t,U(t,t_{k-1})\varphi_\Delta(t_{k-1}))$ is strongly continuous. Then again we have (7).

Let Δ and Δ' be two subdivisions of I_a. Then, if t is not a subdivision point of either Δ or Δ', then we can take k and j such that $t_{k-1} < t < t_k$ and $t'_{j-1} < t < t'_j$ say.
We have

$$\frac{d}{dt}\Phi(t, \varphi_\Delta(t), \varphi_{\Delta'}(t)) = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial u}(A(t)\varphi_\Delta(t)+f(t,U(t,t_{k-1})\varphi_\Delta(t_{k-1}))$$
$$+ \frac{\partial\Phi}{\partial v}(A(t)\varphi_{\Delta'}(t) + f(t,U(t,t'_{j-1})\varphi_{\Delta'}(t'_{j-1}))$$
$$= [(\frac{\partial\Phi}{\partial t} - \widetilde{\frac{\partial\Phi}{\partial t}}) + \widetilde{\frac{\partial\Phi}{\partial t}}] + [\frac{\partial\Phi}{\partial u}A(t)\varphi_\Delta(t) + \frac{\partial\Phi}{\partial v}A(t)\varphi_{\Delta'}(t)]$$
$$+ [(\frac{\partial\Phi}{\partial u} - \widetilde{\frac{\partial\Phi}{\partial u}}) + \widetilde{\frac{\partial\Phi}{\partial u}}]f(t,U(t,t_{k-1})\varphi_\Delta(t_{k-1}))$$
$$+ [(\frac{\partial\Phi}{\partial v} - \widetilde{\frac{\partial\Phi}{\partial v}}) + \widetilde{\frac{\partial\Phi}{\partial v}}]f(t,U(t,t'_{j-1})\varphi_{\Delta'}(t'_{j-1}))$$

where $\frac{\partial\Phi}{\partial t} = \frac{\partial\Phi}{\partial t}(t, \varphi_\Delta(t), \varphi_{\Delta'}(t))$, $\widetilde{\frac{\partial\Phi}{\partial t}} = \frac{\partial\Phi}{\partial t}(t,U(t,t_{k-1})\varphi_\Delta(t_{k-1}),$ $U(t,t'_{j-1})\varphi_{\Delta'}(t'_{j-1}))$, and similarly for $\frac{\partial\Phi}{\partial u}$, $\widetilde{\frac{\partial\Phi}{\partial u}}$, $\frac{\partial\Phi}{\partial v}$ and $\widetilde{\frac{\partial\Phi}{\partial v}}$.

Since $\left|\varphi_\Delta(t) - U(t,t_{k-1})\varphi_\Delta(t_{k-1})\right|$

$$= \left|\int_{t_{k-1}}^{t} U(t,s)f(s,U(s,t_{k-1})\varphi_\Delta(t_{k-1}))ds\right| \leqq M\,|U|\,|\Delta|,$$

and similarly

$$\left|\varphi_{\Delta'}(t) - U(t,t'_{j-1})\varphi_{\Delta'}(t'_{j-1})\right| \leqq M\,|U|\,|\Delta'|\ ,$$

we see that $\{\varphi_\Delta(t)\}$ is a strongly convergent family. Thus there exists $\varphi(t)$ such that $\varphi_\Delta(t) \to \varphi(t)$. Furthermore, by the assumption (P_2'), we see that this convergence is uniform. Hence

$$f_\Delta(t,s) \to U(t,s)f(s,\varphi(s)).$$

Therefore we have

$$\varphi(t) = U(t,t_0)u_0 + \int_{t_0}^{t} U(t,s)f(s,\varphi(s))ds,$$

which shows that φ is a mild solution of (2).

Furthermore, when $A(t)$ is hyperbolic, we can prove that $\varphi(t)$ thus obtained is a strict solution under some additional conditions. Namely, if we assume that $|A(t)f(t,u)|$ is bounded whenever u remains in a bounded set, we see that there is a positive number L such that $|A(t)f(t,U(t,t_{k-1}))| \leqq L$. Then

$$\begin{aligned}\varphi_\Delta'(t) &= A(t)U(t,t_0)A(t_0)^{-1}A(t_0)u_0 \\ &+ \sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j} A(t)U(t,s)A(s)^{-1}A(s)f(s,t_{j-1})\varphi_\Delta(t_{j-1}))ds \\ &+ \int_{t_{k-1}}^{t} A(t)U(t,s)A(s)^{-1}A(s)f(s),\ U(s,t_{k-1})\varphi_\Delta(t_{k-1}))ds \\ &+ f(t,U(t,t_{k-1})\varphi_\Delta(t_{k-1}))\end{aligned}$$

shows that

$$|\varphi'(t)| \leqq |A(t_0)u_0| + L(t - t_0))A + M,$$

where $A = \sup|A(t)U(t,s)A(s)^{-1}|$. Thus $\{A(t)\varphi_\Delta(t)\}$ is bounded. Since $A(t)$ is closed, this shows that $\varphi(t) \in \mathcal{D}(A(t))$ when X is reflexive.[1] Then by the assumption that $A(t)f(t,\psi(t))$ is strongly continuous in t if $\psi(t) \in \mathcal{D}(A(t))$, it follows that

1) Lemma 5, [2], p. 214.

$\varphi(t)$ is a strict solution of (2). Thus we obtain the following

Theorem 5. Let X be a Banach space and $f(t,u)$ be a continuous function from $I_T \times X$ to X which sends bounded sets into bounded sets. Suppose that there exists a real valued continuous function $\Phi(t,u,v)$ having the properties (P_1), (P_2'), (P_3), (P_4) and (P_5). Then the evolution equation (2) has a mild solution on I_a, where a is a positive number $\leq T$, under the conditions that, a) when $A(t)$ is parabolic, u_0 is in $\mathcal{D}(A(t_0))$ and $f(t,u)$ is Hölder continuous in t and u, and b) when $A(t)$ is hyperbolic, u_0 is in $\mathcal{D}(A(t_0))$, and $A(t)f(t,u(t))$ is defined and strongly continuous whenever $u(t) \in \mathcal{D}(A(t))$. Furthermore, in the latter case, the mild solution thus obtained is actually a strict solution if X is reflexive and if we assume the additional condition that $A(t)f(t,u)$ is bounded whenever u remains in a bounded set.

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