数理解析研究所講究録
第 512 巻 1984 年 99-109

The Nilpotent Subvariety of the Vector Space Associated to A Symmetric Pair

—— Survey ——

Jiro SEKIGUCHI
Tokyo Metropolitan University

§1. Introduction.

1.1. This note is a survey of my paper [S].

1.2. Let $\underline{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $\underline{g}_0$ be its real form. Let θ be a Cartan involution of $\underline{g}_0$ and let $\underline{g}_0 = \underline{k}_0 + V_0$ be the corresponding Cartan decomposition. Extend θ to $\underline{g}$ as a complex linear involution and let $\underline{k}$, V be the complexifications of $\underline{k}_0$, V_0, respectively. In this note, $(\underline{g}, \underline{k})$ is called a symmetric pair and V is the vector space associated to it. Put $G = \mathrm{Int}\,\underline{g}$ and $K_\theta = \{g \in G;\ \theta g = g\}$. Let K be the identity component of K_θ. Let $\underline{\underline{N}}(V)$ be the nilpotent subvariety of V, that is, an element X of V is contained in $\underline{\underline{N}}(V)$ if and only if $\mathrm{ad}_{\underline{g}}(X)$ is nilpotent (cf. [1]). Let $\mathbb{C}[V]$ be the polynomial ring over V and let $\mathbb{C}[V]^K$ be the subring of $\mathbb{C}[V]$ consisting of K-invariant elements. Then there exist homogeneneous polynomials $P_1,\ldots,P_\ell$ of $\mathbb{C}[V]^K$ such that $\mathbb{C}[V]^K = \mathbb{C}[P_1,\ldots,P_\ell]$ (cf. [1]). It follows that $\underline{\underline{N}}(V) = \{X \in V;\ P_1(X) = \cdots = P_\ell(X) = 0\}$ and that $\mathrm{codim}_V \underline{\underline{N}}(V) = \ell$, in other words, $\underline{\underline{N}}(V)$ is a complete intersection. The number ℓ is nothing but the restricted rank of the corresponding Riemannian symmetric pair $(\underline{g}_0, \underline{k}_0)$.

1.3. The subjects of [S] are concerned with the following problems:

(1) Determine the irreducible components of $\underline{N}(V)$.

(2) Construct an analogue of Springer's resolution for $\underline{N}(V)$.

(3) Examine the generic singularities of $\underline{N}(V)$.

We determine the number of the irreducible components of $\underline{N}(V)$ completely in [S] (cf. §2). The cotangent bundle over the comlpete flag manifold of G is regarded as a desingularization of the nilpotent subvariety of the Lie algebra $\underline{g}$. This is called the Springer's resolution (cf. [4]). It seems to be interesting to construct an analogue to the Springer's resolution for $\underline{N}(V)$. This will be done in §2. As to (3), we recall Brieskorn's result (cf. [3]). His famous result states: A simple singularity of type A_ℓ, D_ℓ or E_ℓ appears in the nilpotent subvariety $\underline{N}_{\underline{g}}$ of the corresponding simple Lie algebra $\underline{g}$. More precisely, take a subregular nilpotent element of $\underline{g}$ and a transversal slice S_X to the G-orbit of X at X. Then $S_X \cap \underline{N}_{\underline{g}}$ is nothing but the simple singularity. It is interesting to examine the analogue of Brieskorn's result for $\underline{N}(V)$. This is the precise meaning of (3). Although I do not obtain a complete answer to this at present, I shall explain the result for this problem in §4.

§2. The irreducibility of $\underline{N}(V)$.

The variety $\underline{N}(V)$ is not irreducible in general. In constrast to the fact that the nilpotent subvariety of a simple Lie algebra is irreducible, this is a remarkable difference.

Example 1. Consider the case where $\underline{g} = \underline{sl}(2, \mathbb{C})$ and $\theta(X) = -{}^tX$. Then $\underline{N}(V) = \{\begin{pmatrix} x & y \\ y & -x \end{pmatrix}; x^2 + y^2 = 0\}$. It is clear that $\underline{N}(V)$ has two irreducible components defined by the equations $x + \sqrt{-1}y = 0$ and $x - \sqrt{-1}y = 0$.

In general, we show the following ([S, Th.1]).

Theorem 1. Let $(\underline{g}, \underline{k})$ be a symmetric pair, $\underline{g}$ simple.

(1) If the corresponding Riemannian symmetric pair $(\underline{g}_0, \underline{k}_0)$ is Hermitian symmetric and the restricted root system of $\underline{g}_0$ is reduced, then $\underline{N}(V)$ has two irreducible components. Each irreducible components is also a complete intersection.

(2) If $(\underline{g}, \underline{k})$ is one of the following. If d is the number of irreducible components of $\underline{N}(V)$, then d is given in the right column.

$(\underline{g}, \underline{k})$	d
$(\underline{sl}(2n, \mathbb{C}), \underline{so}(2n, \mathbb{C}))$	2
$(\underline{so}(2n+1, \mathbb{C}), \underline{so}(n+1, \mathbb{C}) + \underline{so}(n, \mathbb{C}))$	2
$(\underline{so}(4n, \mathbb{C}), \underline{so}(2n, \mathbb{C}) + \underline{so}(2n, \mathbb{C}))$	4
$(\underline{so}(4n+2, \mathbb{C}), \underline{so}(2n+1, \mathbb{C}) + \underline{so}(2n+1, \mathbb{C}))$	2
$(\underline{so}(4n+k, \mathbb{C}), \underline{so}(2n+k, \mathbb{C}) + \underline{so}(2n, \mathbb{C}))$ $(k, n \geq 2)$	2
$(\underline{e}_7^{\mathbb{C}}, \underline{sl}(8, \mathbb{C}))$	2

(3) If $(\underline{g}, \underline{k})$ is not the one treated in (1) and (2), then $\underline{N}(V)$ is irreducible.

<u>Problem</u>. In the case of (2), each irreduicible component of $\underline{\underline{N}}(V)$ is not a complete intersection. Determine the defining ideal of each irreducible component.

I have no idea to this problem at present.

Put

$$\underline{\underline{N}}(V)_r = \{X \in \underline{\underline{N}}(V);\ dP_1, \cdots, dP_\ell \text{ are linearly independent at } X\}.$$

$$\underline{\underline{N}}(V)_{pr} = \{X \in \underline{\underline{N}}(V);\ K\cdot X \text{ is open in } \underline{\underline{N}}(V)\}.$$

An element of $\underline{\underline{N}}(V)_{pr}$ is called principal nilpotent (cf. [1]). It is clear that $\underline{\underline{N}}(V)_{pr}$ is contained in $\underline{\underline{N}}(V)_r$. But in general they do not coincide. We now give an example of such a pair that $\underline{\underline{N}}(V)_{pr} \subsetneqq \underline{\underline{N}}(V)_r$.

<u>Example 2</u>. $(\underline{sl}(n+1, \mathbb{C}), \underline{gl}(n, \mathbb{C}))$ $(n > 1)$.

Let $\underline{g} = \underline{sl}(n+1, \mathbb{C})$ and let θ be an involution of $\underline{g}$ defined by $\theta(X) = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} X \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$. Then $\underline{k} = \{ X \in g;\ \theta(X) = X\}$ is isomorphic to the Lie algebra $\underline{gl}(n, \mathbb{C})$. We identify $\mathbb{C}^n \times \mathbb{C}^n$ with $V = \{X \in g;\ \theta(X) = -X\}$ by the map $(x_1, \cdots, x_n, y_1, \cdots, y_n) \mapsto \begin{bmatrix} 0 & \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \\ y_1 \cdots y_n & 0 \end{bmatrix}$. Under the identification, we find that

$$\underline{\underline{N}}(V) = \{ (x,y) \in \mathbb{C}^n \times \mathbb{C}^n ;\ x_1 y_1 + \cdots + x_n y_n = 0 \} .$$

By direct calculation, we also find that $\underline{\underline{N}}(V)$ has four K-orbits O_i $(i = 1, \cdots, 4)$ defined by

$$O_1 = \{ (x,y) \in \mathbb{C}^n \times \mathbb{C}^n \ ; \ x_1y_1 + \cdots + x_ny_n = 0, \quad x \neq 0, \ y \neq 0 \}$$

$$O_2 = \{ (x,y) \in \mathbb{C}^n \times \mathbb{C}^n \ ; \ x = 0, \ y \neq 0\}$$

$$O_3 = \{ (x,y) \in \mathbb{C}^n \times \mathbb{C}^n \ ; \ x \neq 0, \ y = 0\}$$

$$O_4 = \{ (x,y) \in \mathbb{C}^n \times \mathbb{C}^n \ ; \ x = y = 0 \}$$

and that $O_1 = \underline{\underline{N}}(V)_{pr}$, $O_1 \cup O_2 \cup O_3 = \underline{\underline{N}}(V)_r$. Hence $\underline{\underline{N}}(V)_{pr} \subsetneqq \underline{\underline{N}}(V)_r$ in this case.

In general, we have the following.

Theorem 2 ([S, Th.4]). Let $(\underline{g}, \underline{k})$ be a symmetric pair and let Σ be the restricted root system of the corresponding Riemannian symmetric pair $(\underline{g}_0, \underline{k}_0)$.

(1) If Σ is reduced, then $\underline{\underline{N}}(V)_{pr} = \underline{\underline{N}}(V)_r$.

(2) If Σ is not reduced, then $\underline{\underline{N}}(V)_{pr} \subsetneqq \underline{\underline{N}}(V)_r$. Moreover, in this case, if X is in $\underline{\underline{N}}(V)_r - \underline{\underline{N}}(V)_{pr}$, then $Ad(G)X \cap V = \underline{\underline{N}}(V)_r - \underline{\underline{N}}(V)_{pr}$.

§3. A resolution of the nilpotent subvariety.

Take $X_0 \in \underline{\underline{N}}(V)_{pr}$ and fix it. If follows from [1] that there exist $H_0 \in \underline{k}$ and $Y_0 \in V$ such that

$$[H_0, X_0] = 2X_0, \quad [H_0, Y_0] = -2Y_0, \quad [X_0, Y_0] = H_0.$$

We define

$$\underline{g}(j) = \{ A \in \underline{g};\ [H_0, A] = jA\}$$

$$\tilde{\underline{l}} = \bigoplus_{j \geq 0} \underline{g}(j), \qquad \tilde{\underline{n}} = \bigoplus_{j > 0} \underline{g}(j)$$

$$\underline{l} = \tilde{\underline{l}} \cap \underline{k}, \qquad \underline{n} = \tilde{\underline{n}} \cap V.$$

We note here that $\tilde{\underline{l}}$ is a parabolic subalgebra of $\underline{g}$, that $\tilde{\underline{n}}$ is its nilpotent radical and that $[\tilde{\underline{l}}, \tilde{\underline{n}}] \subset \tilde{\underline{n}}$. Let $\tilde{L}$ be the parabolic subgroup of G with lie algebra $\tilde{\underline{l}}$ and put $L_\theta = \tilde{L} \cap K_\theta$.

Every element p of L_θ induces an automorphism of $K_\theta \times \underline{n}$ in the following way: $(k, X) \to (kp, \mathrm{Ad}(p^{-1})X)$. We denote by $\tilde{\underline{N}}(V)$ the quotient of $K_\theta \times \underline{n}$ by the action of L_θ and put $k*X = (k, X)L_\theta$ for any $(k, X) \in K_\theta \times \underline{n}$. Let π be the canonical mapping of $\tilde{\underline{N}}(V)$ to $\underline{N}(V)$. By the construction, connected components of $\tilde{\underline{N}}(V)$ correspond to irreducible components of $\underline{N}(V)$. Hence in general $\tilde{\underline{N}}(V)$ is not connected (cf. Th.1).

The following theorem shows that $\tilde{\underline{N}}(V)$ is an analogue of the Springer's resolution of the nilpotent subvariety of a simple Lie algebra (cf. [4]).

<u>Theorem 3 ([S, Th.5])</u>. The mapping $\pi : \tilde{\underline{N}}(V) \to \underline{N}(V)$ has the following properties.

(a) $\tilde{\underline{N}}(V)$ is smooth.

(b) π is proper and surjective.

(c) π induces an isomorphism $\pi^{-1}(\underline{N}(V)_{pr}) \to \underline{N}(V)_{pr}$.

We give here examples which illustrate the resolution of the nilpotent variety $\underline{\underline{N}}(V)$.

Example 3. $(\underline{\underline{so}}(n+1, \mathbb{C}), \underline{\underline{so}}(n, \mathbb{C}))$ $(n \geq 2)$.

In this case, V is identified with $\mathbb{C}^n$ and the nilpotent subvariety $\mathbb{C}(V)$ with the set

$$S = \{ x \in \mathbb{C}^n;\ x_1^2 + \ldots + x_n^2 = 0 \}.$$

Then the resolution $\tilde{\underline{\underline{N}}}(V)$ is identified with

$$\tilde{S} = \{ (x,\xi) \in \mathbb{C}^n \times \mathbb{P}^{n-1};\ x_1^2 + \ldots + x_n^2 = 0, \quad \xi_1^2 + \ldots + \xi_n^2 = 0, \quad x \,/\!/\, \xi \}.$$

Define $S_1 = \{x \in S;\ x \neq 0\}$ and $S_2 = \{ 0 \}$. Then the K-orbits of S are S_1 and S_2. If x is contained in S_1, then $\pi^{-1}(x)$ is clearly a single point. On the other hand, if $x = 0$, then $\pi^{-1}(x) = \mathbb{P}^{n-1}$.

Example 2 (continued).

In this case, V is identified with $\mathbb{C}^{2n}$ and $\underline{\underline{N}}(V)$ is with the set

$$S = \{ (x,y) \in \mathbb{C}^n \times \mathbb{C}^n;\ x_1y_1 + \ldots + x_ny_n = 0 \}.$$

The resolution $\tilde{\underline{\underline{N}}}(V)$ is identified with

$$\tilde{S} = \{((x,y),(\xi,\eta)) \in S \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1};\ \xi_1\eta_1 + \ldots + \xi_n\eta_n = 0, \quad x \,/\!/\, \xi, \quad y \,/\!/\, \eta \}.$$

We may regard O_i $(i = 1, 2, 3, 4)$ as subsets of S. Then S_i $(i = 1,\dots,4)$ are the K-orbits of S. In particular, S_1 is the totality of the principal nilpotent elements and $S' = S_1 \cup S_2 \cup S_3$ is non-singular and identified with $\underline{\underline{N}}(V)_r$. If (x,y) is in S_1, then $\pi^{-1}((x,y))$ consists of a single point. On the other hand, for any $x \in \mathbb{C}^n$ $(x \neq 0)$, we have $\pi^{-1}((x,0)) = \mathbb{P}^{n-1}$. Moreover $\pi^{-1}(0,0) = \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

Remark. Put $L = L_\theta \cap K$. Then L is a parabolic subgroup of K. But L is not the Borel subgroup of K in general.

Problem. Does there exist an analogue of Grothendieck's simultaneous resolution for V ?

§4. The generic singularities of $\underline{\underline{N}}(V)$.

Put

$$\underline{\underline{N}}(V)_s = \{X \in \underline{\underline{N}}(V);\ dP_1,\dots,dP_\ell \text{ are linearly dependent at } X\}.$$

It is pathological that $\underline{\underline{N}}(V)_s$ is neither irreducible nor equidimensional in general. Let $O_1,\dots,O_r$ be K-orbits of $\underline{\underline{N}}(V)$ such that $O_i \cap O_j = \emptyset$ $(i \neq j)$ and $\underline{\underline{N}}(V)_s = \bigcup_{i=1}^{r} \overline{O}_i$. In particular, each $\overline{O}_i$ is an irreducible component of $\underline{\underline{N}}(V)_s$. Put $\underline{\underline{N}}(V)'_s = \bigcup_{i=1}^{r} O_i$. Take $X \in \underline{\underline{N}}(V)'_s$. Let S_X be a transversal slice to the K-orbit of X at X. A standard one is constructed as follows. Let $H \in \underline{k}$ and $Y \in V$ be such that $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$ (cf. [1]). Then $S_X = X + [Y, \underline{k}]$ is a transversal slice. It seems to be interesting to determine the intersection $S_X \cap \underline{N}(V)$. In fact, as stated in the Introduction, a simple singularity is appeared in this manner if we consider a symmetric pair $(\underline{g} \oplus \underline{g}, \underline{g})$.

Let $X \in \underline{N}(V)'_s$. It is not clear whether the intersection $S_X \cap \underline{N}(V)$ is a hypersurface singularity or not. But I conjecture that this is true.

From now on, assume that $(\underline{g}, \underline{k})$ is of the classical type, that is, $\underline{g}$ is simple of the classical type. I give the concrete expression of $S_X \cap \underline{N}(V)$ for such an $X \in \underline{N}(V)'_s$ that $S_X \cap \underline{N}(V)$ is a hypersurface singularity (cf. [S, §§3, 4]).

$(\underline{sl}(n, \mathbb{C}), \underline{so}(n, \mathbb{C}))$	$x^n + y^2 = 0$
$(\underline{sl}(2n, \mathbb{C}), \underline{sp}(n, \mathbb{C}))$	$x^n + u_1v_1 + u_2v_2 = 0$
$(\underline{sl}(2n+k, \mathbb{C}), \underline{sl}(n+k, \mathbb{C})+\underline{sl}(n, \mathbb{C})+\mathbb{C})$	$x^n + yz = 0$
	$x_1y_1 + \cdots + x_{k+1}y_{k+1} = 0$
$(\underline{so}(n+1, \mathbb{C}), \underline{so}(n, \mathbb{C}))$	$x_1^2 + \cdots + x_n^2 = 0$
$(\underline{so}(n+3, \mathbb{C}), \underline{so}(n+1, \mathbb{C}) + \underline{so}(2, \mathbb{C}))$	$xy = 0$
	$x_1^2 + \cdots + x_n^2 = 0$
$(\underline{so}(2n, \mathbb{C}), \underline{so}(n, \mathbb{C}) + \underline{so}(n, \mathbb{C}))$	$x^{n-1} + xy^2 = 0$

$(\underline{so}(2n+1, \mathbb{C}), \underline{so}(n+1, \mathbb{C}) + \underline{so}(n, \mathbb{C}))$	$x^{2n} + y^2 = 0$
	$xy = 0$
$(\underline{so}(2n+k, \mathbb{C}), \underline{so}(n+k, \mathbb{C}) + \underline{so}(n, \mathbb{C}))$ $(k > 1)$	$x^n + y^2 = 0$
	$x_1^2 + \cdots + x_{k+1}^2 = 0$
$(\underline{so}(4n, \mathbb{C}), \underline{gl}(2n, \mathbb{C}))$	$x^n + u_1v_1 + u_2v_2 = 0$
	$xy = 0$
$(\underline{so}(4n+2, \mathbb{C}), \underline{gl}(2n+1, \mathbb{C}))$	$x^n + u_1v_1 + u_2v_2 = 0$
	$u_1v_1 + u_2v_2 + u_3v_3 = 0$
$(\underline{sp}(n, \mathbb{C}), \underline{gl}(n, \mathbb{C}))$	$x^{2n} + y^2 = 0$
	$x^{2n} + xy^2 = 0$
$(\underline{sp}(2n, \mathbb{C}), \underline{sp}(n, \mathbb{C}) + \underline{sp}(n, \mathbb{C}))$	$u_1v_1 + u_2v_2 = 0$
$(\underline{sp}(2n+k, \mathbb{C}), \underline{sp}(n+k, \mathbb{C}) + \underline{sp}(n, \mathbb{C}))$	$x^n + u_1v_1 + u_2v_2 = 0$
	$\sum_{i=1}^{2k+2} u_iv_i = 0$

Remark. In [2], Mr. Y. Shimizu and I treaded the case where $(\underline{g}, \underline{k})$ is of normal type, that is, $\underline{g}_0$ is a normal real form of $\underline{g}$.

References

[S] J. Sekiguchi; The nilpotent subvariety of the vector space associated to a symmetric pair, to appear in Publ. RIMS, Kyoto Univ.

[1] B. Kostant and S. Rallis; Orbits and representations associated with symmetric spaces, Amer. J. Math. 93, 753-809 (1971).

[2] J. Sekiguchi and Y. Shimizu; Simple singularities and infinitesimally symmetric spaces, Proc. of Japan Acad. 57, Ser. A, 42-46 (1981).

[3] P. Slodowy; Simple Singularities and Simple Algebraic Groups, Lecture Notes in Math. 815, Springer Verlag.

[4] T. A. Springer; Th unipotent variety of a semisimple group, Proc. of the Bombay Colloqu. in Algebraic Geometry, ed. S. Abhyanker, London, Oxford Univ. Press, 1969, 373-391.