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On topological Blaschke conjecture I

Cohomological complex projective spaces

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By a Blaschke manifold, we mean a Riemannian manifold (M,g) such that, for any point $m \in M$, the tangential cut locus C_m of m in T_mM is isometric to the sphere of constant radius. There are some equivalent definitions (see Besse[2, 5.43]). The Blaschke conjecture is that any Blaschke manifold is isometric to a compact rank one symmetric space. If the integral cohomology ring of M is equal to the sphere S^k, or the real projective space RP^k, this conjecture is proved by Berger with other mathematiciens [2, Appendix D]). We consider the case where the cohomology ring of M is equal to that of the complex projective space CP^k.

We obtain the following theorem.

Theorem. Let (M,g) be a 2k-dimensional Blaschke manifold such that the integral cohomology ring is equal to that of CP^k. Then M is PL-homeomorphic to CP^k for any k.

Blaschke manifolds with other cohomology rings will be treated in subsequent papers.

If (M,g) is a Blaschke manifold and $m \in M$, Allamigeon [1] has shown that the cut locus $C(m)$ of m in M is the base manifold of a fibration of the tangential cut locus C_m by great spheres. We study the base manifold of such fibration by great circles. We apply the Browder-Novikov-Sullivan's theory in the classification of homotopy equivalent manifolds (see Wall[4]). Calculation of normal invariants gives our theorem. In Appendix, we give examples of non-trivial fibrations of S^3 by great circles. The author thanks to M.Mizutani and K.Masuda for the discussion of results in Appendix.

Detailed proof will appear elsewhere.

§1. Projectable bundles

In the paper [3], we have obtained a method of a calculation of the tangent bundle of the base space of an S^1-principal bundle. We will briefly recall that.

Let X be a smooth manifold and let $\pi : L \longrightarrow X$ be the projection of an S^1-principal bundle.

<u>Definition</u>. A vector bundle $p : E \longrightarrow L$ over L is projectable onto X , if there exists a vector bundle $\hat{p} : \hat{E} \longrightarrow X$ over X such that $\pi^*\hat{E} = E$. The map π induces the bundle map $\pi_! : E \longrightarrow \hat{E}$, which we call the projection. The bundle $\hat{E}$ is called the projected bundle.

Let x be a point in X . For any $a , b \in \pi^{-1}(x) = S^1$, we have a linear isomorphism

$$\Phi_{ab} : p^{-1}(a) \longrightarrow p^{-1}(b)$$

of vector spaces defined by $\Phi_{ab}(u) = v$, where $\pi_!(u) = \pi_!(v)$. Then we have, for $a , b , c \quad \pi^{-1}(x)$,

(1) $$\Phi_{bc}\,\Phi_{ab} = \Phi_{ac} \quad .$$

Let $\pi^*L = \{(a,b) \in L \times L,\ \pi(a) = \pi(b)\}$ be the induced S^1-bundle over L from L . We have two projections π_1 , $\pi_2 : \pi^*L \longrightarrow L$ defined by $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$. let $\pi_i{}^*E$ $(i = 1,2)$ be the induced vector bundle. The map $\Phi : \pi^*L \longrightarrow \mathrm{Iso}(\pi_1{}^*E,\ \pi_2{}^*E)$ defined by $\Phi(a,b) = \Phi_{ab}$ is a continuous cross section of the bundle $\mathrm{Iso}(\pi_1{}^*E,\ \pi_2{}^*E)$ over π^*L .

We call Φ the projecting isomorphism associated with the projectable bundle E .

Proposition 1. Suppose given a vector bundle E over L and a cross section Φ of the bundle $Iso(\pi_1{}^*E, \pi_2{}^*E)$ satisfying (1) . Then we have a vector bundle $\hat{E}$ over X such that $\pi^*\hat{E} = E$ and the projecting isomorphism is equal to Φ .

Now let TL and TX be the tangent bundles of L and X respectively. Let $\rho : S^1 \times L \longrightarrow L$ be the free S^1-action. For each $t \in S^1$, the diffeomorphism $\rho(t) = \rho(t, \cdot)$ induces a budle isomorphism $\rho(t)_* : TL \longrightarrow TL$.

Proposition 2. The collection $\bigcup_{t \in S^1} \rho(t)_*$ induces a projecting isomorphism on the bundle TL such that the projected bundle TL is isomorphic to $TX \oplus 1$.

Proof. Choose a bundle metric on TL . Let TL_1 be the subnundle of TL consisting of tangent vectors normal to the S^1- action. Then TL_1 is projected to TX . The line bundle tangent to the S^1-action is projected to the trivial line bundle on X .

§2. Pontrjagin classes

Let S^{2k-1} be the unit sphere in $\mathbb{R}^{2k}$ and let $\pi : S^{2k-1} \longrightarrow B$ be a fibration of S^{2k-1} by great circles. Thus, for each $b \in B$, $\pi^{-1}(b)$ is the intersection of S^{2k-1} with a 2-plane in $\mathbb{R}^{2k}$. We write the 2-plane by $P(b)$. Let $\rho : S^1 \times S^{2k-1} \longrightarrow S^{2k-1}$ denote the free S^1-action.

Let $V(2k,2)$ and $G(2k,2)$, respectively, be the Stiefel and the Grassmann manifold consisting of orthogonal 2 frames or oriented 2-planes in $\mathbb{R}^{2k}$. Then the natural mapping $\lambda : V(2k,2) \longrightarrow G(2k,2)$ defines a principal S^1-bundle.

The mapping $\theta : B \longrightarrow G(2k,2)$ defined by $\theta(b) = P(b)$ is a smooth embedding. Let $\theta^*(\lambda)$ denote the induced bundle of λ by θ. Since π is also the induced bundle of λ by θ, there exists a natural bundle isomorphism between π and $\theta^*(\lambda)$ inducing the identity on B. Thus we obtain;

Lemma 3. We may suppose that the free S^1-action ρ on S^{2k-1} is equal to the restriction on $\pi^{-1}(b)$ of the linear action on $P(b)$ for every $b \in B$.

In the following, we always assume that ρ is the linear action on each fibre. For each $x \in S^{2k-1}$, let Kx denote the point $\rho(1/4)x$ in S^{2k-1}, where we identify

S^1 with $[0,1]/\ [0] \sim [1]$. Define a mapping $\tilde{\theta}$: $S^{2k-1} \longrightarrow V(2k,2)$ by $\tilde{\theta}(x) = (x,\ Kx)$. This is a smooth embedding and is a bundle map inducing θ on the base manifolds. For an orthogonal 2-frame $w = (x,y)$, let $\tilde{\psi}(w)$ denote the vector $(x/\sqrt{2}\ ,\ y/\sqrt{2}\ )$ in $\mathbb{R}^{2k} \oplus \mathbb{R}^{2k}$. Then the map $\tilde{\psi} : V(2k,2) \longrightarrow \mathbb{R}^{4k}$ is a smooth embedding of $V(2k,2)$ in $S^{4k-1} \subset \mathbb{R}^{4k}$. We identify $\mathbb{R}^{2k} \oplus \mathbb{R}^{2k}$ with $\mathbb{C}^{2k}$ such that the first summand $\mathbb{R}^{2k}$ is the real part and the second pure imaginary. On $\mathbb{C}^{2k} - 0$, we have the free action ρ_0 of S^1 as the multiplication by the complex number of norm one. Then $\tilde{\psi}$ is S^1 - equivalent and we write by ψ the induced map ψ : $G(2k,2) \longrightarrow \mathbb{C}P^{2k-1}$.

Let $\tilde{f} : S^{2k-1} \longrightarrow S^{4k-1}$ be the composition $\tilde{f} = \tilde{\psi}\tilde{\theta}$ and $f = \psi\theta : B \longrightarrow \mathbb{C}^{2k-1}$. The map $\tilde{f}$ is given by $\tilde{f}(x) = (\ x/\sqrt{2}\ ,\ Kx/\sqrt{2}\ )$ for $x \in S^{2k-1}$.

We define a map $\tilde{F} : \mathbb{R}^{2k} - 0 \longrightarrow \mathbb{C}^{2k} - 0$ by $\tilde{F}(tx) = t\tilde{f}(x)$ for $t > 0$ and $x \in S^{2k-1}$. The map $\tilde{F}$ is a smooth embedding . Let E denote the restriction of the tangent bundle $T(\ \mathbb{R}^{2k} - 0\ )$ of $\mathbb{R}^{2k} - 0$ on S^{2k-1} , and we write p for the projection $E \longrightarrow S^{2k-1}$. Then $\tilde{F}$ induces an injective bundle map $\tilde{F}_* : E \longrightarrow \tilde{F}_*(E) \subset T(\ \mathbb{C}^{2k} - 0\ )|_{\tilde{F}(S^{2k-1})}$.

Now define a map $\tilde{G} : \mathbb{R}^{2k} - 0 \longrightarrow \mathbb{C}^{2k} - 0$ by $\tilde{G}(tx) = (\ tx/\sqrt{2}\ ,\ -tK/\sqrt{2}\ )$ for $t > 0$ and $x \in S^{2k-1}$.

Then $\tilde{G}$ is also an embedding and $\tilde{G}$ induces an injective bundle map

$$\tilde{G}_* : E \longrightarrow \tilde{G}_*(E) \subset T(\mathbb{C}^{2k-0} - 0)\,|\,_{\tilde{G}(S^{2k-1})} .$$

If $\bar{\rho}_0$ denote the conjugate action of S^1 on $\mathbb{C}^{2k} - 0$. Then G is S^1-equivariant concerning to this conjugate action.

For any $y \in \mathbb{C}^{2k}$, we naturally identify the tangent space $T_y\mathbb{C}^{2k}$ with $\mathbb{C}^{2k}$ itself. For $x \in S^{2k-1}$, let E_x denote the fiber $p^{-1}(x)$. Then $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are subvector spaces of $\mathbb{C}^{2k}$.

Since $K : S^{2k-1} \longrightarrow S^{2k-1}$ is a diffeomorphism, we obtain;

Lemma 4. The vector spaces $\tilde{F}_*(E_x)$ and $\tilde{G}_*(E_x)$ are transversal. Thus they span $\mathbb{C}^{2k}$.

Let T denote the restriction of the tangent bundle $T(\mathbb{C}^{2k})$ on $\tilde{F}(S^{2k-1})$. Then we have the direct sum decomposition by trivial vector bundles

$$T = \tilde{F}_*(E) \oplus \tilde{G}_*(E) .$$

Notice that $\tilde{G}_*(E)$ on $\tilde{G}(S^{2k-1})$ is identified with the subbundle in T over $\tilde{F}(S^{2k-1})$ by an orientation reversing diffeomorphism of S^{2k-1}.

For any $t \in S^1$, we have the induced bundle isomorphisms $\rho_*(t) : E \longrightarrow E$ and $\rho_{0*}(t) : T \longrightarrow T$.

Lemma 5. The isomorphism $\rho_{0*}(t)$ is equal to the direct sum

$$\rho_*(t) + \rho_*(t) .$$

By Proposition 1, we obtain that the projected bundle $\hat{T}$, defined by the projecting isomorphism $\rho_*(t)$, is isomorphic to the Whitney sum;

$$\hat{T} \cong \hat{E} \oplus \hat{E} .$$

On the other hand, by Proposition 2, we obtain the following.

Lemma 6. The bundle $\hat{T}$ has the complex structure. As a complex vector bundle , $\hat{T}$ is isomorphic to the Whitney sum $T(\mathbb{C}P^{2k-1})|_{f(B)} \oplus 1$.

Lemma 7. As a real vector bundle, $\hat{E}$ is isomorphic to the bundle $T(B) \oplus 2$.

Consequently, we obtain that

$$T(B) \oplus T(B) \oplus 4 \cong (T(\mathbb{C}P^{2k-1})|_{f(B)} \oplus 1)_{\mathbb{R}} .$$

Since the cohomology groups $H^*(B;\mathbb{Z})$ has no torsion element, by the product formula of Pontrjagin classes, we obtain the following .

Proposition 8. The Pontrjagin classes of the smooth manifold B is equal to that of $\mathbb{C}P^{k-1}$, for any k .

§3. Z_2-invariants and proof of Theorem

Let $\mathcal{S}(CP^{k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to CP^{k-1}. The following results are due to Sullivan (cf. [4, §14 C]).

Suppose that $k > 3$.

Proposition 9. For any $N \in \mathcal{S}(CP^{k-1})$, there are invariants $s_{4i+2}(N) \in Z_2$ and $s_{4j}(N) \in Z$, for all integers i,j satisfying $6 \leq 4i+2 < 2(k-1)$, $4 \leq 4j < 2(k-1)$. The invariants define a bijection of $\mathcal{S}(CP^{k-1})$ with

$$(\bigoplus_i Z_2) \oplus (\bigoplus_j Z) .$$

The invariants s_{4j} satisfy the following relations.

Proposition 10. If all the Pontrjagin classes of N in $\mathcal{S}(CP^{k-1})$ coincide with that of CP^{k-1}, then $s_{4j}(N) = 0$ for all j.

Concerning Z_2-invariants s_{4i+2}, the following holds. Let $\mathcal{S}(RP^{2k-1})$ denote the set of PL-homeomorphism classes of closed PL-manifolds homotopy equivalent to RP^{2k-1}. This set is known to be equal to the isomorphism classes of homotopy triangulations of RP^{2k-1}. Any $N \in \mathcal{S}(CP^{k-1})$ is the base manifold of a PL free S^1-action on S^{2k-1}. By restricting the action to $Z_2 = S^0 \subset S^1$, we obtain a manifold homotopy equivalent to RP^{2k-1}.

This defines a map

$$\pi^b : \mathcal{S}(CP^{k-1}) \longrightarrow \mathcal{S}(RP^{2k-1}) .$$

The following holds ([4, §14D.3]).

Proposition 11. Let N be an element in $\mathcal{S}(CP^{k-1})$ such that $\pi^b(N)$ is PL-homeomorphic to RP^{2k-1}. Then

$$s_{4i+2}(N) = 0 ,$$

for all i .

Now let $B \in \mathcal{S}(CP^{k-1})$ be the base manifold of the fibration of S^{2k-1} by great circles. Then, obviously, the image $\pi^b(B) \in \mathcal{S}(RP^{2k-1})$ is PL-homeomorphic to RP^{2k-1} .

Combining the result of §2 with Propositions, we obtain ;

Proposition 12. The base manifold B of a fibration of S^{2k-1} by great circles is PL-homeomorphic to CP^{k-1} if $k \neq 3$.

Now let us prove Theorem. Since the integral cohomology ring of M is equal to that of CP^k , M is simply connected ([2, 7.23]). Thus M is homotopy equivalent to CP^k . By Allamigeon's theorem, we know that M is PL-homeomorphic to the union of the disc D^{2k} with the D^2-bundle associated with the fibration of S^{2k-1} by great circles. We write B for the base manifold of the fibration. If $k = 3$, by Proposition 9, M is

PL-homeomorphic to CP^3 if and only if $s_4(M) = 0$. The invariant $s_4(M)$ is calculated from the first Pontrjagin class $p_1(B)$ of B . By Proposition 8 of §2, $p_1(B)$ is equal to $p_1(CP^2)$. Thus we have $s_4(M) = 0$ and M is PL-homeomorphic to CP^3 . Now suppose that $k \neq 3$. According to Proposition 12, B is PL-homeomorphic to CP^{k-1} . The Euler class of the S^1-bundle is equal to a generator of $H^2(CP^{k-1};Z) = Z$. Thus the total space of the D^2-bundle is PL-homeomorphic to the tubular neighborhood of CP^{k-1} in CP^k . Any orientation preserving PL-homeomorphism of S^{2k-1} is isotopic to the identity. The attached manifold M is PL-homeomorphic to CP^k, which completes the proof of Theorem.

§4. Appendix

If $\pi : S^{2k-1} \longrightarrow B$ is a fibration by great circles, we get the embedding $\theta : B \longrightarrow G(2k,2)$. Since the planes $\theta(b)$ for all $b \in B$ give a foliation of S^{2k-1}, we have the following property.

(*) For two different points b and b' in B, the planes $\theta(b)$ and $\theta(b')$ are transverse.

The converse holds.

Lemma 13. Let $\pi : S^{2k-1} \longrightarrow B$ be a principal S^1-bundle induced from the S^1-bundle $\lambda : W(2k,2) \longrightarrow G(2k,2)$ by a smooth embedding $\theta : B \longrightarrow G(2k,2)$. Suppose that, for any different points b and b' in B, the planes $\theta(b)$ and $\theta(b')$ are transversal. Then the bundle π is a fibration of S^{2k-1} by great circles.

Proof. Consider the union $\bigcup_b (\theta(b) \cap S^{2k-1})$.

Then it covers S^{2k-1} and give a fibration by great circles.

Now we consider the case where $k = 2$. For the following discussion, see [2 , p.55] . Let $\Lambda^2 \mathbb{R}^4$ denote the space of skew-symmetric 2-tensors. The Grassmann manifold $G(4,2)$ is naturally identified with the set of decomposable elements of norm one in $\Lambda^2 \mathbb{R}^4$. We have the Hodge operator $*$ from $\Lambda^2 \mathbb{R}^4$ onto itself. The space $\Lambda^2 \mathbb{R}^4$ is decomposed to two orthogonal subsets E_1 and E_{-1} associated to the eigenvalue 1 and -1 of $*$.

let S^2_1 and S^2_{-1} be the sphere in E_1 and E_{-1} of radius $1/\sqrt{2}$. Then $G(4,2)$ is equal to the product $S^2_1 \times S^2_{-1}$. Define a bilenear map $\zeta : \Lambda^2 \mathbb{R}^4 \times \Lambda^2 \mathbb{R}^4 \longrightarrow \mathbb{R}$ by $\zeta(a,b) = \| a \wedge b \|$, where $\| \ \|$ is the norm on $\Lambda^2 \mathbb{R}^4 \cong \mathbb{R}$. Two planes P_1 and P_2 in $G(4,2)$ are transversal if and only if $\zeta(P_1, P_2) = 0$. Represent P_1 and P_2 by (x_1, x_2) and (y_1, y_2), where $x_1, y_1 \in S^2_1$ and $x_2, y_2 \in S^2_{-1}$. Then we have

$$\zeta(P_1, P_2) = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle ,$$

where $\langle \ \rangle$ is the inner product of the vector space E_1 or E_{-1}.

For a smooth map $\theta : S^2 \longrightarrow G(4,2)$, we define a smooth function $Z(\theta)$ on S^2 by $Z(\theta)(x) = \zeta(\theta(x), \theta(x'))$, by fixing x' in S^2. Thus the principal S^1-bundle $\pi : S^3 \longrightarrow S^2$ induced by an embedding $\theta : S^2 \longrightarrow G(4,2)$ is a fibration by great circles if $Z(\theta)(x) = 0$ only when $x = x'$. Obviously $Z(\theta)(x) = 0$ at $x = x'$. We have;

Lemma 14. For a smooth map $\theta : S^2 \longrightarrow G(4,2)$, the function $Z(\theta)$, for fixed $x' \in S^2$, is critical at $x = x'$.

Proof. Fix P_2 in $G(4,2)$. The function $\zeta(P_1, P_2)$ on $G(4,2)$ is critical at $P_1 = P_2$. Thus $Z(\theta)$ is also critical at $x = x'$.

Now consider the Hopf fibration $\pi_0 : S^3 \longrightarrow S^2$.

The associated map $\theta_0 : S^2 \longrightarrow G(4,2) = S^2{}_1 \times S^2{}_{-1}$ is given by $\theta_0(x) = (1/\sqrt{2}\ x\ ,\ \alpha_0)$, where $\alpha_0 = (1/\sqrt{2}\ ,\ 0,\ 0)$. For two points $x = (x_1\ ,\ x_2,\ x_3)$ and $x' = (x_1', x_2', x_3')$ in S^2 , we have

$$\zeta(\theta_0(x),\ \theta_0(x')) = \langle x,\ x' \rangle - 1/2$$
$$= -1/2\ \Sigma\ (x_i - x_i')^2\ .$$

Thus the function $Z(\theta_0)$ is critical if and only if $x = x'$. The symmetric matrix $(\partial^2 Z(\theta_0)/\partial x_i \partial x_j)$ is positive definite.

Let $\mathrm{Emb}(S^2,\ G(4,2))$ denote the set of smooth embeddings of S^2 in $G(4,2)$ with C^2-topology. Since S^2 is compact, we obtain the following.

Proposition 15. There exists a neighborhood U of θ_0 in $\mathrm{Emb}(S^2,\ G(4,2))$ such that the function $Z(\theta)(x,x') = \zeta(\theta(x),\ \theta(x'))$ is equal to zero if and only if $x = x'$, for any $x,\ x' \in S^2$ and $\theta \in U$.

Corollary 16. In each direction in $\mathrm{Emb}(S^2\ ,\ G(4,2))$, there is a deformation of fibrations of S^3 by great circles starting from the Hopf fibration.

The group of diffeomorphisms of S^2, denoted by Diff S^2, acts naturally on $\mathrm{Emb}(S^2,G(4,2))$. We denote by Diff $S^2\backslash \mathrm{Emb}(S^2,G(4,2))$ the quotient space. Let $\pi : S^3 \longrightarrow B$ be a fibration of S^3 by great circles. The B is diffeomorphic to S^2. Thus we have the class $\{\theta\}$ in Diff $S^2\backslash \mathrm{Emb}(S^2,G(4,2))$.

Let π_1 and π_2 be two fibrations of S^3 by great circles, and let $\{\theta_1\}$, $\{\theta_2\} \in$ Diff $S^2\backslash \mathrm{Emb}(S^2,G(4,2))$ be the associated classes. We say that π_1 and π_2 are isometric if there exists a budle map F from π_1 to π_2 such that F is an isometry of S^3 onto itself.

The group $O(4)$ acts naturally on $G(4,2)$ and on Diff $S^2\backslash \mathrm{Emb}(S^2,G(4,2))$. We denote by Diff $S^2\backslash \mathrm{Emb}(S^2,G(4,2))/O(4)$ the quotient space.

Proposition 17. Two fibrations π_1 and π_2 of S^3 by great circles are isometric if and only if the classes $\{\theta_1\}$ and $\{\theta_2\}$ in Diff $S^2\backslash \mathrm{Emb}(S^2,G(4,2))/O(4)$ are equal.

Remark that we can choose the neighborhood U in Proposition 15 such that U is invariant by the actions of Diff S^2 and $O(4)$. The space Diff $S^2\backslash U / O(4)$ is of infinite "dimension".

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