数理解析研究所講究録
第 325 巻 1978 年 78-83



# An inequality for finite permutation groups

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## 0. Introduction

Let $(G,\Omega)$ be a permutation group of degree n. For any subset X of G, we put

$$F(X):=\{\alpha\in\Omega \mid \forall x\in X \ \ \alpha^{x}=\alpha\}$$
$$f(X):=|F(X)| .$$

For $x\in G$, we use $f(x)$ instead of $f(\{x\})$.

(Definition 1) Let $\ell_i$ $(i=1,\cdots,r)$ be integers such that $0\leq\ell_1<\cdots<\ell_r<n$. We say that $(G,\Omega)$ is an $\{\ell_1,\cdots,\ell_r\}$-group, if $\{f(x) \mid x\in G,\ x\neq 1\}\subseteq\{\ell_1,\cdots,\ell_r\}$ .

E. Bannai and M. Deza posed us the following conjecture ; if $(G,\Omega)$ is an $\{\ell_1,\cdots,\ell_r\}$-group of degree n, then $|G|\leq\prod_{i=1}^{r}(n-\ell_i)$. In §1 this conjecture is proved. In §§ 2,3 we consider the case $|G|=\prod(n-\ell_i)$. Finally in § 4, using the same method as in Theorem 1, we give a proof of the Burnside-Brauer Theorem.

## 1. Proof of the conjecture

Here we prove the conjecture mentioned above.

<u>Theorem 1</u>. [5] Let $(G,\Omega)$ be an $\{\ell_1,\cdots,\ell_r\}$-group of degree n. Then $|G|$ divides $\prod_{i=1}^{r}(n-\ell_i)$.

Proof. Let $\theta$ be the permutation character of G, and let $1_G$ be the principal character of G. Then it is well known that

$$\hat{\theta} := \prod_{i=1}^{r}(\theta - \ell_i 1_G)$$

is a generalized character of G. By the definition of $\hat{\theta}$, we have $\hat{\theta}(g)=0$ for all $g \in G$, $g \neq 1$. Hence, the multiplicity of $1_G$ in $\hat{\theta}$ is given by

$$(\hat{\theta}, 1_G) = \frac{1}{|G|}\sum_{g \in G}\hat{\theta}(g) = \frac{1}{|G|}\hat{\theta}(1) = \frac{1}{|G|}\prod_{i=1}^{r}(n-\ell_i).$$

Thus, we get the desired result.

<u>Corollary 2</u>. Assume the hypothesis of Theorem 1. Then we have that $|G| = \prod_{i=1}^{r}(n-\ell_i)$ if and only if $\hat{\theta}$ is the regular character of G, where $\hat{\theta}$ is defined in the proof of Theorem 1.

## 2. $\{\ell_1, \cdots, \ell_r\}$ -sharp groups

(Definition 2) Assume the hypothesis of Theorem 1. We say that $(G,\Omega)$ is an $\{\ell_1, \cdots, \ell_r\}$ -sharp group, if $|G| = \prod_{i=1}^{r}(n-\ell_i)$.

We remark that $\{0, 1, \cdots, r-1\}$-sharp group is sharply r-transitive (see Corollary 4). Hence our concept is a generalization of sharply transitivity. It is natural that one hopes to classify all $\{\ell_1, \cdots, \ell_r\}$ -sharp groups. But in general it seems to be difficult. So we must study special cases at first.

Now we state some examples and known results.

<u>Example 1</u>. $Z_\ell \wr Z_2$ is a $\{0, \ell\}$-sharp group of degree $2\ell$.

<u>Example 2</u>. $\{1, 3\}$-sharp groups

(1) $G=S_4$ ; $\Omega=\Delta\cup\Gamma$, $G^{\Delta}=S_3$, $G^{\Gamma}=S_4$.

(2) $G=PSL(2,7)$ ; $\Omega=\Delta\cup\Gamma$, $G^{\Delta}$ is 2-transitive of degree 7,
$G^{\Gamma}$ is 2-transitive of degree 8.

Known results. For the following $L=\{\ell_1,\cdots,\ell_r\}$, L-sharp groups have been classified.

| | |
|---|---|
| $L=\{2\}$ | Iwahori [3] |
| $L=\{3\}$ | Iwahori and Kondo [4] |
| $L=\{0,2\}$ | Tsuzuku [6] |

The following lemma is due to E. Bannai.

Lemma 3. Let G be a $\{0,\ell_2,\cdots,\ell_r\}$ -sharp group on $\Omega$. Then G is transitive on $\Omega$, and $G_\alpha$ is an $\{\ell_2-1,\cdots,\ell_r-1\}$ -sharp group on $\Omega-\{\alpha\}$, where $\alpha$ is any element of $\Omega$.

Applying Theorem 1 to $G_\alpha$ , we can easily get the proof of Lemma 3.

Corollary 4. Let G be a $\{0,1,\cdots,r-1\}$ -sharp group. Then G is sharply r-transitive.

The following Theorem, due to T. Ito, is an extension of Corollary 4.

Theorem 5. [2] Let G be an $\{\ell,\ell+1,\cdots,\ell+r-1\}$-sharp group on $\Omega(r\geq 2)$. Then $f(G)=\ell$ and G is sharply r-transitive on $\Omega-F(G)$.

Remark. It looks very likely that every $\{\ell_1,\cdots,\ell_r\}$ -sharp group has $\ell_1+1$ orbits. Note that Lemma 3 is a special case where $\ell_1=0$.

## 3. The case r=2

Now we consider the case r=2 i.e. $\{\ell,\ell+s\}$ -sharp groups. In this case we can show that f(G) is considerably large and that $\ell-f(G)$ is bounded by a function of s. Hence the essential parameter is s alone. More precisely we have

Theorem 6. [2] Let G be an $\{\ell,\ell+s\}$ -sharp group.
Put $s':=\max\left\{1, \left[\frac{s-1}{2}\right]\right\}$, $m:=\ell+(1-s)s'+s'^2-1$.
Then we have $f(G)\geq m$.

For s=1,2,3,4 this inequality is best possible. For $s\geq 5$ we guess that f(G)=m does not occur. But I can not prove it yet.

Using Theorem 6, we can classify all $\{\ell,\ell+s\}$ -sharp groups for s=1,2,3,4. For example, the $\{\ell,\ell+2\}$ -sharp groups are the following groups ; $G=D_8$, $S_4$, GL(2,3), PSL(2,7). These groups are determined up to permutation isomorphism. For more details see [2]. The case $s\geq 5$ is very difficult.

## 4. Final remark

We give another example which can be proved by the same method as in the proof of Theorem 1. Let G be a finite group, and let $\theta$ be a faithful character of G. Let $\theta(1)=\alpha_1, \alpha_2, \cdots, \alpha_m$

be the distinct values taken by $\theta$. We put $\hat{\theta} := \prod_{i=2}^{m}(\theta-\alpha_i)$.
Since $\theta$ is faithful, we have

$$\hat{\theta} = \alpha \cdot \rho_G = \sum_{\chi \in Irr(G)} \alpha\,\chi(1)\chi, \text{ where } \alpha = \frac{1}{|G|}\hat{\theta}(1) \in \mathbb{C}.$$

Since $\hat{\theta}(1) \neq 0$, we have $\alpha \neq 0$. On the other hand $\hat{\theta}$ is a $\mathbb{C}$-linear combination of $\theta^j$ for $0 \leq j < m$, as it can be seen from the definition of $\hat{\theta}$. Then every $\chi \in \mathrm{Irr}(G)$ must be a constituent of some $\theta^j$. Thus we obtain

__Theorem__.(Burnside-Brauer cf. [1] p49) Let $\theta$ be a faithful character of G and suppose $\theta(g)$ takes exactly m different values for $g \in G$. Then every $\chi \in \mathrm{Irr}(G)$ is a constituent of one of the characters $\theta^j$ for $0 \leq j < m$.

If some $\alpha_i = 0$, then $\hat{\theta}$ is a $\mathbb{C}$-linear combination of $\theta^j$ for $0 < j < m$. Thus we obtain

__Corollary__. Assume the hypothesis of the Theorem. Suppose that $\theta(g) = 0$ for some $g \in G$. Then every $\chi \in \mathrm{Irr}(G)$ is a constituent of one of the characters $\theta^j$ for $0 < j < m$.

We remark that every non-linear faithful irreducible character of G satisfies the hypothesis of the Corollary.

References

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