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HIGHLY NONCONTINUABLE FUNCTIONS ON POLYNOMIALLY CONVEX SETS

by

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Globevnik and Stout [2] proved the following theorem:

If D is a bounded strictly convex domain in C^N with a $\mathcal{C}^2$ boundary then there exists a holomorphic function f in the Nevanlinna class on D with the following "high noncontinuation property":

For every complex line L the plane domain $L \cap D$ is the natural domain of existence of $f|L \cap D$.

This theorem gives an affirmative answer to a question asked by N.Sibony. Globevnik and Stout [2] and W.Rudin [5] asked if it is possible to construct such a noncontinuable function with continuous or smooth boundary values. Their technique could not yield such functions. In this paper we answer their question in the affirmative by the method of the extremal function Φ_K ([6], [7]).

Let us recall that for every compact subset K of C^N the extremal function Φ_K is defined by the formula

(1) $$\Phi_K(x) := \sup_{n \geq 1} \left(\sup\{|P(x)|;\ P \in \mathcal{P}_n,\ \|P\|_K \leq 1\}\right)^{1/n}, \quad x \in C^N,$$

where $\mathcal{P}_n = \mathcal{P}_n(C^N, C)$ is the set of all complex-valued polynomials of N complex variables of degree at most n, and

$$\|P\|_K := \sup\{|P(x)|;\ x \in K\}$$

is the supremum norm of P on K.

We shall need the following properties of Φ_K ([6],[7]):

1.1. $|P(x)| \leq \|P\|_K \Phi_K^n(x)$, $x \in C^N$, $P \in \mathcal{P}_n(C^N, C)$.

1.2. $\Phi_K = \Phi_{\hat{K}}$, where $\hat{K}$ is the polynomially convex hull of K.

1.3. $\Phi_K(x) = 1$ on $\hat{K}$, $\Phi_K(x) > 1$ on $C^N - \hat{K}$.

1.4. $\Phi_A(x) \leq \Phi_B(x)$ in C^N, if $B \subset A$.

1.5 Φ_K is locally bounded in C^N if and only if K is not pluripolar.

Let us recall that a subset E of C^N is called <u>pluripolar</u> if there exists a plurisubharmonic function W in C^N such that $W = -\infty$ on E.

1.6. If K_j is a compact subset of C^{N_j} $(j = 1, 2)$ then

$$\Phi_{K_1 \times K_2}(x,y) = \max\{\Phi_{K_1}(x), \Phi_{K_2}(y)\}, \quad (x,y) \in C^{N_1} \times C^{N_2} .$$

1.7. If $N = 1$ and $K \subset C^1 = C$ is a compact set with positive logarithmic capacity then $\log \Phi_K$ is the Green function of $C - \hat{K}$ with pole at infinity.

1.8. If $\|\cdot\|$ is any norm in C^N and $B(a,r) = \{x \in C^N ; \|x-a\| \leq r\}$ is a closed ball then

$$\Phi_{B(a,r)}(x) = \max\{1, \|x-a\|/r\}, \quad x \in C^N.$$

The crucial role in our considerations is played by the following Lemmas.

<u>Lemma 1.</u> If K is a compact subset of C^N then there exist an increasing sequence of positive integers $(n_j)_{j \geq 1}$ and a sequence of polynomials $(P_j)_{j \geq 1}$ of N complex variables such that

(i) $\lim (\sqrt{n_{j+1}}/n_j) = +\infty$;

(ii) $\deg P_j \leq n_j$;

(iii) $\Phi_K(x) = \sup_j |P_j(x)|^{1/n_j} = \limsup_{j \to \infty} |P_j(x)|^{1/n_j}, x \in C^N$.

Lemma 2. (An extension of the Ostrowski theorem on lacunary power series (see [1]) to the case of series of polynomials of N complex variables with lacunary sequence of degrees, see [3], [4], [7]). Let f be a holomorphic function in a domain $D \subset C^N$ and let E be a nonpluripolar compact subset of D. Assume that $(n_j)_{j \geq 1}$ is a sequence of positive integers and let $(P_j)_{j \geq 1}$ be a sequence of polynomials of N variables such that

(a) $\deg P_j \leq n_j$,

(b) $\lim_{s \to \infty} \| f - \sum_{j=1}^{s} P_j \|_E^{1/n_s} = 0$.

Then for every compact subset F of D

$$\lim_{s \to \infty} \| f - \sum_{j=1}^{s} P_j \|_F^{1/n_s} = 0;$$

in particular

$$f(x) = \sum_{j=1}^{\infty} P_j(x) \quad \text{for all } x \text{ in } D.$$

Moreover, if G denotes the maximal open subset of C^N in which the series $\sum P_j$ is locally uniformly convergent, then the maximal Riemann domain of existence S of f is identical with the connected component of G conataining D. In particular S is univalent.

Now, if K is a polynomially convex compact subset of C^N, the required function $f: K \to C$ with a strong noncontinuation property may be defined by the formula

$$(\&) \qquad f(x) := \sum_{j=1}^{\infty} \Theta^{\sqrt{n_j}} P_j(x), \qquad x \in K,$$

where Θ is a fixed real number with $0 < \Theta < 1$, and $(n_j)_{j \geq 1}$ $(P_j)_{j \geq 1}$ are sequences satisfying the statements (i), (ii) and (iii) of Lemma 1.

It is clear that the series (&) is uniformly convergent on K, because $\|P_j\|_K \leq 1$ and the number series $\sum_{j=1}^{\infty} \theta^{\sqrt{n_j}}$ is convergent; the function f is continuous on K and holomorphic in the interior of K. The series (&) is divergent at each point x of $C^N - K$, because

$$\limsup_{j\to\infty} \left(\theta^{\sqrt{n_j}} |P_j(x)| \right)^{1/n_j} = \Phi_K(x) > 1.$$

Observe that

$$\left\| f - \sum_{j=1}^{s} \theta^{\sqrt{n_j}} P_j \right\|_K \leq \sum_{j=s+1}^{\infty} \theta^{\sqrt{n_j}} \leq M \theta^{\sqrt{n_{s+1}}}, \quad M = \text{const.}$$

Therefore

$$(+) \qquad \lim_{s\to\infty} \left\| f - \sum_{j=1}^{s} \theta^{\sqrt{n_j}} P_j \right\|_K^{1/n_s} = 0.$$

Hence as a direct consequence of Lemmas 1 and 2 we get the following main result of this paper

THEOREM 1. If K is a polynomially convex compact subset of C^N then the function f defined by (&) has the following strong noncontinuation property:

(SNCP) If $\gamma: C^M \to C^N$ is any polynomial mapping of C^M into C^N with $\deg \gamma \geq 1$, $M \geq 1$, and if h is a holomorphic function on a ball $B(a,r) \subset C^M$ such that $h = f\circ\gamma$ on a nonplucripolar compact subset E of the set $F := \gamma^{-1}(K)$, then $B(a,r) \subset F$. In particular $f\circ\gamma$ has no analytic continuation from the interior of F.

Under some additional assumptions on K we shall study differentiability properties of the function f. First we get

THEOREM 2. If K is a polynomially convex compact subset of C^N such that

$$(*) \qquad \Phi_K(x) \leq 1 + \varkappa \delta^{\mu}, \quad x \in C^N, \quad \operatorname{dist}(x,K) \leq \delta, \quad 0 < \delta \leq 1,$$

where $\varkappa$, μ are positive constants, then for every multiindex $\alpha \in Z_+^N$ the series $\sum_{j=1}^{\infty} \Theta^{\sqrt{n_j}} D^{\alpha} P_j$ is uniformly convergent on K; here $D^{\alpha} = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$.

This theorem is a direct consequence of the following Lemma 3 and of the fact that the series $\sum_{j=1}^{\infty} n_j^k \Theta^{\sqrt{n_j}}$ is convergent for every positive number k.

Lemma 3. (Markov's inequality). If K is a compact subset of C^N such that the extremal function Φ_K satisfies the continuity condition (*) then for every polynomial $P \in \mathcal{P}_n(C^N, C)$ and for all $\alpha \in Z_+^N$

$$\|D^{\alpha} P\|_K \leqslant e^{|\alpha|} (\varkappa n)^{|\alpha|/\mu} \|P\|_K$$

with $|\alpha| := \alpha_1 + \ldots + \alpha_N$.

The inequality (*) is true with $\mu = 1/2$ for all K satisfying the following

Geometrical Condition. There exists $r > 0$ such that for every point $b \in K$ one can find a point $a \in K$ such that the convex hull of the set $\{b\} \cup B(a,r)$ is contained in K.

If K is a subset of R^N (identified with the subset $R^N + i0$ of C^N) then it is sufficient if the Geometrical Condition is satisfied with balls $B(a,r)$ in R^N.

Lemma 3, Theorem 1, Theorem 2 and the Whitney's extension theorem imply the following

THEOREM 3. If D is a bounded convex domain, or if D is a bounded domain with Lipschitz boundary such that $K := \overline{D}$ is polynomially convex, then the function f defined by (&) has the following properties

(i) f is $\mathcal{C}^{\infty}$ on $\overline{D}$,

(ii) f is holomorphic in D,

(iii) f has the SNCP.

EXAMPLE. Let $D = B(0,1)$ be the unit Euclidean ball in C^N and let (a_j) be a sequence of unit vectors dense in ∂D. Let (n_j) be an increasing sequence of positive integers with $\lim_{j\to\infty} n_j/\sqrt{n_{j+1}} = 0$ (e.g. $n_j = 2^{j!}$). Finally let θ be a fixed real number with $0<\theta<1$. Put

$$(\#) \qquad f(x)=\sum_{j=1}^{\infty}\theta^{\sqrt{n_j}}\langle x , a_j\rangle^{n_j} , \quad x \in \bar{D},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in C^N.

We claim that f is $\mathcal{C}^\infty$ on $\bar{D}$, holomorphic in D and has SNCP. Indeed, if $P_j(x) :=\theta^{\sqrt{n_j}}\langle x,a_j\rangle^{n_j}$, then for all $\alpha\in Z_+^N$

$$|D^\alpha P_j(x)| \leq n_j^{|\alpha|}\theta^{\sqrt{n_j}} , \quad j\geq 1, \quad x\in\bar{D}.$$

Hence f is $\mathcal{C}^\infty$ on $\bar{D}$, as the series $\sum n_j^{|\alpha|}\theta^{\sqrt{n_j}}$ is convergent for all α. It is obvious that f is holomorphic in D. Now, if $\gamma : C^M \to C^N$ is any polynomial mapping with $\deg\gamma =: d\geq 1$, then

$$\|f\circ\gamma - P_j\circ\gamma\|_E^{1/dn_s} \leq \Big(\sum_{j=s+1}^{\infty}\|P_j\|_D\Big)^{1/dn_s} \leqslant \Big(\sum_{j=s+1}^{\infty}\theta^{\sqrt{n_j}}\Big)^{1/dn_s} \to 0, \text{ as } s\to\infty,$$

where $E := \gamma^{-1}(\bar{D})$. If $\lambda_0\in C^M - E$ then the series $\sum P_j(\gamma(\lambda_0))$ diverges, because $\limsup_{j\to\infty}|P_j(\gamma(\lambda_0))|^{1/n_j} = \|\gamma(\lambda_0)\| > 1$. Hence by Lemma 2 each connected component S of the interior of E is the maximal domain of existence of the holomorphic function $f\circ\gamma|_S$, i.e. f has the SNCP. In particular, if $\gamma(\lambda)= \mathring{x} + \lambda a$, $\lambda\in C$, where $\mathring{x}$ and a are fixed points of C^N with $\|\mathring{x}\|<1$ and $\|a\| = 1$, respectively, then $S :=\{\lambda\in C;\ x+\lambda a\in D\}$ is the natural doain of existence of $\lambda\to f(\mathring{x}+\lambda a)$.
This example shows that Lemma 2 implies a positive solution of the Problem 19.3.5 in W.Rudin's book [5].

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