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ON THE TOTAL CURVATURE OF A LINK

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INTRODUCTION

The total curvature $\kappa(K) = \int_K |\xi'(s)|\,ds$ of a knot K (a simple closed connected space curve) of class C'', a quantity which measures the total turning of the tangent unit vector, was studied by W. Fenchel [2] who proved, in 1929, $\kappa(K) \geqq 2\pi$. In 1950, J. Milnor [4] generalized the concept of the total curvature to that of a topological knot, and proved that if K is knotted, then $\kappa(K) \gtrsim 4\pi$. Furthermore, he associated to the ambient isotopy class k of a knot K an invariant "crookedness of k" $\mu(k)$ and showed that

$$2\pi\mu(k) = \underset{K\in k}{\text{g.l.b.}}\,\kappa(K).$$

Soon after Milnor, R,Fox [3] succeeded in giving a lower bound of the crookedness of a knot type k; $\mu(k) \geqq \rho(k)$, where $\rho(k)$ is the rank of the fundmental group of the complement of a knot belonging to k. As for an upper bound he conjectured that

$$\nu(k) \geqq 3\ (\ \mu(k)-1\ )$$

for a knot type k, where $\nu(k)$ is the minimal number of crossing points of k. Milnor's arguments concerned with the crookedness of a knot work equally well for the crookedness of a link.

However, in the case of a link, Fox' conjecture is false! (see Fig. 1)

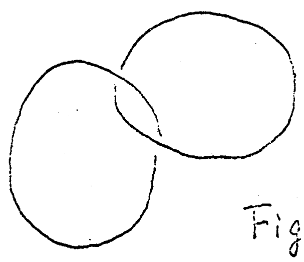


Fig. 1.

We shall prove the following:

Main Theorem.

For a smooth link type ℓ we have that

$$\nu(\ell) \geqq 2\,(\,\mu(\ell)-1\,),$$

where $\mu(\ell)$ is the crookedness of ℓ and $\nu(\ell)$ is the minimal number of crossing points of projections of ℓ.

For the proof of this, we make use of the graph representation of a link [5] and prove a certain basic property of a planar graph which generalizes I. Fary's result [1] in some context.

An implication of this is as follows;

Corollary.

For a smooth link type ℓ, there is a link $L \in \ell$ such that $\kappa(L) = (\,\nu(\ell)+2\,)\pi$.

§1. Definition and statement of Theorem 1.

Let $G = \{V, E\}$ be a graph where $V = \{$ vertices of G $\}$ and $E = \{$ edges of G $\}$. In the following, we shall assume that G is a finite and planar graph with no loop. Here we recall that a finite graph is planar if it is represented on the plane R^2 ((x, y)-plane) with simple arcs as edges meeting only at their end points which are the vertices of the graph.

A graph G is straight, if each edge of G is a straight segment. Two distinct edges e_1 and e_2 are multiple edges, if they have common end points. A graph having no multiple edges will be called a linear graph. The vertices which can be joined to the infinite point will be called the outer most vertices.

The plane is divided by the edgea of the graph into a number of regions called the regions of the graph. A vertex is adjoining to an edge, if it is an endpoint of the edge. A vertex or an edge is adjoining to a region, if it lies on the boundary of the region.

The graphs G and G′ will be called congruent, if there can be established a one-to-one correspondence between the vertices, edges and regions of G and G′ such that (1) adjoining elements correspond to adjoining elements, and (2) the regions containing the point at infinity correspond to each other.

We would like to extend the following result by Fary in some context.

Proposition 1. (Fary) [1].

A linear graph G is congruent to a straight graph.

For our purpose, we define the notion of normal position of a graph as follows;

Let G be a graph in a plane R^2((x,y)-plane).
Let $\pi: R^2 \to$ x-axis be the orthogonal projection $\pi(x,y) = x$ for $(x,y) \in R^2$.

A graph G is in normal position with respect to the x-axis, if the following conditions are satisfied,

(1) (vertices are in general position with respect to the x-axis); for distinct vertices v and v', $\pi(v) \neq \pi(v')$, and

(2) (edges are monotone with respect to the x-axis); for each edge e with endpoints v and v' such that $\pi(v) < \pi(v')$ there is a smooth function $f: [\ \pi(v),\ \pi(v')\ ] \to R$ such that e is a graph of f;
$e = \{ (x, f(x)) \in R^2 \mid x \in [\ \pi(v),\ \pi(v')\ ] \}$

Theorem 1

Any planar graph G with no loop is congruent to a graph in normal position.

§2. Proof of Theorem 1.

First of all we define a notion of multiplicity of a graph. Let G be a graph. Recall that we restricted ourselves to a planar graph with no loop. Picking up two vertices v and v′ of G, we define a multiplicity m (v,v′) of an unordered pair (v, v′) (= (v, v′)) by m (v, v′) = (the number of edges with common endpoints v and v′) − 1.

Thus there is no edge with endpoints v and v′ if and only if m (v, v′) = -1, and there is exactly one edge with endpoints v and v′ if and only if m (v, v′)=0. The multiplicity m(G) of G is defined by

$$m (G) = \Sigma\, m (v', v),$$

where (v, v′) ranges over all unordered pairs of vertices of G with m (v, v′) $\geq$ 0.

It follows from the definition that m (G) = 0 if and only if G has no multiple edges; namely G is linear.

We shall prove Theorem 1 by induction on the multiplicity m = m (G) of a graph G.

[0]. Suppose that the multiplicity m = m (G) is equal to zero. Then G is linear. It follows from Proposition 1 (Fary) that G is congruent to a straight graph. Hence we may regard G as a simplicial complex of dimension 1 in a plane.

By the general position arguments, without less of generality, we may assume that all the vertices of the graph are in general position with respect to the x-axis;

namely $\pi(v) \neq \pi(v')$, if v and v' are distinct vertices. Since each edge e is linear, this implies that $\pi|e : e \to$ the x-axis is injective. Therefore, G is in normal position, completing the proof.

$[k] \Rightarrow [k+1]$. Assuming inductively that Theorem 1 holds for a graph with multiplicity $m(G) \leq k$, we shall prove that Theorem 1 holds for a graph with $m(G) = k+1$ ($k \geq 0$). Let G be a graph with $m(G) = k+1$. If e and e' are multiple edges with common endpoints v and v', then $e \cup e'$ is a loop, which will be called a <u>loop with two vertices</u> of G. A loop with two vertices is <u>inner most,</u> if the closed region bounded by the loop contains no loop with two vertices of G except for the loop. Let $e \cup e'$ be an inner most loop with two vertices v and v' of G. By $[e, e']$ we shall denote a subgraph of G on the closed region bounded by the loop $e \cup e'$. Then $[e, e']$ is planar and the multiplicity $m([e, e']) = 1$. We take a point v'' in the interior of the edge e'. This point v'' divides e' into two edges e_1 and e_2. Let H be a new graph obtained from $[e, e']$ by removing the edge e' and adding a new vertex v'' and new edges e_1 and e_2. Then H is planar and of the multiplicity $m(H) = o$. Hence H is congruent to a straight graph. Since three edges e, e_1 and e_2 bound a closed region on which H lies, it follows that the straight graph is on a triangle. Let $h: R^2 \to R^2$ be a homeomorphism such that $h(H) = H'$ is the straight graph. Up to a further transformation of R^2 we may assume that $h(e)$ is the closed interval

[0, 2] on the x-axis, $h(v'') = (1, 1) \in R^2$ and H' is a straight graph in normal position. Let F be graph obtained from G by removing the subgraph H and adding an edge e and vertices v and v′. Then F is still planar and of the multiplicity $m(F) \leq m(G) - 1 = k$, since an edge e′ which is one of multiple edges of G has been removed, Then by the induction hypothesis F is congruent to a graph in normal position. So we may assume that F itself is a graph in normal position. We may also assum that edges are smooth arcs and intersect at vertices transversally each other. In particular, since F is in normal position, we may assume that $\pi(e) = [0, 2]$ and e is a graph of a smooth function $f : [0, 2] \to R$. Note that e separates locally the plane into two parts ; above and below. We may assume that the subgraph [e, e′] of G is located above e.

Now let $p_m : R^2 \to R^2$ be a map defined by $p_m(x, y) = (x, \frac{1}{m}y)$. Taking sufficiently large positive number m, each edge d of a straight graph $p_m(H')$ is almost parallel to the x-axis and is a graph of a linear map g restricted on a closed subinterval [u, u′] of [o, 2] such that $f(t) + g(t)$ is contained in the region above e for all $t \in [u, u']$. The resulting graph of f + g on [u, u′] is denoted by $e + \dot{d}$. Then we have a graph $e + p_m(H') = \{ e + d \mid d \in p_m(H') \}$ in the region above e of the graph F. This graph $e + p_m(H')$ is congruent to H, and hence if we forget the vertex v″, we may regard $F \cup (e + p_m(H'))$ be congruent to G.

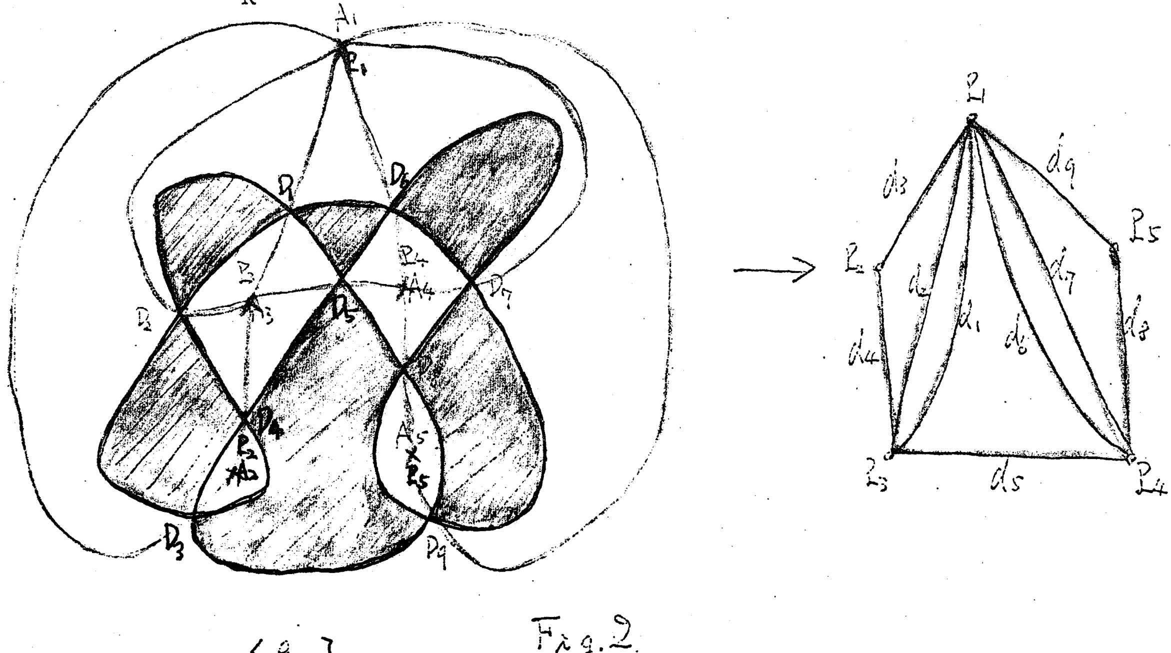
Since e and p_m (H´) are in normal position, it follows that e + p_m (H´) is so. By moving some vertices of F, if necessary, we may assume that F ∪ (e + p_m(H´)) is in normal position. This implies that G is congruent to a graph F ∪ (e + p_m(H´)) in normal position, completing the proof.

§3. The graph of a link.

Let π be a regular normal projection of a link L on a plane R^2. If π has ν double points $D_1, \cdots\cdots, D_\nu$, then it divides R^2 into $\nu + 2$ regions, each of which is homeomorphic to an open disk except for the region containing the point at infinity.

We can separate these regions into two classes, say "white" and "black", in such a way that each segment of the graph π, i.e. an arc from a double point to the next one, is always the common boundary of white and black regions. Then a graph of L with respect to π is obtained as follows:

Let $A_1, \cdots\cdots\cdots, A_\alpha$, be, for example, black regions on R^2. Take points $P_i \in A_i$ ($i = 1, \cdots\cdots, \alpha$) and connected these points by ν non-intersecting arcs $d_1, d_2, \cdots\cdots, d_\nu$ in such a way that each d_k corresponds to D_k ($k = 1, \cdots\cdots, \nu$), and P_i and P_j are connected by d_k if and only if A_i and A_j have a common double point D_k on their boundaries.



[9_{23}]

Fig. 2

We shall call a graph thus obtained a black graph of L with respect to π. In the same way we obtain a white graph of L with respect to π.

As is immediately seen, the black and white graphs are dualy related each other.

Conversely, given a graph, then we have a link by reversing the process above so that the given graph is a graph of a normal projection of the link. For this, first, we take a disk sufficiently small radius centered at each vertex. Then along each edge we connect two disks centered at endpoints of the edge by pasting a half twisted band along arcs on their boundaries.

Thus we have a surface with boundary, whose boundary is a link required. Note that the result is not unique.

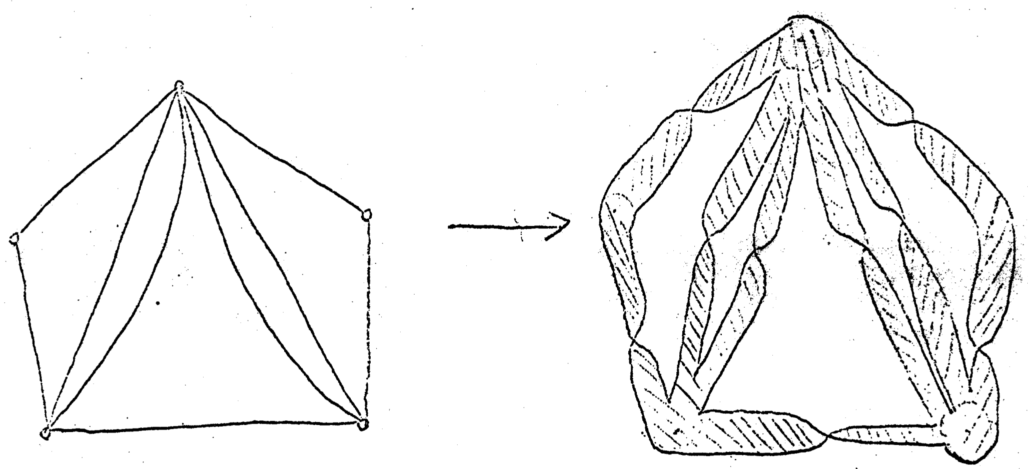


Fig. 3

In order to obtain unique link type from a graph, it is only necessary to give signs on each edge which assign under or over crossing at double (crossing) points. Such a graph will be called an oriented graph. Two oriented graphs are congruent, if they are congruent proserving orientations of edges. Thus a congruence class of oriented graph determine unique link type.

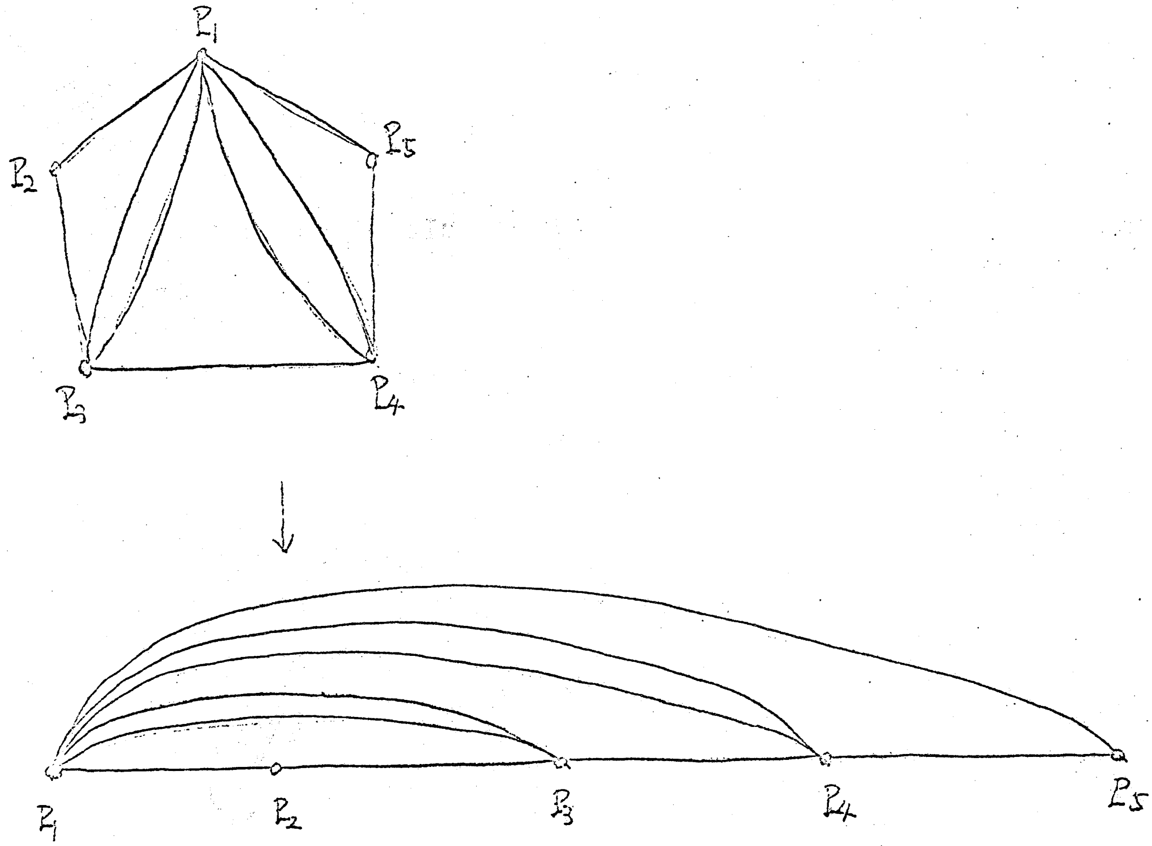


Fig. 4

§4. Croockedness of a link and proof of the main theorem.

For each closed connected curve C parametrized by a vector function $\xi(t)$, $t \varepsilon R$, and each unite vector b, we define $\mu(C, b)$ to be the number of maxima of the function $b \cdot \xi(t)$ (i.e. the number of parameter values t_0 for which $b \cdot \xi(t_0) \geq b \cdot \xi(t)$ for t within some neighborhood of t_0) in a fundamental period. We define $\mu(C) = \min_b \{\mu(C, b)\}$.

For a link L with m components $K_1, \cdots, K_m$ we define $\mu(L, b) = \sum_{i=1}^{m} \mu(K_i, b)$

and $\mu(L) = \min_b \{\mu(L, b)\}$. We shall call $\mu(L)$ to be the crookedness of L. Hence the crookdness of a link is always positive integers. Let $\mathfrak{L}$ be a link type (the ambient isotopy class of a link). Then the crookedness $\mu(\mathfrak{L})$ of the type $\mathfrak{L}$ is defined by $\mu(\mathfrak{L}) = \min_{L \varepsilon \mathfrak{L}} \mu(L)$.

The following was essentially proved by Milnor.

Proposition 2 (Milnor).

Let $\mathfrak{L}$ be a link type. Then the total curvatures $\kappa(L) = \int_L |\xi''(s)| ds$ of a link L of the type $\mathfrak{L}$ and the crookedness of $\mathfrak{L}$ are related in a formula:

$$2\pi\mu(\mathfrak{L}) = \underset{\ell \in L}{\text{g.l.b.}}\ \kappa(L).$$

where g.l.b. stands for the greatest lower bound of $\kappa(L)$ when L ranges over all links of type $\mathfrak{L}$.

Theorem 2.

Let ℓ be a link type of a link L. Suppose that L has a graph with n vertices with respect to a regular normal projection π of L. Then $\mu(\ell) \leqq n$.

Proof

Let G be a graph of L with respect to π. If G has a loop, then L is ambient isotopic to a link L' of which graph with respect to a regular normal projection of L' is congruent to a planar graph obtained from G by removing loops, see [5]. Thus we may assume that G is a finite and planar graph with no loop. By Theorem 1 we may assume that G is in normal position.

We would like to get a link L' which is ambient isotopic to L and the crookedness $\mu(L', b) = n$, where b is the unit vector on the y-axis of the (x, y)-plane on which a regular normal projection lies.

For this, we may assume that the regular normal projection π lies on the (x, y)-plane in the (x, y, z)-space. We orient G so that the oriented graph G gives rise to the link L. By making use of the fact that G is in normal position with respect to the x-axis, we construct L' as follows:

First of all, we stretch small disks centered at vertices of G along the y-axis so long that bands connecting disks can be spanned mutually disjoint and the center lines are parallel to the x-axis. Since each edge is monotone along the x-axis, for a vertex v we may disguish an edge adjoining to v to be the left or right hand side of v. This implies that each band starts from the right hand side of a stretched disk and ends at the left hand side of a stretched disk. Those bands should be half twisted according to the sign of the corresponding edges. Thus we obtain from an oriented graph G a link L'. We may assume that the corners of bands are smoothed and the edges of bands are monotone along the y-axis.

Fig. 5.

This implies that if a component K of L' is parametrized by a vector function $\xi(t)$, then the function $b \cdot \xi(t)$ attains maxima only at the tops of stretched disks, where b is the unit positive vector on the y-axis.

This implies that $\mu(L', b) = n$, and therefore $\mu(\mathfrak{L}) \leqq n$. This completes the proof of Theorem 2.

Let L be a link. We define the number of crossing points of L to be the minimum number of double points of regular normal projection of L into planes in the space. For a link type $\mathfrak{L}$, we define $\nu(\mathfrak{L})$ by

$$\nu(\mathfrak{L}) = \min_{L \in \mathfrak{L}} \nu(L).$$

Fig. 5.

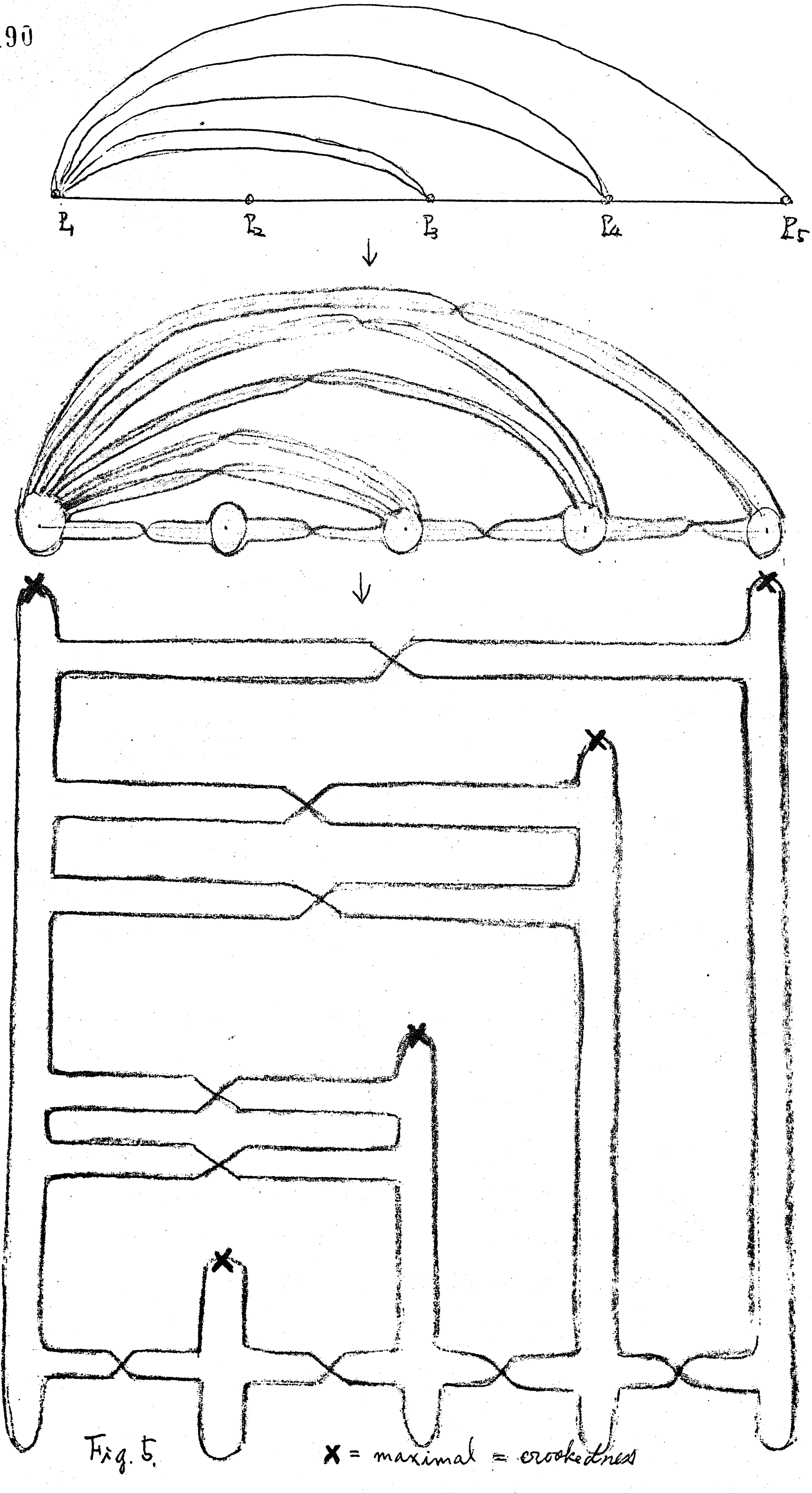


Fig. 5. X = maximal = crookedness

Now we are ready to prove the

Main Theorem

For a link type $\mathfrak{L}$, we have that

$$\nu(\mathfrak{L}) \geqq 2(\mu(\mathfrak{L}) - 1).$$

Proof.

Let L be a link of type $\mathfrak{L}$ with a regular normal projection Π into a plane having $\nu = \nu(\mathfrak{L})$ double points. Note that the number of regions on the plane separated by the graph is equal to $\nu + 2$. By α and β we denote the numbers of black and white regions, respectively, so that $\alpha + \beta = \nu + 2$. We may assume that $\alpha \leqq \beta$. Hence the number of vertices of the black graph of L with respect to Π is equal to α.

It follows from Theorem 2 that $\mu(\mathfrak{L}) \leqq \alpha$. Since $\alpha \leqq \beta$, we have that $2\alpha \leqq \nu(\mathfrak{L}) + 2$, and hence that $2(\mu(\mathfrak{L}) - 1) \leqq \nu(\mathfrak{L})$.

This completes the proof of the Theorem.

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