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Relation between Kaup's and Mikhailov's Equations, their Exact Solutions and Stories How It Discovered

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We know that

Kaup's equation

$$u_t + u_{5x} + 30(u_{3x}u + \frac{5}{2} u_{xx}u_x) + 180u^2u_x = 0 ,$$

Sawada - Kotera's equation

$$u_t + u_{5x} + 15(u_{3x}u + u_{xx}u_x) + 45u^2u_x = 0$$

and Lax's 5-th order K-dV equation

$$u_t + u_{5x} + 10(u_{3x}u + 2u_{xx}u_x) + 30u^2u_x = 0$$

are reduce, using the potentials

$$u = \frac{1}{2} w_x, \quad u = w_x \quad \text{and} \quad u = \sqrt{3/2}\, w_x,$$

to

$$w_t + w_{5x} + 15(w_{3x}w_x + \frac{3}{4} w_{xx}^2) + 15w_x^3 = 0 ,$$

$$w_t + w_{5x} + 15w_{3x}w_x + 15w_x^3 = 0$$

and

$$w_t + w_{5x} + \sqrt{3/2}\,10(w_{3x}w_x + \frac{1}{2}w_{xx}^2) + 15w_x^3 = 0 \ .$$

Kaup found the inverse scattering transform for his equation (more than 2 years ago)

$$\begin{cases} \psi_{xxx} + 6u\psi_x + 3u_x\psi = \lambda\psi \ , \quad (\lambda : \text{eigenvalue}) \\ \psi_t = 9\lambda\psi_{xx} - 3(u_{xx} + 12u^2)\psi_x + 3(u_{xxx} + 12\lambda u + 24u_x u)\psi \ . \end{cases}$$

I found that the form is transformed into the bilinear form (2 years ago)

$$D_x^3 f'\cdot f + 3D_x g'\cdot f = 4\lambda f'f \ ,$$

$$D_x^2 f'\cdot f = g'f \ ,$$

$$D_t f'\cdot f = -\frac{3}{8}D_x^5 f'\cdot f + \frac{15}{8}D_x^3 g'\cdot f + \frac{15}{2}\lambda D_x^2 f'\cdot f \ ,$$

through the transformation

$$\psi = f'/f \ , \qquad u = \frac{1}{4}\,\frac{D_x^2 f\cdot f}{f^2} \ .$$

A. Ramani found at my request that Kaup's equation satisfies the resonance criterion that is the necessary condition for the equation to be of Painlevé - type (about 4 monthes ago).

Kaup found a one-soliton solution to it. (2 ~ 3 monthes ago)

$$u = 2p^2\,\frac{e^{\eta} + e^{-\eta} + 1}{[e^{\eta} + e^{-\eta} + 4]^2} \ ,$$

$$\eta = px + \Omega t + \text{const}, \quad \Omega + p^5 = 0 \ .$$

Ramani pointed out that Kaup's one-soliton solution can be obtained using the bilinear form (about 2 monthes ago)

He found, for $\lambda = 0$,

$$f = e^{\eta} + e^{-\eta} + 4 ,$$

$$f' = e^{\eta} + e^{-\eta} - 2 ,$$

$$g' = 2p^2 ,$$

are the solutions to the bilinear form.

About a month ago, I found that Kaup's equation is transformed into the bilinear form

$$D_x(D_t + \frac{1}{16} D_x^5) f \cdot f + \frac{15}{16} D_x^2 g \cdot f = 0 ,$$

$$D_x^4 f \cdot f = gf ,$$

through the transformation

$$u = \frac{1}{4} \frac{D_x^2 f \cdot f}{f^2} ,$$

and found 2-soliton solution (Oct. 4, '79)

$$f = 1 + 4(e^{\eta_1} + e^{\eta_2}) + e^{2\eta_1} + 2(1 + \alpha_{12})e^{\eta_1+\eta_2} + e^{2\eta_2}$$
$$+ 4\beta_{12}(e^{\eta_1+2\eta_2} + e^{2\eta_1+\eta_2}) + \beta_{12}^2 e^{2\eta_1+2\eta_2} ,$$

$$g = 8(p_1^4 e^{\eta_1} + p_2^4 e^{\eta_2}) + 16\gamma_{12} e^{\eta_1+\eta_2}$$
$$+ 8\beta_{12}(p_1^4 e^{\eta_1+2\eta_2} + p_2^4 e^{2\eta_1+\eta_2}) ,$$

where

$$\alpha_{12} = (p_1 - p_2)^2 [\frac{3}{(p_1 + p_2)^2} + \frac{4}{p_1^2 + p_1 p_2 + p_2^2}] ,$$

$$\beta_{12} = (\frac{p_1 - p_2}{p_1 + p_2})^2 (\frac{p_1^2 - p_1 p_2 + p_2^2}{p_1^2 + p_1 p_2 + p_2^2}) ,$$

$$\gamma_{12} = (p_1 - p_2)^2 (\frac{2p_1^4 + 3p_1^2 p_2^2 + 2p_2^4}{p_1^2 + p_1 p_2 + p_2^2}) ,$$

$$\eta_i = p_i x + \Omega_i t + \eta_i^0 , \quad \Omega_i + p_i^5 = 0 , \quad \text{for} \quad i = 1, 2.$$

In his letter dated Oct. 18, 1979, Ramani wrote me that Shabat and Mikhailov found "L,A" pair for the equation

$$u_{x\tau} = e^u - e^{-2u}$$

(Correspondence to him by Mark Ablowitz).

Ramani was able to transform it to the third Painlevé equation, and write it in bilinear form

$$D_x D_\tau \hat{f} \cdot \hat{f} = 2\hat{f}(\hat{f} - \hat{g})$$

$$D_x D_\tau \hat{g} \cdot \hat{g} = - 2(\hat{f}^2 - \hat{g}^2) ,$$

through the transformation

$$u = \log(\hat{g} / \hat{f}) .$$

He and I found two-soliton solution to it independently. I found it on Oct. 30, '79.

$$\hat{f} = h^2 ,$$

$$h = 1 + e^{\eta_1} + e^{\eta_2} + \beta_{12} e^{\eta_1 + \eta_2} ,$$

$$\hat{g} = 1 - 4(e^{\eta_1} + e^{\eta_2}) + e^{2\eta_1} + b_{12} e^{\eta_1 + \eta_2} + e^{2\eta_2}$$

$$- 4\beta_{12}(e^{2\eta_1 + \eta_2} + e^{\eta_1 + 2\eta_2}) + \beta_{12}^{\ 2} e^{2\eta_1 + 2\eta_2} ,$$

where $\eta_i = p_i x + w_i \tau + \eta_i^{\ 0}$, $p_i w_i = 3$ for $i = 1, 2$,

$$\beta_{12} = \left(\frac{p_1 - p_2}{p_1 + p_2} \right)^2 \left(\frac{p_1^{\ 2} - p_1 p_2 + p_2^{\ 2}}{p_1^{\ 2} + p_1 p_2 + p_2^{\ 2}} \right) ,$$

$$b_{12} = 8 \frac{2p_1^{\ 4} - p_1^{\ 2} p_2^{\ 2} + 2p_2^{\ 4}}{(p_1 + p_2)^2 (p_1^{\ 2} + p_1 p_2 + p_2^{\ 1})} .$$

To our surprise, the functional form of $\hat{g}$ is equal to that of f, the solution to Kaup's equation, namely

$$f(\eta_1, \eta_2) = \hat{g}(\eta_1 + i\pi, \eta_2 + i\pi) .$$

Furthermore, h is equal to two-soliton solution to Sawada - Kotera's equation

*) $D_x(D_t + D_x^{\ 5}) h \cdot h = 0$

and to the equation

$$D_x^{\ 2}(D_x D_\tau - 3) h \circ h = 0 ,$$

which is the special case of

$$D_x(D_\tau - D_x^{\ 2} D_\tau + D_x) f \circ f = 0 ,$$

which is the bilinear form of the model equation for shallow water waves (R. Hirota and J. Satsuma, J. Phys. Soc. Japan. 40 (1976) 611) ,

$$u_x - u_{xx\tau} - 3uu_\tau + 3u_x \int_x^\infty u_\tau dx' + u_x = 0 ,$$

where

$$u = 2(\log f)_{xx} .$$

Suggested by these facts, I found that the solution of Mikhailov's equation is expressed with the solution of eq. *) (Nov. 14, 1979)

$$u = \log(1 - S_\tau) ,$$

$$S = 2(\log h)_x ,$$

where u and S are the solutions to

$$u_{x\tau} = e^u - e^{-2u} , \quad \text{Mikhailov's equation,}$$

$$S_{xx\tau} + 3S_x S_\tau - 3S_x = 0$$

or $D_x^2(D_x D_\tau - 3)h \cdot h = 0$, Shallow water wave eq., respectively.

Furthermore, the inverse scattering form for Kaup's equation (for λ = 0), namely

$$D_x^3 f' \cdot f + 3D_x g' \cdot f = 0 ,$$

$$D_x^2 f' \cdot f = g'f ,$$

$$D_t f' \cdot f = -\frac{3}{8} D_x^5 f' \cdot f + \frac{15}{8} D_x^3 g' \cdot f ,$$

is satisfied by

$$f' = h^2, \quad g' = -D_x^2 h \cdot h, \quad f = h^2 - D_x D_\tau h \cdot h,$$

provided that h satisfies

$$D_x(D_t + D_x^5) h \cdot h = 0, \quad \text{Sawada - Kotera's eq.}$$

Hence, N - Soliton solutions to Kaup's and Mikhailov's equation are expressed with h:

$$u = \frac{1}{2} (\log f)_{xx}, \quad f = h^2 - D_x D_\tau h \cdot h$$

and

$$u = \log (1 - S_\tau), \quad S_\tau = 2(\log h)_{x\tau},$$

respectively, where

$$h = \sum_{\underset{\sim}{\mu}=0,1} \exp \left\{ \sum_{i=1}^{N} \mu_i \eta_i + \sum_{i>j}^{(N)} \beta_{ij} \mu_i \mu_j \right\}$$

$$\exp (\beta_{ij}) = \frac{(p_i - p_j)^2 (p_i^2 - p_i p_j + p_j^2)}{(p_i + p_j)^2 (p_i^2 + p_i p_j + p_j^2)},$$

$$\eta_i = p_i x + \Omega_i t + w_i \tau + i\pi + \eta_i^0 .$$

$$\Omega_i + p_i^5 = 0, \quad w_i p_i = 3 \quad \text{for} \quad i = 1, 2, \ldots, N.$$