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ON THE FRACTIONAL CALCULUS

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I. INTRODUCTION.

There are many definitions of the fractional calculus. In 1832, J. Liouville defined the fractional integral of order α in [2]. Recently, T. J. Osler defined the fractional derivative of order α in [5] and [6]. Moreover, K. Nishimoto defined the fractional derivative and integral of order α in [4]. And in 1978, M. Saigo defined the integral operators in [12]. Furthermore in 1978, S. Owa gave the following definitions for the fractional calculus in [8].

DEFINITION I. The fractional integral of order α is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\alpha}},$$

where α is greater than 0, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\ln(z-\zeta)$ to be real when $(z-\zeta)$ is greater than 0. Moreover

$$f(z) = \lim_{\alpha \to 0} D_z^{-\alpha} f(z) .$$

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DEFINITION 2. The fractional derivative of order α is defined by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^\alpha},$$

where $0 < \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z - \zeta)^{-\alpha}$ is removed by requiring $\ln(z - \zeta)$ to be real when $(z - \zeta) > 0$. Moreover,

$$f(z) = \lim_{\alpha \to 0} D_z^\alpha f(z)$$

and

$$f'(z) = \lim_{\alpha \to 1} D_z^\alpha f(z) .$$

REMARK I. The fractional derivative of order $(n + \alpha)$ is defined by

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n} D_z^\alpha f(z) ,$$

where $0 < \alpha < 1$ and $n \in N \cup \{0\}$.

DEFINITION 3. Let E be a domain in the extended complex plane. The function $f(z)$ is called univalent in E if and only if it is analytic except for at most one pole and $f(z_1) \neq f(z_2)$ for $z_1 \in E$, $z_2 \in E$ and $z_1 \neq z_2$. Let S denote the class of function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is analytic and univalent in the unit disk $U = \{|z| < 1\}$, S^* denote the subclass of S which is univalent starlike with respect to the origin in the unit disk U, and C denote the subclass of S^* which is univalent convex in the unit disk U.

THEOREM 1 ([7]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class S. If the Bieberbach conjecture on the coefficients of $f(z)$ is true for any $n \geqq 2$, then

$$|f^{(n)}(z)| \leqq \frac{n!(n + |z|)}{(1 - |z|)^{n+2}}$$

for $z \in U$.

REMARK 2. For $n = 1$, Theorem 1 means the Koebe distortion inequality. And Theorem 1 is already shown by F. Marty [3] for $n = 2, 3$ and by Y. Komatu and H. Nishimiya [1] for $n = 4$.

2. A CONJECTURE.

S. Owa gave the following conjecture in [7].

CONJECTURE. Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $\mathcal{S}$. Then, for any non-negative α and $z \in \mathcal{U}$,

$$|D_z^{\alpha} f(z)| \leq \frac{\Gamma(\alpha + 1)(\alpha + |z|)}{(1 - |z|)^{\alpha+2}} \quad .$$

Now, for the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad ,$$

we put

$$F(z) = \Gamma(2 + \alpha) z^{-\alpha} D_z^{-\alpha} f(z) \qquad (\alpha > 0)$$

and

$$G(z) = \Gamma(2 - \alpha) z^{\alpha} D_z^{\alpha} f(z) \qquad (0 < \alpha < 1) \; .$$

Let $\mathcal{S}_G^*$ denote the class of univalent starlike functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the unit disk $\mathcal{U}$ such that $G(z) \in \mathcal{S}^*$ and $\mathcal{C}_G$ denote the class of univalent convex functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the unit disk $\mathcal{U}$ such that $G(z) \in \mathcal{C}$.

The following results hold for Conjecture.

THEOREM 2. If $f(z) \in C_G$, then for $0 < \alpha < 1$ and

$$\frac{\alpha^2(\alpha - 1) + \sqrt{\alpha^6 - 2\alpha^5 + \alpha^4 - 4\alpha + 4}}{2\alpha(1 - \alpha)} \leqq |z| < 1,$$

$$|D_z^\alpha f(z)| \leqq \frac{\Gamma(\alpha + 1)(\alpha + |z|)}{(1 - |z|)^{\alpha+2}} .$$

THEOREM 3. If $f(z)$ is in the class S_G^*, then for $0 < \alpha < 1$ and

$$\frac{\alpha^2(\alpha - 1) + \sqrt{\alpha^6 - 2\alpha^5 + \alpha^4 - 4\alpha + 4}}{2\alpha(1 - \alpha)} \leqq |z| < 1,$$

$$|D_z^\alpha f(z)| \leqq \frac{\Gamma(\alpha + 1)(\alpha + |z|)}{(1 - |z|)^{\alpha+2}} .$$

3. APPLICATION OF THE FRACTIONAL CALCULUS FOR D(K).

DEFINITION 4. Let $D(k)$ denote the class of function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which is analytic in the unit disk U and satisfying

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < k$$

for $0 < k \leqq 1$ and $z \in U$.

THEOREM 4 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk U such that

$$\sum_{n=2}^{\infty} n^2 |a_n| < k \qquad (0 < k \leqq 1).$$

Then, for $0 < \alpha < 1$ and $z \in U$,

$$|D_z^{\alpha} f(z)| \geqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}} \left\{ -|z| + \frac{2(2-k)}{k} \log\left(1 + \frac{k}{2-k}|z|\right)\right\},$$

$$|D_z^{\alpha} f(z)| \leqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}} \left\{ -|z| - \frac{2(2-k)}{k} \log\left(1 - \frac{k}{2-k}|z|\right)\right\},$$

and

$$|D_z^{1+\alpha} f(z)| \leqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}} \left\{ \frac{2-k+k|z|}{2-k-k|z|} - \alpha - \frac{2\alpha(2-k)}{k|z|} \log\left(1 - \frac{k}{2-k}|z|\right)\right\}.$$

THEOREM 5 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk $\mathbb{U}$ such that

$$\sum_{n=2}^{\infty} n|a_n| \leqq k \qquad (0 < k \leqq 1).$$

Then, for $\alpha > 0$ and $z \in \mathbb{U}$,

$$|D_z^{-\alpha} f(z)| \geqq \frac{|z|^\alpha}{\Gamma(2+\alpha)} \left\{ -|z| + \frac{2(2-k)}{k} \log\left(1 + \frac{k}{2-k}|z|\right)\right\},$$

$$|D_z^{-\alpha} f(z)| \leqq \frac{|z|^\alpha}{\Gamma(2+\alpha)} \left\{ -|z| - \frac{2(2-k)}{k} \log\left(1 - \frac{k}{2-k}|z|\right)\right\},$$

and

$$|D_z^{1-\alpha} f(z)| \leqq \frac{|z|^\alpha}{\Gamma(2+\alpha)} \left\{ \frac{2-k+k|z|}{2-k-k|z|} - \alpha - \frac{2\alpha(2-k)}{k|z|} \log\left(1 - \frac{k}{2-k}|z|\right)\right\}.$$

THEOREM 6 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk U such that

$$\sum_{n=2}^{\infty} n^2 |a_n| < k \qquad (0 < k \leqq 1).$$

Then, for $0 < \alpha < 1$, $0 < K < (2 - k - k|z|)/(2 - k + k|z|)$, and $z \in U$,

$$|D_z^{2+\alpha} f(z)| \leqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}} \left\{ \frac{2(1-K)}{(1-|z|)^2} + \frac{2\alpha(2-k+k|z|)}{(2-k-k|z|)|z|} - \frac{\alpha(1+\alpha)}{|z|} - \frac{2\alpha(1+\alpha)(2-k)}{k|z|^2} \log\left(1 - \frac{k}{2-k}|z|\right) \right\}$$

THEOREM 7 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk U such that

$$\sum_{n=2}^{\infty} n^2 |a_n| < k \qquad (0 < k \leqq 1).$$

Then, for $0 < \alpha < 1$, $0 < K < (2 - k - k|z|)/(2 - k + k|z|)$, and $z \in U$,

$$|D_z^{2+\alpha}f(z)| \leqq \frac{1}{\Gamma(2-\alpha)|z|^{\alpha}}\left\{\frac{2\alpha(2-k+k|z|)}{(2-k-k|z|)|z|}\right.$$

$$+\frac{2(1-K)(2-k+k|z|)}{(1-|z|)\{1+(1-2K)|z|\}(2-k-k|z|)}$$

$$\left.-\frac{\alpha(1+\alpha)}{|z|}-\frac{2\alpha(1+\alpha)(2-k)}{k|z|^2}\log\left(1-\frac{k}{2-k}|z|\right)\right\}.$$

THEOREM 8 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk $\mathbb{U}$ such that

$$\sum_{n=2}^{\infty} n|a_n| < k \qquad (0 < k \leqq 1).$$

Then, for $\alpha > 0$, $0 < K < (2-k-k|z|)/(2-k+k|z|)$, and $z \in \mathbb{U}$,

$$|D_z^{2-\alpha}f(z)| \leqq \frac{|z|^{\alpha}}{\Gamma(2+\alpha)}\left\{\frac{2(1-K)}{(1-|z|)^2}+\frac{2\alpha(2-k+k|z|)}{(2-k-k|z|)|z|}\right.$$

$$\left.-\frac{\alpha(1+3\alpha)}{|z|}-\frac{2\alpha(1+3\alpha)(2-k)}{k|z|^2}\log\left(1-\frac{k}{2-k}|z|\right)\right\}$$

THEOREM 9 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk $\mathcal{U}$ such that

$$\sum_{n=2}^{\infty} n|a_n| < k \qquad (0 < k \leqq 1).$$

Then, for $\alpha > 0$, $0 < K < (2 - k - k|z|)/(2 - k + k|z|)$, and $z \in \mathcal{U}$,

$$|D_z^{2-\alpha} f(z)| \leqq \frac{|z|^{\alpha}}{\Gamma(2+\alpha)} \left\{ \frac{2\alpha(2 - k + k|z|)}{(2 - k - k|z|)|z|} - \frac{\alpha(1+3\alpha)}{|z|} \right.$$

$$+ \frac{2(1-K)(2 - k + k|z|)}{(1 - |z|)\{1 + (1 - 2K)|z|\}(2 - k - k|z|)}$$

$$\left. - \frac{2\alpha(1+3\alpha)(2-k)}{k|z|^2} \log\left(1 - \frac{k}{2-k}|z|\right) \right\}.$$

THEOREM 10 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk $\mathcal{U}$ such that

$$\sum_{n=2}^{\infty} n^2|a_n| < k \qquad (0 < k \leq 1)$$

and $a_2 \geq 0$. Then, for $0 < \alpha < 1$ and $z \in U$,

$$\mathrm{Re}\{G'(z)\} \geq \frac{1 - |z|^2}{1 + \dfrac{4a_2}{2 - \alpha}|z| + |z|^2} .$$

Furthermore, this result is sharp for each value of a_2, $0 \leq a_2 \leq (2 - \alpha)/2$, by considering the functions

$$G'_{a_2}(z) = \frac{1 - z^2}{1 - \dfrac{4a_2}{2 - \alpha} z + z^2} .$$

COROLLARY I ([9]). Under the hypotheses of Theorem 10,

$$\mathrm{Re}\{G'(z)\} > \frac{1 - |z|^2}{1 + k|z| + |z|^2}$$

for $z \in U$.

THEOREM II ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk U such that

$$\sum_{n=2}^{\infty} n|a_n| < k \qquad (0 < k \leq 1)$$

and $a_2 \geq 0$. Then, for $\alpha > 0$ and $z \in U$,

$$\mathrm{Re}\{F'(z)\} \geqq \frac{1 - |z|^2}{1 + \dfrac{4a_2}{2 + \alpha}|z| + |z|^2} .$$

Furthermore, this result is sharp for each value of a_2, $0 \leqq a_2 \leqq (2 + \alpha)/2$, by considering the functions

$$F'_{a_2}(z) = \frac{1 - z^2}{1 - \dfrac{4a_2}{2 + \alpha} z + z^2} .$$

COROLLARY 2 ([9]). Under the hypotheses of Theorem 11,

$$\mathrm{Re}\{F'(z)\} > \frac{1 - |z|^2}{1 + k|z| + |z|^2}$$

for $z \in U$.

THEOREM 12 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be an analytic function in the unit disk U such that

$$\sum_{n=2}^{\infty} n^2|a_n| < k \qquad (0 < k \leqq 1)$$

and $a_2 \geqq 0$. Then, for $0 < \alpha < 1$ and $z \in U$,

$$|D_z^{1+\alpha}f(z)| \leqq \frac{1}{\Gamma(2 - \alpha)|z|^\alpha} \left\{ \frac{1 + \dfrac{4a_2}{2 - \alpha}|z| + |z|^2}{1 - |z|^2} \right.$$

$$- \alpha - \frac{2\alpha(2-k)}{k|z|} \log\left(1 - \frac{k}{2-k}|z|\right)\Bigg\} .$$

Furthermore, this result is sharp for each value of a_2, $0 \leqq a_2 \leqq (2-\alpha)/2$, by considering the functions

$$G'_{a_2}(z) = \frac{1-z^2}{1 - \frac{4a_2}{2-\alpha}z + z^2} .$$

COROLLARY 3 ([9]). Under the hypotheses of Theorem 12,

$$|D_z^{1+\alpha} f(z)| < \frac{1}{\Gamma(2-\alpha)|z|^\alpha} \Bigg\{ \frac{1+k|z|+|z|^2}{1-|z|^2} - \alpha$$

$$- \frac{2\alpha(2-k)}{k|z|} \log\left(1 - \frac{k}{2-k}|z|\right)\Bigg\}$$

for $0 < \alpha < 1$ and $z \in \mathbb{U}$.

THEOREM 13 ([9]). Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk $\mathbb{U}$ such that

$$\sum_{n=2}^{\infty} n|a_n| < k \qquad (0 < k \leqq 1)$$

and $a_2 \geqq 0$. Then, for $\alpha > 0$ and $z \in U$,

$$|D_z^{1-\alpha} f(z)| \leqq \frac{|z|^\alpha}{\Gamma(2+\alpha)} \left\{ \frac{1 + \frac{4a_2}{2+\alpha}|z| + |z|^2}{1 - |z|^2} - \alpha - \frac{2\alpha(2-k)}{k|z|} \log\left(1 - \frac{k}{2-k}|z|\right) \right\} .$$

Furthermore, this result is sharp for each value of a_2, $0 \leqq a_2 \leqq (2+\alpha)/2$, by considering the functions

$$F'_{a_2}(z) = \frac{1 - z^2}{1 - \frac{4a_2}{2+\alpha} z + z^2} .$$

COROLLARY 4 ([9]). Under the hypotheses of Theorem 13, we have

$$|D_z^{1-\alpha} f(z)| < \frac{|z|^\alpha}{\Gamma(2+\alpha)} \left\{ \frac{1 + k|z| + |z|^2}{1 - |z|^2} - \alpha - \frac{2\alpha(2-k)}{k|z|} \log\left(1 - \frac{k}{2-k}|z|\right) \right\}$$

for $\alpha > 0$ and $z \in U$.

4. APPLICATION OF THE FRACTIONAL CALCULUS FOR K_α.

DEFINITION 5. Let A denote the family of functions f(z) analytic in the unit disk U and normalized f(0) = 0 and f'(0) = 1. And let K_n denote the class of functions $f(z) \in A$ satisfying the following conditions

$$(1) \quad \mathrm{Re}\left[\frac{\{z^n f(z)\}^{(n+1)}}{\{z^{n-1} f(z)\}^{(n)}}\right] > \frac{n+1}{2} \qquad (z \in U),$$

where $n \in N \cup \{0\}$.

REMARK 3. In particular, for n = 0 the conditions (1) become

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \qquad (z \in U).$$

Therefore, the class K_0 equals the class $S^*(1/2)$ that denote the class of starlike functions of order 1/2.

DEFINITION 6. Let f*g(z) denote the Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$, and in particular, we put

$$(2) \quad D^\alpha f(z) = \left\{\frac{z}{(1-z)^{\alpha+1}}\right\} * f(z).$$

REMARK 4. In definition 6, the relation (2) implies

$$(3) \quad D^n f(z) = \frac{z\{z^{n-1} f(z)\}^{(n)}}{n!},$$

where $n \in N \cup \{0\}$.

Remark 5. With this notation (3), we have that the necessary and sufficient condition for a function $f(z) \in A$ to be in the class $K_0 \equiv S^*(1/2)$ is

$$\mathrm{Re}\left\{ \frac{D^1 f(z)}{D^0 f(z)} \right\} > \frac{1}{2} \qquad (z \in U),$$

the necessary and sufficient condition for a function $f(z) \in A$ to be in the class $K_1 \equiv K$ is

$$\mathrm{Re}\left\{ \frac{D^2 f(z)}{D^1 f(z)} \right\} > \frac{1}{2} \qquad (z \in U),$$

and the necessary and sufficient condition for a function $f(z) \in A$ to be in the class K_n is

$$\mathrm{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \frac{1}{2} \qquad (z \in U). \tag{4}$$

Moreover, in the notation (4) also a class K_{-1} can be defined as the family of functions $f(z) \in A$ satisfying the condition

$$\mathrm{Re}\left\{ \frac{f(z)}{z} \right\} > \frac{1}{2} \qquad (z \in U).$$

Remark 6. R. Singh and S. Singh showed some results for the subclass R_n of K_n in [13], where the subclass R_n means the class whose members are characterized by the condition

$$\mathrm{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \frac{n}{n+1} \qquad (z \in U).$$

THEOREM 14. Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n .$$

Then, for $0 < \alpha < 1$, we have

$$D^{\alpha} f(z) = \frac{z}{\Gamma(1 + \alpha)} D_z^{\alpha}\{z^{\alpha-1} f(z)\} ,$$

$$D^{0} f(z) = \lim_{\alpha \to 0} D^{\alpha} f(z),$$

and

$$D^{1} f(z) = \lim_{\alpha \to 1} D^{\alpha} f(z).$$

THEOREM 15. Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n .$$

Then, for $0 < \alpha < 1$, we have

$$D^{-\alpha} f(z) = \frac{z}{\Gamma(1 - \alpha)} D_z^{-\alpha}\{z^{-\alpha-1} f(z)\},$$

$$D^{0} f(z) = \lim_{\alpha \to 0} D^{-\alpha} f(z) ,$$

and

$$D^{-1} f(z) = \lim_{\alpha \to 1} D^{-\alpha} f(z) .$$

DEFINITION 7. Let $\mathcal{A}$ denote the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk U. And let K_α and $K_{-\alpha}$ denote the classes of functions $f(z) \in \mathcal{A}$ satisfying the following conditions

$$\mathrm{Re}\left[\frac{D_z^{\alpha+1}\{z^\alpha f(z)\}}{D_z^\alpha\{z^{\alpha-1} f(z)\}}\right] > \frac{1+\alpha}{2} \qquad (z \in U)$$

and

$$\mathrm{Re}\left[\frac{D_z^{1-\alpha}\{z^{-\alpha} f(z)\}}{D_z^{-\alpha}\{z^{-\alpha-1} f(z)\}}\right] > \frac{1-\alpha}{2} \qquad (z \in U)$$

for $0 < \alpha < 1$, respectively.

THEOREM 16. The nacessary and sufficient condition for a function $f(z) \in \mathcal{A}$ to be in the class K_α, $0 < \alpha < 1$, is

$$\mathrm{Re}\left\{\frac{D^{1+\alpha} f(z)}{D^\alpha f(z)}\right\} > \frac{1}{2} \qquad (z \in U).$$

THEOREM 17. The necessary and sufficient condition for a function $f(z) \in \mathcal{A}$ to be in the class $K_{-\alpha}$, $0 < \alpha < 1$, is

$$\mathrm{Re}\left\{\frac{D^{1-\alpha} f(z)}{D^{-\alpha} f(z)}\right\} > \frac{1}{2} \qquad (z \in U).$$

THEOREM 18. Let the function f(z) belong to the family $\mathcal{A}$ and satisfy the condition

$$\sum_{n=2}^{\infty} n(n+2)|a_n| < 1.$$

Then, for $0 < \alpha < 1$, the function f(z) is in the class K_α.

THEOREM 19. Let the function f(z) belong to the family $\mathcal{A}$ and satisfy the condition

$$\sum_{n=2}^{\infty} (2n+1)|a_n| < 1.$$

Then, for $0 < \alpha < 1$, the function f(z) is in the class $K_{-\alpha}$.

Recently, St. Ruscheweyh gave the following problems in [10].

PROBLEM 1. What can be said about the classes K_α, if we replace the natural number n in (4) by an arbitrary real number $\alpha \geqq 1$. Is it perhaps that $K_\alpha \subset K_\beta$ for $\alpha > \beta$?

PROBLEM 2. Is K_α closed under the Hadamard product ?

REMARK 7. The truth of Problem 2 is trivial for $\alpha = -1$ and was proved by St. Ruscheweyh and T. Sheil-Small for $\alpha = 0, 1$ in [11].

Now, we give some results for Problem 1 in a sense.

THEOREM 20. Let the function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

belong to the class $K_{\alpha+\delta}$ and satisfy the condition

$$\sum_{n=2}^{\infty} \frac{(2n + 3\delta + 4)\Gamma(n + \delta + 1)}{(n - 1)!\Gamma(\delta + 3)}|a_n| < 1$$

for $0 < \alpha < 1$ and $0 < \alpha + \delta < 1$. Then the function $f(z)$ is in the class K_α.

COROLLARY 5. There exists the function $f(z)$ of the class $K_{\alpha+\delta}$ such that is in the class K_α, where $0 < \alpha < 1$ and $0 < \alpha + \delta < 1$.

COROLLARY 6. For the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

satisfying the following condition

$$\sum_{n=2}^{\infty} \frac{(2n + 3\alpha - 3\beta + 4)\Gamma(n + \alpha - \beta + 1)}{(n - 1)!\Gamma(\alpha - \beta + 3)}|a_n| < 1 ,$$

if $0 < \beta < \alpha < 1$ and $0 < 2\alpha - \beta < 1$, then $K_\alpha \subset K_\beta$.

THEOREM 21. Let the function $f(z)$ belong to the class $K_{-\alpha+\delta}$ and satisfy the condition

$$\sum_{n=2}^{\infty} \frac{(2n + 3\delta + 1)\Gamma(n + \delta)}{(n - 1)!\Gamma(\delta + 2)}|a_n| < 1$$

for $0 < \alpha < 1$ and $0 < \alpha + \delta < 1$. Then the function f(z) is in the class $K_{-\alpha}$.

COROLLARY 7. There exists the function f(z) of the class $K_{-\alpha+\delta}$ such that is in the class $K_{-\alpha}$, where $0 < \alpha < 1$ and $0 < \alpha + \delta < 1$.

COROLLARY 8. For the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

satisfying the following condition

$$\sum_{n=2}^{\infty} \frac{(2n + 3\alpha - 3\beta + 1)\Gamma(n + \alpha - \beta)}{(n - 1)!\Gamma(\alpha - \beta + 2)}|a_n| < 1,$$

if $0 < \alpha < \beta < 1$ and $0 < 2\alpha - \beta < 1$, then $K_{-\alpha} \subset K_{-\beta}$.

Furthermore, we have the following results for Problem 2 in a sense.

THEOREM 22. Let the function f(z) belong to the family $\mathcal{A}$ and satisfy the condition

$$\sum_{n=2}^{\infty} n(n + 2)|a_n| < 1.$$

Then, for $0 < \alpha < 1$, the Hadamard product $f*f(z)$ is in the class K_{α}.

COROLLARY 9. There exists the function $f(z)$ of the class K_α such that the Hadamard product $f*f(z)$ is in the class K_α, $0 < \alpha < 1$.

COROLLARY 10. If the function $f(z)$ belongs to the class K_α and satisfies the condition

$$\sum_{n=2}^{\infty} n(n + 2)|a_n| < 1,$$

then the Hadamard product $f*f(z)$ is in the class K_α, $0 < \alpha < 1$.

THEOREM 23. Let the function $f(z)$ belong to the family $\tilde{A}$ and satisfy the condition

$$\sum_{n=2}^{\infty} (2n + 1)|a_n| < 1.$$

Then, for $0 < \alpha < 1$, the Hadamard product $f*f(z)$ is in the class $K_{-\alpha}$.

COROLLARY 11. There exists the function $f(z)$ of the class $K_{-\alpha}$ such that the Hadamard product $f*f(z)$ is in the class $K_{-\alpha}$, $0 < \alpha < 1$.

COROLLARY 12. If the function $f(z)$ belongs to the class $K_{-\alpha}$ and satisfies the condition

$$\sum_{n=2}^{\infty} (2n + 1)|a_n| < 1,$$

then the Hadamard product $f*f(z)$ is in the class $K_{-\alpha}$, $0 < \alpha < 1$.

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