数理解析研究所講究録
第 344 巻 1979 年 95-101

On a Transfer Theorem for Schur Multipliers

Hiroki Sasaki

Hokkaido University

In [3], D.F.Holt proved the following theorem.

Theorem*(Holt).

Let P be a Sylow p-subgroup of a finite group G, and suppose that P has nilpotency class at most p/2. Then the Sylow p-subgroups of the Schur multipliers of G and $N_G(P)$ are isomorphic.

In this report we shall show that the above theorem can be obtained by an application of the method of G-functors which was originated by J.A.Green and is developed by T.Yoshida.

Maps and functors will be written on the right in their arguments, with the corresponding convention for writing composites.

Let G be a finite group and k a commutative ring with identity element.

Definition 1.

A G-functor over k is defined to be a quadruple $A = (a, \tau, \rho, \sigma)$ where a, τ, ρ, σ are families of the following kinds :

$a = (a(H))$ gives, for each subgroup H of G, a finitely generated k-module $a(H)$.

$\tau = (\tau_H{}^K)$ and $\rho = (\rho^K{}_H)$ give, for each pair (H , K) of subgroups of G such that $H \leq K$, the respective k-homomorphisms $\tau_H{}^K : a(H) \longrightarrow a(K)$ and $\rho^K{}_H : a(K) \longrightarrow a(H)$.

$\sigma = (\sigma_H^g)$ gives, for each pair (H , g) where H is a subgroup of G and g an element in G, the k-homomorphism $\sigma_H^g : a(H) \longrightarrow a(H^g)$.

These families of k-modules and k-homomorphisms must satisfy the following

Axioms for G-functors. (In these axioms D, H, K, L are any subgroups of G ; g, g' are any elements in G.)

(a) $\tau_H{}^H = 1_{a(H)}, \quad \tau_H{}^K \tau_K{}^L = \tau_H{}^L$ if $H \leq K \leq L$;

(b) $\rho^H{}_H = 1_{a(H)}, \quad \rho^K{}_H \rho^H{}_D = \rho^K{}_D$ if $K \geq H \geq D$;

(c) $\sigma_H^h = 1_{a(H)}$ if $h \in H, \quad \sigma_H^g \sigma_{H^g}^{g'} = \sigma_H^{gg'}$;

(d) $\tau_H{}^K \sigma_K^g = \sigma_H^g \tau_{H^g}{}^{K^g}, \quad \rho^K{}_H \sigma_H^g = \sigma_K^g \rho^{K^g}{}_{H^g}$;

(e) (Mackey axiom) If $H \leq L$, $K \leq L$ and Γ is a transversal of the (H, K)-double cosets in L, then

$$\tau_H{}^L \rho^L{}_K = \sum_{g \in \Gamma} \sigma_H^g \rho^{H^g}{}_{H^g \cap K} \tau_{H^g \cap K}{}^K.$$

The images by the k-homomorphisms $\tau_H{}^K$, $\rho^K{}_H$ and σ_H^g are simply written as follows ;

$\alpha\tau_H{}^K = \alpha^K$ for α in $a(H)$, $\beta\rho^K{}_H = \beta_H$ for β in $a(K)$

and $\alpha\sigma_H^g = \alpha^g$ for α in $a(H)$, respectively.

A G-functor A is naturally considered to be a H-functor for any subgroup H of G. We denote such a H-functor by $A_{|H}$.

Definition 2.

A G-functor $A = (a, \tau, \rho, \sigma)$ is called cohomological if it satisfies the following axiom (C) :

(C) If $H \leq K \leq G$, then

$$\rho^K{}_H \tau_H{}^K = |K : H| 1_{a(G)}.$$

For examples of G-functors, see [2] and [7].

Definition 3.

Let $A = (a, \tau, \rho, \sigma)$ be a cohomological G-functor and let S be a subgroup of G, α an element in $a(S)$, and X a subgroup of G. Then a triple (S, α, X) is called a singularity in G for A provided

(a) $\alpha^G{}_X \neq 0$,

(b) $\alpha_{S \cap Y^u} = 0$ for every proper subgroup Y of X and every element u in G.

The subgroup S is called the singular subgroup of the singularity. If the singular subgroup S is a proper subgroup of G, then the singularity is called proper.

Now we can state a transfer theorem for cohomological G-functors on which our proof of Theorem* depends.

Theorem 1.

Let P be a Sylow p-subgroup of a finite group G and $A = (a, \tau, \rho, \sigma)$ a cohomological G-functor over a commutative ring k. Assume that the ring k is uniquely divisible by $|G : P|$ and P has no proper singularity in P for $A_{|P}$. Then

$$\mathrm{Im}\rho^G{}_P = \mathrm{Im}\rho^N{}_P, \text{ where } N = N_G(P).$$

And therefore

$$a(G) \simeq a(N).$$

We consider the following functor. Let M(G) denote the Schur multiplier $H^2(G, \mathbf{C}^*)$ of a finite group G. For each subgroup H of G, put $a(H) = \Omega_1(M(H)_p)$ where $M(H)_p$ is the Sylow p-subgroup of M(H). For each pair (H, K) of subgroups of G such that $H \leq K$, let τ_H^K and ρ_H^K be $cor_{H,K|a(H)}$ and $res_{K,H|a(K)}$, respectively. For each pair (H, g) of a subgroup H of G and an element g in G, we define $\sigma_H^g = con_{H|a(H)}^g$. Then $A = (a, \tau, \rho, \sigma)$ is a cohomological G-functor over $\mathbf{F}_p$. We call this functor the <u>multiplier functor</u> (with respect to a prime p).

If a Sylow p-subgroup P of G has no proper singularity in P for $A_{|P}$, then by Theorem 1 we have

$$\Omega_1(M(G)_p) \simeq \Omega_1(M(N_G(P))_p).$$

Hence by Tate's theorem it follows that

$$M(G)_p \simeq M(N_G(P))_p.$$

Thus we can establish Theorem*(Holt) by proving the following Theorem 2.

<u>Theorem</u> 2.

Let P be a p-group of nilpotency class at most p/2. Then P has no proper singularity in P for the multiplier functor.

<u>Sketch of the proof</u>. Suppose that P has a proper singularity (S, α, X) in P for the multiplier functor. First we may assume that the singular subgroup S is a maximal in P and X is contained in S by general arguments for singularities in a p-group.

Second we have two extensions

$$1 \to Z \to H \to P \to 1 \quad \text{and} \quad 1 \to Z \to K \to S \to 1,$$

where the latter extension is a central extension of S and the transgression map

$$t : \mathrm{Hom}(Z, C^*) \to M(S)$$

is an isomorphism and hence $Z \simeq M(S)$.

Let ϕ be a unique element in $\Omega_1(\mathrm{Hom}(Z, C^*))$ such that $\alpha = (\phi)t$ and I the kernel of ϕ. Then I is maximal in Z. Let u be an element in $P - S$. Then we have $I \neq I^u$ from the property that $\alpha^G{}_X \neq 0$. Hence if we put $L = \bigcap_{i=0}^{p-1} I^{u^i}$, then a factor group Z/L is an elementary abelian p-group of order p^p that has a basis on which u acts regularly. Let T be a semidirect product of Z by P. Then T involves the wreath product $Z_p \wr Z_p$ since L is normalized by P. Hence the nilpotency class of T is at least p.

On the other hand as in [3] Lemma 7 it follows that the nilpotency class of T is less than p by using the assumption that P has nilpotency class at most p/2. Thus we have a contradiction.

<u>Remark</u>. As we have seen in the above if a p-group P has a proper singularity in P for the multiplier functor, then P has a maximal subgroup S whose Schur multiplier

M(S) has a factor group isomorphic to an elementary abelian p-group of order p^p. Therefore a p-group which has no such maximal subgroup has no proper singularity. For example a 2-group of maximal class has no proper singularity.

References.

[1] A.Babakhanian, Cohomological Methods in Group Theory, Marcel Dekker, Inc., New York, 1972.

[2] J.A.Green, Axiomatic representation theory for finite groups, J. Pure Appl. Algebra, 1 (1971), 41 - 77.

[3] D.F.Holt, On the local control of Schur multipliers, Quart. J. Math. Oxford (2), 28 (1977), 495 - 508.

[4] B.Huppert, Endliche Gruppen I, Springer-Verlag, Berlin/ Hidelberg/New York, 1967.

[5] M.Suzuki, Group Theory, Iwanami, Tokyo, 1977 (Japanese).

[6] K.Yamazaki, Projective representations and ring extensions, J. Fac. Sci. Univ. Tokyo. Sect. 1 vol. 10 (1964), 147 - 195.

[7] T.Yoshida, On G-functors (I) : Transfer theorems for cohomological G-functors, Hokkaido Math. J., to appear.

[8] _________, Transfer theorems in cohomology theory of finite groups, Japanese 23-th Symposium on Algebra, 1977.

[9] H.Sasaki, On a transfer theorem for Schur multipliers, Hokkaido Math. J., to appear.