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Existence of Periodic Solutions of One-dimensional Differential-delay Equations

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1. Introduction

This paper is motivated by the data of numerical computing experiments by Professor Y. Ueda and his colleagues. In the study of phase lock loops which are widely used in communication systems also, in order to utilize the frequency range effectively, it has become necessary to consider phase locked loops acting in the high frequency range. In this case, it is necessary to analyze the acting principles of the system with time delays, since we cannot ignore influences to the system of time delays which arise in the parts of the system. In their studies for this purpose, the following difference-differential equation arises:

$$(1.1) \qquad \dot{u}(t) = \delta - \sin(u(t-h)), \; t \geqq 0, \; \delta \geqq 0, \; h > 0.$$

Roughly speaking, the variables in (1.1) are related with the model in the following way: t is the time, u(t) denotes the phase difference at time t, δ is the difference between signal frequency and free-running frequency of the voltage-controoled oscillator, and h is the sum of the time delays which arise in the parts of the system. In the case where $0 \leqq \delta \leqq 1$, (1.1) has trivial periodic solutions, namely, the constant functions $u(t) \equiv \alpha$, $t \geqq 0$, where α is a number such that $\sin\alpha = \delta$. In their experiments, they observed the existence of a nontrivial periodic solution for $\delta = 0.3$ and $h = 2$, a periodic solution of the second kind for $\delta = 0.8$ and $h = 2$, and solutions which approach asymptotically to a constant solution. Thus there arise the following problems. Find the relation between δ and h so that (1.1) has periodic solutions, or a constant solution is uniformly asymptotically stable. We shall give sufficient conditions for these problems in Sections 2, 3, and 4.

There are various methods and many results for the existence of periodic solutions of functional differential equations [cf. 1, 2, 3, 4, 6, 8]. We shall show the existence of periodic solutions of (1.1) by using a fixed point theorem for the truncated cones by Krasnosel'skii in [2, 3] (see Section 2). Especially, for the existence of a periodic solutions of the second kind of (1.1), we consider also the case where $\delta > 1$ (see Section 3). On the other hand, there are many results on the stability of solutions of functional differential equations [cf. 5, 7]. Using these results, in Section 4, we shall discuss the uniform asymptotic stability of a constant solution, the nonexistence of periodic solutions, and the nonexistence of periodic solutions of the second kind.

For a given $h > 0$, C denotes the space of continuous function mapping the interval $[-h, 0]$ into R, and for $\phi \in C$, $|\phi| = \sup_{-h \leqq \theta \leqq 0} |\phi(\theta)|$. For any continuous function $x(u)$ defined on $-h \leqq u < A$, $A > 0$, and any fixed t, $0 < t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t-h, t]$, i.e., x_t is an element of C defined by $x_t(\theta) = x(t+\theta)$, $-h \leqq \theta \leqq 0$.

2. Existence of a nontrivial periodic solution

Consider equation (1.1) for $0 \leqq \delta < 1$. Let $\alpha = \sin^{-1}\delta$. Then (1.1) has a constant solution $u(t) \equiv \alpha$. Substituting $x(t) = u(t) - \alpha$, $t \geqq -h$ into (1.1), we have an equivalent equation

$$\dot{x}(t) = \delta - \sin(x(t-h)+\alpha), \quad t \geqq 0, \quad 0 \leqq \delta < 1, \tag{2.1}$$

which has the zero solution $x(t) \equiv 0$. Define $f(x) = \delta - \sin(x+\alpha)$. Then $f(x)$ satisfies the following conditions for $-\pi - 2\alpha < x < \pi - 2\alpha$.

(H1) $f(x)$ is continuous, $xf(x) < 0$ for $x \neq 0$, $f(-\frac{\pi+2\alpha}{2}) = 1 + \delta$,

$f(\frac{\pi-2\alpha}{2}) = \delta - 1$ and $\delta - 1 \leqq f(x) \leqq 1 + \delta$.

(H2) $f(x)$ is nonincreasing in $-\frac{\pi+2\alpha}{2} < x < \frac{\pi-2\alpha}{2}$, and $f(x) \geqq -\frac{2(1-\delta)}{\pi-2\alpha}x$ for $-\frac{\pi+2\alpha}{2} \leqq x \leqq 0$, and $f(x) \geqq -\frac{2(1-\delta)}{\pi-2\alpha}x$ for $0 \leqq x \leqq \frac{\pi-2\alpha}{2}$.

(H3) $f(x) = -Lx + M(x)$ for $L = \sqrt{1-\delta^2} = \cos\alpha > 0$, where $M(x)$ is the higher order part and satisfies $|M(x) - M(y)| \leqq \mu(\sigma)|x - y|$ for $|x|$, $|y| \leqq \sigma$ and $\mu(\sigma)$ is continuous and nondecreasing with $\mu(0) = 0$.

Definition 2.1. Let E be a Banach space. A set $K \subset E$ is a cone if

(i) K is closed and convex,

(ii) if ϕ is in K, then $\lambda\phi \in K$, $\lambda \geqq 0$,

(iii) for any $\phi \neq 0$ in E, both ϕ and $-\phi$ cannot belong to K.

A truncated cone is the intersection of a cone with a convex neigborhood of zero. The neighborhood does not need to be closed.

For k_1, $\frac{1}{2} < k_1 < 1$, we define the following truncated cone K by

$$K=\{\phi \in C:\ \phi(-h)=0,\ \phi(\theta) \text{ is nondecreasing on } [-h,0],\ \phi(0) \leqq (\pi-2\alpha)k_1\}.$$

Lemma 2.1. Let

$$F(\alpha)=\frac{11(\pi-2\alpha)}{8a}-\frac{3(\pi-2\alpha)\sin\alpha+8\cos\alpha-8\cos(\frac{14\pi-3\alpha}{8})}{8a^2},\quad 0 \leqq \alpha < \frac{\pi}{2},$$

where $a = \sin\alpha + \cos(\frac{7\alpha}{4} + \frac{\pi}{8})$, and let $G(\alpha) = \min(\frac{3(\pi-2\alpha)}{8(1-\sin\alpha)} + \frac{\pi-2\alpha-2\cos\alpha}{2(1-\sin\alpha)^2}, F(\alpha))$ and $H(\alpha) = \max(\frac{\pi-2\alpha}{1+\sin\alpha} + \frac{\pi+2\alpha-2\cos\alpha}{2(1+\sin\alpha)^2}, G(\alpha))$. If $\frac{\alpha}{2\cos\alpha} < h < H(\alpha)$, then there exist k_0, k_1 such that $\frac{1}{2} < k_0 < k_1 < 1$, and for $\phi \in K\setminus\{0\}$, $x(t) = x(t, \phi)$ oscillates and all zero points are simple, and the minimal values of $x(t)$ are greater than $-(\pi+2\alpha)k_1$, and the maximal values of $x(t)$ are less than $(\pi-2\alpha)k_0$ uniformly for $\phi \in K\setminus\{0\}$. Furthermore, for the second zero point $t_0 = t_0(\phi) > 0$ of $x(t)$, we have $x_{\tau(\phi)}(\phi) \in K$, and $\{\tau(\phi)\}$ is bounded uniformly for $\phi \in K\setminus\{0\}$, where $\tau(\phi) = t_0 + h$.

For $\phi \in K\setminus\{0\}$, define the mapping A by

$$A\phi = x_{\tau(\phi)}(\phi).$$

Then, if $\frac{\pi}{2\cos\alpha} < h < H(\alpha)$, A is a positive mapping relative to K, that is, $A(K) \subset K$. Since $\{\tau(\phi)\}$ is bounded uniformly for $\phi \in K \setminus \{0\}$, define $\tau(0) = \limsup_{\phi \to 0} \tau(\phi)$. Then $\tau\colon K \to [0, \infty)$ takes closed bounded sets into bounded sets. Furthermore, since $x(t, \phi)$ is continuous in (t, ϕ), $\tau(\phi)$ is continuous on $K \setminus \{0\}$. On the other hand, A takes bounded sets into bounded sets because $|A\phi| \leqq (\pi-2\alpha)k_0$. Moreover, the following lemmas hold.

Lemma 2.2. If $\frac{\pi}{2\cos\alpha} < h < H(\alpha)$ and if G is an open bounded neighborhood of zero, then

$$\inf_{\phi \in \partial G \cap K} |A\phi| > 0.$$

Lemma 2.3. If $h > \frac{\pi}{2\cos\alpha}$, there is a zero $\lambda = \rho + i\sigma$ of

$$\lambda e^{h\lambda} = -L, \quad L = \cos\alpha \tag{2.2}$$

with $\rho > 0$, $0 < \sigma h < \pi$.

The linear part of (2.1) is

$$\dot{x}(t) = -Lx(t-h), \quad t \geqq 0, \tag{2.3}$$

and (2.2) is the characteristic equation of (2.3). Let $(\lambda_0, \overline{\lambda}_0)$ be the characteristic roots of (2.2) whose existence was guaranteed by Lemma 2.3. We decompose C by $(\lambda_0, \overline{\lambda}_0)$ as $C = U \oplus S$, $e^{\lambda_0 \theta} \in U$, and denote by Π_U the projection operator onto U.

Lemma 2.4. If $h > \frac{\pi}{2\cos\alpha}$, then for any ε, $0 < \varepsilon \leqq (\pi-2\alpha)k_1$, we have

$$\inf_{\phi \in \partial B(\varepsilon) \cap K} |\Pi_U \phi| > 0,$$

where $B(\varepsilon) = \{\phi \in C: |\phi| < \varepsilon\}$.

We are now ready to show the existence of a nontrivial periodic solution of (1.1) in $-\pi < u < \pi$ by the following theorem, which is found in [3].

Suppose K is a cone (or a truncated cone) such that for any ϕ $\in$ K, there is a time $\tau(\phi) > 0$ such that $x_{\tau(\phi)}(\phi) \in K$. If we let $A\phi = x_{\tau(\phi)}(\phi)$, $\phi \in K$, then $A: K \to K$ is a positive operator.

Theorem 2.1. Suppose A is the same as defined above, A is continuous, $\tau(\phi) \geq h$, $\phi \in K$, τ and A take closed bounded sets into bounded sets and the following conditions are satisfied:

(I) For any open bounded set $G \subset C$, $0 \in G$,

$$\inf_{\phi \in \partial G \cap K} |A\phi| > 0.$$

(II) If F is the set of positive eigenvectors of A, there is an $M > 0$ such that $\phi \in F$, $|\phi| = M$, $A\phi = \mu\phi$ implies $\mu < 1$.

(III) For any $\varepsilon > 0$,

$$\inf_{\phi \in \partial G \cap K} |\Pi_U \phi| > 0.$$

Under these conditions, there exists a nontrivial periodic solution of (1.1) with period greater than h. In (II), $\phi \neq 0$ is called a positive eigenvector if $A\phi = \mu\phi$ for a positive operator A.

Among the assumptions of Theorem 2.1, the continuity of A is given by the continuity of solutions for the initial conditions. Also we have that τ and A take closed bounded sets into bounded sets. Furthermore, if $\frac{\pi}{2\cos\alpha} < h < H(\alpha)$, then (I) holds by Lemma 2.2, (II) holds for $M > k_0 X_2$ by Lemma 2.1, and (III) also holds by Lemma 2.4. Hence we have the following theorem.

Theorem 2.2. If $\frac{\pi}{2\cos\alpha} < h < H(\alpha)$, then there exists a nontrivial periodic solution of (1.1) in $-\pi < u < \pi$, and its period is greater than 2h.

Remark. In particular, for $\delta = 0.3$ and $h = 2$, Ueda and his colleagues have observed the existence of a nontrivial periodic solution. We can conclude from Theorem 2.2 that there exists a nontrivial periodic solution for $\delta = 0.3$, $1.81 \leqq h \leqq 2.45$, and $|\delta| \leqq 0.445$, $h = 2$.

3. Existence of a periodic solution of the second kind

Employing the following theorem, which is found in [3], we show the existence of a periodic solution of the second kind for (1.1), where $x(t)$ is called a periodic solution of the second kind if there exist $X \neq 0$ and $T > 0$ such that $x(t+T) = x(t) + X$ for $t \geqq 0$.

Theorem 3.1. Suppose K is a cone (or a truncated cone), A is positive with respect to K, is completely continuous and F is the set of positive eigenvectors of A. If

(i) for any open set $G \subset C$, $0 \in G$,

$$\inf_{\phi \in \partial G \cap K} |A\phi| > 0,$$

(ii) there exists an $M > 0$ such that $\phi \in F$, $|\phi| = M$, $A\phi = \mu\phi$ implies $\mu < 1$,

(iii) there exists an open neighborhood H of zero, $\overline{H} \subset B(M)$, such that $\phi \in \partial H \cap F$, $A\phi = \mu\phi$ implies $\mu > 1$,

then A has a fixed point in $K \cap (B(M) \setminus \overline{H})$.

First, consider equation (2.1) for $0 < \delta \leqq 1$. Let $-\frac{\pi}{2} - 2\alpha < \xi_0(\alpha) < -\frac{\pi}{2} - \alpha$. For a fixed α and $\xi_0 = \xi_0(\alpha)$, let

$$K_0 = \{\psi \in C: \psi(-h) = \xi_0,\ \psi(\theta) \text{ is nondecreasing on } [-h,0],\ \psi(0) \leqq -\tfrac{\pi}{2}\},$$

and

$$K = \{\phi \in C: \phi = \psi - \zeta^{\xi_0} \text{ for some } \psi \in K_0\},$$

where $\zeta^{\nu}(\theta) = \nu$, $-h \leqq \theta \leqq 0$. Then K is a truncated cone. The following lemma holds.

Lemma 3.1. There exist $\xi_0 = \xi_0(\alpha)$ and δ_0, $0 < \delta_0 < 1$, such that for $\xi_0 \leqq \delta \leqq 1$,

$$\frac{\pi}{2(\delta+\sqrt{1-\delta^2})} < \min\left(\frac{\xi_0+2\pi}{1+\delta} + \frac{\pi+2\alpha-2\cos\alpha}{2(1+\delta)^2}, \frac{\pi+2\xi_0}{2(\sin(\xi_0+\alpha)-\delta)}\right), \tag{3.1}$$

and

$$\pi + \frac{6(\pi-2\alpha)(1-\delta)}{3\delta+\pi-3\sqrt{3}} < 2\alpha + \frac{(\pi-2\alpha+\delta)\delta}{2(\delta+\sqrt{1-\delta^2})}. \tag{3.2}$$

Furthermore, if h satisfies

$$\frac{\pi}{2(\delta+\sqrt{1-\delta^2})} \leqq h < \min\left(\frac{\xi_0+2\pi}{1+\delta} + \frac{\pi+2\alpha-2\cos\alpha}{2(1+\delta)^2}, \frac{\pi+2\xi_0}{2(\sin(\xi_0+\alpha)-\delta)}\right), \tag{3.3}$$

then any solution $x(t, \psi)$ of (2.1) reaches $\xi_0+2\pi$ in finite time uniformly for $\psi \in K_0$. Moreover, $\pi - 2\alpha \leqq x(t, \psi) \leqq \xi_0 + 2\pi$ for $t_0 - h \leqq t \leqq t_0$, where $t_0 = \inf\{t: x(t, \psi) = \xi_0 + 2\pi\}$, and consequently $x_{t_0+h}(\theta)$ is nondecreasing, and for some $\eta = \eta(\delta, h) > 0$ and k_0, $0 \lesssim k_0 < 1$, sufficiently near 1, we have $\eta \leqq x(t_0+h, \psi) - x(t_0, \psi) \leqq \frac{(\pi+2\xi_0)k_0}{2}$ uniformly for $\psi \in K_0$.

For $\phi \in K$ and $\psi = \phi + \zeta^{\xi_0}$, let $\tau(\phi) = t_0 + h$, and let $A: \phi \to \tilde{\phi} = x_\tau(\psi) - \zeta^{\xi_0+2}$. Then, under the assumptions of Lemma 3.1, A satisfies the assumptions of Theorem 3.1. Thus we have the following proposition.

Proposition 3.1. Under the assumptions of Lemma 3.1, (1.1) has a periodic solution of the second kind.

Next, consider equation (1.1) for $\delta > 1$. Let $x(t)$ be a periodic solution of the second kind of (1.1), and let $T > 0$ be the smallest period. Then it is easy to see that $x(t+T) - x(t) = 2p\pi$ for some integer p. We consider the case $p = 1$. Then T must be less than $\frac{2\pi}{\delta-1}$ because $\delta > 1$, and consequently it is sufficient to consider only $h < \frac{2\pi}{\delta-1}$. Let

$K_1 = \{\psi \in C: \psi(-h) = \frac{\pi}{2}, \psi(\theta) \text{ is nondecreasing on } [-h, 0], \psi(0) \leqq \frac{3\pi}{2}\}$,

and

$$K = \{\phi \in C: \phi = \psi - \zeta^{\frac{\pi}{2}} \text{ for some } \psi \in K_1\},$$

then K is a truncated cone. We have the following lemma.

Lemma 3.2. Let $\delta > 1$ in (1.1). Then any solution $x(t, \psi)$ of (1.1) is increasing and reaches $\frac{5\pi}{2}$ till the time $\frac{2\pi}{\delta-1}$. If $h < \frac{\pi}{1+\delta} + \frac{3\pi+2}{4(1+\delta)^2}$, then for k_0, $0 < k_0 < 1$, sufficiently near 1, $(\delta-1)h \leq x(t_0+h, \psi) - x(t_0, \psi) \leq k_0\pi$ uniformly for $\psi \in K_1$, where $x(t_0, \psi) = \frac{5\pi}{2}$.

If we define the mapping A similarly to the case $\delta \leq 1$, A satisfies the assumptions of Theorem 3.1 by Lemma 3.2, and hence we have the following proposition.

Proposition 3.2. Under the assumptions of Lemma 3.2, (1.1) has a periodic solution of the second kind.

Remark. (3.2) is true for $\delta_0 = \sin\frac{3\pi}{7}$, where $0.974 < \delta_0 < 0.975$. (3.1) holds for a wider region of δ than (3.2) for $\xi_0(\alpha) = -\frac{\pi}{2} - \frac{7\alpha}{2}$.

4. Stability of a constant solution

In this section, using a theorem of Yorke [6], we consider the nonexistence of nontrivial periodic solutions of (1.1) for $-\pi < u < \pi$ and the uniform asymptotic stability of the zero solution of (2.1) when h is smaller than that in Theorem 2.2. Furthermore, we consider the nonexistence of periodic solutions of the second kind.

Definition 4.1. We say 0 is uniformly stable for (2.1) if for any $\eta > 0$ there exists a $\rho = \rho(\eta)$ in $(0, \eta]$ such that for any $\phi \in C$ we have for all $t \geq 0$

$$|\phi| < \rho \text{ implies } |x(t, \phi)| < \eta.$$

Definition 4.2. Let $\gamma > 0$. We say 0 is uniformly asymptotically stable with attraction radius γ for (2.1) if

(i) 0 is uniformly stable,

(ii) there exists $T = T(\gamma_1)$ for each $\gamma_1 \in (0, \gamma)$ such that for any solution $x(t, \phi)$ of (2.1) with $|\phi| \leq \gamma_1$, $|x(t_0+s, \phi)| \leq \frac{\gamma_1}{2}$ for all $s \geq T(\gamma_1)$.

Consider a nonlinear one-dimensional differential-delay equation

$$(4.1) \qquad \dot{x}(t) = F(t, x_t), \ t \geq 0.$$

Let $C_\beta = \{\phi \in C: |\phi| < \beta\}$ and let $F: [0, \infty) x C_\beta \to R$ be continuous. For $\phi \in C_\beta$, define $M(\phi) = \max\{0, \sup_{-h \leq \theta \leq 0} \phi(\theta)\}$. The following theorem can be found in [7].

Theorem 4.1(Yorke). Let $\beta > 0$ and $h > 0$. Let $F: [0, \infty) x C_\beta \to R$ be continuous. Assume for some $c \geq 0$

$$(4.2) \qquad -cM(\phi) \leq F(t, \phi) \leq cM(-\phi) \text{ for all } \phi \in C_\beta.$$

(i) Assume $ch \leq \frac{3}{2}$. Then $x(t) = 0$ is a solution and is uniformly stable.

(ii) Assume $0 < ch < \frac{3}{2}$ and

$$(4.3) \begin{cases} \text{for all sequences } t_n \to \infty \text{ and } \phi_n \in C_\beta \text{ converges to a constant} \\ \text{nonzero function in } C_\beta, F(t_n, \phi_n) \text{ does not converge to } 0. \end{cases}$$

Then 0 is uniformly asymptotically stable, and if $|\phi| \leq \frac{2\beta}{5}$, then $x(t) \to 0$ as $t \to \infty$.

Remark. (i) and (ii) can be made more specific as follows [7].

$$(4.4) \begin{cases} \text{If } |\phi| < \frac{2\beta}{5}, \text{ then the solution } x(t, t_0, \phi) \text{ is defined and} \\ \text{satisfies } |x(t, t_0, \phi)| \leq \frac{5}{2}|\phi| \text{ for all } t \geq t_0. \end{cases}$$

$$(4.5) \begin{cases} V(t) \overset{\text{def}}{=} \sup_{t \leq s \leq t+3h} |x(s)| \text{ is a monotonic non-increasing function} \\ \text{for } t \geq t_0, \text{ and if } 0 < ch < \frac{3}{2}, \text{ then } V(t) \to 0. \end{cases}$$

For equation (2.1), let $f(x) = \delta - \sin(x+\alpha)$. Let $y = -c(\alpha)x$ be tangential to the curve $y = f(x)$ at the point $(-\xi(\alpha), f(-\xi(\alpha)))$ for $-\frac{\pi}{2} - \alpha < -\xi(\alpha) < -\alpha$. Moreover, let $\zeta_1(\alpha) = \min(\xi(\alpha), \frac{1-\sin\alpha}{\cos\alpha})$, $\zeta_2(\alpha) = \min(\xi(\alpha), \pi-2\alpha)$, $H(x) = \frac{3x}{2f(-x)}$ $(x \neq 0)$, and let $H(0) = \frac{3}{2}$. Corresponding to $f(x)$, define

$$f_1(x) = \begin{cases} f(-(\pi+2\alpha)k), & x < -(\pi+2\alpha)k, \\ f(x), & -(\pi+2\alpha)k \leqq x \leqq (\pi-2\alpha)k, \\ f((\pi-2\alpha)k), & x > (\pi-2\alpha)k, \end{cases}$$

where $0 < k < 1$. Then $F(t, \phi) = f_1(\phi(-h))$ is continuous and satisfies (4.3) for any $\beta > 0$. Moreover, since condition (4.2) for $F(t, \phi) = f_1(\phi(-h))$ is equivalent to

$$\frac{f_1(x)}{x} \geqq -c, \ x \neq 0, \tag{4.6}$$

we obtain the following proposition by applying Theorem 4.1.

Proposition 4.1. (i) If $h < H(\zeta_1(\alpha))$, then (1.1) has no nontrivial periodic solution in $-\pi < u < \pi$.

(ii) If $h < H(\zeta_2(\alpha))$, then for any γ, $0 < \gamma < \frac{2(\pi-2\alpha)}{5}$, 0 is uniformly asymptotically stable with attraction radius γ for (2.1).

(iii) If $h < \min(H(\zeta_2(\alpha), \frac{4(\pi-2\alpha)}{5(1+\sin\alpha)})$, then (1.1) has no periodic solution of the second kind.

Remark 1. If $0 < h < \frac{3}{2\cos\alpha}$, then for sufficiently small $\gamma > 0$, 0 is uniformly asymptotically stable with attraction radius γ for (1.1).

Remark 2. Since $\frac{1-\sin\alpha}{\cos\alpha} \leqq \frac{\pi-2\alpha}{\pi} \leqq \pi-2\alpha$, we have $\zeta_1(\alpha) \leqq \zeta_2(\alpha) \leqq \xi(\alpha)$ and consequently $H(\zeta_1(\alpha)) \geqq H(\zeta_2(\alpha))$. On the other hand, for any given $h > 0$, α, $0 < \alpha < \frac{\pi}{2}$, sufficiently near $\frac{\pi}{2}$, satisfies conditions in (i) and (ii), since $\lim_{\alpha\to\frac{\pi}{2}-0} H(\zeta_2(\alpha)) = \infty$.

Remark 3. Since we can take $c = \frac{\sin\alpha+1}{\alpha+1}$ in (4.2) for $f(x)$, if $0 < h < \frac{3(\alpha+1)}{2(\sin\alpha+1)}$, then (i) and (ii) hold. Moreover, since $\xi(\alpha) \geqq \alpha$ and $\frac{1-\sin\alpha}{\cos\alpha}$ is decreasing, if we let $\alpha_0\cos\alpha_0 = 1-\sin\alpha_0$, $\frac{\pi}{6} < 0.555 < \alpha_0 < 0.556 < \frac{\pi}{5}$, then $\zeta_1(\alpha) = \frac{1-\sin\alpha}{\cos\alpha} \leqq \alpha$ and consequently $H(\zeta_1(\alpha)) \geqq H(\alpha) = \frac{3\alpha}{2\sin\alpha}$ for $\alpha_0 \leqq \alpha < \frac{\pi}{2}$. Therefore, if $0 < h < \frac{3\alpha}{2\sin\alpha}$, then the condition in (i) holds for $\alpha_0 \leqq \alpha < \frac{\pi}{2}$. Similarly, if $0 < h < \frac{3\alpha}{2\sin\alpha}$, then the condition in (ii) holds for $\alpha \geqq \frac{\pi}{3}$.

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