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Uniqueness of Normalized forms of Heegaard diagrams

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Let $(V, W; F)$ be a Heegaard splitting of a 3-manifold M, that is, V and W are handlebodies of some fixed genus n such that $M = V \cup W$ and $V \cap W = \partial V \cap \partial W = F$. A properly embedded 2-disk D in a handlebody X of genus n is called a meridian-disk of X if the closure $cl(X - N(D,X))$ is a handlebody of genus $n-1$, and a collection of mutually disjoint n meridian-disks, D_1, $D_2, \cdots, D_n$, in X is called a complete system of meridian-disks of X if $cl(X - \bigcup_{i=1}^{n} N(D_i,X))$ is a 3-ball. Moreover a collection of mutually disjoint circles on the boundary of X is called a complete system of meridians of X if it bounds a complete system of meridian-disks of X.

Let $\{v\} = \{v_1, \cdot\cdot, v_n\}$ and $\{w\} = \{w_1, \cdot\cdot, w_n\}$ be complete systems of meridians of V and W, respectively. Then a triad $(F; v, w)$ is called a Heegaard diagram of $(V, W; F)$ (or M). Hereafter we assume that all meridian circles of Heegaard diagrams considered in this paper transversely intersect. Two Heegaard diagrams (F; v, w) and (F; v',w') of a Heegaard splitting (V, W; F) are isomorphic if there exists a homeomorphism f of F such that f(v) = v' and f(w) = w'. The isomorphism relation is denoted by the symbol ≃ . An isomorphism class of Heegaard diagrams is also called a Heegaard diagram.

A diagram (F; v, w) gives rise to a partition of F into a set of domains, Γ, that is, cutting F along v and w the set Γ is obtained as a result. Then the Heegaard diagram (F; v, w) is said to be normal if the set of domains Γ has no domain whose boundary consists of exactly two edges (remark that the arcs of the meridians of the diagram that lie on between the points where the meridians intersect are called the edges of the diagram). If Γ has a domain whose boundary consists of exactly two edges, such a domain is eliminated by an isotopy deformation on F of one of the two complete systems of meridians with respect to the other and as a result of applying this isotopy deformation to a diagram we obtain again a new diagram whose manifold is homeomorphic to the original one. An operation of applying a finite sequence of such isotopy deformations to a Heegaard diagram is called a normalization of a Heegaard diagram if the resulting diagram is normal. The normal diagram obtained by a normalization of a Heegaard diagram is called the normalized form of the original diagram.

Two Heegaard diagrams $(F;\ v,\ w)$ and $(F;\ v',w')$ of a Heegaard splitting $(V,\ W;\ F)$ are equivalent if there are two ambient isotopies $H_i : F \times I \to F \times I$ $(i = 1,2)$ such that $H_1(v',1) = v$ and $H_2(w',1) = w$. The equivalence relation is denoted by the symbol $\sim$. Then we have;

Lemma 1. Let $(F;\ v,\ w)$ and $(F;\ v',w')$ be two normal Heegaard diagrams of a Heegaard splitting $(V,\ W;\ F)$. Then they are isomorphic if $(F;\ v,\ w) \sim (F;\ v',w')$.

Proof. Let $H_i : F \times I \to F \times I$ $(i = 1,2)$ be two isotopies on F such that $H_1(v',1) = v$ and $H_2(w',1) = w$. Let $v'' = H_2(v',1)$. Then the triad $(F;\ v'',\ w)$ is a Heegaard diagram and this diagram is isomorphic to $(F;\ v',w')$ and equivalent to $(F;\ v,\ w)$. Let $H ; F \times I \to F \times I$ be an ambient isotopy on F such that $H(v'',1) = v$. We may assume that $H(v''_i, 1) = v_i$ $(i = 1,2,\cdots,n)$ where $v = v_1 \cup \cdots \cup v_n$ and $v'' = v''_1 \cup \cdots \cup v''_n$. If $v_i \cap v''_i = \Phi$ for all i, then by Epstein [1] there exist n annuli $A_1, \ldots, A_n$ such that $\partial A_i = v_i \cup v''_i$ $(i = 1,2,\ldots,n)$, because of v_i being disjoint from v''_i and isotopic on F to v''_i. Since $(F;\ v,\ w)$ and $(F;\ v'',\ w)$

are normal and $v_i \cap v_j = \Phi$, $v_i'' \cap v_j'' = \Phi$ for all i and j, where $i \neq j$ and i, j = 1,2,..,n, we may assume that all of annuli, A_1 ,..., A_n, are mutually disjoint. For if $A_i \cap A_j \neq \Phi$ for some i and j, then there is a 2-disk D in A_i, whose boundary consists of an arc k_1 in v_i and an arc k_2 in v_j'', with $\mathring{k}_1 \cap v'' = \Phi$. Let $\tilde{v}_j''$ be a circle obtained by a slightly isotopy modification of the circle $k_1 \cup (v_j'' - k_2)$ around k_1 and let $\tilde{v}'' = v_1'' \cup .. \cup v_{j-1}'' \cup \tilde{v}_j'' \cup .. \cup v_n''$. It is clear that $(F; \tilde{v}_j'' w) \sim (F; v'', w)$. Since (F; v, w) and (F;v'', w) are normal, each component of $w \cap A_i$ (resp. $w \cap D$) is an arc joining v_i and v_i'' (resp. k_1 and k_2) and so we have that $(F; \tilde{v}'', w) \simeq (F; v'', w)$. It is clear that the intersection of v and $\tilde{v}''$ is less than that of v and v''. Thus we can assume that $A_i \cap A_j = \Phi$ for all i and j where $i \neq j$ and i, j = 1,2,..,n. Then (F; v, w) is isomorphic to (F; v'', w), because each component of $w \cap A_i$ for all i (i = 1,2,..,n) is an arc joining v_i and v_i''.

Next let us suppose that $v_i \cap v_i'' \neq \Phi$ for some i. Then by Epstein [1] there exists a 2-disk D' in F, whose boundary consists of an arc k_1' in v_i and an arc k_2' in v_i'', because v_i is isotopic on F to v_i''. We may assume that $\mathring{k}_1' \cap v'' = \Phi$. Let $v_i(1)$ be a meridian obtained by a slightly isotopy modification of the circle $k_1' \cup (v_i'' - k_2')$ around k_1' and let $v(1) = v_1'' \cup .. \cup v_{i-1}''$ $\cup\, v_i(1) \cup .. \cup v_n''$. It is clear that $(F; v(1), w) \sim (F; v'', w)$. Since each component of $D' \cap w$ is an arc joining k_1' and k_2', we have that $(F; v(1), w) \simeq (F; v'', w)$. It is clear that the intersection of v_i and $v_i(1)$ is less than that of v_i and v_i''. Thus we can assume that $v_i \cap v_i'' = \Phi$ for all i (i = 1,2,..,n). This completes the proof of the lemma.

By Lemma 1, we have the following basic result;

Theorem 1. All normalized forms of a Heegaard diagram are isomorphic.

References

[1] D. B. A. Epstein, Curves on 2-manifolds and isotopies, Acta Math., 115, 83-107, 1966.