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Vector bundles on an abelian variety

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Let k be an algebraically closed field of characteristic $p \geq 0$, and let X be an abelian variety over k.

We have been able to answer the following questions, when $\dim(X) = 1$ and $p \neq 0$, posed by R.Hartshorne:

(I) Is $E^{(p)}$ indecomposable, when E is an indecomposable vector bundle on X?

(II) Is the Frobenius map $F^* : H^1(X,E) \longrightarrow H^1(X,E^{(p)})$ injective?

We can also partly answer the following question posed by D.Mumford:

(III) Classify, or at least say anything about, vector bundles on an abelian variety X, when $\dim(X) \neq 1$.

Let us summarize our results first.

When the Hasse invariant of X is not zero, the answers to (I) and (II) are both affirmative. When the Hasse invariant of X is zero and E is an indecomposable vector bundle on X of rank r and of degree d, then

$E^{(p)}$ is indecomposable, if either $(r, d) = 1$ or $(r, d) \neq 1$ with $r/(r, d)$ divisible by p. Otherwise $E^{(p)}$ decomposes into a direct sum of $\min\{(r, d), p\}$ indecomposable components. As for (II), the Frobenius map F^* is not injective (and in fact the zero map), if and only if $r < p$, $d = 0$ and E has a non-zero section (i.e. in Atiyah's notation $E = E_{r,0}$ with $r < p$).

Questions (I) and (II) themselves may not look interesting. However, R.Hartshorne recently proved the following: A vector bundle E on an elliptic curve X is ample, if and only if every quotient bundle of E has positive degree. He uses Atiyah's "multiplicative structure" when $p = 0$, and our answer to (II) when $p \neq 0$.

When $\dim(X) = 1$, Atiyah classified all the indecomposable vector bundles on X. He also gave the "multiplicative structure" in case $p = 0$. His construction of indecomposable vector bundles is essentially by successive extensions of line bundles. To answer (I) and (II), however, it is very hard to keep track of these extensions after we pull them back by the Frobenius map.

We give an entirely different way, inspired by Schwarzenberger's results,of constructing indecomposable vector bundles, which is very easy to handle and which gives us a clearer picture especially in characteristic $p \neq 0$. This construction can also be generalized to higher dimension, and thus partly answers the question (III).

Before going further, let us review Atiyah's results. Let $\underline{E}_X(r,d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and of degree d on an elliptic curve X .

(i) If we fix one E in $\underline{E}_X(r,d)$, then every other vector bundle is of the form $E \otimes L$ with L in $\mathrm{Pic}^0(X) = \underline{E}_X(1,0)$. Moreover, $E \otimes L_1 \cong E \otimes L_2$ if and only if $L_1^{\otimes r'} \cong L_2^{\otimes r'}$ where $r' = r/(r, d)$.

(ii) In $\underline{E}_X(r,0)$ there is a unique element $E_{r,0}$ such that $H^0(X,E_{r,0}) \neq 0$.

(iii) Let $h^i(E)$ be the dimension of $H^i(X,E)$. Then for E in $\underline{E}_X(r,d)$, we have

$$h^0(E) = d \quad \text{and} \quad h^1(E) = 0 \quad \text{when} \quad d > 0$$
$$h^0(E) = 0 \quad \text{and} \quad h^1(E) = |d| \quad \text{when} \quad d < 0$$
$$h^0(E) = h^1(E) = 0 \quad \text{when} \quad d = 0 \quad \text{and} \quad E \neq E_{r,0}$$
$$h^0(E) = h^1(E) = 1 \quad \text{when} \quad E = E_{r,0} .$$

(iv) Suppose $p = 0$. For E in $\underline{E}_X(r,d)$ with $(r, d) = 1$, $\mathrm{End}_{\underline{O}_X}(E) = k$. In fact such E is "stable" .

(v) Suppose $p = 0$. For E in $\underline{E}_X(r,d)$ with $(r, d) = 1$, $E \otimes E_{h,0}$ is in $\underline{E}_X(rh,dh)$.

(vi) When $p = 0$, $E_{r,0}$ is isomorphic to the (r-1)-st symmetric power $S^{r-1}(E_{2,0})$.

(vii) When k is the field of complex numbers, a vector bundle has a holomorphic integrable connection if and only if it is a direct sum of those in $\underline{E}_X(r,0)$ for various r.

For (i), (iv) and (v), the key is his Lemma 7 to the effect that for E in $\underline{E}_X(r,d)$ with $(r, d) = 1$ and $p \not| r$,

$$\underline{\mathrm{End}}_{\underline{0}_X}(E) = \bigoplus L$$

where L runs over all the line bundles on X with $L^{\otimes r} \cong \underline{0}_X$.

What we do now is to generalize these results to arbitrary characteristic and dimension if possible. The first key result is the following. For a line bundle L on an abelian variety Y, we denote by $\Lambda(L) : Y \longrightarrow Y^t$ = the dual abelian variety, the homomorphism which sends a point y in Y to $T_y{}^*L \otimes L^{-1}$, where $T_y : Y \longrightarrow Y$ is the translation.

<u>Theorem</u>

Let $\varphi : Y \longrightarrow X$ be an isogeny of g-dimensional abelian varieties over k, and let L be a line bundle on Y such that the restriction of $\Lambda(L)$ to the (scheme-theoretic) kernel of φ is an isomorphism. Then $\mathrm{End}_{\underline{0}_X}(\varphi_* L) = k$. Especially $\varphi_* L$ is an indecomposable vector bundle on X.

Corollary

Let $\varphi : Y \longrightarrow X$ be an isogeny of degree r between elliptic curves, and let L be a line bundle of degree d on Y with $(r, d) = 1$. Then $\varphi_* L$ is in $\underline{E}_X(r,d)$. For E in $\underline{E}_X(r,d)$, we have $\mathrm{End}_{\underline{0}_X}(E) = k$.

In view of Atiyah's classification, we thus get all the stable vector bundles on an elliptic curve in this way. We next generalize Atiyah's key Lemma 7 mentioned above. We denote by $\underline{P}$ the normalized Poincaré line bundle on the product abelian variety $X \times X^t$, that is, the universal family of line bundles algebraically equivalent to zero parametrized by X^t , and so normalized that $\underline{P} \mid X \times \{0\} \cong \underline{0}_X$ and $\underline{P} \mid \{0\} \times X^t \cong \underline{0}_{X^t}$.

Proposition

Let $\varphi : Y \longrightarrow X$ be an isogeny of abelian varieties over k with (scheme-theoretic) $\ker(\varphi) = G$. Let L be a line bundle on Y such that $\Lambda(L)$ induces an isomorphism on G . Then $\underline{\mathrm{End}}_{\underline{0}_X}(\varphi_* L) = p_{1*}(\underline{P} \mid X \times \tilde{G})$, where $\tilde{G}$ is the total inverse image by the dual isogeny $\varphi^t : X^t \longrightarrow Y^t$ of the subgroup scheme $\Lambda(L)(G)$ of Y^t . Especially if both φ and φ^t are separable, $\underline{\mathrm{End}}_{\underline{0}_X}(\varphi_* L) = \bigoplus L'$, where L' runs over all the line bundles L' on X such that

$$\varphi^* L' \cong T_a^* L \otimes_{\underline{0}_Y} L^{-1}$$

for some closed point a in $\ker(\varphi) = G$.

Corollary

Let X be an elliptic curve over k of characteristic exponent p. Let E be an element of $\underline{E}_X(r,d)$ with $(r, d) = 1$. Suppose $r = r'q$, with $(r', p) = 1$ and q a power of p. Then

$$\underline{\mathrm{End}}_{\underline{0}_X}(E) = \bigoplus (L \otimes E_{q,0}) \quad \text{if Hasse}(X) \neq 0$$

where L runs over all the line bundles on X with $L^{\otimes r} \cong \underline{0}_X$ (there are r'^2q of those).

$$\underline{\mathrm{End}}_{\underline{0}_X}(E) = \bigoplus (L \otimes E_{q^2,0}) \quad \text{if Hasse}(X) = 0$$

where L runs over all the line bundles on X with $L^{\otimes r'} \cong \underline{0}_X$.

To deduce this corollary from the proposition, we need:

Lemma

Let S be the h-th order neighborhood of the origin of X, i.e. $S = \mathrm{Spec}(\underline{0}_{X,0}/\underline{m}^h_{X,0})$. Then $p_{1*}(\underline{P}|X \times S) = E_{h,0}$ where $\underline{P}$ is the normalized Poincaré line bundle on $X \times X$. Moreover $\mathrm{End}_{\underline{0}_X}(E_{h,0})$ is isomorphic to the truncated polynomial ring $k[t]/(t^h)$.

Although we do not mention the exact form here, this lemma has an obvious generalization to higher dimension. This lemma is interesting, since it says that we get a non-simple vector bundle $E_{h,0}$ as the direct image of a line bundle $\underline{P}|X \times S$ via the covering $p_1 : X \times S \longrightarrow X$, whose top space is <u>not reduced</u>.

When k is the field of complex numbers, Morikawa recently characterized those simple vector bundles on a complex torus X which we get as in our theorem as follows:

Let $\tilde{X}$ be the universal covering space of X , that is, a g-dimensional vector space over k . Let Γ be the fundamental group, i.e. a lattice in $\tilde{X}$. Then a vector bundle E of rank r on X corresponds to a cohomology class of a 1-cocycle (a factor of automorphy or a matric multiplier) $h(\alpha, z)$ in $H^1_{gr}(\Gamma, GL_r(\underline{H}))$, where $\underline{H}$ is the ring of all holomorphic functions on $\tilde{X}$, and $h(\alpha, z)$ in $GL_r(\underline{H})$ for α in Γ and z in $\tilde{X}$ satisfies

$$h(\alpha + \beta, z) = h(\beta, z + \alpha)\cdot h(\alpha, z) .$$

Then E is of the form in our theorem, if and only if the corresponding 1-cocycle is cohomologous to one of the form

$$h(\alpha, z) = \exp(B(\alpha, z))\cdot C(\alpha)$$

where $B(\alpha, z)$ is a bilinear form, k-linear in z , $C(\alpha)$ is a constant matrix in $GL_r(k)$, and the linear envelope of $C(\alpha)$ with α running over Γ is the full matrix ring $M_r(k)$.

We can re-interpret Morikawa's result as follows: There exists an isogeny $\varphi : Y \longrightarrow X$ and a line bundle L on Y such that $\Lambda(L)$ induces an isomorphism on $\ker(\varphi)$ and that $E = \varphi_* L$, if and only if the canonical inclusion $\underline{0}_X \longrightarrow \underline{End}_{\underline{0}_X}(E)$ induces an isomorphism

$$H^i(X, \underline{0}_X) \xrightarrow{\sim} H^i(X, \underline{End}_{\underline{0}_X}(E))$$

for $i = 0$ and 1.

It is well known that $H^1(X, \underline{End}_{\underline{0}_X}(E))$ measures the infinitesimal deformation of E on X, that is, it is isomorphic to the tangent space at E of the moduli of vector bundles on X. Our characterization above says that the vector bundle of the form $\varphi_* L$ moves essentially in a g-dimensional family. We can actually construct such a family by moving L in its universal algebraic family. In fact to answer our question (I), we construct a universal family of indecomposable vector bundles of rank r and of degree d on an elliptic curve. Then we do as in our lemma to construct all the elements in $\underline{E}_X(rh, dh)$ with $(r, d) = 1$.

We believe there are lots of other simple vector bundles on a higher dimensional abelian variety, although we have not been able to produce them yet. Possibly we need direct image of line bundles by a finite ramified coverings. But then we no longer have abelian varieties above.