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The Order of the Attaching Class of the Suspended Quaternionic Quasi-Projective Space

By

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§ 0. Introduction

In this note, F denotes the field of the complex numbers C or the field of the quaternions H. We denote by FP^n the F-projective space of n F-dimensions and by $Q_n(F)$ the quasi-F-projective space. $G_n(F)$ denotes the unitary group U(n) or the symplectic group Sp(n) according as F is C or H. Let d be the dimension of F over the field of the real numbers R and S^{dn-1} the unit sphere in F^n. Let T'_n: $S^{dn-2} \longrightarrow G_{n-1}(F)$ be the characteristic map for the normal form of the principal $G_{n-1}(F)$-bundle over S^{dn-1}. Then, as is well known ([2], [3] and [9]), Im $T'_n = Q_{n-1}(F)$, precisely, the following diagram commutes:

$$\begin{array}{ccc} S^{dn-2} & \xrightarrow{T_n} & Q_{n-1}(F) \\ & {\scriptstyle T'_n}\searrow \quad \swarrow{\scriptstyle j_{n-1}} & \\ & G_{n-1}(F), & \end{array}$$

where j_{n-1} is the canonical reflection map. $Q_n(F) = Q_{n-1}(F) \cup_{T_n} e^{dn-1}$ and $Q_n(C) = E(CP^{n-1}_+)$, where E( ) denotes the reduced suspension and CP^{n-1}_+ a disjoint union of CP^{n-1} and { one point }.

Let $\omega_{n-1} = \omega_{n-1}(F)$ be the homotopy class of T_n and p: $Q_n(C) \longrightarrow Q_n(C)/Q_1(C) = ECP^{n-1}$ the collapsing map. In the previous paper [6], we proved that the k-th suspension $E^k(p_*\omega_n(C))$ is of order n! for $k \geqslant 0$.

The purpose of this note is to examine the order of $E^k\omega_{n-1}(H)$.

Let α be an element of a homotopy group $\pi_n(\ )$ and $E^\infty\alpha \in \pi_n^S(\ )$ the stable element of α. $o(\beta)$ denotes the order of β. Then, our result is the following

Theorem. i). $o(E^k\omega_{n-1}(H)) = 2\cdot(2n-1)!$ _for_ $k \geqslant 0$ _if_ n _is_ _even_.
ii). $o(E^\infty\omega_{n-1}(H)) = (2n-1)!$ _if_ n _is_ _odd_.

Our method is essentially to use the K-theory. To examine $o(\omega_{n-1}(H))$, we use the Toda's theorem about the generator of $\pi_{2n-1}(U(n))$ [6] and the group structure of $\pi_{4n+2}(Sp(n))$ [4]. To determine the lower bound of $o(E^k\omega_{n-1}(H))$, we use the standard method of D. M. Segal [7] from the unstable viewpoint, exactly, we use the Hurewicz homomorphism $h: \pi_{k+4n-1}(E^kQ_n(H)) \longrightarrow H_{k+4n-1}(E^kQ_n(H); Z)$. A powerful tool is the Toda-Kozima's map $\tilde{t}_n: Q_n(H) \longrightarrow Q_{2n}(C)$ [8].

Our result overlaps partially with the works of K. Morisugi [5] and G. Walker [9].

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§1. Determination of $o(\omega_{n-1}(H))$ for even n

First we recall the definition of the quasi-projective space and the reflection map. $S(F^n)$ denotes the unit sphere in F^n. $Q_n(F)$ is the space obtained from $S(F^n) \times S(F)$ by imposing the equivalence relation: $(u, q) \sim (ug, g^{-1}qg)$ for $g \in S(F)$ and collapsing $S(F^n) \times \{1\}$ to a point. The reflection map $j_n = j_n(F): Q_n(F) \longrightarrow G_n(F)$ is defined as follows:

$$j_n([u, q])(v) = v + u(q-1)\langle u, v\rangle$$

for $u \in S(F^n)$, $q \in S(F)$ and $v \in F^n$, where $\langle u, v\rangle = \Sigma_{k=1}^n \bar{u}_k v_k$ for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

Let $z = x + jy \in H$, where $x, y \in C$. By regarding $x \in C$ as $x + j0 \in H$, we have the injection $C \hookrightarrow H$. Obviously, this induces the canonical maps i_n: $Q_n(C) \longrightarrow Q_n(H)$ and i'_n: $U(n) \longrightarrow Sp(n)$. From the definition, the following diagram commutes:

$$\text{(1.1)} \qquad \begin{array}{ccc} Q_n(C) & \xrightarrow{i_n} & Q_n(H) \\ \downarrow j_n & & \downarrow j_n \\ U(n) & \xrightarrow{i'_n} & Sp(n). \end{array}$$

In the complex case, we can define the reduced reflection map [6]:

$$\tilde{j}_n = \tilde{j}_n(C)\colon ECP^{n-1} \cong Q_n(C)/Q_1(C) \longrightarrow U(n)/U(1) \cong SU(n).$$

By abuse of notation, we often use the same letter j_n for the reduced case.

Lemma 1.2. i). <u>If</u> n <u>is even</u>, $j_{n*}\colon \pi_{4n-1}(Q_n(H)) \longrightarrow \pi_{4n-1}(Sp(n))$ <u>is an epimorphism</u>.

ii). <u>If</u> n <u>is odd</u>, $\mathrm{Im}\, j_{n*} = a\pi_{4n-1}(Sp(n))$, <u>where</u> $a = 1$ <u>or</u> 2.

<u>Proof</u>. Let $p\colon Q_{2n}(C) \longrightarrow Q_{2n}(C)/Q_1(C) \cong ECP^{2n-1}$ be the collapsing map, $k\colon Q_n(H) \longrightarrow Q_{2n}(H)$ and $k'\colon Sp(n) \longrightarrow Sp(2n)$ the inclusion maps, respectively. Then, by (1.1), the following diagram commutes for $r = 4n - 1$:

$$\begin{array}{ccccccc} \pi_r(ECP^{2n-1}) & \xleftarrow{p_*} & \pi_r(Q_{2n}(C)) & \xrightarrow{i_{2n*}} & \pi_r(Q_{2n}(H)) & \xleftarrow{k_*} & \pi_r(Q_n(H)) \\ \downarrow \tilde{j}_{2n}(C)_* & & \downarrow j_{2n}(C)_* & & \downarrow j_{2n*} & & \downarrow j_{n*} \\ \pi_r(SU(2n)) & = & \pi_r(U(2n)) & \xrightarrow{i'_{2n*}} & \pi_r(Sp(2n)) & \xleftarrow{k'_*} & \pi_r(Sp(n)). \end{array}$$

p_* is an epimorphism since $Q_{2n}(C) \simeq ECP^{2n-1} \vee S^1$. By Theorem 4.1 of [6], $\tilde{j}_{2n}(C)_*$ is an epimorphism. So, $j_{2n}(C)_*$ is an epimorphism. k_* and k'_* are isomorphisms respectively. As is well known, i'_{2n*} is an isomorphism if n is even and $\mathrm{Im}\, i'_{2n*} = 2\pi_{4n-1}(Sp(2n))$ if n is odd. Therefore, the above commutative diagram leads us to the assertion. This completes the proof.

Proposition 1.3. i). $o(\omega_{n-1}) = 2\cdot(2n - 1)!$ <u>for even</u> n.

ii). $o(\omega_{n-1}) = a\cdot(2n - 1)!$ <u>for odd</u> n, <u>where</u> a <u>is the same number as in</u> Lemma 1.2.

<u>Proof</u>. Let $p: (Q_n(H), Q_{n-1}(H)) \longrightarrow (S^{4n-1}, *)$ be the collapsing map. We consider the natural homomorphism between the exact sequences for $r = 4n - 1$:

$$\begin{array}{ccccccc} \pi_r(Q_n(H)) & \xrightarrow{j_*^!} & \pi_r(Q_n(H), Q_{n-1}(H)) & \xrightarrow{\partial} & \pi_{r-1}(Q_{n-1}(H)) & \longrightarrow & \pi_{r-1}(Q_n(H)) \\ \downarrow j_{n*} & & \downarrow p_* & & \downarrow j_{n-1*} & & \downarrow j_{n*} \\ \pi_r(Sp(n)) & \xrightarrow{p_*^!} & \pi_r(S^{4n-1}) & \xrightarrow{\Delta'} & \pi_{r-1}(Sp(n-1)) & \longrightarrow & \pi_{r-1}(Sp(n)), \end{array}$$

where the mappings are canonical and ∂ and Δ' are the connecting homomorphisms.

As is well known, $\pi_{4n-1}(Sp(n)) \approx Z$, $\pi_{4n-2}(Sp(n)) \approx 0$ and $\pi_m(S^m) = \{\iota_m\} \approx Z$. By the Blakers-Massey theorem [1], p_* is an isomorphism. By the definition, $\omega_{n-1} = \Delta(\iota_{4n-1})$, where $\Delta = \partial\circ p_*^{-1}$. So, by Theorem 2.2 of [4], j_{n-1*} is an epimorphism and the following holds:

(1.4) $\quad \pi_{4n-2}(Sp(n-1)) = \{j_{n-1*}\omega_{n-1}\} \approx Z_{b\cdot(2n-1)!}$, where $b = 1$ for odd n and $b = 2$ for even n.

By the exactness of the upper sequence, $o(\omega_{n-1})$ is equal to the order of the cokernel of $j_*^!$. Hence, by (1.4), Lemma 1.2 and by the above commutative diagram, we have the assertion. This completes the proof.

By inspecting the above proof, we have the following

Proposition 1.5. $j_{n*}: \pi_{4n-1}(Q_n(H)) \longrightarrow \pi_{4n-1}(Sp(n))$ <u>is an epimorphism if and only if</u> $o(\omega_{n-1}) = b\cdot(2n - 1)!$, <u>where</u> b <u>is the same number as in</u> (1.4).

2. Some fundamental facts

For $n \geqslant 0$, X_n denotes a connected finite CW complex such that $X_0 = \{*\}$ and $X_n = e^0 \cup e^{r_1} \cup \ \dots \ \cup e^{r_n}$ for $n \geqslant 1$. Here $r = r_n = dn - \varepsilon$ with $\varepsilon = 0$ or 1 and $d - \varepsilon \geqslant 2$. θ_{n-1}: $S^{r-1} \longrightarrow X_{n-1}$ denotes the attaching map, and so $X_n = X_{n-1} \cup_{\theta_{n-1}} e^r$. For example, $X_n = FP^n$ ($d = 2$ or 4 and $\varepsilon = 0$) and $X_n = Q_n(H)$ ($d = 4$ and $\varepsilon = 1$).

Let p: $X_n \longrightarrow X_n/X_{n-1} = S^r$ and p': $(X_n, X_{n-1}) \longrightarrow (S^r, *)$ be the collapsing maps. Let ∂: $\pi_{r+m}(E^m X_n, E^m X_{n-1}) \longrightarrow \pi_{r+m-1}(E^m X_{n-1})$ be the connecting homomorphism. Then, $(E^m p')_*$: $\pi_{r+m}(E^m X_n, E^m X_{n-1}) \longrightarrow \pi_{r+m}(S^{r+m})$ is an isomorphism for $m \geqslant 0$ [1], and we define a homomorphism Δ: $\pi_{r+m}(S^{r+m}) \longrightarrow \pi_{r+m-1}(E^m X_{n-1})$ by the composition $\partial \circ (E^m p')_*^{-1}$. By the definition, $\Delta(\iota_{r+m}) = E^m \theta_{n-1}$, where the same letter is used for a mapping and its homotopy class.

Let $h = h_m$: $\pi_{r+m}(E^m X_n) \longrightarrow H_{r+m}(E^m X_n;\ Z) \approx Z$ for $m \geqslant 0$ be the Hurewicz homomorphism and $h(n, m)$ the non-negative integer such that $\mathrm{Im}\ h = h(n, m) H_{r+m}(E^m X_n;\ Z)$. Then we have the following

Lemma 2.1. $\quad o(E^m \theta_{n-1}) = h(n, m)$.

<u>Proof</u>. j: $(E^m X_n, *) \longrightarrow (E^m X_n, E^m X_{n-1})$ denotes the inclusion. Then, we consider the commutative diagram:

$$\begin{array}{ccccc} \pi_{r+m}(E^m X_n) & \xrightarrow{j_*} & \pi_{r+m}(E^m X_n, E^m X_{n-1}) & \xrightarrow{\partial} & \pi_{r+m-1}(E^m X_{n-1}) \\ \downarrow h & & \downarrow h' & & \\ H_{r+m}(E^m X_n;\ Z) & \xrightarrow{j_*} & H_{r+m}(E^m X_n, E^m X_{n-1};\ Z), & & \end{array}$$

where h' denotes the relative Hurewicz homomorphism and the upper sequence is exact. From the cell structure of X_n, the lower j_* is an isomorphism. By the relative Hurewicz theorem, h' is an isomorphism. This completes the proof.

According to [8], a representative element of $Q_n(H)$ can be taken as $(x + jy, e^{i\pi t})$, where $x, y \in C^n$ satisfying $x + jy \in S(H^n)$ and $0 \leqslant t \leqslant 1$. Toda and Kozima defined $\tilde{t}_n: Q_n(H) \longrightarrow Q_{2n}(C)$ by the equation

$$\tilde{t}_n[(x + jy, e^{i\pi t})] = [(x \oplus y, e^{2i\pi t})].$$

We define $t_n: Q_n(H) \longrightarrow ECP^{2n-1}$ by the composition $p \circ \tilde{t}_n$, where $p: Q_{2n}(C) \longrightarrow ECP^{2n-1}$ is the collapsing map. From the definition, the following diagram commutes for $k < n$:

$$(2.2) \qquad \begin{array}{ccc} Q_k(H) & \xrightarrow{t_k} & ECP^{2k-1} \\ \downarrow i & & \downarrow i' \\ Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1}, \end{array}$$

where i and i' are the canonical inclusions.

The following lemma is a reduced version of Proposition 2.5 of [8].

Lemma 2.3 (Toda-Kozima). The following diagram commutes up to homotopy:

$$\begin{array}{ccc} Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1} \\ \downarrow j_n & & \downarrow j_{2n} \\ Sp(n) & \xrightarrow{c} & SU(2n), \end{array}$$

where c is the complexification map.

Let $p: Q_n(H) \longrightarrow Q_n(H)/Q_{n-1}(H) = S^{4n-1}$ for $n \geqslant 1$ and $p': ECP^{2n-1} \longrightarrow ECP^{2n-1}/ECP^{2n-3} \simeq S^{4n-3} \vee S^{4n-1}$ for $n \geqslant 2$ be the collapsing maps. Then, by (2.2), there exists a mapping $t'_n\colon S^{4n-1} \longrightarrow S^{4n-3} \vee S^{4n-1}$ for $n \geqslant 2$ such that the following diagram commutes:

$$\begin{array}{ccc} Q_n(H) & \xrightarrow{t_n} & ECP^{2n-1} \\ \downarrow p & & \downarrow p' \\ S^{4n-1} & \xrightarrow{t'_n} & S^{4n-3} \vee S^{4n-1}. \end{array} \qquad (2.4)$$

Let $p_2\colon S^{4n-3} \vee S^{4n-1} \longrightarrow S^{4n-1}$ for $n \geqslant 2$ be the projection map. Then, we have the following

Lemma 2.5. $\deg t_1 = -1$ <u>and</u> $\deg (p_2 t'_n) = (-1)^n$ <u>for</u> $n \geqslant 2$.

<u>Proof</u>. We define $g_n\colon S(H^n) \longrightarrow S(C^{2n})$ by the equation

$$g_n(x + jy) = x \oplus y$$

for $x, y \in C^n$. It is clear that g_n is a homeomorphism and $\deg g_n = (-1)^n$. By Lemma 2.3, $t_1 \simeq g_1$ and $p_2 t'_n \simeq g_n$ for $n \geqslant 2$. This completes the proof.

Hereafter the same letter is often used for a mapping and its homotopy class. Let $\gamma_n = \gamma_n(F)\colon S(F^{n+1}) \longrightarrow FP^n$ be the projection map. Let $i\colon ECP^{2n-1} \longrightarrow ECP^{2n}$ be the inclusion map. Then, we have the following

Proposition 2.6. $\quad (-1)^{n+1} E\gamma_{2n}(C) = i t_n \omega_n(H)$.

<u>Proof</u>. By (2.2) and (2.4), the following diagram commutes for $r = 4n + 3$:

$$\begin{array}{ccccc}
\pi_r(S^{4n+3}) & \xleftarrow{p_*} & \pi_r(Q_{n+1}(H), Q_n(H)) & \xrightarrow{\partial} & \pi_{r-1}(Q_n(H)) \\
\downarrow t'_{n+1*} & & \downarrow t_{n+1*} & & \downarrow t_{n*} \\
\pi_r(S^{4n+1} \vee S^{4n+3}) & \xleftarrow{p'_*} & \pi_r(ECP^{2n+1}, ECP^{2n-1}) & \xrightarrow{\partial'} & \pi_{r-1}(ECP^{2n-1}) \\
\downarrow p_{2*} & & \downarrow i'_* & & \downarrow i_* \\
\pi_r(S^{4n+3}) & \xleftarrow{p''_*} & \pi_r(ECP^{2n+1}, ECP^{2n}) & \xrightarrow{\partial''} & \pi_{r-1}(ECP^{2n}),
\end{array}$$

where the mappings are canonical.

p_* and p''_* are isomorphisms [redacted] [1]. We note that $\omega_n(H) = \partial p_*^{-1}(\iota_{4n+3})$ and $E\gamma_{2n}(C) = \partial'' p''^{-1}_*(\iota_{4n+3})$. So, by Lemma 2.5 and the above commutative diagram, we have the assertion. This completes the proof.

Remark 1. Owing to Proposition 2.6, it suffices to take $(-1)^{n+1} t_n \omega_n$ as λ_{2n} in Proposition 6.5.ii) of [6]. By Theorem 1.2 of [6] and Proposition 1.3, $o(\lambda_{2n}) = (2n + 1)!$ or $2 \cdot (2n + 1)!$. In the last section, we shall show that $o(\lambda_4) = 5!$ (cf. Lemma 11.1 of [6]).

§ 3. Determination of the lower bound of $o(E^m \omega_{n-1}(H))$.

Let $v \in \tilde{K}(CP^{2n-1})$ be the stable isomorphism class of the canonical line bundle over CP^{2n-1}. We denote by $I_C: \tilde{K}(\ ) \longrightarrow \tilde{K}(E^2\ )$ the Bott periodicity isomorphism. The following Lemma is well known (cf. Lemma 2.2 of [8]).

Lemma 3.1. $I_C(v) \in \tilde{K}(E^2 CP^{2n-1})$ <u>is represented by the adjoint of the composite of the canonical maps</u>:

$$ECP^{2n-1} \xrightarrow{j_{2n}} SU(2n) \xrightarrow{i} U(2n) \xrightarrow{k} \Omega BU(2n),$$

<u>where</u> k <u>is the homotopy equivalence</u>.

Hereafter, Z or the rational number field Q is taken as the coefficients of the homology or cohomology groups, unless otherwise stated.

Let ch^k: $\tilde{K}(\ ) \longrightarrow H^{2k}(\ ;Q)$ be the k-th Chern character and $ch = \Sigma_k ch^k$ the total Chern character. Let σ: $\tilde{H}^i(E\ ) \longrightarrow \tilde{H}^{i-1}(\ )$ be the suspension isomorphism. Then, as is well known, the following diagram commutes:

$$\begin{array}{ccc} \tilde{K}(CP^{2n-1}) & \xrightarrow{I_C} & \tilde{K}(E^2CP^{2n-1}) \\ \downarrow ch & & \downarrow ch \\ H^*(CP^{2n-1};Q) & \xrightarrow{\sigma^{-2}} & H^*(E^2CP^{2n-1};Q). \end{array} \qquad (3.2)$$

We denote by y a generator of $H^2(CP^{2n-1})$. It is also well known that

$$ch^{2n-1}v = 1/(2n-1)!y^{2n-1}. \qquad (3.3)$$

Proposition 3.4. $o(E^m\omega_{n-1})$ <u>is a multiple of</u> $(2n-1)!$ <u>for</u> $m \geqslant 0$.

<u>Proof</u>. The assertion is a direct consequence of Theorem 1.2 of [6] and Proposition 2.6. For the later use, we give another proof for even m.

By (2.4) and Lemma 2.5, t_n^*: $H^{4n-1}(ECP^{2n-1}) \longrightarrow H^{4n-1}(Q_n(H))$ is an isomorphism. So, $y' = t_n^*\sigma^{-1}y^{2n-1}$ is taken as a generator of $H^{4n-1}(Q_n(H))$. We choose a generator x of $H_{4n-1}(Q_n(H))$ satisfying $\langle y', x\rangle = 1$, where $\langle\ ,\ \rangle$ denotes the Kronecker index.

Put $o(E^m\omega_{n-1}) = k(n)$. Denote by s: $\tilde{H}_i(\ ) \longrightarrow \tilde{H}_{i+1}(E\ )$ the suspension isomorphism. Then, by Lemma 2.1, there exists an element $\alpha \in \pi_{m+4n-1}(E^mQ_n(H))$ satisfying $h_m(\alpha) = k(n)s^mx$. By the definition of the Hurewicz homomorphism, $h_m(\alpha) = \alpha_*s^m\xi_n$, where ξ_n denotes a generator of $H_{4n-1}(S^{4n-1})$. So, we have $k(n) = \langle\sigma^{-m}y', \alpha_*s^m\xi_n\rangle = \langle\alpha^*\sigma^{-m}y', s^m\xi_n\rangle$. Choose a generator τ_n of $H^{4n-1}(S^{4n-1})$ satisfying $\langle\tau_n, \xi_n\rangle = 1$. Then, we have $\alpha^*\sigma^{-m}y' = k(n)\sigma^{-m}\tau_n$.

Put $m = 2t$ and $u = I_C^t(Et_n)^*I_C(v) \in \tilde{K}(E^{m+1}Q_n(H))$. Then, by (3.2), (3.3) and by the naturality of the Chern character, we have the following:

$\sigma ch^{2n+t}(E\alpha)^*u = \alpha^*\sigma^{-m}t_n^*\sigma^{-1}ch^{2n-1}(v) = 1/(2n-1)!\alpha^*\sigma^{-m}y'$.

So, we have $ch^{2n+t}(E\alpha)^*u = k(n)/(2n-1)!\sigma^{-m-1}\tau_n$. As is well known, Im ch^{2n+t} $= H^{4n+m}(S^{4n+m}; Z)$. Hence, $k(n)/(2n-1)!$ is an integer. This completes the proof.

Lemma 3.5. $(Et_n)^*I_C(v)$ <u>belongs to the image of the complexification homomorphism</u> $c': \widetilde{KSp}(EQ_n(H)) \longrightarrow \tilde{K}(EQ_n(H))$.

<u>Proof</u>. By Lemmas 2.3 and 3.1, $u' = (Et_n)^*I_C(v) = (\mathrm{adj}\ (k\circ i\circ j_{2n}(C)))_*(Et_n)$ $= (\mathrm{adj}\ k)_*(Ec)_*(Ej_n(H))$.

Let ρ_c: $BSp(n) \longrightarrow BU(2n)$ be the mapping induced from c: $Sp(n) \longrightarrow U(2n)$ and k': $Sp(n) \longrightarrow \Omega BSp(n)$ be the canonical homotopy equivalence. Then, it is well known that $k\circ c \simeq \Omega\rho_c\circ k'$. So, we have $(\mathrm{adj}\ k)_*(Ec)_* = (\rho_c)_*(\mathrm{adj}\ k')_*$. Hence, $u' = (\rho_c)_*(\mathrm{adj}\ k')_*(Ej_n(H)) \in \mathrm{Im}\ c'$. This completes the proof.

As is well known, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{KSp}(\ ) & \xrightarrow{c'} & \tilde{K}(\ ) \\ \downarrow I_H & & \downarrow I_C^4 \\ \widetilde{KSp}(E^8\ ) & \xrightarrow{c'} & K(E^8\ ), \end{array} \tag{3.6}$$

where I_H denotes the Bott periodicity isomorphism.

Proposition 3.7. <u>If</u> n <u>is even and</u> $m \equiv 0 \bmod 8$, $o(E^m\omega_{n-1})$ <u>is a multiple of</u> $2\cdot(2n-1)!$.

<u>Proof</u>. As is well known, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{KSp}(E^{m+1}Q_n(H)) & \xrightarrow{(E\alpha)^*} & \widetilde{KSp}(S^{4n+m}) \\ \downarrow c' & & \downarrow c \\ \tilde{K}(EQ_n(H)) & \xrightarrow{(E\alpha)^*} & \tilde{K}(S^{4n+m}), \end{array}$$

and $\operatorname{Im} c = 2\tilde{K}(S^{4n+m})$ if n is even. So, by Lemma 3.5, (3.6) and by the proof of Proposition 3.4, $(E\alpha)^* u = (E\alpha)^* I_C^t (Et_n)^* I_C(v) \in 2\tilde{K}(S^{4n+m})$ and $ch^{2n+t}(E\alpha)^* u \in 2H^{4n+m}(S^{4n+m}; Z)$. Therefore, $k(n)/(2n-1)!$ is an even integer. This completes the proof.

Remark 2. By the similar arguments, we have the following for $k \geqslant 1$ (cf. [7]):

(1). $o(E^k \gamma_{n-1}(C))$ is a multiple of $n!$ for even k.

(2). $o(E^k \gamma_{n-1}(H))$ is a multiple of $(2n)!/2$ for even k. If n is even and $k \equiv 0 \mod 8$, $o(E^k \gamma_{n-1}(H))$ is a multiple of $(2n)!$.

§ 4. Proof of the theorem

To prove ii) of our theorem, we use the following [3]:

Theorem 4.1 (James). _The stunted quasi-projective space_ $Q_n(F)/Q_{n-k}(F)$ _is an S-retract of the factor space_ $G_n(F)/G_{n-k}(F)$ _for_ $k \leqslant n$. _In particular_, $j_{n*}^S : \pi_i^S(Q_n(H)) \longrightarrow \pi_i^S(Sp(n))$ _is a monomorphism for_ $i \geqslant 0$.

Now we are ready to prove the theorem. The assertion i) is a direct consequence of Propositions 1.3.i) and 3.7.

By Theorem 4.1, $j_{n-1*}^S : \pi_{4n-2}^S(Q_{n-1}(H)) \longrightarrow \pi_{4n-2}^S(Sp(n-1))$ is a monomorphism. So, we have $o(E^\infty \omega_{n-1}) = o(E^\infty j_{n-1*}\omega_{n-1})$. Therefore, (1.4) and Proposition 3.4 lead us to the assertion. This completes the proof of the theorem.

Remark 3. We can give an improved proof of Theorem 1.2 of [6]. We use the first half of the proof of Theorem 1.2 of [6] and Remark 2.(1). We have

(1). $o(E^k \gamma_{n-1}(C)) = n!$ for $k \geqslant 1$.

By (1) and Remark 2.(2), we have the following:

(2). If n is even, $o(E^k \gamma_{n-1}(H)) = (2n)!$ for $k \geqslant 1$.

By Theorem 1.1 of [7] and by Lemma 2.1,

(3). $o(E^{\infty}\gamma_{n-1}(H)) = (2n)!/2$ if n is odd.

In this case, the Adams spectral sequence is used for the 2-primary stable homotopy of quaternionic and complex projective spaces [7].

§ 5. An example

An open problem is to determine the order of $\omega_n(H)$ completely. The author hopes that an affirmative answer is given to the following

Conjecture. $o(\omega_{n-1}(H)) = (2n - 1)!$ *if* n *is odd*.

In this section, we determine the group structure of $\pi_{10}(Q_2(H))$ and we show that the conjecture is true for n = 3. We use the following: $\pi_{11}(S^{10}) \approx Z_2$, $\pi_{10}(S^7) = \{\nu_7\} \approx Z_{24}$, $\pi_{11}(S^7) \approx 0$, $\pi_9(S^3) \approx Z_3$ and $\pi_{10}(S^3) \approx Z_{15}$.

Example. $\pi_{10}(Q_2(H)) \approx Z_{5!} + Z_2$ *and* $o(\omega_2(H)) = 5!$.

Proof. Let p: $(Q_2(H), S^3) \longrightarrow (S^7, *)$ be the collapsing map. Then, p_*: $\pi_7(Q_2(H), S^3) \longrightarrow \pi_7(S^7)$ is an isomorphism [1]. We choose a generator α of $\pi_7(Q_2(H), S^3) \approx Z$ such that $p_*\alpha = \iota_7$.

Sp(2) is regarded as the cell complex $Q_2(H) \cup e^{7,3}$. Let p': $(Sp(2), Q_2(H)) \longrightarrow (S^{10}, *)$ be the collapsing map. Then, p'_*: $\pi_n(Sp(2), Q_2(H)) \longrightarrow \pi_n(S^{10})$ is an isomorphism for $n \leqslant 11$ [1].

We consider the following commutative diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & & \pi_{11}(Sp(2), S^3) \approx 0 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & \pi_{11}(Sp(2), Q_2(H)) & = & \pi_{11}(Sp(2), Q_2(H)) \approx Z_2 & & & & \\
 & & & & \downarrow \partial & & \downarrow \partial' & & & & \\
 & & \pi_{10}(S^3) & \xrightarrow{i_*} & \pi_{10}(Q_2(H)) & \xrightarrow{j_*} & \pi_{10}(Q_2(H), S^3) & \xrightarrow{\partial''} & \pi_9(S^3) \approx Z_3 & & \\
 & & \| & & \downarrow j_{2*} & & \downarrow p_* & & \| & & \\
\pi_{11}(S^7) & \longrightarrow & \pi_{10}(S^3) & \xrightarrow{i'_*} & \pi_{10}(Sp(2)) & \xrightarrow{p'_*} & \pi_{10}(S^7) & \xrightarrow{\Delta'} & \pi_9(S^3) & \longrightarrow & \pi_9(Sp(2)), \\
\approx 0 & & \approx Z_{15} & & \approx Z_{5!} & & \approx \pi_{10}(Sp(2), S^3) & & & & \approx 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & \pi_{10}(Sp(2), Q_2(H)) \approx Z & & & &
\end{array}
$$

where the mappings are canonical and the horizontal and perpendicular sequences are exact respectively.

p_* is a split epimorphism since $p_*(\alpha\nu_7) = \nu_7$. So, we have $\pi_{10}(Q_2(H), S^3) \approx Z_{24} + Z_2$. By the commutativity of the above diagram, i_* is a monomorphism and ∂'' is an epimorphism. Therefore, by the upper horizontal sequence, $\pi_{10}(Q_2(H)) \approx Z_{5!} + Z_2$. Hence, by Proposition 1.3.ii), we have $o(\omega_2) = 5!$. This completes the proof.

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