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SOLENOIDS IN GENERIC HAMILTONIAN DYNAMICS

Seeking solenoids
elliptic orbits whirl fog
up dim tori-i

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1. Recurrence in Hamiltonian Dynamics. A Hamiltonian dynamical system is defined in $\mathbb{R}^{2n}$ by the differential equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x^i} \qquad i = 1, \ldots, n,$$

or in matrix notation

1) $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J \begin{pmatrix} H_x \\ H_y \end{pmatrix},$$

where $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ is a standard skew-symmetric $2n \times 2n$ matrix with blocks of unit matrices E_n, and the gradient $dH = (H_x, H_y)$ has been transposed as a column vector. The given Hamiltonian function $H : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}$ is suitably differentiable in the state space $\mathbb{R}^{2n}$ which is written $\mathbb{R}^n \oplus \mathbb{R}^n$ to emphasize the use of the canonical coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n)$.

A change of local coordinates $(x, y) \to (q, p)$ in $\mathbb{R}^n \oplus \mathbb{R}^n$ preserves the form of the Hamiltonian system 1), with $H(q, p) = H(x(q, p), y(q, p))$ as Hamiltonian function, provided the Jacobian matrix $T = \frac{\partial(q, p)}{\partial(x, y)}$ at each point satisfies the identity $TJT' = J$. In this case T belongs to the real symplectic group $Sp(2n, \mathbb{R})$, (q, p) are called canonical coordinates in $\mathbb{R}^n \oplus \mathbb{R}^n$, and the map $(x, y) \to (q, p)$ is symplectic, see [1,5].

The recurrent trajectories of such dynamical systems have been studied intensively since the age of Lagrange and Hamilton. At first attention was restricted to critical points and periodic orbits, but recent theory deals with almost periodic orbits that are dense in minimal tori.

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In this paper we define new types of recurrent trajectories that fill minimal solenoids. Our approach to these solenoidal structures follows a geometric path through a progression of massive tori-i[1] ($\pi \approx S^{2n-2} \times S^1$), centered on elliptic orbits that wind ever higher. Furthermore, we shall show that the existence of such minimal solenoids is a typical feature that occurs for generic Hamiltonian dynamical systems.

We recall the technical definition of a solenoid Σ_a, as specified by a sequence of positive integers $a = (a_0, a_1, a_2, a_3, \ldots)$, each $a_j \geqslant 2$. Consider a sequence of circles $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, with maps of degree a_j, say $z \to z^{a_j}$,

$$S \xleftarrow{a_0} S^1 \xleftarrow{a_1} S^1 \xleftarrow{a_2} S^1 \longleftarrow \cdots \quad .$$

Then the inverse or projective limit is the solenoid

$$\Sigma_a = \varprojlim \{ S^1, (a_j) \} \; .$$

In more explicit notation Σ_a is the subset of the countable Cartesian product $S^1 \times S^1 \times S^1 \times \cdots$ consisting of sequences $(z_0, z_1, z_2, z_3, \ldots)$ for which $z_{j+1}^{a_j} = z_j$, $j = 0, 1, 2, 3, \ldots$. It is known [3,4,7] that each such solenoid is a topological curve, that is, Σ_a is a compact, connected, separable metric space of 1-dimension; but Σ_a is not locally connected. There are noncountably many topological types of such solenoids, but we have some choice in the construction of any particular solenoid Σ_a. In fact, solenoids Σ_a and Σ_b are topologically homeomorphic provided:

for every prime power p^r;
p^r divides some product $(a_0 a_1 \cdots a_k)$ if and only if
p^r divides some product $(b_0 b_1 \cdots b_\ell)$.

We shall prove the existence of solenoidal minimal sets, for generic

1) See haiku above.

Hamiltonian systems on a compact symplectic manifold M^{2n}, by considering a limit of long period elliptic orbits, as explained below. The space $\mathcal{H}^{k+2}$ of Hamiltonians is the set of all real C^{k+2}-functions on M^{2n} (for any fixed integer $k \geq 0$). We impose the C^{k+2}-topology on $\mathcal{H}^{k+2}$, consistent with a complete metric, and consider generic (residual) subsets of $\mathcal{H}^{k+2}$. We can now state our principal theorem, whose proof appears later.

Theorem 1. Let $\mathcal{H}^{k+2}$ be the space of Hamiltonians on a compact symplectic manifold M^{2n} ($n \geq 2$ and any fixed integer $k \geq 4$). Then there exists a generic set $\mathcal{M}_\Sigma \subset \mathcal{H}^{k+2}$ such that: for each Hamiltonian $H \in \mathcal{M}_\Sigma$, and each solenoid Σ_a, there is a minimal set of H that is homeomorphic to Σ_a.

2. Local Theory of Hamiltonian Dynamical Systems. Consider a Hamiltonian function $H : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}$ in class C^{k+2} (fixed integer $k \geq 0$), and the corresponding Hamiltonian differential system in the (x, y) canonical coordinates

$$1) \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J \begin{pmatrix} H_x \\ H_y \end{pmatrix} .$$

Let p_0 be a critical point (where $dH = 0$) of 1). Say $p_0 = (0, 0)$, and take $H(0, 0) = 0$, so that near the origin we have

$$H(x, y) = \frac{1}{2}(x, y)\ S \begin{pmatrix} x \\ y \end{pmatrix} + \cdots$$

for a real symmetric matrix $S = S'$. The Hamiltonian differential system is then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = JS \begin{pmatrix} x \\ y \end{pmatrix} + \cdots$$

where the Hamiltonian matrix $A = JS$, in the Lie algebra $sp(2n, \mathbb{R})$ of

$Sp(2n, \mathbb{R})$, has the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n, -\lambda_1, -\lambda_2, \ldots, -\lambda_n)$.

Definition. The critical point p_0, and the Hamiltonian matrix A, are elliptic in case all the eigenvalues are pure imaginary, that is, $\operatorname{Re} \lambda_j = 0$ $j = 1, \ldots, n$. Further p_0, and A, are generic elliptic in case $\lambda_1, \lambda_2, \ldots, \lambda_n$ (say with $\operatorname{Im} \lambda_j > 0$) are linearly independent over the rational field.

Next let γ be a periodic orbit of the Hamiltonian differential system 1), and take a transversal $(2n-1)$-section Σ through some point $p_0 \in \gamma$. The Poincaré map P around γ maps Σ (or some neighborhood of $p_0 \in \Sigma$) into Σ by following the trajectories of 1) once around a tube encircling γ. Choose local canonical coordinates $(\bar{x}, \bar{y})$ about p_0 in $\mathbb{R}^n \oplus \mathbb{R}^n$ so that Σ is defined by $\bar{x}^1 = 0$, the energy levels $H = h$ are $\bar{y}_1 = h$, and the Hamiltonian vector field is just $\frac{\partial}{\partial \bar{x}^1}$ near p_0. Then Σ admits the parameter-symplectic coordinates $(h, \bar{x}^2, \ldots, \bar{x}^n, \bar{y}_2, \ldots, \bar{y}_n)$ as a neighborhood of the origin in the parameter-symplectic space $\mathbb{R} \times (\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})$, see [9] for precise definitions and details.

In this situation the Poincaré map

$$P : \Sigma \longrightarrow \mathbb{R} \times (\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}) ,$$

is a parameter-symplectic map, that is

$$\bar{y}_1 \to \bar{y}_1 = h \quad \text{(each energy level } h \text{ conserved)}$$

and on each level $\bar{y}_1 = h$ we have a symplectic map

$$P_h : (\bar{x}^j, \bar{y}_j) \longrightarrow (\bar{X}^j, \bar{Y}_j) \quad \text{for } j = 2,3,\ldots,n .$$

The map P_h gives all the information of the Poincaré map around γ, and we shall be interested in the k-jet $P_h^{(k)}$ at the origin p_0. Each such $P_h^{(k)}$ specifies an element of $J^k(2n-1)$, which is the space of all k-jets of parameter-symplectic maps around a fixed origin in $\mathbb{R} \times (\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})$. In particular the 1-jet $P_h^{(1)}$ at p_0 is specified by a symplectic matrix $dP_0 = \dfrac{\partial(\bar{X}, \bar{Y})}{\partial(\bar{x}, \bar{y})} \in Sp(2n-2, \mathbb{R})$ and a vector $\dfrac{\partial}{\partial h}\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \in \mathbb{R}^{2n-2}$. The eigenvalues of dP_0 are the (nontrivial) characteristic multipliers $(\mu_2, \mu_3, \ldots, \mu_n, \mu_2^{-1}, \mu_3^{-1}, \ldots, \mu_n^{-1})$.

Definition. The periodic orbit γ is elliptic in case all characteristic multipliers have modulus of one, that is, $|\mu_j| = 1$, $j = 2, \ldots, n$. In this case each $\mu_j = e^{2\pi i w_j}$ defines a real frequency w_j (mod 1).

A periodic orbit γ is non-degenerate in case all $\mu_j \neq 1$ for $j, 2, 3, \ldots, n$. Then γ lies in a geometric 2-cylinder or band of periodic orbits $\gamma(h)$, with least period varying differentiably with h. If γ is an elliptic orbit with distinct frequencies $w_2, \ldots, w_n$, then all $\gamma(h)$ are elliptic, and the distinct frequencies $w_j(h)$ vary differentiably and none vanishes (provided $\gamma(h)$ is suitably restricted). At this stage we do not specify any further notions of genericity of periodic orbits.

3. Generic Hamiltonian Dynamics on Symplectic Manifolds. A symplectic manifold M^{2n} is a differentiable 2n-manifold (connected, separable, metrizable C^∞-manifold without boundary), together with a C^∞-atlas of symplectic charts or local canonical coordinates (x, y) covering M^{2n}, and having coordinate transformation Jacobians T lying pointwise in the symplectic group $Sp(2n, \mathbb{R})$. For each Hamiltonian function $H : M^{2n} \to \mathbb{R}$ in C^{k+2} write the local vector field or differential system

1)
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J\begin{pmatrix} H_x \\ H_y \end{pmatrix}$$

in each canonical coordinate chart (x, y). Since $T \in Sp(2n, \mathbb{R})$ it is immediate that any two such local vector fields coincide on the intersection of symplectic charts, and thus a global Hamiltonian vector field X_H, or global Hamiltonian differential system is defined on M^{2n}.

For the same reason, the symplectic 2-form $\Omega = \sum_{j=1}^{n} dx^j \wedge dy_j$, in each symplectic chart (x, y). is a globally defined C^∞-form that is nonsingular $(\det J = +1)$, and closed $(d\Omega = 0)$ everywhere on M^{2n}. Using the symplectic form Ω we introduce a duality between covariant and contravariant vectors at each point of M^{2n}. In particular, a Hamiltonian function H has the gradient 1-form dH, and thus the contravariant or tangent vector X_H (also written $dH^\#$) defined by $\langle X_H, \Omega \rangle = dH$. In each symplectic chart (x, y) the Hamiltonian vector field $dH^\#$ is expressed by the Hamiltonian differential system 1).

We assume known the basic theory [1,5] of global Hamiltonian differential systems $dH^\#$ on symplectic manifolds M^{2n}. In particular, near each critical point or periodic orbit of $dH^\#$ the local analysis of section 2 above is valid, since those constructions were independent of the choice of local canonical coordinate chart. In the classical case of mechanics M^{2n} is the cotangent bundle of the positional configuration manifold, but we shall restrict attention to the case when M^{2n} $(n \geq 2)$ is compact (for example T^{2n}). Then each Hamiltonian differential system has trajectories (solution or integral curves) defined for all times $t \in \mathbb{R}$.

Definition. Let $\mathcal{H}^{k+2}$ be the set of all real functions of class C^{k+2} (any fixed integer $k \geq 0$) on a compact symplectic manifold M^{2n}. We impose the C^{k+2}-topology on $\mathcal{H}^{k+2}$.

The C^{k+2}-topology on $\mathcal{H}^{k+2}$ can be defined by a complete metric, with norm on $F \in \mathcal{H}^{k+2}$

$$\|F\| = \sum_{\alpha} \sum_{|\ell|=0}^{k+2} |D^{\ell} F|_{\alpha} .$$

Here D^{ℓ} is any partial derivative of total order $|\ell|$. Further the local norms $|D^{\ell} F|_{\alpha}$ are computed in $\overline{W}_{\alpha}$, where $\{W_{\alpha}\}$ is a finite open covering of M^{2n} and each compact $\overline{W}_\alpha$ lies in one coordinate chart wherein the derivatives are computed. A change of the covering $\{W_\alpha\}$ introduces an equivalent norm, and hence the C^{k+2}-topology is well-defined on $\mathcal{H}^{k+2}$. Note that $\mathcal{H}^{k+2}$ is a Baire space — each countable intersection of open-dense subsets is dense in $\mathcal{H}^{k+2}$.

<u>Definition</u>. A subset $\mathcal{P} \subset \mathcal{H}^{k+2}$ (or the logical property defining the set $\mathcal{P}$) is generic, or residual, in case $\mathcal{P}$ contains a countable intersection of open-dense subsets of $\mathcal{H}^{k+2}$.

In an earlier paper [5] the authors prove that the following set $\mathcal{S}_0$ is generic in $\mathcal{H}^{k+2}$ (for every $k \geq 0$):

$\mathcal{S}_0$: all $H \in \mathcal{H}^{k+2}$ such that

(i) H has a unique point at which it assumes its minimum on M^{2n}, and

(ii) at the unique minimum point H has a generic elliptic critical point.

Takens [9] has given a procedure for constructing generic subsets of $\mathcal{H}^{k+2}$ in terms of the k-jets of the Poincaré maps around periodic orbits. We next paraphrase a particular result of his general theory.

<u>Proposition</u>. Let Q be an intrinsic, analytic subset of the jet space $J^k(2n-1)$, fixed integer $k \geq 0$.

If codim $Q = 1$, then there exists a generic set $\mathcal{T}_Q \subset \mathcal{H}^{k+2}$ such that: each $H \in \mathcal{T}_Q$ has at most a countable number of periodic orbits whose k-jets have the property Q.

If codim $Q \geq 2$, then each $H \in \mathcal{T}_Q$ has no periodic orbit whose k-jet has the property Q.

Example 1. Consider $Q_1 \subset J^1(2n-1)$ defined by the discriminant of the characteristic polynomial $F(\mu) = \det |(dP_0)-\mu I|$ of the Jacobian matrix dP_0. Namely, Q_1 is the set specified by the condition: discrim $(F) = 0$. Then codim $Q_1 = 1$ in $J^k(2n-1)$, for every $k \geq 1$, and hence there exists a generic set $\tau_{Q_1} \subset \mathcal{H}^{k+2}$ such that: $H \in \tau_{Q_1}$ has only a countable number of periodic orbits possessing repeated (nontrivial) multipliers. In particular, H has only a countable number of degenerate periodic orbits. Since this result was obtained earlier by Robinson [8], we shall denote the generic set τ_{Q_1} by $\mathcal{R}$.

Example 2. In this same spirit define a set $Q_2 \subset J^1(2n-1)$ by the condition $F(1) = 0$, $F'(1) = 0$, $F''(1) = 0$, $F'''(1) = 0$. Then $\mu = +1$ is a characteristic multiplier of multiplicity 4, that is, $+1$ is a nontrivial characteristic multiplier of multiplicity > 2. In this case codim $Q_2 \geqslant 2$ and so each $H \in \tau_{Q_2}$ has no periodic orbit of this higher degeneracy. In particular, among the elliptic orbits of H no two frequencies $w_2, \cdots, w_n$ are simultaneously zero.

4. Generic Constructions for Minimal Solenoids. Fix a compact symplectic manifold M^{2n} $(n \geq 2)$ and consider the Baire space $\mathcal{H}^{k+2}$ (any fixed integer $k \geq 0$) of Hamiltonian functions on M^{2n}. We shall construct the required generic set $\mathcal{M}_\Sigma$ within $\mathcal{R} \cap \mathcal{S}_0$ (so, $k \geq 1$). Recall that the intersection of a finite or countable number of generic sets is still generic.

Lemma 1. Each Hamiltonian $H \in \mathcal{R} \cap \mathcal{S}_0 \subset \mathcal{H}^{k+2}$ (for $k \geq 1$) has an elliptic periodic orbit γ, with distinct frequencies $w_2,\ldots,w_n$, and γ lies within any prescribed neighborhood of the point where H achieves its minimum.

Proof

Since $H \in \mathcal{S}_0$ the minimum, say $H(p_0) = 0$, occurs at a generic elliptic critical point. By the theorem of Liapunov, as modified for the differentiable case [1], there are n families of periodic orbits concentric about p_0. Take γ to be any such Liapunov orbit with small energy $H = h_0$. The further details of the proof will appear in the completed paper in a later publication.

Note that γ is a nondegenerate elliptic *periodic* orbit, and so lies in a geometric 2-cylinder or band of elliptic orbits $\gamma(h)$, with $\gamma(h_o) = \gamma$. Furthermore, the corresponding frequencies $w_2(h),\ldots,w_n(h)$ are distinct for each h near h_o, and these vary differentiably. These properties will be relevant when we impose the further generic restrictions $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathcal{H}^{k+2}$ $(k \geq 2)$, by means of Takens polynomials in $J^2(2n-1)$.

$\mathcal{S}_1^{(r)}$: The characteristic polynomial $F^{(r)}(\mu)$ of the r-th power $(dP_o)^r$ satisfies the conditions $F^{(r)}(1) = 0$, $F^{(r)'}(1) = 0$, $F^{(r)''}(1) = 0$, $F^{(r)'''}(1) = 0$.

$$\mathcal{S}_1 = \bigcap_{r=1}^{\infty} \mathcal{S}_1^{(r)}.$$

Then $H \in \mathcal{S}_1$ has no elliptic periodic orbit that has two rational characteristic frequencies.

$\mathcal{S}_2$: $\frac{d}{dh}(\text{Trace } P_h^{(1)}) = 0.$

Then each elliptic orbit γ of $H \in \mathcal{S}_2$, with distinct characteristic frequencies $w_2,\ldots,w_n$, has a rational frequency — or else γ can be arbitrarily closely approximated within the band $\gamma(h)$ by some γ' with a rational frequency.

Lemma 2. Let $H \in \mathcal{S}_1 \cap \mathcal{S}_2 \subset \mathcal{H}^{k+2}$ (for $k \geq 2$) have an elliptic orbit γ with distinct characteristic frequencies $w_2,\ldots,w_n$. Then γ, or some arbitrarily nearby γ' within the band $\gamma(h)$, has exactly one frequency, say w_2, that is rational and all the other $w_3,\ldots,w_n$ are irrational.

The last generic condition $\mathcal{S}_3 \subset \mathcal{H}^{k+2}$ (for $k \geq 4$) is imposed by a Takens polynomial in $J^4(2n-1)$. In order to understand $\mathcal{S}_3$ consider the Birkhoff normal form [2] of the Poincaré map $\underline{P}$ around an elliptic periodic orbit γ of H. Assume that the frequencies $w_2, \ldots, w_n$ of γ are distinct, and furthermore they are "linearly independent over small integers", see [2,5,6]. Then there exist real canonical coordinates (x^i, y_i) $i = 2, \ldots, n$ in terms of which $\underline{P} : (x^i, y_i) \to (X^i, Y_i)$ has the form:

$$X^i = x^i \cos 2\pi \tilde{w}_i - y_i \sin 2\pi \tilde{w}_i + \hat{X}^i(x,y)$$

$$Y_i = x^i \sin 2\pi \tilde{w}_i + y_i \cos 2\pi \tilde{w}_i + \hat{Y}_i(x,y).$$

Here $\tilde{w}_i = w_i + \sum_{j=2}^{n} c_{ij} u_j + \ldots$ are real polynomials in the symplectic polar radii

$$u_i = \frac{(x^i)^2 + (y_i)^2}{2}$$. The degree of $\tilde{w}_i(u)$ can be pre-selected, and if

we require $\tilde{w}_i$ to be polynomials of degree 2, then the remainder terms $\hat{X}^i$, $\hat{Y}_i$ are of order 5 as $|x| + |y| \to 0$. The determinant $\Delta = \det (c_{ij})$ is a symplectic invariant of the orbit γ in H.

$\mathcal{S}_3$: $\quad \Delta = 0$ and $\dfrac{d\Delta}{dh} = 0$.

Then each appropriate elliptic orbit γ of $H \in \mathcal{S}_3$ will have a nonzero "twist coefficient", as explained later. A proof that the conditions defining $\mathcal{S}_3$ are described by intrinsic analytic subsets of $J^4(2n-1)$ will be given in the final complete paper.

Definition. In $\mathcal{H}^{k+2}$ (each fixed integer $k \geq 4$) define the generic set $\mathcal{M}_\Sigma = \mathcal{R} \cap \mathcal{S}_o \cap \mathcal{S}_1 \cap \mathcal{S}_2 \cap \mathcal{S}_3$.

The proof of our principle theorem 1, as stated in section 1 above, will now follow from a lemma concerning the existence of a periodic orbit γ_1 with a high encircling multiplicity q around a given periodic orbit γ. That is γ_1 is a closed curve homotopic to q times γ, within a tubular neighborhood about γ.

Lemma 3. Let $H \in \mathcal{M}_\Sigma \subset \mathcal{H}^{k+2}$ (any fixed integer $k \geq 4$) have an elliptic periodic orbit γ with distinct characteristic frequencies $w_2, \ldots, w_n$. Let $\Pi \approx S^{2n-2} \times S^1$ be a tori-i through which passes a tubular neighborhood of γ, and let q_o, s_o be primes ≥ 2.

Then there exists a periodic orbit γ_1 of H such that:

i) γ_1 lies inside Π and encircles this tube exactly q times before completing its least period.

Here q can be required to have the factorization $q = q_o^a s_o$, for some suitably large integer $a \geq 1$, and $a = 1$ is permissible if s_o is suitably large.

ii) γ_1 is an elliptic orbit with distinct characteristic frequencies.

Proof

Since γ is nondegenerate, it is embedded in a 2-band of periodic orbits $\gamma(h)$, and we choose the energy constant so $\gamma = \gamma(0)$. From lemma 2 we can assume that w_2 is rational and $w_3, \ldots, w_n$ are all irrational. In fact, using $\frac{dw_2}{dh}(0) \neq 0$, we can assume that $w_2 = p/q$ for any integers p,q provided that p/q lies in a suitable prescribed real interval. But then it is easy to choose $(p,q) = 1$ with q of the required form. (For instance, suppose $q_o = 2$, $s_o = 3$

and $\mathcal{P}_o/2^a\cdot 3$ lies within the allowed real interval, for some large $a \geq 1$. Take p_o odd and choose $p = p_o$ if 3 does not divide p_o; otherwise, choose $p = p_o + 2)$.

The first part of the proof of this lemma concerns the existence of a periodic orbit γ_1 of H which encircles Π exactly q times before closing. As a preliminary step we must rule out the possibility of periodic orbits in Π with encircling multiplicity $1 < \ell < q$. We give here only a sketch of the full argument; the details will appear later.

Consider the Poincare maps $\underline{P}$: $(x^i, y_i) \to (X^i, Y_i)$ $i = 2,\ldots,n$ for the trajectories of H upon encircling once around Π (or Π suitably narrowed). We seek to solve the functional equation $\underline{P}^q = \mathrm{Id}$ for some nontrivial q-periodic point of $\underline{P}$, and also to show that $P^\ell = \mathrm{Id}$ has no nontrivial solution for $1 \leq \ell < q$. Since all frequencies $w_2,\ldots,w_n$ of γ are distinct, there exist real canonical coordinates (still denoted (x^i, y_i)) diagonalizing the linear terms of $\underline{P}$ as a product of 2-dimensional rotations. Thus, for each $\ell = 1,2,\ldots,q$ the map $\underline{P}^\ell$ (denoted $(x,y) \to (X^\ell, Y^\ell)$) has fixed points that are solutions of the $(2n-2)$ equations $\underline{P}^\ell = \mathrm{Id}$, or:

$$x^2 \cos 2\pi\ell \frac{p}{q} - y_2 \sin 2\pi\ell \frac{p}{q} + \ldots = x^2$$

$$x^2 \sin 2\pi\ell \frac{p}{q} + y_2 \cos 2\pi\ell \frac{p}{q} + \ldots = y_2$$

$$x^3 \cos 2\pi\ell w_3 - y_3 \sin 2\pi\ell w_3 + \ldots = x^3$$

$$x^3 \sin 2\pi\ell w_3 + y_3 \cos 2\pi\ell w_3 + \ldots = y_3$$

$$\vdots$$

$$x^n \cos 2\pi\ell w_n - y_n \sin 2\pi\ell w_n + \ldots = x^n$$

$$x^n \sin 2\pi\ell w_n + y_n \cos 2\pi\ell w_n + \ldots = y_n$$

Here the omitted terms are of order $[|h| + \sum_{i=2}^{n} (x^i)^2 + (y_i)^2]$

near the point $h = 0$, $x = y = 0$ on γ.

a) Absence of small periodic orbits. For each $\ell = 1,2,3,\ldots,q-1$ the matrix of the linear terms in (x,y) is nonsingular. In this case the implicit function theorem guarantees the uniqueness of the solution $x = y = 0$, which corresponds to the known periodic orbit $\gamma(h)$. Thus the tube Π (suitably narrowed) contains no periodic orbits of H of encircling multiplicity less than q —— excepting $\gamma(h)$ which have encircling multiplicity of one.

b) Existence of q-periodic orbits : reduction to 2-surface $S(h)$.

Next consider the existence of a nontrivial solution (not $\gamma(h)$) of the equations $\underline{P}^q = \mathrm{Id}$. Here the first pair of equations reduce to $X^q = x^2 + \ldots = x^2$, $Y^q = y_2 + \ldots = y_2$ so the implicit function theorem is not applicable to the set of $(2n-2)$ equations.

However, consider the last $(2n-4)$ equations and treat (x^2, y_2, h) as parameters near zero. Then solve for differentiable functions

$$x^j = \sigma^j(x^2, y_2, h), \quad y_j = \tau^j(x^2, y_2, h) \qquad j = 3,4,\ldots,n.$$

For each small $|h|$ this locus in the ball $B^{2n-1} = \Sigma$ (transversal to γ within the tori-i $S^{2n-2} \times S^1$) is a 2-surface $S(h)$, covering a neighborhood of the origin in the (x^2, y_2)-plane. Each point of $S(h)$ is moved by $\underline{P}^q$ so that only the (x^2, y_2) coordinates are changed, and the other coordinates are unchanged.

c) Existence of q-periodic orbits: fixed points on $S(h)$

Introduce the symplectic polar coordinates $u_i = \dfrac{(x^i)+(y_i)^2}{2}$ $\theta_i = \arctan y_i/x_i$ and write the q-th itterate $\underline{P}^q$: $(u,\theta) \to (\mathcal{U}^q, \Theta^q)$. Restrict $\underline{P}^q$ to the surface $S(h)$, use the polar coordinates $u = u_2$, $\theta = \theta_2$, and then compute

$$\mathcal{U}^q = u + \ldots\ldots, \qquad \Theta^q = \theta + 2\pi p + q(\alpha h - \beta u) + \ldots\ldots$$

where the omitted terms are of order $(h^2 + u^2)$.

The constants $\alpha = \frac{dw_2}{dh}$ and $\beta = -\frac{\partial \Theta}{\partial u}$ (at $h = 0$, $u = 0$) are assumed nonzero ($\alpha \neq 0$ since $H \in \mathcal{S}_2$ and the twist coefficient $\beta \neq 0$ since $H \in \mathcal{S}_3$). We assume $\alpha > 0$, $\beta > 0$ and the other cases involve similar arguments.

Thus we seek a solution (u, θ) with $u > 0$ for the pair of equations

$$q(\alpha h - \beta u) + \ldots = 0, \quad u + \ldots = u.$$

Clearly, for $h \leq 0$ there is no solution with $u > 0$.

However, for each $h > 0$, the implicit function theorem applied to the first equation yields

$$u = u(\theta, h) = \frac{\alpha}{\beta} h + \ldots$$

which specifies a curve $\mathcal{C}(h)$ on the surface $S(h)$.

Each point of $\mathcal{C}(h)$ is moved by P^q so that (at most) the radial coordinate u changes. If P^q mapped $S(h)$ into itself, then area conservation would yield a fixed point on $\mathcal{C}(h)$. But, in any case, P^q is a symplectic map and so the differential 1-form $\sum_{j=2}^{n} (\mathcal{U}_j^q \, d\Theta_j^q - u_j d\theta_j)$ is closed. Within the simply-connected ball B^{2n-1} this 1-form is the differential dW of a real function W. But on $\mathcal{C}(h)$ we have $\Theta_j^q = \theta_j$ for $j = 2,3,\ldots n$ and $\mathcal{U}_j^q = u_j$ for $j = 3,\ldots,n$ so we obtain $dW = (\mathcal{U}^q - u)d\theta$.

Since W must have critical points at its maximum and minimum on the compact set $\mathcal{C}(h)$, we obtain at least two fixed points of P^q on each such curve $\mathcal{C}(h)$. Each of these points produces a nontrivial q-periodic orbit $\gamma_1^{(1)}(h)$ and $\gamma_1^{(2)}(h)$.

d) <u>Existence of elliptic orbit</u> γ_1. Since $H \in \mathcal{R}$, we can select $h > 0$ so that all q-periodic orbits of H in Π are nondegenerate, with distinct characteristic multipliers.

The method for this part of the proof is to show that the nondegeneracy of the q-periodic orbits implies the nondegeneracy of the critical points of W on $\mathcal{C}(h)$. Then there are only a finite number of such critical points of W and classical index-theoretic arguments show that at least one of these is elliptic, and so yields the required elliptic periodic orbit γ_1 with distinct characteristic frequencies.

We defer the details of the above proofs, and the application of the three lemmas to the principal theorem, to the completed paper that will appear in a later publication.

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