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Difference Analogue of Volterra's Equation

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In the classic works of Volterra and Lotka the following coupled nonlinear differential equation

$$\begin{cases} \dfrac{dx}{dt} = (\alpha - y)x \ , \\ \dfrac{dy}{dt} = -(\beta - x)y \ , \end{cases} \tag{1}$$

is presented to discribe the growth of populations of two species, prey x and predator y[1], where α and β are positive parameters.

The differential mapping (1) is known to exhibit the following invariant curve

$$x + y - \beta \log x - \alpha \log y = \text{const.}$$

We look for a difference analogue of eq.(1) which exhibits an invariant curve. For this purpose we transform eq.(1) into the bilinear form and construct a difference analogue of the bilinear form using the dependent variable transformation.[2),3)]

Let $x(t) = g(t)/f(t)$ and $y(t) = h(t)/f(t)$, then eq.(1) is transformed into the following bilinear form

$$\begin{cases} D_t g(t)\cdot f(t) = \alpha\, g(t)f(t) - g(t)h(t) \\ D_t h(t)\cdot f(t) = -\beta h(t)f(t) + g(t)h(t) \ , \end{cases} \tag{2}$$

where the bilinear operator D_t^n operating on $a \cdot b$ is defined, for an integer n, by

$$D_t^n\, a(t) \cdot b(t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(t) b(t') \Big|_{t=t'} .$$

A difference analogue of eq.(2) is obtained by replacing the bilinear operators in eq.(2) by the difference analogues of them, namely D_t by δ^{-1} [exp $(\delta D_t/2)$ - exp $(-\delta D_t/2)$] and 1 (unit operator) by $(1 - \varepsilon_i)$ exp $(\delta D_t/2) + \varepsilon_i$ exp $(-\delta D_t/2)$, where δ is the time difference and ε_i is a parameter, and the difference operator exp $(\delta D_t/2)$ operating on $a(t) \cdot b(t)$ is defined by

$$\exp (\delta D_t/2) a(t) \cdot b(t) = a(t + \delta/2) b(t - \delta/2) .$$

By these replacements, eq.(2) becomes

$$\delta^{-1} [g(t+\delta/2)f(t-\delta/2) - g(t-\delta/2)f(t+\delta/2)]$$
$$= \alpha[(1-\varepsilon_1)g(t+\delta/2)f(t-\delta/2) + \varepsilon_1 g(t-\delta/2)f(t+\delta/2)]$$
$$- [(1-\varepsilon_2)g(t+\delta/2)h(t-\delta/2) + \varepsilon_2 g(t-\delta/2)h(t+\delta/2)] ,$$

$$\delta^{-1} [h(t+\delta/2)f(t-\delta/2) - h(t-\delta/2)f(t+\delta/2)]$$
$$= - \beta[(1-\varepsilon_3)h(t+\delta/2)f(t-\delta/2) + \varepsilon_3 h(t-\delta/2)f(t+\delta/2)]$$
$$+ [(1-\varepsilon_2)g(t+\delta/2)h(t-\delta/2) + \varepsilon_2 g(t-\delta/2)h(t+\delta/2)] .$$

Dividing the above equations by $f(t+\delta/2)f(t-\delta/2)$, we obtain

$$\left.\begin{aligned} \delta^{-1} [x(t+\delta/2) - x(t-\delta/2)] = {} & \alpha[(1-\varepsilon_1)x(t+\delta/2) + \varepsilon_1 x(t-\delta/2)] \\ & - [(1-\varepsilon_2)x(t+\delta/2)y(t-\delta/2) \\ & + \varepsilon_2 x(t-\delta/2)y(t+\delta/2)] , \end{aligned}\right\} \quad (3)$$

$$\delta^{-1}[y(t+\delta/2) - y(t-\delta/2)] = -\beta[(1-\varepsilon_3)y(t+\delta/2) + \varepsilon_3 y(t-\delta/2)]$$
$$+ [(1-\varepsilon_2)x(t+\delta/2)y(t-\delta/2)$$
$$+ \varepsilon_2 x(t-\delta/2)y(t+\delta/2)] ,$$

Equation (3) is a candidate of difference analogue of Volterra's equation. Now we impose the physical condition on eq.(3) that for arbitrary value of positive δ, the populations $x(t)$ and $y(t)$ should be non-negative for all time if they were positive at a time. We shall select parameters ε_1, ε_2 and ε_3 to satisfy the condition. For small values of x and y, eq.(3) is approximated by the linear equations,

$$x(t + \delta/2) = \frac{1 + \delta\alpha\varepsilon_1}{1 - \delta\alpha(1-\varepsilon_1)} x(t - \delta/2) ,$$

$$y(t + \delta/2) = \frac{1 - \delta\beta\varepsilon_3}{1 + \delta\beta(1-\varepsilon_3)} y(t - \delta/2) ,$$

which show that $x(t + \delta/2)$ and $y(t + \delta/2)$ become negative for large values of δ unless $\varepsilon_1 = 1$ and $\varepsilon_3 = 0$.

Hereafter we put $x(t + \delta/2) = x_{t+1}$, $x(t - \delta/2) = x_t$, $y(t + \delta/2) = y_{t+1}$, $y(t - \delta/2) = y_t$, $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon$ and $\varepsilon_3 = 0$, and rewrite eq.(3) as

$$x_{t+1} - x_t = \delta[\alpha x_t - (1-\varepsilon)x_{t+1}y_t - \varepsilon x_t y_{t+1}]$$

$$y_{t+1} - y_t = \delta[-\beta y_{t+1} + (1-\varepsilon)x_{t+1}y_t + \varepsilon x_t y_{t+1}] . \quad (4)$$

Equation (4) can be transformed into an explicit scheme for x_{t+1} and y_{t+1}

$$x_{t+1} = \frac{[1 - \delta\varepsilon(1+\delta\beta)^{-1} x_t](1+\delta\alpha) - \delta\varepsilon(1+\delta\beta)^{-1} y_t}{1 - \delta\varepsilon(1+\delta\beta)^{-1} x_t + \delta(1-\varepsilon)y_t} x_t , \quad (5)$$

$$y_{t+1} = \frac{1 + \delta(1-\varepsilon)x_{t+1}}{1 + \delta\beta - \delta\varepsilon x_t} \; y_t \; . \qquad (6)$$

Equation (5) shows that x_{t+1} becomes negative when x_t and y_t satisfy the following conditions

$$\begin{cases} 1 - \delta\varepsilon(1+\delta\beta)^{-1} \, x_t < 0 \, , \\ 1 - \delta\varepsilon(1+\delta\beta)^{-1} \, x_t + \delta(1-\varepsilon)y_t > 0 \, , \end{cases} \qquad (7)$$

or

$$\begin{cases} 1 - \delta\varepsilon(1+\delta\beta)^{-1} \, x_t > 0 \, , \\ [1 - \delta\varepsilon(1+\delta\beta)^{-1} \, x_t](1+\delta\alpha) - \delta\varepsilon(1+\delta\beta)^{-1} \, y_t < 0 \, , \end{cases} \qquad (8)$$

for positive values of x_t and y_t. Hence ε must be zero. Accordingly we have a difference analogue of Volterra's equation

$$\begin{cases} x_{t+1} - x_t = \delta(\alpha x_t - x_{t+1}y_t) \\ y_{t+1} - y_t = \delta(-\beta y_{t+1} + x_{t+1}y_t) \, , \end{cases} \qquad (9)$$

which reduces to eq.(1) in the small limit of δ.

Several numerical experiments carried on eq.(9) show, within experimental errors, ($\sim 10^{-8}$), that there exisit invariant curves of the mapping eq.(9). We plot a typical example of them in Fig. 1.

References

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3) Ryogo Hirota: J. Phys. Soc. Japan, No.1, 46 (1979).