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On an exterior problem for fully non-linear wave equation
(非線形波動方程式の外部問題について)

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§1. Introduction. Klainerman [1], [2] proved that one and only one C^∞ solution exists for Cauchy problem of a fully nonlinear wave equation of the form: $\Box u + F(t,x,u',u") = 0$, with sufficiently small and smooth initial data under the following assumptions:

(1) $F=F_{u'} = F_{u"} = 0$ for $u' = u" = 0$ if $n \geqq 6$,

(2) $F=F_{u'}=F_{u"}=F_{u'u'}=F_{u'u"}=F_{u"u"} = 0$ for $u' = u" = 0$ if $3 \leqq n \leqq 5$.

Here n is the space dimension and $\Box$ denotes the d'Alembertian, u' represents the vector of first derivatives, u" that of second derivatives with respect to the $x = (x_1,\ldots,x_n)$ and t. See also Klainerman-Ponce [3] and Shatah [4]. The case n=3 is, of course, of special importance for applications. Under the assumption (1) Klainerman's theorem cannot be extended to n=3, which was showed by John [5]. According to John's result, for example we have that every non-tivial C^2-solution of the equation: $\Box u = 2u_t u_{tt}$ for which $u(0,x), u_t(0,x)$ are of compact support and $\int_{R^3} [u_t(0,x)-u_t^2(0,x)]\, dx \geqq 0$ blows up in finite time. Thererfore, when n=3, the assumption (2) is needed to get a global existence theorem. For example, a classical non-linear wave operator: $u_{tt} - \Delta u(1+ \Sigma_{j=1}^n (u_{x_j})^2)^{-1/2}$, satisfies the assumption (2).

In this note, our purpose is to extend Klainerman's theorems to an exterior problem. Our results stated here will be published elsewhere (see Shibata and Tsutsumi [6] and [7]). Let Ω be an unbounded in R^n, its boundary $\partial\Omega$ being C^∞ and compact. We denote a time variable by t or x_0 and a space variable by $x = (x_1,\dots,x_n)$, respectively. We abbreviate $\partial/\partial t$, $\partial/\partial x_j$ and $(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$ to ∂_t or ∂_0, ∂_j and ∂_x^α, respectively, where α is a multi-index with $|\alpha| = \alpha_1 +\cdots+ \alpha_n$, and $j = 1,\dots,n$. We consider the following problem:

$$\text{(M.P)}\quad \begin{cases} \Box u + F(t,x,u',u'') = f(t,x) & \text{in } [0,T)\times\Omega, \\ u = 0 & \text{on } [0,T)\times\partial\Omega, \\ u(0,x) = u_0(x),\ (\partial_t u)(0,x) = u_1(x) & \text{in } \Omega. \end{cases}$$

Roughly speaking, under the assumptions (1) and (2) we can establish unique existence theorem of time global C^∞ solutions for (M.P) ($T = \infty$) if data u_0, u_1 and f are sufficiently small and smooth. In §2, we consider the local existence theorem of C^∞ solutions for (M.P). In §3, we consider the global existence theorem of C^∞ solutions for (M.P) under the additional assumption: Ω is non-trapping.

To conclude we list notations. For p with $1 \leqq p \leqq \infty$ we denote the standard L^p space defined on Ω and its norm by $L^p(\Omega)$ and $\|\cdot\|_p$, respectively. For a vector valued function $h = (h_1,\dots,h_s)$, we put $\|h\|_p = \Sigma_{j=1}^s \|h_j\|_p$. For a positive integer N we put $\|f\|_{p,N} = \Sigma_{|\alpha|\leqq N} \|\partial_x f\|_p$ and $\|h\|_{p,N} = \Sigma_{j=1}^s\Sigma_{|\alpha|\leqq N} \|\partial_x^\alpha h_j\|_p$. We set $H_p^N(\Omega) = \{ f \in L^p(\Omega);\ \|f\|_{p,N} < \infty \}$. By $\mathring{H}_p^N(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in $H_p^N(\Omega)$. $H_p^\infty(\Omega)$ denotes $\bigcap_{j=0}^\infty H_p^j(\Omega)$. For $-\infty \leqq a < b \leqq \infty$, a non-negative integer k and a Banach space E, $C^k([a,b];E)$ denotes the set of all E-valued functions having all derivatives of order $\leqq k$ continuous in [a,b]. For $u \in \bigcap_{j=0}^L C^j([a,b];H_p^{L-j}(\Omega))$, $v \in \bigcap_{j=0}^L C^j([0,\infty);H_p^{L-j}(\Omega))$, we put $|u|_{L,[a,b],p} = \sup_{a\leqq t\leqq b} \sum_{j+|\alpha|\leqq L} \|\partial_t^j\partial_x^\alpha u(t,\cdot)\|_p$, and $|u|_{L,k,p} =$

$\sup_{t \geq 0} (1+t)^k \sum_{j+|\alpha| \leq L} \|\partial_t^j \partial_x^\alpha u(t,\cdot)\|_p$, respectively.

§2. Local existence theorem. In this section, for finite $T > 0$ we consider the problem (M.P) under the following assumption.

Assumption 2.1. (1) $F(t,x,u',u'')$ is a real-valued function defined on $[0,T] \times \bar{\Omega} \times \{(u',u'') \in R^{(n+1)(n+2)};\ |u'| + |u''| \leq 3\lambda_0\}$ such that $F = F_{u'} = F_{u''} = 0$ for $u' = u'' = 0$.
(2) Let us define functions F_2^j, F_2^{ij}, F_1^j and F_0 by the formula:
$(dF/d\theta)(t,x,u'+\theta u'')|_{\theta=0} = \Sigma_{j=0}^n F_2^j(t,x,u',u'')\partial_t\partial_j v -$
$- \Sigma_{ij=1}^n F_2^{ij}(t,x,u',u'')\partial_i\partial_j v + \Sigma_{j=0}^n F_1^j(t,x,u',u'')\partial_j v + F_0(t,x,u',u'')v.$
Then there exists a positive number d such that
$\Sigma_{ij=1}^n [\delta_{ij} + F_{ij}^2(t,x,u',u'')]\xi_i\xi_j \geq d|\xi|^2$, $1 + F_2^0(t,x,u',u'') \geq d$
for all $(t,x) \in [0,T]\times\bar{\Omega}$, $|u'| + |u''| \leq 3\lambda_0$ and $\xi \in R^n$. Here $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$.//

Before stating main results in this section, we define a certain class of data and compatibility condition.

Definition 2.2. We shall say that a pair of functions $(u_0(x), u_1(x), f(t,x))$ with $u_0 \in \mathcal{B}^2(\bar{\Omega})$, $u_1 \in \mathcal{B}^1(\bar{\Omega})$ and $f(0,x) \in \mathcal{B}^0(\bar{\Omega})$ belongs to $\mathcal{D}$ if there exists a $u_2(x) \in \mathcal{B}^0(\bar{\Omega})$ such that $\|u_0\|_{\infty,2} + \|u_1\|_{\infty,1} + \|u_2\|_\infty \leq \lambda_0$ and $u_2(x) - \Delta u_0(x) + F(0,x,D_x^2 u_0(x), D_x^1 u_1(x), u_2(x)) = f(0,x)$ in Ω, where $D_x^2 u_0 = (\partial_1 u_0, \ldots, \partial_n u_0, \partial_i\partial_j u_0; i,j=1,\ldots,n)$ and $D_x^1 u_1 = (\partial_1 u_1, \ldots, \partial_n u_1)$.//

Of course, it follows from Assumption 2.1 that u_2 is unique if exists. If u_0, u_1 and f are sufficiently small, u_2 exists near 0 by implicit function theorem. To discuss the compatibility condition, we have to introduce some notations as follows. For a smooth function $v(t,x)$, we put $(\partial_t^p v)(0,x) = v_p(x)$. We define functions G_{p-2} as follows:

$$\partial_t^{p-2} F(t,x,v'v")\Big|_{t=0} = F_2^0(0,x,\overline{D}_x^2 v_0, \overline{D}_x^1 v_1, v_2)v_p + $$
$$+ G_{p-2}(x,\overline{D}_x^1 v_{p-1}, \overline{D}_x^2 v_{p-2},\ldots,\overline{D}_x^p v_0)$$

where $\overline{D}_x^j v = (\partial_x^\alpha v,\ |\alpha| \leqq j)$. If $(u_0,u_1,f) \in \mathcal{D}$, it follows from assumption 2.1 that $1+F_2^0(0,x,\overline{D}_x^2 u_0,\overline{D}_x^1 u_1,u_2) \geqq d > 0$. Therefore, we can define $u_p, p \geqq 3$, successively as follows:

$$u_p(x) = [1+F_2^0(0,x,\overline{D}_x^2 u_0,\overline{D}_x^1 u_1,u_2)]^{-1}[\Delta u_{p-2} - G_{p-2}(x,\overline{D}_x^1 u_{p-1},\ldots,\overline{D}_x^p u_0) +$$
$$+ (\partial_t^{p-2} f)(0,x)].$$

Now, we define the compatibility condition in terms of u_p.

Definition 2.3. We assume that the assumption 2.1 holds. Let L be an integer $\geqq 2$. We shall say that data $(u_0,u_1,f) \in \mathcal{D}$ satisfy the compatibility condition of order L if $u_j \in \overset{\circ}{H}{}_2^1(\Omega)$, $0 \leqq j \leqq L$. //

We state the local existence theorem.

Main Theorem I. We assume that Assumption 2.1 holds. Let Ω be a domain in R^n, its boundary $\partial\Omega$ being C^∞ and compact. If data u_0,u_1,f belong to class $\mathcal{D}$ and satisfy the compatibility condition of order infinity and u_0, $u_1 \in H_2^\infty(\Omega)$ and $f \in C^\infty([0,T];H_2^\infty(\Omega))$, then there exists one and only one solution $u \in C^\infty([0,T'];H_2^\infty(\Omega))$ of the problem (M.P) for some positive $T' \leqq T$. Here T' depends essentially only on n, Ω, F and the bound for $\|u_0\|_{2L+2}$, $\|u_1\|_{2L+1}$, $|f|_{2L,[0,T],2}$ and $L = \max(2[\frac{n}{2}]+4,\ [\frac{n}{2}]+7)$.//

§3. Global existence theorem. We consider the global existence theorem of (M.P) with $T = \infty$ under the assumptions (1) and (2) and additional assumption: Ω is non-trapping. More precisely, we introduce the following assumption.

Assumption 3.1. (1) The spatial dimension $n \geqq 3$.

(2) The nonlinear mapping F is a real-valued function belonging to $\mathcal{B}^\infty([0,\infty)\times\bar{\Omega}\times\{(u',u'') \in R^{(n+1)(n+2)};\ |u'| + |u''| \leq 1\})$.

(3) Put $\lambda = (u',u'')$.

$$F(t,x,\lambda) = \begin{cases} O(|\lambda|^2) & \text{near } \lambda = 0 \text{ if } n \geqq 6, \\ O(|\lambda|^3) & \text{near } \lambda = 0 \text{ if } 3 \leqq n \leqq 5. \end{cases}$$

(4) The exterior domain Ω is "non-trapping" in the following sense: Let $G(t,x,y)$ be the Green function for the following problem:

$$\Box G = 0 \text{ in } [0,\infty) \quad , \quad \lim_{t\to 0+} (\partial/\partial t)^j G = \begin{cases} 0 & \text{if } j=0, \\ \delta(x-y) & \text{if } j=1, \end{cases} \quad G|_{x\in\partial\Omega} = 0.$$

where y is an arbitrary point in Ω and $\Box$ is the d'Alembertian with respect to the x and t. Let a and b be arbitrary positive constants such that $b \geqq a \geqq r_0$ with $\partial\Omega \subset \{ x \in R^n;\ |x| \leqq r_0\}$. For any $v \in L^2(\Omega)$ with $\operatorname{supp} v \subset \{ x \in R^n;\ |x| \leqq a \}$ we put

$$(\mathbb{G}v)(t,x) = \int_\Omega G(t,x,y)v(y)dy.$$

Then there exists a $T_0 > 0$ such that $(\mathbb{G}v)(t,x) \in C^\infty([T_0,\infty)\times\bar{\Omega}_b)$ for any $v \in L^2(\Omega)$ with $\operatorname{supp} v \subset \{ x \in R^n;\ |x| \leqq a\}$, where T_0 depends only on n, a, b and Ω , and Ω_b denotes the set $\Omega_b = \{ x \in \Omega;\ |x| \leqq b\}$.

Remark 3.2. (1) It is well known that if the complement of Ω is convex, then Assumption 3.1 (4) is satisfied. (see e.g. Melrose [8]). (2) It is well known that under the Assumption 3.1. (4) the local energy of solutions of wave equations of the form: $\Box u = 0$

decays exponentially if $n \geqq 3$ and n is odd (see e.g. Lax-Phillips [9]). Melrose [8] proved that if $n \geqq 4$ and n is even, the order of local energy of solutions of wave equations is -n. But, as byproduct of our proof in Shibata and Tsutsumi [6] we proved that its order is -2n+2. This is a sharper result than Melrose [8]. Furthermore, comparing with the order of local enery in the whole space R^n, our result is best.//

Now, we shall state our global existence theorem.

Main Theorem II. Let m be an arbitrary integer $m \geqq 0$. Let Assumption 1.1 be all satisfied. (1) Put $m° = 2\max(4[\frac{n}{2}]+7, m+1)+4[\frac{n}{2}]+8$. If $n \geqq 6$, then there exist positive constants a and δ_0 having the following properties: If $\phi_0 \in \mathcal{B}^{2m°+[\frac{n}{2}]+3}(\bar{\Omega})$, $\phi_1 \in \mathcal{B}^{2m°+[\frac{n}{2}]+2}(\bar{\Omega})$ and $f \in \mathcal{B}^{2m°+[\frac{n}{2}]+1}([0,\infty)\times\bar{\Omega})$ satisfy for some δ with $0 < \delta \leqq \delta_0$

$$\|\phi_0\|_{4/3,2m°} + \|\phi_1\|_{4/3,2m°-1} + |f|_{4/3,(n-1)/4,2m°-2} \leqq a\delta,$$

$$\|\phi_0\|_{4,2m°+2} + \|\phi_1\|_{4,2m°+1} + |f|_{4,0,2m°} \leqq a\delta,$$

$$\|\phi_0\|_{\infty,2m°+2} + \|\phi_1\|_{\infty,2m°+1} + |f|_{\infty,0,2m°} \leqq a\delta$$

and the compatibility condition of order m°, then Problem (M.P) has a solution $u \in C^{m+2}([0,\infty)\times\bar{\Omega})$ satisfying

$$|(u',u'')|_{2,0,m} + |(u',u'')|_{4,(n-1)/4,m} \leqq \delta.$$

(2) Put $m° = 2\max(3[\frac{n}{2}]+6, m-1) + 3[\frac{n}{2}]+7$. If $4 \leqq n \leqq 5$, then there exist positive constants a and δ_0 having the following properties: If $\phi_0 \in \mathcal{B}^{2m°+2}(\bar{\Omega})$, $\phi_1 \in \mathcal{B}^{2m°+1}(\bar{\Omega})$ and $f \in \mathcal{B}^{2m°}([0,\infty)\times\bar{\Omega})$ satisfy for some δ with $0 < \delta \leqq \delta_0$

$$\|\phi_0\|_{1,2m°} + \|\phi_1\|_{1,2m°-1} + |f|_{1,(n-1)/2,2m°-2} \leqq a\delta,$$

$$\|\phi_0\|_{2,2m°+2} + \|\phi_1\|_{2,2m°+1} + |f|_{2,(n-1)/2,2m°} \leqq a\delta,$$

$$\|\phi_0\|_{\infty,2m^\circ+2} + \|\phi_1\|_{\infty,2m^\circ+1} + |f|_{\infty,0,2m^\circ} \leqq a\delta$$

and the compatibility condition of order m°, then Problem (M.P) has a solution $u \in C^{m+2}([0,\infty)\times\bar{\Omega})$ satisfying

$$|(u',u'')|_{2,0,m} + |(u',u'')|_{\infty,(n-1)/2,m} < \delta.$$

(3) Let σ be a positive constant with $0 < \sigma \leqq 1/(7m+18)$, and m° an integer with $m^\circ \geqq \frac{7}{\sigma}[\frac{3}{2} + (3m+7)\sigma] + 3[\frac{n}{2}] + 6$. If $n = 3$, then there exist positive constants a and δ_0 having the following properties: If $\phi_0 \in \mathcal{B}^{2m^\circ+2}(\bar{\Omega})$, $\phi_1 \in \mathcal{B}^{2m^\circ+1}(\bar{\Omega})$ and $f \in \mathcal{B}^{2m^\circ}([0,\infty)\times\bar{\Omega})$ satisfy for some δ with $0 < \delta \leqq \delta_0$

$$\|\phi_0\|_{1,2m^\circ} + \|\phi_1\|_{1,2m^\circ-1} + |f|_{1,1+\sigma,2m^\circ-2} \leqq a\delta,$$

$$\|\phi_0\|_{2,2m^\circ+2} + \|\phi_1\|_{2,2m^\circ+1} + |f|_{2,1+\sigma,2m^\circ} \leqq a\delta,$$

$$\|\phi_0\|_{\infty,2m^\circ+2} + \|\phi_1\|_{\infty,2m^\circ+1} + |f|_{\infty,0,2m^\circ} \leqq a\delta$$

and the compatibility condition of order m°, the Problem (M.P) has a solution $u \in C^{m+2}([0,\infty)\times\bar{\Omega})$ satisfying

$$|(u',u'')|_{2,0,m} + |(u',u'')|_{\infty,\frac{1}{2}+\sigma,m} \leqq \delta.$$

Furthermore, there exists a small constant $\delta_1 > 0$ such that if $u, v \in C^3([0,\infty)\times\bar{\Omega})$ are two solutions of Problem (M.P) for the same data with $|(u',u'')|_{\infty,0,0} \leqq \delta_1$ and $|(v',v'')|_{\infty,0,0} \leqq 1$, then $u = v$.//

Combining Main Theorems I and II, we have

<u>Main Theorem III</u>. The same assumption and conditions as in Main Theorem II are satisfied by data ϕ_0, ϕ_1 and f, nonlinea term F and the domain Ω. Furthermore, assume that ϕ_0, $\phi_1 \in H_2^\infty(\bar{\Omega})$ and $f \in C^\infty([0,\infty);H_2^\infty(\bar{\Omega}))$ and that ϕ_0, ϕ_1 and f satisfy the compatibility condition of infinite order. Then there exists one and only one solution of Problem (M.P) belonging to $C^\infty([0,\infty);H_2^\infty(\bar{\Omega}))$.

<u>Concluding Remarks.</u> (1) The local existence theorem can be extended to more general fully nonlinear 2nd order hyperbolic operators (see Shibata and Tsutsumi [7]).

(2) We can also obtain the analogous results for the mixed problems of the nonlinear Klein-Gordon equation and the nonlinear Schrödinger equation.//

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