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FIBONACCI SEQUENCE OF STABLE PERIODIC ORBITS FOR ONE-PARAMETER FAMILIES OF C^1-UNIMODAL MAPPINGS

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ABSTRACT

For one-parameter families of C^1-unimodal mappings, pay the appearaces of stable periodic orbits, whose periods increase as Fibonacci sequences, is studied. The convergence rate of parameter values where they appear is calculated numerically. As a (numerical) result, a kind of universality, different from that of the period doubling phenomenon, is suggested.

1. Introduction

One parameter families of unimodal mappings have some remarkable properties, one of which is presented in the accumlation of period-doublings. When we vary a parameter, superstable periodic orbits of periods $1,2,4,\ldots,2^n,\ldots$ appear at parameter values

$$\mu_0<\mu_1<\mu_2<\ldots<\mu_n<\ldots.$$

By numerical calculations of some one-parameter families, Feigenbaum[5,6] discovered the following property:

$$\lim\ (\mu_n-\mu_\infty)/(\mu_{n+1}-\mu_\infty)= 4.669\ldots$$

where this number is called the Feigenbaum constant, which is independent of the families. For this independence, such phenomenon is called universal. Feigenbaum proposed an explanation for the universality, and mathematical support for his explanation is given by

Collet, Eckmann, Lanford III[3], Campanino, Epstein[1], and Eckmann, Epstein, Wittwer[4].

Tsuda[7] studied the bifurcation structure in the Belouzov-Zhabotinsky reaction system, which is one of the chemical systems. He constructed a one-parameter family of unimodal mappings (Figure 2) by looking at the Lorenz plot of experimental data (Figure 1). Studying numerically this one-parameter family, he noticed the following fact. Vary the parameter starting with the value where the 9-periodic orbit is stable. Taking notice of periodic orbits stable in relatively wide ranges of parameters, their periods are 18,36,54,90,144 in this order, or characterized by the Fibonacci numbers ($p_n=p_{n-1}+p_{n-2}$). He proposed that this phenomenon, which he called as Fibonacci bifurcations, has another kind of universality, which is different from that of the period doubling phenomenon.

Figure 1

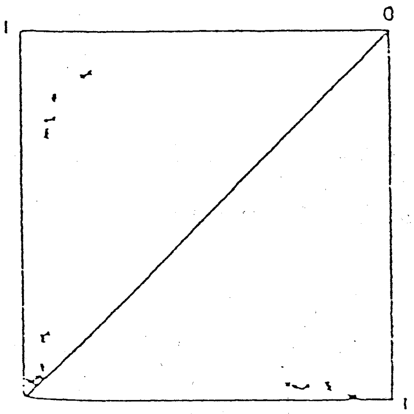
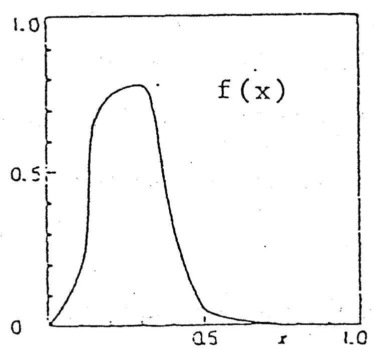


Figure 2



$F_b(x)=f(x)+b$

In this note, we show the existence of the recursive appearance of stable periodic orbits whose periods increase as Fibonacci numbers, and calculate the convergence rate numerically for some families. In section 2, we express the stable periodic orbits by symbolic sequences. In section 3, we show the existence of sequences of symbolic sequences which correspond to the superstable periodic orbits whose periods increase as Fibonacci numbers, and give their formation rule. In section 4, for above sequences some numerical results are given which suggest the universality.

2. Preliminaries

We give some definitions on symbolic dynamical systems.

Definition 2.1 (C^1-unimodal mappings)

A C^1-mapping $f:[-1,1]\to[-1,1]$ is called C^1-unimodal if

(1) $f(0)=1$,

(2) $f'(x)>0$ for $x<0$,

(3) $f'(x)<0$ for $x>0$.

Definition 2.2 (symbolic sequences)

A sequence of symbols L,C,R is called admissible if it is either

(1) an infinite sequence of symbols L's and R's,

or

(2) a finite sequence of symbols L's and R's followed by C.

For $x\varepsilon[-1,1]$, the admissible sequence $\underline{I}_f(x)=I_0I_1\ldots I_n\ldots$ is defined by

$$I_n = \begin{cases} L & \text{if } f^n(x)\varepsilon[-1,0) \\ C & \text{if } f^n(x)=0 \\ R & \text{if } f^n(x)\varepsilon(0,1]. \end{cases} \qquad (n=0,1,2,\ldots)$$

Definition 2.3 (ordering)

Let $\underline{A}=A_0A_1\ldots$ and $\underline{B}=B_0B_1\ldots$ be admissible sequences. We say $\underline{A}<\underline{B}$ if

(1) $L < C < R$,

(2) if $A_0\ldots A_{k-1}=B_0\ldots B_{k-1}$, $A_k\neq B_k$, either

(a) $A_0\ldots A_{k-1}$ is even and $A_k<B_k$,

or

(b) $A_0\ldots A_{k-1}$ is odd and $B_k<A_k$,

where a finite symbolic sequence is called even or odd according to the parity of the number of R's included.

Definition 2.4 (shift operator)

For admissible sequences, the shift operator is defined by

$$\sigma(\underline{A})=\sigma(A_0A_1A_2\ldots)=A_1A_2\ldots.$$

Note that σC is not defined.

Clearly $\sigma\underline{I}_f(x)=\underline{I}_f(f(x))$ holds.

Definition 2.5 (maximality)

An admissible sequence $\underline{A}$ is called maximal if

$$\sigma^k\underline{A} \leqq \underline{A}$$

for all k, $0\leqq k<|\underline{A}|-1$. Where $|\underline{A}|$ denotes the length of the sequence $\underline{A}$ $(0\leqq|\underline{A}|\leqq+\infty)$.

Definition 2.6 (one-parameter family)

Let C be the space of C^1-unimodal mappings with C^1-topology. A continuous curve in C

$$\mu \mapsto f_\mu$$

is called one-parameter family of C^1-unimodal mappings.

By the definition of the ordering of admissible sequences, if $x \leqq x'$ then $\underline{I}_f(x) \leqq \underline{I}_f(x')$. Because $f^k(1) \leqq 1$ for all k, we have

$$\sigma^k \underline{I}_f(1) = \underline{I}_f(f^k(1)) \leqq \underline{I}_f(1).$$

So $\underline{I}_f(1)$ is maximal. The converse is also true in the sense of the following theorem.

Theorem 2.7

Let f_μ be a one-parameter family of C^1-unimodal mappings. If a maximal sequence $\underline{A}$ satisfies

$$\underline{I}_{f\mu_0}(1) < \underline{A} < \underline{I}_{f\mu_1}(1), \tag{1}$$

then there is a parameter value $\mu \varepsilon (\mu_0, \mu_1)$ such that

$$\underline{I}_{f\mu}(1) = \underline{A}.$$

For the proof, see [2]. In the next section, we shall construct finite symbolic sequences and it will be necessary to show the existence of parameter value μ's such that f_μ's have stable periodic orbits whose symbolic expressions coincide with such sequences. Above theorem says that this is done only by checking the maximality and the condition (1) of the constructed sequences. For the sake of simplicity, we introduce the following family.

Definition 2.8 (full family)

A one-parameter family of C^1-unimodal mappings f_μ is called full if

(1) $f_{\mu 0}(1)=0$,

(2) $f_{\mu 1}(-1)=f_{\mu 1}(1)=-1$.

For full families, an arbitrary maximal sequence starting with RL... satisfies the condition (1). Throughout this note, we shall deal with only full families.

3 Main Results

In this section, we give the rule to construct recursively symbolic sequences which correspond to superstable periodic orbits of period p_n, satisfying $p_n=p_{n-1}+p_{n-2}$.

Theorem 3.1

Let $\underline{A}C$ and $\underline{B}C$ be maximal sequences which satisfy

(1) $\underline{A}C<\underline{B}C$,

(2) $|\underline{A}|<|\underline{B}|$,

(3) there is no maximal sequence $\underline{D}C$ such that $\underline{A}C<\underline{D}C<\underline{B}C$ and $|\underline{D}|<|\underline{B}|$.

Then

(a) $\underline{B}\beta\underline{A}C$ is a maximal sequence where

$$\beta=\begin{cases} R & \text{if } \underline{B} \text{ is even,} \\ L & \text{if } \underline{B} \text{ is odd.} \end{cases}$$

(b) There is no maximal sequence $\underline{D}C$ such that $\underline{B}C<\underline{D}C<\underline{B}\beta\underline{A}C$ and $|\underline{D}|\leq 2|\underline{B}|$.

The proof is not difficult but tedious, so it is omitted.

On the one-parameter family, choose two superstable periodic orbits so that corresponding symbolic sequences $\underline{A}C$, $\underline{B}C$ satisfy the hypothesis of Theorem 3.1. By Theorem 2.7 and this theorem, there exists a parameter value μ such that f_μ has a superstable periodic orbit of the form $\underline{B}\beta\underline{A}C$. As $|\underline{B}\beta\underline{A}C|=|\underline{B}C|+|\underline{A}C|$, the period of this stable periodic orbit is the sum of the periods of periodic orbits of the forms $\underline{A}C$ and $\underline{B}C$.

Next, execute the same procedure taking $\underline{B}C$ and $\underline{B}\beta\underline{A}C$ in place of $\underline{A}C$ and $\underline{B}C$ respectively. Theorem 3.1(b) says that the hypothesis(3) of Theorem 3.1 for new $\underline{A}C$ and $\underline{B}C$ is fulfilled. So we can adapt this theorem repeatedly, and we get a sequence of maximal sequences, and then a sequence of parameter values

$$\mu_1<\mu_2<\mu_3<\cdots<\mu_n<\cdots$$

such that $f\mu$ have superstable periodic orbits whose period increase as Fibonacci numbers p_n.

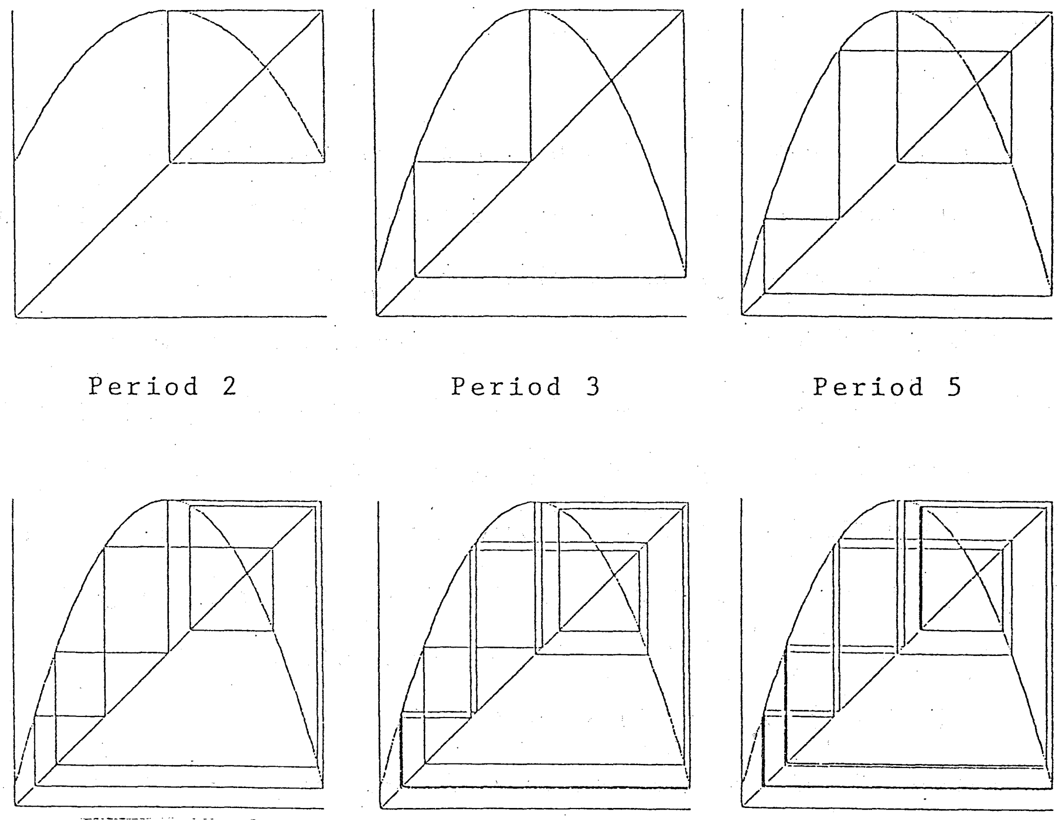
Example 3.2

Take the 2-periodic maximal sequence RC as $\underline{A}C$ and the 3-periodic maximal sequence RLC as $\underline{B}C$. It is easy to check that these sequences satisfy the hypothesis of Theorem 3.1. A sequence of maximal sequences is given as follows.

period 2	RC
period 3	RLC
period 5	RLLRC
period 8	RLLRRRLC
period 13	RLLRRRLRRLLRC
......	

For the one-parameter family $f_{\mu}(x)=1-\mu x^2$, parameter values corresponding to above maximal sequences calculated by a digital computer are given in Table 1 of section 4. Graphs of $f_{\mu n}(x)$ and periodic orbits of period p_n's are found in Figure 3.

Figure 3



Period 2 Period 3 Period 5

Period 8 Period 13 Period 21

Theorem 3.1(b) gives a characterization of the stable periodic orbits. Varying the parameter μ from μ_n to μ_{n+1}, no superstable periodic orbits whose periods are less than p_{n+1} appear. Of course there are stable periodic orbits with longer periods. But the parameter ranges of the existence of these orbits seems narrower. A conspicuous periodic orbit next to μ_n appears near μ_{n+1}. This agrees with Tsuda's observation of mentioned in the introduction.

4. Numerical Calculations

We give some numerical results. Tables 1-4 are the results for the family

$$f_\mu(x)=1-\mu x^2.$$

In Table 1, RC and RLC are taken as the first two stable periodic orbits, and parameter value μ_n's are computed. In Table 2, with the same first two stable periodic orbits, $x_n=f_{\mu n}^{p_n}(0)$ is computed. x_n is the nearest point to zero among the stable periodic points except zero, which is considered to be related to the scaling of the mapping when some renormalization group acts. In Tables 3 and 4, RLC and RLLC are taken as the first two stable periodic orbits and μ_n's and x_n's are computed. These calculations suggest the following relations:

$$(\mu_n-\mu_\infty)(\mu_{n+2}-\mu_\infty)/(\mu_{n+1}-\mu_\infty)^2 \to \text{const.}=0.62\ldots$$

$$(x_n/x_{n+1})/(x_{n+2}/x_{n+3}) \to \text{const.}=0.62\ldots.$$

In Tables 5 and 6, analogous computations are performed for the family

$$f_\mu(x)=1-\mu(1-\cos x),$$

and in Tables 7 and 8, for the family

$$f_\mu(x)=1-\mu\{1-(1+x)\exp(-x)\}.$$

The results for these families are the same as the quadratic family, i.e. the above constant seems universal.

TABLE 1

p_n	μ_n	$\frac{(\mu_n-\mu_{n+1})(\mu_{n+2}-\mu_{n+3})}{(\mu_{n+1}-\mu_{n+2})^2}$
2	1.0000000000000000000000000	
3	1.7548776662466927600495089	
5	1.8607825222048548712322420	
8	1.8700038808287653615732964	0.6206399698
13	1.8705121633603489261766659	0.6330401179
21	1.8705283128137401674361262	0.5764244197
34	1.8705286283554136039953836	0.6149591304
55	1.8705286321362691562023694	0.6132456326
89	1.8705286321645129022778814	0.6234460129
144	1.8705286321646445036234603	0.6237431922

TABLE 2

p_n	x_n	$\frac{x_n \cdot x_{n+3}}{x_{n+1} \cdot x_{n+2}}$
2	1.0000000000000000000000000	
3	− 0.7548776662466927600495089	
5	− 0.3787403911204367026322966	
8	0.1421921094565950263857493	0.4973444745
13	0.0411988128313606376425583	0.5774895873
21	− 0.0094368646525993418083511	0.6101114152
34	− 0.0016530891925859864285224	0.6045877672
55	0.0002285705958130745381894	0.6036442238
89	0.0000248494566961974303003	0.6206231584
144	− 0.0000021385558195833061674	0.6224143815

TABLE 3

p_n	μ_n	$\frac{(\mu_n-\mu_{n+1})(\mu_{n+2}-\mu_{n+3})}{(\mu_{n+1}-\mu_{n+2})^2}$
3	1.7548776662466927600495089	
4	1.9407998065294847522320910	
7	1.9537058942843962454276222	
11	1.9541658574001347490618013	0.5134106696
18	1.9541773904863448139279304	0.7035486561
29	1.9541775537957952538170802	0.5647331058
47	1.9541775552881271856422488	0.6453396297
76	1.9541775552964610050286351	0.6111172430
123	1.9541775552964904385061277	0.6324392860
199	1.9541775552964905033977621	0.6242371988

TABLE 4

p_n	x_n	$\frac{x_n \cdot x_{n+3}}{x_{n+1} \cdot x_{n+2}}$
3	- 0.7548776662466927600495089	
4	- 0.7178102075530693384661318	
7	- 0.1795175081528058258324280	
11	0.0475914528792719269796573	- 0.2787976435
18	0.0088372065873681808466447	0.7424870152
29	- 0.0013811542387749336427763	0.5895287928
47	- 0.0001629881104776992804876	0.6355177721
76	0.0000155005169535893664708	0.6085035055
123	0.0000011545830545478994339	0.6311974742
199	- 0.0000000684762614191080928	0.6236265618

TABLE 5

p_n	μ_n	$\frac{(\mu_n-\mu_{n+1})(\mu_{n+2}-\mu_{n+3})}{(\mu_{n+1}-\mu_{n+2})^2}$
2	2.1753426496700214107767868	
3	3.7920997531838516783867029	
5	4.0282127958185425487468834	
8	4.0491784070170441065405361	0.6080122789
13	4.0503597921513332162474767	0.6345946758
21	4.0503981756677342795597632	0.5765928905
34	4.0503989434168407439074041	0.6156321666
55	4.0503989528230562032625769	0.6125210835
89	4.0503989528949036531836562	0.6234488569
144	4.0503989528952458402327116	0.6235277326

TABLE 6

p_n	x_n	$\frac{x_n \cdot x_{n+3}}{x_{n+1} \cdot x_{n+2}}$
2	1.0000000000000000000000000	
3	- 0.7432195124566131196910582	
5	- 0.3749910811506306926800573	
8	0.1452586604110911812343931	0.5211995033
13	0.0426666124379217568037609	0.5821599073
21	- 0.0098847023322658459258556	0.5980731924
34	- 0.0017505738355439900403163	0.6029352975
55	0.0002448452150744761924395	0.6037202671
89	0.0000269124418880461561215	0.6206469797
144	- 0.0000023417892621450129409	0.6221333994

TABLE 7

p_n	μ_n	$\frac{(\mu_n-\mu_{n+1})(\mu_{n+2}-\mu_{n+3})}{(\mu_{n+1}-\mu_{n+2})^2}$
2	3.7844223823546656287531058	
3	5.6787272022080630926769816	
5	5.8539324182048029150218630	
8	5.8669314456327530653031447	0.8021704540
13	5.8675498106529981524703770	0.6411656178
21	5.8675669675372330778239102	0.5832563185
34	5.8675672588242034511022906	0.6119122269
55	5.8675672618774833577389425	0.6173947377
89	5.8675672618974313561176014	0.6232856719
144	5.8675672618975128116010784	0.6250119154

TABLE 8

p_n	x_n	$\frac{x_n \cdot x_{n+3}}{x_{n+1} \cdot x_{n+2}}$
2	0.9999999999999999999999999	
3	− 0.5005532227813223097150958	
5	− 0.2705740860102237384028748	
8	0.0977101905398183818625134	0.7214452829
13	0.0256199772194124794149590	0.4850683527
21	− 0.0054056051095749607096027	0.5842677527
34	− 0.0008878202159429125581342	0.6263856998
55	0.0001146196687206414011172	0.6118832178
89	0.0000116644833144632950411	0.6196203089
144	− 0.0000009392846067266152384	0.6237313595

5. Problems

(1) Though numerical results suggest the existence of the universality, we have no proof at present.

(2) As an extension of Theorem 3.1, we have the following. Before we state it, we define the *-product.

Definition 5.1 (*-product)

Let $\bar{R}=L$, $\bar{L}=R$. We define the *-product of $\underline{A}$ and $\underline{E}$ by

$$\underline{A}*\underline{E} = \underline{A}\tilde{E}_1\underline{A}\tilde{E}_2\cdots\underline{A}\tilde{E}_{|\underline{E}|}$$

where

$$\tilde{E}_k = \begin{cases} E_k & \text{if } \underline{A} \text{ is even,} \\ \bar{E}_k & \text{if } \underline{A} \text{ is odd.} \end{cases}$$

Theorem 5.2

Let $\underline{A}C$ and $\underline{B}C$ be maximal sequences which satisfy

(1) $\underline{A}C<\underline{B}C$,

(2) $|\underline{A}|<|\underline{B}|$,

(3) if a maximal sequence $\underline{D}C$ satisfies $\underline{A}C<\underline{D}C<\underline{B}C$ and $|\underline{D}|\leq|\underline{B}|$, then $\underline{D}C=\underline{A}*(\underline{E}C)$ for some $\underline{E}C$.

(4) $\underline{B}C\neq\underline{A}*(\underline{E}C)$ for any $\underline{E}C$.

Then

(a) $(\underline{B}*(RL^{q-1}))\underline{A}C$ is maximal,

(b) if a maximal sequence $\underline{D}C$ satisfies $\underline{B}C<\underline{D}C<(\underline{B}*(RL^{q-1}))\underline{A}C$ and $|\underline{D}|\leq(|\underline{B}|+1)q+|\underline{A}C|$, then $\underline{D}C=\underline{B}*(\underline{E}C)$ for some $\underline{E}C$.

This theorem shows the existence of sequences of stable periodic orbits whose periods are defined recursively by $p_n=qp_{n-1}+p_{n-2}$. We give numerical results for q=2. The family studied here is

$$f_\mu(x)=1-\mu x^2,$$

and as the first two stable periodic orbits we take RC, RLC (Tables 9,10), and RLC, RLLC (Tables 11,12).

TABLE 9

p_n	μ_n	$\frac{(\mu_n-\mu_{n+1})(\mu_{n+2}-\mu_{n+3})}{(\mu_{n+1}-\mu_{n+2})^2}$
2	1.0000000000000000000000000	
3	1.7548776662466927600495089	
8	1.8100013857280120727260324	
19	1.8103300282549967479199316	0.0816438187
46	1.8103303412225741687989131	0.1597314152
111	1.8103303412693563390411911	0.1569659417
268	1.8103303412693575501617510	0.1731912692

TABLE 10

p_n	x_n	$\frac{x_n \cdot x_{n+3}}{x_{n+1} \cdot x_{n+2}}$
2	1.0000000000000000000000000	
3	− 0.7548776662466927600495089	
8	− 0.1875459724144121537071649	
19	0.0188156948892531444673692	0.1329033547
46	0.0007324718848547660057375	0.1566893691
111	− 0.0000116518635123701203477	0.1585593692
268	− 0.0000000774735988222213193	0.1707999127

TABLE 11

p_n	μ_n	$\frac{(\mu_n-\mu_{n+1})(\mu_{n+2}-\mu_{n+3})}{(\mu_{n+1}-\mu_{n+2})^2}$
3	1.7548776662466927600495089	
4	1.9407998065294847522320910	
11	1.9449852110535572801956194	
26	1.9449892785382417485245866	0.0431700108
63	1.9449892793560979774548605	0.2069009822
152	1.9449892793561242743828449	0.1599105113
367	1.9449892793561242745366315	0.1818802626

TABLE 12

p_n	x_n	$\frac{x_n \cdot x_{n+3}}{x_{n+1} \cdot x_{n+2}}$
3	− 0.7548776662466927600495089	
4	− 0.7178102075530693384661318	
11	− 0.0560716962082892925722888	
26	0.0026329869497044668187552	− 0.0493823734
63	0.0000462823732305058101222	0.2250260773
152	− 0.0000003468174783031502269	0.1595806918
367	− 0.0000000010979744692936536	0.1801044120

These computations suggest

$$(\mu_n-\mu_\infty)(\mu_{n+2}-\mu_\infty)/(\mu_{n+1}-\mu_\infty) \to \text{const.}=0.17...$$

$$(x_n/x_{n+1})/(x_{n+2}/x_{n+3}) \to \text{const.}=0.17....$$

In the case of q=1, this constant seems near $(\sqrt{5}-1)/2$ (golden mean), but in this case 0.17... is not near $\sqrt{2}-1$. What is the meaning of these constants ?

(3) In the period-doubling phenomenon, the limit of μ's plays an important role as the boundary across which the chaos appears. In our case, we do not have such characterization of the limit system.

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