数理解析研究所講究録
第 545 巻 1985 年 138-151

Yang-Mills equations and holomorphic structures of vector bundles

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1. Main subject of Yang-Mills theory is to study variational problem associated to connections on a principal fibre bundle. Yang-Mills functional is defined over a set of connections on a given bundle in terms of square norm of curvature. Euler-Lagrange equation of this functional is called Yang-Mills equation. With respect to this equation we have two problems;

1. when are there solutions ?
2. what sort of geometric structure does the moduli space (parameter space) of solutions admit ?

In this lecture we discuss these over a four-dimensional manifold, especially over a complex 2-dim complex manifold with positive definite metric.

2. Let P be a principal fibre bundle with gauge group $SU(n)$ over a compact oriented 4-dim manifold M with metric h.

Definition. A connection on P is a system $A = \{A_\alpha, A_\beta, \ldots\}$ where each A_α is an $\mathfrak{su}(n)$-valued 1-form defined over a trivializing neighborhood U_α of P such that these A_α's satisfy over $U_\alpha \cap U_\beta \neq \phi$

$$(*) \qquad A_\beta = g^{-1}\cdot dg + g^{-1}.A_\alpha\cdot g.$$

Here $g = g_{\alpha\beta}$ denotes the transition function of P: $U_\alpha \cap U_\beta \xrightarrow{C^\infty} SU(n)$ and dot $\cdot$ means multiplication of matrices.

We have another global definition of connection, equivalent to the above. But we adopt the above for a convenience.

For each Lie algebra valued function on M the covariant derivative ∇_A is defined: $\nabla_A\phi = d\,\phi + [A,\phi] = d\,\phi + A\cdot\phi - \phi\cdot A$ and we have also the covariant exterior derivative $d_A\,\psi = d\,\psi + [A\wedge\psi] = d\,\psi + A \wedge \psi + (-1)^{p+1}\cdot \psi \wedge A$ (ψ is a p-form).

A 2-form $F(A) = dA + A\wedge A$ is said curvature form of A. The curvature form takes values in $\mathfrak{su}(n)$. In local expression F(A) is represented by $F(A) = \frac{1}{2}\sum_{\mu,\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ ($F_{\mu\nu} = -F_{\nu\mu}$), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

We have an identity $d_A\nabla_A\phi = [F(A),\phi]$, called Ricci formula which will be used later to define an elliptic complex.

<u>Definition</u>. Over the set of connections $\mathcal{C}_P$ Yang-Mills functional $\mathcal{YM}$ is defined by

$$\mathcal{YM}(A) = \frac{1}{2} \int_M |F(A)|^2 \, dv,$$

where $|F(A)|^2 = \frac{1}{2} \sum h^{\mu\sigma} h^{\nu\tau} (- \operatorname{Tr} F_{\mu\nu} \cdot F_{\sigma\tau})$ ($(h^{\sigma\tau})$ is the inverse matrix of the metric $(h_{\mu\nu})$).

Its Euler-Lagrange equation is written locally by

$$\sum_{\sigma,\tau} h^{\sigma\tau} \nabla_\sigma F_{\tau\mu} = \sum h^{\sigma\tau} (D_\sigma F_{\tau\mu} + [A_\sigma, F_{\tau\mu}]) = 0$$

where D_σ denotes the covariant derivative with respect to Levi-Civita connection of M.

<u>Definition</u>. A connection is called Yang-Mills if it is a critical point of $\mathcal{YM}$, that is, it is a solution to the Yang-Mills equation.

We have assumed that M is oriented and 4-dimensional. Then we have the operator $*$ which is an involutive endomorphism of the bundle of 2-forms Λ^2. Hence Λ^2 splits into $\Lambda^2_+ + \Lambda^2_-$ corresponding to eigenvalues. A 2-form α is said (anti-)self-dual if $\alpha \in \Lambda^2_+$ (or Λ^2_-). Curvature form splits also into $F(A) = F_+ + F_-$.

<u>Definition</u>. A connection is said self-dual (or anti-self-dual) if it is a solution of self-dual equation $F_- = 0$ (or anti-self-dual equation $F_+ = 0$).

From Chern-Weil homomorphism theorem $\mathcal{YM}$ has the lowest bounds;

$$\mathcal{YM}(A) = \frac{1}{2}\int_M \{|F_+|^2 + |F_-|^2\} dv \geqq \frac{1}{2}\left|\int_M \{|F_+|^2 - |F_-|^2\} dv\right|$$

for all $A \in \mathcal{C}_P$ and the integral of the right hand side represents $(-c)\, c_2(P)[M]$ for a universal constant $c > 0$ and the second Chern number $c_2(P)[M]$. Then the equality " = " holds if and only if A is self-dual or anti-self-dual according to $c_2(P)[M] \leqq 0$ or $\geqq 0$.

Thus we have

Proposition 2.1. Every self-dual connection (or anti-self-dual connection) minimizes $\mathcal{YM}$. Hence it is a Yang-Mills connection.

Note. The Yang-Mills equation is a non-linear second order equation with respect to unknown connection. But the (anti-) self-duality equation is of first order.

For Problem 1 $c_2(P)[M] \geqq 0$ is a necessary condition for P to admit an anti-self-dual connection. Conversely we have

Theorem (Taubes [11]). If $H^2_+(M) = \{$ harmonic 2-form which is anti-self-dual $\}$ vanishes, then P with $c_2(P)[M] \geqq 0$ admits an irreducible anti-self-dual connection provided that a principal fibre bundle over S^4 with identical Chern number carries an irreducible anti-self-dual connection.

The irreducibility of connection will be defined at § 4. It is seen that a connection is irreducible if and only if the gauge group can not be compressed into $S(U(n_1)\times U(n_2))$, ($n_1 + n_2 = n$).

With respect to Problem 2 we have by fundamental ideas of Atiyah-Hitchin-Singer [2] and also of Donaldson [3] the moduli space of anti-self-dual connections carries a structure of real analytic set. Donaldson applied the structure to 4-dimensional topology.

3. Assume that the base space M is a complex surface with complex coordinates z^1, z^2 and of a Kähler metric $h = \sum_{\mu,\nu} h_{\mu\bar{\nu}}(z^1,z^2)dz^{\mu}.dz^{\bar{\nu}}$. Then the space M carries an orientation induced naturally from the complex structure.

Fact(Atiyah[1],Itoh[6]) A connection over a complex surface is anti-self-dual if and only if $F(A)$ is of type (1,1) and primitive, that is, $F(A)$ is written by $F(A) = \sum_{\mu,\nu} F_{\mu\bar{\nu}}\, dz^{\mu}\wedge dz^{\bar{\nu}}$ and satisfies

$$\sum_{\sigma,\tau} h^{\sigma\bar{\tau}} F_{\sigma\bar{\tau}} = 0.$$

This fact is a starting point of research of the moduli space $\mathcal{M}_a$ over a complex surface.

Koiso generalizes this anti-self-duality to higher dimensional complex manifolds [9].

Theorem (Kobayashi[8]). If the bundle P admits an anti-self-dual connection, then the associated vector bundle $E = P \times_{\rho} \mathbb{C}^n$ must be a (semi-)stable holomorphic vector bundle in the sense of Mumford and Takemoto.

Recently Donaldson proved in his preprint [4] that over an algebraic surface the stability is indeed a sufficient condition.

Over a Kähler surface the following is an answer to Problem 2.

Theorem [7]. The moduli space of irreducible anti-self-dual connections admits over a compact Kähler surface a structure of complex analytic set, that is, it is a zero point set of local several holomorphic functions.

Remarks. 1. An anti-self-dual connection is just an Einstein Hermitian structure of the associated vector bundle in the sense of Kobayashi.

2. Over an algebraic surface the moduli of stable holomorphic structure of a smooth vector bundle is a quasi-projective variety (Maruyama[10]).

Example. In the case of $M = P_2(\mathbb{C})$, $SU(2)$ and $c_2(P) = k$ (> 1) the moduli space is a complex manifold without singularity of $\dim_{\mathbb{C}}$ $4k - 3$.

The theorem is first conjectured by Atiyah.

A sketch of proof of Theorem will be given in § 5.

4. To define precisely the moduli space over a general 4-manifold we need notion of gauge transformations. A bundle automorphism f of P (that is, $f ; P \longrightarrow P$ is a diffeomorphism and satisfies $f(ua) = f(u)a$) which induces on M the identity transformation is called a guage transformation. A gauge transformation can be identified with a global section of the automorphism bundle $G_P = P \times_{conj} SU(n)$. A gauge transformation operates on a connection A to induce a new connection $f^*A = f^{-1}.df + f^{-1}.A.f$ with curvature form $F(f^*A) = f^{-1}.F(A).f$. Therefore $\mathcal{YM}$ is gauge-invariant, because $|F(f^*A)|^2 = |F(A)|^2$ and an anti-self-dual connection is transformed into new one which is also anti-self-dual. But in physical and geometrical meaning they represent the same thing.
So we have a quotient space of the set of anti-self-dual connections modulo gauge transformations and call it moduli space $\mathcal{M}_a$ of anti-self-dual connections:

$$\{ \text{ anti-self-dual connections } \} \xrightarrow{\pi} \mathcal{M}_a .$$

The adjoint representation of the gauge group induces a vector bundle associated to P which we call the adjoint bundle and denote it by $\mathfrak{g}_P$.

Almost everything of Yang-Mills theory must be discussed over this bundle because curvature forms are $\mathfrak{g}_P$-valued 2-forms and an infinitesimal gauge transformation is just a global section of $\mathfrak{g}_P$ etc.

Now we assume that a connection A is anti-self-dual over a general compact oriented Riemannian 4-manifold M. By the aid of the Ricci formula we have the sequence:

$$(*) \quad 0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{\nabla_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^+ = P_+ \circ d_A} \Omega^2_+(\mathfrak{g}_P) \longrightarrow 0$$

is an elliptic complex, that is, $d_A^+ \circ \nabla_A = 0$ and the symbol sequence of this is exact at any covector $\xi \neq 0$. Here $\Omega^k(\mathfrak{g}_P)$ denotes the space of smooth $\mathfrak{g}_P$-valued k-forms (k = 1, 2) and $\Omega^2_+(\mathfrak{g}_P)$ the space of $\mathfrak{g}_P$-valued self-dual 2-forms. Further P_+ is the orthogonal projection of Λ^2 onto Λ^2_+. Cohomologies H^0, H^1 and H^2_+ associated to (*) are all finite dimensional. The Atiyah-Singer index theorem shows that the index of (*), that is, $\dim H^0 - \dim H^1 + \dim H^2_+$ is given by $\int_M$ { characteristic classes of M and P }

According to either $H^0 = 0$ or $H^0 \neq 0$ and either $H^2_+ = 0$ or $H^2_+ \neq 0$ we have four cases:

(i) $H^0 = 0$ and $H^2_+ = 0$; the moduli space $\mathcal{M}_a$ is smooth at $[A] = \pi(A)$ and has dimension equal to $-$ index (*)

(ii) $H^0 = 0$ and $H^2_+ \neq 0$; $\mathcal{M}_a$ is at [A] a zero point set of a C^ω-map Φ from a 0-neighborhood of H^1 to H^2_+

(iii) $H^0 \neq 0$ and $H^2_+ = 0$; $\mathcal{M}_a$ is written as a Γ_A-quotient of a 0-neighborhood of H^1

(iv) $H^0 \neq 0$ and $H^2_+ \neq 0$; $\mathcal{M}_a$ has at [A] a structure of Γ_A-quotient of zero point set of a C^ω-map Φ from a 0-neighborhood of H^1 to H^2_+

Here Γ_A is the isotropy group of A, that is, Γ_A = { gauge transformations f; $f^*A = A$ }.

We call a connection to be irreducible if $\operatorname{Ker} \nabla_A = 0$ and call a connection reducible when it is not irreducible. Then by combining (i) —— (iv) we have

<u>Proposition</u> 4.1. The moduli space { $[A] \in \mathcal{M}_a$; A is irreducible } is a real analytic set and the whole moduli space is around reducible connection a Γ_A-quotient of a real analytic set.

Those properties of $\mathcal{M}_a$ stated above are based on the following facts:

<u>Fact</u> 1. Since $F(A+\alpha) = F(A) + d_A\alpha + \alpha\wedge\alpha \quad (\alpha \in \Omega^1(\mathfrak{g}_P))$, for a fixed anti-self-dual connection A a connection $A + \alpha$ is anti-self-dual if and only if $d_A^+\alpha + P_+(\alpha\wedge\alpha) = 0$.

Fact 2. Because $\mathrm{Ker}\ d_A^* = \{ \alpha \in \Omega^1(\mathfrak{g}_P);\ d_A^*\alpha = 0\}$ is transversal in $\mathcal{C}_P$ to the orbit through A of gauge transformations, where d_A^* is the formal adjoint of d_A, $\mathcal{M}_a$ has a neighborhood homeomorphic to a slice $\mathcal{S}_A = \{ \alpha \in \Omega^1(\mathfrak{g}_P);\ d_A^*\alpha = 0,\ d_A^+\alpha + P_+(\alpha\wedge\alpha) = 0 \}$ or a Γ_A- quotient $\mathcal{S}_A/\Gamma_A$ according to the irreducibility of A.

Fact 3. We combine these facts to derive so-called Kuranishi's map from the slice to H^1. Define a map $\Phi;\ \Omega^1(\mathfrak{g}_P) \longrightarrow \Omega^1(\mathfrak{g}_P)$, $\Phi(\alpha) = \alpha + (d_A^+)^* \circ G_A(P_+\alpha\wedge\alpha)$ where G_A is the Green operator of the Laplacian $d_A^+ \circ (d_A^+)^*$.
Since $\frac{d}{d\alpha}\Phi|_{\alpha=0} =$ identity, Φ has an inverse over a neighborhood of 0. We give a map $\psi;\ \{ \beta \in \mathrm{Ker}\ d_A^*;\ \|\beta\|_{L^2_k} < \varepsilon \} \longrightarrow H^2$ by $\psi(\beta) = -$ harmonic part of $P_+\alpha\wedge\alpha$ for $\alpha = \Phi^{-1}(\beta)$ to assert that $\Phi;\ \mathcal{S}_A \longrightarrow \{ \beta \in H^1;\ \|\beta\|_{L^2} < \varepsilon,\ \psi(\beta) = 0 \}$ is a homeomorphism.

5. Now we assume again in this section that the base space is Kähler. In order to give the moduli space $\mathcal{M}_a$ a complex manifold structure we need finer considerations than the discussion of § 4. For this purpose we complexify every real structures, SU(n) to SL(n;**C**), the principal fibre bundle, the adjoint bundle, gauge transformations etc, except but the transition functions of the bundle.
A connection A then splits into the (1,0)-part A' and the (0,1)-part A" so that A'' satisfies

$$A''_\beta = g^{-1}.\bar{\partial}g + g^{-1}.A''_\alpha.$$

Definition. A system $\{\tilde{A}_\alpha, \tilde{A}_\beta, \ldots\}$ of locally defined $\mathfrak{sl}(n;\mathbb{C})$-valued (0,1)-forms compatible with the transition functions of P,

$$\tilde{A}_\beta = g^{-1}.\bar{\partial}g + g^{-1}.\tilde{A}_\alpha.g \qquad \text{on} \quad U_\alpha \cap U_\beta,$$

is called a (0,1)-connection.

A (0,1)-connection $\tilde{A}$ induces the partial covariant derivative $\bar{\partial}_{\tilde{A}}: \Omega^0(\mathfrak{g}_P^{\mathbb{C}}) \longrightarrow \Omega^{0,1}(\mathfrak{g}_P^{\mathbb{C}})$, $\phi \longmapsto \partial\phi + [A, \phi]$ and also the covariant exterior derivative $\bar{\partial}_A: \Omega^{p,q}(\mathfrak{g}_P^{\mathbb{C}}) \longrightarrow \Omega^{p,q+1}(\mathfrak{g}_P^{\mathbb{C}})$ where $\mathfrak{g}_P^{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}_P$.

Definition. A (0,1)-connection $\tilde{A}$ is said holomorphic if its curvature $F(\tilde{A}) = \bar{\partial}\tilde{A} + \tilde{A} \wedge \tilde{A}$ vanishes.

Remark. Each holomorphic (0,1)-connection induces by integrability condition a holomorphic structure on $\mathfrak{g}_P^{\mathbb{C}}$ such that $\bar{\partial}_A$ is just $\bar{\partial}$-operator(Atiyah-Hitchin-Singer [2] and Griffiths [5]).

Note. Holomorphic (0,1)-connection can be also defined over a higher dimensional complex manifold.

In a way similar to the case of anti-self-dual connections we can define moduli space $\mathcal{M}_h$ of holomorphic (0,1)-connections with respect to complex gauge transformations.

Since each anti-self-dual connection A has curvature form of type $(1,1)$, its $(0,1)$-component A'' gives automatically a holomorphic $(0,1)$-connection because $F(A'')$ is just the $(0,2)$-component of full curvature form $F(A)$. Then we have a canonical map ϕ from $\mathcal{M}_a$ to $\mathcal{M}_h$ by assigning $[A'']$ to $[A]$.

When $\tilde{A}$ is holomorphic, the sequence (**) is an elliptic complex

$$(**) \quad 0 \longrightarrow \Omega^0(\mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{\bar\partial_{\tilde{A}}} \Omega^{0,1}(\mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{\bar\partial_A} \Omega^{0,2}(\mathfrak{g}_P^{\mathbb{C}}) \longrightarrow 0.$$

<u>Proposition</u> 5.1. The moduli space of irreducible holomorphic $(0,1)$-connections is a complex analytic set.

Here a $(0,1)$-connection is called irreducible when $\mathrm{Ker}\,\{\bar\partial_{\tilde{A}}: \Omega^0(\mathfrak{g}_P^{\mathbb{C}}) \longrightarrow \Omega^{0,1}(\mathfrak{g}_P^{\mathbb{C}})\} = 0$. This proposition is shown by a way similar to the anti-self-dual case, while the complex analyticity is assured by the fact that the Kuranishi's map is in this case holomorphic.

<u>Proposition</u> 5.2. i) The spaces $\mathcal{M}_a$ and $\mathcal{M}_h$ of generic connections (that is, connections with $H^0 = 0$ and $H^2 = 0$, respectively) are smooth manifolds with same real dimension $2\{c_2(\mathfrak{g}_P^{\mathbb{C}}) - \dim SU(n)\cdot p(M)\}$ where $p(M)$ is the arithmetic genus of M and ii) the canonical map ϕ is smooth and of maximal rank over $\mathcal{M}_{a,gen} = \{[A] \in \mathcal{M}_a;\ A \text{ is generic}\}$

Note. We use the moment map due to Donaldson [4] to conclude that ϕ is one-to-one over $\mathcal{M}_{a,ir} = \{ [A] \varepsilon \mathcal{M}_a;$ A is irreducible $\}$

As a consequence of these results the moduli space $\mathcal{M}_{a,ir}$ has a structure of complex analytic set which around a generic one is smooth and is sigular at one with $H_+^2 = 0$. This is just the statement of Theorem.

We remark that around a reducible anti-self-dual connection the moduli space $\mathcal{M}_a$ is a Γ_A-quotient of a real analytic set.

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