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# A proof-theoretic approach to Paris-Harrington's results

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## § 1. Introduction

In this paper, we make a proof-theoretic study of Paris-Harrington's independence results for Peano arithmetic [6]. First, we give a characterization of provably recursive functions in natural fragments of Peano arithmetic. Then, we give an alternative proof of Paris' result in [5] on the provability of statements related to Paris-Harrington's principle, in fragments of Peano arithmetic. While Paris used a model-theoretic method, our method is of a purely proof-theoretic character. We owe our proof much to the close examination of rapidly growing functions, due to Ketonen and Solovay [1]. We also mention explicitly how the provability or the unprovability of these statements depends on their representation in formal systems.

In §2, we give some basic facts on ordinal recursive functions and Wainer's hierarchy [9]. In §3, we state our theorem on the characterization of provably recursive functions. In §4, we give a proof of Paris' result in [5].

## § 2. Preliminaries

Define the ordinal $\omega_n(m)$ for each $m,n < \omega$ by

$$\omega_0(m) = m, \qquad \omega_{n+1}(m) = \omega^{\omega_n(m)} .$$

We abbreviate $\omega_n(1)$ to $\omega_n$ . As usual, $\varepsilon_0$ denotes the first ordinal $\alpha$ such that $\alpha = \omega^{\alpha}$.

For $0 < k < \omega$, $<_k$ denotes the elementary recursive well-ordering of natural numbers of order-type $\omega_k$, which is defined in §3 of Wainer [9]. Let $\alpha < \varepsilon_0$ be any ordinal and n be the smallest natural number such that $\alpha < \omega_n$. Following [9], we define $U(\alpha)$ to be the smallest class of functions containing all primitive recursive functions, which is closed under substitution and the following (unnested) $\alpha$-recursion;

$$f(0,\mathbf{z}) = g_1(\mathbf{z}),$$
$$f(x+1,\mathbf{z}) = g_2(x+1,\mathbf{z},f(h(x+1,\mathbf{z}),\mathbf{z})),$$

where $h(x,\mathbf{z}) <_n x$ for each $0 <_n x <_n \mathrm{num}_n(\alpha)$ (= the number represented by $\alpha$ in the well-ordering $<_n$ ), and $h(x,\mathbf{z}) = 0$ otherwise. A function f is said to be $\alpha$-*ordinal recursive* if f belongs to $U(\alpha)$.

Suppose that $\alpha < \varepsilon_0$ and $\alpha$ is of the form $\omega^\beta\cdot(\gamma+1)$. Then, $\{\alpha\}(n) = \omega^\beta\cdot\gamma + \omega^\delta\cdot n$ if $\beta = \delta + 1$, and $= \omega^\beta\cdot\gamma + \omega^{\{\beta\}(n)}$ if $\beta$ is a limit ordinal. When $\alpha = \varepsilon_0$, $\{\varepsilon_0\}(n)$ is $\omega_n$ for each n.

Now, the functions $F_\alpha$ ($\alpha \leqq \varepsilon_0$) are defined inductively as follows;

$$F_0(x) = x + 1,$$
$$F_1(x) = (x + 1)^2,$$
$$F_{\beta+1}(x) = F_\beta^{x+1}(x) \quad (= F(\cdots(F(x))\cdots) \ x+1\ F\text{'s}) \quad \text{if } \beta > 0,$$
$$F_\sigma(x) = F_{\{\sigma\}(x)}(x) \text{ if } \sigma \text{ is a limit ordinal.}$$

Let $\mathcal{F}_\alpha$ ($\alpha \leqq \varepsilon_0$) be the smallest class of functions containing $F_\alpha$, the zero function, addition and projection functions, which is closed under substitution and limited primitive recursion.

Proposition 2.1 For each $\alpha \leqq \varepsilon_0$,

1) $F_\alpha$ is strictly increasing,

2) if $\beta < \alpha$ then $F_\beta$ is dominated by $F_\alpha$ (i.e. there exists a k such that $F_\beta(x) < F_\alpha(x)$ whenever $k \leqq x$),

3) if $\beta < \alpha$ then $F_\beta$ is elementary recursive in $F_\alpha$ (in the Csillag-Kalmar sense).

Proposition 2.2 For each $\alpha \leqq \varepsilon_0$,

1) if $\beta < \alpha$ then every function in $\mathcal{F}_\beta$ is dominated by $F_\alpha$,

2) if $\beta < \alpha$ then $\mathcal{F}_\beta \subsetneqq \mathcal{F}_\alpha$,

3) $\mathcal{F}_\alpha$ is equal to the class of functions elementary recursive in $F_\alpha$.

The following result by Wainer [9] shows a relation between ordinal recursive functions and Wainer's hierarchy $\{\mathcal{F}_\alpha\}_{\alpha \leqq \varepsilon_0}$.

Proposition 2.3 For each ordinal $\alpha$ such that $0 < \alpha < \varepsilon_0$,

$$U(\omega^\alpha) = \bigcup_{\beta < \alpha \cdot \omega} \mathcal{F}_\beta .$$

In particular, if $n \geqq 1$ then

$$\bigcup_{m<\omega} U(\omega_n(m)) = \bigcup_{\beta < \omega_n} \mathcal{F}_\beta .$$

Let PA be Peano's first order arithmetic. The language of our PA contains function symbols for primitive recursive functions. Our system PA is obtained from LK by adding 1) the axioms for defining equations for each primitive recursive function and 2) a rule of inference which represents the mathematical induction.

PA* is obtained from PA by adding all sequents of the form $\longrightarrow C$, where C is any true $\Pi_1$-formula, as its new initial sequents. For each $k \geqq 0$, $PA_k$ (or $PA_k^*$) is obtained from PA (or

PA*) by restricting the induction formulas of the mathematical induction to formulas containing at most k quantifiers.

A n-ary recursive function f is said to be *provably recursive* in $PA_k^*$ (or PA*) if there exists a Gödel number e of f such that

$$\forall x_1 \cdots \forall x_n \exists y T_n(\bar{e}, x_1, \cdots, x_n, y)$$

is provable in $PA_k^*$ (or PA*), where $T_n$ is the $\Pi_0$-formula representing Kleene's T-predicate [2]. We omit the subscript n of $T_n$.

## § 3. Provably recursive functions in $PA_k^*$

We characterize the class of provably recursive functions in $PA_k^*$. A formula A is called a $\Delta_1$-*formula* in $PA_m$, if there exist a $\Sigma_1$-formula B and a $\Pi_1$-formula C, each of which is equivalent to A in $PA_m$.

Theorem 3.1 Let $n \geqq 1$. Then, the following three conditions are equivalent;

1) f is provably recursive in $PA_n^*$,
2) there exists a $\Delta_1$-formula $S(\mathbf{x}, y)$ in $PA_n$ such that $f(\mathbf{x}) = \mu y S(\mathbf{x}, y)$ and $PA_n \vdash \forall \mathbf{x} \exists ! y S(\mathbf{x}, y)$,
3) f is $\omega_n(m)$-ordinal recursive for some $m < \omega$.

Proof. We give here the outline of the proof of our Theorem. For the detail of the proof, see Ono and Kadota [3]. We can show the following Lemma, by using Corollary 12.16 of Takeuti [8] p.114.

Lemma 3.2 Let $n \geqq 1$. Suppose that $R(\mathbf{x}, y)$ is a $\Pi_0$-formula and $A_1, \cdots, A_t$ be true $\Pi_1$-formulas such that

$$PA_n \vdash A_1, \cdots, A_t \longrightarrow \forall \mathbf{x} \exists y R(\mathbf{x}, y).$$

Then, the function defined by $f(\mathbf{x}) = \mu y R(\mathbf{x}, y)$ is $\omega_n(m)$-ordinal recursive for some $m < \omega$.

Proof of 1) $\Rightarrow$ 3). Suppose that $PA_n^* \vdash \forall x \exists y T(\bar{e},x,y)$ for some Gödel number e of f. Then there exist true $\Pi_1$-formulas $A_1,\cdots,A_t$ such that

$$PA_n \vdash A_1,\cdots,A_t \longrightarrow \forall x \exists y T(\bar{e},x,y).$$

By Lemma 3.2, a function $\mu y T(e,x,y)$ is $\omega_n(m)$-ordinal recursive for some $m < \omega$. Moreover

$$f(x) = U(\mu y T(e,x,y)),$$

where U is the primitive recursive function in Kleene [2]. Hence, f is also $\omega_n(m)$-ordinal recursive.

We take the canonical, primitive recursive well-ordering $\prec$ on nutural numbers which is of order-type $\varepsilon_0$. For each $x < \omega$, define ord(x) to be the ordinal represented by x in the ordering $\prec$ and for each $\alpha < \varepsilon_0$, define num($\alpha$) to be the nutural number x such that ord(x) = $\alpha$. Let $h(u,v,x) = F_{ord(v)}^{u+1}(x)$ and let e be a Gödel number of h. Then, we can give the following lemma, by using the definition of $F_\alpha$ and Shirai's results in [7].

Lemma 3.3 Let $\alpha < \omega_n$ for $n \geqq 2$. Then,

$$PA_n^* \vdash v \preceq num(\alpha) \longrightarrow \forall x \exists y T(\bar{e},u,v,x,y).$$

Using Lemma 3.3, we can show the following.

Lemma 3.4 For $n \geqq 1$, if $\alpha < \omega_n$ then $F_\alpha$ is provably recursive in $PA_n^*$.

Proof of 3) 1). From Lemma 3.4, it follows that every function in the class $\mathcal{F}_\alpha$ is provably recursive in $PA_n^*$, if $\alpha < \omega_n$. By Proposition 2.3, $\bigcup_{m<\omega} U(\omega_n(m)) = \bigcup_{\beta<\omega_n} \mathcal{F}_\beta$, so we can derive that every $\omega_n(m)$-ordinal recursive function is provably recursive in $PA_n^*$.

Proof of 2)$\Longrightarrow$3). We can derive 2)$\Rightarrow$3) from Lemma 3.2.

Proof of 3)$\Longrightarrow$2). See §4 in [3].

Using Theorem 3.1, we can show the following.

Theorem 3.5 Suppose that $R(\mathbf{x},y)$ is a $\Pi_0$-formula such that $\forall\mathbf{x}\exists yR(\mathbf{x},y)$ is true and f is a function satisfying $f(\mathbf{x}) = \mu yR(\mathbf{x},y)$. For $n \geqq 1$, f is $\omega_n(m)$-ordinal recursive for some $m < \omega$ if and only if $PA_n^* \vdash \forall x\exists yR(\mathbf{x},y)$.

## § 4. Paris-Harrington's principle in fragments of Peano arithmetic

For a set $A \subsetneqq \omega$ and $n < \omega$, define $A^{[n]} = \{B \subsetneqq A;\ \mathrm{card}(B)=n\}$, and for $k,m < \omega$, define $[k,m] = \{x;\ k \leqq x \leqq m\}$. For $c,k,m,n < \omega$, the expression

$$[k,m] \xrightarrow[*]{} (n+1)^n_c$$

means that for every function $f:[k,m]^{[n]} \longrightarrow c$, there exists a subset $H \subsetneqq [k,m]$ such that 1) H is *homogeneous* for f (i.e. f is constant on $H^{[n]}$), 2) H is *relatively large* (i.e. $\mathrm{card}(H) \geqq \min(H)$) and 3) $\mathrm{card}(H) \geqq n+1$. We remark that $[k,m] \xrightarrow[*]{} (n+1)^n_c$ is a primitive recursive relation with respect to c,k,m,n.

We can define a recursive function $\sigma_{n,c}$ by

$$\sigma_{n,c}(k) = \mu y([k,y] \xrightarrow[*]{} (n+1)^n_c).$$

In [1], Ketonen and Solovay obtained a sharp estimation of functions $\sigma_{n,c}$ and using it, they gave an alternative proof of Paris-Harrington's theorem which says that

(1) $\qquad \forall w\forall x\forall z\exists y([x,y] \xrightarrow[*]{} (w+1)^w_z)$

is not provable in Peano arithmetic. On the other hand, it is pointed out that

(2) $\forall x \forall z \exists y([x,y] \xrightarrow[*]{} (n+1)^n_z)$

is provable in Peano arithmetic for each $n < \omega$ (cf. [4] and [5]). We investigate the provability of the formula (2) in fragments of Peano arithmetic, by utilizing results in [1].

Ketonen and Solovay have introduced a sequence $\{G_\alpha\}_{\alpha \leqq \varepsilon_0}$ of functions similarly as Wainer's $\{F_\alpha\}_{\alpha \leqq \varepsilon_0}$. (In [1], $G_\alpha$ is written as $F_\alpha$.) The functions $G_\alpha$ are defined inductively as follows;

$$G_0(x) = x + 1,$$

$$G_{\beta+1}(x) = G_\beta^{x+1}(x),$$

$$G_\sigma(x) = G_{\{\sigma\}(x)}(x) \quad \text{if } \sigma \text{ is a limit ordinal.}$$

Then, we can show that for each $\alpha < \varepsilon_0$ and each $x < \omega$,

$$G_\alpha(x) \leqq F_\alpha(x) \leqq G_{\alpha+1}(x).$$

The next two propositions proved in [1] are essential in the following discussion.

Proposition 4.1 Let $n \geqq 2$, $c \geqq 2$ and $k \geqq 4$. Then,

$$\sigma_{n,c}(k) \leqq G_{\omega_{n-2}(c+5)}(k).$$

Proposition 4.2 Let $n \geqq 2$. For any weakly monotone increasing function f, f is dominated by $\sigma_{n,c}$ for some c if and only if f is dominated by $G_\alpha$ for some $\alpha < \omega_{n-1}$.

We call the relation $[x,y] \xrightarrow[*]{} (w+1)^w_z$, the *Ramsey relation*, and the relation $\sigma_{w,z}(x) = y$, the *strong Ramsey relation*. We can give an alternative proof of the following result by Paris [5].

Theorem 4.3 For $n \geqq 2$, if $P(x,z,y)$ is a formula containing only bounded quantifiers which represents the Ramsey relation $[x,y] \xrightarrow[*]{} (n+1)^n_z$, then

1) $PA_n^* \vdash \forall x \forall z \exists y P(x,z,y)$,

2) $PA_{n-1}^* \nvdash \forall x \forall z \exists y P(x,z,y)$.

Proof. 1) Let $\delta_n(x) = \sigma_{n,L(x)}(K(x))$. (We take $J:\omega\times\omega \longrightarrow \omega$, primitive recursive bijection such that $J(K(x),L(x)) = x$, $K(x) \leq x$, $L(x) \leq x$.) By Proposition 4.1,

$$\delta_n(x) \leq \sigma_{n,x}(x) \leq \sigma_{n,x+2}(x+4) \leq G_{\omega_{n-2}(x+7)}(x+4) \leq G_{\omega_{n-1}}(x+7)$$
$$\leq F_{\omega_{n-1}}(x+7).$$

So, $\delta_n(w) = \mu y \leq F_{\omega_{n-1}}(w+7)[P(K(w),L(w),y)]$. This means that $\delta_n \in \mathcal{F}_{\omega_{n-1}}$, because $F_{\omega_{n-1}} \in \mathcal{F}_{\omega_{n-1}}$. By Theorem 3.5, $PA_n^* \vdash \forall w \exists y P($ $K(w),L(w),y)$. So, $PA_n^* \vdash \forall x \forall z \exists y P(x,z,y)$.

2) Suppose $PA_{n-1}^* \vdash \forall x \forall z \exists y P(x,z,y)$. Then, $PA_{n-1}^* \vdash \forall u \exists y P(u,u,y)$. Let $\gamma_n(u) = \mu y P(u,u,y)$, i.e. $\gamma_n(u) = \sigma_{n,u}(u)$. Then, $\gamma_n$ is $\omega_{n-1}(m)$-ordinal recursive for some $m < \omega$ by Theorem 3.5. So, $\gamma_n \in \mathcal{F}_\beta$ for some $\beta < \omega_{n-1}$ by Proposition 2.3. By Proposition 2.2 1), $\gamma_n$ is dominated by $F_{\beta+1}$, and therefore $\gamma_n$ is dominated by $G_{\beta+2}$. Thus, $\gamma_n$ is dominated by $\sigma_{n,c}$ for some c by Proposition 4.2. Hence, there exists a k such that for every $u \geq k$,

$$\sigma_{n,u}(u) = \gamma_n(u) < \sigma_{n,c}(u).$$

Let $d = \max\{c+1,k\}$. Then,

$$\sigma_{n,d}(d) < \sigma_{n,c}(d).$$

This contradicts that $\sigma_{n,c}(u)$ is monotone increasing with respect to c. Therefore, $PA_{n-1}^* \nvdash \forall x \forall z \exists y P(x,z,y)$.

Corollary 4.4 1) For any $\Sigma_0$-representation of the Ramsey relation, $PA^* \nvdash \forall w \forall x \forall z \exists y([x,y] \xrightarrow[*]{} (w+1)_z^w)$.

2) For any $\Sigma_1$-representation of the strong Ramsey relation, $PA^* \nvdash \forall w \forall x \forall z \exists y(\sigma_{w,z}(x) = y)$.

Theorem 4.5 There exists a $\Sigma_1$-formula $P(x,z,y,w)$ such that for each $n \geqq 2$,

1) $\sigma_{n,z}(x) = \mu y P(x,z,y,n)$,

2) $PA_n \vdash \forall x \forall z \exists ! y P(x,z,y,n)$,

3) $PA_{n-1} \nvdash \forall x \forall z \exists ! y P(x,z,y,n)$.

Proof. Similarly as Theorem 4.3, we can obtain that

$$\sigma_{w,z}(x) = \mu y \leqq F_{\omega_{n-1}}(J(x,z)+7)[f^*(x,z,y,w) = 0],$$

where f* denotes the characteristic function of the Ramsey relation. Define

$$j(x,z,w,v) = \mu y \leqq v[f^*(x,z,y,w) = 0].$$

Then, j is primitive recursive. Next, by the proof of Theorem 3.1 (cf. §4 in [3]), we can take a $\Sigma_1$-formula $R(v,x,y)$,

(1)' $F_{ord(v)}(x) = \mu y R(v,x,y)$,

(2)' $PA_n \vdash \forall x \exists ! y R(\overline{g(n)},x,y)$,

where $g(n)$ is the number such that $ord(g(n)) = \omega_{n-1}$. Now we define

$$P(x,z,y,w) \equiv \mu v(R(g(w),J(x,z)+7,v) \wedge j(x,z,w,v) = y).$$

Then, we can affirm 1) and 2) by using (1)' and (2)'. We can show that for $n \geqq 2$, and for any $\Sigma_1$-representation of the strong Ramsey relation,

$$PA^*_{n-1} \nvdash \forall x \forall z \exists ! y(\sigma_{n,z}(x) = y),$$

similarly as Theorem 4.3. Thus, $PA_{n-1} \nvdash \forall x \forall z \exists ! y P(x,z,y,n)$.

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