数理解析研究所講究録
第 537 巻 1984 年 123-131

On the Discrepancy and Uniform Distribution of Sequences

Takeshi Kano (Okayama Univ.)

This article is a survey of recent results obtained by us and others. For full proofs of them and related results the reader should consult the original papers indicated in the References.

1. We define the counting function of the interval $J = [a, b)$ in $[0, 1)$, $A_N(x, J) = \#\{n, 1 \leq n \leq N : \langle x_n \rangle \subset J\}$.*)
Then we call

$$D_N(x) = \sup_J \left| \frac{A_N(x, J)}{N} - |J| \right|$$

as the "Discrepancy" of the sequence (x_n). The sequence (x_n) is said to be uniformly distributed mod 1 if $D_N(x) \to 0$ as $N \to \infty$. There is a criterion due to Weyl, i.e.

(x_n) is uniformly distributed mod 1 iff

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i h x_n) = 0 \tag{1}$$

for all fixed natural numbers h.

*) For real numbers t, $\langle t \rangle$ denotes the fractional part of t.

The following theorem of Erdős and Turán (cf. [12] Chap. I) is often useful to obtain estimates for $D_N(x)$.

Theorem 1. For some numerical constants c_1 and c_2, and for any natural number m, we have

$$(2) \qquad ND_N(x) \leq c_1 \frac{N}{m} + c_2 \sum_{h \leq m} \frac{1}{h} \left| \sum_{n \leq N} e^{2\pi i h x_n} \right| .$$

Obviously, (2) shows in particular that (1) is a sufficient condition for (x_n) being u.d. mod 1.

Weyl showed that if (x_n) is a monotone sequence of integers, then the sequence (θx_n) is u.d. mod 1 for almost all real numbers θ. Erdős and Koksma [5] and Cassels [3] proved that

$$(3) \qquad ND_N(x) \ll \sqrt{N} (\log N)^{5/2 + \epsilon}$$

holds for almost all θ, if (x_n) is a monotone sequence of integers. Erdős [4] stated the conjecture that for some positive constant c and for almost all θ, we have

$$(4) \qquad ND_N(x) \ll \sqrt{N} (\log\log N)^c ,$$

which is true if (x_n) is a lacunary sequence of integers.

We remark that (4) does not necessarily hold for the uniformly distributed sequences of non-integral numbers. For example, let us take

$$(5) \qquad x_n = \theta (\log n)^{\alpha} , \qquad (\alpha > 1).$$

This sequence is u.d. mod 1 for all $\theta \neq 0$, and actually

$$(6) \qquad \left| \sum_{n \leq N} \exp(2\pi i h \theta (\log n)^{\alpha}) \right| \asymp N (\log N)^{1-\alpha}.$$

On the other hand it is known (cf.[12] Chap.I)that we have for any real sequence (y_n),

$$\left|\sum_{n\le N} e^{2\pi i y_n}\right| \ll N D_N(y). \tag{7}$$

Thus it follows from (6) and (7) that

$$N D_N(x) \gg N(\log N)^{1-\alpha},$$

which contradicts to (4).

2. Recently R.C.Baker [1] succeeded in improving (3) to

$$N D_N(x) \ll \sqrt{N}(\log N)^{3/2+\epsilon} \tag{8}$$

for almost all θ, provided (x_n) is a strictly increasing sequence of natural numbers. He applied a deep L^2-theorem of L. Carleson on Fourier series. Independently of Baker, I have found (cf.[8][9]) that Carleson's result can be adapted to improve (3) so as to obtain the following theorem [10].

Theorem 2. Let (x_n) be a sequence of real numbers such that

$$\inf_n (x_{n+1} - x_n) > 0. \tag{9}$$

Then for a.a. θ, the discrepancy $D_N(x)$ of (θx_n) satisfies

$$N D_N(x) \ll \sqrt{N}(\log N)^{3/2}(\log\log N)^{1/2+\epsilon}. \tag{10}$$

My argument which leads to (10) is different from that of Baker, and seems more direct and simpler. In effect, I applied the following theorem which is proved by Carleson's theorem [2].

Theorem 3. If the sequence (x_n) of real numbers satisfies (9) and the sequence (a_n) of real numbers is such that

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty ,$$

then

$$\sum_{n=1}^{\infty} a_n e^{i\theta x_n}$$

converges for a.a. θ .

It will be worth noting that if (9) is replaced by

$$\inf_{n \neq m} |x_n - x_m| > 0, \tag{11}$$

then (3) still holds for a.a. θ , however, at present I cannot prove a much better estimate like (10) in this case[8].

3. It is known (cf.[12] Chap.I) that if (x_n) is u.d. mod 1, then necessarily

$$\limsup_{n \to \infty} n|\Delta x_n| = \infty , \tag{12}$$

where $\Delta x_n = x_{n+1} - x_n$. This in fact implies that any concave or convex real sequence (x_n) such that $x_n = O(\log n)$ is not u.d. mod 1. Recently Niederreiter proved among other things the following strong result[13].

Theorem 4. If (x_n) is a monotone sequence of real numbers such that it is u.d. mod 1, then it holds that

$$\lim_{n \to \infty} \frac{|x_n|}{\log n} = \infty \tag{13}$$

We remark that this theorem is in fact a corollary of a general result proved w.r.t. probability measures and weighted means. Moreover, this is sharp in the sense that in (13) we cannot replace log n by a function with much faster speed of tending to infinity.

As an application of this theorem, we see that both of $(\log p_n)$ and $(\log \gamma_n)$ are not u.d. mod 1, where p_n denotes n-th prime and γ_n is the imaginary part of the zero of Riemann zeta-function.

In case (x_n) is not necessarily monotone, the following result will be sometimes useful[8].

Theorem 5. If a real sequence (x_n) satisfies the condition

$$(14) \qquad \sum_{n \leq N} n\,|\Delta x_n| \ll N ,$$

then (x_n) has no continuous distribution function.

This theorem also shows that $(\log p_n)$ and $(\log \gamma_n)$ are not u.d. mod 1. We can generalize this theorem to weighted means[6].

Theorem 6. If a real sequence (x_n) satifies

$$(15) \qquad \sum_{n \leq N} \Lambda_n\,|\Delta x_n| \ll \Lambda_N ,$$

then (x_n) has no continuous (M,λ_n) - distribution function mod 1, where (λ_n) is a positive decreasing sequence and

$$\Lambda_N = \lambda_1 + \lambda_2 + \ldots + \lambda_N .$$

We say that (x_n) has (M, λ_n) - distribution function mod 1 $g(x)$ if

$$\lim_{N\to\infty} \frac{1}{\Lambda_N} \sum_{n=1}^{N} \lambda_n f(x_n) = \int_0^1 f(x)\, dg(x)$$

holds for all continuous functions $f(x)$ defined on $[0, 1]$ with period 1.

We applied for the proof the following theorem due to Karamata [11].

Theorem 7. If the series

$$\sum^{\infty} u_n \tag{16}$$

is (M, λ_n) - summable to s and satisfies

$$\sum_{n \le N} \Lambda_n |u_{n+1}| \ll \Lambda_N ,$$

then (16) is (M, λ_n) - strongly summable to s.

We say that (16) is (M, λ_n) - summable to s if

$$\sigma_N = \frac{1}{\Lambda_N} \sum_{n=1}^{N} \lambda_n s_n \to s \quad (N \to \infty),$$

where

$$s_n = \sum_{k=1}^{n} u_k .$$

Also (16) is said to be (M, λ_n) - strongly summable to s if

$$\lim_{N\to\infty} \frac{1}{\Lambda_N} \sum_{n \le N} \lambda_n |s_n - s| = 0.$$

4. It is known that $((\log p_n)^{\alpha})$ is u.d. mod 1 if $\alpha > 1$. We shall state here two theorems both containing this result as a particular case[7].

Theorem 8. Let $f(t)$ be in $C^2[1, \infty)$ such that

(i) $f(t)$ is increasing for $t \geq t_0$,

(ii) $t^2 |f''(t)| \to \infty \quad (t \to \infty)$,

(iii) $f(n)/(\log n)^c \to 0$, for some constant $c > 1$.

Then the sequence $(f(p_n))$ is u.d. mod 1.

Theorem 9. Let $f(t)$ be in $C^1[1, \infty)$ such that

(i) $f(t)$ tends to infinity monotonically,

(ii) $n f'(n) \to \infty$,

(iii) $f'(t) \cdot \log t$ is decreasing for $t \geq t_0$,

(iv) $f(n)/(\log n)^c \to 0$, for some constant $c > 1$.

Then the sequence $(f(p_n))$ is u.d. mod 1.

References

[1] R.C. Baker: Metric number theory and the large sieve, J. London Math. Soc.(2),24 (1981), 34-40.

[2] L. Carleson : On convergence and grouth of partial sums of Fourier series, Acta Math. 116 (1966),135-157.

[3] J.W.S. Cassels : Some metrical theorems in diophantine approximation. III., Proc. Camb.Phil.Soc. 46(1950), 219-225.

[4] P. Erdős : Problems and results on diophantine approximations, Compositio Math. 16(1964), 52-65.

[5] P.Erdős and J.F.Koksma : On the uniform distribution modulo 1 of sequences (f(n,a)), Indag. Math.11 (1949), 299-302.

[6] K. Goto and T. Kano : Some necessary conditions for (M, λ_n) uniform distribution mod 1, to appear.

[7] K. Goto and T. Kano : Uniform distribution of special sequences, to appear in Proc. Japan Acad..

[8] T. Kano : Sur la théorie de l'équirépartion mod 1, Monographies Sci. Maison Franco-Japonaise, Fasc.3(1981).

[9] T. Kano : Some applications of Fourier analysis to uniform distribution mod 1, Lecture notes of Research Inst. Math. Sci. Kyoto Univ., no.496 (1983),18-23.

[10] T. Kano : On convergence and size of exponential sums, to appear in the Proceedings of the Symposium in number theory (Tokyo-Okayama).

[11] J. Karamata : Sur la sommabilité forte et la sommabilité absolue. Mathematica (Cluj), 15 (1939), 119-124.

[12] L. Kuipers and H. Niederreiter : Uniform Distribution of Sequences, Wiley, 1974.

[13] H. Niederreiter : Distribution mod 1 of monotone sequences, to appear in Indag. Math..